ASSIGNMENT 3: DUE ON FRIDAY, SEPTEMBER 29: INDUCTION AND THE BINOMIAL THEOREM

Warm-up problems

(1) A chessboard in real life is an 8×8 grid of squares, but for this question we'll take an $m \times n$ chessboard to be a grid of squares with m rows and n columns. Let n be a positive integer. Show that a $2^n \times 2^n$ chessboard with one square removed can be completely covered by non-overlapping "L" pieces, where an L piece, is three squares shaped like an L (so a corner, a top, and a right-side). You're allowed to rotate the pieces however you see fit. (Remarks: you should be able to show this regardless of which square is removed. Can you figure out for which m, n you can do the same thing with an $m \times n$ chessboard with one square removed?

(2) Show that for $n \geq 1$ we have
$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

(3) The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that $F_n^2 = F_{n+1}F_{n-1} + (-1)^{n+1}$ for $n \geq 1$.

(4) Show by induction that every number ≥ 40 can be written as a sum of 5's and 8's. (Hint: you might need a few base cases.)

(5) Let r be a real number such that $r + 1/r$ is an integer. Prove that $r^n + 1/r^n$ is an integer for every natural number n.

(6) Let $n \geq 2$ be a positive integer. When we expand $(1 + x + x^2)^n$ we obtain a polynomial of degree $2n$. Show that for some $i = 0, 1, \ldots, 2n$, the coefficient of x^i in $(1 + x + x^2)^n$ is even.

(7) For $n \geq 1$, we let
$$a_n = \sum_{j=0}^{n} \binom{n}{j}^2.$$
Compute the values of a_n for small n and try to find a conjecture for a closed form for a_n (not involving $\sum$ notation or $\ldots$) and then prove that your conjecture is correct.

(8) (Bernoulli's inequality) Let $x > -1$ be a real number with $x \neq 0$. Prove by induction that $(1 + x)^n > 1 + nx$ for all positive integers $n \geq 2$.

(9) Prove that for $n \geq 1$ we have
$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.$$

(10) Prove by induction that $n! > 2^n$ for $n \geq 4$.

Assignment questions

(1) Let n and k be positive integers and let S denote the number of ordered k-tuples of nonnegative integers $(a_1, \ldots, a_k)$ such that
$$a_1 + a_2 + \cdots + a_k = n.$$

Show that S has size

$$\binom{n+k-1}{k-1}.$$

(Remark: the word ordered means that the order matters. So, for example, when $n = 4$ and $k = 3$. We get the following 15 ordered 3-tuples:

$$(0,0,4),(0,4,0),(4,0,0),(0,1,3),(0,3,1),(1,0,3),(3,0,1),(1,3,0),$$
$$(3,1,0),(0,2,2),(2,0,2),(2,2,0),(1,1,2),(1,2,1),(2,1,1).$$

(2) For $n \geq 1$, we let

$$a_n = \sum_{j=0}^{n} \binom{n}{j} j.$$

Compute the values of a_n for small n and try to find a conjecture for a closed form for a_n (not involving $\sum$ notation or $\ldots$) and then prove that your conjecture is correct.

(3) The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Show by induction that $F_{2n+1} = F_n^2 + F_{n+1}^2$ for $n \geq 0$.