Extragalactic Astronomy

Equilibrium of a Plummer's Sphere - Numerical Study

Efrem Maconi
814088

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1 About the simulations

The simulations are made through the Joshua E. Barnes' *Tree Code* (for more details see `https://www.ifa.hawaii.edu/~barnes/treecode/treeguide.html`).

The system is initialized through the jupyter notebook *Plummer Sphere Equilibrium - Initial Conditions Code (Barnerstreecode)*. The execution of that notebook give as output two files,

```
initial_conditions_file.txt
system_properties.txt,
```

where the first one in used by the *Barnes' Tree Code* and contains the number of particles, the number of dimensions, the initial simulation time and the particles' mass, initial positions and velocities; the second one is used by the jupyter notebook made for the analysis.

The execution of the initial condition notebook has to be followed by the run of the *Barnes' Tree Code* compiled from the terminal writing, for example,

```
./treecode in=initial_conditions_file.txt out=output_data.txt dtime=value
 eps=value theta=value options=out-phi tstop=value dtout=value
 > system_description.txt
```

where it is needed to specify the parameters' values. In the *Plummer Sphere Equilibrium - Initial Conditions Code (Barnerstreecode)* notebook and in this relation is present a section that gives hints on the choice of the parameters' values.

2 Introduction

We study how a Plummer's Sphere, with a given distribution of velocities, evolves over time in order to verify if it's at equilibrium or not.

The sphere has a mass M and a scale radius b and it is approximated with a finite number of particles, $Nparticles$, which are equal mass ($m = M/Nparticles$). The particles are distributed as the Plummer's Density,

$$\rho = \frac{3M_{tot}}{4\pi b^3}\Big(\frac{1}{1+\frac{r^2}{b^2}}\Big)^{\frac{5}{2}}, \tag{1}$$

from which we can compute the probability distribution function of the radius r,

$$pdf(r)dr = \frac{3r^2}{b^3}\Big(1+\frac{r^2}{b^2}\Big)^{-\frac{5}{2}} \tag{2}$$

whose integration gives the $cdf(r)$.

Since the problem is spherical symmetric for the angular components, θ and ϕ, we use the probability densities

$$pdf(\phi)d\phi = \frac{d\phi}{2\pi} \tag{3}$$

$$pdf(\theta)d\theta = \frac{\sin\theta}{2}d\theta \tag{4}$$

and the relative cdfs.

From the cumulative distribution functions we can generate, through a *Monte Carlo method*, the position of our particles and, before evolving the system, we have to assign to each particle a velocity. Extracting the velocities from the distribution function

$$f(\epsilon)d\epsilon = \left(\psi - \frac{v^2}{2}\right)^{\frac{7}{2}} v^2 dv, \tag{5}$$

where ϵ is the *relative energy* ($\epsilon = \psi - \frac{v^2}{2}$) and ψ the *relative potential* ($\psi = -\phi_{Plum}$), will initialize a Plummer Sphere which is at equilibrium and both the positions and the velocities distribution do not change over time: we are now going to verify this numerically.

3 Numerical parameters' values

3.1 dtime

The *dtime* parameter is the integration time-step and it's convenient to use a value that has an exact representation as a floating-point number; a small value of *dtime* will result into a better precision but a greater computational cost. For the simulations presented in this relation, we choose it as a fraction of the system dynamical time t_{dyn}, roughly equal to $\frac{b}{v_{circ}(r=b)} = b\left(\frac{GM_{tot}}{b}\right)^{-\frac{1}{2}}$ (set, for example, $dtime = t_{dyn}/100$).

3.2 eps

The *eps* parameter is the *Gravitational Softening Parameter*; for more details see the previous relation *Collapse of an Homogeneous Sphere - Numerical Study* where we explained in details what is *eps* and, from the analysis did, found that the better value for the *gravitational softening parameter* has to be chosen in the interval

$$10^{-1}\Delta_{mean} \leq \epsilon \leq \Delta_{mean}, \tag{6}$$

where Δ_{mean} is the mean separation between particles.

The Plummer Sphere has a spherical symmetry and an in-homogeneous density but, since the *Barners' Tree Code* is non adaptive, we need to choice a single value for *eps*. We decided to compute Δ_{mean} dividing the sphere into shells, computing the particles' mean distance in each shell and then selecting the smallest one.

3.3 theta

The *theta* parameter is the *opening angle* used to adjust the accuracy of the force calculation; values less than unity produce more accurate forces, albeit at greater computational expense. We usually set $\theta = 0.1$.

3.4 tstop

The *tstop* parameter is the time at which the N-Body integration terminates; we chosen it as a multiple of the dynamical fall time ($tstop = 10t_{dyn}$).

3.5 dtout

The *dtout* parameter is the time interval between the saving of two complete description of the system.

4 Study of the system's equilibrium

We are now going to analyze if the system remains at equilibrium or not: to do that we set the initial conditions, evolve the system, and than analyze the results at a given time. if the system is at equilibrium already from $t = 0$ the position and the velocity distributions shouldn't evolve and so be the same at any given time.

The simulation is made for a Plummer's Sphere of mass $M = 10$ and scale radius $b = 1$; unfortunately we have been able to execute only a simulation with 1000 particles (we tried to execute a simulation using 10'000 particles but the *Plummer Sphere Equilibrium - Analysis*

Code(Barnerstreecode), due to our pc performances, was too slow in analyzing the *output-data.txt*, even reducing the number of snapshots taken into consideration (in more than 10 hours there haven't been results yet)).

For our simulation ($M = 10$, $r_{scale} = 1$, $Nparticle = 1000$) we runned the Barners' treecode from the terminal with

```
./treecode in=initial_conditions_file.txt out=output_data.txt dtime=0.002
 eps=0.015 theta=0.1 options=out-phi tstop=3.0 dtout=0.01
 > system_description.txt
```

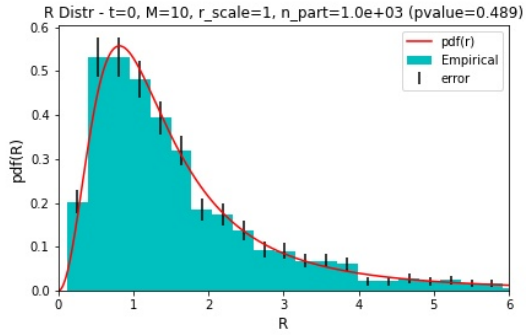
4.1 Position

4.1.1 Position Distributions and Density Function

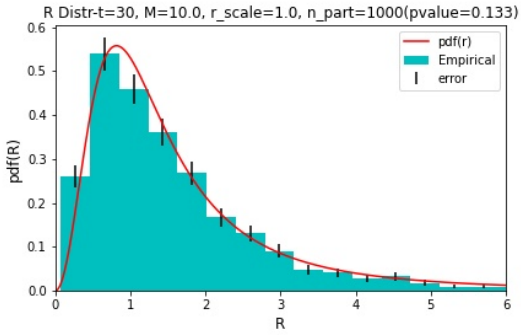
We know that if the system is at equilibrium already from $t = 0$, the position distributions should be the same at any given time; to check this we compare at $t = 0$ and at another time (in our case, $t = 10 * t_{dyn} = 3$), the empirical distributions with the theorical ones, both qualitatively and quantitatively. For the qualitative test we plot the normalized histograms of the positions and the theorical pdf; for the quantitative test we used a chi square distribution to verify how well the theoretical prediction fitted the empirical one: in each plot we reported the *p-value* computed (for other details, see the relative Section on the notebook devoted to the analysis).

From the theory we know the probability density functions of the particles' positions in spherical coordinates; their pdf have been already reported in the *Introduction* Section.

We now report the plots obtained in the simulation.

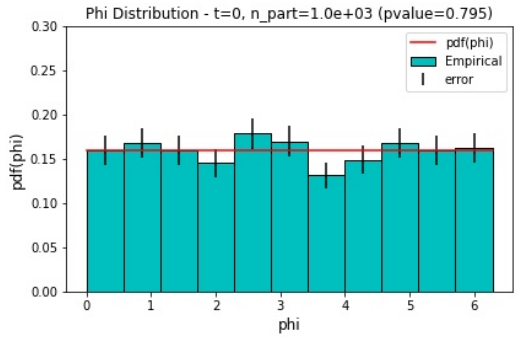


(a) t=0

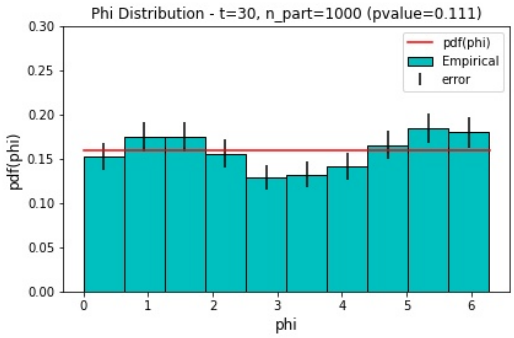


(b) t=3

Figure 1: R distribution

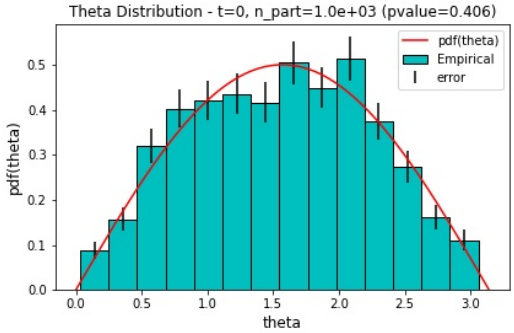


(a) t=0

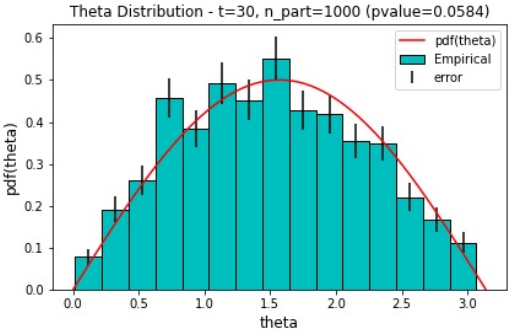


(b) t=3

Figure 2: ϕ distribution



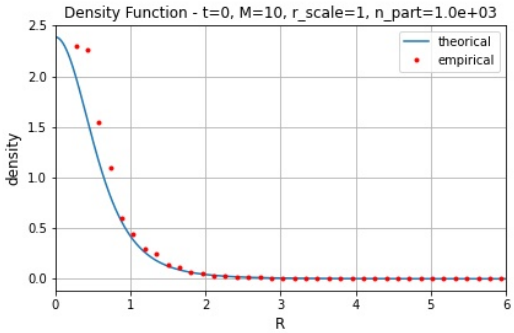
(a) t=0



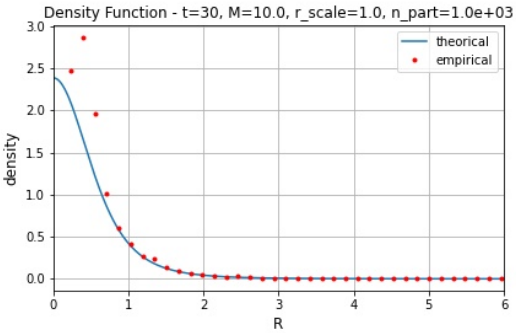
(b) t=3

Figure 3: θ distribution

We can conclude, analyzing the plots above, that the position distributions do not change over time.
We then computed the density, at $t = 0$ and $t = 3$, of the generated Plummer's Sphere and compared it qualitatively to the expected one (the functional form can be seen in the *Introduction* Section). To compute the empirical density we divide the sphere into shells, calculate the mass embedded in each shell and then divide it by the shell's volume. As can be seen in the plots below, the empirical density is, at least qualitatively, in accordance with the predicted one.



(a) t=0



(b) t=3

Figure 4: Density distribution

4.1.2 Lagrangian Radius and Sample Distance

We then analyze the evolution over time of the so called *Lagrangian Radius*, which is the radius that contains a certain percentage of the mass (for more details about their computation, see the previous relation *Collapse of an Homogeneous Sphere - Numerical Study*).

In the following plot are presented the evolution for the Lagrangian radius that embed the 20%, 40%, 60% and 80% of the mass; since the system should be at equilibrium, the Lagrangian radius should be constant at any time and this is, indeed, what we have obtained.

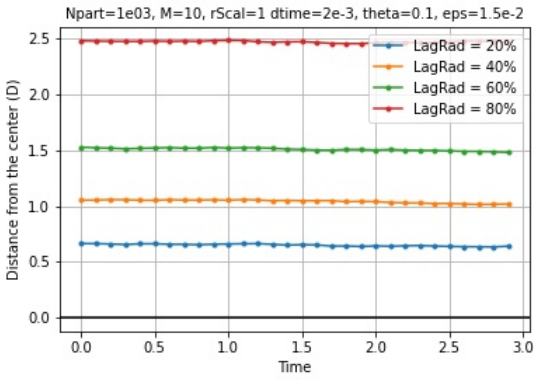


Figure 5: Lagrangian Radius

For a better visualization of the fact that the Lagrangian radius are constant, we focused our attention only on two of them, plotting for each one also a dotted horizontal line that corresponds to the mean value assumed by the Lagrangian radius between $0 < t < 3$.

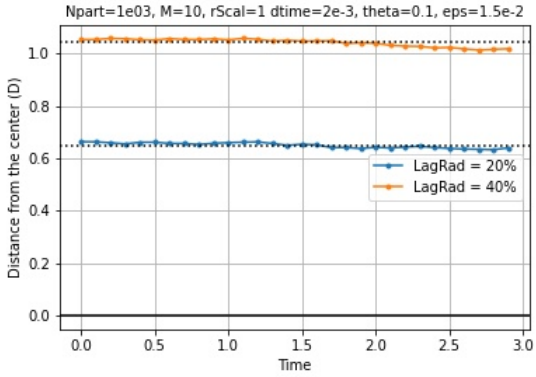


Figure 6: Lagrangian Radius and Mean

We report now the evolution over time of 5 samples' distance from the center of the sphere. You can observe that they made a periodical motion around the center (some of them, at $t = 3$, have not already computed a period).

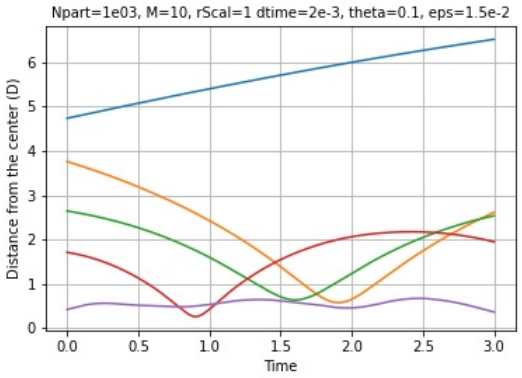


Figure 7: Distance from the center

4.1.3 Center of Mass

We computed the evolution over time of center of mass' distance respect to the centre of the sphere; in principle, since we are dealing with a spherical symmetric and isolated system, its center of mass

should be and remain in $(x, y, z) = (0, 0, 0)$: the results obtained are in agreement with what we have just stated.

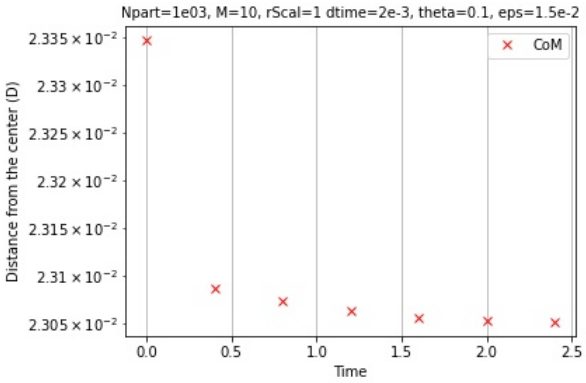


Figure 8: Center of Mass

4.2 Velocity

The initial velocities are distributed as the distribution function

$$f(\epsilon)d\epsilon = \left(\psi - \frac{v^2}{2}\right)^{\frac{7}{2}} v^2 dv, \tag{7}$$

that can be rewritten as

$$f(q)dq = \left(1 - q^2\right)^{\frac{7}{2}} q^2 dq, \tag{8}$$

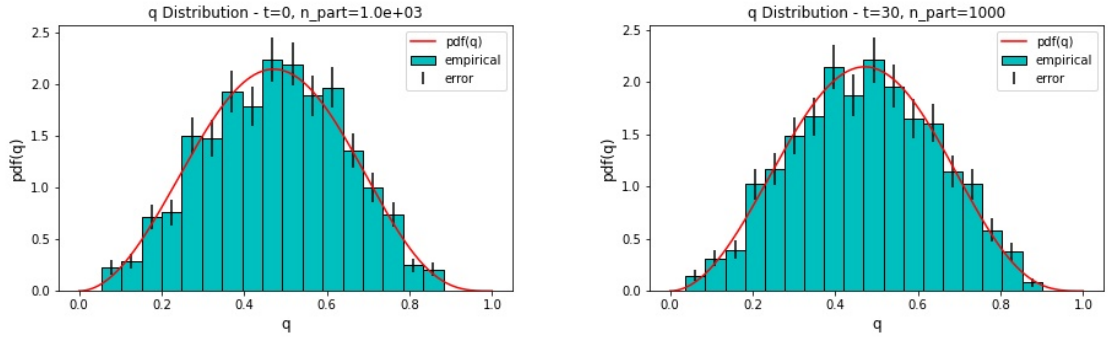
where we have introduce q, which is defined as

$$q = \frac{v}{v_{escape}} = \frac{v}{\sqrt{2\psi}}. \tag{9}$$

To assign a velocity to a particle at $t = 0$, we extract through the *Indirect Monte Carlo method* (since the integral of the q distribution can't be analytically inverted) a number distributed as q and then we find out the modulus of the velocity, $v_{mod} = v_{max} * q$ (where $v_{max} = \sqrt{2 * \psi}$). The velocities have a spherical distribution in the *velocity space* and so, knowing the modulus and generating θ and ϕ with distributions equal to the ones reported in the *Introduction* Section, we can compute the cartesian components of the velocity.

In order to find if the velocity distribution is preserved at a given time different from 0, we do the above process in the inverse way: from the Cartesian components we compute the modulus of the velocity; from the particle's relative potential we find v_{max} and, at the end, we are able to compute q.

In the following plots we reported our results: qualitatively the q distribution is preserved and so it's the velocity one.



(a) t=0

(b) t=3

Figure 9: q distribution

4.3 Energy

We report here the energy plot for our simulation. If the system is at equilibrium, its position and its velocity distributions do not evolve in time and so do the potential and the kinetic energies. We observe indeed that the energies are constant over time.

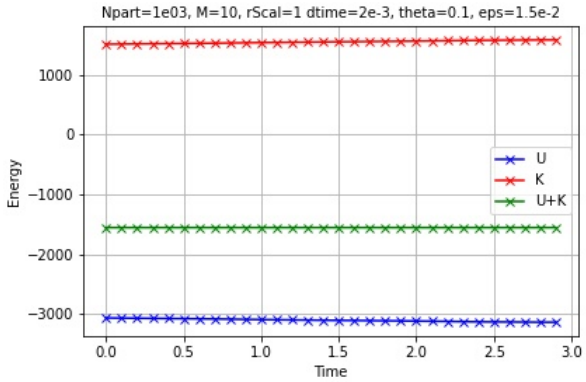


Figure 10: Energies

From the analysis made we can conclude that the Plummer's Sphere under analysis is at equilibrium.

(There is an error in the plots reported: in thei titles is written t=30, but it should be t=3)