Efficient Computational Algorithms

Luca Pernigo

Università della Svizzera italiana[1]

Monte Carlo and MCMC
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Outline

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Monte Carlo

Random numbers generator: A mechanism for producing a sequence of random variables $U_1, U_2, ...$ with the following two properties:

- each U_i follows the uniform distribution between 0 and 1
- the U_i are mutually independent

Monte Carlo

The idea behind Monte Carlo sampling is to use a random number generator in order to sample from the desired distribution. In this presentation the Inverse Transform Method will be considered.

Inverse Transform Method

Choosing the uniform distribution is handy because it enables us to generate random sample from any other distribution using the Inverse Transform Method.

- First simulate observations $U_i \sim U[0, 1]$
- Second calculate $Y = F^{-1}(U_i)$, where F^{-1} is the quantile function of the desired distribution

It follows that the random variable $Y_i = F^{-1}(U_i)$ is distributed according to F

Inverse Transform Method

Before illustrating the procedure, it is important to remind a useful property of the uniform distribution [0,1]:

$$F(X) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1 \\ 1 & x > 1 \end{cases}$$

Hence the property $F(c) = P(X) = c$

Inverse Transform Method

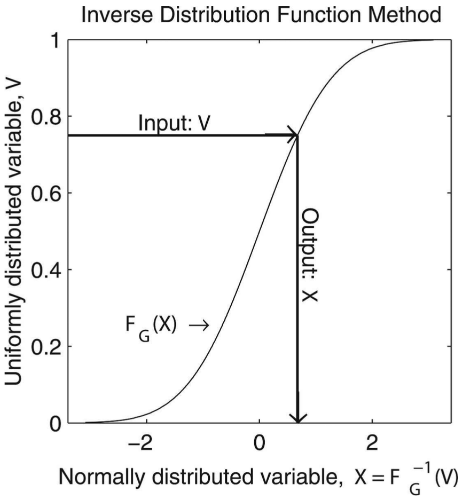
To show why $Y_i \sim F$ consider:

$P(Y_i \leq c) = P(F^{-1}(U_i) \leq c)$ Definition of Y
$P(F^{-1}(U_i) \leq c) = P(U_i \leq F(c))$ Apply F to both side of the disequality

$P(U_i \leq F(c)) = F(c)$

Finally, use the fact that U_i is a random variable following the uniform distribution [0,1]

Inverse Transform Method



Random sample from an exponential distribution

Consider the exponential distribution $f(y) = \frac{e^{-y/\theta}}{\theta}$
its cumulative distribution is $F(x) = 1 - e^{-y/\theta}$
Our goal is to obtain its quantile distribution F^{-1}

$$F(y) = 1 - e^{-y/\theta} = F(F^{-1}(U)) = U$$
$$U = 1 - e^{-y/\theta}$$
$$1 - U = e^{-y/\theta}$$
$$log(1 - U) = \frac{-y}{\theta}$$
$$y = -log(1 - U)\theta$$

Pluggin into y the observation from a random uniform [0,1] we get a random sample from an exponential with parameter θ

approximate Inverse Transform Method Normal

Some distributions do not even have a close form quantile function.

approximate Inverse Transform Method Normal

Nevertheless, we can approximate it and apply the same logic to the approximation of the quantile function.

approximate Inverse Transform Method Normal

Input: u between 0 and 1
Output: x, approximation to $\Phi^{-1}(u)$.
$y \leftarrow u - 0.5$
if $|y| < 0.42$
    $r \leftarrow y * y$
    $x \leftarrow y * (((a_3 * r + a_2) * r + a_1) * r + a_0) /$
        $((((b_3 * r + b_2) * r + b_1) * r + b_0) * r + 1)$
else
    $r \leftarrow u;$
    if $(y > 0)$ $r \leftarrow 1 - u$
    $r \leftarrow \log(-\log(r))$
    $x \leftarrow c_0 + r * (c_1 + r * (c_2 + r * (c_3 + r * (c_4 +$
        $r * (c_5 + r * (c_6 + r * (c_7 + r * c_8)))))))$
    if $(y < 0)$ $x \leftarrow -x$
return x

algorithm

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Conclusion

Thank you