

Efficient Computational Algorithms

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Monte Carlo and MCMC

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Outline

- 1 Monte Carlo
- 2 MCMC
- 3 Conclusion

Table of Contents

1 Monte Carlo

2 MCMC

3 Conclusion

Monte Carlo

Random numbers generator: A mechanism for producing a sequence of random variables U_1, U_2, \dots with the following two properties:

- each U_i follows the uniform distribution between 0 and 1
- the U_i are mutually independent

Monte Carlo

The idea behind Monte Carlo sampling is to use a random number generator in order to sample from the desired distribution. In this presentation the Inverse Transform Method will be considered.

Inverse Transform Method

Choosing the uniform distribution is handy because it enables us to generate random sample from any other distribution using the Inverse Transform Method.

- First simulate observations $U_i \sim U[0, 1]$
- Second calculate $Y = F^{-1}(U_i)$, where F^{-1} is the quantile function of the desired distribution

It follows that the random variable $Y_i = F^{-1}(U_i)$ is distributed according to F

Inverse Transform Method

Before illustrating the procedure, it is important to remind a useful property of the uniform distribution $[0,1]$:

$$F(X) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1 \\ 1 & x > 1 \end{cases}$$

Hence the property $F(c) = P(X) = c$

Inverse Transform Method

To show why $Y_i \sim F$ consider:

$$\begin{aligned}P(Y_i \leq c) &= P(F^{-1}(U_i) \leq c) \\P(F^{-1}(U_i) \leq c) &= P(U_i \leq F(c))\end{aligned}$$

$$P(U_i \leq F(c)) = F(c)$$

variable

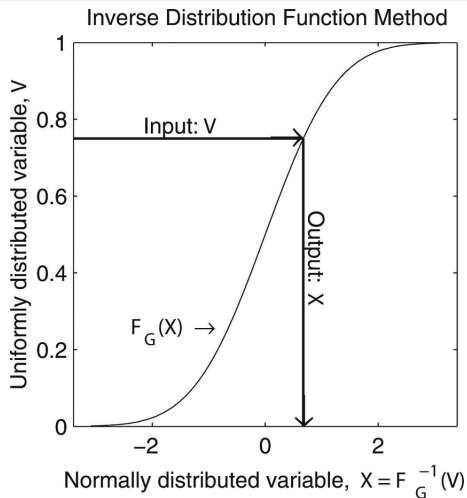
following the uniform distribution $[0,1]$

Definition of Y

Apply F to both side of the disequality

Use the fact that U_i is a random

Inverse Transform Method



Random sample from an exponential distribution

Consider the exponential distribution $f(y) = \frac{e^{-y/\theta}}{\theta}$
its cumulative distribution is $F(x) = 1 - e^{-y/\theta}$
Our goal is to obtain its quantile distribution F^{-1}

$$F(y) = 1 - e^{-y/\theta} = F(F^{-1}(U)) = U$$

$$U = 1 - e^{-y/\theta}$$

$$1 - U = e^{-y/\theta}$$

$$\log(1 - U) = \frac{-y}{\theta}$$

$$y = -\log(1 - U)\theta$$

Plug into y the observation from a random uniform $[0,1]$ we get a random sample from an exponential with parameter θ

approximate Inverse Transform Method Normal

Some distributions do not even have a close form quantile function.

approximate Inverse Transform Method Normal

Nevertheless, we can approximate it and apply the same logic to the approximation of the quantile function.

approximate Inverse Transform Method Normal

```
Input:  $u$  between 0 and 1
Output:  $x$ , approximation to  $\Phi^{-1}(u)$ .
 $y \leftarrow u - 0.5$ 
if  $|y| < 0.42$ 
     $r \leftarrow y * y$ 
     $x \leftarrow y * (((a_3 * r + a_2) * r + a_1) * r + a_0) /$ 
         $((((b_3 * r + b_2) * r + b_1) * r + b_0) * r + 1)$ 
else
     $r \leftarrow u$ ;
    if  $(y > 0)$   $r \leftarrow 1 - u$ 
     $r \leftarrow \log(-\log(r))$ 
     $x \leftarrow c_0 + r * (c_1 + r * (c_2 + r * (c_3 + r * (c_4 +$ 
         $r * (c_5 + r * (c_6 + r * (c_7 + r * c_8))))))$ 
    if  $(y < 0)$   $x \leftarrow -x$ 
return  $x$ 
```

algorithm

Table of Contents

1 Monte Carlo

2 MCMC

3 Conclusion

MCMC

MCMC

This section

MCMC

This section

MCMC

This section

Metropolis Hastings

This section

Metropolis Hastings

This section

Metropolis Hastings

This section

HMC

This section

HMC

This section

HMC

This section

Table of Contents

1 Monte Carlo

2 MCMC

3 Conclusion

Conclusion

Thank you