Course Notes for AMATH 251

Introduction to Differential Equations (Advanced)

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Contents

Chapter 1

Introduction

This course provides an introduction to ordinary differential equations (ODEs) and mathematical modelling based on ODEs. It includes material on non-dimensionalization of ODEs, which in general simplifies problems by reducing the number of parameters, and dimensional analysis. An introduction to Laplace Transforms provides additional tools for solving linear constant-coefficient ODEs.

The first documented mathematical model was developed by Claudius Ptolemy (c. 100–178 CE) to describe and predict the motions of the sun, moon and planets about the Earth (which at that time was the centre of motion) [3]. His model was described in his *Mathematicki Syntaxis*, a work in 13 books, better known as the *Almagest* after Islamic scientists, who translated the book from Greek to Arabic, began calling it *al-magisti*. Models based on differential equations had to wait until the 1660s and 1670s for the invention of Calculus by Sir Isaac Newton (1642–1727) and Gottried Leibniz (1646–1716).

1.1 Mathematical Modelling

Mathematical modelling requires

(i) A problem statement which requires a detailed and thorough description of the problem.

(ii) Use of relevant guiding principles to relate various quantities resulting in equations that need to be solved. These comprise the mathematical model.

(iii) Finding a solution of the equations.

(iv) Interpretation of the solution.

A wide range of mathematical models covering a vast range of fields, including the physical sciences, economics, biology, health and social sciences have since been developed that are based on differential equations. Since Newton was one of the inventors of Calculus let's start with an example from Newtonian Mechanics.

PROBLEM: We wish to develop a model for the motion of an object moving perpendicular to the Earth's surface for three situations:

1. At distances far from the Earth where the atmosphere is negligible but the force of gravity varies;
2. For an object moving near the Earth but with negligible air resistance (drag);
3. For an object moving near the Earth such that air resistance affects its motion.

We are going to make several simplifying assumptions. For example, we will assume that the object is much denser than the air so that only gravitational and drag forces need to be considered (e.g., we ignore the buoyancy force which is crucial to model a bubble of air rising through water). We ignore relativistic effects, the gravitational forces exerted by other planets etc.

1.1.1 Falling objects

To proceed we need two laws from Newtonian Mechanics and one empirical law which provides an approximate model for air drag. The two laws from Newtonian Mechanics are:

1. *Newton's Universal Law of Gravitation:* Two bodies of masses M_1 and M_2 exert a mutually attractive gravitational force on one another of magnitude $\frac{GM_1M_2}{r^2}$ where G is a constant and r is the distance between their centres of mass.
2. *Newton's 2nd Law:* For a body of mass m moving with velocity $\vec{v}$ the time rate of change of its momentum $m\vec{v}$ is equal to the net force $\vec{F}$ acting on the body. Mathematically

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\vec{v}) = \vec{F} \tag{1.1}$$

or, in one dimension,

$$\frac{\mathrm{d}}{\mathrm{d}t}(mv) = F. \tag{1.2}$$

If m is constant then

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = F. \tag{1.3}$$

Model (a): *Varying gravity, no drag.*

(i) Let m be the mass of the object a height $s > 0$ above the Earth's surface (Figure 1.1).

(ii) Let M be the mass of the Earth which we assume to be spherical with radius R with centre of mass at the geometric centre.

(iii) The gravitational force on the object, whose centre of mass is a distance $R+s$ from the centre of the earth, is

$$F_{gr} = -\frac{GmM}{(R+s)^2} \tag{1.4}$$

(iv) From Newton's 2nd law

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = F_{gr} = -\frac{GmM}{(R+s)^2}. \tag{1.5}$$

or, using $v = \frac{ds}{dt}$,.

$$m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -\frac{GmM}{(R+s)^2}. \tag{1.6}$$

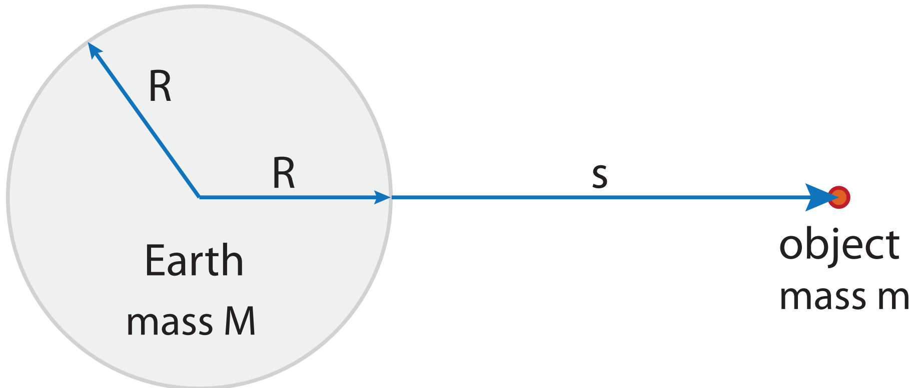

Figure 1.1: Schematic of the coordinate system for object moving perpendicular to the Earth's surface. s is the height of the centre of mass of an object of mass m above the Earth's surface. The Earth is assumed to be spherical with radius R, mass M and centre of gravity at the geometric centre.

Equation (1.6), a second-order nonlinear differential equation, is our mathematical model. It is an approximate model as many simplifications have been made.

Note that here, and below, $v > 0$ if s is increasing, i.e., if the object is moving upward (away from the Earth's surface) and $v < 0$ if it is moving downward.

Model (b): *Close to the Earth's surface, no drag.*

The gravitational force is

$$F_{gr} = -\frac{GmM}{(R+s)^2} = -\frac{GmM}{R^2}\frac{1}{(1+\frac{s}{R})^2} \approx -\frac{GmM}{R^2} \tag{1.7}$$

if $s/R \ll 1$. This is what is meant by 'close to the Earth's surface'. Now $R \approx 6400$ km so for $s \leq 64$ km the error in approximating $R+s$ by R is less than 1%. With this approximation

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = F_{gr} = -\frac{GmM}{R^2}. \tag{1.8}$$

Definition: The gravitational acceleration at the Earth's surface is

$$g = \frac{MG}{R^2}. \tag{1.9}$$

Thus our model is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg, \tag{1.10}$$

or

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g, \tag{1.11}$$

which is a linear first-order differential equation for $v(t)$. In terms of s we have

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -g \tag{1.12}$$

a linear second-order differential equation for $s(t)$. This is the constant graviational acceleration, no drag model. (*Aside: the gravitational acceleration measured at the Earth's surface includes a* *component associated with the Earth's rotation because it is measured in an accelerating reference frame. The gravitational acceleration varies with latitude in part because of latitudinal variations of the effects of rotation and also because the distance from the centre of the Earth to the Earth's surface varies).*

Model (c): *Constant gravitational acceleration with air resistance.*

Now our model equation has the form

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = F_{gr} + F_d = -mg + F_d \tag{1.13}$$

where F_d is the drag force. The drag force is very complicated. It depends on the object's velocity and on its shape — in particular on how fluid flows over the object's surface. F_d always acts to oppose the motion of the object. The force exerted by a fluid flowing around an object can have a component perpendicular to the direction of motion so an object can accelerate sideways (or tangentially). There are different mechanisms that give rise to tangential forces one of which gives the lift that allows airplanes to fly.

We consider two *Empirical Laws*

(i) At 'low' speeds $F_d \propto v$:

$$F_d = -\gamma v \qquad \gamma > 0 \text{ a constant.} \tag{1.14}$$

(ii) At 'high' speeds $|F_d| \propto v^2$:

$$F_d = -\beta v|v| \qquad \beta > 0 \text{ a constant.} \tag{1.15}$$

More about what 'low' and 'high' mean later. The coefficients γ and β are dimensional drag coefficients. More commonly a dimensionless drag coefficient is used which we will discuss later. Note that in both cases $F_d > 0$ if $v < 0$ and $F_d < 0$ if $v > 0$ so that the drag force always acts to oppose the motion of the object by decreasing its speed ($|v|$) and hence its kinetic energy.

Our models are

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g - \frac{\gamma}{m}v \qquad \textit{linear drag law} \tag{1.16}$$

and

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g - \frac{\beta}{m}v|v| \qquad \textit{quadratic drag law} \tag{1.17}$$

The first model equation is a first-order **linear** DE. The second is a first-order **nonlinear** DE.

Example Solutions:

1. *Constant acceleration, no drag*

   Have

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -g \tag{1.18}$$

   which can be integrated twice to give

$$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0 \tag{1.19}$$

   where $v_0 = \frac{ds}{dt}(0)$ and $s_0 = s(0)$ are the initial height and velocity of the object.

2. *Constant gravitational acceleration, linear drag:*

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g - \frac{\gamma}{m}v = -\frac{\gamma}{m}\big(v + \frac{gm}{\gamma}\big). \tag{1.20}$$

   Some insight before solving:

   (i) $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$ when $v = -\frac{gm}{\gamma}$. Since this is a constant $v = -\frac{gm}{\gamma}$ is a solution of the DE.

   (ii) If $v > -\frac{gm}{\gamma}$, $\frac{\mathrm{d}v}{\mathrm{d}t} < 0$ so v decreases.

   (iii) If $v < -\frac{gm}{\gamma}$, $\frac{\mathrm{d}v}{\mathrm{d}t} > 0$ so v increases.

   (iv) The previous two points show that if initially $\frac{dv}{dt} \neq 0$ then $v \to -\frac{gm}{\gamma}$ as t increases.

   The DE is separable and easily solved:

$$\frac{\mathrm{d}v}{v + \frac{mg}{\gamma}} = -\frac{\gamma}{m}\mathrm{d}t \tag{1.21}$$

   so

$$\ln|v + \frac{mg}{\gamma}| = -\frac{\gamma}{m}t + C \tag{1.22}$$

   or

$$v + \frac{mg}{\gamma} = Ae^{-\frac{\gamma}{m}t}. \tag{1.23}$$

   Why could we drop the absolute value on the left hand side? This is an important question whose answer you should understand. Thus, the solution of the DE is

$$v = -\frac{mg}{\gamma} + Ae^{-\frac{\gamma}{m}t}. \tag{1.24}$$

Interpretation:

- As $t \to \infty$ $v \to -\frac{mg}{\gamma}$. This is called the **terminal velocity** (often the terminal velocity is taken as the absolute value of this, technically the terminal speed). The terminal velocity increases in magnitude as the drag coefficient gets smaller (i.e., drag is reduced for fixed velocity), as the object's mass increases and as the gravitational acceleration increases, all of which make sense. This is part of the interpretation of the solution — does your solution make sense (if not, the model may be a poor one).
- The velocity relaxes to the terminal velocity exponentially with time scale m/γ. The larger the mass or the weaker the drag coefficient the longer it takes the object's velocity to relax to the terminal velocity.

1.1.2 *Plotting solutions, scaling and non-dimensional variables:*

For a specific problem, with given values of m, g, and γ the solutions are easily plotted. In general, however, if these values are given as undetermined parameters it is convenient to scale the variables and put them in terms of dimensionless variables. Here, for example, we can use $\tilde{v} = v/(\frac{mg}{\gamma})$ as a function of $\tilde{t} = \frac{\gamma}{m}t$. In terms of $\tilde{v}$ and $\tilde{t}$ the solution is

$$\tilde{v} = \tilde{A}e^{-\tilde{t}} - 1 \tag{1.25}$$

which involves only a single parameter $\tilde{A} = \frac{A\gamma}{mg}$. Sample solutions for different values of $\tilde{A}$ are shown in Figure 1.2. The variables $\tilde{v}$ and $\tilde{t}$ are *dimensionless*: $\frac{mg}{\gamma}$ has dimensions of velocity so dividing v by this quantity results in a dimensionless variable.

We could have non-dimensionalized the problem from the beginning and this is often worth doing because it simplifies the equation. Consider again the DE

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g - \frac{\gamma}{m}v. \tag{1.26}$$

Introduce a time scale T (with dimensions of time) and set $t = T\tilde{t}$ so that $\tilde{t}$ is a dimensionless time. Then introduce a velocity scale U (with dimensions of length divided by time) and set $v(t) = U\tilde{v}\big(\tilde{t}(t)\big)$. The time and velocity scales will be chosen later to make the dimensionless problem as simple as possible.

By the chain rule we have

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}(U\tilde{v})}{\mathrm{d}t} = U\frac{\mathrm{d}}{\mathrm{d}t}\tilde{v}\big(\tilde{t}(t)\big) = U\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\tilde{t}}\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} = \frac{U}{T}\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\tilde{t}}. \tag{1.27}$$

The differential equation becomes

$$\frac{U}{T}\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\tilde{t}} = -g - \frac{\gamma}{m}U\tilde{v}, \tag{1.28}$$

or

$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\tilde{t}} = -\frac{T}{U}g - \frac{T\gamma}{m}\tilde{v}, \tag{1.29}$$

Setting $T = \frac{m}{\gamma}$ makes the coefficient of $\tilde{v}$ on the right hand side equal to 1. Next choose U so that $\frac{T}{U}g = 1$. This gives $U = Tg = \frac{mg}{\gamma}$. The result is the dimensionless differential equation

$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\tilde{t}} = -1 - \tilde{v}, \tag{1.30}$$

You should convince yourself that the dimensionless variables $\tilde{v}$ and $\tilde{t}$ are the same as those used to plot the solutions in Figure 1.2 and that the general solution of the dimensionless DE is

$$\tilde{v} = -1 + \tilde{A}e^{-\tilde{t}}. \tag{1.31}$$

We will learn more about non-dimensionalizing equations and dimensional analysis in Chapter 3.

Comments on the solutions:

- For the constant gravitational force without drag we solved a second-order DE for $s(t)$. The solution has two arbitrary constants v_0 and s_0 which can be determined from initial conditions.
- For the constant gravitational force, linear drag problem we had a first order DE for the velocity v. The solution has one arbitrary constant which can be determined from the initial conditions.

Example: Find the height $s(t)$ of a ball thrown vertically upward from height $s_0 = 1$ m with velocity $v_0 = 10$ m s^{-1}. Ignore air resistance and assume a constant gravitational acceleration.

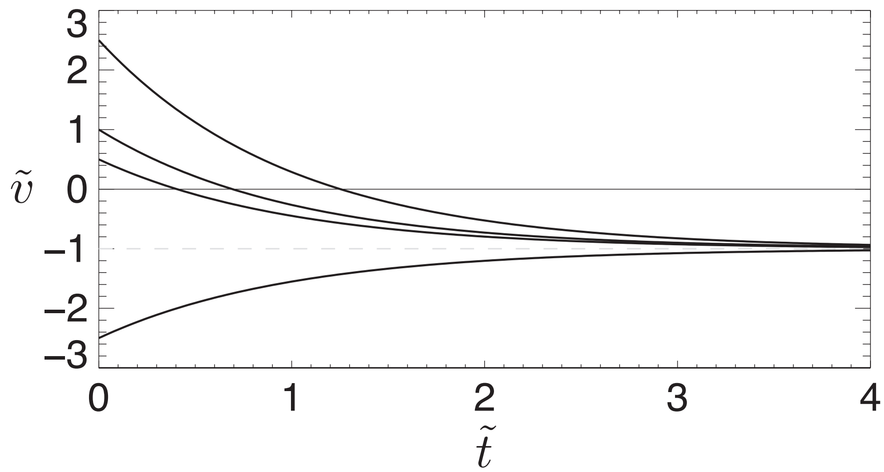

Figure 1.2: Solutions of the linear drag law for an object moving perpendicular to the Earth's surface plotted using non-dimensional variables $\tilde{t}$ and $\tilde{v}$.

Solution: From the above solution

$$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0. \tag{1.32}$$

Thus $s(0) = s_0 = 1$ m and $v(t) = -gt + v_0 \implies v(0) = v_0 = 10$ m s^{-1}. Thus the solution is

$$s(t) = -\frac{1}{2}gt^2 + 10t + 1 \tag{1.33}$$

where s is in meters, t in seconds and $g \approx 9.81$ m s^{-2}.

Note that the maximum value of s is $100/2g \approx 5$ m so s is always much smaller than R and hence the approximation of a constant gravitational acceleration is OK.

In general, the solution of an n$^{\text{th}}$ order ODE involves n arbitrary constants that are determined by n conditions (e.g., initial or boundary conditions). This is not always the case: for some n$^{\text{th}}$ order ODEs not all solutions involve n constants.

1.2 Population Growth: the Logistic Equation

Let $P(t)$ represent the population of some species at time t. Populations take integer values and jump at discrete times. We assume that the population is so large that we can make the *continuum approximation* that assumes that the population varies continuously. The population could be measured in thousands so values like 1.234 make sense. The continuous function $P(t)$ is expected to be a 'good fit' to the discretely varying population.

The growth rate per individual, $\frac{1}{P}\frac{dP}{dt}$ is the difference between the average birth rate and the average death rate. We assume the following:

1. the average birth rate (number of births per unit population) is constant, say β;

2. the average death rate due to the effects of crowding and increased competition for food, is proportional to the size of the population. Let δ be its constant of proportionality.

This gives a simple model for population growth, called the *logistic equation*:

$$\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = \beta - \delta P,$$

or

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(\beta - \delta P). \tag{1.34}$$

The population increases if $P < \frac{\beta}{\delta}$ and decreases if $P > \frac{\beta}{\delta}$. In both cases the population approaches the steady state (equilibrium) solution $P = \frac{\beta}{\delta}$ as $t \to \infty$.

This equation is separable and easily solved. The solution will depend on the two parameters α and β.

1.3 Classification of Differential Equations

Equations of the form

$$F\big(x, y, y', y'', \ldots, y^{(n)}\big) = 0 \tag{1.35}$$

to be solved for the function $y(x)$ are called ordinary differential equations, or ODEs. Here $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$ is the first derivative of y, $y'' = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is the second derivative and $y^{(n)}$ denotes the n^{th} derivative. That is F is a function of x, $y(x)$ and the first n derivatives of $y(x)$.

Definition: The *order of the ODE* is the order of the highest derivative.

Definition: An n^{th} order ODE is linear if it has the form

$$a_n(x)\frac{\mathrm{d}^ny}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + a_{n-2}(x)\frac{\mathrm{d}^{n-2}y}{\mathrm{d}x^{n-2}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x) \tag{1.36}$$

where the $a_k(x)$ and $g(x)$ are known functions. If an ODE is not linear it is said to be nonlinear.

Definition: An n^{th} order linear ODE is **homogeneous** linear if $g(x) = 0$. Otherwise it is called an **inhomogeneous**, or **forced** ODE.

Definition: In general an n^{th} order ODE requires n conditions to uniquely determine a solution. The problem of solving (1.35) subject to the n conditions

$$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_0' \\ &\vdots \\ y^{(n)}(x_0) &= y_0^{(n)} \end{aligned} \tag{1.37}$$

where the $y_0^{(j)}$ are given constants, is called an **initial value problem**, or **IVP**. Not that all of the conditions are given at the same value of x. Problems arise in which conditions are specified at different locations. This situation is more complicated and solutions may not exist. For nonlinear ODEs there may be problems for which it is inappopriate to specify n initial conditions. We will see some examples below.

Examples

(i) $y'(x) - y = e^x$ is a first-order linear ODE for $y(x)$;

(ii) $x^2y'' - 2y' + 3xy = \sin(x)$ is a second-order linear ODE for $y(x)$;

(iii) $x^2(y')^2 + 2y = 6$ is a first-order nonlinear ODE for $y(x)$;

(iv) $t^2\frac{\mathrm{d}^3x}{\mathrm{d}t^3} + tx = 0$ is a third-order linear ODE for $x(t)$;

(v) $v(u)\frac{\mathrm{d}^4v}{\mathrm{d}u^4} + v^3(u) = 6u$ is a fourth-order nonlinear ODE for $v(u)$;

(vi) $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \tan(x) = 2t$ is a second-order nonlinear ODE for $x(t)$.

Definition *Partial differential equations* are equations that involve partial derivatives of multi-variable functions, e.g., and equation for the function $z(x, y)$ such as

$$z_{xy} + z_y^2 - xz_x = 0. \tag{1.38}$$

The short form for this type of equation is PDE. Famous examples include

1. The one-dimensional wave equation

$$\eta_{tt} - c^2\eta_{xx} = 0. \tag{1.39}$$

   Here t is time and x is a spatial coordinate. It is used to model waves in a violin string, although a better model would the effects of damping. There are many other wave equations.

2. Laplace's equation which in R^3 has the form

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0. \tag{1.40}$$

   This models the shape of a soap film.

3. The heat equation which in one spatial dimension has the form

$$w_t = \kappa w_{xx} \tag{1.41}$$

   where t is time and x is the spatial coordinate. This, for example, models the flow of heat along a one-dimensional rod. The heat equation modelling the flow of heat in R^3 has the form

$$w_t = \kappa\nabla^2 w. \tag{1.42}$$

ODEs and PDEs are different types of differential equations denoted by DEs. PDEs are generally much more difficult. We will often use the term DEs in this course by which we will mean ODEs.

1.4 Problems

1. Nondimensionalize the ODE for a falling object subjected to a quadratic drag law. Show that it can be written in the form

$$\frac{d\tilde{v}}{d\tilde{t}} = -1 - |\tilde{v}|\tilde{v}. \tag{1.43}$$

2. Nondimensionalize the logistic equation. Show that it can be written in the form

$$\frac{d\tilde{P}}{d\tilde{t}} = \tilde{P}(1 - \tilde{P}). \tag{1.44}$$

Chapter 2

First-Order Ordinary Differential Equations

We begin our study of ODEs by considering the simplest possible cases: first-order ODEs which have the general form

$$F(x, y, y') = 0. \tag{2.1}$$

One of the primary goals in early studies of differential equations was the classification of types of equations that can be solved. There are several different types of first-order ODEs that can be solved, many of which often arise in applications. Hence they are important types of ODEs. In this chapter we consider a few of them.

2.1 Separable DEs

One class of ODEs that can be solve is *separable* first-order differential equations which most of you have seen in first year calculus. To quickly recap, a first-order differential equation of the form

$$y' = f(x, y) \tag{2.2}$$

is said to be separable if the function $f(x, y)$ (sometimes called a forcing function) has the form $f(x, y) = h(x)g(y)$, i.e., it is the product of a function of x with a function of y. The equation can then be written as

$$\frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = h(x). \tag{2.3}$$

Integrating both sides with respect to x gives

$$\int^x \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int^x h(x)\,\mathrm{d}x + C \tag{2.4}$$

or

$$\int^{y(x)} \frac{\mathrm{d}y}{g(y)} = \int^x h(x)\,\mathrm{d}x + C \tag{2.5}$$

Here the integrals represent any anti-derivative of the corresponding functions and C is an arbitrary constant. The equation is said to be solved although in practice it may be impossible to find analytical expressions for the anti-derivatives. This technique for solving separable equations was found by Leibniz in 1691 well before the concept of a function was formalized. Johan Bernoulii (1667-1748) in 1718 and the Leonard Euler (1707-1783) in 1748 put functions front and centre, *Example:* Find the solution of the initial value problem (IVP)

$$\begin{aligned} x^3 y' &= \frac{x}{y} \\ y(1) &= 2 \end{aligned} \tag{2.6}$$

Solution: Writing the DE in the form $y' = f(x, y)$ gives $y' = 1/x^2 y$ which is separable. The DE can be written as

$$yy' = \frac{1}{x^2}. \tag{2.7}$$

Integrating we have

$$\int^y y \, \mathrm{d}y = \int^x \frac{\mathrm{d}x}{x^2} + C \tag{2.8}$$

or

$$\frac{1}{2} y^2 = -\frac{1}{x} + C. \tag{2.9}$$

This is called the *general solution*. The value of C is determined from the condition $y(1) = 2$ which gives

$$\begin{aligned} \frac{1}{2} y(1)^2 &= -\frac{1}{1} + C \\ 2 &= -1 + C \\ C &= 3. \end{aligned} \tag{2.10}$$

Thus the solution of the IVP is

$$\frac{1}{2} y^2 = -\frac{1}{x} + 3. \tag{2.11}$$

The initial condition is satisfied by only one of these, namely

$$y(x) = \sqrt{6 - \frac{2}{x}}. \tag{2.12}$$

Because we only deal with real valued functions in this course the desired solution exists only when $6 - \frac{2}{x} \geq 0$. This includes two ranges of value for x: $x < 0$ and $x \geq 1/3$. The initial condition $y(1) = 2$ tells us that the solution curve must pass through $x = 1$. So our solution curve includes only values of x with $x \geq 1/3$.

The condition $y(1) = 2$ is called an initial condition and the problem consisting of the DE plus the initial condition is called an initial value problem. The solution of an initial value problem does not contain any arbitrary constants. In general an initial value problem for an n^{th}-order ODE requires n inital conditions.

Note: In the preceding example you can write (2.7) immediately without first putting the equation in the form $y' = f(x, y)$. Any equation of the form $a(x)b(y)y' = c(x)d(y)$ is separable.

Examples: Sample solutions of four separable DEs are plotted in figure 2.1. For each DE several points (x_0, y_0) on the x-y plane are indicated along with the solution passing through them.

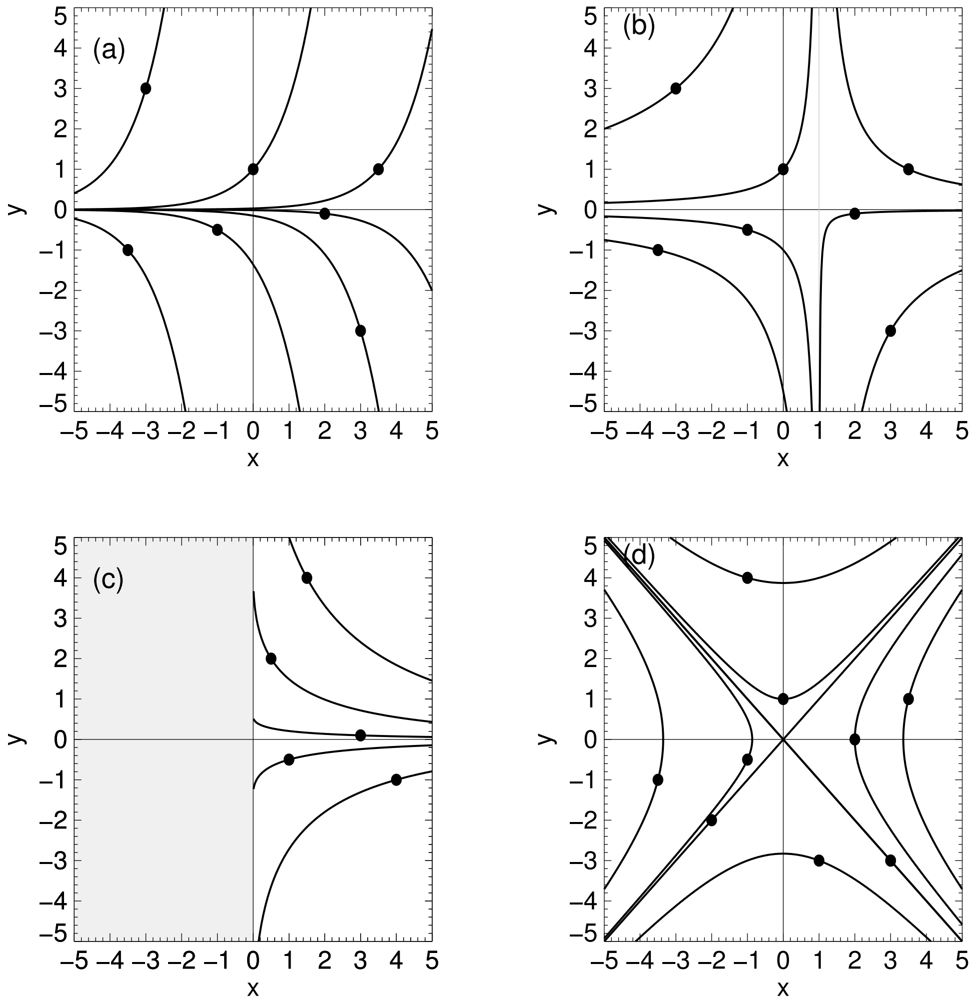

Figure 2.1: Solutions of first-order DEs. Solid circles represent points in R^2. Solutions passing through these points are plotted. (a) $y' = y$. (b) $y' = -\frac{y}{x-1}$ The grey line $x = 1$ is a line along which no solutions exist. (c) $y' = -\frac{y}{2\sqrt{x}}$. No solutions pass through the grey region $x < 0$. (d) $y' = \frac{x}{y}$.

1. Figure 2.1(a) shows solutions $y(x)$ of $y' = y$. The solutions have the form $y = ae^x$ where a is an arbitrary constant (exercise). Any point (x_0, y_0) in R^2 has one and only one solution passing through it, namely the solution with $a = y_0 e^{-x_0}$. The solutions cover the whole plane.

2. Figure 2.1(b) shows solutions $y(x)$ of $y' = -\frac{y}{x-1}$ which have the form $y(x) = \frac{a}{x-1}$ where a is an arbitrary constant (exercise). Any point (x_0, y_0) not lying on the line $x = 1$ has a single solution passing through it, with $a = y_0(x_0 - 1)$. In this case the solutions cover the whole plane minus the line $x = 1$.

   The exclusion of the line $x = 1$ arises because of the choice to seek solutions of the DE of the form $y(x)$. Alternatively we can write the DE as $\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{x-1}{y}$ which leads to solutions $x(y) = 1 + \frac{a}{y}$ as the solutions. Now the solutions $x(y)$ cover the xy-plane minus the line $y = 0$. Thus, the part of the plane covered by the solutions can depend on what type of solutions you want to include. A general form of the DE is $y\,\mathrm{d}x + (x-1)\,\mathrm{d}y = 0$ which does not favour

$y(x)$ or $x(y)$. Solutions of this DE can be written as $y(x-1)=a$ and for $a=0$ there are two curves: $x=1$ and $y=0$. The solutions $y(x-1)=a$ includes the solutions of the form $y(x)$ and $x(y)$ and the solutions now cover the whole plane. Note there are two solutions passing through $(x_0,y_0)=(0,1)$ and a single solution through all other points. The non-uniqueness of solutions passing through $(0,1)$ is related to the singularities in the equation for $y(x)$ written in the form $y'=f(x,y)$ or for the equation for $x(y)$ when written in the form $x'(y)=g(x,y)$.

3. Figure 2.1(c) shows solutions of $y'=-\frac{y}{2\sqrt{x}}$ which have the form $y=ae^{-\sqrt{x}}$ where a is an arbitrary constant (exercise). Because we consider real-valued functions only in this course, solutions only exist for $x>0$. In this example the solutions cover the half-plane $x>0$ with any point (x_0,y_0) with $x_0>0$ having one and only one solution passing through it. Points with $x=0$ have been excluded since the right hand side of the DE is singular at $x=0$. The solutions can, however, be extended to $x=0$ by taking the limit $x\to 0^+$ since $\lim_{x\to 0^+} y=a$. What happens to the derivative as $x\to 0^+$?

4. Figure 2.1(d) shows solutions of $y'=\frac{x}{y}$ which have the form $y^2-x^2=a$ where a is an arbitrary constant (exercise). The solutions are hyperbolae which cover the whole plane as a varies between $\pm\infty$. For each (x_0,y_0) in the upper or lower half plane (i.e., with $y_0>0$ or $y_0<0$ there is a unique solution $y(x)$ passing through the point. For points with $y_0=0$ there are two solutions $y(x)$ passing through the point. For example through $(x_0,y_0)=(1,0)$ the two solutions are $y(x)=\pm\sqrt{x^2-1}$. These two curves join to form a single curve in the x-y plane which has the functional form $x(y)=\sqrt{1+y^2}$. The special nature of points with $y=0$ when solutions $y(x)$ are sought is indicated by the singularity in the right-hand side of the differential equation $y'=\frac{x}{y}$.

In the above we found solutions of the form $y(x)$ in cases 1, 2, and 3. These are called *explicit* solutions: we have given the solution y as a function of x. In the last case the solution $y^2-x^2=a$ has the form $f(x,y)=0$. This defines $y(x)$ *implicitly*.

2.1.1 Autonomous First-order DEs

A special case of a separable first-order DE is the case when the forcing function is independent of the independent variable. Such equations often occur in differential equations that model the evolution of some quantity in time, e.g.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=f(x). \tag{2.13}$$

The reason for this is that the behaviour of many systems does not depend on the start time. Examples we have seen so far include all our models for the motion of an object subject to gravitational and drag forces and for the logistic model. Systems driven by time varying forces will not be autonomous (e.g., a mixing tank problem for which the inflow varies with time). The solution of (2.13) is given implicitly by

$$\int^x \frac{\mathrm{d}s}{f(s)}=t+c \tag{2.14}$$

which you may or may not be able to put in closed form. Much about the behaviour can be determined from the equation, for example:

- Any point where $f(x)=0$ is a solution of the equation. It is called a *fixed point*;

- $x(t)$ is increasing with time in regions where $f(x) > 0$;
- $x(t)$ is decreasing with time in regions where $f(x) < 0$.

Plotting $f(x)$ shows the regions where x is increasing and decreasing. It is easy to show that if the solution $x(t)$ exists on an interval (a, b) then $x(t)$ is either increasing or decreasing monotonically. If $f(x)$ is a continuous function $x(t)$ can not take the same value at two different times unless $x(t)$ is a constant. Proof of this is left as a problem.

2.2 Numerical Methods for ODEs: Construction of approximate solutions numerically

An example of a separable ODE for which a solution $y(x)$ can't be explicitly found is

$$y' = \big(y^4 + 3y^3 - \sin(y)\big)\big(\tanh(x^2 + \sin(x)\big). \tag{2.15}$$

The vast majority of differential equations can not be solved analytically. Such DEs often arise in mathematical models. In this case 'numerical solutions' must be sought. Such solutions are approximate solutions and there are a number of sophisticated methods to compute numerical solutions of sufficient accuracy. A good introductory reference for numerical methods for ODEs is the text by Leveque [4].

2.2.1 Euler's Method

The basis for many numerical methods is Euler's method, which as the name suggests was used by Euler himself to find approximate solutions to problems of interest. This method uses the fact that usually the desired solution is 'close' to other solutions in some sense.

Consider the initial value problem (IVP)

$$y'(x) = f(x, y) \qquad y(x_0) = y_0. \tag{2.16}$$

Euler proposed the following method to find approximate solutions:

1. From the differential equation and initial condition we know the value of y and its derivative at $x = x_0$:

$$\begin{aligned} y(x_0) &= y_0, \\ y'(x_0) &= f(x_0, y_0). \end{aligned} \tag{2.17}$$

2. We can approximate the function $y(x)$ in the vicinity of (x_0, y_0) by its tangent line. This of course assumes the tangent line exists but this is guaranteed if $f(x, y)$ is continuous. Thus, letting $x_1 = x_0 + h$ we can approximate $y(x_1) = y(x_0 + h)$, for small h, as

$$y(x_1) = y(x_0 + h) \approx y_0 + y'(x_0)h = y_0 + f(x_0, y_0)h \equiv y_1. \tag{2.18}$$

This is illustrated in Figure 2.2(a,c). For the value $h = 2$ used in panel (a) the approximated value is not very good. It is much improved for $h = 1$ but still not very good (panel (c)), however for a smaller value of h, say $h = 0.4$, the estimated value would give a good approximation of $y(x_1)$.

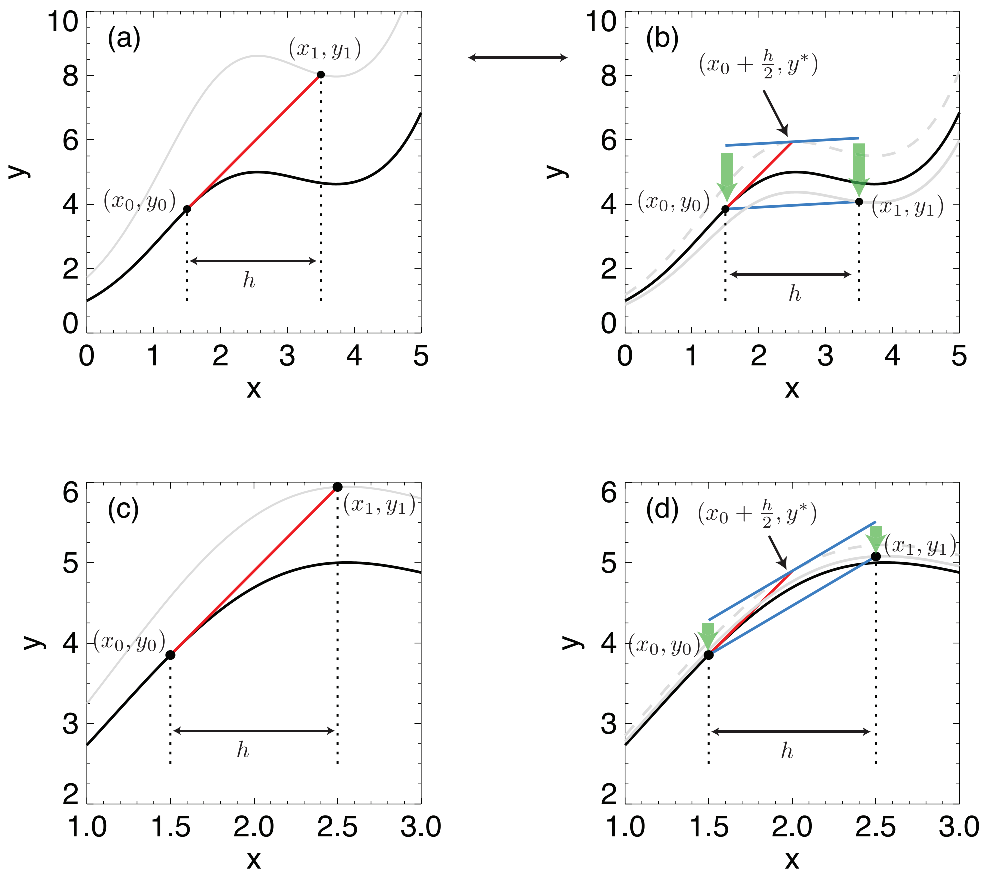

Figure 2.2: Euler's and RK-2 method for estimating $y(x_0 + h)$ starting from (x_0, y_0). The black curves indicate the solution of the DE passing through (x_0, y_0). (a,c) Euler's method. Starting at the point (x_0, y_0) the value of $y(x)$ at $x_1 = x_0 + h$ is approximated by following the tangent line at (x_0, y_0) to (x_1, y_1). The grey curves show the solution curve on which the estimated point lies. (a) Result using $h = 2$. (c) Result using $h = 1$. (b,d) RK-2 method. Taking half a step gives the intermediate points $(x_0 + \frac{h}{2}, y^*)$. The solution curve passing through this point is indicated with the dashed grey curve. The slope of the tangent line (upper blue curve) to the solution curve through the intermediate point is then used as the slope of the straight line approximation from (x_0, y_0) to (x_1, y_1) (lower blue line). The solid grey curve indicates the solution curve passing through the estimated point (x_1, y_1). (b) Result using $h = 2$. (d) Result using $h = 1$.

It should be clear that in general the point (x_1, y_1) is not on the solution curve $y(x)$ but if h is sufficiently small it will be close to it. We then continue by estimating $y(x_2) = y(x_0 + 2h)$ starting from (x_1, y_1):

$$y(x_2) = y(x_1 + h) \approx y_1 + y'(x_1)h \approx y_1 + f(x_1, y_1)h = y_2, \tag{2.19}$$

etc.

Suppose we want to estimate $y(z)$ given initial values at x_0. One approach is the following:

1. Subdivide the interval $[x_0, z]$ into N intervals $[x_j, x_{j+1}]$ with $x_j = x_0 + jh$ for $j = 1, \ldots, N$ where $h = (z - x_0)/N$. Note $x_N = z$.

2. Calculate $y_j = y_{j-1} + f(x_{j-1}, y_{j-1})h, \qquad j = 1, 2, \ldots, N$ to find y_N.

3. The y_j are *estimates* of $y(x_j)$. In particular y_N is an *estimate* of $y(z)$. The hope is that as $N \to \infty$, $y_N \to y(z)$, i.e., the numerical approximation converges to $y(z)$ as the step size goes

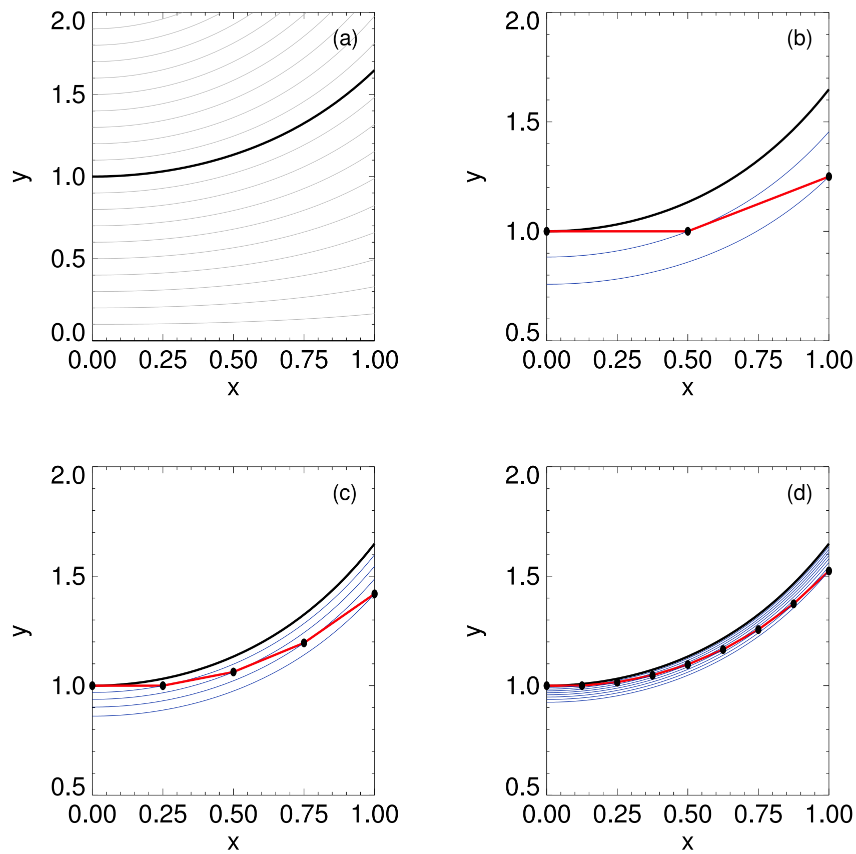

Figure 2.3: Approximate solutions of $y' = xy$, $y(0) = 1$, obtained using Euler's method. (a) Solutions of the DE for different initial conditions. The solution of the IVP $y(0) = 1$ is $y = e^{x^2/2}$ giving $y(1) = \sqrt{e} =$. (b) Approximating $y(1)$ using two steps. (c) Approximating $y(1)$ using 4 steps. (d) Approximating $y(1)$ using 8 steps. The blue curves in panels (b)–(d) are the solution curves passing through the approximated values at $x = n/h$.

to zero (and the number of steps goes to infinity). At the same time the intermediate values approach the solution curve $y(x)$.

This is called Euler's method.

Example: Use Euler's method to find approximations to the solution of

$$y' = xy \qquad y(0) = 1 \tag{2.20}$$

at $x = 1$ taking 1, 2 and 4 steps.

Solution:

Exercise: Show that the general solution of the DE is $y = Ae^{x^2/2}$ and that the solution satisfying the initial condition $y(0) = 1$ is $y(x) = e^{x^2/2}$. Hence $y(1) = \sqrt{e} = 1.64872127\cdots$. Figure 2.3(a) shows a number of solutions for different initial conditions including the solution of the IVP. Now we find the required approximate values of $y(1)$ using Euler's method.

1. *1 step:* Taking one step we have $h = 1$. The approximate solution is

$$y(1) \approx y(0) + f(0, 1)h = 1 + 0 \cdot h = 1. \tag{2.21}$$

This gives a very poor approximation of $y(1) = e^{1/2} \approx 1.6487$.

2. *2 steps:* Now $h = 1/2 = 0.5$. We have

$$\begin{aligned} y(0.5) &\approx 1 + f(0, 1) \cdot \frac{1}{2} \\ &= 1, \\ y(1) &\approx 1 + f\big(\frac{1}{2}, 1)\big) \cdot \frac{1}{2} \\ &= 1 + \frac{1}{4} \\ &= \frac{5}{4}. \end{aligned} \tag{2.22}$$

This approximate solution is illustrated in Figure 2.3(b). The solution curves passing through the points (x_1, y_1) and (x_2, y_2) are included. After one step we are at $(x_1, y_1) = (0.5, 1.0)$ which is on the solution curve $y = e^{-1/8}e^{x^2/2} \approx 0.88e^{x^2/2}$.

3. *4 steps:* Now $h = 1/4 = 0.25$. We have

$$\begin{aligned} y(0.25) &\approx 1 + f(0, 1) \cdot \frac{1}{4} \\ &= 1, \end{aligned} \tag{2.23}$$

$$\begin{aligned} y(0.5) &\approx 1 + f\big(\frac{1}{4}, 1\big) \cdot \frac{1}{4} \\ &= 1 + \frac{1}{4} \cdot \frac{1}{4} \\ &= \frac{17}{16}, \end{aligned} \tag{2.24}$$

$$\begin{aligned} y(0.75) &\approx \frac{17}{16} + f\big(\frac{1}{2}, \frac{17}{16}\big) \cdot \frac{1}{4} \\ &= \frac{17}{16} + \frac{1}{2} \cdot \frac{17}{16} \cdot \frac{1}{4} \\ &= \frac{17}{16}\big(1 + \frac{1}{8}\big) \\ &= \frac{17}{16} \cdot \frac{9}{8} \\ &= \frac{153}{128} = \frac{17}{16} \cdot \frac{18}{16} \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} y(1) &\approx \frac{153}{128} + f\big(\frac{3}{4}, \frac{153}{128}\big) \cdot \frac{1}{4} \\ &= \frac{153}{128}\Big(1 + \frac{3}{16}\Big) \\ &= \frac{153}{128} \cdot \frac{19}{16} \\ &= \frac{17}{16} \cdot \frac{18}{16} \cdot \frac{19}{16} \\ &= 1.4194 \cdots \end{aligned} \tag{2.26}$$

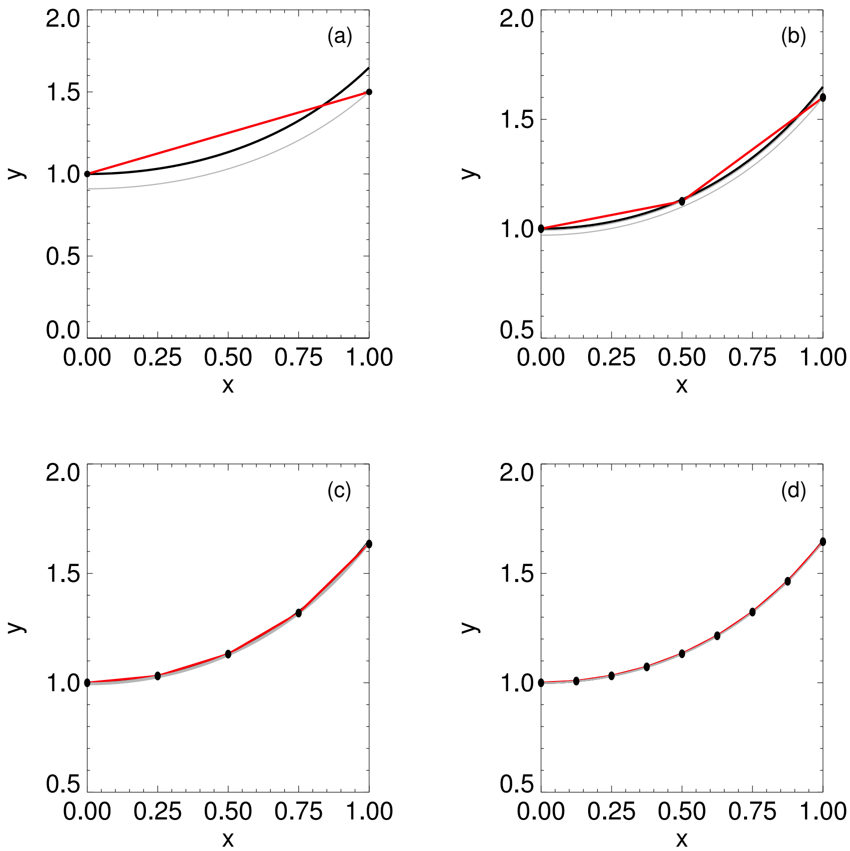

Figure 2.4: Approximate solutions of $y' = xy$, $y(0) = 1$, obtained using the RK-2 method. (a) Solutions of the DE for different initial conditions. The solution of the IVP $y(0) = 1$ is in bold. (b) Approximating $y(1)$ using two steps. (c) Approximating $y(1)$ using 4 steps. (d) Approximating $y(1)$ using 8 steps.

Exercise: Show that for 8 steps with $h = 0.125$ the approximate solution is $y(1) \approx 65 \cdot 66 \cdot \ldots 70 \cdot 71/64^7 \approx 1.5240\cdots$.

2.2.2 The second-order Runga-Kutta or mid-point method

If $f(x, y)$ is smooth Euler's method will converge as the step size goes to zero, that is as the number of steps increases the approximate solution goes to the correct value. Its convergence can be slow requiring a very large number of steps to get an answer of sufficient accuracy. It can be shown rigourously that the error is proportional to the step size h as $h \to 0$ (see problems at the end of the chapter). Each step taken in Euler's method introduces an error of $O(h^2)$, i.e.,

$$y(x_n + h) = y_n + f(x_n, y_n)h + O(h^2) = y_{n+1} + O(h^2). \tag{2.27}$$

After $N = 1/h$ steps the global error is $O(h)$.

Luckily there are much better methods so Euler's method is not used in practice. It is, however, the basis for many better methods. A method for which the error goes to zero like h^p as $h \to 0$ is said to be of order-p.

From Figure 2.3 we can see that it would be much better to estimate the slope at a mid-way point and use it instead of the slope of the tangent line at the starting point. A simple improvement on Euler's method is the second-order Runge-Kutta, or RK-2, method which does just this. It proceeds as follows. First estimate y at the mid-point $x_0 + \frac{h}{2}$ using Euler's method and use the DE to estimate the slope at the mid-point. For the second step go back to the initial point x_0 and use the estimated slope at the mid-point to go from x_0 to $x_0 + h$. Thus first find

$$y^* = y_0 + f(x_0, y_0)\frac{h}{2} \tag{2.28}$$

and then estimate $y(x_0 + h)$ using the slope at $(x_0 + \frac{h}{2}, y^*)$ via

$$y_1 = y_0 + f(x_0 + \frac{h}{2}, y^*)h. \tag{2.29}$$

This is illustrated in Figures 2.2(b,d). The second-order Runga-Kutta method is often written algorithmically as

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1) \\ y_{n+1} &= y_n + k_2. \end{aligned} \tag{2.30}$$

Example: Using RK-2 find approximations to the solution of

$$y' = xy \qquad y(0) = 1 \tag{2.31}$$

at $x = 1$ taking 1 and 2 steps. This is the same example used above to illustrate Euler's method.

Solution:

1. *1 step:* Taking one step we have $h = 1$. First find

$$\begin{aligned} y^* &= y_0 + f(0, 1)\frac{h}{2} \\ &= 1 + 0 \cdot \frac{1}{2} = 1. \end{aligned} \tag{2.32}$$

Then

$$\begin{aligned} y(1) &\approx y_0 + f(x_0 + \frac{h}{2}, y^*)h \\ &= 1 + f(\frac{1}{2}, 1)h \\ &= 1 + \frac{1}{2}h \\ &= \frac{3}{2} \\ &= 1.5. \end{aligned} \tag{2.33}$$

2. *2 steps:* Now $h = 1/2 = 0.5$. For the first step we have

$$\begin{aligned} y^* &= y_0 + f(0,1)\frac{h}{2} \\ &= 1 + 0 \cdot \frac{1}{4} = 1. \end{aligned} \tag{2.34}$$

Then

$$\begin{aligned} y(\frac{1}{2}) &\approx y_0 + f(x_0 + \frac{h}{2}, y^*)h \\ &= 1 + f(\frac{1}{4}, 1)\frac{1}{2} \\ &= 1 + \frac{1}{8} \\ &= \frac{9}{8}. \end{aligned} \tag{2.35}$$

For the second step, starting from $(x_1, y_1) = (\frac{1}{2}, \frac{9}{8})$ first find

$$\begin{aligned} y^* &= y_1 + f(x_1, y_1)\frac{h}{2} \\ &= \frac{9}{8} + \frac{1}{2} \cdot \frac{9}{8} \cdot \frac{1}{4} \\ &= \frac{81}{64}. \end{aligned} \tag{2.36}$$

Then

$$\begin{aligned} y(1) &\approx y_1 + f(x_1 + \frac{h}{2}, y^*)h \\ &= \frac{9}{8} + f(\frac{3}{4}, \frac{81}{64})\frac{1}{2} \\ &= \frac{9}{8} + \frac{3}{4} \cdot \frac{81}{64} \cdot \frac{1}{2} \\ &= \frac{819}{512} \\ &\approx 1.599609375. \end{aligned} \tag{2.37}$$

Note that this approximation is better than the one obtained using Euler's method with 8 steps!

The approximate solution obtained taking 2 steps is illustrated in Figure 2.4(b). We will stop here as these calculations quickly get tedious. They are best left for a computer. Figure 2.4 also shows solutions obtained using 1, 2, 4 and 8 points. Using 8 points it is hard to distinguish the approximate points from the solution curve. Comparing Figures 2.3 and 2.4, the RK-2 method is vastly superior, at least for this example.

Table 2.1 compares our computed estimates for $y(1)$ using Euler's method and the RK-2 method. Recall the correct answer is $y(1) = e^{1/2} \approx 1.6487$. The solution using RK-2 with one step is better than the solution obtained using Euler's method with 4 steps and the solution using RK-2 with two steps is better than that obtained using Euler's method with 8 steps.

RK-2 is an example of a Runga-Kutta method. This family of methods was developed by Carl Runge (1856–1927) and Wilhelm Kutta (1867–1944), two German mathematicians. RK-2 is a second-order method. Much better methods are available, a fourth- or fifth-order Runge-Kutta method is often used in practice (RK4-5 combines fourth and fifth order methods to give an estimate of the error which is used to adjust the step size). Much higher-order methods also exist. You can learn more about the wonderful world of numerically solving ODEs and PDEs in AMATH 342 and 442.

2.2.3 A Fourth-order Runga-Kutta Method

Here we briefly outline a fourth-order Runga-Kutta method. This provides a very good general purpose ODE solver however for highly accurate computations there are better methods. The fourth-order RK4 method is as follows:

$$\begin{aligned}
k_1 &= hf(x_n, y_n) \\
k_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1) \\
k_3 &= h(f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2) \\
k_4 &= hf(x_n + h, y_n + k_3) \\
y_{n+1} &= y_n + \frac{1}{6}\Big(k_1 + 2k_2 + 2k_3 + k_4\Big).
\end{aligned} \tag{2.38}$$

Note that k_1/h is the slope at x_n, k_2/h and k_3/h are estimates of the slope at the midpoint $x_n+h/2$ obtained using different estimates of the value of y at the midpoint, and k_4/h is an estimate of the slope at $x_n + h$. Now

$$y(x_n + h) = y(x_n) + \int_{x_n}^{x_n+h} y'(x)\,\mathrm{d}x \tag{2.39}$$

Using Simpson's rule

$$\int_x^{x+2h} g(s)\,ds \approx \frac{h}{3}\Big(g(x) + 4g(x+h) + g(x+2h)\Big), \tag{2.40}$$

Table 2.1: Approximate values for $y(1)$ of the solution of $y'(x) = xy$ with initial condition $y(0) = 1$ obtained using Euler's method and the second-order RK-2 method. The exact solution is $y(1) = \sqrt{e} = 1.64872127\ldots$.

n	$y(1)$ (Euler)	$y(1)$ (RK-2)
1	1	1.5
2	1.25	$1.5996\cdots$
4	$1.4194\cdots$	$1.6342\cdots$
8	$1.5240\cdots$	$1.6448\cdots$

which is exact for all polynomials of degree $n \leq 3$, to evaluate the integral on the right after splitting the interval into two sub-intervals of length $h/2$ gives

$$y(x_n + h) \approx y(x_n) + \frac{h}{6}\Big(y'(x_n) + 4y'(x_n + \frac{h}{2}) + y'(x_n + h)\Big) \tag{2.41}$$

The fourth-order Runga-Kutta method approximates this with

$$\begin{aligned} hy'(x_n) &\approx k_1 \\ 4hy'(x_n + \frac{h}{2}) &\approx 2k_2 + 2k_3 \\ hy(x_n + h) &\approx k_4 \end{aligned} \tag{2.42}$$

where the k_j have been selected to make the method fourth-order, that is

$$y_{n+1} = y(x_n + h) + O(h^5). \tag{2.43}$$

so that after N steps of size $1/h$ the error is $O(h^4)$. There are many higher-order Runga-Kutta methods. There are many subtleties that we have not addressed. One is that higher-order does not necessarily mean higher accuracy. A discussion of this and many other aspects of finding approximate solutions numerically is left for a course in numerical methods.

2.3 Picard's Method: finding approximate solutions

This method of finding approximate solutions was formalized by Émile Picard (1856–1941) in the 1880's and 1890's but the method was discovered 50 years earlier by Joseph Liouville (1809–1882).

Consider

$$y' = f(x, y) \qquad y(x_0) = Y_0. \tag{2.44}$$

Integrating from x_0 to x gives

$$\int_{x_0}^{x} y'(t)\,\mathrm{d}t = \int_{x_0}^{x} f\big(t, y(t)\big)\,\mathrm{d}t \tag{2.45}$$

so

$$y(x) = y(x_0) + \int_{x_0}^{x} f\big(t, y(t)\big)\,\mathrm{d}t. \tag{2.46}$$

Define the operator T via

$$T[w](x) = y(x_0) + \int_{x_0}^{x} f\big(t, w(t)\big)\,\mathrm{d}t. \tag{2.47}$$

T maps functions on R to functions on R: given the function $w(x)$, $T[w](x)$ is a new function. We want to find a function y such that $T[y] = y$, i.e., a fixed point of T. The idea is to iterate to get a sequence of functions $y_0(x)$, $y_1(x)$, $y_2(x), \ldots$, where

$$y_{n+1}(x) = y(x_0) + \int_{x_0}^{x} f\big(t, y_n(t)\big)\,\mathrm{d}t = T[y_n] \tag{2.48}$$

with the hope that y_n converges to a function $y(x)$ as $n \to \infty$. If it does then taking the limit as $n \to \infty$ of $y_{n+1} = T[y_n]$ implies that $y = T[y]$, i.e., we have the solution of the DE (note all of the functions $y_n(x)$ satisfy the initial condition $y_n(x_0) = y(x_0) = Y_0$). Often the initial guess is the initial condition, i.e., take $y_0(x) = Y_0$, but this is not necessary.

Example: Consider $y' = y$ with initial condition $y(0) = 1$. We know the solution is $y = e^x$. Since $f(x, y) = y$ in this case the operator is

$$T[w] = 1 + \int_0^x w(t)\,\mathrm{d}t. \tag{2.49}$$

Iterating, starting from $y_0 = 1$ we have

$$\begin{aligned} y_1 = T[y_0] &= y_0 + \int_0^x y_0\,\mathrm{d}t \\ &= 1 + x, \\ y_2 = T[y_1] &= y_0 + \int_0^x y_1(t)\,\mathrm{d}t \\ &= 1 + \int_0^x (1+t)\,\mathrm{d}t \\ &= 1 + x + \frac{x^2}{2}, \\ y_3 = T[y_2] &= y_0 + \int_0^x y_2(t)\,\mathrm{d}t \\ &= 1 + \int_0^x (1 + t + \frac{t^2}{2})\,\mathrm{d}t \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}, \end{aligned} \tag{2.50}$$

etc. By induction it is straight forward to show that

$$y_n = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^2}{n!}. \tag{2.51}$$

Therefore

$$\lim_{n\to\infty} y_n(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \tag{2.52}$$

which is the solution of the DE. Picard's method can be used to prove the existence-uniqueness theorem for solutions of first-order differential equations.

2.4 Existence-Uniqueness of Solutions of First-Order Differential Equations

Consider the general first-order initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \qquad y(x_0) = y_0. \tag{2.53}$$

THEOREM I (Picard's Theorem): *If f and $\frac{\partial f}{\partial y}$ are continuous functions in an open rectangle*

$$R = \Big\{(x, y) : a < x < b;\ c < y < d\Big\} \tag{2.54}$$

that contains (x_0, y_0) then the initial value problem (2.53) has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$ where $\delta > 0$.

There are stronger versions of this theorem. We mention one. First a definition.

Definition: A function $f(x, y)$ is said to be uniformly Lipshitz continuous (or to satisfy a Lipshitz condition) in the variable y on an open or closed rectangle R if there is a constant K such that

$$|f(x, y_2) - f(x, y_1)| < K|y_2 - y_1| \tag{2.55}$$

for all x, y_1 and y_2 in R.

The Lipschitz condition bounds the difference in slopes of solution curves of $y' = f(x, y)$ and says that as $y_2 \to y_1$ for fixed x the difference in slopes $f(x, y_2)$ and $f(x, y_1)$ goes to zero at least linearly in the difference in y values. There are many functions $f(x, y)$ that satisfy the Lipschitz condition for which f_y is not continuous.

Note that if $f_y(x, y)$ is continuous on a closed rectangle then it satisfies a Lipshitz condition since on R we have $|f(x, y_2) - f(x, y_1)| \leq \max_R\{|f_y\}|y_2 - y_1|\}$ where $\max_R$ is the maximum on R which exists if R is closed. So take $K = \max_R\{|f_y|\}$.

THEOREM II (Picard-Lindelöf Theorem): *Let $f(x, y)$ be a continuous function on the closed rectangle*

$$R = \Big\{(x, y) : x_0 \leq x \leq x_0 + a;\ |y - y_0| \leq b\Big\} \tag{2.56}$$

with $f(x, y)$ uniformly Lipschitz continuous with respect to y on R. Let M be a bound for $|f(x, y)|$ on R and let $\alpha = \min\{a, b/M\}$. Then the initial value problem (2.53) has a unique solution $y = \phi(x)$ on $\big[x_0, x_0 + \alpha\big]$.

Comments:

- The value of α is natural:
  - it is clear that the solution can't be guaranteed to exist for $x - x_0 > a$ because we know nothing about the function $f(x, y)$ for $x - x_0 > a$;
  - Since $y' = f(x, y)$ we have $|y'| = |f(x, y)| \leq M$ on R. This implies that $|y(x) - y(x_0)| \leq M(x - x_0)$ which can not exceed b if $x - x_0 \leq \alpha$.
- There is a corresponding result for the rectangle with $x_0 - a \leq x \leq x_0$ and hence for $|x - x_0| \leq a$ with existence and uniqueness in $|x - x_0| \leq \alpha$.
- The Picard-Lindelöf Theorem and Picard's Theorem both imply there is one and only one solution passing through (x_0, y_0);
- Picard's Theorem gives no information on how large δ is. All we know is that it is greater than zero.

Example: What does the Picard's Theorem tell us about solution of

$$\begin{aligned} 3\frac{\mathrm{d}y}{\mathrm{d}x} &= x^2 - xy^3, \\ y(1) &= 6? \end{aligned}. \tag{2.57}$$

Putting the DE in the form $y' = f(x,y)$ we have $f = \frac{1}{3}x^2 - \frac{1}{3}xy^3$. Both f and $\frac{\partial f}{\partial y} = -xy^2$ are continuous on R^2. The hypothesis of the theorem is satisfied $\implies$ there is a unique solution on an interval $(1-\delta, 1+\delta)$ for some $\delta > 0$.

Example: What does the Picard-Lindelöf Theorem tell us about solution of

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= y, \\ y(1) &= 1? \end{aligned}. \tag{2.58}$$

Here $f = y$ so f_y is continuous everywhere and hence is uniformly Lipshitz continuous on any closed rectangle R. Or, trivially, $|f(x,y_2) - f(x,y_1)| = |y_2 - y_1|$ is uniformly Lipshitz continuous with constant $K = 1$ for all rectangles.

On $x_0 \le x \le x_0 + a$ and $|y - y_0| \le b$ we have $|f(x,y)| = |y| \le y_0 + b$ so $M = y_0 + b = 1 + b$. Take a larger than $b/M = \frac{b}{1+b}$ so that $\alpha = \frac{b}{1+b}$. The theorem says there is a unique solution on $1 \le x \le 1 + \frac{b}{1+b} = \frac{1+2b}{1+b}$. Since b is arbitrary we can choose b to be as large as we want so the solution exists and is unique on $x \in [1,2)$. The solution is $y = e^x$ which exists everywhere. Thus the range over which the theorem guarantees existence and uniqueness may be much smaller that the range on which a unique solution actually does exist.

Example: What does the Theorem I tell us about solution of

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{y}{x-1}, \\ y(1) &= 1? \end{aligned} \tag{2.59}$$

This DE was considered above, solutions of which are plotted in Figure 2.1(b). Here $f(x,y) = -\frac{y}{x-1}$ is *not* continuous on an open neighbourhood containing the initial point $(x_0,y_0) = (1,1)$ so the theorem does not tell us anything. In this case there are no solutions of the form $y(x)$ passing through the initial point. Rewriting the differential equation as $\frac{dx}{dy} = -\frac{x-1}{y}$ both $f(x,y) = -\frac{x-1}{y}$ and $f_y = \frac{x-1}{y^2}$ are continuous at $(x_0,y_0) = (1,1)$ and the theorem tells us that a solutions $x(y)$ exists. It is the straight line $x = 1$.

Here is a weaker theorem stated by Guiseppe Peano (1858–1932). He gave a proof of his theorem in 1886 but his proof was found wanting and was not properly proved until many years later.

THEOREM III: (Peano's Theorem): *If f is continuous in an open rectangle*

$$R = \Big\{(x, y) : a < x < b;\ c < y < d\Big\} \tag{2.60}$$

that contains (x_0, y_0) then the initial value problem (2.53) has a solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$ where $\delta > 0$.

Comment: Uniqueness has been lost. There may be more than one solution. See problems for an example.

2.5 Linear First-Order Differential Equations

A second, very important, class of differential equations that can always be solved are linear first-order DEs. By 'can always be solved' we mean that the solution can be written in terms of integrals. In many cases it will be impossible to find analytic expressions for the integrals.

Linear first-order (and higher-order) differential equations arise in many problems, for example the projectile problem above for the case of motion of an object near the Earth's surface subject to linear drag and the radioactive decay problem discussed below. There is a great deal of theory for linear differential equations and much of this course will be focussed on first- and second-order equations of this type.

Some of you may have learned how to solve linear first-order DEs by finding an integrating factor. Here I use a different approach that can be generalized to higher-order linear DEs. In the following section the integrating factor method is described and we will see that both methods are essentially the same.

2.5.1 Illustrative example

We begin by consider a very simple inhomogeneous linear first-order DE. Finding its solution will illustrate most of the key points of the general problem. Consider

$$y' + y = 1. \tag{2.61}$$

We wish to find the general solution, that is, all solutions. First note that $y_p(x) = 1$ is a solution. This method of finding a solution is referred to as 'by inspection'. If you can solve a DE 'by inspection' do so. The subscript 'p' indicates that we have a **particular solution**. A particular solution is any solution of the inhomogeneous DE that does not involve any undetermined constants. There are infinitily many particular solutions. $y_p = 1$ is the simplest.

Next let $y(x) = y_p + v(x) = 1 + v(x)$ and substitute into the DE. This gives

$$(1 + v)' + (1 + v) = 1 \tag{2.62}$$

or

$$v' + v = 0. \tag{2.63}$$

Not that the DE for v is the homogeneous version of the original DE for y. That is, it is the DE for y with the forcing term removed. Now the solution of (2.62) can be obtained using the method of separation of variables. The solution is

$$v = ae^{-x} \tag{2.64}$$

where a is an arbitrary constant. All solutions of (2.62) are of this form. With this solution for v we have

$$y = 1 + ae^{-x}. \tag{2.65}$$

This is the general solution of (2.61). That is *all* solutions have this form. There is one undetermined constant, a, whose value can be determined if an initial condition $y(x_0) = y_0$ is provided.

The function $y_h = e^{-x}$ is a non-zero solution of the homogeneous equation. The general solution has the form of a particular solution y_p, which is the function 1 in this case, plus an arbitrary constant times the homogeneous solution y_h. This is a general feature of linear first-order DEs as we will show below.

Before moving on, note that because $y_h = e^{-x}$ is never zero we could write the solution as

$$y(x) = e^{-x}(e^x + c) = y_h(e^{-x} + c). \tag{2.66}$$

With this in mind let's solve (2.61) in a slightly different manner. First find a non-zero solution of the homogeneous problem

$$y' + y = 0. \tag{2.67}$$

One such solution is $y_h = e^{-x}$. Next let $y = y_h(x)v(x)$. Substituting into (2.61) we have

$$y' + y = \big(y_h v\big)' + y_h v = \big(y_h' + y_h\big)v + y_h v' = 1 \tag{2.68}$$

or

$$v' = \frac{1}{y_h} = e^x. \tag{2.69}$$

Integrating gives the general solution $v' = e^x + a$ where c is an arbitrary constant. Thus

$$y = y_h v = e^{-x}\big(e^x + c\big) = 1 + ae^{-x}. \tag{2.70}$$

This recovers our previous solution. Here we chose e^{-x} as the non-zero homogeneous solution. We could have chosen ce^{-x} for any non-zero constant c. For example we could have chosen $y_h = 2e^{-x}$, or $y_h = -101.1287e^{-x}$ etc. We would end up with the same result. You should convince yourselves of this. Note also that $y_h = 0$ is also a solution of the homogenous problem but this choice for y_h does not work. Why?

The latter approach is what we will use in general because it is generally much harder to find a particular solution than to find the homogeneous solution. Normally you need to find the homogeneous solution first.

2.5.2 Standard form for linear first-order DEs

The general linear first-order differential equation has the form

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = b(x) \tag{2.71}$$

where $a_1(x)$ is non-zero.

Case a: $a_0 = 0$. Then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x)}{a_1(x)}$ and we can integrate to get

$$y = \int^x \frac{b(x)}{a_1(x)}\,\mathrm{d}x. \tag{2.72}$$

So there is nothing to discuss in this case.

Case b: $a_0 \neq 0$. In this case we divide by $a_1(x)$ and write the DE in the standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x). \tag{2.73}$$

This is the case we are interested in.

2.5.3 Solving linear first-order equations

Now we consider the general problem. It will simplify our study of linear DEs, particularly when we consider second-order DEs, to write the equation in terms of a differential operator. So we begin by introducing this concept.

Some formalism: Differential Operators

Definition: *An* **operator** *maps a function to another function. That is, if L is an operator and f is a function then $g = L[f]$ is a new function. f must be in a space of functions that the operator L can act on.*

Example: *A simple example of an operator is the differential operator which we will call D. If $f(x)$ is a function then $D[f]$ is the derivative of f, that is $D[f] = \frac{df}{dx}$. Here the space of functions the operator D can act on is the space of functions that are differentiable on a domain of interest.*

Definition: *An operator L is said to be a* **linear operator** *if $L[af + g] = aL[f] + L[g]$ for all functions f and g in the domain of the operator and all constants a.*

Example: *The differential operator $D[]$ is a linear operator:*

$$D[af + g] = \frac{d}{dx}\big(af + g\big) = a\frac{df}{dx} + \frac{dg}{dx} = aD[f] + D[g]. \tag{2.74}$$

We work with linear first-order DEs in the standard form

$$y' + P(x)y = Q(x). \tag{2.75}$$

Define the *linear differential operator* $L[]$, which maps differentiable functions to functions, via

$$L[f] \equiv \frac{\mathrm{d}f}{\mathrm{d}x} + P(x)f \equiv D[f] + P(x)f. \tag{2.76}$$

Thus, given a differentiable function f, L returns the function $f' + Pf$. L is a *linear* operator, that is

$$L[af(x) + g(x)] = aL[f] + L[g] \tag{2.77}$$

for any two differentiable functions f and g and any constant a. The proof is left as an exercise.

Our goal is to find the solution of

$$L[y] = Q(x). \tag{2.78}$$

First we consider the general structure of the solution. Afterward we will consider one method for finding the general solution.

General structure of the solution

Let y_p be any solution of $L[y] = Q$. Let y_g be any other solution. Both y_g and y_p are solutions of the DE so

$$L[y_g] = L[y_p] = Q(x) \tag{2.79}$$

and, by linearity,

$$L[y_g - y_p] = L[y_g] - L[y_p] = 0. \tag{2.80}$$

That is, the difference between any two solutions of (2.78) is a solution of the *homogeneous* problem

$$L[y] = 0. \tag{2.81}$$

Let y_h be any non-zero solution ($y = 0$ is always a solution of the homogeneous problem but it is not a useful one). Then by linearity so is Cy_h for any constant C. By linearity of the operator L we have

$$L[y_p + Cy_h] = L[y_p] + CL[y_h] = Q + C \cdot 0 = Q. \tag{2.82}$$

Thus $y_p + Cy_h$ is a solution of (2.78) for all values of C. At this point this procedure does not tell us that all solutions of (2.78) have this form for our chosen y_p and y_h because, for example, we have not shown that there could not be another solution y_{h2} of the homogeneous problem that is not a constant multiple of y_h. We will soon see that there is no such solution and indeed, our previous solution shows that *all* solutions of the homogeneous problem are multiplies of a selected non-zero solution, that is, if y_{h2} is a solution of $L[y] = 0$ then $y_{h2} = Cy_h$ for some constant C. The existence-uniqueness theorem discussed below also shows that this is the case.

In summary, all solutions of $L[y] = Q$ have the form

$$y = y_p + Cy_h. \tag{2.83}$$

for some C, where y_p is any solution of $L[y] = Q$ and y_h is any non-zero solution of $L[y] = 0$.

Finding the solution

Using the formalism introduced above we now find the general solution.

First we find a non-zero solution of the homogeneous DE

$$L[y] = y' + Py = 0. \tag{2.84}$$

Writing this as

$$y' = -Py \tag{2.85}$$

we have

$$y_h = e^{-\int^x P(s)\,\mathrm{d}s} \tag{2.86}$$

as a homogeneous solution. By construction all solutions of the homogeneous problem have this form, so **any solution of the homogeneous problem is a constant multiple of** $e^{-\int^x P(x)\,\mathbf{d}s}$. As mentioned above any anti-derivative of P can be used. Changing it (i.e., by varying the lower limit of integration) simply results in multiplying y_h by a constant.

This gives us one piece of the solution. Next we need to find a particular solution y_p. To do this we are going to set $y_p(x) = v(x)y_h(x)$, find an equation for v and find its solution. This idea will be used again when we consider second-order differential equations. We have

$$\begin{aligned} L[vy_h] &= \big(vy_h\big)' + Pvy_h \\ &= v'y_h + vy_h' + Pvy_h \\ &= v'y_h + v\Big(y_h' + Py_h\Big) \\ &= v'y_h \end{aligned} \tag{2.87}$$

since y_h is a solution of the homogeneous problem, i.e., $y_h' + Py_h = 0$. Thus v is a solution of the differential equation

$$v' = \frac{Q(x)}{y_h(x)}. \tag{2.88}$$

The right-hand side is a known function so we can integrate:

$$v = \int^x \frac{Q(s)}{y_h(s)}\,\mathrm{d}s. \tag{2.89}$$

The solution is then

$$y(x) = v(x)y_h(x) + Cy_h(x) = y_h(x)\int^x \frac{Q(s)}{y_h(s)}\,\mathrm{d}s + Cy_h(x). \tag{2.90}$$

In the above I am taking v given by (2.89), and the corresponding integral in (2.90) to be a particular anti-derivative of Q/h_h. Any one will do. Thus vh_h is a particular solution. You could also view it has being the general anti-derivative which involves an undetermined constant in which case vh_h would be the general solution however this hides the structure of the solution and I expect all general solutions to involve an arbitrary constant.

EXAMPLE: Find the solution of

$$\cos(x)y' - \sin(x)y = \cos(x) \tag{2.91}$$

SOLUTION:

Step 1: Put in standard form by dividing by the coefficient of y' to get

$$y' - \tan(x)y = 1. \tag{2.92}$$

Step 2: Solve the homogeneous problem

$$y_h' = \tan(x)y_h. \tag{2.93}$$

This gives

$$y_h = e^{\int^x \tan(s)\,\mathrm{d}s} = e^{-\ln|\cos(x)|} = \frac{1}{|\cos(x)|}. \tag{2.94}$$

We can take

$$y_h = \frac{1}{\cos(x)} \tag{2.95}$$

because if $\cos(x)$ happens to be negative we can multiply $y_h = -1/\cos(x)$ by -1 to get a new homogeneous solution. Note that y_h is undefined at points where $\cos(x) = 0$, i.e., where the function changes sign so our solution will in general only be valid for a range of x values for which $\cos(x)$ is only positive or only negative.

Step 3. Set $y_p = v(x)y_h(x)$ and substitute into (2.92) to get

$$v'y_h + vy_h' - \tan(x)vy_h = v'y_h + v\big(y_h' - \tan(x)y_h\big) = v'y_h = 1. \tag{2.96}$$

Thus

$$v' = \frac{1}{y_h} = \cos(x) \tag{2.97}$$

so

$$v = \sin(x). \tag{2.98}$$

Thus a particular solution is $y_p = v(x)y_h(x) = \tan(x)$. The general solution is

$$y = y_p + Cy_h = \tan(x) + \frac{C}{\cos(x)} \tag{2.99}$$

which can be verified by substituting this solution into the original equation (this is always a good idea). Note that the solution is generally singular at $x = \frac{\pi}{2} + n\pi$ for all integers n (for appropriate choices of C some of the singularities can be removed — you should convince yourself of this).

Comment: Other choices for the homogeneous and particular solutions could be made. For example taking $y_h = -2/\cos(x)$ leads to $v' = -\frac{1}{2}\cos(x)$. For v one could then choose $v = -\frac{1}{2}\sin(x) + 10.76$. Then $y = vy_h + Cy_h = \tan(x) - 10.76/\cos(x) - 2C/\cos(x)$. Letting $B = -10.76 - 2C$ we can write this as $y = \tan(x) + B/\cos(x)$ which is the same as our previous solution since C and B are both arbitrary.

EXERCISE: Find the solution of

$$xy' + 2x^2y = x^2 \tag{2.100}$$

following this procedure.

Problems often have forcing functions that can be separated into the sum of simpler functions. In that case it is often convenient to find a particular solution corresponding to each piece and add them together. Consider, for example the problem

$$L[y] \equiv y' + P(x)y = Q(x) = a_1Q_1(x) + a_2Q_2(x) + a_3Q_3(x) \tag{2.101}$$

where the a_j are constants. Suppose you find particular solutions y_{pj} for each $Q_j(x)$. That is, $L[y_j] = Q_j$ for $j = 1, 2, 3$. Then $y_p = \sum_j a_jy_{pj}(x)$ is a particular solution of $L[y] = Q$ since

$$L\big[\sum_j a_jy_{pj}(x)\big] = \sum a_jL[y_{pj}] = \sum a_jQ_j = Q. \tag{2.102}$$

EXAMPLE: Find the solution of

$$\cos(x)y' - \sin(x)y = -2\cos(x) + 5\sin(x). \tag{2.103}$$

Solution: Putting the equation in standard form we have

$$L[y] \equiv y' - \tan(x)y = -2 + 5\tan(x). \tag{2.104}$$

From the previous problem we know that a non-zero homogeneous solution is $y_h = 1/\cos(x)$ and $L[\tan(x)] = 1$ so

$$L[-2\tan(x)] = -2. \tag{2.105}$$

Next we need to find a particular solution of $L[y] = \tan(x)$. From the work done in the previous example we need to solve

$$v' = \frac{\tan(x)}{y_h} = \sin(x) \tag{2.106}$$

so $v = -\cos(x)$ and the particular solution is $vy_h = -1$. Thus $y_{p2} = -1$ is a solution of $L[y] = \tan(x)$ (this can also be seen by inspection). Then

$$y_p = -2\tan(x) - 5 \tag{2.107}$$

is a particular solution of the DE. The general solution is this plus a multiple of the homogeneous solution:

$$y = -2\tan(x) - 5 + \frac{C}{\cos(x)}. \tag{2.108}$$

2.5.4 Existence-Uniqueness

Linear first-order differential equations have the general form

$$y' + P(x)y = Q(x) \tag{2.109}$$

where P and Q are known functions. Equations of this type are a special form of $y' = f(x, y)$ with $f(x, y) = Q(x) - P(x)y$. There is a stronger version of the existence-uniqueness theorem for linear equations.

Consider the Lipschitz condition. We have $|f(x, y_2) - f(x, y_1)| = |P(x)||y_2 - y_1|$ so if $P(x)$ is bounded $f(x, y)$ satisfies a Lipschitz condition in y with $K = \max |P(x)|$ or $\sup P$ for $a \leq x \leq b$ and for all y_1 and y_2. In particular, the Lipshitz condition is satisfied for all y, not just for values in a finite interval (c, d) as above. This strengthens the existence-uniqueness theorem:

THEOREM IV: *Suppose $f(x, y)$ is a continuous function that satisfies a Lipschitz condition in y on an infinite strip $a \leq x \leq b$ and $-\infty < y < \infty$ If (x_0, y_0) is any point on the strip then the IVP*

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{2.110}$$

has a unique solution $y(x)$ on the interval $a \leq x \leq b$.

Comment:

- With this stronger condition on $f(x, y)$ (i.e., Lipschitz on $-\infty < y < \infty$), the solution exists on the whole interval $[a, b]$.
- Consider the linear linear equation (2.109), if $P(x)$ and $Q(x)$ are continuous on $[a, b]$ then $f(x, y) = -Py + Q$ satisfies the conditions of the theorem hence a unique solution $y(x)$ exists on $[a, b]$.

For a proof see [7].

2.5.5 Radioactive Decay

Before discussing some theoretical aspects of linear first-order differential equations we consider another physical example to provide further motivation.

Example: Radioactive Decay. A rock contains a radioactive isotope RA_1 which decays into a second radioactive isotope RA_2. The latter subsequently decays into stable atoms:

$$RA_1 \rightarrow RA_2 \rightarrow \textit{stable atoms}. \tag{2.111}$$

Let $M_i(t)$ be the mass of RA_i, $i = 1, 2$ and assume the following:

(i) There is no source of RA_1.

(ii) RA_1 decays into RA_2 at a rate αM_1 where $\alpha > 0$ is constant.

(iii) RA_2 decays into stable atoms at a rate kM_2 where $k > 0$ is constant.

The latter two assumptions are based on empirical evidence. The problem: find $M_1(t)$ and $M_2(t)$.

We need some equations to solve: our mathematical model. For this we use *conservation of mass* for each isotope.

1. First consider RA_1. Conservation of mass says that:

$$\begin{aligned} \frac{\mathrm{d}M_1}{\mathrm{d}t} &= \text{rate of creation of } \mathrm{RA}_1 - \text{rate of decay of } \mathrm{RA}_1 \\ &= 0 - \alpha M_1. \end{aligned} \tag{2.112}$$

   or

$$\frac{\mathrm{d}M_1}{\mathrm{d}t} + \alpha M_1 = 0. \tag{2.113}$$

   This has the solution

$$M_1(t) = Re^{-\alpha t} \tag{2.114}$$

   where $R = M_1(0)$ is the initial mass of RA_1.

2. Next consider RA_2. We use conservation of mass again noting that M_1 *decreases* at the rate αM_1 so RA_2 is created at the rate $\alpha M_1 = \alpha Re^{-\alpha t}$. Thus

$$\begin{aligned} \frac{\mathrm{d}M_2}{\mathrm{d}t} &= \text{rate of creation of } \mathrm{RA}_2 - \text{rate of decay of } \mathrm{RA}_2 \\ &= \alpha M_1 + kM_2 \\ &= \alpha Re^{-\alpha t} - kM_2. \end{aligned} \tag{2.115}$$

   Thus, M_2 satisfies the DE

$$\frac{\mathrm{d}M_2}{\mathrm{d}t} + kM_2 = \alpha Re^{-\alpha t}. \tag{2.116}$$

   Note the similarity of equations (2.113) and (2.116). The form of the left hand side is the same:

$$\frac{\mathrm{d}M_i}{\mathrm{d}t} + a_i M_i \tag{2.117}$$

   where a_i is a constant. The right hand sides differ and this has an important effect on the solution.

To solve (2.116) we multiply the equation by e^{kt} giving

$$e^{kt}\frac{\mathrm{d}M_2}{\mathrm{d}t} + ke^{kt}M_2 = \alpha Re^{(k-\alpha)t}. \tag{2.118}$$

Noting that $ke^{kt} = \frac{d}{dt}(e^{kt})$ (which is precisely why we multiplied by e^{kt}), this can be rewritten as

$$e^{kt}\frac{\mathrm{d}M_2}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}(e^{kt})M_2 = \alpha Re^{(k-\alpha)t}. \tag{2.119}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{kt}M_2\right) = \alpha Re^{(k-\alpha)t}, \tag{2.120}$$

the point being that the left hand side is an exact derivative. We can now integrate both sides giving

$$e^{kt}M_2 = \begin{cases} \frac{\alpha R}{k-\alpha}e^{(k-\alpha)t} + C, & \text{if } k \neq \alpha; \\ \alpha R t + C, & \text{if } k = \alpha, \end{cases} \tag{2.121}$$

where C is an undetermined constant. The solution is then

$$M_2(t) = \begin{cases} \frac{\alpha R}{k-\alpha}e^{-\alpha t} + Ce^{-kt}, & \text{if } k \neq \alpha; \\ \alpha R t e^{-\alpha t} + Ce^{-\alpha t}, & \text{if } k = \alpha, \end{cases} \tag{2.122}$$

Discussion: Equation (2.116) is a linear constant coefficient first-order differential equation. We solved the DE by multiplying it by the function e^{kt} which put the left hand side in the form of an exact derivative. It was then straightforward to solve the equation. This procedure can be used to find a closed form solution of any first-order linear DE with *non-constant* coefficients as we discuss next.

2.5.6 Solving linear first-order equations using an integrating factor

First consider the procedure discussed above that. First we found a non-zero homogeneous solution y_h of $L[y] = 0$. Then we set the general solution y equal to $v(x)y_h(x)$ for an unknown function v. Substituting into the inhomogeneous problem $L[y] = Q$ led to

$$v'(x) = \frac{Q(x)}{y_h(x)}. \tag{2.123}$$

which we integrated to find v which gives a particular solution after multiplying by y_h. We can write this equation as

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{y}{y_h}\Big) = \frac{Q(x)}{y_h(x)} \tag{2.124}$$

or, setting $\mu(x) = 1/y_h(x)$, as

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\mu(x)y(x)\Big) = \mu(x)Q(x). \tag{2.125}$$

This is the equation in the form of an exact derivative. The right-hand side is know so we can integrate immediately to find $\mu(x)y(x)$ from which we get the solution $y(x)$.

Let us start from scratch and rederive (2.125) in another way. Start with the linear first-order DE in standard form:

$$L[y] \equiv y' + Py = Q. \tag{2.126}$$

Multiplying by an as yet unknown function $\mu(x)$ we have

$$\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mu(x)P(x)y = \mu(x)Q(x). \tag{2.127}$$

and *we choose* μ so that

$$\mu(x)P(x) = \frac{\mathrm{d}\mu}{\mathrm{d}x}. \tag{2.128}$$

If we can do this the equation becomes

$$\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}\mu}{\mathrm{d}x}y = \mu(x)Q(x) \tag{2.129}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\mu y\big) = \mu(x)Q(x) \tag{2.130}$$

which is (2.125). The left hand side is an exact derivative and we can now integrate to get

$$\mu y = \int^x \mu(s)Q(s)\,\mathrm{d}s + C \tag{2.131}$$

or

$$y = \frac{1}{\mu}\Big[\int^x \mu(s)Q(s)\,\mathrm{d}s + C\Big]. \tag{2.132}$$

Thus, if we can find a function $\mu(x)$ satisfying (2.128) then in principal we have found the solution.

In turns out that find $\mu(x)$ is easy. The equation for $\mu(x)$ is

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = \mu P(x) \tag{2.133}$$

which is a separable linear DE. Rewriting as

$$\frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}x} = P(x) \tag{2.134}$$

and integrating gives

$$\ln|\mu| = \int^x P(s)\,\mathrm{d}s + C \tag{2.135}$$

or

$$\mu = Ae^{\int^x P(s)\,\mathrm{d}s}. \tag{2.136}$$

Because the multiplication of (2.73) by $\mu(x)$ makes the left hand side an exact derivative, the function $\mu(x)$ is called an *integrating factor*.

In the solution

$$y = \frac{1}{\mu}\Big[\int^x \mu(s)Q(s)\,\mathrm{d}s + C\Big]. \tag{2.137}$$

multiplying $\mu(x)$ by a constant just changes C, which is an arbitrary constant, so we can take $A = 1$ (or any other non-zero number). Thus, the general solution of

$$a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = b \tag{2.138}$$

is

$$y = e^{-\int^x P(s)\,\mathrm{d}s}\Big[\int^x e^{\int^s P(t)\,\mathrm{d}t}Q(s)\,\mathrm{d}s + C\Big], \tag{2.139}$$

where

$$P(x) = \frac{a_0(x)}{a_1(x)}, \qquad Q(x) = \frac{b(x)}{a_1(x)} \tag{2.140}$$

and C is an arbitrary constant. While this gives a closed form solution in many cases it may be impossible to find the integrals analytically.

Comments:

- This was the method first used to find the general solution of a first-order linear DE. It was found by Leibniz in 1694 using his 'infinitesimal' notation.

- *The lower limit of integration has not been specified because it does not matter.* Any antiderivative can be used. Changing the lower limit of $\int^x P(s)\,\mathrm{d}s$ merely changes the value of the integral by an additive constant. The exponential of this is changed by a multiplicative constant (i.e., changes the value of A in (2.136) and you should convince yourself that this multiplicative constant cancels in the first term. In the second term, involving C, it doesn't matter because C is arbitrary.

- **YOU SHOULD NOT MEMORIZE THIS FORMULA!** You should understand the process and follow the steps used to derive (2.139) for a given problem (or the method discussed below).

- You should know that the solution of (2.133) is (2.136). There is no need to go through the intermediate steps every time.

- The solution (2.139) has the form

$$y = y_p(x) + Cy_h(x) \tag{2.141}$$

where

$$y_p =^{-\int^x P(s)\,ds} \int^x e^{\int^s P(t)\,dt} Q(s)\,\mathrm{d}s \tag{2.142}$$

and

$$y_h = e^{-\int^x P(s)\,\mathrm{d}s}. \tag{2.143}$$

We call the first a *particular solution*. It is one solution of the DE, the solution obtained from (2.139) by setting $C = 0$ (other particular solutions can be obtained by choosing other values for C). The second function, y_h, is a solution when $Q = 0$. It is a solution of the *homogeneous problem*

$$y' + Py = 0. \tag{2.144}$$

EXERCISE: Consider (2.133). Seek a solution of the form $\mu = e^{S(x)}$. Substitute into the DE for μ and find a DE for $S(x)$. Solve the DE (i.e., express S in terms of P). This gives a solution that is always positive. How can you use this solution to get the general solutions (2.136) with an arbitrary constant A of either sign?

EXAMPLE: Find the solution of

$$xy' + 2x^2y = x^2. \tag{2.145}$$

Solution:

Step 1: *First put in standard form by dividing the equation by the coefficient of y' giving*

$$y' + 2xy = x. \tag{2.146}$$

Step 2: *Multiply by μ:*

$$\mu y' + 2x\mu y = x\mu. \tag{2.147}$$

Step 3: *Choose μ so that the coefficient of y is μ':*

$$\mu' = 2x\mu. \tag{2.148}$$

The solution of this is

$$\mu = Ae^{\int^x 2x\,dx} = Ae^{x^2}. \tag{2.149}$$

We can take $A = 1$ and use $\mu = e^{x^2}$. Note than using another antiderivative of $2x$, e.g., $x^2 + 10$ gives $\mu = Ae^{10}e^{x^2}$ and the factor e^{10} can be absorbed into the arbitrary constant A. Any antiderivative of the function P will do.

Step 4: *The DE is now*

$$\frac{d}{dx}\Big(e^{x^2}y\Big) = xe^{x^2}. \tag{2.150}$$

Integrating both sides gives

$$e^{x^2}y = \frac{1}{2}e^{x^2} + C \tag{2.151}$$

so

$$y = \frac{1}{2} + Ce^{-x^2}. \tag{2.152}$$

Step 5: *Check your solution:*

$$\begin{aligned} xy' + 2x^2y &= x \cdot \left(-2xCe^{-x^2}\right) + 2x^2\Big(\frac{1}{2} + Ce^{-x^2}\Big) \\ &= -2x^2Ce^{-x^2} + x^2 + -2x^2Ce^{-x^2} \\ &= x^2 \end{aligned} \tag{2.153}$$

so indeed have a solution of the DE.

EXERCISE: Find the solution using $\mu = -2e^{x^2}$ as the integrating factor and show that you get the same result.

2.5.7 Applications of first-order linear ODEs

Newton's Law of Cooling

Consider an object with temperature $\theta(t)$ surrounded by fluid with an ambient (constant) temperature T_a. Newton's Law of Cooling states that the rate of change of the object's temperature is proportional to the temperature difference $\theta - T_a$:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k\Big(\theta - T_a\Big) \tag{2.154}$$

where k is a positive constant. This is a simple model that captures gross averaged features of cooling.

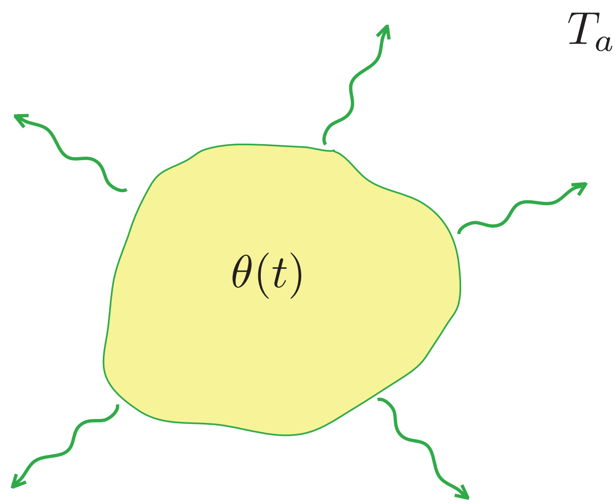

Figure 2.5: Newtonian cooling

Example: At one stage in the process of brewing beer the wort has to be cooled from boiling temperature to 25° C, ideally within about 45 minutes. Assume the pot of wort is put in a large sink of cold water and ice which is continually replenished so that the temperature of the water is maintained at a temperature of $T_w = 5°$ C. The wort is continually stirred to maintain a uniform temperature. The temperature of the wort decreases from 100.0° C to 80.0° C in 5.00 minutes. How long does it take to reach a temperatures of 25° C assuming Newton's Law of Cooling applies.

What if $T_w = 10°$ C and k is unchanged. Now how long does it take the wort to cool to 25° C?

Solution: Let $\theta(t)$ be the temperature of the wort at time t where t is in minutes. Our model equation is

$$\frac{d\theta}{dt} + k\theta = kT_w. \tag{2.155}$$

A homogeneous solution is $\theta_h(t) = e^{-kt}$. By inspection a particular solution is $\theta_p(t) = T_w$. Thus the general solution of the DE is

$$\theta(t) = T_w + Ae^{-kt} \tag{2.156}$$

where A is unknown. From the initial condition $\theta(0) = 100°$ C we have

$$\theta(t) = T_w + (100 - T_w)e^{-kt}. \tag{2.157}$$

1. For the first case $T_w = 5°$C and we are told that $\theta_1 = 80°$C at time $t_1 = 5.00$ min. Thus

$$\begin{aligned} \theta_1 &= T_w + (100 - T_w)e^{-kt_1} \\ \implies e^{kt_1} &= \frac{100 - T_w}{\theta_1 - T_w} \\ \implies k &= \frac{1}{t_1}\ln\Big(\frac{100 - T_w}{\theta_1 - T_w}\Big). \end{aligned} \tag{2.158}$$

This determines k:

$$k = \frac{1}{5}\ln\Big(\frac{95.0}{75.0}\Big) \approx 0.0472778 \text{ min}^{-1} \approx 0.0473 \text{ min}^{-1}. \tag{2.159}$$

The wort reaches the final temperature $T_f = 25°C$ at the time t satisfying

$$T_w + (100 - T_w)e^{-kt} = T_f \tag{2.160}$$

i.e., when

$$e^{kt} = \frac{100 - T_w}{T_f - T_w} \tag{2.161}$$

or when

$$t = \frac{1}{k}\ln\Big(\frac{100 - T_w}{T_f - T_w}\Big) = \frac{1}{k}\ln\Big(\frac{95.0}{20.0}\Big) \approx 33 \text{ min.} \tag{2.162}$$

2. If $T_w = 10°C$ and k is unchanged the time to cool to 25°C is

$$t = \frac{1}{k}\ln\Big(\frac{100 - T_w}{T_f - T_w}\Big) = \frac{1}{k}\ln\Big(\frac{90.0}{15.0}\Big) \approx 38 \text{ min.} \tag{2.163}$$

Comments:

1. It is best to find the solution in terms of the variables T_w, T_f, k etc., until the final stage rather than substituting there values early on. This makes it possible to answer the second question easily rather than working out the whole solution again. It also shows how the solution depends on the various parameters. For example, double k halves the cooling time.

2. How many significant figures should be kept in the value for k and in the value for the cooling time? Increasing θ_1 by 0.05°C changes k to $\frac{1}{5}\ln\Big(\frac{95}{75.05}\Big) = 0.0471445\cdots$ instead of $0.0472778\cdots$. Decreasing θ_1 by 0.05°C changes k to $0.0474111\cdots$. Thus if θ_1 is known to within an error of 0.05°C then $k = 0.047 \pm 1.4 \times 10^{-4}$ min^{-1} which leads to an error in the final time of about 0.09 min or 5/6 s. How accurate is your thermometer and clock? To what precision do you want the cooling time to be known?

Mixing Tank Problem

Consider a tank of volume V. A liquid flows in at a rate V_f. This is called the volume flux. It has units of volume per unit time. The inflowing liquid has a dissolved chemical with concentration c_{in} (mass per unit volume). The concentration of the chemical in the tank is $c(t)$. The liquid inside the tank is well stirred so that the concentration is uniform throughout the tank. The well mixed liquid flows out of the tank with the same rate as the inflow so the volume of fluid in the tank is constant in time. Determine how the chemical concentration in the tank $c(t)$ varies with time.

We use the principal of conservation of mass: the rate of change of the mass of chemical in the tank is equal to the rate mass of chemical enters the tank minus the rate mass of chemical leaves the tank. The mass of chemical in the tank is $M(t) = Vc(t)$. The rate mass enters is equal to the product of the incoming volume flux times the concentration in the inflow, i.e., $V_f c_{in}$ (units of mass per unit time). The rate mass leaves is $V_f c(t)$. Thus, conservation of mass gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(Vc(t)\big) = V_f c_{in} - V_f c(t) \tag{2.164}$$

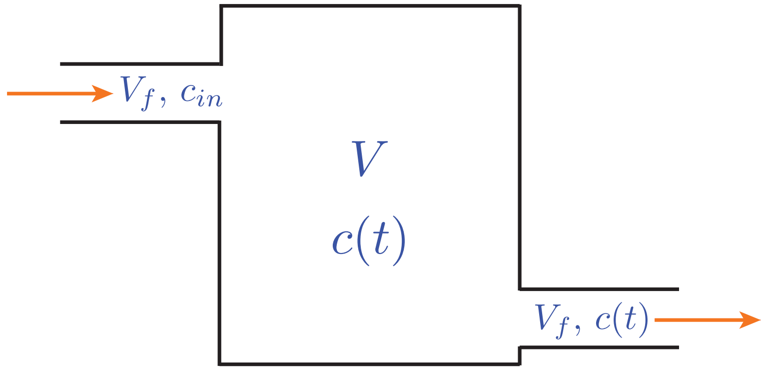

Figure 2.6: Mixing tank of volume V. $c(t)$ is the chemical concentration in the tank, V_f is the volume flux into and out of the tank and c_{in} is the concentration of the chemical in the inflowing liquid.

or

$$\frac{\mathrm{d}c}{\mathrm{d}t} + \frac{V_f}{V} c(t) = \frac{V_f}{V} c_{in}. \tag{2.165}$$

A homogeneous solution is

$$c_h = e^{-\frac{V_f}{V} t} \tag{2.166}$$

and by inspection a particular solution is the constant solution

$$c_p = c_{in}. \tag{2.167}$$

Thus the general solution is

$$\begin{aligned} c &= c_p + A c_h \\ &= c_{in} + A e^{-\frac{V_f}{V} t} \end{aligned} \tag{2.168}$$

or, in terms of the initial concentration $c(0)$,

$$c(t) = c_{in} + \big(c(0) - c_{in}\big) e^{-\frac{V_f}{V} t} \tag{2.169}$$

Note that according to the solution after a long time the concentration $c(t)$ is equal to that in the inflow which makes physical sense. This is the particular solution.

2.6 Nonlinear First-order Differential Equations

2.6.1 The clepsydra

A clepsydra, or water clock, measures time by the regulated flow of a liquid into or out of a vessel.

Consider the following problem (Figure 2.7). A rotationally symmetric container of the form $z = f(r)$ or $r = r(z)$, where z is the vertical coordinate and r is the radius of the container, is filled with water of depth $h(t)$ where t is time. Water flows out through a hole at the bottom of the container. We wish to find an equation for $h(t)$ and solve for some special shapes.

Daniel Bernoulli (1700–1782) studied simple fluid flows and derived Bernoulli's Law (see Bernoulli's 'Hydrodynamica' published in 1738). Ignoring viscosity and assuming steady laminar flow this states that

$$p + \rho g z + \frac{1}{2}\rho q^2 = \text{constant} \tag{2.170}$$

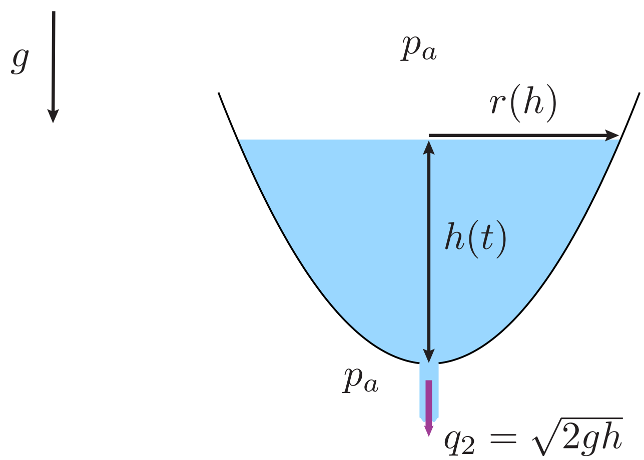

Figure 2.7: A clepsydra. Fluid in the container is exposed to atmospheric pressure p_a at the fluid surface and at the exit. The container is radially symmetric with radius r a function of the height h above the bottom. Bernouill

where p is the pressure, ρ the fluid density, g the gravitational acceleration, z the vertical coordinate, q the speed of the fluid. If the water level in the container is dropping slowly enough we can treat the flow in the container as steady. Relating the flow at the surface (at $z = h(t)$) to the flow exiting the hole at $z = 0$ we have

$$p_1 + \rho g z_1 + \frac{1}{2}\rho q_1^2 = p_2 + \rho g z_2 + \frac{1}{2}\rho q_2^2 \tag{2.171}$$

where the subscript 1 refers to the value at point 1 at the surface ($z_1 = h$) and subscript 2 refers to point 2 at the hole at the bottom of the container ($z_2 = 0$). Assuming $q_1 = h'(t)$ is negligibly small (which requires that the surface area at the top is much larger than the cross-sectional area of the hole) and that the pressure at the surface (p_1) and at the exit point (p_2) are both equal to the atmospheric pressure we get

$$q_2 = \sqrt{2gh}. \tag{2.172}$$

This is known as *Toricelli's Law* after Evangelista Toricelli (1608–1697). In reality viscous effects reduce the outflow speed somewhat, particularly if the hole is very tiny:

$$q_2 = \gamma\sqrt{2gh} \tag{2.173}$$

where $0 < \gamma < 1$ depends on the fluid viscosity and the shape and length of the hole.

Now the rate at which fluid exits the container must be equal to the rate at which the volume of fluid in the container is decreasing (conservation of mass assuming the fluid density is constant). The volume flux through the hole is $aq_2 = a\alpha\sqrt{2gh}$ where a is the cross-sectional area of the hole. The downward volume flux at the surface is $A\frac{\mathrm{d}h}{\mathrm{d}t}$ (note it is negative) where $A = \pi r^2(h)$ is the area of the surface. Conservation of mass gives our model equation

$$\pi r^2 \frac{\mathrm{d}h}{\mathrm{d}t} = -a\gamma\sqrt{2gh} \qquad \text{both sides negative} \tag{2.174}$$

or

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{a\gamma\sqrt{2g}}{\pi}\frac{h^{\frac{1}{2}}}{r^2(h)}. \tag{2.175}$$

This is a first-order nonlinear differential equation for $h(t)$ (in general - the equation is linear for what special container shapes?). It is separable so in principle it is easy to solve.

Note that fluid viscosity is temperature dependent, hence the time taken for a vessel to empty will depend on the fluid temperature. We have ignored the effects of surface tension which become very important when the water level is low or the container is very narrow (e.g., a syringe for which neglecting q_1 in Bernoulli's equation would also not be valid).

Examples:

1. If the bowl is a parabola, i.e., $h = \alpha r^2$ for some constant $\alpha > 0$, we have

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -ch^{-1/2} \tag{2.176}$$

where $c = \frac{a\gamma\alpha}{\pi}\sqrt{2g}$. Thus

$$h^{\frac{1}{2}}\,\mathrm{d}h = -c\,\mathrm{d}t \tag{2.177}$$

so

$$h(t) = \left(h_0^{3/2} - \frac{3}{2}ct\right)^{2/3} \tag{2.178}$$

where $h_0 = h(0)$. The bowl is empty when $h = 0$, i.e., after time $t = \frac{2}{3}\frac{h_0^{3/2}}{c}$.

2. For what special shape does is the water surface drops at a constant rate, i.e., $\frac{dh}{dt}$ is constant. This requires $r^2 \propto \sqrt{h}$. Setting $h = kr^4$ the differential equation becomes

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{a\gamma\sqrt{2gk}}{\pi}. \tag{2.179}$$

2.6.2 Clairaut Equations and Singular Solutions

Alexis Clairaut (1713–1765) wrote a book in 1731 (at the age of 13!) on curves, one of his main interests [3]. In particular, he considered envelopes of a set of tangent lines.

To derive Clairaut's equation (see [8]) consider a family of straight lines

$$y = mx + b. \tag{2.180}$$

This is a two parameter family which includes all straight lines on the plane. Suppose that we take b to be a function of the slope m via $b = f(m)$. Then we have a one parameter family of straight lines:

$$y = mx + f(m). \tag{2.181}$$

Now $y' = m$ so we can write this as a first-order differential equation

$$y = xy' + f\big(y'\big) \tag{2.182}$$

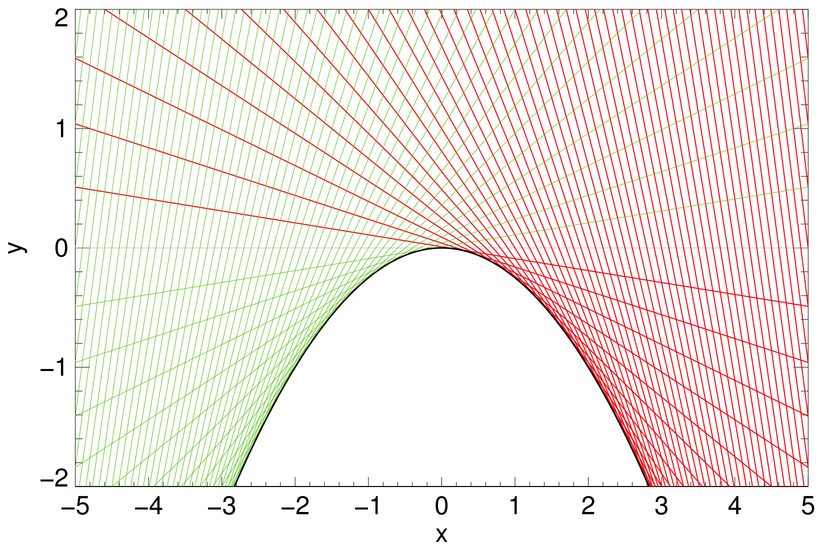

Figure 2.8: Solutions of the Clairault equation $y = xy' + \left(y'\right)^2$.

where in general $f(y')$ is a *nonlinear function*. This is known as **Clairaut's equation**. Note that in (2.182), y' need not be constant so this is a generalization of $y = mx + f(m)$. There will in general be solutions that are not lines in the plane.

Differentiating both sides gives

$$y' = y' + xy'' + f'\big(y'\big)y''. \tag{2.183}$$

Letting $p = y'$ we have

$$\big(x + f'(p)\big)p' = 0 \tag{2.184}$$

so either $p' = 0$ or $f'(p) = -x$. The first case says $p = y'$ is constant, i.e., y is linear in x. The general solution is $p = m$ and $y = mx + b$ where b and m are unrelated. This is not what we started with. This discrepancy has arisen because we have found the solution of (2.184), not of (2.182). That is, differentiating (2.182) has resulted in a loss of information. Substitituting $y = mx + b$ into (2.182) gives $b = f(m)$, so we have recovered $y = mx + f(m)$.

The second solution given by $f'(p) = -x$ is new. This will give the envelope to a family of straight lines as illustrated in the following example.

Example: Consider the problem with $f(y') = y'^2$

(i) The first solution has $p = y' = a$, where a is a constant, hence $y = ax + b$ where a and b are constants.

(ii) The second solution has $f'(p) = 2p = -x \implies 2y' = -x \implies y = -\frac{1}{4}x^2 + d$, where d is a constant.

Now our solutions need to satisfy the original DE. What we have found so far are solutions of the derivative of the original DE so we need to substitute our solutions into the original DE which in this example is

$$y = xy' + y'^2. \tag{2.185}$$

(i) Substituting $y = ax + b$ into (2.185) gives

$$ax + b = ax + a^2 \tag{2.186}$$

hence we must have $b = a^2$.

(ii) Substituting $y = -x^2/4 + d$ into the DE gives

$$-\frac{1}{4}x^2 + d = x\big(-\frac{1}{2}x\big) + \Big(-\frac{1}{2}x\Big)^2 = -\frac{1}{4}x^4 \tag{2.187}$$

so $d = 0$.

The solutions of the Clairaut Equation (2.185) are

$$y = ax + a^2 \tag{2.188}$$

and

$$y = -\frac{1}{4}x^2. \tag{2.189}$$

The first has a single arbitrary constant, the second solution has no arbitrary constants. *Thus not all solutions of a first-order differential equation involve an arbitrary constant.* The solutions are shown in Figure 2.8. Here the green lines are the straight line solutions with positive slope and the red lines are those with negative slope. The black curve is the second solution $y = \frac{1}{4}x^2$ which is the envelope of the family of straight line solutions.

Exercise: Consider the IVP

$$\begin{aligned} y &= xy' + y'^2, \\ y(0) &= y_0 \end{aligned} \tag{2.190}$$

How many real valued solutions are there if (i) $y_0 < 0$, (ii) $y_0 = 0$, or (iii) $y_0 > 0$?

What does the Existence-Uniqueness theorem say about solutions of this equation? To apply the theorem we must write the equation in the form $y' = f(x, y)$. For our example this gives

$$y' = \frac{-x \pm \sqrt{x^2 + 4y}}{2}. \tag{2.191}$$

Then

$$f_y(x, y) = \pm\frac{1}{\sqrt{x^2 + 4y}} \tag{2.192}$$

which is not continuous along the curve $x^2 + 4y = 0$, i.e., along the solution curve $y = -\frac{x^2}{4}$.

Exercise: For initial points on the curve $y = -x^2/4$ how many solutions are there (i.e., solutions of the initial value problem $y(x_0) = y_0 = -x_0^2/4$)?

To read more about the Clairaut equation and a number of applications see [8]. [3] provides a short biography on Clairaut and briefly discusses some of his other work.

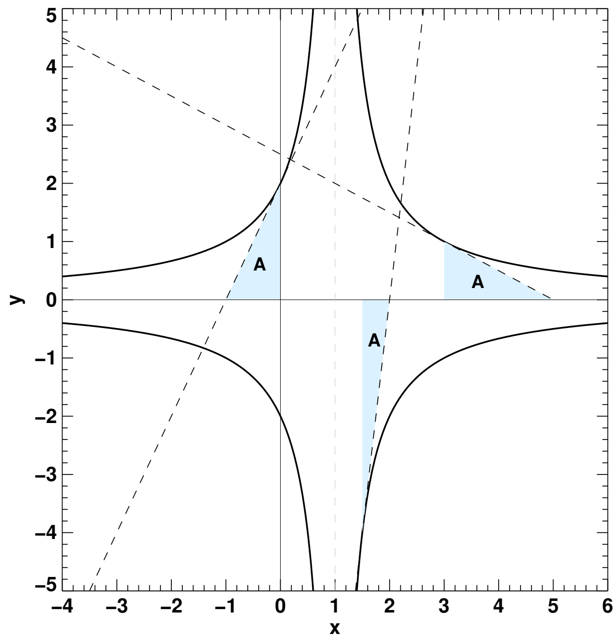

Figure 2.9:
Equal area triangles for $A = 1$ and $c = 1$.

2.6.3 A geometry problem

Find the family of curves for which the area of the triangle bounded by the segment of the tangent line between a point (x, y) on the curve and the x-axis, the perpendicular from the point and the x-axis and the segment of the x-axis between these two lines is equal to a given constant A.

Solution: Let (x_0, y_0) be a point on the curve $y = f(x)$. The equation for the tangent line is $y = y_0 + f'(x)(x - x_0)$. The x intercept is $x = x_0 - \frac{y_0}{y'(x_0)}$. The area of the triangle is

$$A = \frac{1}{2}|y_0|\Big|\Big(x_0 - \big(x_0 - \frac{y_0}{f'(x_0)}\big)\Big| = \pm\frac{1}{2}\frac{f(x_0)^2}{f'(x_0)}. \tag{2.193}$$

The $\pm$ sign arises because the the area A must be constant and y_0 may be positive or negative and the x-intercept of the tangent line could lie to the left or right of x_0. Thus the curve is given by solutions of the nonlinear DE

$$f' = \pm\frac{1}{2}\frac{f^2}{A} \tag{2.194}$$

which has solutions

$$f = \pm\frac{2A}{x - c} \tag{2.195}$$

where c is arbitrary. The solution curves for $A = 1$ and $c = 1$ are shown in Figure 2.9. Changing the value of c simply translates the curves left or right.

2.7 Exact First-Order Differential Equations

We have seen how the general solution of a linear first-order DE can be found. Many first-order differential equations, however, are nonlinear and other solutions methods are requied. There are no general methods to solve nonlinear DEs but there are methods that can be used to solve certain types of nonlinear DEs (e.g., separable equations). Here we consider equations that are exact or that can be make to be exact.

Implicit function are given by equations of the form

$$F(x,y) = C \tag{2.196}$$

where C is a constant. This equation defines curves, the contours of F, on the x-y plane. We can view this as defining a function $y(x)$ or $x(y)$ (there may be several such functions satisfying (2.196)). Taking the former, we can differentiate with respect to x to get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{2.197}$$

'Multiplying' by $\mathrm{d}x$ leads to the *total differential*

$$\mathrm{d}F = \frac{\partial F}{\partial x}\,\mathrm{d}x + \frac{\partial F}{\partial y}\,\mathrm{d}y = 0. \tag{2.198}$$

which for simplicity we will usually write as

$$\mathrm{d}F = F_x\,\mathrm{d}x + F_y\,\mathrm{d}y = 0. \tag{2.199}$$

This states that the infinitesimal vector (dx, dy), which is tangent to the solution curve, is perpendicular to the gradient of F.

Definition: A differential form $M(x,y)\,dx + N(x,y)\,dy$ is said to be **exact** in a rectangle R if there is a function $F(x,y)$ such that

$$M = F_x \quad \text{and} \quad N = F_y \tag{2.200}$$

for all $(x,y) \in R$. Then

$$\mathrm{d}F \equiv F_x\,\mathrm{d}x + F_y\,\mathrm{d}y = M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y. \tag{2.201}$$

The solution of $\mathrm{d}F = 0$ is $F =$ constant.

Example: The differential form

$$\Big(2x + \frac{y^3}{x^2}\Big)\,dx - 3\frac{y^2}{x}\,dy = 0 \tag{2.202}$$

is exact since

$$\begin{aligned} 2x + \frac{y^3}{x^2} &= \frac{\partial}{\partial x}\Big(x^2 - \frac{y^3}{x}\Big), \\ -3\frac{y^2}{x} &= \frac{\partial}{\partial y}\Big(x^2 - \frac{y^3}{x}\Big). \end{aligned} \tag{2.203}$$

Thus

$$\begin{aligned} &\Big(2x + \frac{y^3}{x^2}\Big)\,dx - 3\frac{y^2}{x}\,dy = d\Big(x^2 - \frac{y^3}{x}\Big) = 0 \\ &\implies x^2 - \frac{y^3}{x} = C. \end{aligned} \tag{2.204}$$

This is the implicit solution of the DE.

2.7.1 Test For Exactness

Given the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.205}$$

how do we know if it is exact?

Theorem IV: *Let $f(x,y)$ be a scalar field such that the partial derivatives f_x, f_y, and f_{yx} exist on an open rectangle R. Then if f_{yx} is continuous on R the derivative f_{xy} exists and $f_{xy} = f_{yx}$ on R.*

You should see the proof of this MATH 237 or a similar course.

Theorem V (Euler 1734): *Suppose the first partial derivatives of $M(x,y)$ and $N(x,y)$ are continuous on a rectangle R. Then*

$$M = F_x \quad \textit{and} \quad N = F_y \tag{2.206}$$

is exact on R iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.207}$$

for all $(x,y) \in R$.

Note: With $F_x = M$ and $F_y = N$ this means $F_{xy} = F_{yx}$.

Example: Find the solution of

$$\frac{dy}{dx} = -\frac{3x^2 + 6xy^2}{6x^2y + 4y^3}. \tag{2.208}$$

Solution: We can write this as

$$\left(3x^2 + 6xy^2\right) \mathrm{d}x + \left(6x^2y + 4y^3\right) \mathrm{d}y = 0. \tag{2.209}$$

Here $M = 3x^2 + 6xy^2$ and $N = 6x^2y + 4y^3$ and

$$\begin{aligned} M_y &= 12xy, \\ N_x &= 12xy, \end{aligned} \tag{2.210}$$

Since $M_y = N_x$ are continuous everywhere the equation is exact.

$$\begin{aligned} F_x &= M = 3x^2 + 6xy^2 \\ \implies F &= x^3 + 3x^2y^2 + g(y). \end{aligned} \tag{2.211}$$

Next $F_y = N$ gives

$$\begin{aligned} F_y = 6x^2y + g'(y) &= N = 6x^2y + 4y^3 \\ \implies g'(y) &= 4y^3 \\ \implies g(y) &= y^4. \end{aligned} \tag{2.212}$$

Thus

$$F = x^3 + 3x^2y^2 + y^4 \tag{2.213}$$

and the solutions of the DE are given implicitly by

$$x^3 + 3x^2y^2 + y^4 = C. \tag{2.214}$$

Sketch of a Proof of Theorem V: We need to show there exists a function $F(x, y)$ such that $M = F_x$ and $N = F_y$ iff $M_y = N_x$.

1. If there is a function F such that $M = F_x$ and $N = F_y$ we know that F, F_x, F_y, F_{xy} and F_{yx} exist and are continuous since the first partial derivatives of M and N are continuous. By Theorem IV $F_{xy} = F_{yx} \implies M_y = N_x$.

2. Suppose $M_y = N_x$. If F exists such that $F_x = M$ and $F_y = N$ then

$$F_x = M \implies F(x, y) = \int_{x_0}^{x} M(t, y)\,\mathrm{d}t + g(y) \tag{2.215}$$

so

$$F_y = N = \int_{x_0}^{x} M_y(t, y)\,\mathrm{d}t + g'(y) \tag{2.216}$$

or

$$g'(y) = N(x, y) - \int_{x_0}^{x} M_y(t, y)\,\mathrm{d}t. \tag{2.217}$$

This is only possible if the right hand side is a function of y only, i.e., if

$$\begin{aligned} \frac{\partial}{\partial x}\Big[N(x, y) - \int_{x_0}^{x} M_y(t, y)\,\mathrm{d}t\Big] &= 0 \\ \implies N_x(x, y) - M_y(x, y) &= 0. \end{aligned} \tag{2.218}$$

That is, if $N_x = M_y$ the right hand side of (2.217) is a function of y only so $g(y)$ exists. By construction $N_x = M_y \implies F(x, y)$ exists.

2.7.2 Inexact differential equations

The equation

$$\Big(2x + \frac{y^3}{x^2}\Big)\,\mathrm{d}x - 3\frac{y^2}{x}\,\mathrm{d}y = 0 \tag{2.219}$$

is exact. Multiplying it by x^2 gives

$$\Big(2x^3 + y^3\Big)\,\mathrm{d}x - 3y^2x\,\mathrm{d}y = 0 \tag{2.220}$$

which is not exact since

$$\frac{\partial}{\partial y}\Big(2x^3 + y^3\Big) = 3y^2 \neq \frac{\partial}{\partial x}\Big(-3y^2x\Big) = -3y^2. \tag{2.221}$$

Clearly we can make this inexact DE exact by multiplying by the *integrating factor* $\mu(x,y) = \frac{1}{x^2}$. This is a generalization of the method discussed above for finding solutions of linear first-order DEs. We are now extending it to nonlinear DEs.

In general it can be very difficult to find an integrating factor. If

$$\mu M\,\mathrm{d}x + \mu N\,\mathrm{d}y = 0 \tag{2.222}$$

is exact then

$$\frac{\partial}{\partial y}\Big(\mu M\Big) = \frac{\partial}{\partial x}\Big(\mu N\Big) \tag{2.223}$$

so

$$N\mu_x - M\mu_y + (N_x - M_y)\mu = 0. \tag{2.224}$$

This is a first-order partial differential equation to solve for μ. Usually one guesses at a simple form for μ, for example that μ is a function of x or y only.

Example: Find the solution of

$$\Big(3\frac{x^2}{y^2} + 1 + 2xy\Big)\,dx + \Big(2\frac{x}{y} + 3x^2\Big)\,dy = 0. \tag{2.225}$$

Solution: Here $M = 3\frac{x^2}{y^2} + 1 + 2xy$ and $N = 2\frac{x}{y} + 3x^2$ so

$$\begin{aligned} M_y &= -6\frac{x^2}{y^3} + 2x, \\ N_x &= \frac{2}{y} + 6x. \end{aligned} \tag{2.226}$$

Since $M_y \neq N_x$, the equation is not exact. Multiplying by μ to make it exact leads to

$$\frac{\partial}{\partial y}\Big(\mu(3\frac{x^2}{y^2} + 1 + 2xy)\Big) = \frac{\partial}{\partial x}\Big(\mu(2\frac{x}{y} + 3x^2)\Big) \tag{2.227}$$

or

$$\big(3\frac{x^2}{y^2} + 1 + 2xy\big)\mu_y - \big(2\frac{x}{y} + 3x^2\big)\mu_x + \big(-6\frac{x^2}{y^3} + 2x - \frac{2}{y} - 6x\big)\mu = 0 \tag{2.228}$$

or

$$\big(3\frac{x^2}{y^2} + 1 + 2xy\big)\mu_y - \big(2\frac{x}{y} + 3x^2\big)\mu_x - 2\big(3\frac{x^2}{y^3} + \frac{1}{y} + 2x\big)\mu = 0 \tag{2.229}$$

or

$$\big(3\frac{x^2}{y^2} + 1 + 2xy\big)\mu_y - \big(2\frac{x}{y} + 3x^2\big)\mu_x - \frac{2}{y}\big(3\frac{x^2}{y^2} + 1 + 2xy\big)\mu = 0. \tag{2.230}$$

Now, since *any* solution of this equation will do, you need to notice that if you assume μ is independent of x this reduces to

$$\mu_y = \frac{2}{y}\mu \tag{2.231}$$

which has the solution $\mu = y^2$. Using this as the integrating factor puts the DE in the form

$$\Big(3x^2 + y^2 + 2xy^3\Big)\,\mathrm{d}x + \Big(2xy + 3x^2y^2\Big)\,\mathrm{d}y = 0 \tag{2.232}$$

which is exact. The solution is then easily found to be $x^3 + y^2x + y^3x^2 = C$ (exercise).

2.8 Solution by Substitution

There are several types of first-order DEs that can be solved by a suitable variable transformation. We consider two: Bernoulii equations and homogeneous equations. Then we look at the Brachistochrone problem which is also solved via a substitution.

Bernoulli Equations

Bernoulli equations have the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^{\alpha}. \tag{2.233}$$

If $\alpha = 0$ or 1 the equations is linear and easy to solve so here we are only interested in $\alpha \neq 0, 1$. Let

$$v = y^{1-\alpha}. \tag{2.234}$$

Then

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}x} &= (1-\alpha)y^{-\alpha}\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= (1-\alpha)\frac{v}{y}\Big[-P(x)y + Q(x)y^{\alpha}\Big] \\ &= (1-\alpha)v\Big[-P(x) + Q(x)\frac{1}{v}\Big] \\ &= -(1-\alpha)P(x)v + (1-\alpha)Q, \end{aligned} \tag{2.235}$$

so v is a solution of the linear first-order DE

$$\frac{\mathrm{d}v}{\mathrm{d}x} + \tilde{P}(x)v = \tilde{Q}(x) \tag{2.236}$$

where

$$\tilde{P} = (1-\alpha)P \qquad \text{and } \tilde{Q} = (1-\alpha)Q. \tag{2.237}$$

Example: *Find the solution of*

$$y' + xy = xy^{1/3}. \tag{2.238}$$

Solution: This is a Bernoulii equation with $\alpha = 1/3$. Let

$$v = y^{1-\alpha} = y^{2/3}. \tag{2.239}$$

Then

$$\begin{aligned} v' &= \frac{2}{3}y^{-1/3}y' \\ &= \frac{2}{3}\frac{v}{y}\Big(-xy + xy^{1/3}\Big) \\ &= -\frac{2}{3}xv + \frac{2}{3}x. \end{aligned} \tag{2.240}$$

or

$$v' + \frac{2}{3}xv = \frac{2}{3}x. \tag{2.241}$$

A homogeneous solution is

$$v_h = e^{-x^2/3}. \tag{2.242}$$

To find a particular solution let $v = u(x)v_h(x)$ to get

$$u' = \frac{\frac{2}{3}x}{v_h(x)} = \frac{2}{3}xe^{x^2/3} \tag{2.243}$$

which has

$$u = e^{x^2/3} \tag{2.244}$$

as a solution. Then a particular solution for v is $v_p = uv_h = 1$ as could have been seen by inspection. If you can spot this derectly when looking at the DE for $v(x)$ you do not need to go through the above steps to find a particular solution. The general solution is

$$v = 1 + Ae^{-x^2/3} \tag{2.245}$$

giving

$$y = v^{3/2} = \left(1 + Ae^{-x^2/3}\right)^{3/2}. \tag{2.246}$$

Homogeneous equations

A first-order differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right) \tag{2.247}$$

is called a homogeneous equation. Leibniz discovered how to solve problems of this type in the early 1690's.

Let $y = xz$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}z}{\mathrm{d}x} + z = f(z). \tag{2.248}$$

This is a separable equation for z:

$$\frac{\mathrm{d}z}{f(z) - z} = \frac{\mathrm{d}x}{x}. \tag{2.249}$$

Example: *Solve* $(x + y)\,dx - (x - y)\,dy = 0.$

Solution: Write the DE as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + y}{x - y} = \frac{1 + \frac{y}{x}}{1 - \frac{y}{x}}. \tag{2.250}$$

The equation is homogeneous. Letting $y = xz$ leads to

$$\frac{\mathrm{d}z}{\frac{1+z}{1-z} - z} = \frac{\mathrm{d}x}{x}, \tag{2.251}$$

or

$$\frac{1-z}{1+z^2}\,\mathrm{d}z = \frac{\mathrm{d}x}{x}. \tag{2.252}$$

Integrating:

$$\int \frac{1}{1+z^2}\,\mathrm{d}z - \int \frac{z}{1+z^2}\,\mathrm{d}z = \ln x + c \tag{2.253}$$

or

$$\tan^{-1}(z) - \frac{1}{2}\ln(1+z^2) = \ln x + c. \tag{2.254}$$

Using $z = y/x$ we have

$$\tan^{-1}\big(\frac{y}{x}\big) - \ln\sqrt{1+\big(\frac{y}{x}\big)^2} = \ln x + c. \tag{2.255}$$

The Brachistochrone

Consider a bead sliding down a frictionless wire joining points A and B. What is the shape of the curve which minimizes the travel time? This problem is known as the Brachistochrone problem (from the Greek *brachistos*, shortest + *chronos*, time). The Brachistochrone problem is an historically interesting and important problem. It ultimately led to the development of the Calculus of Variations. The problem was posed by Johann Bernoulli in the June 1696 issue of the *Acta eruditorum* as a challenge to other mathematicians of the time. Solutions by Newton, Leibniz, Jakob Bernoulii and by Johann Bernoulli himself were published the following year [3, 7].

To setup the problem we first consider a related problem in optics. Suppose a ray of light travels through a medium comprised of layers of different material in which the light has different velocities (Figure 2.10). It was experimentally observed by Snell that at the interface between two layers (e.g., at the point x_{j-1} in the figure) the ratio of the sines of the angles the ray makes with the normal on the two sides of the interface is constant, i.e., $\sin(\alpha_{j-1})/\sin(\alpha_j)$ is constant. That is, as the angle α_{j-1} varies, the angle α_j changes to keep this ratio constant. A rational explanation of this is that the ray of light takes the path between two points which minimizes the travel time. This is known as Fermat's principle of least time.

Consider a material comprised of N layers. Layer j has thickness d_j and light travels in it at speed v_j. Light travels from $x = x_0$ at the top of the upper layer to point x_N at the bottom of the N^{th} layer. What path minimizes the travel time?

The travel time ΔT between x_{j-1} and x_{j+1} is

$$\Delta T = \frac{\sqrt{(d_j^2 + (x_j - x_{j-1})^2}}{v_j} + \frac{\sqrt{(d_{j+1}^2 + (x_{j+1} - x_j)^2}}{v_{j+1}}. \tag{2.256}$$

For fixed x_{j-1} and x_{j+1} what value of x_j minimizes the travel time ΔT? It is given by the value of x_j for which

$$\frac{\mathrm{d}\Delta T}{\mathrm{d}x_j} = \frac{x_j - x_{j-1}}{\sqrt{(d_j^2 + (x_j - x_{j-1})^2}v_j} - \frac{x_{j+1} - x_j}{\sqrt{(d_{j+1}^2 + (x_{j+1} - x_j)^2}v_{j+1}} = 0. \tag{2.257}$$

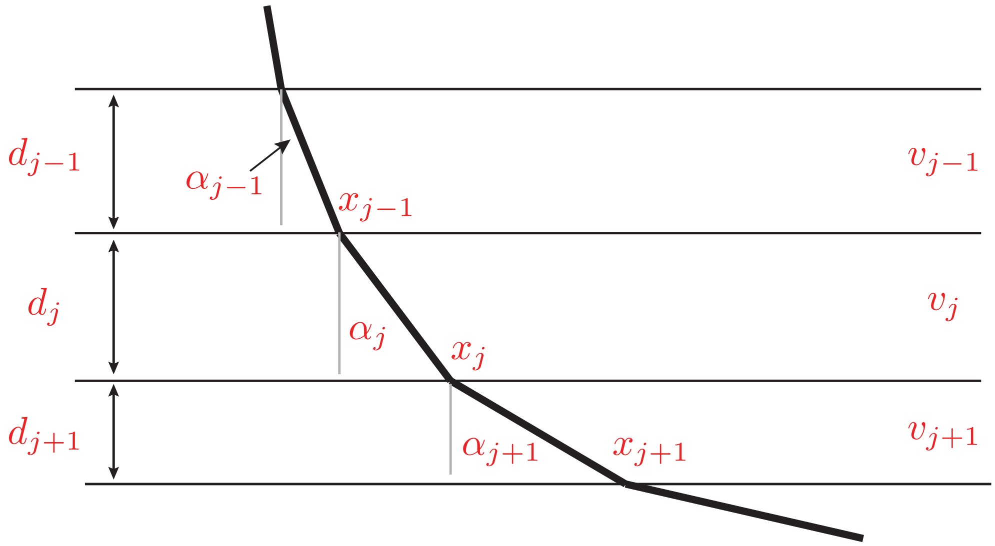

Figure 2.10:

which we can write as

$$\frac{\sin(\alpha_j)}{v_j} = \frac{\sin(\alpha_{j+1})}{v_{j+1}}. \tag{2.258}$$

Thus, we must have

$$\frac{\sin(\alpha_j)}{v_j} = \textit{constant} \text{ for all } j, \tag{2.259}$$

for if

$$\frac{\sin(\alpha_j)}{v_j} \neq \frac{\sin(\alpha_{j+1})}{v_{j+1}} \tag{2.260}$$

for some j we can move x_j until these are equal and the travel time is reduced.

Taking the limit as the number of layers goes to zero so that the properties of the medium vary continuously with height y, including, for example, the light speed $v(y)$, the path taken by the ray of light becomes a smooth curve along which (2.11)

$$\frac{\sin(\alpha)}{v(y)} = \textit{constant} \tag{2.261}$$

where α is the angle between the vertical and the direction the ray of light is travelling.

Johann Bernoulli's solution of the Brachistochrone problem was based on this principle. Consider a frictionless wire in the shape of a curve $y(x)$ joining the point A at the origin to a point B below and to the right (see Figure 2.11). A bead moves along the curve starting from rest at the origin. Conservation of energy gives $\frac{1}{2}v^2 + gy = 0$ so the bead moves with speed $v = \sqrt{-2gy}$. Now

$$\sin(\alpha) = \frac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}} = \frac{1}{\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}} = \frac{1}{\sqrt{1 + (y')^2}}. \tag{2.262}$$

Hence, the curve is given by

$$\frac{\sin(\alpha)}{v(y)} = \frac{1}{\sqrt{-2gy}\sqrt{1 + (y')^2}} = \textit{constant} \tag{2.263}$$

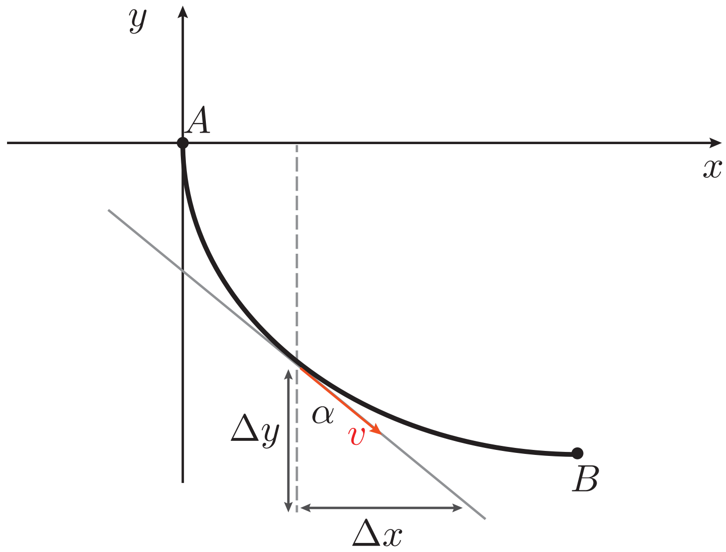

Figure 2.11:

or

$$\frac{1}{\sqrt{-y}\sqrt{1+(y')^2}} = \tilde{c}. \tag{2.264}$$

Rearranging we have

$$1 + (y')^2 = -\frac{c}{y}, \tag{2.265}$$

where c is a new constant, positive as $y < 0$. Solving for y' we have

$$y' = -\sqrt{-\frac{c}{y} - 1} \tag{2.266}$$

where the negative square root is used because $y' < 0$ (at least at the beginning, y' can change sign). This is a separable equation:

$$-\sqrt{\frac{-y}{c+y}}\,\mathrm{d}y = \mathrm{d}x. \tag{2.267}$$

To solve this we make the substitution

$$\tan\frac{\theta}{2} = \sqrt{\frac{-y}{c+y}}. \tag{2.268}$$

Then

$$\frac{-y}{c+y} = \tan^2\frac{\theta}{2} \tag{2.269}$$

from which we find $y = -c\sin^2\frac{\theta}{2}$. Then $dy = -c\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta$ and the DE becomes

$$\begin{aligned} \mathrm{d}x = \left(-\tan\frac{\theta}{2}\right)\left(-c\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,\mathrm{d}\theta\right) &= c\sin^2\frac{\theta}{2}\,\mathrm{d}\theta \\ &= \frac{c}{2}\Big(1-\cos\theta\Big)\,\mathrm{d}\theta \end{aligned} \tag{2.270}$$

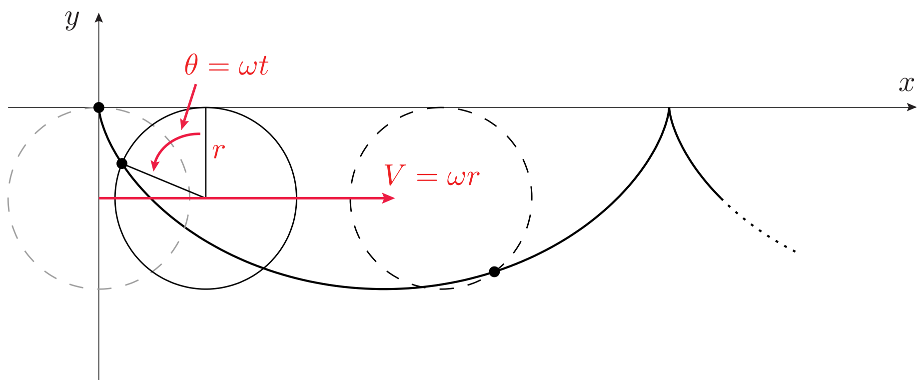

Figure 2.12: A cycloid of radius r rolling beneath the x-axis with angular velocity ω.The center of the wheel moves with speed V.

so

$$x = \frac{c}{2}\Big(\theta - \sin\theta\Big) + A. \tag{2.271}$$

The initial point is $(x, y) = 0$. From $y = -c\sin^2\frac{\theta}{2}$ we have $\theta = 0$ at the initial point. Hence $A = 0$. The final solution is

$$\begin{aligned} x &= \frac{c}{2}\Big(\theta - \sin\theta\Big) \\ y &= -\frac{c}{2}\Big(1 - \cos\theta\Big). \end{aligned} \tag{2.272}$$

This curve is a cycloid: the path followed by a point on the circumference of a wheel rolling along a flat surface, in this case the surface lies above the wheel. Here $c/2$ is the radius of the wheel.

Cycloid: Consider a wheel of radius r which lies below and tangent to the x-axis. It rolls along the x-axis with constant angular velocity ω. A point on the circumference of the wheel is initially at the origin. If the wheel spins counter-clockwise about its centre, which is initially at $(x_c, y_c) = (0, -r)$, while its centre remains stationary a point on the circumference has position $(x, y) = \big(-r\sin(\omega t), r(1 - \cos(\omega t)\big)$. The period of rotation is $\tau = \frac{2\pi}{\omega}$. If the wheel rolls along the x-axis it will move a distance equal to the circumference in one period, so the centre of the wheel moves with speed $V = \frac{2\pi r}{\tau} = \omega r$. So taking into account the motion of the wheel centre the point has position $(x_p(t), y_p(t))$ given by

$$\begin{aligned} x_p(t) &= \omega r t - r\sin(\omega t) \\ y_p(t) &= r\big(1 - \cos(\omega t)\big). \end{aligned} \tag{2.273}$$

In terms of the counter-clockwise angle of rotation $\theta = \omega t$ the path followed by the point is

$$\begin{aligned} x_p(\theta) &= r\big(\theta - \sin(\theta)\big) \\ y_p(\theta) &= r\big(1 - \cos(\theta)\big). \end{aligned} \tag{2.274}$$

This is (2.272) with $r = c/2$.

The cycloid has many interesting properties. The time taken for a bead released from rest at any point on the curve to reach the bottom under the influence of gravity is independent of the release point. Hence, the cycloid is also an *isochrone*. This result was proved by Hygens in 1659 using infinitesimals and a geometric arguement. Jacob Bernoulli proved this result in 1690 by setting up an appropriate differential equation and solving it. Huygens used this idea in his invention of a pendulum clock realising that if the pendulum moved along a cycloid it would keep time more accurately as the period of oscillation did not depend on the amplitude of the oscillation as it does for a simple pendulum. This is useful because friction and damping associated with the motion of the pendulum through air will decrease the amplitude of oscillation over time.

2.9 Problems

1. Find the solutions of the following separable differential equations.

   (a) $\frac{dy}{dx} = y^2 x^3, \quad y(0) = -1;$

   (b) $\frac{dx}{dt} = (1 - x)\sin t;$

   (c) $\frac{dp}{dt} = (p-1)(p-2);$

   (d) $\frac{dq}{ds} = \tan(s)q^3 \quad q(\pi) = 1.$

   On what intervals are the solutions defined?

2. For the following autonomous equations plot the forcing functions and find the fixed points. Are the fixed point stable, i.e., if the solution is perturbed away from the fixed point does it return to the fixed point? Is it globally stable, i.e., does the solution go to the fixed point no matter where it starts from? Answer this without using the solution of the equation.

   (a)
   $$\frac{dv}{dt} = -1 - v. \tag{2.275}$$

   (b)
   $$\frac{dv}{dt} = -1 - |v|v. \tag{2.276}$$

   (c)
   $$\frac{dp}{dt} = -p(p-1)(p-5). \tag{2.277}$$

3. Consider the autonomous differential equation
   $$\frac{dy}{dx} = f(y) \tag{2.278}$$
   where $f(y)$ is a continous function in any interval of interest. Suppose $y(x)$ is a solution of the differential equation on some interval (a, b). Prove that the solution $y(x)$ is a monotonic function on (a, b). If the solution is exists on $(-\infty, \infty)$ show that $y(x)$ goes to a constant or to $\pm\infty$ as $x \to \pm\infty$.

4. Consider the IVP
   $$y' = y, \qquad y(0) = 1. \tag{2.279}$$
   Estimate the value of $y(1)$ using Euler's method with N steps of size $h = 1/N$.

(a) Show that the estimate is $y_N = (1+h)^N$.

(b) Show that the error $E = e - y_N$ in this estimate satisfies

$$\lim_{h\to 0} \frac{E}{h} = \frac{1}{2}e \approx 1.359 \tag{2.280}$$

and hence the error E goes to zero linearly in h as $h \to 0$. That is, Euler's method gives a first-order approximation for $y(1)$.

5. Consider the IVP

$$y' = y, \qquad y(0) = 1. \tag{2.281}$$

Estimate the value of $y(1)$ using the RK-2 method with N steps of size $h = 1/N$.

(a) Show that the estimate is $y_N = (1 + h + \frac{h^2}{2})^N$.

(b) Show that the error $E = e - y_N$ in this estimate satisfies

$$\lim_{h\to 0} \frac{E}{h^2} = \frac{1}{6}e \approx 0.453 \tag{2.282}$$

and hence the error E goes to zero quadratically in h as $h \to 0$. That is, the RK-2 method gives a sedond-order approximation for $y(1)$.

6. Consider the IVP

$$y' = x + y, \qquad y(0) = 1. \tag{2.283}$$

Using Picard's method find the n^{th} iterate $y_n(x)$ using induction. What does $y_n(x)$ converge too? Show that this is the solution of the IVP. Do this using the following starting functions:

(a) $y_0(x) = 1$;

(b) $y_0(x) = e^x$;

(c) $y_0(x) = 1 + x$;

(d) $y_0(x) = \cos(x)$.

7. Find the exact solution of the IVP

$$y' = y^2, \qquad y(0) = 1. \tag{2.284}$$

Starting with $y_0(x) = 1$ use Picard's method to calculate $y_1(x)$, $y_2(x)$, and $y_3(x)$. Compare your results with the exact solution. Maple would be helpful in going further. Find the exact solution and compare with the Picard iterates.

8. Consider the system of first-order DEs

$$x' = y \tag{2.285}$$
$$y' = -x \tag{2.286}$$

with initial conditions $x(0) = 0$ and $y(0) = 1$. This can be written in vector form as

$$\begin{aligned} \frac{d\vec{v}}{dt} &= \vec{f}(t, \vec{v}) \\ \vec{v}(0) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{2.287}$$

where $\vec{v} = (x, y)^T$ and $\vec{f} = (y, -x)^T$ (T being the transpose). Picard's method can be applied to this equation first-order vector equation. Doing so find $x(t)$ and $y(t)$.

9. Show that $f(x,y) = x^2|y|$ satisfies a Lipschitz condition on the rectangle $|x| < 1$ and $|y| < 1$ but that f_y fails to exist at many points in the rectangle.

10. Show that $f(x,y) = xy^2$

    (a) satisfies a Lipschitz condition on any rectangle $a \le x \le b$ and $c \le y \le d$.

    (b) does not satisfy a Lipschitz condition in y on any rectangle $a \le x \le b$ and $-\infty < y < \infty$.

11. Show that $f(x,y) = xy$

    (a) satisfies a Lipschitz condition on any rectangle $a \le x \le b$ and $c \le y \le d$.

    (b) satisfies a Lipschitz condition in y on any rectangle $a \le x \le b$ and $-\infty < y < \infty$.

    (c) does not satisfy a Lipschitz condition in y on the entire plane.

12. Consider the problem

$$y' = |y|, \qquad y(x_0) = y_0. \tag{2.288}$$

    (a) For what points (x_0, y_0) does Theorem I imply a unique solution $y(x)$ exists on a non-zero interval containing x_0?

    (b) For what points (x_0, y_0) does Theorem II imply a unique solution $y(x)$ exists on a non-zero interval containing x_0?

    (c) What is the solution of the DE?

13. Consider the problem

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -1 - v|v|, \qquad v(0) = v_0. \tag{2.289}$$

    For what points $(0, v_0)$ does Theorem I imply a unique solution $v(t)$ exists on a non-zero interval $|t| < \delta$ for some $\delta > 0$?

14. Show that $f(x,y) = y^{\frac{1}{2}}$

    (a) does not satisfy a Lipschitz condition in y on the rectangle $|x| \le 1$ and $0 \le y \le 1$;

    (b) does satisfy a Lipschitz condition in y on any rectangle $|x| \le 1$ and $c \le y \le d$ with $0 < c < d$.

15. Consider the problem

$$y' = y^{\frac{1}{2}}, \qquad y(x_0) = y_0 \tag{2.290}$$

    with $y_0 \ge 0$.

    (a) For what points (x_0, y_0) does Theorem I imply a unique solution $y(x)$ exists on a non-zero interval containing x_0?

    (b) For what points (x_0, y_0) does Theorem II imply a unique solution $y(x)$ exists on a non-zero interval containing x_0?

    (c) For what points (x_0, y_0) does Theorem III imply that at least one solution $y(x)$ exists on a non-zero interval containing x_0?

    (d) Find all the solutions. For what points is the solution unique? For what points is it non-unique?

16. Consider the problem
$$y' = |y|^{\frac{1}{2}}, \qquad y(x_0) = 0. \tag{2.291}$$
There are four solutions on $-\infty < x < \infty$. What are they?

17. Solve the following DEs by finding an integrating factor

    (a) $y' + 3x^2 y = xe^{x^2-x^3}$;

    (b) $y' + 3y = e^{-3x} \sin x \cos x$;

    (c) $y' + t^2 y = te^{t^2-t^3/3}$;

    (d) $\frac{\mathrm{d}s}{\mathrm{d}t} + 2s = t\sin(t)$.

18. Solve the following DEs by finding a homogeneous solution $y_h(x)$ and finding a particular solution $y_p(x)$ by setting $y_p = v(x)y_h(x)$ and solving a DE for $v(x)$.

    (a) $xy' - y = x^3 \cos(x)$;

    (b) $2y' + y = \frac{1}{1+e^x}$;

    (c) $3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = t^2 e^{2t}$;

    (d) $3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = t^2 e^{-2t/3}$.

19. Constant coefficient linear first-order DEs arise in many problems. They have the general form $y' + \alpha y = f(x)$ where α is a constant. Show that by a suitable scaling of x this equation can always be put in the form $y' + y = g(x)$ (where x is different from in the first equation). Sometimes, e.g., if the independent variable is time, you might want to scale only by a positive scale in which case there are two possible forms: $y' + y = g(x)$ or $y' - y = g(x)$. Find the general solution of the following DEs. In each case n is a positive integer.

    (a) $y' + y = e^{-x}$;

    (b) $y' + y = e^{kx}$ with $k \neq -1$.

    (c) $y' + y = x^n$. Find a particular solution y_p by quessing it is a polynomial in x;

    (d) $y' + y = x^n e^{-x}$. Find a particular solution y_p by setting $y_p = f(x)e^{-x}$, finding a DE for $f(x)$ and then finding a particular solution for f;

    (e) $y' + y = x^n e^{kx}$ $k \neq -1$, Follow procedure from previous problem;

    (f) $y' + y = e^{-x} + 2e^x + x^2 - x^3$.

    Note the difference in the results when the exponential terms in the forcing functions are solutions of the homogeneous equation and when they are not. More on this when we study second-order DEs.

20. Solve the following DEs.

    (a) $y' - \frac{1}{\tan(x)} y = \sin(2x)$;

    (b) $y' + xy = x$;

    (c) $t^2 \frac{dx}{dt} - 3x = 2t^3 e^{3t-3t^{-1}}$.

    (d) $t\frac{dv}{dt} - (1+t)v = t^3$.

21. Consider the DE
$$y' + y = \cos(x). \tag{2.292}$$

    (a) Solve this by guessing that there is a particular solution of the form $y = a\cos(x) + b\sin(x)$ and substituting into the differential equation.

    (b) Find the particular solution by using the fact that $\cos(x) = \Re\{e^{ix}\}$. That is, solve $y' + y = e^{ix}$ and take the real part of the solution. Explain why this gives the solution of the DE invoking linearity of the operator $L = \frac{d}{dx} + 1$.

    (c) In a similar manner find the solution of the following:

i. $y' + y = \sin(x)$;

ii. $y' + y = \cos(2x)$;

iii. $y' + 2y = \sin(3x)$.

22. Use the solution of the previous problem to find the general solution of (see end of section on the alternative approach)

$$y' + y = \sin(x) + e^x - e^{-x} - \pi\cos(2x) + \frac{3}{2}\sin(2x). \tag{2.293}$$

23. Without solving the DEs, for what initial points does the existence-uniqueness theory say that the following IVPs have a unique solution. On what domains do the solutions exist according to the theorem?

(a) $y' + \frac{1}{x-1}y = \frac{x}{x-2}, \qquad y(x_0) = y_0.$

(b) $\sin(t)\frac{\mathrm{d}x}{\mathrm{d}t} + tx = t^2, \qquad x(t_0) = x_0.$

24. Suppose the functions $x_1(t) = e^{t^2} + te^{t^3}$ and $x_2(t) = \frac{1}{5}e^{t^2} + te^{t^3}$ are solutions of the same first-order linear DE. What is a homogeneous solution for the DE? What is the general solution? What is the differential equation?

25. Find the one-parameter family of straight lines and their envelope for each of the following Clairaut equations:

(a) $y = xy' + [1 + (y')^2]$;

(b) $y = xy' - (y')^3$;

(c) $y = xy' - e^{y'}$;

(d) $y = xy' - (y')^{2/3}$.

26. Which of the following equations are exact? Solve those that are.

(a) $\left(x - \frac{1}{y}\right)\mathrm{d}y + y\,\mathrm{d}x = 0$;

(b) $\left(\sin x \tan y + 1\right)\mathrm{d}x + (x + y^3)\,\mathrm{d}y = 0$;

(c) $\cos x \cos^2 x\,\mathrm{d}x + 2\sin x \sin y \cos y\,\mathrm{d}y = 0$;

(d) $\big[\cos(xy) - xy\sin(xy)\big]e^y\,\mathrm{d}x + \big[-x\sin(xy) + x\cos(xy)\big]e^y\,\mathrm{d}y = 0$;

(e) $(2s + s^2)e^{x+s}\,\mathrm{d}s + s^2e^{x+s}\,\mathrm{d}x = 0$;

(f) $\left(3x^2y + 2x^2y + 4x^3y^3\right)\mathrm{d}x + \left(x^3 + 2x^2y + 3x^4y^2 + 1 + x\right)\mathrm{d}y = 0$;

(g) $\left(x^3 + 2x^4t + 3t^2\right)\mathrm{d}t + \left(3x^2t + 4x^3t^2\right)\mathrm{d}x = 0$;

(h) $dx = \frac{y}{1-x^2y^2}\,\mathrm{d}x + \frac{x}{1-x^2y^2}\,\mathrm{d}y$;

(i) $\frac{1}{2}\sin\left(\frac{x}{y}\right)\mathrm{d}x - \frac{1}{y}\sin\left(\frac{x}{y}\right)\mathrm{d}y = 0$.

27. Suppose that

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{2.294}$$

is exact. Find all possible functions $\mu(x,y)$ for which

$$\mu M(x,y)\,\mathrm{d}x + \mu N(x,y)\,\mathrm{d}y = 0 \tag{2.295}$$

is also exact (Note: this is a challenge and requires ideas from multi-variable calculus. Will not be on quiz/midterm/final exam).

28. Consider the Brachistochrone problem. If the initial point is the origin $(0, 0)$ and the final point has coordinates (x_b, y_b) where $y_b < 0$, determine the values of θ and c. Is there always a solution? Under what conditions does the path go below y_b before terminating at (x_b, y_b)?

29. Solve the following problems

    (a) $t\frac{\mathrm{d}x}{\mathrm{d}t} = x + (t^2 - x^2)^{1/2}$.

    (b) $x' = \frac{x}{t} + tx^3; \qquad x(\frac{1}{2}) = \sqrt{\frac{2}{15}}$.

    (c) $tx' + x = t\sec(tx)$.

Chapter 3

Non-dimensionalization and Dimensional Analysis

In this course we will use *non-dimensionalization* to simplify problems by introducing scaled dimensionless variables in order to make the problems look simple by reducing the number of parameters that appear in the problem. In more advanced courses, e.g., on perturbation theory, a further goal is to scale variables so that the dimensionless variables and their derivatives have typical sizes of about 1 in order to introduce small parameters that indicate which terms may be neglected.

Dimensional analysis is a useful method for deducing functional relationships between variables and simplifying problems.

In an equation modelling a physical process all of the terms separated by a '+' sign must have the same units. Otherwise we'd have something silly like 'oranges + avocados = houses', or $2\ \mathrm{kg} + 4\ \mathrm{m} = 10\ \mathrm{m\ s^{-1}}$. For example, in a mixing tank the equation for the mass of salt $A(t)$ in the tank is, based on the principle of conservation of mass,

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= \text{rate mass enters} - \text{rate mass exits}, \\ &= c_{in} V_{f_{in}} - c_{out} V_{f_{out}}. \end{aligned} \tag{3.1}$$

Here

- $\frac{\mathrm{d}A}{\mathrm{d}t}$ has units of kg s^{-1}
- c_{in} has units of kg m^{-3}
- $V_{f_{in}}$ has units of $\mathrm{m}^3\ \mathrm{s}^{-1}$

so $c_{in} V_{f_{in}}$ has units of kg s^{-1}, the same as those of $\frac{\mathrm{d}A}{\mathrm{d}t}$. Similarly for $c_{out} V_{f_{out}}$.

Checking the dimensions of terms in your equation can be helpful. It is always a good idea and often reveals simple mistakes.

We will use square brackets to denote the *dimensions* of a quantity. All scientific units can be written in terms of *primary dimensions* such as mass [M], length [L], time [T], electric current [I], temperature [θ] etc. The units for these quantities depend on the measurement system being used. The SI units for these quantities are kilograms (kg), meters (m), seconds (s), amperes (A) and Kelvin (K), although $^{\circ}C$ may be used for the latter. Other systems include the British foot-pound system and the cgs system (grams-centimeters-seconds).

Other *secondary* dimensions can be written in terms of primary dimensions. For example,

- force F = mass × acceleration has dimensions [F] = [M][L][T]$^{-2}$
- pressure P = force per unit area has dimensions [P] = [F][L]$^{-2}$ = [M][L]$^{-1}$[T]$^{-2}$
- energy E has dimensions [E] = [M][L]^2[T]$^{-2}$

etc.

Use of dimensionless parameters can reduce the number of parameters in a problem, simplify the presentation and analysis of results and improve understanding. For experimentalists or numerical modellers the use of dimensionless parameters can significantly reduce the number of parameters that need to be varied to explore the behaviour of a system in parameter space.

3.1 Non-dimensionalization

Problems are non-dimensionalized by scaling variables by a characteristic scale. We illustrate the process via a few examples.

Example: Consider the mixing tank problem above. We assume the volume fluxes into and out of the tank are equal and constant, i.e., $V_{f_{in}} = V_{f_{out}} = V_f$. Let V be the constant volume of the tank and $c(t) = A(t)/V$ be the concentration of salt in the tank. The fluid in the tank is well mixed and $c_{out} = c$. The governing equation is then

$$\frac{\mathrm{d}c}{\mathrm{d}t} = c_{in}\frac{V_f}{V} - c\frac{V_f}{V} \tag{3.2}$$

and the IVP is

$$\begin{aligned} \frac{\mathrm{d}c}{\mathrm{d}t} + \frac{V_f}{V}c &= c_{in}\frac{V_f}{V}, \\ c(0) &= c_0. \end{aligned} \tag{3.3}$$

The problem involves two variables c and t and three parameters V, V_f and c_{in}. We introduce a characteristic concentration C_c and time scale T_c and non-dimensional concentration $\tilde{c}$ and time $\tilde{t}$ via

$$c = C_c\tilde{c} \qquad t = T_c\tilde{t} \tag{3.4}$$

where C_c and T_c are currently unknown. That is, we haven't decided what to use yet. We will choose them to make the problem look relatively simple. Under our scaling time derivatives scale according to

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} = \frac{1}{T_c}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}. \tag{3.5}$$

Note that the factor $1/T_c$ carries the dimensions of the time derivative operator. Using this

$$\frac{\mathrm{d}c}{\mathrm{d}t} = \frac{1}{T_c}\frac{\mathrm{d}c}{\mathrm{d}\tilde{t}} = \frac{1}{T_c}\frac{\mathrm{d}\big(C_c\tilde{c}\big)}{\mathrm{d}\tilde{t}} = \frac{C_c}{T_c}\frac{\mathrm{d}\tilde{c}}{\mathrm{d}\tilde{t}}. \tag{3.6}$$

Note here that on the right-hand side C_c/T_c has dimensions of concentration over time, the dimensions of the left hand side. The term $\frac{\mathrm{d}\tilde{c}}{\mathrm{d}\tilde{t}}$ is dimensionless.

Substituting this, along with $c = C_c\tilde{c}$ into the IVP gives

$$\begin{aligned} \frac{C_c}{T_c}\frac{\mathrm{d}\tilde{c}}{\mathrm{d}\tilde{t}} + \frac{V_f}{V}C_c\tilde{c} &= c_{in}\frac{V_f}{V}, \\ C_c\tilde{c}(0) &= c_0. \end{aligned} \tag{3.7}$$

or

$$\begin{aligned} \frac{\mathrm{d}\tilde{c}}{\mathrm{d}\tilde{t}} + T_c\frac{V_f}{V}\tilde{c} &= \frac{T_c}{C_c}c_{in}\frac{V_f}{V}, \\ \tilde{c}(0) &= \frac{c_0}{C_c}, \end{aligned} \tag{3.8}$$

where now all the terms are dimensionless. **We now choose the scales C_c and T_c to put the problem in a simple form.** There are many ways to do this. The DE includes two 'messy' coefficients. We have two scalings to play around with so we can make both of them equal to one. That is, choose

$$T_c = \frac{V}{V_f} \tag{3.9}$$

so that the coefficient of $\tilde{c}$ in the DE is equal to one and then choose C_c to set the right hand side equal to one. That is, set

$$\frac{T_c}{C_c}c_{in}\frac{V_f}{V} = \frac{c_{in}}{C_c} = 1 \qquad \Longrightarrow \qquad C_c = c_{in} \tag{3.10}$$

(note we have used $T_c = V/V_f$). We can then define

$$\tilde{c}_0 = \frac{c_0}{C_c} \tag{3.11}$$

to get the final dimensionless form of the problem:

$$\begin{aligned} \frac{d\tilde{c}}{d\tilde{t}} + \tilde{c} &= 1, \\ \tilde{c}(0) &= \tilde{c}_0. \end{aligned} \tag{3.12}$$

This involves only one parameter, $\tilde{c}_0$. The solution of this IVP problem is

$$\tilde{c}(\tilde{t}) = 1 + (\tilde{c}_0 - 1)e^{-\tilde{t}}. \tag{3.13}$$

To get the solution of the dimensional problem you can now use $c = C_c\tilde{c}$ and $t = T_c\tilde{t}$ to get

$$\begin{aligned} c(t) &= C_c\Big(1 + (\tilde{c}_0 - 1)e^{-\tilde{t}}\Big) \\ &= C_c + \big(c_0 - C_c\big)e^{-t/T_c} \\ &= c_{in} + \big(c_0 - c_{in}\big)e^{-\frac{V_f}{V}t}. \end{aligned} \tag{3.14}$$

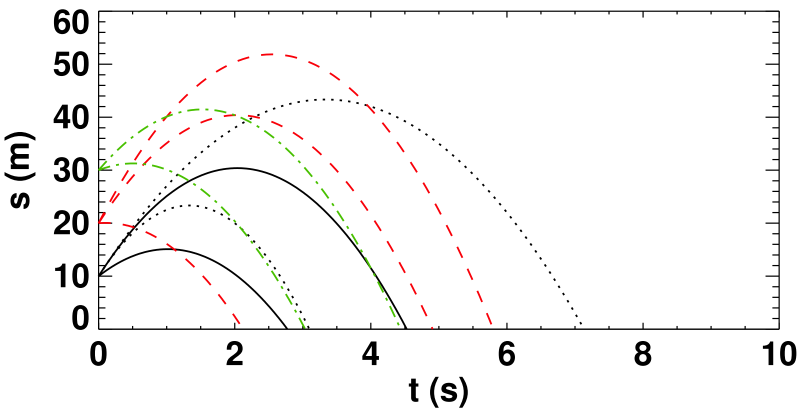

Figure 3.1: Height vs time for various initial heights s_0, initial velocities v_0 and different gravitational accelerations g.

Example: The height of an object thrown up from height $s_0 > 0$ with velocity $v_0 > 0$ at time t is governed by

$$\begin{aligned} \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} &= -g \\ s(0) &= s_0 \\ \frac{\mathrm{d}s}{\mathrm{d}t}(0) &= v_0. \end{aligned} \tag{3.15}$$

which has the solution

$$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0. \tag{3.16}$$

This problem involves the two variables s and t, and three parameters g, v_0, and s_0. To plot solutions we need to vary g (for different planets), v_0 and s_0. Sample trajectories are shown in Figure 3.1.

We can make the problem dimensionless. Here we introduce characteristic length and time scales L_c and T_c along with dimensionless variables $\tilde{s}$ and $\tilde{t}$ given by $s = L_c\tilde{s}$ and $t = T_c\tilde{t}$. Using (3.5) we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big) = \frac{1}{T_c}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\Big(\frac{1}{T_c}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\Big) = \frac{1}{T_c^2}\frac{\mathrm{d}^2}{\mathrm{d}\tilde{t}^2}. \tag{3.17}$$

Thus

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{T_c}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\Big(L_c\tilde{s}\Big) = \frac{L_c}{T_c}\frac{\mathrm{d}\tilde{s}}{\mathrm{d}\tilde{t}}, \tag{3.18}$$

and

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{1}{T_c^2}\frac{\mathrm{d}^2}{\mathrm{d}\tilde{t}^2}\Big(L_c\tilde{s}\Big) = \frac{L_c}{T_c^2}\frac{\mathrm{d}^2\tilde{s}}{\mathrm{d}\tilde{t}^2}. \tag{3.19}$$

In terms of the dimensionless variables, the IVP is

$$\begin{aligned}\frac{L_c}{T_c^2}\frac{\mathrm{d}^2\tilde{s}}{\mathrm{d}\tilde{t}^2} &= -g \\ L_c\tilde{s}(0) &= s_0 \\ \frac{L_c}{T_c}\frac{\mathrm{d}\tilde{s}}{\mathrm{d}\tilde{t}}(0) &= v_0,\end{aligned} \tag{3.20}$$

or

$$\begin{aligned}\frac{\mathrm{d}^2\tilde{s}}{\mathrm{d}\tilde{t}^2} &= -g\frac{T_c^2}{L_c} \\ \tilde{s}(0) &= \frac{s_0}{L_c} \\ \frac{\mathrm{d}\tilde{s}}{\mathrm{d}\tilde{t}}(0) &= v_0\frac{T_c}{L_c},\end{aligned} \tag{3.21}$$

where now all the terms are dimensionless.

There are many ways to choose the scalings T_c and L_c. If you are dropping an object from a height s_0 above the ground (at $s = 0$) then choosing $L_c = s_0$ is very sensible as then $\tilde{s}$ varies from an initial value of 1 to a final value of 0 when it hits the ground. If you fire a cannonball vertically upward from the ground at $s = 0$ (so $s_0 = 0$) then it will rise to a maximum height of $s_{max} = v_0^2/2g$ in which case using $L_c = v_0^2/2g$ is a good choice because then $\tilde{s}$ increases from its initial value of 0 to a maximum value of 1 before decreasing back to 0 as the cannonball returns to the ground. There are likewise many possible choices for T_c. As a general rule, the best choices for the scales will depend on your problem.

Here I will assume $s_0 \neq 0$ and $v_0 \neq 0$ and take $L_c = s_0$ and $T_c = v_0/g$. The latter is the time taken for the object to reach its maximum height. With these choices

$$g\frac{T_c^2}{L_c} = v_0\frac{T_c}{L_c} = \frac{1}{2}\frac{v_0^2}{gs_0}. \tag{3.22}$$

and the problem becomes

$$\begin{aligned}\frac{\mathrm{d}^2\tilde{s}}{\mathrm{d}\tilde{t}^2} &= -2\lambda \\ \tilde{s}(0) &= 1 \\ \frac{\mathrm{d}\tilde{s}}{\mathrm{d}\tilde{t}}(0) &= 2\lambda,\end{aligned} \tag{3.23}$$

where

$$\lambda = \frac{1}{2}\frac{v_0^2}{gs_0}. \tag{3.24}$$

The solution in dimensionless terms is

$$\tilde{s}(\tilde{t}) = -\lambda\tilde{t}^2 + 2\lambda\tilde{t} + 1. \tag{3.25}$$

This involves three variables ($\tilde{s}$, $\tilde{t}$ and λ) instead of five and to plot solutions we just need to vary the value of a single variable, λ, instead of three. There is now a 1-parameter family of solutions. All dimensionless solutions have the properties that

- the initial height is $\tilde{s}(0) = 1$.
- the initial vertical velocity is $\frac{d\tilde{s}}{d\tilde{t}}(0) = 2\lambda$.
- $\frac{d\tilde{s}}{d\tilde{t}}(\tilde{t}) = 0$ when $\tilde{t} = 1$.
- the maximum height is $\tilde{s}(1) = 1 + \lambda$.
- the object reaches the ground ($\tilde{s}(\tilde{t}) = 0$) at time $\tilde{t}_f = 1 + \sqrt{\frac{1+\lambda}{\lambda}}$.

Some sample trajectories are plotted in Figure 3.2.

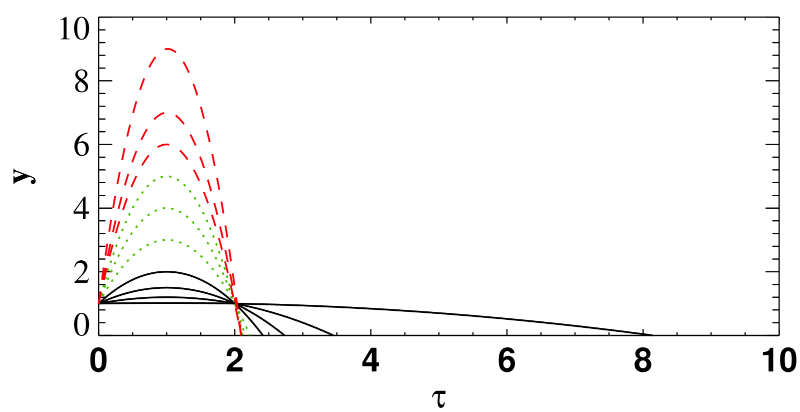

Figure 3.2: Nondimensional height vs time ($y = \tilde{s}$ and $\tau = \tilde{t}$) for several values of λ.

3.2 Dimensional Analysis: The Buckingham-π Theorem

The Buckingham-π Theorem is used to determine the number of dimensionless parameters in a problem and can simplify relationships among variables by reducing the number of variables. It is named after E. Buckingham who proved a general version of the theorem in 1914 however the theorem was first proved in 1878 by J. Bertrand and the technique was made widely known by Rayleigh in the 1890's.

Suppose there are n physical variables and parameters Q_1, Q_2, ..., Q_n and the solution of a mathematical model gives one in terms of the others:

$$Q_n = f(Q_1, Q_2, \ldots, Q_{n-1}). \tag{3.26}$$

Suppose that there are r independent basic physical dimensions [M], [L], [T], etc. Suppose that k dimensional quantities π_j for $j = 1, 2, \ldots, k$ can be constructed via multiples of powers of the Q_j. We will see that there are at least $n - r$ of them. Then

$$\pi_k = h(\pi_1, \pi_2, \ldots, \pi_{k-1}). \tag{3.27}$$

To see how the Buckingham-π Theorem is used we consider a simple illustrative example: the simple nonlinear pendulum.

Example: Consider a simple pendulum of mass m in the absence of air damping. Assume the mass is attached to a frictionless pivot via an inextensible massless wire of length l. Assuming Newtonian mechanics applied (e.g., ignore relativistic effects). The pendulum is released from rest at an angle α. How does the period τ of oscillation depend on the parameters α, m, l and the gravitational acceleration g?

The first step is to determine the dimensions of the various parameters. Here we have

- $[\tau]$ = [T]
- [g] = [L][T]$^{-2}$
- [l] = [L]
- $[\alpha] = 1$ (the angle is dimensionless)
- [m] = [M]

There are $n = 5$ independent variables. These variables involve $r = 3$ dimensions: [L], [T], and [M]. In this case there are $k = n - r = 5 - 3 = 2$ independent dimensionless combinations of the parameters. Since α is dimensionless it can be taken as one of the dimensionless parameters. To find a second let

$$\pi = \tau^a m^b g^c l^d \tag{3.28}$$

(note: since the fifth parameter α is dimensionless there is no point in including it. Multiplying π by any power of α will yield another dimensionless parameter). The dimension of π is

$$\begin{aligned}
[\pi] &= [\tau]^a[m]^b[g]^c[l]^d, \\
&= [T]^a[M]^b\big([L][T]^{-2}\big)^c[L]^d, \\
&= [T]^{a-2c}[M]^b[L]^{c+d}.
\end{aligned} \tag{3.29}$$

Table 3.1: The international system of units: fundamental (or primary, or basic) dimensions. Sometimes angles in radians is included as a dimension but not in the ISU.

Physical Quantity	unit symbol	unit name (MKS)
mass	kg	kilogram
length	m	meter
time	s	second
electric current	A	ampere
luminosity	C	candela
temperature	K	degree Kelvin
amount	mol	mole

This is dimensionless iff the exponent of each of the primary dimensions is zero, that is iff

$$a - 2c = b = c + d = 0, \tag{3.30}$$

which gives $c = a/2$, $b = 0$ and $d = -c = -a/2$. Thus, all dimensionless combinations of m, l, g and τ have the form

$$\pi = \left(\tau\sqrt{\frac{g}{l}}\right)^a. \tag{3.31}$$

We can choose a to have any non-zero value. For simplicity we take $a = 1$. Our two dimensionless parameters are $\pi_1 = \tau\sqrt{\frac{g}{l}}$ and $\pi_2 = \alpha$. Any other dimensionless parameter must have the form $\pi_1^\alpha \pi_2^\beta$ and hence is not independent of π_1 and π_2. According to the theorem

$$\pi_1 = h(\pi_2) \tag{3.32}$$

for some function h, or

$$\tau\sqrt{\frac{g}{l}} = h(\alpha), \tag{3.33}$$

which can be written as

$$\tau = \sqrt{\frac{l}{g}}h(\alpha). \tag{3.34}$$

Without even writing down any differential equations, let alone solving them, we have learnt several important things:

1. the period of oscillation does not depend on the mass m;
2. the period of oscillation is proportional to $\sqrt{l}$;
3. the period of oscillation is inversely proportional to $\sqrt{g}$;
4. $\tau\sqrt{\frac{g}{l}}$ depends only on the initial angle α.

Thus we know that doubling the length of the pendulum increases the period of oscillation by a factor of $\sqrt{2}$ and increasing the gravitational acceleration by a factor of two decreases the period of oscillation by a factor of $\sqrt{2}$. If one wanted to conduct a series of experiments to determine how the period τ depends on g, m, l and α, instead of doing a large set of experiments by varying all four parameters one only needs to vary α. This sort of reduction in the number of parameters that need to be varied is very powerful for both laboratory experiments and numerical simulations.

For small angles of oscillation, i.e., in the limit $\alpha \to 0$, the period of oscillation becomes

$$\tau_{lin} = \sqrt{\frac{l}{g}}h(0). \tag{3.35}$$

Mathematically the governing equation (a second-order ODE) is a nonlinear DE for arbitrary initial angles but in the limit $\alpha \to 0$ the equation becomes a linear DE hence we refer to the period of oscillation in this limit as the linear period τ_{lin}. As we will see later the period is $2\pi\sqrt{\frac{l}{g}}$, i.e., $h(0) = 2\pi$.

Comment: There are an infinite number of choices for the dimensionless parameters. For example

$$\begin{aligned}\pi_1 &= \left(\tau\sqrt{\frac{g}{l}}\right)^{-2}, \\ \pi_2 &= \left(\tau\sqrt{\frac{g}{l}}\right)^{3}\alpha^2.\end{aligned} \tag{3.36}$$

A direct application of the theorem gives

$$\left(\tau\sqrt{\frac{g}{l}}\right)^{-2} = H\left(\left(\tau\sqrt{\frac{g}{l}}\right)^{3}\alpha^2\right) \tag{3.37}$$

for some function H. Inverting we have

$$\alpha^2 = H^{-1}\left(\left(\tau\sqrt{\frac{g}{l}}\right)^{-2}\right)\Big/\left(\tau\sqrt{\frac{g}{l}}\right)^{3} \tag{3.38}$$

where H^{-1} is the inverse of H. Thus $\alpha = f\left(\tau\sqrt{\frac{g}{l}}\right)$. Inverting this leads to (3.34).

3.2.1 Proof of the Buckingham-π Theorem

The following is based on the discussion in [2]. See also [6] which is a very useful applied math text covering a broad range of subjects. A book all Applied Mathematicians should be familiar with.

Assumptions

Dimensional analysis is built upon the following assumptions:

(i) A quantity u is determined in terms of n measurable quantities (variables or parameters) $(W_1, W_2, \ldots, W_n)$ by a functional relationship of the form

$$u = f(W_1, W_2, \ldots, W_n) \tag{3.39}$$

(ii) The quantities $(u, W_1, W_2, \ldots, W_n)$ involve m fundamental dimensions which are labelled L_1, L_2, …, L_m.

(iii) Let Z represent any of $(u, W_1, W_2, \ldots, W_n)$. Then the dimensions of Z, denoted by [Z], is a product of powers of the fundamental dimensions via

$$[Z] = L_1^{\alpha_1} L_2^{\alpha_2} \cdots L_m^{\alpha_m}. \tag{3.40}$$

The dimension vector of Z is the column vector

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \tag{3.41}$$

A quantity Z is dimensionless if and only if all its dimension exponents are zero. Let

$$\vec{b}_i = \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{mi} \end{bmatrix} \tag{3.42}$$

be the dimension vector of W_i, $i = 1, 2, 3, \ldots, n$, and let

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \tag{3.43}$$

be the $m \times n$ dimensional matrix of the problem.

Example: Before our final assumption consider the period of a simple pendulum τ which is released from rest at an initial angle α. The period is assumed to depend on the pendulum mass m, the length of the pendulum l, the gravitational acceleration g and the initial angle α. This gives $\tau = f(m, l, g, \alpha)$. The quantities $u = \tau$, $W_1 = m$, $W_2 = l$, $W_3 = g$ and $W_4 = \alpha$ involve three dimensions: $L_1 = $ [M], $L_2 = $ [L], and $L_3 = $ [T]. Hence $n = 4$ and $m = 3$. Now $[m] = [M] = L_1$, $l = [L] = L_2$, $[g] = [L][T[^{-2} = L_2 L_3^{-2}$ and $[\alpha] = 1$. Thus, for this problem

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \tag{3.44}$$

(iv) For any set of fundamental dimension one can choose a system of units for measuring the value of any quantity Z. For example, the SI system, the cgs system or the British foot-pound system. Changing from one system to another involves a positive scaling of each fundamental unit. E.g., 1 m = 100 cm, 1 cm = 2.54 inches, or 1 kg = 2.2 lbs. Secondary quantities are scaled accordingly. This results in a corresponding scaling of each quantity Z. *The final assumption is that the relationship between u and the variables $(W_1, W_2, \ldots, W_n)$ is invariant under any scaling of the fundamental units.* For example, the relationship between kinetic energy K and the mass m and velocity $\vec{v}$ of an object is $K = \frac{1}{2} m\vec{v} \cdot \vec{v}$ in all measurement systems, or Newton's second law of motion $\vec{F} = m\vec{a}$ is independent of the measurement system.

The invariance of the functional relationship means the following. Suppose under a scaling of the dimension L_j $L_j \to L_j^* = e^{\epsilon} L_j$ and under this scaling $(u, W_1, W_2, \ldots, W_n) \to (u^*, W_1*, W_2^*, \ldots, W_n^*)$. Then $u = f(W_1, W_2, \ldots, W_n)$ becomes $u^* = f(W_1^*, W_2^*, \ldots, W_n^*)$.

Illustration of the Buckingham-π theorem

Before proving the theorem we re-consider our simple pendulum which has period

$$\tau = f(m, l, g, \alpha). \tag{3.45}$$

This relationship is assumed to be independent of scalings of the fundamental dimensions. Suppose we scale the mass dimension by e^{ϵ} so $L_1 \to L_1^* = e^{\epsilon} L_1$. Under this scaling $m \to m^* = e^{\epsilon} m$ but as the other variables do not involve dimensions $L_1 = [M]$, $\tau \to \tau^* = \tau$, $g \to g^* = g$ and $l \to l^* = l$ (α, being dimensionless does not change). Thus, $\tau = f(m, l, g, \alpha)$ becomes $\tau^* = f(m^*, l^*, g^*, \alpha)$ according to assumption (iv). This means that

$$\tau = f(e^{\epsilon} m, l, g, \alpha). \tag{3.46}$$

This is true for all ϵ and the only way (3.45) and (3.46) can both hold is if τ does not depend on m, i.e., $\tau = f(l, g, \alpha) = f(W_2, W_3, W_4)$.

We now tackle the next dimension $[L] = L_2$. Scale L_2 by e^{ϵ}. Then $\tau^* = \tau$ is unchanged (it does not involve dimensions of length), $l^* = e^{\epsilon} l$, and $g* = e^{\epsilon} g$ so $\tau^* = f(l^*, g^*, \alpha)$ gives $\tau = f(e^{\epsilon} l, e^{\epsilon} g, \alpha)$. The appearance of e^{ϵ} in two term makes things slightly more complicated. To proceed we eliminate dimension L_2 from all but one variable by choosing two new independent variables. Let us remove it from the second variable, W_3, by defining new variables $X_2 = W_2$ and $X_3 = W_3/W_2 = g/l$. Here X_3 has been chosen so that $[X_3] = [\mathrm{T}]^{-2} = L_3^{-2}$ does not include the dimension L_2. In terms of the new variables $\tau = f(l, g) = \tilde{f}(X_2, X_3, \alpha) = \tilde{f}(l, g/l, \alpha)$. If we now scale l by e^{ϵ} we obtain

$$\tau = \tilde{f}(e^{\epsilon} X_2, X_3, \alpha) \tag{3.47}$$

which again is true for all ϵ. Hence $\tilde{f}$ must be independent of X_2, i.e.,

$$\tau = \tilde{f}(X_3, \alpha) = \tilde{f}\big(\frac{g}{l}, \alpha\big). \tag{3.48}$$

For the final step we note that $[\tau] = [T]$ and $[g/l] = [T]^{-2}$. Since $\tau = \tilde{f}(g/l, \alpha)$ it follows that $v = \tau\sqrt{\frac{g}{l}} = \tilde{f}((g/l, \alpha)\sqrt{\frac{g}{l}} \equiv h(g/l, \alpha) = h(X_3, \alpha)$ is dimensionless. Now $[v] = 1$ and $[X_3] = [T]^{-2}$. We now scale time by $L_3^* = e^{\epsilon} L_3$. Under this scaling $v \to v^* = v$ and $X_3 \to X_3^* = e^{-2\epsilon} X_3$. By the assumption $v^* = h(X_3^*, \alpha)$ or $v = h(e^{-2\epsilon} X_3, \alpha)$. The only way this can be true is if h is independent of X_3, i.e., $v = h(\alpha)$, or $\tau/\sqrt{g/l} = h(\alpha)$ recovering our previous result.

Statement of the Buckingham-π theorem

Consider

$$u = f(W_1, W_2, \ldots, W_n) \tag{3.49}$$

and let B be the dimension matrix for $W_1, W_2, \ldots, W_n$. Let

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \tag{3.50}$$

be the dimension vector of u. As in the simple pendulum example we want to introduce a new *dimensionless* dependent variable v by multiplying u by appropriate powers of the W_j. Thus, choose $(y_1, y_2, \ldots, y_n)$ so that

$$\pi = u W_1^{y_1} W_2^{y_2} \cdots W_n^{y_n} \tag{3.51}$$

is dimensionless. Recalling that the dimensions of W_i are $[W_i] = L_1^{b_{1i}} L_2^{b_{2i}} \cdots L_m^{b_{mi}}$ it follows that the dimensions of the expression on the right hand side is

$$\begin{aligned} &L_1^{a_1 + y_1 b_{11} + y_2 b_{12} + \cdots + y_n b_{1n}} L_2^{a_2 + y_1 b_{21} + y_2 b_{22} + \cdots + y_n b_{2n}} \cdots \\ &\quad \cdots L_j^{a_j + y_1 b_{j1} + y_2 b_{j2} + \cdots + y_n b_{jn}} \cdots L_m^{a_m + y_1 b_{m1} + y_2 b_{m2} + \cdots + y_n b_{mn}}. \end{aligned} \tag{3.52}$$

This is dimensionless if

$$a_j + y_1 b_{j1} + y_2 b_{j2} + \cdots + y_n b_{jn} = 0 \tag{3.53}$$

for all j, i.e., if $\vec{y}$ is a solution of

$$B\vec{y} = -\vec{a}. \tag{3.54}$$

Note that in B is often not invertible, e.g., it is not invertible if $m \neq n$ (for example if there are more variables W_k than dimensions L_i as in our simple pendulum example). There are in general many solutions. Indeed, if π is dimensionless and π_1 is any other dimensionless variables (e.g., $\tau/\sqrt{\frac{g}{l}}$ and α in our simple pendulum example) then $\pi^a \pi_1^b$ is an alternative dimensionless replacement for u for any nonzero a and b. It is also possible that there are no solutions which means the problem is poorly formulated. For example u may include a dimension that is not a dimension of any of the variables W_i. A scaling of this dimension leads to $e^{\epsilon} u = f(W_1, W_2, \ldots, W_n) = u$ which can't be true for all ϵ. For example, one may speculate that the period of revolution of the moon around the Earth, τ_r, depends on the mass of the Earth M_E, the mass of the moon M_m and the distance R_m of the moon from the centre of the Earth giving $\tau_r = f(M_E, M_m, R_m)$. M_E, M_m and R_m do not involve dimensions of time so the period τ_r can not be a function of these three variables only. Something is missing (in this case Newton's Universal Gravitational Constant). Henceforth we assume that (3.54) has a solution.

Next, we want to construct dimensionless variables by taking appropriate combinations of the W_j. Suppose

$$\pi_i = W_1^{x_{1i}} W_2^{x_{2i}} \cdots W_n^{x_{ni}} \tag{3.55}$$

is dimensionless. Then it follows that

$$B\vec{x_i} = 0 \tag{3.56}$$

which follows from (3.51) and (3.54) after replacing $\vec{a}$ by the zero vector. There are $k = n - r(B)$ linearly independent solutions of (3.56). Here $r(B)$ is the rank of the dimension matrix B (i.e., the number of linearly independent columns, or equivalently, the number of linearly independent rows). Let $\vec{x_i}$ $i = 1, 2, \cdots, k$ be any such set and let π_i given by (3.55) be the corresponding k dimensionless variables. Then the Buckingham-π theorem states that

$$\pi = g(\pi_1, \pi_2, \ldots, \pi_k). \tag{3.57}$$

Example: Consider again the simple pendulum. The dimension matrix (3.44) has rank $r = 3$. Hence there is a dimensionless dependent variable π, which we can take as $\tau\sqrt{g/l}$, and a single independent variable $\pi_1 = \alpha$. The theorem says that $\pi = h(\alpha)$ for some function h.

Proof of the Buckingham-π theorem

The proof of the Buckingham-π theorem follows procedures illustrated in the simple pendulum example. Suppose dimension L_1 is scaled by e^{ϵ}. Under this scale $u = f(W_1, W_2, \ldots, W_n)$ becomes $e^{\epsilon a_1} u = f(e^{\epsilon b_{11}} W_1, e^{\epsilon b_{12}} W_2, \ldots, e^{\epsilon b_{1n}} W_n)$. There are two cases to consider.

CASE I: If $b_{11} = b_{12} = \cdots = b_{1n} = a_1 = 0$ it follows the L_1 is not a fundamental dimension of the problem, i.e., the problem is independent of this dimension. We can assume this is not the case.

CASE II: If $b_{11} = b_{12} = \cdots = b_{1n} = 0$ and $a_1 \neq 0$ we have $e^{\epsilon a_1} u = f(W_1, W_2, \ldots, W_n)$ for all ϵ hence it follows that $u \equiv 0$, a situation that is not of interest.

CASE III: At least one of $b_{11},\ b_{12},\ \ldots,\ b_{1n}$ is non-zero. Wlog assume $b_{11} \neq 0$. Define new quantities

$$X_i = W_i W_1^{-b_{1i}/b_{11}}, \quad i = 2, 3, \ldots, n \tag{3.58}$$

and a new unknown

$$v = u W_1^{-a_i/b_{11}}. \tag{3.59}$$

The formula $u = f(W_1, W_2, \ldots, W_n)$ is equivalent to $v = F(W_1, X_2, X_3, \ldots, X_n)$. By construction v and the X_i are independent of dimension L_1. Scaling L_1 by e^{ϵ} gives $v = F(e^{\epsilon b_{11}} W_1, X_2, X_3, \ldots, X_n)$. Since this is true for all ϵ it follows that F is independent of W_1. That is, $v = G(X_2, X_3, \ldots, X_n)$ where v and the quantities $X_2,\ X_3,\ \ldots,\ X_n$ are independent of the dimension L_1.

Note that under the change of variables $W_j \to W_u W_1^{-b_{1i}/b_{11}}$ the dimension matrix is changed by subtracting $(b_{1i}/b_{11})\vec{b_1}$ from the ith column which makes the entries in row 1 zero apart from b_{11}:

$$\begin{aligned}
B &= \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \\
&= \begin{bmatrix} \vec{b_1} & \vec{b_2} & \cdots & \vec{b_n} \end{bmatrix} \\
&\to \begin{bmatrix} \vec{b_1} & \vec{b_2} - \frac{b_{12}}{b_{11}}\vec{b_1} & \cdots & \vec{b_n} - \frac{b_{1n}}{b_{11}}\vec{b_1} \end{bmatrix} \\
&= \begin{bmatrix} b_{11} & b_{12} - \frac{b_{12}}{b_{11}}b_{11} & \cdots & b_{1n} - \frac{b_{1n}}{b_{11}}b_{11} \\ b_{21} & b_{22} - \frac{b_{12}}{b_{11}}b_{21} & \cdots & b_{2n} - \frac{b_{1n}}{b_{11}}b_{21} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} - \frac{b_{12}}{b_{11}}b_{m1} & \cdots & b_{mn} - \frac{b_{1n}}{b_{11}}b_{m1} \end{bmatrix} \\
&= \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} - \frac{b_{12}}{b_{11}}b_{21} & \cdots & b_{2n} - \frac{b_{1n}}{b_{11}}b_{21} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} - \frac{b_{12}}{b_{11}}b_{m1} & \cdots & b_{mn} - \frac{b_{1n}}{b_{11}}b_{m1} \end{bmatrix} \\
&= \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & \tilde{b}_{22} & \cdots & \tilde{b}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & \tilde{b}_{m2} & \cdots & \tilde{b}_{mn} \end{bmatrix}.
\end{aligned} \tag{3.60}$$

We continue with the remaining $m-1$ dimensions. In turn we eliminate each dimension from all but one variable and reduce the number of independent variables. As discussed above, after the m dimensions have been eliminated we are left with $k+1 = n+1-r(B)$ dimensionless quantities π and $\pi_1,\ \pi_2,\ \ldots,\ \pi_k$ where

$$\pi = g(\pi_1, \pi_2, \ldots, \pi_k). \tag{3.61}$$

or

$$u = W_1^{-y_1} W_2^{-y_2} \cdots W_n^{-y_n} g(\pi_1, \pi_2, \dots, \pi_k). \tag{3.62}$$

Comments:

1. This proof makes no assumptions about the continuity of the functions f and g.

2. The assumed relationship $u = f(W_1, W_2, \dots, W_n)$ may be incorrect. This can manifest itself in several ways. The non-dimensionalization procedure may fail (e..g, the orbital period of the moon example). Alternatively, the nondimensionalization procedure may work but the resulting $\pi = g(\pi_1, \pi_2, \dots, \pi_n)$ may be incorrect because u depends on parameters or variables not included in the analysis. This may, however, still give a useful results in certain limiting cases. For example, if the period of the idealized simple pendulum is assumed to depend on l, g and m only the analysis follows through and results in $\tau = k\sqrt{\frac{l}{g}}$ for some constant k instead of the correct[1] result $\tau = f(\alpha)\sqrt{\frac{l}{g}}$ where f is an undetermined function to be found via experiments or mathematical analysis. The approximation $f(\alpha) = k$, a constant, is only valid in the small amplitude limit (which may be what you are interested in and hence OK).

3. The scaling $L_i^* = e^{\epsilon} L_i$ which induces the transformation $u \to u^* = e^{\epsilon a_i}$, and $W_j \to W_j^* = e^{\epsilon b_{ij}} W_j$ for $j = 1, 2, ..., n$ defines a one-parameter (ϵ) Lie group of scaling transformations of the $n + 1$ quantities $(u, W_1, W_2, \dots, W_n)$ with $\epsilon = 0$ corresponding to the identity transformation. Assumption (iv) behind dimensional analysis states that the relationships $u = f(W_1, W_2, \dots, W_n)$ is invariant under this Lie group transformation. The Buckingham-π theorem is exploiting these symmetries to reduce the dimensionality of the problem. There is a whole field of study devoted to Lie group transformations which has given rise to many useful results in the study of differential equations.

3.3 Problems

1. Introduce suitable non-dimensional population $\tilde{P}$ and time $\tilde{t}$ to write the logistic equation in the form

$$\frac{d\tilde{P}}{dt} = \tilde{P}(1 - \tilde{P}). \tag{3.63}$$

   What is the general solution $\tilde{P}(\tilde{t})$? Use your dimensionless solution to find the solution $P(t)$ of the dimensional problem.

[1] Even this is an approximation as we have ignored lots of other effects, e.g., relativistic effects, assuming them to be negligible.

Chapter 4

Objects Falling/Rising Through a Fluid

We are now going to reconsider objects moving through a fluid. We begin with a discussion of drag forces and consider an example problem of a falling parachutist. We then consider the inclusion of buoyancy and 'added mass' (or 'accelaration-reaction') forces which are important if the object's density is not a lot larger than the fluid density.

4.1 Drag Force on an Object Moving Through a Newtonian Fluid

Consider an object moving through a Newtonian fluid (e.g., air or water the situation is much more complicate for more non-Newtonian fluids such as paint, ketchup or egg white). We assume that the density of the object is much larger than the density of the surrounding fluid so buoyancy and added mass forces, discussed below, can be ignored. The governing equation is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg + F_d(v) \tag{4.1}$$

where the drag force F_d depends on the objects velocity and shape (orientation being part of the latter). We assume the object is rotationally symmetric about its vertical axis and is not spinning so the drag force acts to oppose the objects motion. That is, there is no tangential component of the force (as there is, for example, on an airplane wing or a spinning soccer ball). The drag force depends on the fluid density ρ_f, the size of the object given by, for example, a cross-sectional diameter D, the object's velocity V and the fluid viscosity μ. The dimensions are

- $[F_d] = [M][L][T]^{-2}$;
- $[\rho_f] = [M][L]^{-3}$;
- $[D] = [L]$;
- $[V] = [L][T]^{-1}$;
- $[\mu] = [M][L]^{-1}[T]^{-1}$;

There are five variables involving three dimensions, hence two independent dimensionless variables. These can be taken as

$$\pi_1 = \frac{F_d}{\frac{1}{2}\rho_f D^2 V^2}, \tag{4.2}$$

$$Re = \frac{UD}{\mu/\rho_f}. \tag{4.3}$$

The factor 1/2 in the first is traditional. The latter dimensionless parameter is called the Reynold's number after Osborne Reynolds (1842-1912) who was led to define this fundamentally important dimensionless parameter based on his studies of the conditions in which flow through a pipe transitions from laminar to turbulent flow. The transition depends on Re.

The drag force (technically, we are just considering the magnitude of the drag force) also depends on the shape of the object which is dimensionless. Thus, according to the Buckingham-π Theorem $\pi_1 = f(Re, shape)$, or

$$F_d = \frac{1}{2}\rho_f D^2 V^2 C_d(Re, shape) \tag{4.4}$$

where C_d is called the drag coefficient. It depends on the Reynolds number and on the shape of the object. Since D^2 has dimensions of area this is often written in the form

$$F_d = \frac{1}{2} C_d \rho_f A V^2 \tag{4.5}$$

where of course the function $C_d(Re, shape)$ will in general be different as A is not necessarily equal to D^2.

The drag force can be expressed in other ways. For example

$$F_d = \frac{1}{2} C_d \frac{\rho_f D V}{\mu}\frac{\mu A V}{D} = \frac{1}{2} C_d Re \frac{\mu A V}{D} = \frac{\tilde{C}_d}{2}\frac{\mu A V}{D} \tag{4.6}$$

where $\tilde{C}_d = Re\, C_d(Re, shape)$. In parameter regimes where the drag force varies linearly with V this is more appropriate as $\tilde{C}_d$ is constant. When the drag force varies quadratically with V (4.5) is more appropriate because in this regime C_d is constant.

The drag on a solid sphere of radius a for low $Re = \frac{Ua}{\rho/\mu}$ (less than about 0.1–1) can be shown to be (Stokes' Law)

$$F_d \approx 6\pi\mu a U. \tag{4.7}$$

We can write this as in terms of the cross-sectional area $A = \pi a^2$ as

$$F_d = \frac{1}{2} 12\pi\mu \frac{A}{\pi a^2}\frac{a}{U} U^2 = \frac{1}{2}\rho_f \frac{24\mu/\rho}{2Ua} A U^2 = \frac{1}{2}\frac{24}{Re}\rho_f A U^2 \tag{4.8}$$

where $Re = 2aU/(\mu/\rho)$ is based on the sphere diameter. Thus, the drag coefficient in this case is $C_d = 24/Re$, i.e., it varies inversely with the Reynolds number. In this regime the drag force varies linearly with speed. For a solid sphere the drag coefficient is roughly constant for Re between about 10^3 and 10^5. In this regime a quadratic drag law is appropriate.

Examples:

1. Consider a pollen grain in air. Here $\nu = \mu/\rho \approx 1.5 \times 10^{-5}$ m^2 s^{-1}, $D \approx 100$ μm or 10^{-4} m, and $U_{term} \approx 0.1$ m s^{-1} giving
$$Re \approx \frac{10^{-4} \times 10^{-1}}{1.5 \times 10^{-5}} \approx 0.7, \tag{4.9}$$
so drag on a pollen grain varies approximately linearly with U.

2. Pitched baseball in air: $\mu \approx 1.5 \times 10^{-5}$ m^2 s^{-1}, $D \approx 0.07$ m. Then $Re \approx 5 \times 10^3 U$. A quadratic drag law is valid for Re between about 10^3 and 10^5 or for U between about 0.2 and 20 m s^{-1} (professional pitchers throw at 25–40 m s^{-1} depending on the type of pitch).

4.1.1 The Parachute Problem

Consider a parachutist falling through the air. A quadratic drag law is generally appropriate for this situation. The drag force is
$$F_d = \frac{1}{2} C_d \rho_a S V^2 \tag{4.10}$$
where $C_d \approx 0.75$ for a parachute, S is the surface area of the parachute and ρ_a is the air density. The governing equation is
$$m\frac{dv}{dt} = -mg + \frac{1}{2} C_d \rho_a S v^2. \tag{4.11}$$
Note that by writing this down we are assuming the parachutist is always falling, that is $v < 0$ and the drag force acts up in the opposite direction to gravitational force. The terminal speed, obtained by setting $\frac{dv}{dt} = 0$ is
$$U_{term} = \sqrt{\frac{2mg}{C_d \rho_a S}}. \tag{4.12}$$
Scale the DE:
$$v = V_c u \qquad t = T_c \tau \tag{4.13}$$
where u is the dimensionless velocity and τ is the dimensionless time. The dimensionless governing equation is
$$\frac{V_c}{T_c}\frac{\mathrm{d}u}{\mathrm{d}\tau} = -g + \Big(\frac{C_d \rho_a S V_c^2}{2m}\Big) u^2. \tag{4.14}$$
or
$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = -g\frac{T_c}{V_c} + \Big(\frac{C_d \rho_a S V_c^2}{2mg}\Big)\frac{gT_c}{V_c} u^2. \tag{4.15}$$

To put this equation in a simple form we set
$$\begin{aligned} g\frac{T_c}{U_c} &= 1, \\ \frac{C_d \rho_a S V_c^2}{2mg} &= 1, \end{aligned} \tag{4.16}$$
which simplifies to
$$V_c = U_{term} \quad \text{and} \quad T_c = \frac{U_{term}}{g}. \tag{4.17}$$
Since we are using the terminal speed as the velocity scaling, as $\tau \to \infty$ we expect that $u \to -1$, a useful check on our solution after we have found it. The nondimensional DE is
$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = -1 + u^2. \tag{4.18}$$

This is separable:

$$\begin{aligned}\frac{du}{1-u^2} &= -d\tau \\ \implies \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| &= -\tau + c. \\ \implies \left|\frac{1+u}{1-u}\right| &= Ae^{-2\tau} \\ \implies \frac{1+u}{1-u} &= Be^{-2\tau}\end{aligned} \tag{4.19}$$

where $B = \pm A$, the sign depending on the sign of the left-hand side. Solving for u we obtain

$$u(\tau) = \frac{Be^{-2\tau} - 1}{Be^{-2\tau} + 1}. \tag{4.20}$$

If we are given an initial condition $u(0) = u_0$ we find

$$B = \frac{1+u_0}{1-u_0}. \tag{4.21}$$

Note that since $u < 0$ the denominator is non-zero. The solution can be written as

$$u(\tau) = \frac{(1+u_0)e^{-2\tau} - (1-u_0)}{(1+u_0)e^{-2\tau} + (1-u_0)}. \tag{4.22}$$

As $\tau \to \infty$, $u \to -1$ as expected.

Example: *A sky diver is falling at 40 m s^{-1} when she opens her parachute.*

1. *What is her terminal velocity?*
2. *Nondimensionalize the problem. What are the velocity and time scales? What are the initial conditions?*
3. *How long does it take her to be falling at twice her terminal velocity?*
4. *How far has she fallen during this time?*

Take $\rho_a = 1.2$ kg m^3, $S = 20$ m^2, $m = 100$ kg and $C_d = 0.75$.

Solution:

1. The terminal speed is

$$\begin{aligned}U_{term} &= \sqrt{\frac{2mg}{C_d\rho_a S}} \\ &= \sqrt{\frac{2\times 100\text{kg}\times 9.81\frac{\text{m}}{\text{s}^2}}{0.75\times 1.2\frac{\text{kg}}{\text{m}^3}\times 20\text{m}^2}} \\ &\approx 10.44\frac{\text{m}}{\text{s}}.\end{aligned} \tag{4.23}$$

2. The velocity and time scales are

$$U_{term} \approx 10.44 \ \frac{\text{m}}{\text{s}}, \qquad T_c = \frac{U_{term}}{g} \approx 1.06 \text{ s}. \tag{4.24}$$

The initial dimensional velocity is $v_0 = -40$ m s^{-1}. The initial dimensionless velocity is

$$u(0) = \frac{v(0)}{U_{term}} \approx \frac{-40}{10.44} \approx -3.83. \tag{4.25}$$

3. We need to find the time at which $v = 2U_{term}$, or $u = -2$. From

$$u(\tau) = \frac{(1+u_0)e^{-2\tau} - (1-u_0)}{(1+u_0)e^{-2\tau} + (1-u_0)}. \tag{4.26}$$

we find

$$e^{2\tau} = \Big(\frac{1+u_0}{1-u_0}\Big)\Big(\frac{1-u}{1+u}\Big). \tag{4.27}$$

Using $u_0 = -3.83$ and setting $u = -2$ we have

$$e^{2\tau} = \Big(\frac{-2.83}{4.83}\Big)\Big(\frac{3}{-1}\Big) = 1.76 \tag{4.28}$$

so

$$\tau = \frac{1}{2}\ln(1.76) = 0.28. \tag{4.29}$$

The corresponding dimensional time is $t = T_c\tau = 1.06 \times 0.28$ s, or $t = 0.29s$.

4. The (nondimensional) distance travelled in time τ is

$$h = -\int_0^\tau u(\tau)\, d\tau. \tag{4.30}$$

It is convenient to use $u(\tau)$ in the form

$$u = \frac{\frac{1+u_0}{1-u_0}e^{-2\tau} - 1}{\frac{1+u_0}{1-u_0}e^{-2\tau} + 1} \tag{4.31}$$

For our problem $(1+u_0)/(1-u_0) = -2.83/4.83 \approx -0.586 < 0$. Set it equal to $-e^{-2\alpha}$. Note that $\alpha > 0$.

$$\begin{aligned} u = \frac{-e^{-2\alpha-2\tau} - 1}{-e^{-2\alpha-2\tau} + 1} &= -\frac{e^{-(\tau+\alpha)} + e^{\tau+\alpha}}{-e^{-\tau-\alpha} + e^{\tau+\alpha}} \\ &= -\frac{e^{\tau+\alpha} + e^{-(\tau+\alpha)}}{e^{\tau+\alpha} - e^{-\tau-\alpha}} \\ &= -\coth\big(\tau+\alpha\big), \end{aligned} \tag{4.32}$$

where

$$\begin{aligned} \coth(x) &= \frac{\cosh(x)}{\sinh(x)}, \\ \cosh(x) &= \frac{e^x + e^{-x}}{2}, \\ \sinh(x) &= \frac{e^x - e^{-x}}{2}. \end{aligned} \tag{4.33}$$

Now

$$\int \coth(x)\,dx = \ln|\sinh(x)| \tag{4.34}$$

so

$$s(\tau) = \int_0^\tau u(\tau)\,d\tau = \ln|\sinh(\tau+\alpha)| - \ln|\sinh(\alpha)| = \ln\left|\frac{\sinh(\tau+\alpha)}{\sinh(\alpha)}\right|. \tag{4.35}$$

For our problem $\alpha = -0.5\ln(0.586) \approx 0.267$ and $\tau = 0.28$ so the distance travelled is

$$s(0.28) \approx \ln\left|\frac{\sinh(0.55)}{\sinh(0.27)}\right| \approx 0.75. \tag{4.36}$$

In dimensional term we need to scale by the length scale $L_c = U_{term}T_c = 10.44 \times 1.06 = 11.07$ m. The dimensional distance travelled is about 8.3 m.

4.2 Buoyancy Forces and Added Mass

Other forces act on objects moving through a fluid.

4.2.1 Buoyancy Force

Consider an object of density ρ_o and volume V_o immersed in a fluid of density ρ_f. The object is in place by a rigid wire and both the fluid and the object are at rest. Pressure in the fluid increases with depth due to the influence of gravity hence the pressure exerted on the object by the surrounding fluid will be larger at the bottom of the object than at the top of it. The net result is an upward force exerted on the object by the surrounding fluid. This is called the **buoyancy force** (see Figure 4.1). If you imaging replacing the object with fluid identical to the surrounding fluid, the fluid occupying the space taken by the object will be in equilibrium with the surrounding fluid. The buoyancy force acting on it is exactly balanced by the force of gravity $-\rho_f V\hat{k}$ acting on it. Here $\hat{k}$ is the vertical unit vector in the direction opposing gravity and g is the gravitational acceleration. Thus the buoyancy force acting on the object is

$$\vec{F}_b = \rho_o V\hat{k}. \tag{4.37}$$

Including this force, the equation of motion for a vertically moving object is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg + \rho_f V_o g + F_d = (\rho_f - \rho_o)V_o g + F_d. \tag{4.38}$$

4.2.2 Acceleration-reaction or added mass

Something is missing in (4.38). Consider an air bubble of volume V_a and density $\rho_a \approx 1.2$ kg m^{-3} in water of density $\rho_w = 1000$ kg m^{-3} released from rest. Its initial velocity is zero so initially the drag force is zero. The above equation predicts an initial acceleration of

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\rho_w - \rho_a}{\rho_a}g \approx 1000g. \tag{4.39}$$

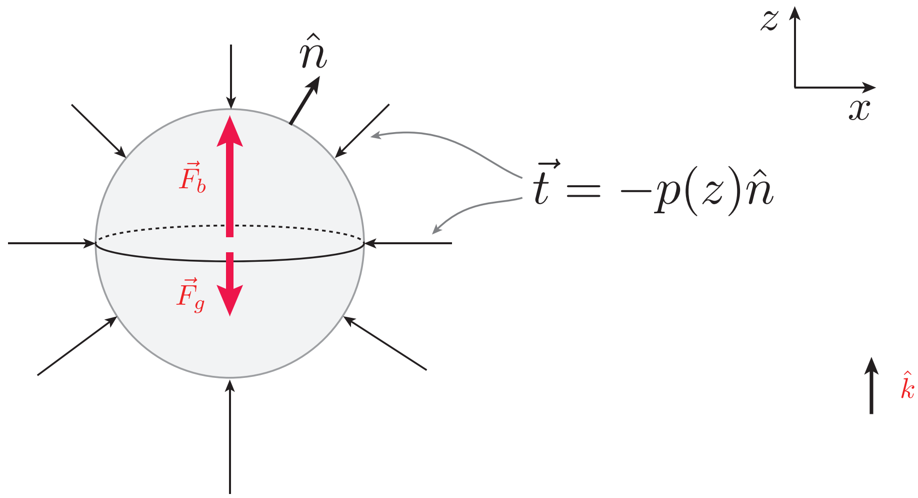

Figure 4.1: Buoyancy force on an object at rest. The surrounding fluid exerts a surface stress (force per unit area) $\vec{t}$ on the surface of the object in the inward, $-\hat{n}$, direction of magnitude $p(z)$ where p is thefluid pressure and z the vertical coordinate. p increases with depth. The fluid pressure is unchanged if the object is replaced by more of the same fluid hence the net result of the surface forces is a buoyancy force that balances the force of gravity that would act on the replacement fluid. The force of gravity acting on the object is $\vec{F}_g = -\rho_o V_o g\hat{k}$ where ρ_o is the density of the object, V_0 its volume, g the gravitational acceleration and $\hat{k}$ is the unit vector in the vertical z direction. The buoyancy force is $\vec{F}_b = \rho_f V_0 g\hat{k}$ where ρ_f is the density of the surrounding fluid. If the object is less dense that the surrounding fluid the buoyancy force is larger than the gravitational force and the object accelerates upward.

This is an enormous acceleration and is far from correct. As the density of the air goes to zero it predicts an acceleration that goes to infinity.

What is missing is the fact that for the air bubble to rise the surrounding fluid has to move and, more particularly, as the air bubble accelerates the surrounding fluid must accelerate too. So work is being done on the fluid, not just on the air bubble. From an energetics point of view, if the air bubble rises a height h a parcel of fluid of the same volume has in effect been moved down a distant h. The potential energy of the system has been decreased by $(\rho_w - \rho_a)V_a gh$. Equation (4.38) says, ignoring the drag force, that this decrease in potential energy has been converted to kinetic energy of the air bubble. In actual fact, most of the potential energy is converted to kinetic energy of the surrounding water.

So, we need to take the acceleration of the surrounding fluid into account when an object accelerates. In terms of the object, the acceleration of the surrounding fluid is associated with a force exerted by the object on the fluid. There is an equal and opposite force exerted by the fluid on the object. This is called the acceleration-reaction.

For a spherical object it can be shown that the force exerted on the object by a surrounding inviscid fluid is

$$F_a = -\frac{1}{2}M_f\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{4.40}$$

This assumes irrotational incompressible flow (in particular the vorticity $\vec{\nabla} \times \mathbf{u} = 0$.) Here $M_f = \rho_f V_o$ is the mass of the displaced fluid. The negative sign indicates that this force opposes the acceleration of the object. It makes it harder to speed up an object but also makes it harder to slow it down. Thus, including this force, the equation of motion for a vertically moving *spherical* object is

$$\big(m + \frac{1}{2}M\big)\frac{dv}{dt} = (M - m)g + F_d, \tag{4.41}$$

or

$$\big(\rho_o + \frac{1}{2}\rho_f\big)\frac{dv}{dt} = (\rho_f - \rho_0)g + \frac{F_d}{V_o}. \tag{4.42}$$

In other words, the object accelerates as if its mass has been increased by half the mass of the displaced fluid. Hence the term 'added mass'. For more general objects the acceleration-reaction can have a tangential component and is described by a second-order tensor. This model for the acceleration-reaction term does not include modifications due to the presence of a boundary layer over the surface of the object, which reduces the fluids kinetic energy, or the effects of flow separation. It is a very good model for estimating the object's initial acceleration from rest but once the object starts moving some additional terms may be required. The motion of an object through a fluid can be very complicated!

Assuming a spherical air bubble, we now get an initial acceleration of

$$\frac{dv}{dt} = \frac{\rho_w - \rho_a}{\rho_a + \frac{1}{2}\rho_w}g \approx 2g. \tag{4.43}$$

Much more reasonable!

Both the buoyancy and added mass forces are neglible if the density of the object is much larger than the density of the surrounding fluid, as is the case for a parachutist.

Chapter 5

Second-Order Linear Ordinary Differential Equations

5.1 Some general theory

The general linear second-order equation has the form

$$L[y] \equiv y'' + P(x)y' + Q(x)y = R(x). \tag{5.1}$$

The operator L maps twice differentiable functions to a new function. As for the first-order case L is a linear operator (exercise). We consider only real-valued functions $P(x)$ and $Q(x)$. For now we assume $R(x)$ is real as well and we seek real valued solutions $y(x)$. Later on we will see that can at times be useful to consider complex-valued forcing functions $R(x)$ in which case the solution $y(x)$ is complex-valued too.

Theorem I. Existence-Uniqueness of Solutions: *Let $P(x)$, $Q(x)$ and $R(x)$ be continuous functions on a closed interval $[a, b]$. If $x_0 \in [a, b]$ and y_0 and y_0' are any real numbers then (5.1) has a unique solution $y(x)$ on $[a, b]$ satisfying the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$.*

For a proof see [7]. We can write this theorem in terms similar to the corresponding theorem for first-order linear DEs. Let $y_1(x) = y(x)$ and $y_2(x) = y'(x)$ and define the vector

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{5.2}$$

Then

$$\begin{aligned} \frac{\mathrm{d}\vec{y}}{\mathrm{d}x} = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_1'' \end{pmatrix} &= \begin{pmatrix} y_2 \\ R - Py_1' - Qy_1 \end{pmatrix} \\ &= \begin{pmatrix} y_2 \\ R - Py_2 - Qy_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ R \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= \vec{F}(x, \vec{y}). \end{aligned} \tag{5.3}$$

The initial conditions are

$$\vec{y}(x_0) = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}. \tag{5.4}$$

The second-order DE has been replaced by a system of two first-order DEs (note we can also do this for a general second-order DE of the form $y'' = F(x, y, y')$). The system has the same form as the general first-order DE $y' = f(x, y)$ with y replaced by a vector. In a similar manner an n$^{\text{th}}$ order DE can be replaced by a first-order DE for an n-vector. The existence-uniqueness proof for first-order DEs can be extended to the case where y is a vector. This is also the basis of many numerical methods used to find approximate numerical solutions of differential equations.

As we have seen, the general solution of a first-order linear DE can be expressed as the sum of a particular solution and a multiple of any non-zero homogeneous solution. This idea can be extended to higher-order DEs: the general solution of an n^{th}-order linear DE has the form of the sum of a particular solution plus a linear combination of n linearly-independent solutions of the homogeneous problem. Here we consider second-order DEs so we will need two linearly-independent solution. This concept is defined below.

If $y_p(x)$ is a particular solution of (5.1) and $y(x)$ is any other solution of (5.1) then the difference $y_h = y - y_p$ is a solution of the homogeneous equation

$$L[y] = 0 \tag{5.5}$$

since by linearity

$$L[y - y_p] = L[y] - L[y_p] = R - R = 0. \tag{5.6}$$

Thus, the problem of find the general solution of (5.1) reduces to finding *any* particular solution of (5.1) plus the general solution of the homogeneous equation.

Theorem II: *If $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation (5.5) then so is $c_1 y_1(x) + c_2 y_2(x)$ for any constants c_1 and c_2.*

Proof: Follows trivially from linearity of the operator L.

Definition: If two functions $f(x)$ and $g(x)$ are defined on an interval $[a, b]$ and have the property that one is a constant multiple of the other then they are said to be **linearly dependent** *on $[a, b]$. Otherwise they are* **linearly independent**.

Example:

- $y_1 = x^2$ and $y_2 = 0$ are linearly dependent on any interval since $y_2 = 0 \cdot y_1$.
- $y_1 = x^2$ and $y_2 = x^3$ are linearly independent on any interval since there is no constant k such that $y_1 = k y_2$ or $y_2 = k y_1$.
- $y_1 = x^2$ and $y_2 = \pi x^2$ are linearly dependent on any interval since $y_2 = \pi y_1$.
- $\cos(x)$ and $\sin(x)$ are linearly independent on any interval since neither is a constant multiple of the other on any interval.

Theorem III: *Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogeneous equation (5.5) on $[a, b]$. Then*

$$c_1 y_1(x) + c_2 y_2(x) \tag{5.7}$$

is the general solution of (5.5) on $[a,b]$. That is any solution of (5.5) is a linear combination of y_1 and y_2.

Before proving this theorem a definition and a couple of lemmas.

Definition: The Wronskian $W(y_1, y_2)$ of two functions $y_1(x)$ and $y_2(x)$ is the new function

$$W(y_1, y_2)(x) = y_1 y_2' - y_2 y_1'. \tag{5.8}$$

The Wronskian is linear in each of its arguements(exercise). It is an example of a bi-linear operator.

Lemma 1: If $y_1(x)$ and $y_2(x)$ are any two solutions of (5.5) on $[a,b]$ then their Wronskian is either identically zero or never zero on $[a,b]$.

Proof:

$$\begin{aligned}
\frac{\mathrm{d}W}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}(y_1 y_2' - y_2 y_1') \\
&= y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' \\
&= y_1 y_2'' - y_2 y_1'' \\
&= y_1(-P y_2' - Q y_2) - y_2(-P y_1' - Q y_1) \\
&= -P(y_1 y_2' - y_2 y_1') - Q(y_1 y_2 - y_2 y_1) \\
&= -PW
\end{aligned} \tag{5.9}$$

The general solution of this first-order DE for $W(x)$ is

$$W(x) = c e^{-\int^x P(t)\,\mathrm{d}t}. \tag{5.10}$$

$W(x)$ is identically zero if $c = 0$ or never zero if $c \neq 0$.

Lemma 2: If $y_1(x)$ and $y_2(x)$ are any two solutions of (5.5) on $[a,b]$ then they are linearly dependent on $[a,b]$ iff their Wronskian is identically zero on $[a,b]$.

Proof:

(a) Suppose y_1 and y_2 are linearly dependent. If $y_1 = 0$ or $y_2 = 0$ then $W(y_1, y_2) = 0$. If neither are identically zero on $[a,b]$ then $y_1(x) = k y_2(x)$ for some constant k and $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = k y_1 y_1' - k y_1' y_1 = 0$. Thus if y_1 and y_2 are linearly dependent $W(y_1, y_2) = 0$.

(b) Suppose $W(y_1, y_2) = 0$ on $[a,b]$. If y_1 is identically zero on $[a,b]$ then y_1 and y_2 are linearly dependent. Otherwise, since y_1 is continuous (it is twice differentiable on $[a,b]$) there is a subinterval $[c,d] \subset [a,b]$ on which $y_1 \neq 0$. Thus we can divide $W(y_1, y_2) = 0$ by y_1^2 on $[c,d]$ to get
$$\frac{W}{y_1^2} = \frac{y_1 y_2' - y_1' y_2}{y_1^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y_2}{y_1}\right) = 0. \tag{5.11}$$
Thus y_2/y_1 is constant on $[c,d]$ or $y_2 = k y_1$ on $[c,d]$. We also have $y_2' = k y_1'$ on $[c,d]$ so by the existence-uniqueness Theorem I $y_1 = k y_2$ on $[a,b]$. That is, if $W(y_1, y_2) = 0$ then y_1 and y_2 are linearly dependent.

We now prove Theorem III.

Proof of Theorem III: Let $y(x)$ be any solution of (5.5). We need to show that we can find constants c_1 and c_2 such that $y = c_1y_1 + c_2y_2$. By Theorem I we pick a point x_0 and values $y_0 = y(x_0)$ and $y'_0 = y'(x_0)$. We next find c_1 and c_2 such that

$$\begin{aligned} c_1y_1(x_0) + c_2y_2(x_0) &= y(x_0) = y_0 \\ c_1y'_1(x_0) + c_2y'_2(x_0) &= y'(x_0) = y'_0 \end{aligned} \tag{5.12}$$

If we can do this then by uniqueness (Theorem I) $y = c_1y_1 + c_2y_2$. Equation (5.12) has a solution iff

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix} = y_1(x_0)y'_2(x_0) - y'_1(x_0)y_2(x_0) \neq 0. \tag{5.13}$$

In other words, iff $W(y_1, y_2)(x_0) \neq 0$. Since y_1 and y_2 are linearly independent by supposition, $W(y_1, y_2)(x_0) \neq 0$ so we can solve system (5.12) for c_1 and c_2. Thus we can find c_1 and c_2 such that y and $c_1y_1 + c_2y_2$ and their derivatives are equal at x_0. By the existence-uniqueness theorem $y = c_1y_1 + c_2y_2$ on $[a, b]$.

Example: Show that $y = c_1 \sin(x) + c_2 \cos(x)$ is the general solution of $y'' + y = 0$ on any interval and find the solution of the initial value problem $y(0) = 2$ and $y'(0) = 3$.

Solution Let $y_1 = \sin(x)$ and $y_2 = \cos(x)$. Both are solutions of $y'' + y = 0$. Their Wronskian is

$$W(y_1, y_2) = y_1y'_2 - y'_1y_2 = \sin(x) \cdot \big(-\sin(x)\big) - \big(\cos(x)\big) \cdot \cos(x) = -1. \tag{5.14}$$

which is non-zero. Thus $\sin(x)$ and $\cos(x)$ are linearly independent (of course this should be obvious to you but this illustrates that $W \neq 0$ anywhere). Since $P(x) = 0$ and $Q(x) = 1$ are continuous theorem III says that $c_1y_1 + c_2y_2 = c_1 \sin(x) + c_2 \cos(x)$ is the general solution of $y'' + y = 0$ on any interval $[a, b]$. The constant are determined via the initial conditions:

$$\begin{aligned} c_1y_1(0) + c_2y_2(0) &= c_1 \sin(0) + c_2 \cos(0) = c_2 = 2 \\ c_1y'_1(0) + c_2y'_2(0) &= c_1 \cos(0) - c_2 \sin(0) = c_1 = 3. \end{aligned} \tag{5.15}$$

Thus, the solution of the IVP is $y = 3\sin(x) + 2\cos(x)$.

We finish this section with a very useful result.

Theorem IV: *If $f(x)$ is a complex valued solution of $L[y] = 0$ (with $P(x)$ and $Q(x)$ real) then so is $y_1 = \Re\big\{f(x)\big\}$ and $y_2 = \Im\big\{f(x)\big\}$.*

The proof is left as an exercise.

Example: $y(x) = e^{ix}$ is a solution of $y'' + y = 0$ since $y'' = (i)^2e^{ix} = -e^{ix} = -y$. Then $\Re\big\{e^{ix}\big\} = \cos(x)$ and $\Im\big\{e^{ix}\big\} = \sin(x)$ are also solutions.

5.1.1 Finding a second homogeneous solution

There is a techinque for finding solutions of the inhomogeneous equation (5.1) once you have two linearly independent homogeneous solutions which is a generalization of the technique for first-order linear DEs. So the real problem is finding a pair of homogeneous solutions. If you can find one, say $y_1(x)$, finding a second turns out to be straight forward.

Let $y_2(x) = v(x)y_1(x)$ be a second linearly independent solution. Then

$$\begin{aligned} L[y_2] &= L[vy_1] \\ &= \big(vy_1\big)'' + P\big(vy_1\big)' + Qvy_1 \\ &= y_1v'' + 2y_1'v' + y_1''v + P\big(v'y_1 + vy_1'\big) + Qvy_1 \\ &= y_1v'' + \big(2y_1' + Py_1\big)v' + vL[y_1] \\ &= y_1v'' + \big(2y_1' + Py_1\big)v' \\ &= 0. \end{aligned} \tag{5.16}$$

This is a linear first-order homogeneous DE for v'. The solution is

$$v' = e^{-\int^x 2\frac{y_1'}{y_1} + P\,\mathrm{d}t} = e^{-2\ln|y_1|}e^{-\int^x P(t)\,\mathrm{d}t} = \frac{e^{-\int^x P(t)\,\mathrm{d}t}}{y_1^2}. \tag{5.17}$$

Integrating again

$$v = \int^x \frac{e^{-\int^x P(t)\,\mathrm{d}t}}{y_1^2}\,\mathrm{d}x. \tag{5.18}$$

Any integral will do. Adding a constant merely adds a multiple of y_1 to $y_2 = vy_1$. I do not recommend memorizing this formula. Understand the process and apply it from first principles.

Example: By inspection $y_1(x) = x$ is a solution of

$$x^2y'' + xy' - y = 0. \tag{5.19}$$

To find a second solution let $y_2 = v(x)y_1 = xv(x)$. Then $y_2' = v + xv'$ and $y_2'' = 2v' + xv''$. Substituting into the DE gives

$$x^2y_2'' + xy_2' - y_2 = x^2(2v' + xv'') + x(v + xv') - xv = x^2(2v' + xv'') + x^2v' \tag{5.20}$$

so the DE gives

$$x^3v'' + 3x^2v' = 0 \tag{5.21}$$

or

$$v'' = -\frac{3}{x}v' \implies v' = \frac{1}{x^3} \implies v = -\frac{1}{2x^2} \tag{5.22}$$

We can multiply v by any constant to get another solution of the equation for v. It is convenient to multiply by -2 to get $v(x) = \frac{1}{x^2}$. Then $y_2 = xv = xx^{-2} = x^{-1}$ is a second linearly independent solution.

The general solution of the DE is

$$y = c_1x + c_2x^{-1}. \tag{5.23}$$

Note: We did not have to divide the DE by x^2 to put the DE in the general form $y''+Py'+Qy=0$ to apply this technique. To use formula (5.18) you do have to be careful to divide the DE by the coefficient of y'', x^2 in this case, to determine $P(x)$.

5.2 Constant Coefficient Equations

We now consider the special case of constant coefficient equations using t as the independent variable, i.e.,

$$L[y] \equiv y'' + P(t)y' + Q(t)y = 0 \tag{5.24}$$

where differentiation is with respect to t and $P(t)$ and $Q(t)$ are constant. Such problems arise in many applications. We have already seen several examples in our study of first-order DEs.

To find the general solution of a constant coefficient linear DE of the form

$$L[y] \equiv ay'' + by' + cy = f(t) \tag{5.25}$$

where a, b and c are constants with $a \neq 0$ we must find any particular solution y_p and two linearly independent solutions of the homogeneous problem

$$L[y] = ay'' + by' + cy = 0. \tag{5.26}$$

Because techniques for finding particular solutions of the inhomogeneous problem make use of solutions of the homogeneous equation we consider the homogeneous problem first. If $a=0$ the equation is a first order DE which we know how to solve so in the following we assume that $a \neq 0$.

5.2.1 Homogeneous Constant Coefficient DEs

Since $a \neq 0$ we can divide through by a to get

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0. \tag{5.27}$$

The behaviour of the solutions depends fundamentally on the signs of b/a and c/a.

Case I: $b=0$ **and** $c/a = \omega_0^2 > 0$

Wlog take $\omega_0 > 0$. This is the equation for an undamped linear mass-spring system or an LC-circuit. The equation

$$y'' + \omega_o^2 y = 0 \tag{5.28}$$

has the general solution

$$y(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t). \tag{5.29}$$

We will often consider equations in dimensionless form. To this end scale y via $y = L_c\tilde{y}$ where L_c is a characteristic scale for y that carries the dimensions, and $\tilde{y}$ is dimensionless. Let $\tau = \omega_0 t$ be the dimensionless time. Then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = L_c\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tau} = L_c\omega_0\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tau} \tag{5.30}$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = L_c \omega_0^2 \frac{\mathrm{d}^2 \tilde{y}}{\mathrm{d}\tau^2}. \tag{5.31}$$

The differential equation for $\tilde{y}$ is

$$\tilde{y}'' + \tilde{y} = 0 \tag{5.32}$$

where here it is understood that the primes denote differentiation with respect to τ. Note that because the equation is linear and homogeneous L_c does not appear: the scaling does not affect the form of the differential equation. It will affect the initial conditions for the dimensionless problem. The general solution of the dimensionless equation is

$$\tilde{y}(\tau) = \tilde{a}\cos(\tau) + \tilde{b}\sin(\tau). \tag{5.33}$$

Using $y = L_c\tilde{y}$ and $\tau = \omega_0 t$

$$\begin{aligned} y(t) = L_c\tilde{y}\big(\tau(t)\big) &= L_c\tilde{a}\cos(\omega_0 t) + L_c\tilde{b}\sin(\omega_0 t) \\ &= a\cos(\omega_0 t) + b\sin(\omega_0 t). \end{aligned} \tag{5.34}$$

The solution can be put in an alternative form using $\sin(\alpha+\beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)$. To this end write

$$\begin{aligned} \tilde{y}(\tau) &= \tilde{a}\cos(\tau) + \tilde{b}\sin(\tau) \\ &= \sqrt{\tilde{a}^2+\tilde{b}^2}\left(\frac{\tilde{a}}{\sqrt{\tilde{a}^2+\tilde{b}^2}}\cos(\tau) + \frac{\tilde{b}}{\sqrt{\tilde{a}^2+\tilde{b}^2}}\sin(\tau)\right). \end{aligned} \tag{5.35}$$

Defining an angle ϕ via

$$\cos(\phi) = \frac{\tilde{b}}{\sqrt{\tilde{a}^2+\tilde{b}^2}}; \quad \sin(\phi) = \frac{\tilde{a}}{\sqrt{\tilde{a}^2+\tilde{b}^2}}. \tag{5.36}$$

and setting

$$\tilde{A} = \sqrt{\tilde{a}^2+\tilde{b}^2} \tag{5.37}$$

we can write the solution in the form

$$\tilde{y}(\tau) = \tilde{A}\sin\big(\tau+\phi\big). \tag{5.38}$$

This form of the solution also involves two undetermined constants: the phase ϕ and amplitude A. In dimensional terms the solution is

$$y(t) = A\sin\big(\omega_0 t + \phi\big) \tag{5.39}$$

where $A = L_c\tilde{A}$.

Exercise: Show that the solution can be written in the forms $\tilde{A}\sin(\tau\pm\phi)$ or $\tilde{A}\cos(\tau\pm\phi)$. How is ϕ defined in each of these four cases?

Case II: $b = 0$ **and** $c/a = -r^2 < 0$

Wlog take $r > 0$. Setting $\tau = rt$ the non-dimensionalized equation is

$$y'' - y = 0. \tag{5.40}$$

Two linearly independent solutions are $y_1 = e^{-\tau}$ and $y_2 = e^{\tau}$. The general solution

$$y = c_1 e^{-\tau} + c_2 e^{\tau} \tag{5.41}$$

is the sum of an exponentially decaying term and an exponentially growing term. If $c_2 = 0$ the solution decays to zero as $\tau \to +\infty$. If $c_2 \neq 0$ then the solution is unbounded, going to $\pm\infty$ as $\tau \to +\infty$ depending on the sign of c_2.

Case I revisited.

The equation

$$\tilde{y}'' + \tilde{y} = 0 \tag{5.42}$$

can also be solved using exponentials. Let $\tilde{y} = e^{\gamma\tau}$. Substituting into the governing equation leads to

$$\gamma^2 + 1 = 0 \tag{5.43}$$

or

$$\gamma = \pm i. \tag{5.44}$$

Since γ is imaginary $y = e^{\gamma\tau}$ is a complex-valued function. We are interested in real-valued solutions.

Consider the general homogeneous second-order differential operator $L[y] = y'' + P(t)y' + Q(t)y = 0$ where $P(t)$ and $Q(t)$ are real valued functions. Consider any solution $y(t)$. Using the linearity of the operator and the fact that the coefficients of the DE are real-valued, we have

$$L[y] = L[y_r + iy_i] = L[y_r] + iL[y_i] \tag{5.45}$$

where y_r and y_i are the real and imaginary parts of y and $L[y_r]$ and $L[y_i]$ are the real and imaginary parts of $L[y]$. If $L[y] = 0$ then both $L[y_r] = 0$ and $L[y_i] = 0$. In our case, the real and imaginary parts are $\cos(\tau)$ and $\sin(\tau)$ so

$$L[e^{\gamma\tau}] = L[\cos(\tau)] + iL[\sin(\tau)] = 0 \tag{5.46}$$

gives

$$L[\cos(\tau)] = L[\sin(\tau)] = 0. \tag{5.47}$$

Both $\cos(\tau)$ and $\sin(\tau)$ are solutions of $L[y] = 0$. Note that using the solution $e^{-i\tau}$ gives the same two real-valued solutions. *The use of complex-valued solutions in this manner is very useful as we will shortly see.*

The General Case

We now consider the linear constant coefficient second-order DE

$$L[y] = ay'' + by' + cy = 0 \tag{5.48}$$

in its general form. Here a, b and c are constant with $a \neq 0$. It will prove convenient to work in terms of the differential operator

$$D \equiv \frac{\mathrm{d}}{\mathrm{d}t}. \tag{5.49}$$

In terms of D the differential equation operator L is

$$L = aD^2 + bD + c. \tag{5.50}$$

We look for solutions of (5.48) of the form

$$y = e^{\lambda t}. \tag{5.51}$$

Substituting $e^{\lambda t}$ into the differential equation gives

$$L[e^{\lambda t}] = \Big(a\lambda^2 + b\lambda + c\Big)e^{\lambda t} = 0. \tag{5.52}$$

Definition: $P(\lambda) = a\lambda^2 + b\lambda + c$ is called the **characteristic polynomial** *of the linear differential equation (5.48).*

Our trial function $e^{\lambda t}$ is a solution of the differential equation if

$$P(\lambda) \equiv a\lambda^2 + b\lambda + c = 0. \tag{5.53}$$

i.e., if λ is a root of the characteristic polynomial.

You can think of the operator L as a polynomial in the differential operator D, in particular $L = P(D)$ (see (5.50)). For functions of the form $e^{\lambda t}$ each derivative brings down a factor of λ so $P(D)[e^{\lambda t}] = P(\lambda)e^{\lambda t}$. So setting $P(D)[e^{\lambda t}] = 0$ gives

$$P(\lambda) = a\lambda^2 + b\lambda + c = 0. \tag{5.54}$$

Solving gives

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \sqrt{\Big(\frac{b}{2a}\Big)^2 - \frac{c}{a}}. \tag{5.55}$$

There are three cases to consider.

CASE A: $b^2 - 4ac > 0$. For this case there are two real distinct roots:

$$\begin{aligned} \lambda_1 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \\ \lambda_2 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \end{aligned} \tag{5.56}$$

It is easy to show that $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are linearly independent (exercise) so the general solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}. \tag{5.57}$$

Note that if $ac < 0$ one root is negative and one is positive: $\lambda_1 < 0 < \lambda_2$. If $ac > 0$ the two roots have the same sign as $-b$ has. If $ac = 0$ then $c = 0$, since we assume $a \neq 0$, in which case one root is equal to zero. In this case the solution is a constant which is indeed a solution of the equation $ay'' + by' = 0$.

We can write the characteristic polynomial as $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$ and the DE as

$$P(D)[y] = (D - \lambda_1)(D - \lambda_2)[y] = (\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_1)(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_2)y = 0. \tag{5.58}$$

Defining the operators

$$L_i \equiv D - \lambda_i \qquad \text{for } i = 1, 2 \tag{5.59}$$

we can write the DE as

$$L[y] = L_1 \circ L_2[y] = L_1\big[L_2[y]\big] = 0. \tag{5.60}$$

Because the λ_i are constants the first-order operators commute:

$$L_1 \circ L_2 = L_2 \circ L_1. \tag{5.61}$$

Note that $y_1 = e^{\lambda_1 t}$ is in the null space of L_1 and $y_2 = e^{\lambda_2 t}$ is in the null space of L_2. That is $L_1[e^{\lambda_1 t}] = 0$ and $L_2[e^{\lambda_2 t}] = 0$.

CASE B: $b^2 - 4ac < 0$. In this case there are two complex conjugate roots of the form $\alpha \pm i\beta$. The simplest way to obtain two real-valued linearly independent solutions is to take the real and imaginary parts of $e^{\lambda_1 t}$ (exercise):

$$\begin{aligned} y_1 &= \Re\Big\{e^{\alpha t + i\beta t}\Big\} = e^{\alpha t}\cos(\beta t), \\ y_2 &= \Im\Big\{e^{\alpha t + i\beta t}\Big\} = e^{\alpha t}\sin(\beta t), \end{aligned} \tag{5.62}$$

where

$$\alpha = -\frac{b}{2a} \qquad \text{and} \qquad \beta = \frac{\sqrt{4ac - b^2}}{2a}. \tag{5.63}$$

An alternative approach which does not involve complex-valued functions is the following. First write the DE as

$$\Big(D^2 + \frac{b}{a}D + \frac{b^2}{4a^2} + \big(\frac{c}{a} - \frac{b^2}{4a^2}\big)\Big)[y] = 0, \tag{5.64}$$

or

$$\Big(D + \frac{b}{2a}\Big)^2[y] + \big(\frac{c}{a} - \frac{b^2}{4a^2}\big)y = 0. \tag{5.65}$$

The operator $D + \frac{b}{2a}$ annihilates $e^{-\frac{b}{2a}t}$, that is

$$\big(D + \frac{b}{2a}\big)\big[e^{-\frac{b}{2a}t}\big] = 0 \tag{5.66}$$

so let $y = e^{-\frac{b}{2a}t}f(t) = e^{\alpha t}f(t)$ where $\alpha = -\frac{b}{2a}$ as in (5.63). Now

$$\Big(D + \frac{b}{2a}\Big)\big[e^{-\frac{b}{2a}t}f(t)\big] = -\frac{b}{2a}e^{-\frac{b}{2a}t}f(t) + e^{-\frac{b}{2a}t}f'(t) + \frac{b}{2a}e^{-\frac{b}{2a}t}f(t) = e^{-\frac{b}{2a}t}f'(t), \tag{5.67}$$

so

$$\Big(D + \frac{b}{2a}\Big)^2\big[e^{-\frac{b}{2a}t}f(t)\big] = e^{-\frac{b}{2a}t}f''(t), \tag{5.68}$$

hence

$$L\big[e^{-\frac{b}{2a}t}f(t)\big] = e^{-\frac{b}{2a}t}f''(t) + \big(\frac{c}{a} - \frac{b^2}{4a^2}\big)e^{-\frac{b}{2a}t}f(t) = 0 \tag{5.69}$$

which simplifies to

$$f''(t) + \frac{4ac - b^2}{4a^2}f(t) = 0. \tag{5.70}$$

Since $b^2 < 4ac$ this equation has the form of $f'' + \beta^2 f = 0$. As we've seen above the general solution is

$$f = a\cos\big(\beta t\big) + b\sin\big(\beta t\big) \tag{5.71}$$

where β is as in (5.63). The solutions $e^{-\frac{b}{2a}t}f(t)$ are the same as those given in (5.62).

CASE C: $b^2 - 4ac = 0$. For this case there is only one root,

$$r = -\frac{b}{2a} \tag{5.72}$$

which is real. Thus only one solution of the form $e^{\lambda t}$ exists. We need to find a second linearly independent solution of the homogeneous DE. There are several ways to find the second root. This simplest is to set $y = e^{-\frac{b}{2a}t}f(t)$ as we did above. This gives equation (5.70) without the second term since $b^2 - 4ac = 0$, that is

$$f'' = 0 \implies f(t) = c_1 + c_2 t \tag{5.73}$$

so

$$y = e^{-\frac{b}{2a}t}f(t) = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t} \tag{5.74}$$

is the general solution.

SUMMARY: In summary, two linearly independent solutions of the homogeneous DE $ay'' + by' + cy = 0$ have the forms shown in table 5.1.

Table 5.1: Forms of solutions of $ay'' + by' + cy = 0$.

$b^2 - 4ac$	y_1	y_2
> 0	$e^{\lambda_1 t}$	$e^{\lambda_2 t}$
$= 0$	e^{rt}	te^{rt}
< 0	$e^{\alpha t}\cos(\beta t)$	$e^{\alpha t}\sin(\beta t)$

5.2.2 The Unforced Linear Oscillator

Consider a mass m attached to a wall via a spring. It is free to slide over a horizontal smooth surface. The mass is subjected to two forces in the horizontal: the force exerted by the spring (F_s) and a drag force (F_d). Let y be the position of the mass relative to its position when the spring is at its equilibrium length (i.e., neither stretched or compressed). The equation of motion is

$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = F_s + F_d. \tag{5.75}$$

For small motion the force exerted by the spring is linearly proportional to the amount the spring has been stretched from its equilibrium position (a 'linear' spring), so $F_s = -ky$ where k is a positive constant, called the spring constant. We assume the drag force, due to friction between the mass and the surface and possibly air drag, varies linearly with the mass' velocity. Thus $F_d = -\beta\frac{\mathrm{d}y}{\mathrm{d}t}$ for some positive drag coefficient β which is constant. Thus the governing equation is

$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \beta\frac{\mathrm{d}y}{\mathrm{d}t} + ky = 0. \tag{5.76}$$

Equations of this type arise in many problems.

The unforced (no external forces) damped mass-spring system is governed by an equation of the form

$$ay'' + by' + cy = 0 \tag{5.77}$$

in which all the coefficients are positive. When $b = 0$ and $c/a > 0$ we have seen that solutions oscillate with frequency $\omega_0 = \sqrt{c/a}$. Many systems in nature undergo oscillations and some of these can be modelled with a linear ODE of this form. These systems are typically damped ($b/a > 0$), however in unstable situations oscillations may grow in time (e.g., wind blowing over the surface of a lake generating surface waves) in which case $b/a < 0$. This requires an energy source because as the oscillations grow in amplitude their energy increases and this energy must come from somewhere.

The behaviour of solutions of (5.77) is easiest to explore using an appropriately non-dimensionalized equation. Thus we non-dimensionalize by setting $y = L_c x$ and $\tau = \omega_0 t$. The differential equation becomes

$$y'' + \frac{b}{a}y' + \omega_0^2 y = L_c\omega_0^2 x'' + L_c\frac{b}{a}\omega_0 x' + L_c\omega_0^2 x = 0 \tag{5.78}$$

where $x = x(\tau)$. Dividing through by $L_c\omega_0^2$ gives (using dots to denote differentiation with respect to τ)

$$\ddot{x} + 2\delta\dot{x} + x = 0 \tag{5.79}$$

where

$$\delta = \frac{b}{2a\omega_0} = \frac{b}{2\sqrt{ac}}. \tag{5.80}$$

is the dimensionless damping coefficient. The behaviour of the solutions of equation (5.79) depends on the single parameter δ. Because the equation is linear and homogeneous the scaling L_c does not play a role however it does affect the initial conditions.

Setting $x = e^{\lambda\tau}$ gives

$$\lambda = -\delta \pm \sqrt{\delta^2 - 1}. \tag{5.81}$$

There are three cases: $\delta^2 < 1$, $\delta^2 = 1$ and $\delta^2 > 1$. The factor of 2 in (5.79) was introduced merely for the convenience of having $\delta^2 = 1$ being the critical value at which the roots change from being complex to real.

Energetics

Multiplying (5.79) by $\dot{x}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2\Big) \equiv \frac{\mathrm{d}E}{\mathrm{d}\tau} = -2\delta\dot{x}^2. \tag{5.82}$$

We call

$$E = \frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2 \tag{5.83}$$

the (nondimensional) energy. For a linear mass-spring system it is the sum of kinetic and potential energy. If $\delta > 0$ the energy can only decay. This is the case of a linearly damped system. If $\delta < 0$ then the energy grows with time. This implies an energy source. A model of this type arises in the study of some unstable systems.

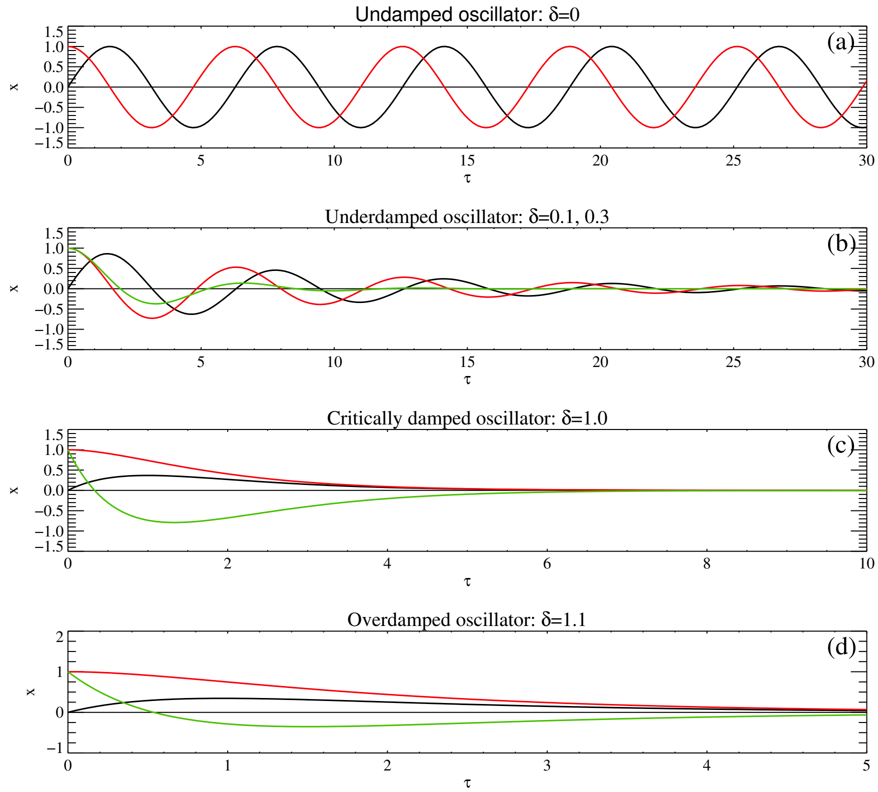

Figure 5.1: Solutions of $\ddot{x} + 2\delta\dot{x} + x = 0$ for various values of δ and different initial conditions.

The linear damped oscillator: $\delta > 0$

The three cases of interest are $\delta < 1$, $\delta = 1$ and $\delta > 1$. Figure 5.1 shows several solutions of $\ddot{x} + 2\delta\dot{x} + x = 0$ for various values of δ.

(i) **Case 1:** $\delta > 1$. This is the overdamped oscillator. For this case there are two real negative roots and the general solution is

$$x = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} = c_1 e^{-\left(\delta + \sqrt{\delta^2 - 1}\right)\tau} + c_2 e^{-\left(\delta - \sqrt{\delta^2 - 1}\right)\tau}. \tag{5.84}$$

$x(\tau)$ can pass through zero at most once (e.g., if the initial conditions have $x(0)$ very small compared to $\dot{x}(0)$. This is called the *overdamped* case. It is desirable for a swinging door on a hinge to be overdamped so it doesn't flap back and forth. See Figure **??**(d).

(ii) **Case 2:** $\delta < 1$. This is the underdamped or undamped ($\delta = 0$ oscillator. For this case there are two complex conjugate roots. The general solution has the form

$$x = Ae^{-\delta\tau} \cos\left(\sqrt{1 - \delta^2}\tau + \phi\right). \tag{5.85}$$

The solution oscillates with frequency $\omega = \sqrt{1 - \delta^2}$ and decays with a decay time scale of $1/\delta$. For $\delta \ll 1$ the frequency is close to 1 and the system undergoes a large number of oscillations before decaying appreciably. For δ close to 1 the oscillation frequency is very small and the decay scale is close to one. The solution decays by a large factor over one oscillation. This case is called the *underdamped* case. Natural systems which undergo unforced oscillations generally have some damping and hence fall into this category. See Figure 5.1(a,b) and 5.2(b).

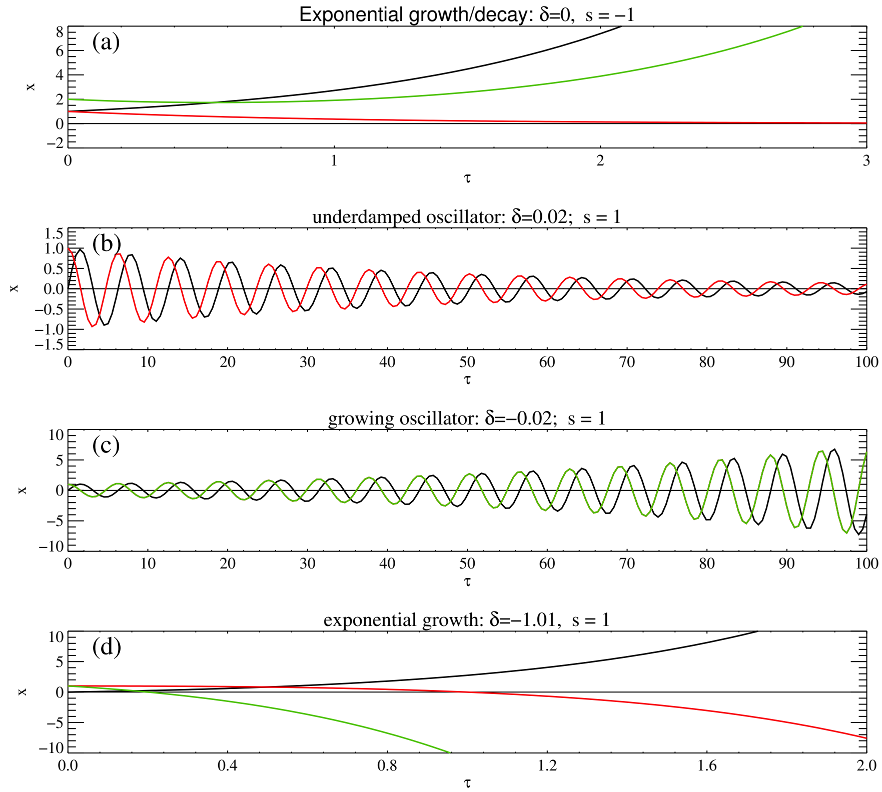

Figure 5.2: Solutions of $\ddot{x} + 2\delta\dot{x} + sx = 0$ for various values of δ and different initial conditions. (a) $s = -1$. (b)–(d) $s = 1$.

(iii) **Case 3:** $\delta = 1$**.** This transitional case is called *critically* damped. The general solution is

$$x = c_1 e^{-\tau} + c_2 \tau e^{-\tau}. \tag{5.86}$$

Solutions are qualitatively similar to the overdamped case. See Figure 5.1(c)

The unstable oscillator: $\delta < 0$

In this case the energy of the system grows in time. If $-1 < \delta < 0$ the solution has the form

$$x = Ae^{-\delta\tau} \cos\left(\sqrt{1-\delta^2}\tau + \phi\right) \tag{5.87}$$

where now $-\delta > 0$ so the amplitude of the oscillations grows in time. In many systems that have natural modes of oscillations (e.g., waves), an unstable situation can result in growing oscillations. An example is the generation of surface water waves by wind blowing over the water surface. If the wind speed is large enough the flow may be unstable and perturbations in the interface displacement (triggered by air pressure fluctuations associated with eddies) can grow in time. The kinetic energy in the wind is the energy source for the growing surface water waves. The difference in the velocities of the air and water means there is a source of kinetic energy to feed the growing waves. Of course the waves will not grow indefinitely. There is a finite amount of energy available to feed the waves and when waves become large enough they break and dissipate energy. The system rapidly becomes nonlinear and very complicated. This is still an active area of research. See Figure **??**(c) for sample solutions.

The equation $\ddot{x} + 2\delta\dot{x} - x = 0$

Changing the sign of the last term fundamentally changes the behaviour of the solutions: when $\delta = 0$ the solutions grow or decay exponentially rather than oscillating (Figure 5.2(a)). Solutions for $\delta = -1.01$ (panel (d)) all grow exponentially in time.

5.2.3 Inhomogeneous Equations: Method of Undetermined Coefficients for Constant Coefficient ODEs

We now consider particular solutions of

$$ay'' + by' + cy = g(t). \tag{5.88}$$

I. Powers of t

For

$$L[y] \equiv ay'' + by' + cy = t^n \tag{5.89}$$

there is a particular solution of the form

$$y_p = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n. \tag{5.90}$$

Note that since y has higher powers of t than y' and y'' do there can be no terms in y_p with powers higher than n.

Example: Find a particular solution for

$$y'' + 2y' - 6y = t^2. \tag{5.91}$$

Solution: Since derivatives of powers of t yield powers of t we guess that the solution is a polynomial in t. Since differentiation reduces the powers the term $-6y$ on the left-hand side will have the highest power. Thus we should try

$$y_p = a_0 + a_1 t + a_2 t^2. \tag{5.92}$$

Then

$$y_p' = a_1 + 2a_2 t \tag{5.93}$$

and

$$y_p'' = 2a_2, \tag{5.94}$$

so

$$\begin{aligned} y_p'' + 2y_p' - 6y_p &= 2a_2 + 2\big(a_1 + 2a_2 t\big) - 6(a_0 + a_1 t + a_2 t^2\big) \\ &= \big(2a_2 + 2a_1 - 6a_0\big) + \big(4a_2 - 6a_1\big)t - 6a_2 t^2. \end{aligned} \tag{5.95}$$

Setting this equal to t^2 gives

$$\begin{aligned} 2a_2 + 2a_1 - 6a_0 &= 0 \\ 4a_2 - 6a_1 &= 0 \\ -6a_2 &= 1. \end{aligned} \tag{5.96}$$

Thus

$$a_2 = -\frac{1}{6}; \qquad a_1 = \frac{2}{3}a_2 = -\frac{1}{9}; \qquad a_0 = \frac{a_1 + a_2}{3} = -\frac{5}{54}. \tag{5.97}$$

A particular solution is

$$y_p = \frac{1}{6} - \frac{1}{9}t - \frac{5}{54}t^2. \tag{5.98}$$

Comments:

(i) If $c = 0$ the problem reduces to a first-order problem for y'. Look for a particular solution with $y_p' = a_0 + a_1 t + a_2 t^2 + \cdots a_n t^n$. Find the a_n and integrate to get y_p in the form $y_p = b_0 t + b_1 t^2 + \cdots b_n t^{n+1}$.

(i) If $b = c = 0$ then you can integrate twice to get $y_p = \frac{at^{n+2}}{(n+2)(n+1)}$.

II. Exponentials e^{qt}

For

$$L[y] \equiv ay'' + by' + cy = e^{qt} \tag{5.99}$$

start by looking for a solution of the form $y_p = Ae^{qt}$ because all derivatives of y_p will then be proportional to e^{qt}. Substituting into the DE gives

$$P(q)Ae^{qt} = e^{qt} \tag{5.100}$$

where $P(q)$ is the characteristic polynomial introduced above. If $P(q) \neq 0$, i.e., if e^{qt} is not a solution of the homogeneous equation, then we have $A = 1/P(q)$ so the solution is

$$y_p = \frac{e^{qt}}{P(q)} = \frac{e^{qt}}{aq^2 + bq + c}. \tag{5.101}$$

If $P(q) = 0$, i.e., if e^{qt} is a solution of the homogeneous equation (which is why it helps to find the homogeneous solution first) then you need to do something else. If this is the case set

$$y_p = f(t)e^{qt}. \tag{5.102}$$

Note that when substituting this into the left had side of the differential equation all terms of $L[y_p]$ will be include a factor e^{qt} which we can then cancel. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(f(t)e^{qt}\Big) = (f' + qf)e^{qt} \tag{5.103}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(f(t)e^{qt}\Big) = (f'' + 2qf' + q^2 f)e^{qt} \tag{5.104}$$

so the differential equation gives

$$L[y_p] = L[f(t)e^{qt}] = \Big[a\big(f'' + 2qf' + q^2 f\big) + b\big(f' + qf\big) + cf\Big]e^{qt} = e^{qt} \tag{5.105}$$

or

$$af'' + (2aq + b)f' + (aq^2 + bq + c)f = 1. \tag{5.106}$$

Now the coefficient of f is $P(q) = 0$ so this reduces to

$$af'' + (2aq + b)f' = 1 \tag{5.107}$$

which is a first-order DE for f'. It is an example of the first problem we considered: a second-order constant coefficient with a power of t as the forcing term. In this case the power is 0.

Case (i): If $2aq + b \neq 0$ take $f' = \frac{1}{2aq+b}$. Then $f = \frac{t}{2aq+b}$.

Case (ii): If $2aq + b = 0$ have $f'' = \frac{1}{a}$ so take $f = \frac{t^2}{2a}$.

In both cases we just need one particular solution and have taken the simplest.

Note: From $aq^2 + bq + c = 0$ we have $q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If $b^2 - 4ac = 0$ then $q = -\frac{b}{2a}$ or $2aq + b = 0$. Thus case (ii) is the case when q is a double root of $P(q) = 0$.

Summary: A particular solution of

$$L[y] = e^{qt} \tag{5.108}$$

has the form

1. $y_p = Ae^{qt}$ if $L[e^{qt}] \neq 0$. That is, if q is not a root of $P(q) = 0$.
2. $y_p = Ate^{qt}$ if $L[e^{qt}] = 0$ and q is a single root of $P(q) = 0$. That is if te^{qt} is not a solution of the homogeneous equation $L[y] = 0$.
3. $y_p = At^2e^{qt}$ if $L[e^{qt}] = 0$ and q is a double root of $P(q) = 0$. That is, if both te^{qt} and te^{qt} are solutions of the homogeneous equation $L[y] = 0$.

Comment: In equation (5.106) for f note that the coefficient of f is $P(q)$, the coefficient of f' is $P'(q)$ and the coefficient of f'' is $\frac{1}{2}P''(q)$. Remembering this can speed things up in some of the following.

III. Exponentials $g(t) = t^m e^{qt}$

More generally consider

$$L[y] \equiv ay'' + by' + cy = t^m e^{qt}. \tag{5.109}$$

For this case set

$$y_p = f(t)e^{qt}. \tag{5.110}$$

to get (see above)

$$af'' + (2aq + b)f' + (aq^2 + bq + c)f = t^m. \tag{5.111}$$

This is a linear, constant-coefficient, second-order DE for f with forcing t^m. This is case I considered above.

Example: Find the general solution of

$$y'' + 3y' + 2y = t^2 e^{-t}. \tag{5.112}$$

Solution: First find the homogeneous solution. Substituting $y = e^{\lambda t}$ gives $\lambda^2 + 3\lambda + 2 = 0$ or $(\lambda+2)(\lambda+1) = 0$. Thus e^{-2t} and e^{-t} are two linearly independent homogeneous solutions.

Next look for a particular solution $y_p = f(t)e^{-t}$. Then

$$\begin{aligned} y_p' &= (f' - f)e^{-t} \\ y_p'' &= (f'' - 2f' + f)e^{-t} \end{aligned} \tag{5.113}$$

Then $L[y_p]$ is

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= \Big(f'' - 2f' + f + 3(f' - f) + 2f\Big)e^{-t} \\ &= (f'' + f')e^{-t}. \end{aligned} \tag{5.114}$$

From the differential equation we have

$$f'' + f' = t^2. \tag{5.115}$$

Solve by setting $f' = a_0 + a_1 t + a_2 t^2$. Substituting gives

$$(a_1 + 2a_2 t) + (a_0 + a_1 t + a_2 t^2) = a_1 + a_0 + (2a_2 + a_1)t + a_2 t^2 = t^2. \tag{5.116}$$

Hence

$$\begin{aligned} a_2 &= 1 \\ a_1 &= -2a_2 = -2 \\ a_0 &= -a_1 = 2 \end{aligned} \tag{5.117}$$

so

$$f' = 2 - 2t + t^2 \implies f = 2t - t^2 + \frac{1}{3}t^3 \tag{5.118}$$

and the particular solution is

$$y_p = \big(2t - t^2 + \frac{1}{3}t^3\big)e^{-t}. \tag{5.119}$$

Hence the general solution is

$$y = \big(2t - t^2 + \frac{1}{3}t^3\big)e^{-t} + c_1 e^{-2t} + c_2 e^{-t}. \tag{5.120}$$

Comments:

1. Because the exponential e^{-t} is a solution of the homogeneous equation the coefficient of f in (5.115) was zero. The coefficient of f' was non-zero because $\lambda = -1$ was a single root of the characteristic equation $P(\lambda) = 0$.

2. An alternative derivation of (5.115) is the following. We have $P(q) = q^2+3q+2$, $P'(q) = 2q+3$ and $P''(q) = 2$. Then $P(-1) = 0$, $P'(-1) = 1$ and $\frac{1}{2}P''(-1) = 1$. So the differential equation for $f(t)$ is $\frac{1}{2}P''(-1)f'' + P'(-1)f' + P(1)f = t^2$ or $f'' + f' = t^2$.

3. When integrating the expression for f' to get f we did not need to include a constant of integration as *any* solution will do. Our goal is to find a particular solution. If we did we would have had $f = a + 2t - t^2 + \frac{1}{3}t^3$ which would add a term ae^{-t} to the final solution. We already have a $c_1 e^{-t}$ term from the homogeneous solution so the new term can just be absorbed into the homogeneous solution. Remember, if y_p is a particular solution so is y_p plus any combination of the homogeneous solutions.

Example: Find the general solution of

$$y'' + 4y' + 4y = 2t^2 e^{-t} - 6t^6 e^{-2t}. \tag{5.121}$$

Solution:

Step 1. First find the homogeneous solution. Substituting $y = e^{\lambda t}$ gives $\lambda^2 + 4\lambda + 4 = 0$ or $(\lambda + 2)^2 = 0$. Thus e^{-2t} and te^{-2t} are two linearly independent homogeneous solutions.

Step 2. For the particular solutions we split the problem into two pieces. First find a particular solution y_{p1} of $L[y] = t^2 e^{-t}$ and then a particular solution y_{p2} for $L[y] = t^6 e^{-2t}$. Then by linearity $y_p = 2y_{p1} - 6y_{p2}$ will be a particular solution of the problem.

Step 2(a): Let $y_{p1} = f(t)e^{-t}$. Then

$$\begin{aligned} y'_{p1} &= (f' - f)e^{-t} \\ y''_{p1} &= (f'' - 2f' + f)e^{-t} \end{aligned} \tag{5.122}$$

Then $L[y_{p1}]$ is

$$\begin{aligned} y''_{p1} + 4y'_{p1} + 4y_{p1} &= \Big(f'' - 2f' + f + 4(f' - f) + 4f\Big)e^{-t} \\ &= \big(f'' + 2f' + f\big)e^{-t}. \end{aligned} \tag{5.123}$$

The coefficient of f is non-zero this time because e^{-t} is not a solution of the homogeneous equation. From the differential equation we have

$$f'' + 2f' + f = t^2. \tag{5.124}$$

Solve by setting $f = a_0 + a_1 t + a_2 t^2$. Substituting gives

$$2a_2 + 2(a_1 + 2a_2 t) + (a_0 + a_1 t + a_2 t^2) = 2a_2 + 2a_1 + a_0 + (4a_2 + a_1)t + a_2 t^2 = t^2. \tag{5.125}$$

Hence

$$\begin{aligned} a_2 &= 1 \\ a_1 &= -4a_2 = -4 \\ a_0 &= -2a_2 - 2a_1 = -2 + 8 = 6 \end{aligned} \tag{5.126}$$

so

$$f = 6 - 4t + t^2 \tag{5.127}$$

and the particular solution is

$$y_{p1} = \big(6 - 4t + t^2\big)e^{-t}. \tag{5.128}$$

Step 2(b): Next find y_{p2}. Let $y_{p2} = f(t)e^{-2t}$. Have

$$\begin{aligned} y_p' &= (f' - 2f)e^{-t} \\ y_p'' &= (f'' - 4f' + 4f)e^{-t} \end{aligned} \tag{5.129}$$

Then $L[y_{p1}]$ is

$$\begin{aligned} y_{p2}'' + 4y_{p2}' + 4y_{p2} &= \Big(f'' - 4f' + 4f + 4(f' - 4f) + 4f\Big)e^{-t} \\ &= f''e^{-t}. \end{aligned} \tag{5.130}$$

From the differential equation we have

$$f'' = t^6 \tag{5.131}$$

which is easily integrated to get

$$f = \frac{t^8}{72}. \tag{5.132}$$

Note that the coefficients of f' and f in (5.131) are zero because $\lambda = -2$ is a double root of the characteristic equation $P(\lambda) = 0$ so the exponential part of the forcing function e^{-2t} is a solution of the homogeneous solution as is te^{-2t}. The coefficients of f and f', $P(\lambda)$ and $P'(\lambda)$, are zero.

Step 3. The particular solution of the differential equation is $2y_{p1} - 6y_{p2}$, hence the general solution of the differential equation is

$$y = \big(12 - 8t + 2t^2\big)e^{-t} - \frac{t^8}{12}e^{-2t} + ae^{-2t} + bte^{-2t}. \tag{5.133}$$

Comment: You could also proceed by finding particular solutions y_{p1} and y_{p2} of $L[y] = 2t^2e^{-t}$ and $L[y] = -6t^6e^{-2t}$ directly which would lead to $f'' + 2f' + f = 2t^2$ in step 2(a) and $f'' = -6t^6$ in step 2(b) with the final solution being $y_p = y_{p1} + y_{p2}$.

III. Sinusoidal forcing functions $g(t) = e^{\alpha t}\cos(\beta t)$ or $e^{\alpha t}\cos(\beta t)$ with α and β real.

For this we note that

$$\begin{aligned} e^{\alpha t}\cos(\beta t) &= \Re\big\{e^{qt}\big\} \\ e^{\alpha t}\cos(\beta t) &= \Im\big\{e^{qt}\big\} \end{aligned} \tag{5.134}$$

where

$$q = \alpha + i\beta. \tag{5.135}$$

To proceed we are going to find solutions of

$$L[y] \equiv ay'' + by' + cy = e^{qt}. \tag{5.136}$$

The solution will be complex valued since the forcing term e^{qt} is. Setting $y = y_r + iy_i$ where y_r and y_i are the real and imaginary parts of y we have

$$L[y] = L[y_r] + iL[y_i] = e^{qt} = e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t). \tag{5.137}$$

Thus the real part, y_r, is a particular solution of $L[y] = e^{\alpha t}\cos(\beta t)$ and the imaginary part, y_i, is a particular solution of $L[y] = e^{\alpha t}\sin(\beta t)$.

The procedure for finding solutions is identical to the case when q is real. One just has to take the real or imaginary parts at the end.

Example: Find the general solution of

$$y'' + 4y' + 13y = \frac{1}{2}e^{-t}\cos(2t) + 3te^{-2t}\sin(3t) + 2t^2. \tag{5.138}$$

Solution:

Step 1. First find the homogeneous solution. Substituting $y = e^{\lambda t}$ gives $\lambda^2 + 4\lambda + 13 = 0$ so

$$\lambda = \frac{-4 \pm \sqrt{16 - 4\cdot 13}}{2} = -2 \pm \sqrt{4 - 13} = -2 \pm 3i. \tag{5.139}$$

Thus two linearly independent homogeneous solutions are

$$\begin{aligned} y_1 &= e^{-2t}\cos(3t), \\ y_2 &= e^{-2t}\sin(3t). \end{aligned} \tag{5.140}$$

Note the the second forcing function in (5.138) involves y_2.

Step 2. For the particular solutions we split the problem into three pieces. We need to find particular solutions: (i) y_{p1} of $L[y] = e^{-t}\cos(2t)$; (ii) y_{p2} of $L[y] = e^{-2t}\sin(3t)$; and (iii) y_{p3} of $L[y] = t^2$. Then by linearity $y_p = \frac{1}{2}y_{p1} + 3y_{p2} + 2y_{p3}$ will be a particular solution of (5.138) .

Step 2(a): Since $e^{-t}\cos(2t) = \Re\big\{e^{(-1+2i)t}\big\}$ we first solve

$$L[y] = e^{qt} \tag{5.141}$$

where $q = -1 + 2i$ and then take the real part of the solution. Since $e^{-t}\cos(2t)$ is not a solution of the homogeneous problem we simply let $y_{p1} = Ae^{qt}$ which leads to

$$P(q)Ae^{qt} = e^{qt} \tag{5.142}$$

where $P(q)$ is the characteristic polynomial. $P(q) \neq 0$ in this case since $e^{-t}\cos(2t)$ is not a solution of the homogeneous problem so

$$A = \frac{1}{P(q)} = \frac{1}{q^2 + 4q + 13} = \frac{1}{(-1+2i)^2 + 4(-1+2i) + 13} = \frac{1}{6+4i} = \frac{3-2i}{26}. \tag{5.143}$$

Next take the real part to get the particular solution y_{p1}:

$$\begin{aligned}\Re\Big\{Ae^{qt}\Big\} &= \Re\Big\{\frac{3-2i}{26}e^{-t}\Big(\cos(2t)+i\sin(2t)\Big)\Big\}\\ &= \frac{e^{-t}}{26}\Big(3\cos(2t)+2\sin(2t)\Big).\end{aligned} \tag{5.144}$$

Note the forcing involved $\cos(2t)$ and the solution involves both $\cos(2t)$ and $\sin(2t)$.

Step 2(b): Next find y_{p2}. We now consider

$$L[y] = te^{qt} \tag{5.145}$$

where now $q = -2+3i$ and take the imaginary part of the solution because the forcing involves $\sin(3t)$. Now e^{qt} is a solution of the homogeneous problem this time so we need to set $y = f(t)e^{qt}$. Doing so leads to

$$f'' + (2q+4)f' = t. \tag{5.146}$$

Thus we try:

$$f' = a_0 + a_1 t \tag{5.147}$$

which gives

$$a_1 + (2q+4)(a_0 + a_1 t) = t \tag{5.148}$$

so

$$\begin{aligned}a_1 &= \frac{1}{2q+4} = \frac{1}{6i} = -\frac{i}{6}\\ a_0 &= -\frac{a_1}{2q+4} = -\frac{a_1}{6i} = \frac{1}{36}.\end{aligned} \tag{5.149}$$

Thus

$$f' = \frac{1}{36} - \frac{i}{6}t \implies f = \frac{1}{36}t - \frac{i}{12}t^2. \tag{5.150}$$

Hence

$$\begin{aligned}y_{p2} &= \Big(\frac{1}{36}t - \frac{i}{12}t^2\Big)e^{qt}\\ &= \Big(\frac{1}{36}t - \frac{i}{12}t^2\Big)e^{-2t}\Big(\cos(3t)+i\sin(3t)\Big)\\ &= \Big(\frac{1}{36}t\cos(3t) + \frac{t^2}{12}\sin(3t)\Big)e^{-2t} + i\Big(\frac{1}{36}t\sin(3t) - \frac{t^2}{12}\cos(3t)\Big)e^{-2t}\end{aligned} \tag{5.151}$$

which is a complex-valued particular solution of $L[y] = te^{qt}$. Taking the imaginary part we get

$$y_{p2} = \frac{1}{36}t\sin(3t)e^{-2t} - \frac{t^2}{12}\cos(3t)e^{-2t} \tag{5.152}$$

as the desired particular solution of $L[y] = e^{-2t}\sin(3t)$.

Step 2(c): Next find y_{p3}. We consider

$$L[y] = t^2. \tag{5.153}$$

For this we set $y_{p3} = a_0 + a_1 t + a_2 t^2$. Substituting into $L[y] = t^2$ gives

$$\begin{aligned} 2a_2 + 4(a_1 + 2a_2 t) &+ 13(a_0 + a_1 t + a_2 t^2) \\ &= 2a_2 + 4a_1 + 13a_0 + (8a_2 + 13a_1)t + 13a_2 t^2 \\ &= t^2 \end{aligned} \tag{5.154}$$

so

$$\begin{aligned} a_2 &= \frac{1}{13} \\ a_1 &= -\frac{8}{13}a_2 = -\frac{8}{269} \\ a_0 &= -\frac{2a_2 + 4a_1}{13} = \frac{6}{13^3}. \end{aligned} \tag{5.155}$$

Thus

$$y_{p3} = \frac{6}{13^3} - \frac{8}{269}t + \frac{1}{13}t^2. \tag{5.156}$$

Step 3. Putting all the parts together, the particular solution of the differential equation is $\frac{1}{2}y_{p1} + 3y_{p2} + 2y_{p3}$, hence the general solution of the differential equation is

$$\begin{aligned} y = \frac{e^{-t}}{52}\Big(3\cos(2t) + 2\sin(2t)\Big) &+ \frac{1}{12}t\sin(3t)e^{-2t} - \frac{t^2}{4}\cos(3t)e^{-2t} \\ + \frac{12}{13^3} - \frac{16}{269}t &+ \frac{2}{13}t^2 + c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t). \end{aligned} \tag{5.157}$$

5.3 Resonance

We now consider the effects of sinusoidal forcing on a linearly damped oscillator in detail.

Many systems have unforced modes of oscillation that are subjected to weak damping, musical instruments being obvious examples. Other examples include

- wine glasses
- waves in a bathtub, lake, bay, swimming pool
- mass-spring systems
- RLC circuits
- pendulums

etc., etc., etc.

If such a system is forced periodically with a period close to one of its natural modes of oscillation then the system can respond with a large amplitude oscillation. This phenomena is called resonance. Examples include the proverbial opera singer shattering a wine glass when she sings at the right pitch. This occurs when sound waves of frequency close to the resonance frequency of the wine glass reflect off it. Wires hum in a high wind when eddies are shed off the wire at a frequency close to a natural frequency of standing waves in the wire. This process was blamed for the failure of the Tacoma Narrows Bridge in 1940 under high wind conditions (lots of videos on the web). Many young children (and maybe university students?) have fun generating large waves in a bathtub by moving back and forth at just the right frequency. A much larger scale example of this is the huge tides in the Bay of Fundy and Ungava Bay in northern Quebec which are generated by small amplitude tidal waves propagating along the coast. These force waves in the bays with frequency close to that of a standing wave in the bay. There is lots of information on this phenomena in the Bay of Fundy available on the web.

5.3.1 The simple harmonic oscillator

The simplest model of resonant-type phenomena is the linear, weakly-damped simple harmonic oscillator subject to sinusoidal forcing, e.g., the linear mass-spring system with weak damping or an RLC circuit with low resistance.

So, we now study in detail the solutions of

$$mx'' + \gamma x' + kx = F\cos(\Omega t). \tag{5.158}$$

As it stands there are five parameters: the mass m, the damping coefficient γ, the spring constant k, the forcing amplitude A and the forcing frequency Ω. All except the forcing amplitude are positive (the first three must be, taking $\Omega > 0$ is a choice). As we know we can reduce the number of parameters by non-dimensionalizing the problem. First divide by m and set

$$\omega_0 = \sqrt{k/m} \tag{5.159}$$

to write the equation as

$$x'' + \frac{\gamma}{m}x' + \omega_0^2 x = \frac{F}{m}\cos(\Omega t). \tag{5.160}$$

Then introduce a nondimensional time via

$$\tau = \omega_0 t \tag{5.161}$$

to obtain

$$\omega_0^2\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + \frac{\gamma}{m}\omega_0\frac{\mathrm{d}x}{\mathrm{d}\tau} + \omega_0^2 x = \frac{F}{m}\cos\left(\frac{\Omega}{\omega_0}\tau\right) \tag{5.162}$$

Dividing by ω_0^2 and defining $\tilde{\Omega} = \Omega/\omega_0$ gives

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + \frac{\gamma}{\omega_0 m}\frac{\mathrm{d}x}{\mathrm{d}\tau} + x = \frac{F}{m\omega_0^2}\cos\big(\tilde{\Omega}\tau\big). \tag{5.163}$$

Next define

$$2\delta = \frac{\gamma}{\omega_0 m} \tag{5.164}$$

and let

$$x = \frac{F}{m\omega_0^2}\tilde{x} \tag{5.165}$$

to get our final equation

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\delta\frac{d\tilde{x}}{d\tau} + \tilde{x} = \cos\big(\tilde{\Omega}\tau\big). \tag{5.166}$$

There are only two parameters: the nondimensional damping coefficient δ and the non-dimensional forcing frequency $\tilde{\Omega}$ which is the ratio of the forcing frequency to the natural frequency of the undamped system. Note that because the governing differential equation is linear we were able to scale x to eliminate the forcing amplitude parameter. Note also that to completely understand the behaviour of solutions of the linear damped oscillator we need only vary these two parameters, not five different parameters as in the original dimensional equation. This is an example of the power of nondimensionalization.

5.3.2 Solution of the equation for the forced, under-damped simple harmonic oscillator

In the following we will drop the tildes and consider the equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + 2\delta\frac{\mathrm{d}x}{\mathrm{d}\tau} + x = \cos\bigl(\Omega\tau\bigr). \tag{5.167}$$

We are going to focus on the under-damped case for which $0 < \delta < 1$ because for the under-damped case the unforced system has oscillatory solutions that decay in time rather than solutions that decay in time without oscillating. Resonance is a phenomenon that only occurs when the unforced system has oscillatory behaviour. The undamped case $\delta = 0$ is considered below.

The homogeneous solution of the DE is

$$x_h = c_1 e^{-\delta\tau}\cos(\sqrt{1-\delta^2}\tau) + c_2 e^{-\delta t}\sin(\sqrt{1-\delta^2}\tau). \tag{5.168}$$

A particular solution can be found by first finding a solution of

$$\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + 2\delta\frac{\mathrm{d}x}{\mathrm{d}\tau} + x = e^{i\Omega\tau} \tag{5.169}$$

and then taking the real part. Setting $x_p = Be^{i\Omega\tau}$ and substituting into the governing equation gives

$$P(i\Omega)B = \bigl(-\Omega^2 + 2\delta\Omega i + 1\bigr)B = 1 \tag{5.170}$$

where $P(\lambda) = \lambda^2 + 2\delta\lambda + 1$ is the characteristic polynomial for the differential equation. $P(i\Omega)$ is a complex number which we can write as

$$P(i\Omega) = 1 - \Omega^2 + 2\delta\Omega i = re^{i\alpha} \tag{5.171}$$

where

$$\begin{aligned} r = |P(i\Omega)| &= \sqrt{(1-\Omega^2)^2 + 4\delta^2\Omega^2}, \\ r\cos(\alpha) &= 1 - \Omega^2, \\ r\sin(\alpha) &= 2\delta\Omega. \end{aligned} \tag{5.172}$$

Note that for $\delta > 0$, $r\sin(\alpha) > 0$ so $\alpha \in (0, \pi)$ so

$$\alpha = \arctan\bigl(\frac{2\delta\Omega}{1-\Omega^2}\bigr) \text{ with } \alpha \in (0,\pi). \tag{5.173}$$

This is not the usual range for the arctan function. Using this form, α is the phase lag of the response relative to the forcing.

From (5.170) we have

$$B = \frac{1}{r} e^{-i\alpha} = Ae^{-i\alpha}, \tag{5.174}$$

where

$$A = \frac{1}{\sqrt{(1-\Omega^2)^2 + 4\delta^2\Omega^2}}. \tag{5.175}$$

Hence, the solution of (5.169) is

$$x_p = Ae^{i(\Omega t - \alpha)}. \tag{5.176}$$

The real-valued part, which gives the solution of (5.167), is

$$x_p = A\cos(\Omega t - \alpha). \tag{5.177}$$

This is the form of the solution we will use. You will see the solution written in other ways, e.g., as

$$x_p = A\sin\big(\Omega\tau + \theta\big) \tag{5.178}$$

where A is unchanged and $\alpha = -\theta + \frac{\pi}{2}$. One advantage of this form is that the angle θ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and $\theta = \arctan(\frac{1-\Omega^2}{2\delta\Omega})$ uses the standard range for arctan.

Interpretation of the Solution

The general solution of (5.167) is

$$x = x_h + x_p \tag{5.179}$$

where the homogeneous solution x_h is given by (5.168) and the particular solution is given by either (5.177) or (5.178). The two undetermined constants c_1 and c_2 in the homogeneous solution are determined by prescribed initial conditions.

1. If the damping coefficient is non-zero then the homogeneous solution decays with time leaving the particular solution x_p *which is independent of the initial conditions*. After a sufficiently long time the system has no memory of its initial state.

2. Consider the phase lag α.

   (a) Since $\delta\Omega \geq 0$, $\sin\alpha > 0$ hence $\alpha \in (0, \pi)$. Since A is positive the response $A\cos(\Omega t - \alpha)$ lags the forcing $\cos(\Omega t)$ by the angle α (see Figure 5.3). Using $\cos\alpha = A(1-\Omega^2)$ we see that $\alpha \in (0, \frac{\pi}{2})$ if $\Omega < 1$ and $\alpha \in (\frac{\pi}{2}, \pi)$ if $\Omega > 1$. Similarly $\theta \in (-\pi/2, \pi/2)$ and θ is easily found from $\theta = \arctan((1-\Omega^2)/(2\Omega\delta))$. To find α use $\alpha = \frac{\pi}{2} - \theta$ or

   $$\alpha = \frac{\pi}{2} - \arctan\Big(\frac{1-\Omega^2}{2\delta\Omega}\Big) \tag{5.180}$$

   (here the arctan function has range $(-\frac{\pi}{2}, \frac{\pi}{2})$). The phase lag α is plotted as a function of Ω for different values of the damping coefficient in Figure 5.4. Note that when $\Omega = 1$ (forcing frequency equals the natural undamped frequency of oscillation) the phase lag $\alpha = \pi/2$.

   (b) As $\delta \to 0$, $\sin\alpha \to 0$ so $\alpha \to 0$ or $\alpha \to \pi$. If $\Omega < 1$, $\alpha \in (0, \frac{\pi}{2})$ hence we must have $\alpha \to 0$. In this case in the limit $\delta \to 0$ the response is in phase with the forcing. If $\Omega > 1$, $\alpha \to \pi$ and the response is $180°$ out of phase. These limiting behaviours can be seen in Figure 5.4 where $\alpha(\Omega) \to 0$ for $\Omega < 1$ and $\alpha(\Omega) \to \pi$ for $\Omega > 1$ as $\delta \to 0$.

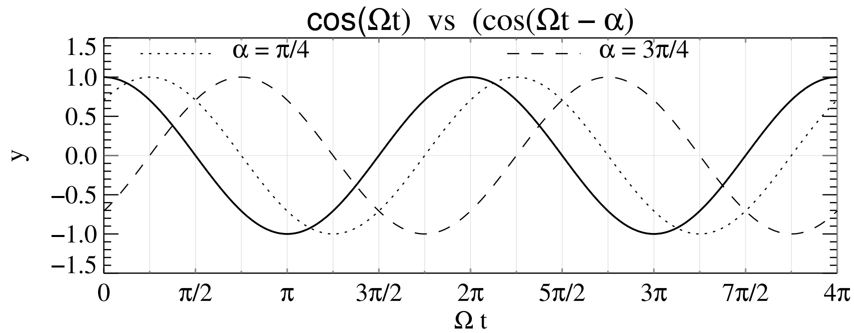

Figure 5.3: Comparisons of $\cos(\Omega t)$ (solid) with $\cos(\Omega t - \alpha)$ for $\alpha = \pi/4$ (dots) and $\alpha = 3\pi/4$ (dashed).

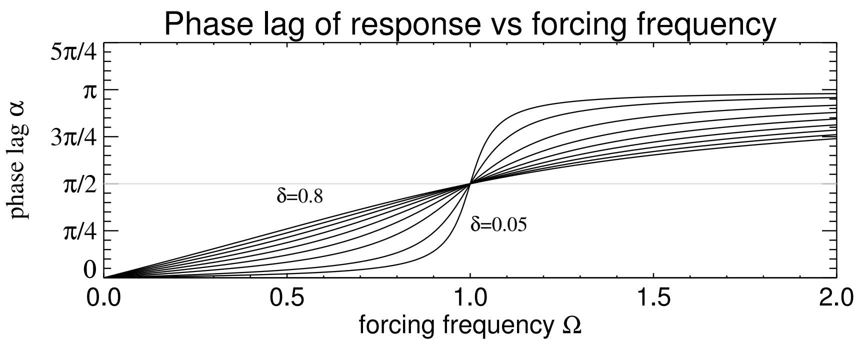

Figure 5.4: Phase lag as a function of frequency for $\delta = 0.05$ and from 0.1 to 0.8 in increments of 0.1.

3. Now consider the amplitude of the response.

   (a) As $\Omega \to 0$ the amplitude goes to 1. As $\Omega \to \infty$

$$A = \frac{1}{\sqrt{\left(1-\Omega^2\right)^2 + 4\delta^2\Omega^2}} \sim \frac{1}{\Omega^2}, \tag{5.181}$$

   since the terms under the square root sign are dominated by Ω^4 for large Ω. Thus, $A \to 0$ like $1/\Omega^2$ as $\Omega \to \infty$.

   (b) As $\delta \to 0$ the amplitude goes to 1 for all values of $\Omega \neq 1$. We will consider this case below.

   (c) At frequencies for which the amplitude of the response has a local maximum (as a function of the frequency) we have

$$\frac{\partial A}{\partial \Omega} = -\frac{1}{2}\frac{-4\Omega(1-\Omega^2) + 8\delta^2\Omega}{\left((1-\Omega^2)^2 + 4\delta^2\Omega^2\right)^{-3/2}} = 0. \tag{5.182}$$

   This gives $\Omega = 0$ or $\Omega^2 = 1 - 2\delta^2$. The second gives a local maximum provided $\delta < 1/\sqrt{2}$. In this case the local maximum is

$$A_{max} = \frac{1}{\sqrt{\left(2\delta^2\right)^2 + 4\delta^2(1-2\delta^2)}} = \frac{1}{2\delta\sqrt{1-\delta^2}}, \tag{5.183}$$

which goes to infinity as $\delta \to 0$ while the frequency of maximum response goes to 1. The maximum amplitude of the response goes to 1 as $\delta \to 1/\sqrt{2}$ as the frequency of maximum response goes to 0. Figure 5.5 shows the amplitude of the response as a function of Ω for different values of the damping coefficient δ. As the damping shrinks the region with a large response near $\Omega = \sqrt{1-2\delta^2}$ get narrower. It is the large response in a narrow frequency range centred near $\Omega = \sqrt{1-2\delta^2}$ that is referred to as resonance: the amplitude of the response is much larger than the forcing amplitude (non-dimensionalized to 1).

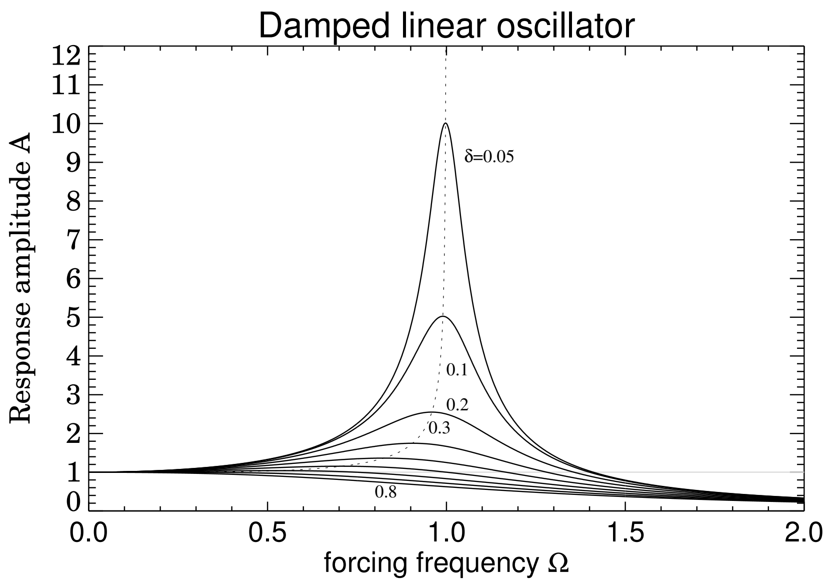

Figure 5.5: Amplitude of the response as a function of frequency for $\delta = 0.05$ and from 0.1 to 0.8 in increments of 0.1. The dotted curve passes through the peaks of the amplitude response curves.

5.4 Undamped Case

If there is no damping the DE reduces to

$$\ddot{x} + x = \cos(\Omega\tau). \tag{5.184}$$

The general solution is

$$x = A\cos\tau + B\sin\tau + \frac{1}{1-\Omega^2}\cos(\Omega\tau) \qquad \text{if } \Omega \neq 1 \tag{5.185}$$

or

$$x = A\cos\tau + B\sin\tau + \frac{\tau}{2}\sin\tau \qquad \text{if } \Omega = 1. \tag{5.186}$$

For $(\delta, \Omega) = (0, 1)$ there is unbounded growth. When $\Omega = 1$ the amplitude of the solution for the under-damped problem with $\delta > 0$ goes to infinity as $\delta \to 0$.

Consider the initial value problem $x(0) = \dot{x}(0) = 0$ for $\Omega \neq 1$. From (5.185) we have

$$x = \frac{1}{1-\Omega^2}\Big[\cos(\Omega\tau) - \cos\tau\Big]. \tag{5.187}$$

if $\Omega \neq 1$. When $\Omega = 1$ the solution (5.186) satisfying $x(0) = \dot{x}(0) = 0$ is

$$x = \frac{\tau}{2} \sin \tau. \tag{5.188}$$

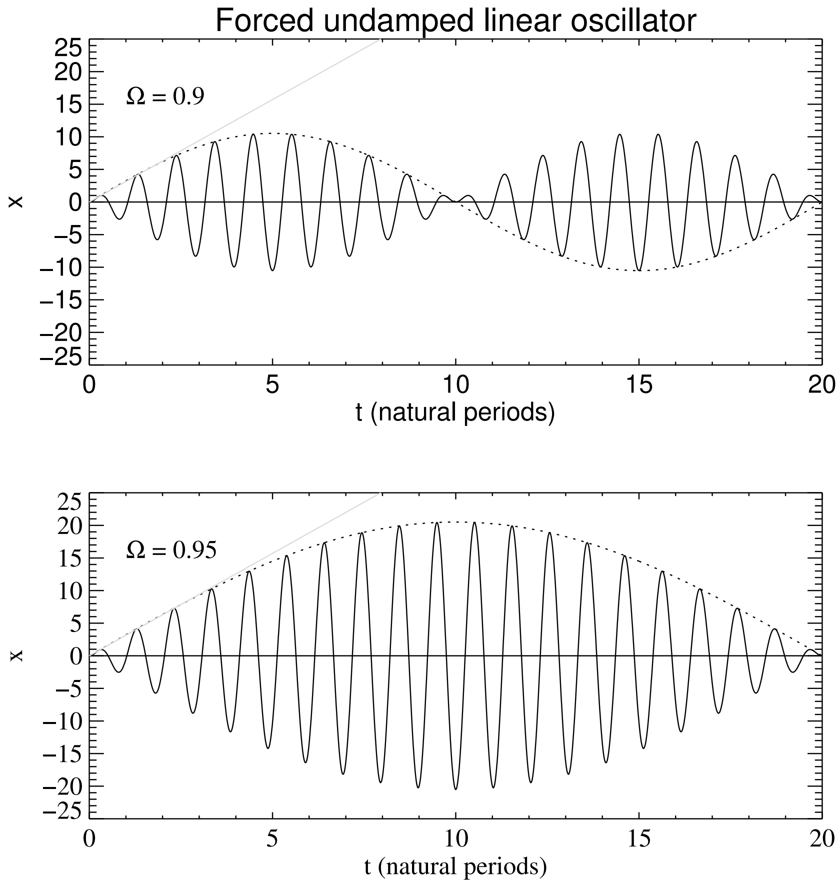

Figure 5.6: Solutions of the IVP $\ddot{x} + x = \cos(\Omega\tau)$ for initial conditions $x(0) = \dot{x}(0) = 0$. The dotted lines is the slowly varying amplitude (envelope) of the fast oscillations. The solid grey line is $\tau/2$. (a) $\Omega = 0.9$. (b) $\Omega = 0.95$

The terms in brackets in the solution for $\Omega \neq 1$ have the form $\cos(\alpha) - \cos(\beta)$. Using

$$\begin{aligned} \cos(\alpha) &= \cos\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) \\ &= \cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - \sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \end{aligned} \tag{5.189}$$

and

$$\begin{aligned} \cos(\beta) &= \cos\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right) \\ &= \cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \end{aligned} \tag{5.190}$$

gives

$$\cos(\alpha) - \cos(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right). \tag{5.191}$$

Thus

$$x = \frac{2}{1-\Omega^2}\sin\big(\frac{1-\Omega}{2}\tau\big)\sin\big(\frac{1+\Omega}{2}t\big). \tag{5.192}$$

Now consider Ω close to 1. To do this let $\Omega = 1-\epsilon$ where $|\epsilon| \ll 1$. Then

$$x = \frac{2}{2\epsilon-\epsilon^2}\sin\big(\frac{\epsilon}{2}\tau\big)\sin\big((1-\frac{\epsilon}{2})\tau\big). \tag{5.193}$$

The last factor is sinusoidal oscillations with frequency $1-\epsilon/2 \approx 1$. The first sine term gives sinusoidal oscillations with frequency $\epsilon/2 \ll 1$. The above expression can be interpreted as sinusoidal oscillations with frequency close to 1 multiplied by a slowly varying amplitude $2\sin(\frac{\epsilon}{2}\tau)/(2\epsilon-\epsilon^2)$. For times which are not too large, i.e., $\epsilon\tau <\approx 0.1$, $\sin((\frac{\epsilon}{2}\tau) \approx \epsilon/2$ and

$$x \approx \frac{1}{2}\tau\sin(\tau). \tag{5.194}$$

This is precisely the solution for $\Omega = 1$.

Two solutions of the undamped case for $\Omega = 0.9$ and $\Omega = 0.95$ are plotted in Figure 5.6. The slowly varying amplitude (also called the envelope) given by $2/(1-\Omega^2)\sin(\frac{1-\Omega}{2}\tau)$ is plotted with the dashed lines. Note the initial near linear growth (given by $\tau/2$) of the amplitude of the fast oscillations indicated by the grey curve. As $\Omega \to 1$ the envelope gets longer and larger in amplitude. The number of fast oscillations for which the approximation $\tau/2$ is valid goes to infinity.

Problems of this type are often cast in terms of multiple time scales by defining a 'slow' time scale $\tilde{\tau} = \epsilon\tau$ and writing the solution as

$$x = A(\tilde{\tau})\sin\big((1-\frac{\epsilon}{2})\tau\big). \tag{5.195}$$

where A is a slowly-varying amplitude as it varies with the slow time scale $\tilde{\tau}$. The method of multiple time-scales, or more generally multiple scales, is a powerful mathematical technique used to find approximate solutions using perturbation theory and asymptotic analysis.

5.5 Method of Variation of Parameters

The method of variation of parameters, due to Lagrange in 1774, is a method that can be used to find a particular solution of

$$L[y] \equiv y'' + P(x)y' + Q(x)y = R(x) \tag{5.196}$$

when two linearly independent solutions y_1 and y_2 of the homogeneous equation are known. Although as yet we don't know how to find two linearly independent homogeneous solutions in general (in fact this is not possible), we do for constant coefficient DEs. We can use this method to find particular solutions for more general forcing functions than those considered above.

The idea is to find a particular solution in the form

$$y_p = v_1(x)y_1(x) + v_2(x)y_2(x) \tag{5.197}$$

and find v_1 and v_2. Note this is an extension of the method we used to find a particular solution of a linear first-order equation. There, given a particular solution y_p, we found a particular solution y_p in the form $v(x)y_h(x)$ where y_h was a known solution of the homogeneous problem and v was determined by a first-order equation for v'. This time we will find a coupled system of first-order equations for v_1' and v_2'. Note also that any function can be expressed in this form and in doing so v_1 can be any function: given v_1 and y_p we can solve for v_2. Similarly we could choose v_2 to be anything and that determines v_1. So there is a lot of freedom in the choice of the functions v_1 and v_2. Finally, note that adding a constant to v_1 or v_2 adds a constant multiple of one of the homogeneous solutions to y_p, hence we have another particular solution. This means we only need find the v_j up to a constant.

Differentiating (5.197) we have

$$y_p' = v_1'y_1 + v_2'y_2 + v_1y_1' + v_2y_2'. \tag{5.198}$$

We now *make the choice* that

$$v_1'y_1 + v_2'y_2 = 0. \tag{5.199}$$

With this constraint on v_1 and v_2, given either v_1 or v_2 the other function is determined up to a constant in a way that does not involve a particular solution y_p. With this choice

$$y_p' = v_1y_1' + v_2y_2'. \tag{5.200}$$

and

$$y_p'' = v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2'. \tag{5.201}$$

Since y_1 and y_2 are solutions of the homogeneous equation

$$\begin{aligned} v_1y_1'' + v_2y_2'' &= -v_1(Py_1' + Qy_1) - v_2(Py_2' + Qy_2) \\ &= -P(v_1y_1' + v_2y_2') - Q(v_1y_1 + v_2y_2) \\ &= -Py_p' - Qy_p. \end{aligned} \tag{5.202}$$

Hence

$$y_p'' = v_1'y_1' + v_2'y_2' - Py_p' - Qy_p. \tag{5.203}$$

Hence y_p is a solution of $L[y] = R$ provided

$$v_1'y_1' + v_2'y_2' = R. \tag{5.204}$$

Equations (5.199) and (5.204) provide a system of two linear equations for v_1' and v_2':

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0, \\ v_1'y_1' + v_2'y_2' &= R. \end{aligned} \tag{5.205}$$

This system has a solution iff

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(y_1, y_2) \neq 0, \tag{5.206}$$

which is always true as y_1 and y_2 are linearly independent solutions of the homogeneous DE. Hence we can solve for v_1' and v_2' giving

$$\begin{aligned} v_1' &= \frac{\begin{vmatrix} 0 & y_2 \\ R & y_2' \end{vmatrix}}{W(y_1, y_2)} = -\frac{y_2(x)R(x)}{W(y_1, y_2)(x)} \\ v_2' &= \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & R \end{vmatrix}}{W(y_1, y_2)} = \frac{y_1(x)R(x)}{W(y_1, y_2)(x)} \end{aligned} \tag{5.207}$$

so

$$v_1 = -\int^x \frac{y_2(t)R(t)}{W(y_1,y_2)(t)}\,\mathrm{d}t$$
$$v_2 = \int^x \frac{y_1(t)R(t)}{W(y_1,y_2)(t)}\,\mathrm{d}t. \tag{5.208}$$

As mentioned above, any anti-derivative will do as adding constants to v_1 or v_2 adds constant multiples of the homogeneous solutions which is just another particular solution.

Example: Find the general solution of $y'' + y = \tan x$.

Solution: Two linearly independent homogeneous solutions are $y_1 = \cos x$ and $y_2 = \sin x$. The Wronskian of these is $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = \cos x \cdot \cos x - (-\sin x)\cdot \sin(x) = 1$. Hence

$$\begin{aligned} v_1 = -\int^x \frac{y_2 \tan x}{W}\,\mathrm{d}x &= -\int^x \sin x\,\tan x\,\mathrm{d}x \\ &= -\int^x \frac{\sin^2 x}{\cos x}\,\mathrm{d}x \\ &= -\int^x \Big(\sec x - \cos x\Big)\,\mathrm{d}x \\ &= \sin x - \ln|\sec x + \tan x| \end{aligned} \tag{5.209}$$

and

$$\begin{aligned} v_2 = \int^x \frac{y_1 \tan x}{W}\,\mathrm{d}x &= \int^x \cos x\,\tan x\,\mathrm{d}x \\ &= \int^x \sin x\,\mathrm{d}x \\ &= -\cos x. \end{aligned} \tag{5.210}$$

Hence a particular solution is

$$y_p = \big[\sin x - \ln|\sec x + \tan x|\big]\cos x - \cos x\,\sin x = -\cos x \ln|\sec x + \tan x| \tag{5.211}$$

and the general solution is

$$y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|. \tag{5.212}$$

5.6 Cauchy-Euler Equations

A linear second-order DE for $y(t)$ of the form

$$L[y] = at^2 y'' + bty' + cy = g(t) \tag{5.213}$$

where a, b, and c are constant, is called a Cauchy-Euler equation or an *equidimensional* equation (the latter because t^2y'', ty' and y all have the same dimensions as do the three constants).

For the homogeneous problem we try solutions of the form $y_h = t^r$ since then y, ty' and t^2y'' will all be proportional to t^r:

$$\begin{aligned} y_h' &= rt^{r-1} \\ y_h'' &= r(r-1)t^{r-2} \end{aligned} \tag{5.214}$$

so

$$L[y_h] = \Big[ar(r-1) + br + c\Big]t^r = 0 \tag{5.215}$$

giving

$$ar^2 + (b-a)r + c = 0. \tag{5.216}$$

There are three cases:

1. If there are two distinct real roots r_1 and r_2 then t^{r_1} and t^{r_2} are two linearly independent solutions.
2. If there are two complex conjugate roots $\alpha \pm i\beta$ take the real and imaginary parts of $t^{\alpha+i\beta} = t^\alpha e^{i\beta \ln|t|}$ to get $t^\alpha \cos(\beta \ln|t|)$ and $t^\alpha \sin(\beta \ln|t|)$ as two linearly independent solutions.
3. If there is a single double root two linearly independent solutions are t^r and $t^r \ln|t|$.

The latter can be derived by setting $y = t^r f(t)$. Substituting in the homogeneous equation gives

$$\big[ar^2 + (b-a)r + c\big]t^r f + \big[2ar + b\big]t^{r+1}f' + at^{r+2}f'' = 0. \tag{5.217}$$

The coefficient of f is zero because r is a root of (5.216). Because r is a double root we also have $2ar + b = a$. Thus we have

$$at^{r+1}f' + at^{r+2}f'' = 0 \implies f'' + \frac{1}{t}f' = 0 \tag{5.218}$$

which has $f' = \frac{1}{t}$ as a solution, hence $f = \ln|t|$.

As you may guess from the form of the soultions, the Cauchy-Euler problem can be solved by changing the independent variable t to s where $|t| = e^s$. This produces a constant coefficient differential equation for $y(s)$. See problems below.

5.7 Problems

1. Suppose $P(x)$ and $Q(x)$ are continuous on an interval $[a,b]$. A solution of

   $$y'' + P(x)y' + Q(x)y = 0, \tag{5.219}$$

   on the interval $[a,b]$ that is tangent to the x-axis at any point on this interval must be identically zero. Why?

2. The pairs of functions $(\sin x, \cos x)$ and $(\sin x, \sin x - \cos x)$ are distinct pairs of linearly independent solutions of $y'' + y = 0$. Thus, pairs of linearly independent solutions of the homogeneous equation $y'' + P(x)y' + Q(x)y = 0$ are not uniquely determined by the equation.

(a) Show that

$$P(x) = \frac{y_1 y_2'' - y_1'' y_2}{W(y_1, y_2)} \tag{5.220}$$

and

$$Q(x) = \frac{y_1' y_2'' - y_2' y_1''}{W(y_1, y_2)} \tag{5.221}$$

so that the equation is uniquely determined by any given pair of linearly independent solutions.

(b) Use this result to reconstruct the equation $y'' + y = 0$ from each of the two pairs of linearly independent solutions mentioned above.

(c) Use this result to find the equation satisfied by $y_1 = xe^x$ and $y_2 = x$.

3. Bessel's equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0 \tag{5.222}$$

with appropriate initial conditions defines new functions called Bessel functions. Verify that $x^{-1/2}\sin x$ is one solution on the interval $x > 0$ for the case $p = 1/2$. Use this to find a second solution. Then find a particular solution of

$$x^2 y'' + xy' + (x^2 - \frac{1}{4})y = 3x^{3/2}\sin x; \qquad x > 0. \tag{5.223}$$

4. Consider the general n^{th}-order constant coefficient linear DE

$$L[y] = P(D)[y] = a_0 + a_1 D + a_2 D^2 + \cdots + a_n D^n = 0. \tag{5.224}$$

where $D = \frac{d}{dx}$. Show that setting $y = f(t)e^{qt}$ gives the following equation for f:

$$P(D + q)[f] = 0. \tag{5.225}$$

Show that this can be written as

$$P(q)f + P'(q)D[f] + \frac{1}{2}D^2[f] + \cdots + \frac{1}{n!}P^{(n)}(q)D^n[f] = 0 \tag{5.226}$$

and that $\frac{1}{n!}P^{(n)}(q) = a_n$.

5. Consider the Cauchy-Euler equation

$$L[y] = at^2 y'' + bty' + cy = g(t) \tag{5.227}$$

were a, b, and c are constants with $a \neq 0$. Convert this to a constant-coefficient differential equation by the substitution $|t| = e^s$ and finding a differential equation for $\tilde{y}(s) = y(t(s))$. Convert the homogeneous solutions of your DE for $\tilde{y}(s)$ to homogeneous solutions of $y(t)$ to recover the solutions of the Cauchy-Euler equation given in section 5.6.

6. Find the general solutions of

(a) $y'' + 9y = 9\sec^2(3t)$;

(b) $y'' - 2y' + y = \frac{e^t}{1+t^2}$;

(c) $y'' + 4y' + 4y = t^{-2}e^{-2t}$;

(d) $x^2 y'' - 3xy' + 4y = x^2 \ln x$ for $x > 0$;

(e) $t^2 y'' - 2ty' + 2y = \frac{t^3}{1+t^2}$.

Chapter 6

Systems of Equations

6.1 Example

Consider a double mixing tank (Figure 6.1). Water flows enters tank 1 and leaves tank 2 at a constant rate u_f. The salt concentration in the inflow, c_{in}, is also constant. There is an exchange between the two tanks, with water flowing from tank 2 to tank 1 at a rate u_e. The volumes of water in the tanks are constant so water flows from tank 1 to tank 2 at a rate $u_f + u_e$. Let $x_1(t)$ and $x_2(t)$ be the masses of salt in the two tanks.

Conservation of salt gives the following two equations:

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= u_f c_{in} + u_e c_2(t) - (u_f + u_e)c_1(t) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= (u_f + u_e)c_1(t) - u_e c_2(t) - u_f c_2(t), \end{aligned} \tag{6.1}$$

or

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{u_f + u_e}{V_1}x_1 - \frac{u_e}{V_2}x_2 = u_f c_{in} \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} + \frac{u_f + u_e}{V_2}x_2 - \frac{u_f + u_e}{V_1}x_1 = 0. \end{aligned} \tag{6.2}$$

This is a system of equations of the form

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} + a_1 x_1 - a_2 x_2 = s \tag{6.3}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} + a_3 x_2 - a_1 x_1 = 0, \tag{6.4}$$

where the a_j and source term s are positive constants. Note that $a_2 < a_3$.

From (6.4) we have

$$x_1 = \frac{1}{a_1}\frac{\mathrm{d}x_2}{\mathrm{d}t} + \frac{a_3}{a_1}x_2. \tag{6.5}$$

Substituting into (6.3) gives

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + (a_3 + a_1)\frac{\mathrm{d}x_2}{\mathrm{d}t} + a_1(a_3 - a_2)x_2 = a_1 s. \tag{6.6}$$

This is a second-order, linear, constant coefficient differential equation.

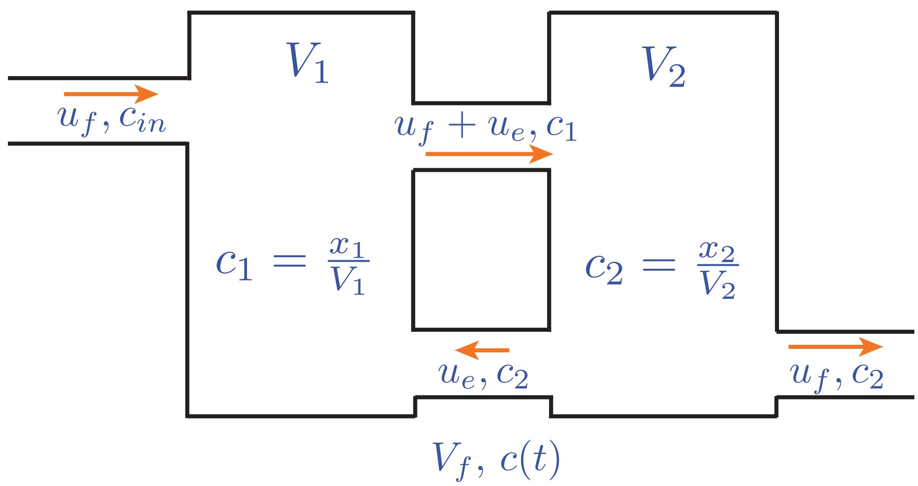

Figure 6.1: Double mixing tanks. V_i, x_i and c_i are the volumes, salt mass and salt concentration in tanks $i = 1, 2$. u_f is the inflow volume flux, u_e is the exchange volume flux and c_{in} is the salt concentration in the inflow.

1. To find the homogeneous solution set $x_{2_h} = e^{\lambda t}$. Substituting into the DE gives

$$\lambda^2 + (a_1 + a_3)\lambda + a_1(a_3 - a_2) = 0, \tag{6.7}$$

or

$$\lambda = -\frac{a_1 + a_3}{2} \pm \frac{1}{2}\sqrt{(a_1 + a_3)^2 + 4a_1(a_2 - a_3)}. \tag{6.8}$$

Note that

(a) $(a_1 + a_3)^2 + 4a_1(a_2 - a_3) = (a_1 - a_3)^2 + 4a_1a_2 > 0$ since the a_j are all positive.

(b) $(a_1 + a_3)^2 + 4a_1(a_2 - a_3) < (a_1 + a_3)^2$ since $a_2 < a_3$.

So the roots are real, distinct, and negative. Call them $-\lambda_1$ and $-\lambda_2$ where

$$\begin{aligned} 0 < \lambda_1 &= \frac{a_1 + a_3}{2} - \frac{1}{2}\sqrt{(a_1 + a_3)^2 + 4a_1(a_2 - a_3)} \\ &< \frac{a_1 + a_3}{2} + \frac{1}{2}\sqrt{(a_1 + a_3)^2 + 4a_1(a_2 - a_3)} = \lambda_2. \end{aligned} \tag{6.9}$$

The homogeneous solution is

$$x_{2_h} = ae^{-\lambda_1 t} + be^{-\lambda_2 t}. \tag{6.10}$$

2. A particular solution, by inspection, is

$$x_{2p} = \frac{s}{a_3 - a_2} = \frac{u_f c_{in}}{u_f/V_2} = V_2 c_{in}. \tag{6.11}$$

The solution is

$$x_2 = ae^{-\lambda_1 t} + be^{-\lambda_2 t} + V_2 c_{in} \tag{6.12}$$

where a and b are arbitrary constants. Now that x_2 is known we can find x_1 using (6.5). This gives

$$x_1 = \frac{a_3 - \lambda_1}{a_1} ae^{-\lambda_1 t} + \frac{a_3 - \lambda_2}{a_1} be^{-\lambda_2 t} + V_1 c_{in}. \tag{6.13}$$

As $t \to \infty$, $x_1 \to V_1 c_{in}$ and $x_2 \to V_2 c_{in}$, i,e., the concentration in each tank goes to the inflow concentration as expected.

6.1.1 Matrix Formulation:

As an alternative derivation we can write the system (6.3)–(6.4) in matrix form as

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} - A\vec{x} = \vec{s} \tag{6.14}$$

where

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A = \begin{pmatrix} -a_1 & a_2 \\ a_1 & -a_3 \end{pmatrix} \tag{6.15}$$

$$\vec{s} = \begin{pmatrix} s \\ 0 \end{pmatrix}.$$

We can look for solutions of the homogeneous problem $\vec{x}' = A\vec{x}$ in the form $\vec{x}_h = \vec{v}e^{\lambda t}$ where $\vec{v}$ is a constant vector. Substituting into the differential equation gives

$$A\vec{v} = \lambda\vec{v} \tag{6.16}$$

so solutions of this form exist if $\vec{v}$ is an eigenvector of the matrix A with eigenvalue λ. The eigenvalues of A are given by

$$\det\Big(\lambda I - A\Big) = \begin{vmatrix} \lambda + a_1 & -a_2 \\ -a_1 & \lambda + a_3 \end{vmatrix} = (\lambda + a_1)(\lambda + a_3) - a_1 a_2 = 0. \tag{6.17}$$

This is identical to (6.7) above, so the eigenvalues are the values $-\lambda_1$ and $-\lambda_2$ found above. The corresponding eigenvectors can be taken as

$$\vec{v_1} = \begin{pmatrix} \frac{a_2}{a_1 - \lambda_1} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{a_3 - \lambda_1}{a_1} \\ 1 \end{pmatrix}, \tag{6.18}$$

and

$$\vec{v_2} = \begin{pmatrix} \frac{a_2}{a_1 - \lambda_2} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{a_3 - \lambda_2}{a_1} \\ 1 \end{pmatrix}. \tag{6.19}$$

A particular solution of (6.14) is the constant vector

$$\vec{x}_p = -A^{-1}\vec{s} = \begin{pmatrix} \frac{a_3}{a_1(a_3 - a_2)} & \frac{a_2}{a_1(a_3 - a_2)} \\ \frac{1}{a_3 - a_2} & \frac{1}{a_3 - a_2} \end{pmatrix} \begin{pmatrix} s \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{a_3 s}{a_1(a_3 - a_2)} \\ \frac{s}{a_3 - a_2} \end{pmatrix} = \begin{pmatrix} V_1 c_{in} \\ V_2 c_{in} \end{pmatrix}. \tag{6.20}$$

The general solution is

$$\vec{x} = a\vec{x}_1 + b\vec{x}_2 + \vec{x}_p = a \begin{pmatrix} \frac{a_1}{a_3 - \lambda_1 t} \\ 1 \end{pmatrix} e^{-\lambda_1} + b \begin{pmatrix} \frac{a_1}{a_3 - \lambda_2} \\ 1 \end{pmatrix} e^{-\lambda_2 t} + \begin{pmatrix} V_1 c_{in} \\ V_2 c_{in} \end{pmatrix}, \tag{6.21}$$

which is identical to our previous solution (6.12)–(6.13).

6.2 Elimination Method for Systems

For this method, one writes the equations in in terms of the differential operator $D \equiv \frac{\mathrm{d}}{\mathrm{d}t}$ and treats the system of equations as you would a system of linear algebraic equations.

Notation: The operator $(D+a)(D+b)$ means apply $(D+b)$ first and than apply $(D+a)$ to the result. If a and b are constant we have

$$\begin{aligned}(D+a)(D+b)[x] &= (D+a)[x'+bx] \\ &= D[x'+bx]+a(x'+bx) \\ &= x''+bx'+ax'+abx \\ &= x''+(b+a)x'+ab\,x.\end{aligned} \tag{6.22}$$

When a and b are constant the operators $D+a$ and $D+b$ commute: $(D+a)(D+b)=(D+b)(D+a)$. Differential operators with variable coefficients do not commute. For example $(D+2t)(D+5) \neq (D+5)(D+2t)$:

$$\begin{aligned}(D+2t)(D+5)[x] &= (D+2t)[x'+5x] \\ &= D[x'+5x]+2t(x'+5x) \\ &= x''+5x'+2tx'+10tx \\ &= x''+(5+2t)x'+10tx,\end{aligned} \tag{6.23}$$

while

$$\begin{aligned}(D+5)(D+2t)[x] &= (D+5)[x'+2tx] \\ &= D[x'+2tx]+5(x'+2tx) \\ &= x''+2x+2tx'+5x'+10tx \\ &= x''+(5+2t)x'+(2+10t)x.\end{aligned} \tag{6.24}$$

Example: Find the general solution of

$$x'+2x-y' = 4t \tag{6.25}$$
$$y'-6x+y = 0. \tag{6.26}$$

Solution: In terms of the differential operator D we write the system as

$$(D+2)[x]-D[y] = 4t \tag{6.27}$$
$$-6x+(D+1)[y] = 0. \tag{6.28}$$

One can solve this by finding x first and then y or visa versa.

Method 1: We can eliminate y to obtain a single equation involving x via $(D+1)[(6.27)]+D[(6.28)]$ which gives

$$\Big((D+1)(D+2)[x]-(D+1)D[y]\Big)+\Big(-6D[x]+D(D+1)[y]\Big)=(D+1)[4t]-D[0] \tag{6.29}$$

or

$$(D^2 - 3D + 2)[x] = 4 + 4t. \tag{6.30}$$

The homogeneous problem is

$$(D - 1)(D - 2)[x] = 0 \tag{6.31}$$

which has the solutions $x_h = ae^t + be^{2t}$. For a particular solution set $x_p = a_0 + a_1 t$ to get

$$x = 5 + 2t + ae^t + be^{2t}. \tag{6.32}$$

Next we need to find y. From (6.26) we have

$$y' = x' + 2x - 4t = 12 + 3ae^t + 4be^{2t}. \tag{6.33}$$

Integrating gives

$$y = c + 12t + 3ae^t + 2be^{2t}. \tag{6.34}$$

This involves three arbitrary constants. The system is comprised of two coupled first order DEs so the general solution should only involve two. At this point equation (6.26) is satisfied, however both equations in the system must be satisfied and the second is satisfied only for a single value of c. So to determine the value of c we must now use equation (6.26). Substituting our expressions for x and y we have

$$\begin{aligned} y' - 6x + y &= (12 + 3ae^t + 4be^{2t}) - 6(5 + 2t + ae^t + be^{2t}) + (c + 12t + 3ae^t + 2be^{2t}) \\ &= (12 - 30 + c) + (-12t + 12t) + (3a - 6a + 3a)e^t + (4b - 6b + 2b)e^{2t} \\ &= c - 18. \end{aligned} \tag{6.35}$$

This should be equal to zero so $c = 18$. Thus, the solution of the system is

$$\begin{aligned} x &= 5 + 2t + ae^t + be^{2t}, \\ y &= c + 12t + 3ae^t + 2be^{2t}. \end{aligned} \tag{6.36}$$

Method 2: Alternatively you can eliminate x to get a single second-order DE for y, namely

$$(D - 1)(D - 2)[y] = 24t. \tag{6.37}$$

Note the operator on the left is identical to the one we had before operating on x. This is no coincidence. The homogeneous solution is $y = ue^t + ve^{2t}$. Next find a particular solution in the form $y_p = a_0 + a_1 t$. The result is the general solution

$$y = 18 + 12t + ue^t + ve^{2t}. \tag{6.38}$$

Next use (6.28) to immediately get

$$x = \frac{y' + y}{6} = 5 + 2t + \frac{u}{3}e^t + \frac{v}{2}e^{2t}. \tag{6.39}$$

Note that letting $u = 3a$ and $v = 2b$ puts the solution in the same form as obtained from in the first solutions. Since these constants are arbitrary either solution is fine.

Comment: The general linear constant coefficient system has the form

$$\begin{aligned} L_1[x] + L_2[y] &= f_1 \\ L_3[x] + L_4[y] &= f_2. \end{aligned} \tag{6.40}$$

Applying L_4 to the first equation and L_2 to the second and subtracting gives $(L_4L_1 - L_2L_3)[x] = L_4[f_1] - L_2[f_2]$ using the fact that $L_2L_4 = L_4L_2$ because the operators have constant coefficients. Alternatively eliminate x by applying L_3 to the first and L_1 to the second and subtracting the first from the second to get $(L_1L_4 - L_3L_2)[y] = L_1[f_2] - L_3[f_1]$. The operators acting on x and y in these two second-order DEs are the same if the operators have constant coefficients.

If $L_1L_4 - L_2L_3$ is identically zero the two equations are not linearly independent and in general have no solution. For example consider

$$\begin{aligned} x' + y' &= 2 \\ x' + y' &= 4. \end{aligned} \tag{6.41}$$

6.3 Introduction to the Phase Plane

The general system of two first-order equations has the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, y, t), \tag{6.42}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, t). \tag{6.43}$$

Definition: If f and g are independent of t the system is said to be *autonomous*.

An example of an autonomous system is the double mixing tank problem:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -a_1x_1 + a_2x_2 + s, \tag{6.44}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = a_1x_1 - a_3x_2. \tag{6.45}$$

Another is the unforced linear damped oscillator which, with $y(t) = x'(t)$, can be written as the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \tag{6.46}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2\delta y - x. \tag{6.47}$$

For the forced linear oscillator the system becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \tag{6.48}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2\delta y - x + \cos(\Omega t). \tag{6.49}$$

This is a non-autonomous system.

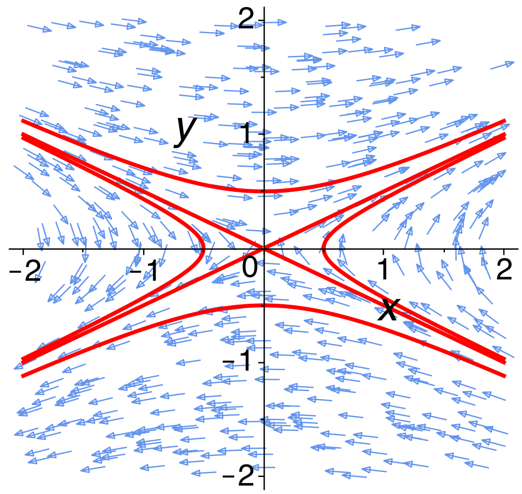

Figure 6.2: Direction fields and sample phase-plane curves for system (6.51) for $s = 0.25$.

Autonomous systems have the property that if $(x(t), y(t))$ is a solution so is $(x(t+t_0), y(t+t_0))$ for any time shift t_0. That is, it doesn't matter when the start time is. The solution of a non-autonomous system for initial conditions $(x(t_0), y(t_0)) = (x_0, y_0)$ depends on the start time t_0.

If f and g are independent of t and $x(t)$ can be inverted to find $t(x)$ we can form the function $\tilde{y}(x) = y\big(t(x)\big)$. As an abuse of notation we normally dispense with the tilde and denote the new function as $y(x)$ even though $y(t)$ and $y(x)$ are really different functions. From the equations for $x(t)$ and $y(t)$ we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)} \equiv h(x,y). \tag{6.50}$$

which does not involve t. This is called the *phase-plane* equation. Solutions give the curves on the x–y plane along which $(x(t), y(t))$ travels, called a trajectory. They do not provide information on how the point $\big(x(t), y(t)\big)$ moves along the curve as t increases. The solution of the system $\big(x(t), y(t)\big)$ does.

Example: Consider

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= sx. \end{aligned} \tag{6.51}$$

The phase-plane equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = s\frac{x}{y} \implies y\,\mathrm{d}y = sx\,\mathrm{d}x \implies y^2 = sx^2 + c. \tag{6.52}$$

Case A: $s > 0$**.** For $s > 0$ the solutions are hyperbolas. The direction of motion along the curve can be inferred from the original system: $\frac{\mathrm{d}x}{\mathrm{d}t} = y \implies x$ is increasing if $y > 0$ and decreasing if $y < 0$. See Figure 6.2.

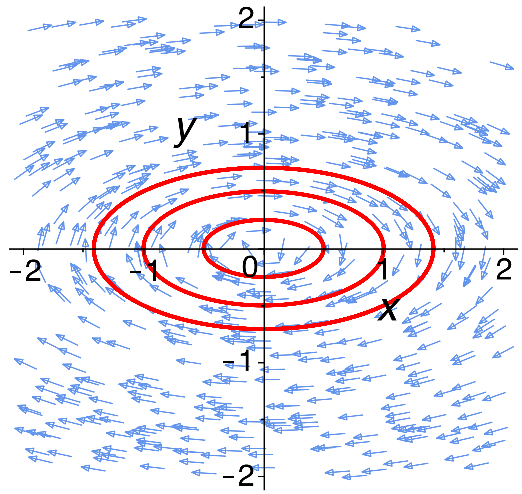

Figure 6.3: Direction fields and sample phase-plane curves for system (6.51) for $s = -0.25$.

Case B: $s < 0$**.** For $s < 0$ the solutions are ellipses centred at $(x, y) = (0, 0)$. As for Case A, x is increasing if $y > 0$ and decreasing if $y < 0$ so the point $(x(t), y(t))$ moves clockwise around the ellipse. See Figure 6.3.

Comments:

1. A trajectory $\big(x(t), y(t)\big)$ of an autonomous system cannot back track along the phase-plane curve since the velocity $(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t})$ at a point (x, y) does not depend on time.

2. A trajectory $\big(x(t), y(t)\big)$ of an autonomous system cannot cross itself for the same reason.

Definition: A *critical point*, or *fixed point*, is a point (x_0, y_0) where $f(x_0, y_0) = g(x_0, y_0) = 0$.

Note that $(x(t), y(t)) = (x_0, y_0)$ is a solution of the differential equation, called an equilibrium solution.

6.3.1 Classification of Critical Points

1. A critical point is said to be **stable** if all trajectories that start in a neighbourhood of the critical point stay nearby for all future times. More precisely, a critical point $\vec{x_0} = (x_0, y_0)$ is stable if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $||\vec{x}(0) - \vec{x}_0|| < \delta$ then $||\vec{x}(t) - \vec{x}_0|| < \epsilon$ for all $t \geq 0$.

2. A critical point that is not stable is said to be **unstable**.

3. A critical point is said to be **asymptotically stable** if trajectories that start sufficiently close to it go to the fixed point as $t \to \infty$. That is, if there exists a $\delta > 0$ such that if $||\vec{x}(0) - \vec{x}_0|| < \delta$ then $\lim_{t\to\infty} \vec{x}(t) = \vec{x}_0$.

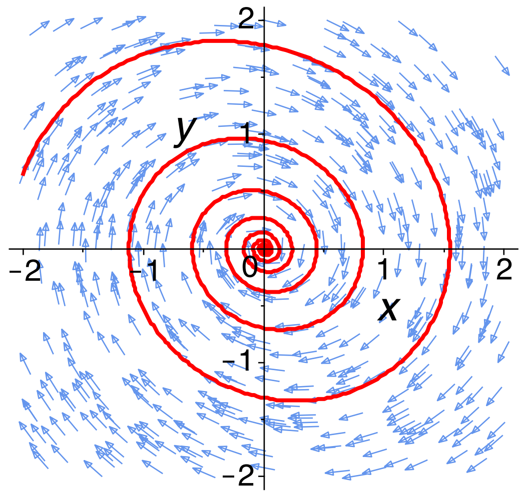

Figure 6.4: Inward (stable) spiral: direction fields and sample trajectory for system (6.53) for $\delta = 0.1$.

4. A critical point is called a **saddle point** if some trajectories go to the fixed point while others go away.

5. A critical point is called a **centre** if trajectories in its neighbourhood are closed curves going around the critical point.

Example: System (6.51) as a single critical point at $(0, 0)$. When $s > 0$ the critical point is unstable. This is an example of a saddle point. Initial points on the line $y = -x/2$ go to the fixed point. All others eventually go off to infinity although they can approach the critical point very closely before doing so (see Figure 6.2). When $s < 0$ the critical point is stable however it is not asymptotically stable. In this case critical point is a centre (see Figure 6.3).

The linear-damped oscillator can be written as a system for $x(t)$ and $y(t) = x'(t)$:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -2\delta y - x. \end{aligned} \tag{6.53}$$

Figure 6.4 shows a sample trajectory for $\delta = 0.1$. This is called an **inward spiral**. When $\delta < 0$ energy is fed into the system and the trajectory spirals out (Figure 6.4).

For linear systems the equations for the critical points, $f(x, y) = 0$ and $g(x, y) = 0$, are the equations for two straight lines. If they are non-parallel there is one critical point. If the lines are distinct and parallel there are none. If they coincide there are an infinite number. Non-linear systems can have a much richer behaviour. We consider such a case next.

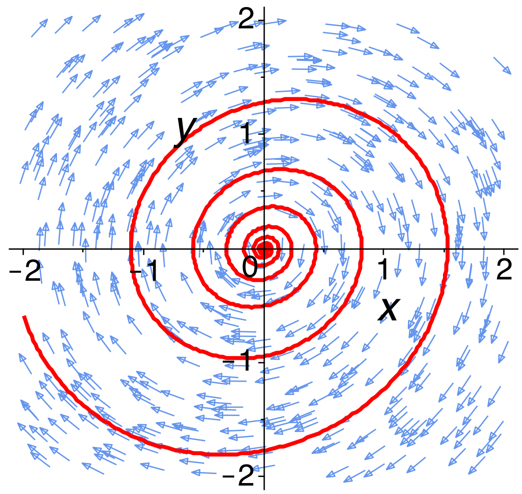

Figure 6.5: Outward (unstable) spiral: direction fields and sample trajectory for system (6.53) for $\delta = -0.1$.

6.4 Competing Species

Here we consider a simple closed system with two species competing for the same food supply, e.g., two species of fish. Let x and y be the populations of the two species. We are assuming that the population is large so that it makes sense to treat the population as a continuous function of time. This is a convenient approximation. Consider a single species, x, on its own. The growth rate per individual, $\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}$ is the difference between the average birth rate and the average death rate. We assume the following:

1. the average birth rate (number of births per unit population) is constant, say β;
2. the average death rate due to the effects of crowding and increased competition for food, is proportional to the size of the population. Let δ be its constant of proportionality.

This gives a simple model for population growth, called the logistic equation:

$$\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t} = \beta - \delta x,$$

or

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(\beta - \delta x). \tag{6.54}$$

As we have already seen, the population increases if $x < \frac{\beta}{\delta}$ and decreases if $x > \frac{\beta}{\delta}$. In both cases the population approaches the steady state (equilibrium) solution $x = \frac{\beta}{\delta}$ as $t \to \infty$.

Now consider our two species with populations x and y. In the absence of the other species, each population is modeled with a logistic equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(\beta_1 - \delta_1 x), \tag{6.55}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(\beta_2 - \delta_2 y), \tag{6.56}$$

where the β_j and δ_j are positive constants. We now add the effects of competition by taking into account the fact that the death rate now depends on the population of both species: the death rate for x is now $\delta_1 x + \alpha_1 y$ and the death rate for y is now $\delta_2 y + \alpha_2 x$. This leads to the coupled nonlinear system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x(\beta_1 - \delta_1 x - \alpha_1 y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y(\beta_2 - \delta_2 y - \alpha_2 x). \end{aligned} \tag{6.57}$$

This is our model for the competition of two species.

To understand the behaviour of the system it is useful to determine the critical points and establish their nature. The critical points are solutions of

$$\begin{aligned} x(\beta_1 - \delta_1 x - \alpha_1 y) &= 0, \\ y(\beta_2 - \delta_2 y - \alpha_2 x) &= 0. \end{aligned} \tag{6.58}$$

There are in general four critical points:

$$\begin{aligned} (x_1, y_1) &= (0, 0), \\ (x_2, y_2) &= \Big(0, \frac{\beta_2}{\alpha_2}\Big), \\ (x_3, y_3) &= \Big(\frac{\beta_1}{\alpha_1}, 0\Big), \\ (x_4, y_4) &= \Big(\frac{\beta_1\alpha_2 - \beta_2\alpha_1}{\delta_1\alpha_2 - \delta_2\alpha_1}, \frac{\delta_1\beta_2 - \delta_2\beta_1}{\delta_1\alpha_2 - \delta_2\alpha_1}\Big). \end{aligned} \tag{6.59}$$

The second and third solutions are the equilibrium solutions for each species in the absence of the other. The fourth solution comes from solving the system

$$\begin{aligned} \delta_1 x + \alpha_1 y &= \beta_1 \\ \delta_2 x + \alpha_2 y &= \beta_2. \end{aligned} \tag{6.60}$$

This system may have (i) no solution (e.g., $\delta_1 = \delta_2$, $\alpha_1 = \alpha_2$ and $\beta_1 \neq \beta_2$); (ii) solutions along a line if the two equations give the same straight line on the x-y plane; or (iii) solutions with $x < 0$ or $y < 0$ which must be rejected as populations can't be negative. If x_4 and y_4 are positive it represents a state in which both species can co-exist.

Note that x, y and t can always be scaled to put it in the form

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x(1 - x - y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y(\beta - \delta y - \alpha x). \end{aligned} \tag{6.61}$$

This is left as an exercise.

Figure (6.6) shows the direction fields and some sample population directories for the case

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x(1 - x - y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y(0.75 - y - 0.5x). \end{aligned} \tag{6.62}$$

The critical points are $(0, 0)$, $(0, 0.75)$, $(1, 0)$ and $(0.5, 0.5)$. Visually we can see that $(0.5, 0.5)$ is a stable critical point. The other critical points are unstable. The critical points $(0, 0.75)$ and $(1, 0)$ are saddle points.

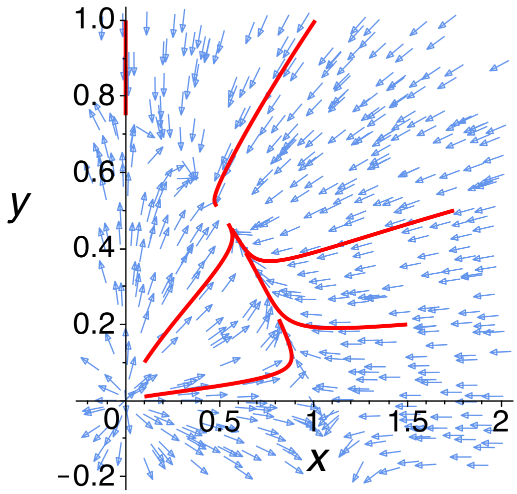

Figure 6.6: Direction fields and sample trajectories for the competing species model (6.53).

6.5 Linearization near a critical point

To study the behaviour more closely in the vicinity of the stable point $(0.5, 0.5)$ in our competing species system (6.62) set $x = 0.5 + \tilde{x}$ and $y = 0.5 + \tilde{y}$. When (x, y) is close to the critical point $\tilde{x}$ and $\tilde{y}$ are small. Since $1 - x - y = 1 - (0.5 + \tilde{x}) - (0.5 + \tilde{y}) = -\tilde{x} - \tilde{y}$ and $0.75 - y - 0.5x = 0.75 - (0.5 + \tilde{y}) - 0.5(0.5 + \tilde{x}) = -\tilde{y} - 0.5\tilde{x}$ we have the system

$$\begin{aligned} \frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} &= -(0.5 + \tilde{x})(\tilde{x} + \tilde{y}), \\ \frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} &= -(0.5 + \tilde{y})(\tilde{y} + 0.5\tilde{x}), \end{aligned} \tag{6.63}$$

or

$$\begin{aligned} \frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} &= -\frac{1}{2}(\tilde{x} + \tilde{y}) - \tilde{x}^2 - \tilde{x}y, \\ \frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} &= -\frac{1}{2}(\tilde{y} + \frac{1}{2}\tilde{x}) - \tilde{y}^2 - \frac{1}{2}\tilde{y}\tilde{x}. \end{aligned} \tag{6.64}$$

So far we have just translated the dependent variables to shift the critical point to $(\tilde{x}, \tilde{y}) = (0, 0)$. Now we assume $\tilde{x}$ and $\tilde{y}$ are small so that the quadratic terms $\tilde{x}^2$, $\tilde{x}\tilde{y}$ and $\tilde{y}^2$ are negligible to a first approximation. This gives the *approximate linearized system*

$$\begin{aligned} \frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} &= -\frac{1}{2}\tilde{x} - \frac{1}{2}\tilde{y}, \\ \frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} &= -\frac{1}{2}\tilde{y} - \frac{1}{4}\tilde{x}. \end{aligned} \tag{6.65}$$

which can be written as

$$\frac{\mathrm{d}\vec{\tilde{x}}}{\mathrm{d}t} = A\vec{\tilde{x}} \tag{6.66}$$

where

$$A = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}. \tag{6.67}$$

The matrix has two real negative eigenvalues: $-\frac{1}{2} \pm \frac{\sqrt{2}}{4}$ so the general solution has the form

$$\vec{x} = a\vec{v}_1 e^{(-\frac{1}{2}-\frac{\sqrt{2}}{4})t} + b\vec{v}_2 e^{(-\frac{1}{2}+\frac{\sqrt{2}}{4})t} \tag{6.68}$$

where $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors for the corresponding eigenvalues. All solutions of the linearized system go to zero as expected. This shows that the fixed point is stable: in this case the model predicts the two populations can co-exist. This is not always the case. For other values of the model parameters the critical point for which both populations are non-zero can be unstable. In that case the model predicts that one population goes extinct.

Writing the linearized system as

$$\begin{aligned} \Big(D + \frac{1}{2}\Big)\tilde{x} + \frac{1}{2}\tilde{y} &= 0, \\ \frac{1}{4}\tilde{x} + \Big(D + \frac{1}{2}\Big)\tilde{y} &= 0 \end{aligned} \tag{6.69}$$

and using the elimination method, a single equation for $\tilde{x}$ is

$$\Big(D + \frac{1}{2}\Big)^2[\tilde{x}] - \frac{1}{8}\tilde{x} = 0, \tag{6.70}$$

or

$$\Big(D^2 + D + \frac{1}{8}\Big)^2[\tilde{x}] = 0. \tag{6.71}$$

This is an over-damped oscillator equation again.

6.5.1 General linearization procedure

In general, to study the behaviour more closely in the vicinity of a critical point (x_0, y_0) of the system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f(x,y) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= g(x,y) \end{aligned} \tag{6.72}$$

we expand f and g in Taylor series about the critical point (assuming the functions are differentiable):

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &\quad + \Big[\frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2\Big] + \cdots \\ g(x,y) &= g(x_0,y_0) + g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0) \\ &\quad + \Big[\frac{1}{2}g_{xx}(x_0,y_0)(x-x_0)^2 + g_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2}g_{yy}(x_0,y_0)(y-y_0)^2\Big] + \cdots . \end{aligned} \tag{6.73}$$

Since $(x_0, y_0$ is a critical point $f(x_0,y_0) = g(x_0,y_0) = 0$. Assuming $x - x_0$ and $y - y_0$ are small we drop the quadratic and higher-order terms to get the approximate linear system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= g(x_0,y_0) + g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0). \end{aligned} \tag{6.74}$$

In matrix form, with $\tilde{x} = x - x_0$ and $\tilde{y} = y - y_0$ as above,

$$\frac{\mathrm{d}\vec{\tilde{x}}}{\mathrm{d}t} = A\vec{\tilde{x}} \tag{6.75}$$

where

$$A = \begin{pmatrix} f_x(\vec{x}_0) & f_y(\vec{x}_0) \\ g_x(\vec{x}_0) & g_y(\vec{x}_0) \end{pmatrix}. \tag{6.76}$$

The eigenvalues of A determine the behaviour of the linear system which is only an approximation for the behaviour in a neighbourhood of the critical point. We can separate the behaviours of the linear system into the following cases:

Major Cases:

A. Two distinct real roots or same sign $\implies$ node;

B. Two distinct real roots of opposite sign $\implies$ saddle point;

C. Two complex conjugate roots, not purely imaginary $\implies$ spiral;

Minor Cases:

D. Real equal roots $\implies$ node;

E. Purely imaginary roots $\implies$ centre.

The behaviour of a nonlinear system in the vicinity of a critical point has the same behaviour as the corresponding linear system if the behaviour is one of the major cases. The behaviour may not be the same globally. For example, for an unstable node the trajectories could go to a stable node rather than off to infinity.

6.6 Nonlinear Oscillations: The Volterra prey-predator model

Vito Volterra (1860–1940) developed a simply model for the populations of two interacting species: a prey (e.g., rabbits) with unlimited food supply, and a predator (e.g., lynxes). Let x be the population of the prey. In the absence of any predators the population of the prey grows with constant birthrate. When predators are present the birth rate is assumed to be unchanged however it now has a death rate that is proportional to the population of the predators. Hence the population of the prey is governed by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bxy,$$

where a and b are positive constants.

In the absence of prey the predators population declines at a constant rate because it has no food. Its death rate (number of deaths per unit population per unit time) is assumed to be constant. When there are prey available its population grows at a rate that is proportional to the prey's population. This gives

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -cy + dxy,$$

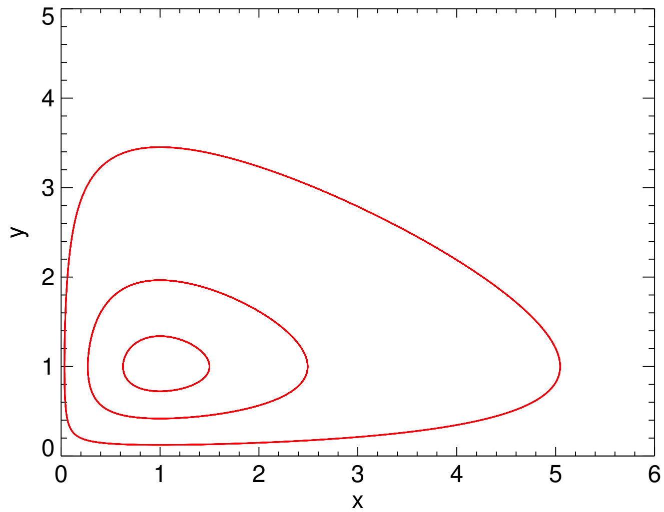

Figure 6.7: Sample orbits for the Volterra model. $a = b = 2$ and $c = d = 1$.

where c and d are positive constants. The combination of the two equations gives a coupled nonlinear system of first-order differential equations:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= ax - bxy,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -cy + dxy.\end{aligned} \tag{6.77}$$

This system cannot be solved in terms of elementary functions.

The Volterra model has two critical points. One is $(x_1, y_1) = (0, 0)$. Linearizing about this point we drop the quadratic terms to get the approximate linear system

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \vec{x}. \tag{6.78}$$

The eigenvalues are $a > 0$ and $-c < 0$ hence $(0, 0)$ is a saddle point.

To study the behaviour near the second critical point $(x_2, y_2) = (\frac{c}{d}, \frac{a}{b})$ we set $\tilde{x} = x - x_2$ and $\tilde{y} = y - y_2$. Letting $f(x, y) = ax - bxy$ and $g(x, y) = -cy + dxy$ we have

$$\begin{aligned}
&f_x = a - by \implies f_x(x_2, y_2) = a - b\frac{a}{b} = 0,\\
&f_y = -bx \implies f_y(x_2, y_2) = -b\frac{c}{d},\\
&g_x = dy \implies g_x(x_2, y_2) = d\frac{a}{b},\\
&g_y = -c + dx \implies g_y(x_2, y_2) = -c + d\frac{c}{d} = 0,
\end{aligned} \tag{6.79}$$

so

$$\frac{\mathrm{d}\vec{\tilde{x}}}{\mathrm{d}t} = \begin{pmatrix} f_x(x_2, y_2) & f_y(x_2, y_2) \\ g_x(x_2, y_2) & g_y(x_2, y_2) \end{pmatrix} \vec{\tilde{x}} = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{da}{d} & 0 \end{pmatrix} \vec{\tilde{x}}. \tag{6.80}$$

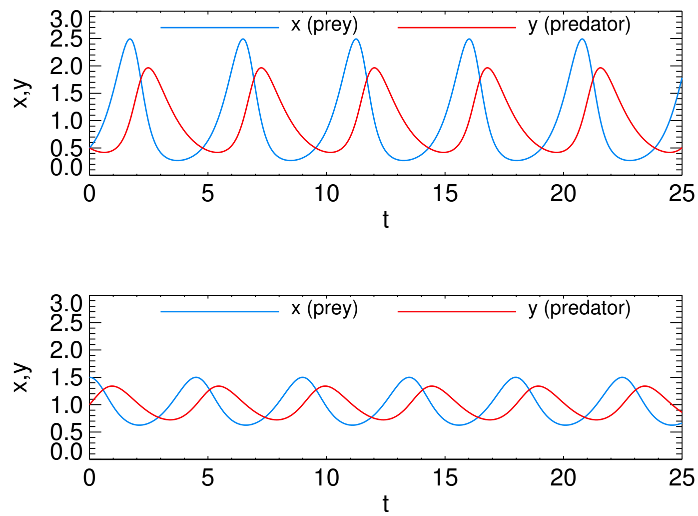

Figure 6.8: Sample orbits for the Volterra model. $a = b = 2$ and $c = d = 1$.

The eigenvalues are imaginary: $\lambda = \sqrt{-\frac{bc}{d} \times \frac{da}{d}} = \pm\sqrt{ac}\, i$. Hence the orbits are ellipses (exercise).

One can show that for the fully nonlinear system the orbits are also closed curves however they are not ellipses. They approach ellipses as the orbit approaches the fixed point. Figure 6.7 shows some example orbits for the case with $a = b = 2$ and $c = d = 1$. Time series for a sample solution are shown in Figure 6.8. Notice that the period for the example with larger amplitude fluctuations is slightly longer than the period of the smaller amplitude oscillations and the shape of the orbit on the phase plane is far from elliptical. The fact that the period of oscillation depends on the amplitude is a common feature of nonlinear oscillations (e.g., the nonlinear pendulum or surface water waves).

6.7 Problems

1. Solve the phase plane equations for the following systems.

(a)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2sy.$$

(b)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2y + 2x$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x.$$

(c)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 6(y - 2x)^2 + 2$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3(y - 2x)^2 - 4.$$

2. Describe the nature of the critical point $(0, 0)$ of each of the following systems and sketch the trajectories. Find $x(t)$ and $y(t)$.

(a)

$$\frac{dx}{dt} = 4x - 3y,$$
$$\frac{dy}{dt} = 5x - 4y.$$

(b)

$$\frac{dx}{dt} = 2y + 2x,$$
$$\frac{dy}{dt} = -x.$$

(c)

$$\frac{dx}{dt} = x + y,$$
$$\frac{dy}{dt} = -x - 3y.$$

(d)

$$\frac{dx}{dt} = 2x + \frac{1}{2}y,$$
$$\frac{dy}{dt} = -\frac{5}{2}x + y.$$

(e)

$$\frac{dx}{dt} = x - 2y,$$
$$\frac{dy}{dt} = 2x - 3y.$$

(f)

$$\frac{dx}{dt} = 4x - 3y,$$
$$\frac{dy}{dt} = 8x + 5y.$$

3. Consider the Volterra system (6.77). Eliminate y from the system and obtain a nonlinear second-order equation for $x(t)$.

4. For the Volterra system shown that $\frac{d^2y}{dt^2} \geq 0$ whenever $\frac{dx}{dt} > 0$. What is the meaning of this result in terms of the time series shown in Figure 6.8?

5. Consider the Volterra system linearized about the critical points $\left(\frac{c}{d}, \frac{a}{b}\right)$. Show that the orbits of the linearized system are ellipses.

Chapter 7

Laplace Transforms

Definition: Let $f(t)$ be a function on $[0, \infty)$. The *Laplace Transform* of $f(t)$ is a function $F(s) = \mathcal{L}[f](s)$ defined by

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t, \tag{7.1}$$

The domain of $F(s)$ is all values of s for which the integral exists.

The Laplace Tranform is another example of an operator that maps a function $f(t)$ to a new function $F(s)$. Another important transform is the Fourier Transform defined by

$$\mathcal{F}[f(x)](k) = \int_{-\infty}^\infty e^{ikx} f(x)\,\mathrm{d}x. \tag{7.2}$$

The definition of the Fourier Transform has different forms which are simple scalings of one another, e..g, replace k by $2\pi k$, divide the above by 2π, etc. You can learn about the Fourier Transform in a future course.

Note: The Laplace Transform is another example of a linear operator. If $\mathcal{L}[f](s)$ and $\mathcal{L}[g](s)$ exist for $s > s_0$ then $\mathcal{L}[f + cg](s) = \mathcal{L}[f](s) + c\mathcal{L}[g](s)$ exists for $s > s_0$ for any constant c. The proof follows from linearity of the integral operator and is left as an exercise.

Example: The Laplace Transform of the constant function $f(t) = 1$ is

$$\mathcal{L}[1](s) = \int_0^\infty e^{-st}\,\mathrm{d}t = \Big(-\frac{e^{-st}}{s}\Big)\Big|_0^\infty = \frac{1}{s} \text{ for } s > 0. \tag{7.3}$$

The transform exists for complex $s = s_r + is_i$ with $\Re\{s\} = s_r > 0$ because we then have $\lim_{t\to\infty} e^{-st} = \lim_{t\to\infty} e^{-s_r t} e^{-is_i t} = 0$.

Example: The Laplace Transform of $f(t) = e^{at}$ for a constant a is

$$\mathcal{L}[e^{at}](s) = \int_0^\infty e^{-st}\, e^{at}\,\mathrm{d}t = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t. \tag{7.4}$$

The integral exists iff $s > a$ in which case

$$\mathcal{L}[e^{at}](s) = \int_0^\infty e^{-(s-a)}\,\mathrm{d}t = \Big(-\frac{e^{-(s-a)t}}{s-a}\Big)\Big|_0^\infty = \frac{1}{s-a}. \tag{7.5}$$

Note that this result can be extended to complex values of a. The transform exists if $s > \Re\{a\}$ since then $\lim_{t\to\infty} e^{-(s-a)t} = \lim_{t\to\infty} e^{-(s-a_r)t}e^{ia_i} = 0$. This result will prove very useful in the next example.

Example: Find $\mathcal{L}[\sin bt](s)$ and $\mathcal{L}[\cos bt](s)$ where b is a nonzero constant.

Solution: First the hard way.

$$\begin{aligned}
\mathcal{L}[\sin bt](s) &= \int_0^\infty e^{-st}\,\sin bt\,\mathrm{d}t \\
&= \int_0^\infty e^{-st}\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(-\frac{\cos bt}{b}\Big)\,\mathrm{d}t \\
&= -\Big(e^{-st}\frac{\cos bt}{b}\Big)\Big|_{t=0}^{t=\infty} - \int_0^\infty \frac{s}{b}e^{-st}\cos bt\,\mathrm{d}t \\
&= \frac{1}{b} - \frac{s}{b}\int_0^\infty e^{-st}\frac{\mathrm{d}}{\mathrm{d}t}\big(\frac{\sin bt}{b}\big)\,\mathrm{d}t \qquad \text{(need } s > 0 \text{ here)} \\
&= \frac{1}{b} - \frac{s}{b}\Big(e^{-st}\frac{\sin bt}{b}\Big)\Big|_0^\infty + \frac{s}{b}\int_0^\infty e^{-st}\sin bt\,\mathrm{d}t\Big) \\
&= \frac{1}{b} - \frac{s^2}{b^2}\int_0^\infty e^{-st}\sin bt\,\mathrm{d}t.
\end{aligned} \tag{7.6}$$

So

$$\Big(1 + \frac{s^2}{b^2}\Big)\int_0^\infty e^{-st}\,\sin bt\,\mathrm{d}t = \frac{1}{b} \tag{7.7}$$

and hence

$$\mathcal{L}[\sin bt](s) = \frac{b}{b^2+s^2} \qquad \text{for } s > 0. \tag{7.8}$$

Similarly one can show that

$$\mathcal{L}[\cos bt](s) = \frac{s}{b^2+s^2} \qquad \text{for } s > 0. \tag{7.9}$$

This is left as an exercise.

Now the easy way, which is a two-for-one deal. Let $f(t) = f_r(t) + if_i(t)$ be a complex valued function with real and imaginary parts f_r and f_i. Since the Laplace operator is linear operator we have $\mathcal{L}[f](s) = \mathcal{L}[f_r + if_i](s) = \mathcal{L}[f_r](s) + i\mathcal{L}[f_i](s)$. In particular, $\mathcal{L}[e^{ibt}](s) = \mathcal{L}[\cos bt](s) + i\mathcal{L}[\sin bt](s)$. So we begin by finding the Laplace Transform of e^{ibt}. Then the real part is the Laplace Transform of $\cos bt$ and the imaginary part is the Laplace Transform of $\sin bt$.

From our second example

$$\mathcal{L}[e^{ibt}](s) = \frac{1}{s-ib} = \frac{s+ib}{s^2+b^2}. \tag{7.10}$$

Taking the real part gives

$$\mathcal{L}[\cos bt](s) = \frac{s}{s^2+b^2}. \tag{7.11}$$

Taking the imaginary part gives

$$\mathcal{L}[\sin bt](s) = \frac{b}{s^2+b^2}. \tag{7.12}$$

Simple!

Example: Use the linearity of the Laplace Transform to find $\mathcal{L}[-\pi + 6e^{\lambda t} + 9\sin(\Omega t)]$.

Solution: Using linearity of the operator gives

$$\begin{aligned}\mathcal{L}[-\pi + 6e^{\lambda t} + 9\sin(\Omega t)] &= -\pi\mathcal{L}[1] + 6\mathcal{L}[e^{\lambda t}] + 9\mathcal{L}[\sin(\Omega t)] \\ &= -\frac{\pi}{s} + \frac{6}{s-\lambda} + 9\frac{\Omega}{\Omega^2+s^2}\end{aligned} \tag{7.13}$$

which is defined for $s > \max\{0, \lambda\}$ (0 for the constant and sine functions, λ for the exponential function).

7.1 Existence of the Laplace Transform

Definition: A function $f(t)$ has a **jump discontinuity** at $t_0 \in (a, b)$ if $f^- = \lim_{t\to t_0^-} f(t)$ and $f^+ = \lim_{t\to t_0^+} f(t)$ exist, are finite and $f^+ \neq f^-$.

Definition: A function $f(t)$ is **piecewise continuous** on a finite interval $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$ except possibly at a finite number of points at which f has a jump continuity.

Definition: A function $f(t)$ is piecewise continuous on $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, T]$ for all $T > 0$. There may be an countable infinity of discrete discontinuities on $[0, \infty)$ with no cluster point. That is, there can be no subset of points of discontinuities $\{t_n\}$ that have a limit point $t_{lim} = lim_{n\to\infty} t_n$.

Examples:

1. The function

$$f(t) = \begin{cases} \cos(t) & 0 \le t < 2\pi; \\ 2 & 2\pi \le t < 8; \\ (t-7)^2 & t \ge 8 \end{cases} \tag{7.14}$$

   is piecewise continuous on $[0, 20]$.

2. The function

$$f(t) = \begin{cases} 1 & 2n \le t < 2n+1; \\ -1 & 2n+1 \le t < 2(n+1); \end{cases} \qquad \text{for } n = 0, 1, 2, \cdots \tag{7.15}$$

is piecewise continuous on $[0, \infty)$.

3. $f(t) = t^{-1}$ is not piecewise continuous on $[0, 2]$ because $\lim_{t \to 0} f(t)$ does not exist.

Definition: A function $f(t)$ is of **exponential order** α if there exist constants T and M such that

$$|f(t)| \le M e^{\alpha t} \tag{7.16}$$

for all $t \ge T$.

Example: $f(t) = at^n$ is of exponential order α for all $\alpha > 0$. To demonstrate we use the fact that

$$\lim_{t \to \infty} \frac{e^{\alpha t}}{|at^n|} = \infty \tag{7.17}$$

for all $\alpha > 0$ as can be deduced by applying L'Hopital's rule n times. Hence for any $M > 0$ there is a T such that $\frac{e^{\alpha t}}{|at^n|} > M^{-1}$ for $t > T$ and $|at^n| \le M e^{\alpha t}$ for $t > T$. This proves the result.

Theorem: Existence of Laplace Transforms. If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α then $\mathcal{L}[f](s)$ exists for $s > \alpha$.

Proof: Since $f(t)$ is of exponential order α there exists constant T and M such that for $t > T$, $|f(t)| < M e^{\alpha t}$. We can split the Laplace Transform integral into two pieces

$$\mathcal{L}[f](s) = \int_0^T e^{-st} f(t)\,\mathrm{d}t + \int_T^\infty e^{-st} f(t)\,\mathrm{d}t. \tag{7.18}$$

The first integral exists because the integral is over a finite interval on which the integrand is piecewise continuous. Hence we just need to show that the second integral exists. On $[T, \infty)$ we have

$$|e^{-st} f(t)| = e^{-st}|f| \le M e^{(\alpha - s)t} \tag{7.19}$$

so

$$\begin{aligned} \Big| \int_T^\infty e^{-st} f(t)\,\mathrm{d}t \Big| &\le \int_T^\infty e^{-st} \big| f(t) \big|\,\mathrm{d}t \\ &\le M \int_T^\infty e^{(\alpha - s)t}\,\mathrm{d}t \\ &= M \frac{e^{(\alpha - s)T}}{s - \alpha} \text{ for } s > \alpha. \end{aligned} \tag{7.20}$$

Hence

$$\int_T^\infty e^{-st} f(t)\,\mathrm{d}t \tag{7.21}$$

exists and so does the Laplace Transform.

Inverse Laplace Transforms can be computed by calculating an integral on the complex plane. This is beyond this course. For our needs we will use Laplace Transform tables.

The Laplace transform is very useful for solving linear constant coefficient differential equations involving piecewise continuous forcing functions.

Example: Find the Laplace Transform of the piecewise continuous function

$$f(t) = \begin{cases} \cos(2t) & 0 < t < \pi; \\ 1 & \pi < t < 4; \\ 0 & t > 4. \end{cases} \tag{7.22}$$

Solution: Using the definition we have

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^\pi e^{-st}\cos(2t)\,\mathrm{d}t + \int_\pi^4 e^{-st}\,\mathrm{d}t. \tag{7.23}$$

Now

$$\begin{aligned} \int_0^\pi e^{-st}\cos(2t)\,\mathrm{d}t &= \Re\Big\{\int_0^p ie^{-st}e^{i2t}\,\mathrm{d}t\Big\} \\ &= \Re\Big\{\int_0^\pi e^{-(s-2i)t}\,\mathrm{d}t\Big\} \\ &= \Re\Big\{-\frac{e^{-(s-2i)t}}{s-2i}\Big|_0^\pi\Big\} \\ &= \Re\Big\{-\frac{e^{-(s-2i)\pi}-1}{s-2i}\Big\} \\ &= \Re\Big\{(1-e^{-s\pi})\frac{s+2i}{s^2+4}\Big\} \\ &= \frac{s(1-e^{-s\pi})}{s^2+4}. \end{aligned} \tag{7.24}$$

and

$$\int_\pi^4 e^{-st}\,\mathrm{d}t = -\frac{e^{-st}}{s}\Big|_\pi^4 = \frac{e^{-s\pi}-e^{-4s}}{s}. \tag{7.25}$$

Hence

$$\mathcal{L}[f(t)] = \frac{s(1-e^{-s\pi})}{s^2+4} + \frac{e^{-s\pi}-e^{-4s}}{s}. \tag{7.26}$$

Since all the integrals are over finite intervals the Laplace Transform is defined for all s. For $s = 0$ note that

$$\lim_{s\to 0}\frac{e^{-s\pi}-e^{-4s}}{s} = \lim_{s\to 0}\big(4e^{-4s} - \pi e^{-s\pi}\big) = 4 - \pi \tag{7.27}$$

which is the result one obtains computing the integral directly for $s = 0$.

7.2 Properties of the Laplaces Transform

1. **Translation in s.**

   Theorem: If $\mathcal{L}[f](s) = F(s)$ exists for $s > \alpha$ then $\mathcal{L}[e^{at} f](s) = F(s-a)$ exists for $s > \alpha + a$.

   Proof:

$$\mathcal{L}[e^{at} f](s) = \int_0^\infty e^{-st} e^{at} f(t)\,\mathrm{d}t = \int_0^\infty e^{-(s-a)t} f(t)\,\mathrm{d}t = F(s-a). \tag{7.28}$$

Examples:

(a) $\mathcal{L}[1](s) = \frac{1}{s} \implies \mathcal{L}[e^{at}](s) = \frac{1}{s-a}$ as we have already seen.

(b) $\mathcal{L}[\sin \beta t](s) = \frac{\beta}{s^2+\beta^2} \implies \mathcal{L}[e^{at} \cos \beta t](s) = \frac{\beta}{(s-a)^2+\beta^2}$.

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2. **Laplace Transform of a Derivative**

   Theorem: If $f(t)$ is continuous on $[0, \infty)$ and $f'(t)$ is piecewise continuous on $[0, \infty)$ with both of exponential order α then for $s > \alpha$

$$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0). \tag{7.29}$$

   Proof:

$$\begin{aligned}
\mathcal{L}[f'](s) &= \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t \\
&= e^{-st} f\Big|_0^\infty - \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\big(e^{-st}\big) f(t)\,\mathrm{d}t \qquad \text{(need continuity of } f \text{ here)} \\
&= -f(0) + s\int_0^\infty e^{-st} f(t)\,\mathrm{d}t \qquad \text{(need } s > \alpha \text{ here)} \\
&= sF(s) - f(0) \qquad \text{(need } s > \alpha \text{ again)}
\end{aligned} \tag{7.30}$$

   where $F(s)$ is the Laplace Transform of $f(t)$.

Examples:

(a) Using the Laplace Transform of $\sin(\beta t)$ we can find the Laplace transform of $\cos(\beta t)$ via

$$\begin{aligned}\mathcal{L}[\cos(\beta t)](s) &= \mathcal{L}\Big[\frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}t}\big(\sin(\beta t)\big)\Big](s)\\ &= \frac{1}{\beta}\mathcal{L}\Big[\frac{\mathrm{d}}{\mathrm{d}t}\big(\sin(\beta t)\big)\Big](s)\\ &= \frac{1}{\beta}\Big(s\mathcal{L}[\sin(\beta t)](s) - \sin(0)\Big)\\ &= \frac{s}{\beta}\frac{\beta}{s^2+\beta^2}\\ &= \frac{s}{s^2+\beta^2}.\end{aligned} \tag{7.31}$$

(b) $\mathcal{L}[1](s) = \mathcal{L}[\frac{\mathrm{d}t}{\mathrm{d}t}](s) = s\mathcal{L}[t](s) - 0$. Hence $\mathcal{L}[t](s) = \frac{1}{s}\mathcal{L}[1](s) = \frac{1}{s^2}$. By induction one can show that

$$\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}}. \tag{7.32}$$

The proof of this is left as an exercise.

(c) Prove the following identity for continuous functions $f(t)$ assuming the transforms exist:

$$\mathcal{L}\Big[\int_0^t f(\tau)\,\mathrm{d}\tau\Big](s) = \frac{1}{s}\mathcal{L}[f](s). \tag{7.33}$$

Proof:

$$\begin{aligned}\mathcal{L}[f](s) &= \mathcal{L}\Big[\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(\tau)\,\mathrm{d}\tau\Big](s)\\ &= s\mathcal{L}\Big[\int_0^t f(\tau)\,\mathrm{d}\tau\Big](s) - \int_0^t f(\tau)\,\mathrm{d}\tau\Big|_{t=0}\\ &= s\mathcal{L}\Big[\int_0^t f(\tau)\,\mathrm{d}\tau\Big](s).\end{aligned} \tag{7.34}$$

---

3. **Higher-order derivatives**

Theorem: Let $f, f', f'', \dots f^{(n-1)}$ (where $f^{(k)}$ denotes the k^{th} derivative of f) be continuous on $[0,\infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0,\infty)$. Let all of these functions be of exponential order α. Then for $s > \alpha$

$$\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0). \tag{7.35}$$

Proof: We prove by induction. We have already shown it is true for $n = 1$. Assume true for $n = k$. Then by the previous result

$$\mathcal{L}[f^{(k+1)}](s) = s\mathcal{L}[f^{(k)}](s) - f^{(k)}(0). \tag{7.36}$$

By assumption

$$\mathcal{L}[f^{(k)}](s) = s^k \mathcal{L}[f](s) - s^{k-1}f(0) - s^{k-2}f'(0) - s^{k-3}f''(0) - \cdots - f^{(k-1)}(0) \tag{7.37}$$

therefore

$$\begin{aligned}
\mathcal{L}[f^{(k+1)}](s) &= s\Big(s^k \mathcal{L}[f](s) - s^{k-1}f(0) - s^{k-2}f'(0) - s^{k-3}f''(0) - \cdots - f^{(k-1)}(0)\Big) - f^{(k)}(0) \\
&= s^{k+1}\mathcal{L}[f](s) - s^k f(0) - s^{k-1}f'(0) - s^{k-2}f''(0) - \cdots - sf^{(k-1)}(0)\Big) - f^{(k)}(0)
\end{aligned} \tag{7.38}$$

hence (7.35) is true for $n = k+1$ if it is true for $n = k$. By induction (7.35) is true for all n.

4. **Derivatives of Laplace Transforms**

Theorem: Let $F(s) = \mathcal{L}[f](s)$ and assume $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α. Then for $s > \alpha$

$$\mathcal{L}[t^n f](s) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s). \tag{7.39}$$

Proof:

$$\begin{aligned}
\frac{\mathrm{d}F}{\mathrm{d}s}(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\Big(e^{-st}\Big) f(t)\,\mathrm{d}t \\
&= -\int_0^\infty e^{-st} t f(t)\,\mathrm{d}t \\
&= -\mathcal{L}[tf](s).
\end{aligned} \tag{7.40}$$

The result follows by induction.

Exercise: Use this to find the Laplace transform of t^n using the Laplace transform of $f(t) = 1$.

Examples:

(a) Find the Laplace Transform of $f(t) = t^2 e^{2t}$.
Solution:

$$\mathcal{L}[t^2 e^{2t}] = (-1)^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{L}[e^{2t}] = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big(\frac{1}{s-2}\Big) = \frac{2}{(s-2)^3}. \tag{7.41}$$

(b) Find the Laplace Transform of $f(t) = t\cos(3t)$.
Solution:

$$\begin{aligned}
\mathcal{L}[t\cos(3t)] = (-1)^1 \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}[\cos(3t)] &= -\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{s}{s^2+9}\Big) \\
&= -\Big(\frac{1}{s^2+9} - \frac{2s^2}{(s^2+9)^2}\Big) \\
&= \frac{s^2-9}{(s^2+9)^2}.
\end{aligned} \tag{7.42}$$

7.3 Inverse Laplace Transforms

Definition: Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0,\infty)$ and satisfies

$$\mathcal{L}[f] = F, \tag{7.43}$$

then $f(t) = \mathcal{L}^{-1}[F]$ is called the inverse Laplace Transform of $F(s)$.

If every function $f(t)$ satisfying (7.43) is discontinuous then one can choose any one of them to be the inverse transform. Two piecewise continuous functions satisfying (7.43) can only differ at the point of discontinuity and this is not physically significant.

Example:

$$f_1(t) = \begin{cases} 0, & 0 \le t \le 1; \\ 1, & t > 1, \end{cases} \tag{7.44}$$

and

$$f_2(t) = \begin{cases} 0, & 0 \le t < 1; \\ 1, & t \ge 1, \end{cases} \tag{7.45}$$

have different values at $t = 1$. They have the same Laplace Transform:

$$\mathcal{L}[f_1] = \mathcal{L}[f_2] = \int_1^\infty e^{-st}\,\mathrm{d}t = \int_0^\infty e^{-s}e^{-s\tilde{t}}\,\mathrm{d}\tilde{t} = \frac{e^{-s}}{s}. \tag{7.46}$$

We can use

$$\mathcal{L}\left[\frac{e^{-s}}{s}\right] = f_1 \qquad \text{or} \qquad \mathcal{L}\left[\frac{e^{-s}}{s}\right] = f_2. \tag{7.47}$$

7.3.1 Linearity of the Inverse Transform

Assume $\mathcal{L}^{-1}[F_1]$ and $\mathcal{L}^{-1}[F_2]$ exist and are continuous on $[0,\infty)$. Then

$$\mathcal{L}^{-1}[F_1 + cF_2] = \mathcal{L}^{-1}[F_1] + c\mathcal{L}^{-1}[F_2] \tag{7.48}$$

for any constant c. To prove this use

$$\begin{aligned} \mathcal{L}\Big[\mathcal{L}^{-1}[F_1] + c\mathcal{L}^{-1}[F_2]\Big] &= \mathcal{L}\Big[\mathcal{L}^{-1}[F_1]\Big] + c\mathcal{L}\Big[\mathcal{L}^{-1}[F_2]\Big] \\ &= F_1 + cF_2. \end{aligned} \tag{7.49}$$

The latter step follows because the two inverse transforms are continuous functions. Hence $\mathcal{L}^{-1}[F_1]+$ $c\mathcal{L}^{-1}[F_2]$ is a continuous function whose Laplace Transform is $F_1 + cF_2$ which proves (7.48).

7.3.2 Examples

1. Find
$$\mathcal{L}^{-1}\Big[-\frac{5}{s^3}+\frac{12}{s^2+5}\Big]. \tag{7.50}$$

   Solution: We known that
$$\mathcal{L}[\sin bt]=\frac{b}{s^2+b^2} \tag{7.51}$$
   and
$$\mathcal{L}[t^2]=\frac{2}{s^3}. \tag{7.52}$$
   Using this
$$\begin{aligned}\mathcal{L}^{-1}\Big[-\frac{5}{s^3}+\frac{12}{s^2+5}\Big]&=\mathcal{L}^{-1}\Big[-\frac{5}{2}\frac{2}{s^3}+\frac{12}{\sqrt{5}}\frac{\sqrt{5}}{s^2+5}\Big]\\&=-\frac{5}{2}\mathcal{L}^{-1}\Big[\frac{2}{s^3}\Big]+\frac{12}{\sqrt{5}}\mathcal{L}^{-1}\Big[\frac{\sqrt{5}}{s^2+5}\Big]\\&=-\frac{5}{2}t^2+\frac{12}{\sqrt{5}}\sin\sqrt{5}t.\end{aligned} \tag{7.53}$$

2. Find
$$\mathcal{L}^{-1}\Big[\frac{3}{2s^2+8s+10}\Big]. \tag{7.54}$$

   Solution: To solve this problem we need to put $F(s)$ in a form, or combination of forms, that look like those that appear in the Laplace Transform Table (see table 7.1). Denominators that are quadratic polynomials in s need to be written in the form $(s+a)^2+b^2$ or factored as $(s-\lambda_1)(s-\lambda_2)$. In this case we have
$$\frac{3}{2s^2+8s+10}=\frac{3}{2}\frac{1}{s^2+4s+5}=\frac{3}{2}\frac{1}{(s+2)^2+1}. \tag{7.55}$$
   Recall the shift theorem:
$$\mathcal{L}[e^{at}f(t)](s)=\mathcal{L}[f](s-a). \tag{7.56}$$
   We know that
$$\mathcal{L}[\sin t]=\frac{1}{s^2+1} \tag{7.57}$$
   hence from the shift theorem with $a=-2$ we have
$$\mathcal{L}[e^{-2t}\sin t]=\frac{1}{(s+2)^2+1}. \tag{7.58}$$
   Thus
$$\mathcal{L}^{-1}\Big[\frac{3}{2s^2+8s+10}\Big]=\frac{3}{2}e^{-2t}\sin t. \tag{7.59}$$

3. Find
$$\mathcal{L}^{-1}\Big[\frac{3}{2s^2+8s+6}\Big]. \tag{7.60}$$

Solution: In the previous example the denominator was always positive: $2s^2+8s+10 = 2\big((s+2)^2+1\big) = 0$ has no real roots. Here $2s^2+8s+6 = 2\big((s+2)^2-1\big) = 0$ has two real roots. We can factor: $2s^2+8s+6 = 2(s+1)(s+3)$. So for this problem we proceed differently:
$$\frac{3}{2s^2+8s+6} = \frac{3}{2}\frac{1}{(s+1)(s+3)} = \frac{3}{2}\Big(\frac{1/2}{s+1} - \frac{1/2}{s+3}\Big) \tag{7.61}$$
so
$$\mathcal{L}^{-1}\Big[\frac{3}{2s^2+8s+6}\Big] = \frac{3}{4}\mathcal{L}^{-1}\Big[\frac{1}{s+1}\Big] - \frac{3}{4}\mathcal{L}^{-1}\Big[\frac{1}{s+3}\Big] = \frac{3}{4}e^{-t} - \frac{3}{4}e^{-3t}. \tag{7.62}$$
The previous problem could have been solved in a similar manner using complex roots (exercise).

4. Find
$$y(t) = \mathcal{L}^{-1}\Big[\frac{3s^3+8s^2+6s+46}{(s+2)^2(s^2-2s+6)}\Big]. \tag{7.63}$$

Solution: Here $s^2-2s+6 = (s-1)^2+5$ so we can't factor the second factor in the denominator of $Y(s)$ unless we use complex roots, which is an option. Write
$$\begin{aligned} Y(s) &= \frac{as+b}{(s+2)^2} + \frac{cs+d}{s^2-2s+6} \\ &= \frac{(a+c)s^3+(-2a+b+4c+d)s^2+(6a-2b+4c+4d)s+6b+4d}{(s+2)^2(s^2-2s+6)}, \end{aligned} \tag{7.64}$$
which gives the system of equations
$$\begin{aligned} a+c &= 3, \\ -2a+b+4c+d) &= 8, \\ 6a-2b+4c+4d &= 6, \\ 6b+4d &= 46, \end{aligned} \tag{7.65}$$
from which we obtain
$$(a,b,c,d) = (2,7,1,1). \tag{7.66}$$
So
$$\begin{aligned} Y(s) &= \frac{2s+7}{(s+2)^2} + \frac{s+1}{(s-1)^2+5} \\ &= \frac{2}{s+2} + \frac{3}{(s+2)^2} + \frac{s-1}{(s-1)^2+5} + \frac{2}{(s-1)^2+5}. \end{aligned} \tag{7.67}$$

Next find the inverses:

$$\begin{aligned}\mathcal{L}^{-1}\Big[\frac{2}{s+2}\Big] &= 2e^{-2t},\\ \mathcal{L}^{-1}\Big[\frac{3}{(s+2)^2}\Big] &= 3\mathcal{L}^{-1}\Big[\frac{1}{(s+2)^2}\Big] = 3te^{-2t},\\ \mathcal{L}^{-1}\Big[\frac{s-1}{(s-1)^2+5}\Big] &= e^t\cos(\sqrt{5}t)\\ \mathcal{L}^{-1}\Big[\frac{2}{(s-1)^2+5}\Big] &= \frac{2}{\sqrt{5}}e^t\sin(\sqrt{5}t).\end{aligned} \tag{7.68}$$

Hence

$$y(t) = \mathcal{L}^{-1}\Big[\frac{3s^3+8s^2+6s+46}{(s+2)^2(s^2-2s+6)}\Big] = 2e^{-2t} + 3te^{-2t} + e^t\cos(\sqrt{5}t) + \frac{2}{\sqrt{5}}e^t\sin(\sqrt{5}t). \tag{7.69}$$

7.4 Solving constant coefficient differential equations. I

Differential equations can be solved using Laplace Transforms. Taking the Laplace Transform of the general constant coefficient second-order DE initial value problem

$$\begin{aligned}L[y] \equiv ay'' + by' + cy &= f(t)\\ y(0) &= y_0\\ y'(0) &= y'_0\end{aligned} \tag{7.70}$$

gives

$$a\Big(s^2\mathcal{L}[y] - sy(0) - y'(0)\Big) + b\Big(s\mathcal{L}[y] - y(0)\Big) + \mathcal{L}[y] = \mathcal{L}[f] \tag{7.71}$$

or

$$(as^2+bs+c)Y(s) = (as+b)y_0 + ay'_0 + F(s) \tag{7.72}$$

where $Y(s) = \mathcal{L}[y]$ and $F(s) = \mathcal{L}[f](s)$ is the Laplace Transform of the forcing function. Thus

$$Y(s) = \frac{(as+b)y_0 + ay'_0}{P(s)} + \frac{F(s)}{P(s)} \tag{7.73}$$

where $P(s)$ is the characteristic polynomial of the differential equation.

Definition: The function

$$G(s) = \frac{1}{P(s)} \tag{7.74}$$

is called the **Transfer Function**. It encapsulates the differential operator L. From (7.73) we see that $Y(s)$ is the sum of the Transfer function multiplied by the function $I(s) = (as+b)y_0 + ay'_0$ which encapsulates the initial conditions, and the Transfer function multiplied by the Laplace Transform of the forcing function.

Write $Y(s)$ as

$$Y(s) = Y_h(s) + Y_p(s) \tag{7.75}$$

where

$$Y_h(s) = \frac{(as+b)y_0 + ay'_0}{P(s)} \tag{7.76}$$

and

$$Y_p(s) = \frac{F(s)}{P(s)}. \tag{7.77}$$

If the forcing function $f(t)$ is zero $Y(s) = Y_h(s)$, hence we see that $y_h(t) = \mathcal{L}^{-1}[Y_h]$ is a solution of the homogeneous problem *that satisfies the initial conditions.* If the initial conditions are zero, i.e., $y_0 = y_0' = 0$, then $Y_h(s) = 0$ and $y_p = \mathcal{L}^{-1}[Y_p]$ is a particular solution for which $y_p(0) = y_0'(0) = 0$, i.e., *y_p is the particular solution satisfying homogeneous initial conditions.*

We now illustrate with some examples.

Example 1: Consider

$$\begin{aligned} y'' - 2y' - y &= 0, \\ y(0) &= 1, \\ y'(0) &= -2. \end{aligned} \tag{7.78}$$

Let $Y(s) = \mathcal{L}[y]$. Taking the Laplace Transform of the ODE we have

$$\mathcal{L}[y'' - 2y' - y] = \mathcal{L}[0] = 0. \tag{7.79}$$

Using the rules for the Laplace Transform of a derivative gives

$$\begin{aligned} \mathcal{L}[y'' - 2y' - y] &= \mathcal{L}[y''] - 2\mathcal{L}[y'] - \mathcal{L}[y] \\ &= s^2\mathcal{L}[y] - sy(0) - y'(0) - 2\Big(s\mathcal{L}[y] - y(0)\Big) - \mathcal{L}[y] \\ &= 0. \end{aligned} \tag{7.80}$$

So

$$(s^2 - 2s - 1)Y(s) = sy(0) + y'(0) - 2y(0) = s - 4, \tag{7.81}$$

or

$$Y(s) = \frac{s-4}{s^2 - 2s - 1}. \tag{7.82}$$

Note that the denominator is the characteristic polynomial for the DE. This is always the case. The numerator comes from the initial conditions. The denominator can be factored:

$$s^2 - 2s - 1 = (s - \lambda_1)(s - \lambda_2) \tag{7.83}$$

where

$$\lambda_1 = 1 - \sqrt{2} < 0 \qquad \text{and} \qquad \lambda_2 = 1 + \sqrt{2} > 0. \tag{7.84}$$

Writing

$$Y(s) = \frac{s-4}{s^2 - 2s - 1} = \frac{a}{s - \lambda_1} + \frac{b}{s - \lambda_2} \tag{7.85}$$

we find $a = (s + 3\sqrt{2})/4$ and $b = 1 - a = (2 - 3\sqrt{2})/4$. Taking the inverse

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\Big[\frac{a}{s - \lambda_1} + \frac{b}{s - \lambda_2}\Big] \\ &= ae^{\lambda_1 t} + be^{\lambda_2 t} \\ &= \frac{s + 3\sqrt{2}}{4}e^{\lambda_1 t} + \frac{s - 3\sqrt{2}}{4}e^{\lambda_2 t}. \end{aligned} \tag{7.86}$$

Example 2: For the next example consider

$$\begin{aligned} y'' + 4y' + 4y &= e^{2t} + \cos(t) \\ y(0) &= 1 \\ y'(0) &= -1. \end{aligned} \tag{7.87}$$

Taking the L.T. of the equation gives

$$s^2Y(s) - sy(0) - y'(0) + 4\big(sY(s) - y(0)\big) + 4Y(s) = \frac{1}{s-2} + \frac{s}{s^2+1}, \tag{7.88}$$

so

$$\begin{aligned} Y(s) &= \frac{sy(0) + y'(0) + 4y(0)}{s^2+4s+4} + \frac{1}{(s-2)(s^2+4s+4)} + \frac{s}{(s^2+1)(s^2+4s+4)} \\ &= \frac{s+3}{s^2+4s+4} + \frac{1}{(s-2)(s^2+4s+4)} + \frac{s}{(s^2+1)(s^2+4s+4)} \\ &= \frac{s+3}{(s+2)^2} + \frac{1}{(s-2)(s+2)^2} + \frac{s}{(s^2+1)(s+2)^2}. \end{aligned} \tag{7.89}$$

Note that $P(s) = s^2+4s+4 = (s+2)^2$ is the characteristic polynomial for the differential equation. We write the equation for $Y(s)$ as

$$Y(s) = Y_h(s) + Y_{p1}(s) + Y_{p2}(s) \tag{7.90}$$

where $Y_h(s)$ is the piece that comes from the initial conditions and Y_{p1} and Y_{p2} are the terms involving the transform of the two forcing functions. We find the inverse transform of each piece separately.

1. Considering the first piece we have

$$\begin{aligned} Y_h(s) = \frac{s+3}{(s+2)^2} &= \frac{s+2+1}{(s+2)^2} \\ &= \frac{1}{s+2} + \frac{1}{(s+2)^2} \end{aligned} \tag{7.91}$$

which has inverse

$$\begin{aligned} y_h(t) = \mathcal{L}^{-1}\Big[Y_h(s)\Big] &= \mathcal{L}^{-1}\Big[\frac{1}{s+2}\Big] + \mathcal{L}^{-1}\Big[\frac{1}{(s+2)^2}\Big] \\ &= e^{-2t} + te^{-2t}. \end{aligned} \tag{7.92}$$

For the latter we have used $\mathcal{L}[t] = 1/s^2$ along with the shift theorem: $\mathcal{L}[te^{-2t}] = 1/(s+2)^2$. Note that this solution, which does not involve the forcing functions, is a linear combination of e^{-2t} and te^{-2t} which are two linearly independent solutions of the homogeneous solution. This solution satisfies the initial conditions: $y_h(0) = 1$ and $y_h'(t) = -2e^{-2t} + e^{-2t} - 2te^{-2t}$ is equal to -1 at $t = 0$.

2. For the second piece we need to use partial fractions to separate it into pieces we can find the inverse transform. Setting

$$\begin{aligned} Y_{p1}(s) = \frac{1}{(s-2)(s+2)^2} &= \frac{a}{(s-2)} + \frac{bs+c}{(s+2)^2} \\ &= \frac{as^2 + 4as + 4a + bs^2 + (c-2b)s - 2c}{(s-2)(s+2)^2}, \end{aligned} \tag{7.93}$$

gives the system

$$\begin{aligned} a + b &= 0 \\ 4a - 2b + c &= 0 \\ 4a - 2c &= 1, \end{aligned} \tag{7.94}$$

from the coefficients of s^2, s and the constant part of the numerator. The solution of this system is

$$(a, b, c) = \Big(\frac{1}{16}, -\frac{1}{16}, -\frac{6}{16}\Big). \tag{7.95}$$

Hence

$$\begin{aligned} Y_{p1}(s) = \frac{1}{(s-2)(s+2)^2} &= \frac{1}{16}\frac{1}{(s-2)} - \frac{1}{16}\frac{s+6}{(s+2)^2} \\ &= \frac{1}{16}\frac{1}{(s-2)} - \frac{1}{16}\Big(\frac{1}{s+2} + \frac{4}{(s+2)^2}\Big). \end{aligned} \tag{7.96}$$

The inverse transform is

$$\begin{aligned} y_{p1}(t) = \mathcal{L}^{-1}\Big[\frac{1}{(s-2)(s+2)^2}\Big] &= \frac{1}{16}e^{2t} - \frac{1}{16}\Big(e^{-2t} + 4te^{-2t}\Big) \\ &= \frac{1}{16}e^{2t} - \frac{1}{16}e^{-2t} - \frac{1}{4}te^{-2t}. \end{aligned} \tag{7.97}$$

This is a particular solution of the differential equation for the forcing term e^{2t}. The first term of $y_{p1}(t)$, $e^{2t}/16$, is a particular solution as

$$(D^2 + 4D + 4)\big[\frac{1}{16}e^{2t}\big] = \frac{1}{16}(4 + 8 + 4)e^{2t} = e^{2t}. \tag{7.98}$$

The second and third terms are a linear combination of two linearly independent homogeneous solutions. This combination is such that the particular solution y_{p1} satisfies homogeneous initial conditions: $y_{p1}(0) = y'_{p1}(0) = 0$. Checking this is a useful check on your solution (exercise).

3. Finally, we consider the third piece. Using partial fractions again we have

$$\begin{aligned} Y_{p2}(s) = \frac{s}{(s^2+1)(s+2)^2} &= \frac{1}{25}\frac{3s+4}{s^2+1} - \frac{1}{25}\frac{3s+16}{(s+2)^2} \\ &= \frac{1}{25}\Big(\frac{3s}{s^2+1} + \frac{4}{s^2+1}\Big) - \frac{1}{25}\frac{3s+6+10}{(s+2)^2} \\ &= \frac{1}{25}\Big(\frac{3s}{s^2+1} + \frac{4}{s^2+1}\Big) - \frac{1}{25}\Big(\frac{3}{s+2} + \frac{10}{(s+2)^2}\Big) \end{aligned} \tag{7.99}$$

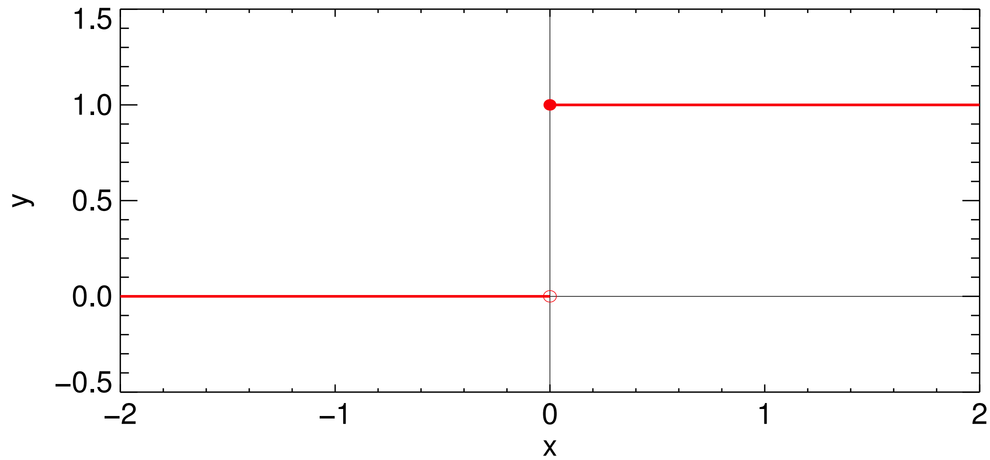

Figure 7.1: The Heaviside function.

The inverse transform is

$$y_{p2}(t) = \mathcal{L}^{-1}\Big[Y_{p2}\Big] = \frac{1}{25}\Big(3\cos(t) + 4\sin(t)\Big) - \frac{1}{25}\Big(3e^{-2t} + 10te^{-2t}\Big). \tag{7.100}$$

The part involving $\cos t$ and $\sin t$ is a particular solution for the forcing $\cos t$. The remaining two pieces are a homogeneous solution that makes the particular solution $y_{p2}(t)$ one that satisfies homogeneous initial conditions, i.e., $y_{p2}(0) = y'_{p2}(0) = 0$. This is a useful check of your solution (exercise).

Putting the three pieces together:

$$\begin{aligned} y &= \Big[e^{-2t} + te^{-2t}\Big] + \left[\frac{1}{16}e^{2t} - \frac{1}{16}e^{-2t} - \frac{1}{4}te^{-2t}\right] \\ &\qquad + \left[\frac{1}{25}\Big(3\cos(t) + 4\sin(t)\Big) - \frac{1}{25}\Big(3e^{-2t} + 10te^{-2t}\Big)\right] \\ &= \frac{327}{400}e^{-2t} + \frac{35}{100}te^{-2t} + \frac{1}{16}e^{2t} + \frac{3}{25}\cos t + \frac{4}{25}\sin t. \end{aligned} \tag{7.101}$$

7.5 The Heaviside Function

Definition: The Heaviside function $H(t)$ (or *unit step function*) is defined by

$$H(t) = \begin{cases} 0 & t < 0; \\ 1 & t \geq 0, \end{cases} \tag{7.102}$$

(see Figure 7.1).

Comments:

1. Sometimes $H(t)$ is defined as 0 for $t \leq 0$ and 1 for $t > 0$.
2. While $H(t)$ is a fairly traditional notation for the Heaviside function other terminology is used, e.g., $u(t)$.

Any piecewise continuous function can be expressed in terms of the Heaviside function. Values may differ at points of discontinuity. The basic idea is that

$$f(t) = f_1(t) + \big(f_2(t) - f_1(t)\big)H(t - t_0) \tag{7.103}$$

is equal to $f_1(t)$ for $t < t_0$ and to $f_1 + f_2 - f_1 = f_2$ for $t > t_0$. Thus the function $f(t)$ given by (7.103) jumps from $f_1(t)$ to $f_2(t)$ at $t = t_0$. We can have any number of jumps as the next example illustrates.

Example: Consider the function

$$y(t) = \begin{cases} \cos t & 0 \leq t < \pi; \\ 2 & \pi \leq t \leq 2\pi; \\ \sin t & 2\pi < t \leq 4\pi; \\ t - 10 & t > 4\pi. \end{cases} \tag{7.104}$$

We can write this as

$$\tilde{y}(t) = \cos t + \big(2 - \cos t\big)H(t - \pi) + \big(\sin t - 2\big)H(t - 2\pi) + \big(t - 10 - \sin t\big)H(t - 4\pi). \tag{7.105}$$

For $t < \pi$ all the Heaviside functions are zero so $\tilde{y}(t) = \cos t$ for $t < \pi$. For $\pi < t < 2\pi$ the first step function is one, the other two are zero so $\tilde{y}(t) = \cos t + 2 - \cos t = 2$, etc. I've called this function $\tilde{y}$ to highlight the fact that this function may not be equal to $y(t)$ at the points of discontinuity and in fact this is the case: $y(2\pi) = 2$ but $\tilde{y}(2\pi) = \sin(2\pi) = 0$.

7.5.1 Laplace Transform of the Heaviside function

For $t_0 \geq 0$:

$$\mathcal{L}\big[H(t - t_0)\big] = \int_0^\infty e^{-st} H(t - t_0)\,\mathrm{d}t = \int_{t_0}^\infty e^{-st}\,\mathrm{d}t = \frac{e^{-st_0}}{s} \tag{7.106}$$

and

$$\mathcal{L}^{-1}\Big[\frac{e^{-st_0}}{s}\Big] = H(t - t_0). \tag{7.107}$$

Translation in t

Let $F(s) = \mathcal{L}[f](s)$ exist for $s > \alpha$. If $a > 0$ is a constant then

$$\mathcal{L}\big[f(t-a)H(t-a)\big](s) = e^{-as}F(s) \tag{7.108}$$

exists for $s > \alpha$ and

$$\mathcal{L}^{-1}\big[e^{-as}F(s)\big] = f(t-a)H(t-a). \tag{7.109}$$

Proof: By definition

$$\begin{aligned}
\mathcal{L}\big[f(t-a)H(t-a)\big](s) &= \int_0^\infty e^{-st}f(t-a)H(t-a)\,\mathrm{d}t \\
&= \int_a^\infty e^{-st}f(t-a)\,\mathrm{d}t \qquad \text{(since } H(t-a)=0 \text{ for } t<a \text{ and } 1 \text{ for } t>a) \\
&= \int_0^\infty e^{-s(\tilde{t}+a)}f(\tilde{t})\,\mathrm{d}\tilde{t} \qquad (\tilde{t} = t-a) \\
&= e^{-sa}\int_0^\infty e^{-s\tilde{t}}f(\tilde{t})\,\mathrm{d}\tilde{t} \\
&= e^{-sa}F(s).
\end{aligned} \tag{7.110}$$

Exercise: What is different if $a < 0$?

Corollary:

$$\mathcal{L}\big[f(t)H(t-a)\big](s) = e^{-as}\mathcal{L}\big[f(t+a)\big]. \tag{7.111}$$

Examples:

1. **Example 1:** Find the Laplace Transform of

$$f(t) = \begin{cases} \sin t & 0 \leq \pi/4; \\ \sin t + \cos(t - \frac{\pi}{4}) & t \geq \frac{\pi}{4}. \end{cases} \tag{7.112}$$

Solution: Write $f(t)$ as

$$f(t) = \sin t + H(t - \frac{\pi}{4})\cos(t - \frac{\pi}{4}). \tag{7.113}$$

Then

$$\begin{aligned}
\mathcal{L}\big[f\big] &= \mathcal{L}\big[\sin t\big] + \mathcal{L}\big[H(t-\frac{\pi}{4})\cos(t-\frac{\pi}{4})\big] \\
&= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\mathcal{L}\big[\cos t\big] \\
&= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\frac{s}{s^2+1}.
\end{aligned} \tag{7.114}$$

2. **Example 2:** Find the Laplace Transform of

$$f(t) = \begin{cases} \sin t & 0 \leq \frac{\pi}{4}; \\ \sin t + \sin^2 t & t \geq \frac{\pi}{4}. \end{cases} \tag{7.115}$$

Solution: Write $f(t)$ as

$$f(t) = \sin t + H\big(t - \frac{\pi}{4}\big)\Big(\frac{1}{2} - \frac{\cos(2t)}{2}\Big) \tag{7.116}$$

so

$$\begin{aligned} \mathcal{L}[f] &= \mathcal{L}[\sin t] + \frac{1}{2}\mathcal{L}\big[H(t - \frac{\pi}{4})\big] - \frac{1}{2}\mathcal{L}\big[H(t - \frac{\pi}{4})\cos 2t\big] \\ &= \frac{1}{s^2+1} + \frac{1}{2}\frac{e^{-\frac{\pi}{4}s}}{s} - \frac{1}{2}\mathcal{L}\big[H(t - \frac{\pi}{4})\cos 2t\big]. \end{aligned} \tag{7.117}$$

We must do some work on the last term to put it in the form $H(t-a)f(t-a)$ so we can use the shift therom. To do this we use

$$\cos 2t = \cos\big(2(t - \frac{\pi}{4}) + \frac{\pi}{2}\big) = -\sin\big(2(t - \frac{\pi}{4})\big). \tag{7.118}$$

Then

$$\mathcal{L}\big[H(t - \frac{\pi}{4})\cos 2t\big] = \mathcal{L}\big[H(t - \frac{\pi}{4})\sin 2(t - \frac{\pi}{4})\big] = e^{-\frac{\pi}{4}s}\mathcal{L}\big[\sin 2t\big] = e^{-\frac{\pi}{4}s}\frac{2}{s^2+4}. \tag{7.119}$$

Thus

$$\mathcal{L}[f](s) = \frac{1}{s^2+1} + \frac{1}{2}\frac{e^{-\frac{\pi}{4}s}}{s} - \frac{e^{-\frac{\pi}{4}s}}{s^2+4}. \tag{7.120}$$

3. **Example 3:** Find $\mathcal{L}[t^2 H(t-1)]$.

Solution:

$$\begin{aligned} \mathcal{L}[t^2 H(t-1)] = \mathcal{L}[(t-1+1)^2 H(t-1)] &= \mathcal{L}[\big((t-1)^2 + 2(t-1) + 1\big)H(t-1)] \\ &= \mathcal{L}[(t-1)^2 H(t-1)] \\ &\quad + 2\mathcal{L}[(t-1)H(t-1)] \\ &\quad\quad + \mathcal{L}[\big)H(t-1)] \\ &= e^{-s}\Big(\frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}\Big). \end{aligned} \tag{7.121}$$

Alternatively, using the corollary (7.111),

$$\mathcal{L}[t^2 H(t-1)] = e^{-s}\mathcal{L}\big[(t+1)^2\big] = e^{-s}\mathcal{L}\big[t^2 + 2t + 1\big] = e^{-s}\Big(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\Big). \tag{7.122}$$

4. **Example 4:** Find

$$f(t) = \mathcal{L}^{-1}\Big[\frac{e^{-2s}}{s^2}\Big]. \tag{7.123}$$

Solution: We use

$$\mathcal{L}\big[f(t-a)H(t-a)\big](s) = e^{-as}\mathcal{L}\big[f\big](s) = e^{-as}F(s). \tag{7.124}$$

Here we identify e^{-2s} with e^{-as}, hence $a=2$, and $\frac{1}{s^2}$ with $F(s)$. Using

$$\mathcal{L}^{-1}\Big[\frac{1}{s^2}\Big] = t, \tag{7.125}$$

we have

$$e^{-2s}\frac{1}{s^2} = e^{-2s}\mathcal{L}[t] = \mathcal{L}\big[H(t-2)(t-2)\big] \tag{7.126}$$

so

$$\mathcal{L}^{-1}\Big[\frac{e^{-2s}}{s^2}\Big] = (t-2)H(t-2). \tag{7.127}$$

5. **Example 5:** The current I in an LC circuit is governed by the initial value problem

$$\begin{aligned} I'' + 4I &= g(t), \\ I(0) = I'(0) &= 0 \end{aligned} \tag{7.128}$$

where

$$g(t) = \begin{cases} 1, & 0 < t < 1; \\ -1, & 1 < t < 2; \\ 0, & t > 2. \end{cases} \tag{7.129}$$

Determine the current $I(t)$.

Solution: Let $J(s) = \mathcal{L}[I](s)$. Taking the Laplace Transform of the equation we have

$$s^2J(s) - sI(0) - I'(0) + 4J(s) = \mathcal{L}[g](s). \tag{7.130}$$

or, after using the iniital conditions,

$$J(s) = \frac{\mathcal{L}[g](s)}{s^2+4}. \tag{7.131}$$

In terms of the Heaviside function

$$\mathcal{L}[g](s) = \mathcal{L}\big[1 - 2H(t-1) + H(t-2)\big] = \frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s}, \tag{7.132}$$

so

$$J(s) = F(s) - 2e^{-s}F(s) + e^{-2s}F(s) \tag{7.133}$$

where

$$F(s) = \frac{1}{s(s^2+4)} = \frac{1}{4s} - \frac{1}{4}\frac{s}{s^2+4}. \tag{7.134}$$

The inverse Laplace Transform of $F(s)$ is

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\big[F(s)\big] &= \frac{1}{4}\mathcal{L}^{-1}\big[\frac{1}{s}\big] - \frac{1}{4}\mathcal{L}^{-1}\big[\frac{1}{s^2+4}\big] \\ &= \frac{1}{4} - \frac{1}{4}\cos(2t) \\ &= \frac{1-\cos(2t)}{4}. \end{aligned} \tag{7.135}$$

Now we use

$$\mathcal{L}^{-1}\big[e^{-as}F(s)\big] = f(t-a)H(t-a) \tag{7.136}$$

where $F(s) = \mathcal{L}[f]$. Thus

$$\begin{aligned} I(s) = \mathcal{L}^{-1}\big[J(s)\big] &= \mathcal{L}^{-1}\big[F(s)\big] - 2\mathcal{L}^{-1}\big[e^{-s}F(s)\big] + \mathcal{L}^{-1}\big[e^{-2s}F(s)\big] \\ &= f(t) - 2f(t-1)H(t-1) + f(t-2)H(t-2) \\ &= \frac{1-\cos(2t)}{4} - 2\frac{1-\cos\big(2(t-1)\big)}{4}H(t-1) \\ &\qquad + \frac{1-\cos\big(2(t-2)\big)}{4}H(t-2). \end{aligned} \tag{7.137}$$

The forcing function and the solution and its first two derivatives are shown in Figure 7.3. The solution and its first derivative are continuous. The second derivative has a jump discontinuity. This can be deduced from the equation:

$$I'' + 4I = 1 - 2H(t-1) + H(t-2). \tag{7.138}$$

The right hand side is piecewise continuous with jumps at $t = 1$ and $t = 2$, hence so does the left hand side. Because differentiation reduces differentiability, i.e., if y is a C^1 function (a continuous function with a continuous first derivative) with a discontinuous second derivative, then the derivative y' is continuous with a discontinuous first derivative. So among the terms on the left-hand side the term with the highest derivative, I'', must have the same level of continuity/differentiability as the right-hand side. In this case I'' must be piecewise continuous with jump discontinuities at $t = 1$ and $t = 2$. Then its integral, I', is a continuous function with jumps in its derivative at $t = 1$ and $t = 2$ and I is a C^1 function with jumps in its curvature at $t = 1$ and $t = 2$.

7.6 Convolutions

Definition: Let $f(t)$ and $g(t)$ be piecewise continuous functions on $[0, \infty)$. The **convolution** of $f(t)$ and $g(t)$ is a new function denoted by $f * g$ defined as

$$f * g(t) = \int_0^t f(t-v)g(v)\,\mathrm{d}v. \tag{7.139}$$

Thus, $*$ is an operator which takes in two functions and returns a new function.

Properties: The convolution operator has the following properties:

(i) $f * g(t) = g * f(t)$;

(ii) $f * (g + ch) = f * g + cf * h$.

(iii) $f * (g * h) = (f * g) * h$;

where f, g and h are piecewise continuous functions on $[0, \infty)$ and c is a constant. The convolution operator is bilinear operator.

Proof:

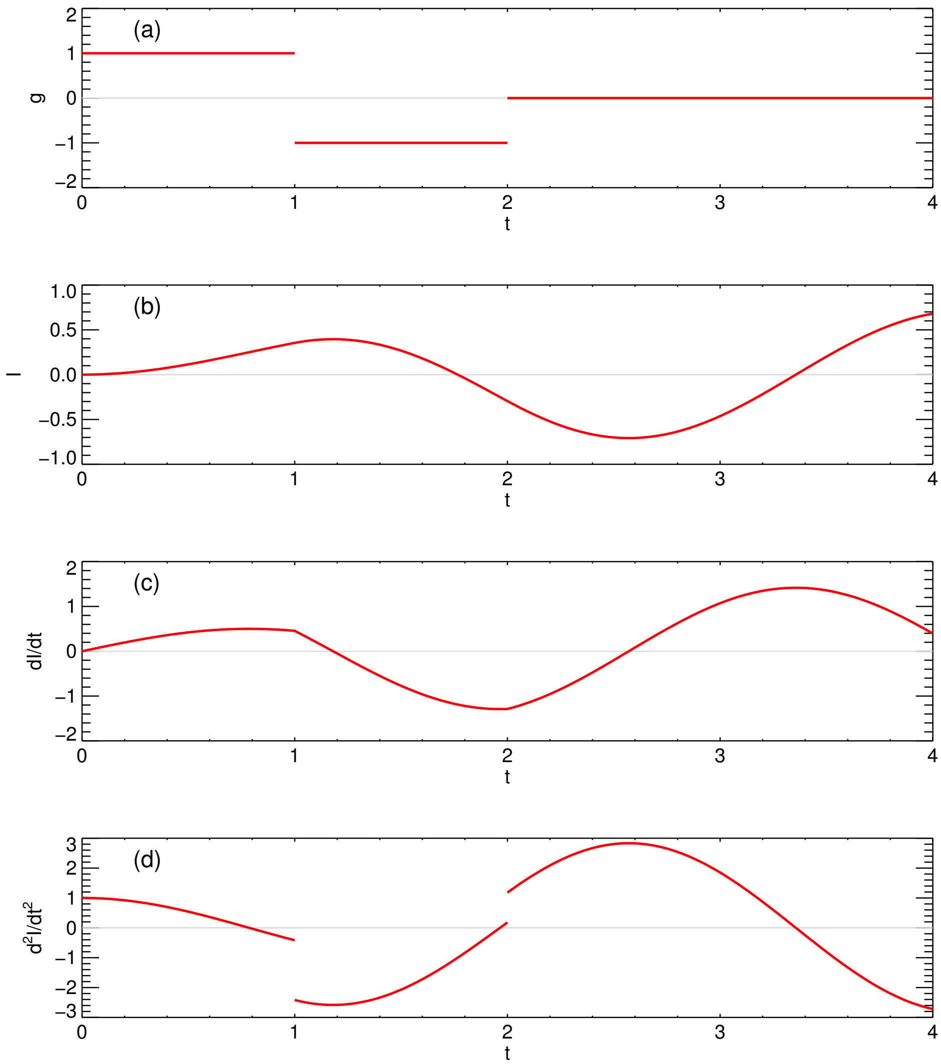

Figure 7.2: Solution of the circuit problem (example 5). (a) The forcing function g. (b) The solution $I(t)$. (c) The first derivative $I'(t)$. (d) The second derivative $I''(t)$.

(i) By definition

$$f * g = \int_0^t f(t-v)g(v)\,\mathrm{d}v. \tag{7.140}$$

Make the substitution $s = t - v$. Then $\mathrm{d}v = -\mathrm{d}s$ and $s = t$ when $v = 0$ and $s = 0$ when $v = t$ so

$$f * g(t) = \int_t^0 f(s)g(t-s)\,(-\mathrm{d}s) = \int_0^t g(t-s)f(s)\,\mathrm{d}s = g * f(t). \tag{7.141}$$

(ii) Follows from linearity of integration.

(iii) By definition

$$(f * g) * h(t) = \int_{v=0}^t \big(f * g\big)(t-v)\,h(v)\,\mathrm{d}v = \int_{v=0}^t \Big[\int_0^{s=t-v} f(t-v-s)g(s)\,\mathrm{d}s\Big]h(v)\,\mathrm{d}v \tag{7.142}$$

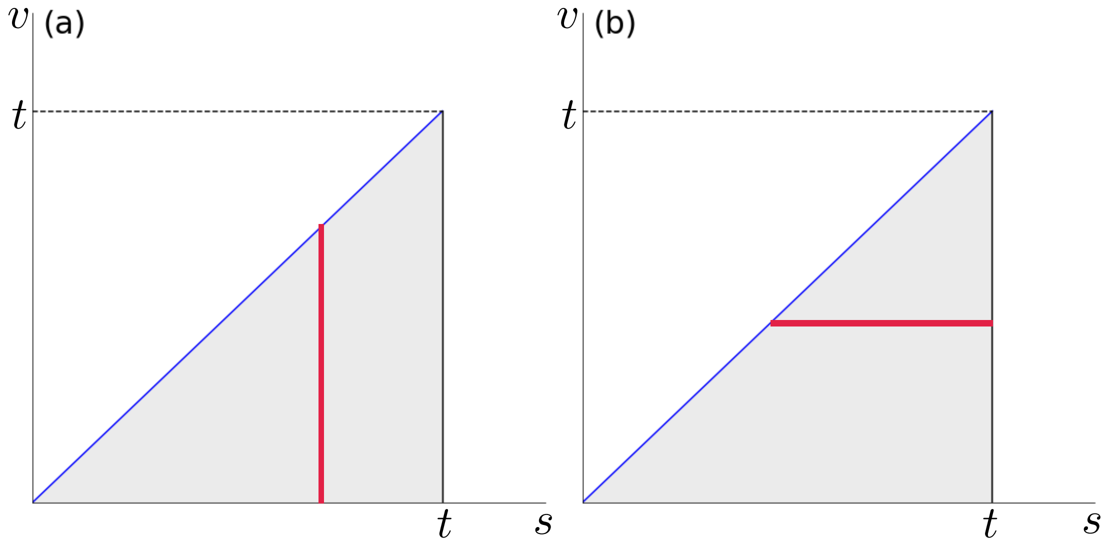

Figure 7.3: Convolution property 3. Integrating over the shaded region you can: (a) integrate with respect to s from $s = 0$ to $s = t$, for each value of s integrate with respect to v from 0 to s (red rectangle); or (b) integrate with respect to v from $v = 0$ to $v = t$, for each value of v integrate with respect to s between $s = v$ and $s = t$ (red rectangle).

and

$$f*(g*h)(t) = \int_{s=0}^{t} f(t-s)\big(g*h\big)(s)\,\mathrm{d}s = \int_{s=0}^{t} f(t-s)\Big[\int_{v=0}^{s} g(s-v)h(v)\,dv\Big]\,\mathrm{d}s. \quad (7.143)$$

The latter integral is over the triangle $0 \le v \le s$, $0 \le s \le t$ in the s-v plane. This is also the triangle $0 \le v \le t$, $v \le s \le t$ so

$$f*(g*h)(t) = \int_{v=0}^{t}\int_{s=v}^{t} f(t-s)g(s-v)h(v)\,\mathrm{d}s\,\mathrm{d}v. \quad (7.144)$$

Making the substitution $q = s - v$ to replace s by q we have

$$f*(g*h)(t) = \int_{v=0}^{t}\int_{q=0}^{t-v} f(t-q-v)g(q)h(v)\,\mathrm{d}q\,\mathrm{d}v. \quad (7.145)$$

Replacing q by s (i.e., change the name of the dummy variable of integration) gives (7.142) proving the result.

Examples:

1. Find the convolution of $f(t) = \cos t$ and $g(t) = 1$.

$$(f*g)(t) = \int_0^t \cos(t-v)\,\mathrm{d}v = -\sin(t-v)\Big|_0^t = -\big(\sin(0) - \sin(t)\big) = \sin(t). \quad (7.146)$$

Note: here, making the identifications $f(t) = \cos t$ and $g(t) = 1$, $g(v)$ in the integral doesn't appear as $g(v) = 1$. Alternatively

$$(f * g)(t) = (g * f)(t) = \int_0^t \cos(v)\,\mathrm{d}v = \sin(v)\Big|_0^t = \sin(t). \tag{7.147}$$

2. Find the convolution of $f(t) = \cos t$ and $g(t) = \sin t$.

$$\begin{aligned}
(f * g)(t) = \int_0^t \cos(t-v)\sin(v)\,\mathrm{d}v &= \int_0^t \frac{1}{2}\Big(\sin(t-v+v) - \sin(t-v-v)\Big)\,\mathrm{d}v \\
&= \frac{1}{2}\int_0^t \sin(t)\,\mathrm{d}v - \frac{1}{2}\int_0^t \sin(t-2v)\,\mathrm{d}v \\
&= \frac{1}{2}t\sin t - \frac{1}{2}\Big(\frac{\cos(t-2v}{2}\Big)\Big|_0^t \\
&= \frac{t\sin t}{2} - \frac{1}{2}\frac{\cos(-t) - \cos(t)}{2} \\
&= \frac{t\sin t}{2}.
\end{aligned} \tag{7.148}$$

7.6.1 The Convolution Theorem

Theorem: Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and be of exponential order α. Let $F(s) = \mathcal{L}[f]$ and $G(s) = \mathcal{L}[g]$. Then

$$\mathcal{L}[f * g] = F(s)G(s), \tag{7.149}$$

which means

$$\mathcal{L}^{-1}[F(s)G(s)] = f * g(t). \tag{7.150}$$

Proof:

$$
\begin{aligned}
\mathcal{L}[f*g] &= \int_0^\infty e^{-st}\Big[\int_0^t f(t-v)g(v)\,\mathrm{d}v\Big]\,\mathrm{d}t\\
&= \int_0^\infty e^{-st}\Big[\int_0^\infty H(t-v)f(t-v)g(v)\,\mathrm{d}v\Big]\,\mathrm{d}t \qquad (H(t-v)=0 \text{ for } v>t)\\
&= \int_0^\infty g(v)\Big[\int_0^\infty e^{-st}H(t-v)f(t-v)\,\mathrm{d}t\Big]\,\mathrm{d}v \qquad (\text{change order of integration})\\
&= \int_0^\infty g(v)\Big[\int_v^\infty e^{-st}f(t-v)\,\mathrm{d}t\Big]\,\mathrm{d}v \qquad (H(t-v)=0 \text{ for } t<v)\\
&= \int_0^\infty g(v)\Big[\int_0^\infty e^{-s(\tilde{t}+v)}f(\tilde{t})\,\mathrm{d}\tilde{t}\Big]\,\mathrm{d}v\\
&= \int_0^\infty g(v)e^{-sv}\Big[\int_0^\infty e^{-s\tilde{t}}f(\tilde{t})\,\mathrm{d}\tilde{t}\Big]\,\mathrm{d}v\\
&= \int_0^\infty g(v)e^{-sv}F(s)\,\mathrm{d}v\\
&= F(s)\int_0^\infty g(v)e^{-sv}\,\mathrm{d}v\\
&= F(s)G(s).
\end{aligned} \tag{7.151}
$$

This result can be useful for finding inverse Laplace Transforms.

Examples:

1. Find

$$\mathcal{L}^{-1}\Big[\frac{a}{s^2(s^2+a^2)}\Big]. \tag{7.152}$$

Solution:

$$\mathcal{L}^{-1}\Big[\frac{a}{s^2(s^2+a^2)}\Big] = \mathcal{L}^{-1}\Big[\frac{1}{s^2}\frac{a}{s^2+a^2}\Big]. \tag{7.153}$$

Using

$$\mathcal{L}^{-1}\Big[\frac{1}{s^2}\Big] = t \tag{7.154}$$

and

$$\mathcal{L}^{-1}\Big[\frac{a}{s^2+a^2}\Big] = \sin(at) \tag{7.155}$$

and the convolution theorem we have

$$\begin{aligned}
\mathcal{L}^{-1}\Big[\frac{a}{s^2(s^2+a^2)}\Big] &= \Big(\sin(at) * t\Big)(t) \\
&= \int_0^t \sin(a(t-v))v \,\mathrm{d}v \\
&= \int_0^t \frac{\mathrm{d}}{\mathrm{d}v}\Big(\frac{\cos(a(t-v))}{a}\Big)v \,\mathrm{d}t, \\
&= \Big(v\frac{\cos(a(t-v))}{a}\Big)\Big|_{v=0}^{v=t} - \frac{1}{a}\int_0^t \cos\big(a(t-v)\big)\,\mathrm{d}v \\
&= \frac{t}{a} + \frac{1}{a}\frac{\sin(a(t-v))}{a}\Big|_{v=0}^{v=t} \\
&= \frac{at-\sin(at)}{a^2}.
\end{aligned} \tag{7.156}$$

2. Find the solution of

$$\begin{aligned}
y'' + 4y &= f(t) \\
y(0) &= 3 \\
y'(0) &= -1
\end{aligned} \tag{7.157}$$

in terms of $f(t)$.

Solution: Taking the Laplace Transform of the differential equation gives

$$(s^2+4)Y(s) = 3s - 1 + F(s) \tag{7.158}$$

where $F(s) = \mathcal{L}[f]$. Hence

$$Y(s) = 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + F(s)G(s) \tag{7.159}$$

where

$$G(s) = \frac{1}{s^2+4} \tag{7.160}$$

is the Transfer Function for the DE. Let

$$g(t) = \mathcal{L}^{-1}[G(s)] = \frac{1}{2}\sin(2t). \tag{7.161}$$

Using the convolution theorem on the last term we have

$$\begin{aligned}
y(t) &= 3\cos(2t) - \frac{1}{2}\sin(2t) + f * g(t) \\
&= 3\cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{2}\int_0^t f(t-v)\sin(2v)\,\mathrm{d}v.
\end{aligned} \tag{7.162}$$

7.7 Periodic Functions

Theorem: If $f(t)$ is piecewise continuous and periodic with period τ then

$$\mathcal{L}[f](s) = \frac{F_\tau(s)}{1 - e^{-s\tau}} \tag{7.163}$$

where

$$F_\tau(s) = \int_0^\tau e^{-st} f(t)\,\mathrm{d}t. \tag{7.164}$$

Proof:

$$\begin{aligned}
\mathcal{L}[f](s) &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^\tau e^{-st} f(t)\,\mathrm{d}t + \int_\tau^\infty e^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^\tau e^{-st} f(t)\,\mathrm{d}t + \int_0^\infty e^{-s(\tau+\tilde{t})} f(\tau+\tilde{t})\,\mathrm{d}\tilde{t} \quad \text{(change of variables: } t = \tau + \tilde{t}) \\
&= \int_0^\tau e^{-st} f(t)\,\mathrm{d}t + e^{-s\tau} \int_0^\infty e^{-s\tilde{t}} f(\tilde{t})\,\mathrm{d}\tilde{t} \quad \text{(using periodicity of } f) \\
&= \int_0^\tau e^{-st} f(t)\,\mathrm{d}t + e^{-s\tau} \mathcal{L}[f](s)
\end{aligned} \tag{7.165}$$

which gives the result.

Example: Find the Laplace Transform of the periodic function

$$f(t) = \begin{cases} 1, & 0 \le t < 1; \\ 0, & 1 \le t < 2; \end{cases} \tag{7.166}$$

with $f(t+2) = f(t)$.

Solution: The function is periodic with period 2 so the Laplace Transform is

$$\mathcal{L}[f](s) = \frac{F_2(s)}{1 - e^{-2s}} \tag{7.167}$$

where

$$F_2(s) = \int_0^2 e^{-st} f(t)\,\mathrm{d}t = \int_0^1 e^{-st}\,\mathrm{d}t = \frac{1 - e^{-s}}{s}. \tag{7.168}$$

Hence

$$\mathcal{L}[f](s) = \frac{1 - e^{-s}}{s(1 - e^{-2s})} = \frac{1}{s(1 + e^{-s})}. \tag{7.169}$$

7.8 Impulses and the Dirac Delta Function

To deal with short very strong impulsive forces (e.g., a hammer blow) the delta function introduced by Dirac is often used.

Consider Newton's Second Law applied to an object of mass m subject to a time varying force $F(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(mv) = F(t). \tag{7.170}$$

Integrating from t_1 to t_2 gives

$$\Delta(mv) = \int_{t_1}^{t_2} F(t)\,\mathrm{d}t \tag{7.171}$$

where $\Delta(mv)$ is the change in momentum between times t_1 and t_2. The integral

$$\int_{t_1}^{t_2} F(t)\,\mathrm{d}t \tag{7.172}$$

is called the impulse.

Consider a hammer blow at time t. The force exerted by the hammer is enormous but it acts over a very short period of time (Figure 7.4). The area under the force-time curve is the impulse imparted by the hammer. A set of functions with the same impulse acting over a progressively narrower base must get higher in order to preserve the area.

Let $F_n(t)$ be a family of piecewise continuous functions with impulse 1 which are non-zero only on $[t_0, t_n]$ for $n = 1, 2, 3, \ldots$ with $t_n \to t_0$ as $n \to \infty$. That is

$$\int_{-\infty}^{\infty} F_n(t)\,\mathrm{d}t = \int_{t_0}^{t_n} F_n(t)\,\mathrm{d}t = 1 \qquad \forall n. \tag{7.173}$$

As $n \to \infty$, $\max\{F_n\} \to \infty$. Otherwise, if the $F_n(t)$ are bounded by, say, M, then $\int_{t_0}^{t_n} |F_n|\,dt \leq M(t_n - t_0) \to 0$ as $n \to \infty$ because $t_n \to t_0$.

There are many possible such sets of functions, for example:

$$F_n(t) = \begin{cases} n, & 0 \leq t \leq \frac{1}{n}; \\ 0, & \text{otherwise.} \end{cases} \tag{7.174}$$

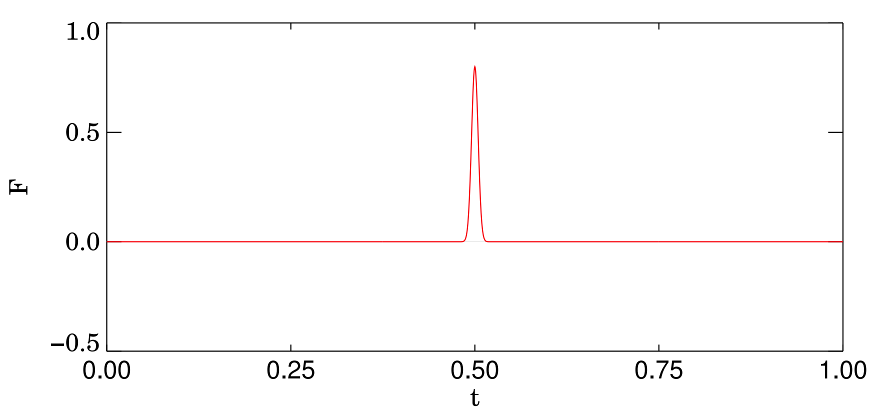

Figure 7.4: An impulse force (non-dimensionalized) F acting for a short duration of time.

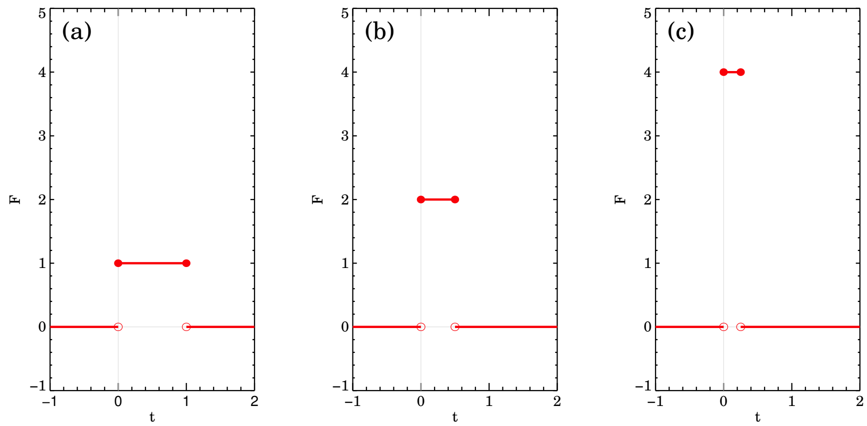

Figure 7.5: Sample functions $F_n(t)$ given by (7.174). (a) $n = 1$. (b) $n = 2$. (c) $n = 4$.

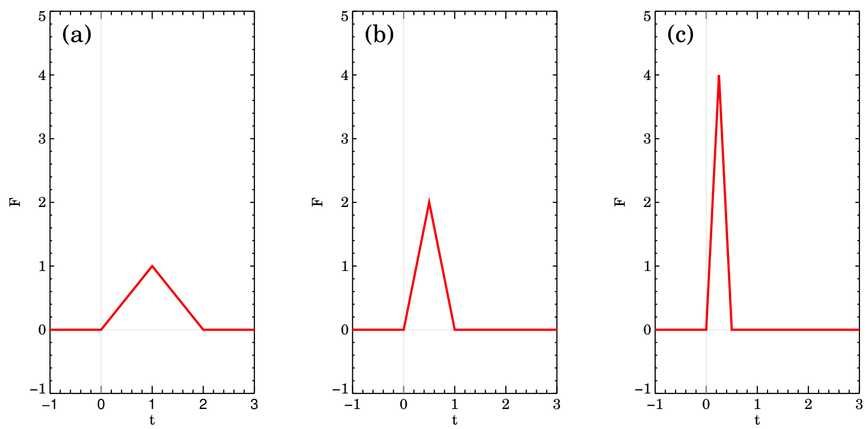

Figure 7.6: Sample functions $F_n(t)$ given by (7.175). (a) $n = 1$. (b) $n = 2$. (c) $n = 4$.

or

$$F_n(t) = \begin{cases} n^2 t, & 0 \le t \le \frac{1}{n}; \\ -n^2(t - \frac{2}{n}) & \frac{1}{n} < t \le \frac{2}{n}, \\ 0, & \text{otherwise.} \end{cases} \tag{7.175}$$

See figures 7.5 and 7.6.

Claim: All of these families of functions have the property that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} F_n(t)\, f(t)\, \mathrm{d}t = f(0) \tag{7.176}$$

for all continuous functions $f(t)$.

Proof: For simplicity we will assume that $F_n(t) \ge 0$. This is not required.

The functions F_n have compact support $[0, t_n]$ so

$$\int_{-\infty}^{\infty} F_n(t) f(t)\, \mathrm{d}t = \int_0^{t_n} F_n(t) f(t)\, \mathrm{d}t. \tag{7.177}$$

Since $f(t)$ is continuous, for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(t) - f(0)| < \epsilon$ for $t < \delta$. Also, since $t_n \to t_0$ as $n \to \infty$, for all $\delta > 0$ there exists an M such that $0 < t_n - t_0 < \delta$ for all $n > M$. Thus, if $n > M$ then $t_n < \delta$ and

$$\int_0^{t_n} F_n(t) f(t)\,\mathrm{d}t < \int_0^{t_n} F_n(t)\big(f(0) + \epsilon\big)\,\mathrm{d}t = \big(f(0) + \epsilon\big) \int_0^{t_n} F_n(t)\,\mathrm{d}t = f(0) + \epsilon, \tag{7.178}$$

since the integral of F_n is 1. Similarly

$$\int_0^{t_n} F_n(t) f(t)\,\mathrm{d}t > f(0) - \epsilon. \tag{7.179}$$

Letting $\epsilon \to 0$ proves the result.

Definition: The **Dirac delta function** $\delta(t)$ is a *distribution* or *generalized function* defined by

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,\mathrm{d}t = f(0). \tag{7.180}$$

for all *continuous* functions $f(t)$.

Comments:

1. The Dirac delta function *is not a function.* You can think of $\delta(t)$ as being zero everywhere except at $t = 0$ where it is infinite in such a way that the integral of $\delta(t)$ is one. You can also think of it as the limit of the families of functions $F_n(t)$ discussed above. Neither of these are precise. Its action on continuous functions when integrated against them is what defines it. To learn more about generalized functions see, for example, the book by Lighthill [5].

2. With a change of variables we see that

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\,\mathrm{d}t = f(a). \tag{7.181}$$

7.8.1 Laplace Transform of the Delta function

For $a > 0$ we have

$$\mathcal{L}[\delta(t-a)](s) = \int_0^{\infty} e^{-st}\delta(t-a)\,\mathrm{d}t = e^{-as}. \tag{7.182}$$

This is not true for $a < 0$. Why? For $a = 0$ the transform is defined as

$$\mathcal{L}[\delta(t)](s) = \lim_{a\to 0^+} \mathcal{L}[\delta(t-a)](s) = \lim_{a\to 0^+} e^{-as} = 1. \tag{7.183}$$

Similarly

$$\int_{-\infty}^{t} \delta(t-a)\,\mathrm{d}t = \begin{cases} 0 & t < a; \\ 1 & t > a; \end{cases} \tag{7.184}$$

and we define

$$\int_{-\infty}^{a} \delta(t-a)\,\mathrm{d}t = \lim_{t\to a^+} \int_{-\infty}^{t} \delta(t-a)\,\mathrm{d}t = 1. \tag{7.185}$$

Hence

$$\int_{-\infty}^{t} \delta(t-a)\,\mathrm{d}t = H(t-a) \tag{7.186}$$

and we can define

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t-a) = \delta(t-a). \tag{7.187}$$

Another way to see this result is the following. Consider $H(t-a)$ to be

$$H(t-a) = \lim_{n\to\infty} f_n(t-a) \tag{7.188}$$

where

$$f_n(t) = \begin{cases} 0, & t < 0; \\ nt, & 0 \le t < \frac{1}{n}; \\ 1, & t \ge \frac{1}{n}. \end{cases} \tag{7.189}$$

This is not exact as $\lim_{n\to\infty} f_n(0) = 0$ where as we defined $H(0)$ to be 1. Other than that they are equivalent.

The functions $f_n(t)$ are continuous with a piecewise continuous derivative:

$$\frac{\mathrm{d}f_n}{\mathrm{d}t}(t) = \begin{cases} 0, & t < 0; \\ n, & 0 < t < \frac{1}{n}; \\ 0, & t > \frac{1}{n}. \end{cases} \tag{7.190}$$

This is the first family of functions F_n considered above (see (7.174)) except here the derivative of f_n is not defined at $t = 0$ and $t = 1/n$. For any continuous function $f(t)$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}f_n}{\mathrm{d}t}(t) f(t)\,\mathrm{d}t = \lim_{n\to\infty} \int_0^{1/n} n\,f(t)\,\mathrm{d}t = f(0) \tag{7.191}$$

so we can say that

$$\delta(t) = \lim_{n\to\infty} \frac{\mathrm{d}f_n}{\mathrm{d}t}. \tag{7.192}$$

Assuming we can reverse the order of differentiation and the limit

$$\delta(t) = \lim_{n\to\infty} \frac{\mathrm{d}f_n}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\lim_{n\to\infty} f_n(t)\Big) = \frac{\mathrm{d}}{\mathrm{d}t} H(t). \tag{7.193}$$

Obviously this is very formal and lacks some rigour. A more careful analysis leads to the same results.

7.8.2 An undamped oscillator subjected to an impulsive force

Consider an undamped oscillator. The system is initially at rest in its equilibrium position. An impulsive unit force acts at $t = 0$:

$$\begin{aligned} \ddot{x} + x &= \delta(t) \\ x(0) &= 0, \\ \dot{x}(0) &= 0. \end{aligned} \tag{7.194}$$

Find $x(t)$ for $t \ge 0$.

We will find the solution in two ways.

1. **Solution 1:** First we find the solution $x_n(t)$ when the forcing is given by

$$F_n = \begin{cases} n, & 0 \le t < \frac{1}{n}; \\ 0, & t \ge \frac{1}{n} \end{cases} \tag{7.195}$$

and then take the limit as $n \to \infty$. This is the family functions (7.174) discussed above. Solving

$$\begin{aligned} \ddot{x}_n + x_n &= F_n(t), \\ x_n(0) &= 0, \\ \dot{x}_n(0) &= 0. \end{aligned} \tag{7.196}$$

gives

$$X_n(s) = \frac{\mathcal{L}[F_n](s)}{s^2+1} \tag{7.197}$$

where $X_n = \mathcal{L}[x_n]$ and

$$\mathcal{L}[F_n](s) = \int_0^{1/n} e^{-st} n \,\mathrm{d}t = \frac{n}{s}(1 - e^{-s/n}). \tag{7.198}$$

Thus

$$X_n(s) = \frac{n}{s(s^2+1)} - e^{-s/n}\frac{n}{s(s^2+1)}. \tag{7.199}$$

We now need to find the inverse transforms. First

$$\mathcal{L}^{-1}\Big[\frac{1}{s(s^2+1)}\Big] = \mathcal{L}^{-1}\Big[\frac{1}{s} - \frac{s}{s^2+1}\Big] = 1 - \cos t. \tag{7.200}$$

Thus,

$$\begin{aligned} x_n(t) &= n(1-\cos t) - nH\big(t - \frac{1}{n}\big)\Big(1 - \cos\big(t - \frac{1}{n}\big)\Big) \\ &= \begin{cases} n(1-\cos t), & 0 \le t < \frac{1}{n}; \\ n\Big(\cos\big(t-\frac{1}{n}\big) - \cos t\Big), & t \ge \frac{1}{n}. \end{cases} \end{aligned} \tag{7.201}$$

We can take the limit at $n \to \infty$ to find $x(t)$. Using

$$\begin{aligned} \cos\big(t - \frac{1}{n}\big) &= \cos t \cos\frac{1}{n} + \sin t \sin\frac{1}{n} \\ &= \Big(1 - \frac{1}{2n^2} + O(n^{-4})\Big)\cos t + \Big(\frac{1}{n} - \frac{1}{6n^3} + O(n^{-5})\Big)\sin t \end{aligned} \tag{7.202}$$

we have

$$\begin{aligned} n\Big(\cos\big(t - \frac{1}{n}\big) - \cos t\Big) &= \Big(-\frac{1}{2n} + O(n^{-3})\Big)\cos t + \Big(1 + O(n^{-2})\Big)\sin t \\ &\to \sin t \text{ as } n \to \infty. \end{aligned} \tag{7.203}$$

Hence

$$x(t) = \lim_{n\to\infty} x_n(t) = \sin t. \tag{7.204}$$

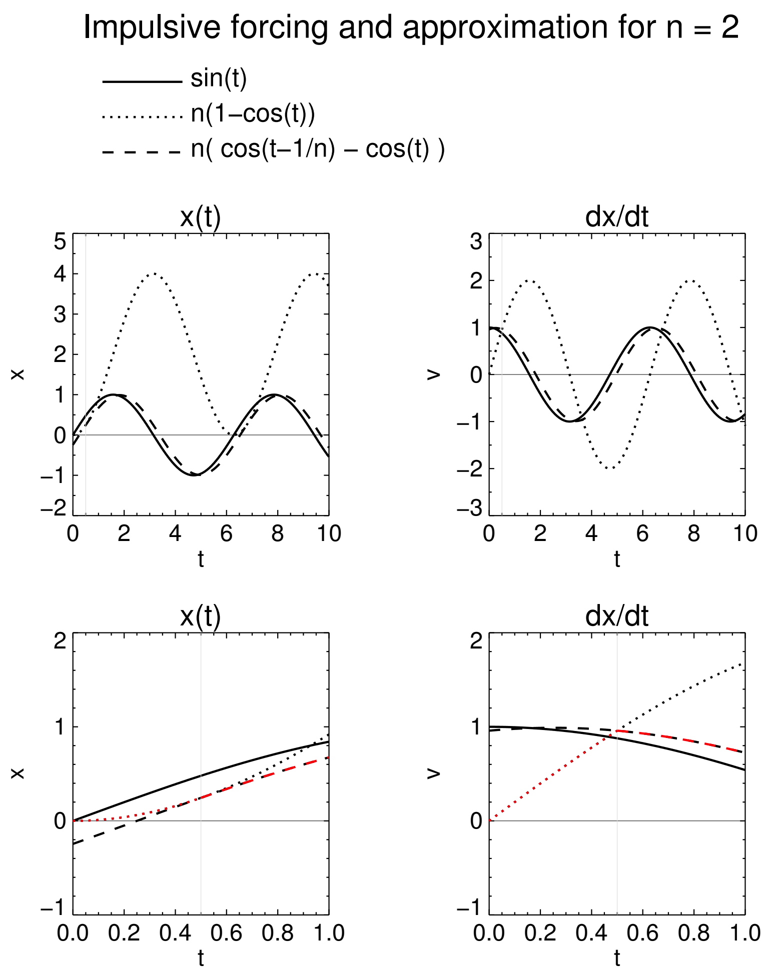

Figure 7.7: Solutions of the impulsively forced oscillator: exact and approximate for $n = 2$. The dotted curves are $n(1 - \cos t)$, the solution of the approximate problem for $0 \le t \le \frac{1}{n} = \frac{1}{2}$. The dashed curves are $n\Big(\cos\big(1 - \frac{1}{n}\big) - \cos t\Big)$, the approximate solution for $t > \frac{1}{n} = \frac{1}{2}$. The solid curve is the solution $\sin t$ when the forcing is modelled by a Dirac delta function. The vertical grey line is $t = \frac{1}{n} = \frac{1}{2}$. The right panels show the corresponding derivatives. Bottom panels show a smaller range of t values and the red shows the solution $x_n(t)$ and $\dot{x}_n(t)$.

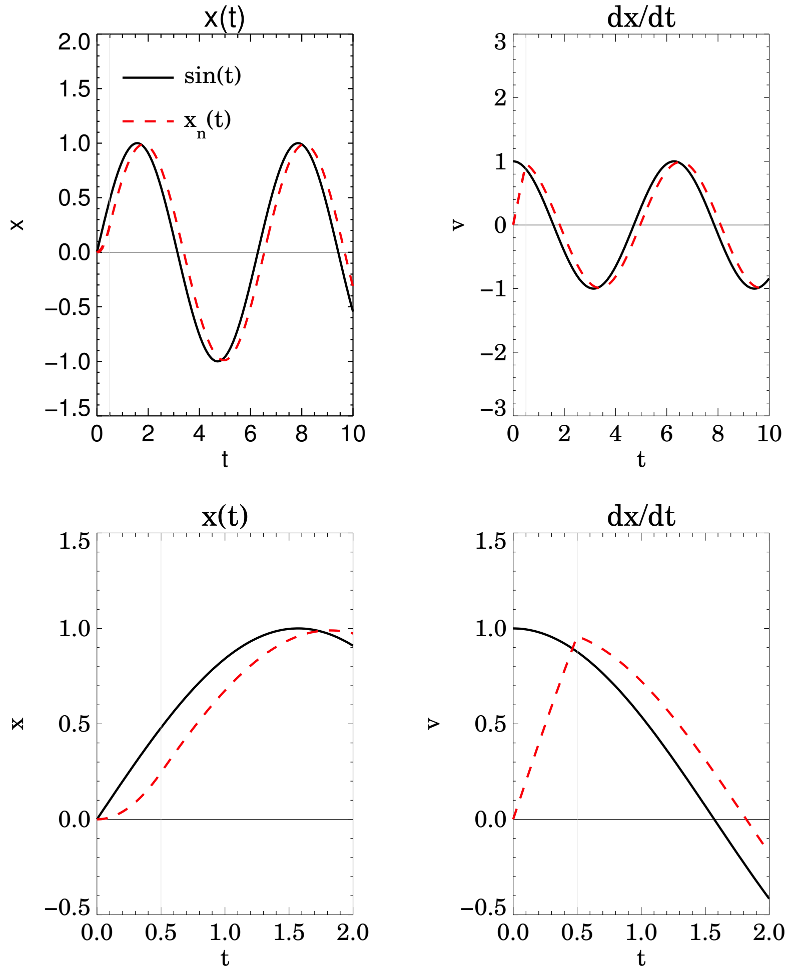

Figure 7.8: Similar to figure 7.7 but only $x(t) = \sin t$ (solid black) and the approximate solution $x_n(t)$ (dashed red) for $n = 2$ are shown.

2. **Solution 2:** Taking the Laplace Transform of the original problem in terms of the delta function we have

$$(s^2 + 1)X(s) = \mathcal{L}[\delta(t)] = 1 \tag{7.205}$$

so

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t. \tag{7.206}$$

Comparing the two solutions the power of using the delta function is compelling. Use of the δ function is only OK if the duration of the forcing is so short that approximating it as a delta function is a good approximation. If that is the case, its use greatly simplifies finding the solution.

The approximate solutions $x_n(t)$ and the exact solution $x(t) = \sin(t)$ are illustrated in Figures 7.7–7.9.

Comments:

1. The solution $x(t) = \sin(t)$ satisfies the initial condition $x(0) = 0$ *but it does not satisfy the second initial condition* $\dot{x}(0) = 0$!

2. The functions $x_n(t)$ and their first derivatives are continuous at $t = 0$ and $1/n$:

$$\dot{x}_n(t) = \begin{cases} 0, & t < 0; \\ n \sin t, & 0 \leq t < \frac{1}{n}; \\ n\Big(\sin t - \sin\big(t - \frac{1}{n}\big)\Big), & t \geq \frac{1}{n}. \end{cases} \tag{7.207}$$

   as can easily be verified. Looking at the equation we solved to find x_n, the forcing function $F_n(t)$ had jump discontinuities so as discussed in an earlier example, $\ddot{x}_n$ has jump discontinuities and x_n and $\dot{x}_n$ are continuous.

3. For large n, on the small interval $0 \leq t \leq 1/n$ we can approximate $\dot{x}(t) = n \sin t$ by $\dot{x}(t) \approx nt$ so $\dot{x}_n(t)$ increases from 0 at $t = 0$ to 1 at $t = 1/n$. In the limit at $n \to \infty$, the derivative undergoes a jump of 1 at $t = 0$. This is why the limiting solution $x(t) = \sin t$ has initial derivative $\dot{x}(0) = 1$.

4. You can think of this in the following way. The equation for the limiting problem with forcing given by the delta function is $\ddot{x}+x = \delta(t)$. The right hand side has a delta function singularity, hence so must the left hand side. This singularity must appear in the term with the highest derivative, i.e., $\ddot{x}$ has a delta function singularity at $t = 0$. Integrating, $\dot{x}$ jumps by one at $t = 0$. This is a reflection of $\delta(t) = \frac{d}{dt}H(t)$.

5. An impulsive forcing results in a jump in $\dot{x}$ at $t = 0$ (instantaneous acceleration) with no change in x.

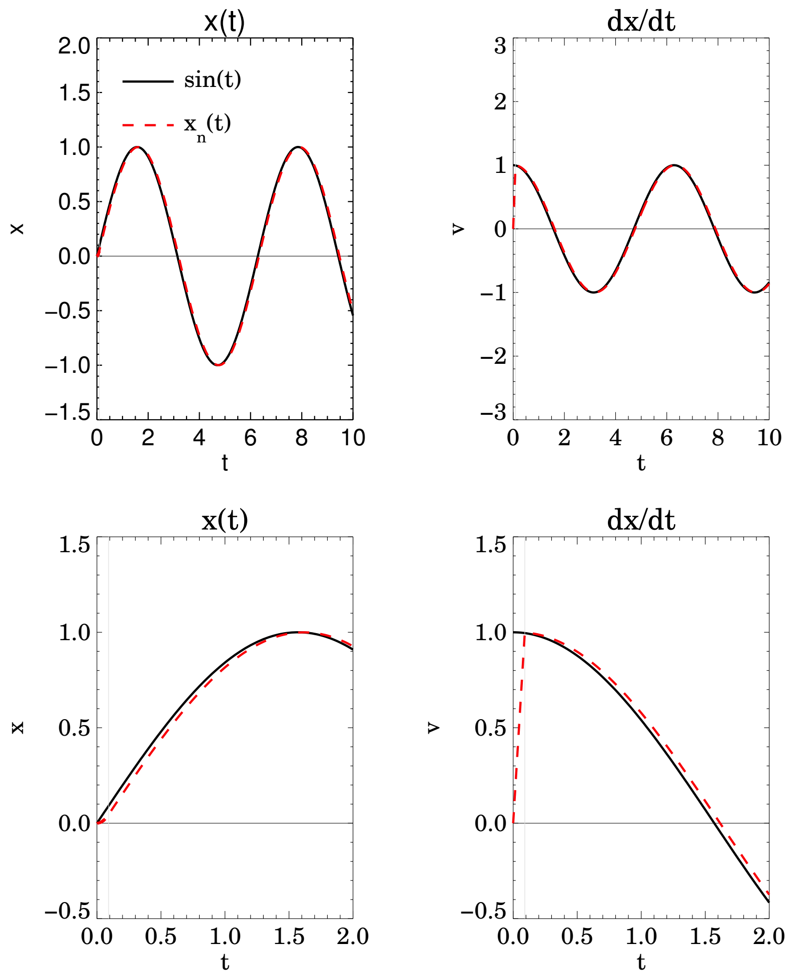

Figure 7.9: Same as figure 7.8 but for $n = 10$.

7.9 Laplace Transform Table

Table 7.1: **Laplace Transforms**

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$e^{at}f(t)$	$\mathcal{L}[f](s-a)$
$\frac{d^n f}{dt^n}$	$s^n\mathcal{L}[f](s) - \sum_{j=0}^{n-1} s^{n-1-j}\frac{d^j f}{dt^j}(0)$
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}\mathcal{L}[f](s)$
$f(t-a)H(t-a)$	$e^{-as}F(s)$
$f * g(t)$	$F(s)G(s)$

Chapter 8

APPENDIX A: Differentiating and Integrating Complex-Valued Functions of a Real Variable

There are a few instances in this course where it will be convenient to integrate and differentiate complex-valued functions of a real variable, in particular the function $f(t) = e^{\lambda t}$ where the constant λ is a complex number while t is a real variable.

8.1 From first principles

Suppose $f(t)$ is a complex-valued function with real and imaginary parts $f_r(t)$ and $f_i(t)$:

$$f(t) = f_r(t) + i f_i(t). \tag{8.1}$$

The derivative of $f(t)$ is defined in the usual way:

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t}(t) &= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \\ &= \lim_{h\to 0} \frac{f_r(t+h) - f_r(t) + i\big(f_i(t+h) - f_i(t)\big)}{h} \\ &= \lim_{h\to 0} \frac{f_r(t+h) - f_r(t)}{h} + i \lim_{h\to 0} \frac{f_i(t+h) - f_i(t)}{h} \\ &= \frac{\mathrm{d}f_r}{\mathrm{d}t}(t) + i \frac{\mathrm{d}f_i}{\mathrm{d}t}(t) \end{aligned} \tag{8.2}$$

To integrate construct the Riemann Sum and take the limit:

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{N\to\infty} \sum_{n=1}^{N} f(t_n)\Delta \tag{8.3}$$

where t_n is a point in $(a + \frac{b-a}{N}(n-1), a + \frac{b-a}{N}n)$ and $\Delta = \frac{b-a}{N}$. Then

$$\begin{aligned}\int_a^b f(t)\,\mathrm{d}t &= \lim_{N\to\infty} \sum_{n=1}^{N} f(t_n)\Delta \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \{f_r(t_n) + i f_i(t_n)\}\Delta \\ &= \lim_{N\to\infty} \Big\{ \sum_{n=1}^{N} f_r(t_n)\Delta + i \sum_{n=1}^{N} f_i(t_n)\Delta \Big\} \\ &= \int_a^b f_r(t)\,\mathrm{d}t + i \int_a^b f_i(t)\,\mathrm{d}t.\end{aligned} \tag{8.4}$$

So to differentiate or integrate a complex-valued function of a real variable simply differentiate and integrate the real and imaginary parts.

Now consider $f(t) = e^{\lambda t}$ where λ is complex with real and imaginary parts λ_r and λ_i. Then

$$e^{\lambda t} = e^{\lambda_r t} e^{i\lambda_i t} = e^{\lambda_r t}\big(\cos(\lambda_i t) + i \sin(\lambda_i t)\big). \tag{8.5}$$

The real part is $f_r(t) = e^{\lambda_r t}\cos(\lambda_i t)$ which has the derivative

$$f_r'(t) = \lambda_r e^{\lambda_r t}\cos(\lambda_i t) - \lambda_i e^{\lambda_r t}\sin(\lambda_i t), \tag{8.6}$$

while the imaginary part is $f_i(t) = e^{\lambda_r t}\sin(\lambda_i t)$ which has the derivative

$$f_i'(t) = \lambda_r e^{\lambda_r t}\sin(\lambda_i t) + \lambda_i e^{\lambda_r t}\cos(\lambda_i t). \tag{8.7}$$

Hence

$$\begin{aligned} f'(t) &= f_r'(t) + i f_i'(t) \\ &= \Big(\lambda_r \cos(\lambda_i t) - \lambda_i \sin(\lambda_i t)\Big)e^{\lambda_r t} + i\Big(\lambda_r \sin(\lambda_i t) + \lambda_i \cos(\lambda_i t)\Big)e^{\lambda_r t} \\ &= \Bigg[\Big(\lambda_r \cos(\lambda_i t) - \lambda_i \sin(\lambda_i t)\Big) + i\Big(\lambda_r \sin(\lambda_i t) + \lambda_i \cos(\lambda_i t)\Big)\Bigg]e^{\lambda_r t} \\ &= \Bigg[\Big(\lambda_r + i\lambda_i\Big)\Big(\cos(\lambda_i t) + i\sin(\lambda_i t)\Big)\Bigg]e^{\lambda_r t} \\ &= \lambda e^{i\lambda_i t} e^{\lambda_r t} \\ &= \lambda e^{\lambda t}. \end{aligned} \tag{8.8}$$

We have shown that

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\lambda t} = \lambda e^{\lambda t} \tag{8.9}$$

for all complex numbers λ. In a future complex analysis course you will learn that this is true for complex-valued t as well.

Now for integration. To integrate $e^{\lambda t}$ you can integrate the real and imaginary parts.

$$\int^t f(t')\,\mathrm{d}t' = \int^t f_r(t')\,\mathrm{d}t' + i\int^t f_i(t')\,\mathrm{d}t'. \tag{8.10}$$

For $f(t) = e^{\lambda t}$ this gives

$$\int^t e^{\lambda t'}\,\mathrm{d}t' = \int^t e^{\lambda t'}\cos(\lambda_i t')\,\mathrm{d}t' + i\int^t e^{\lambda t'}\sin(\lambda_i t')\,\mathrm{d}t'. \tag{8.11}$$

You can now integrate the real and imaginary pieces twice by parts to find the integrals of $e^{\lambda_r t}\cos(\lambda_i t)$ and $e^{\lambda_r t}\sin(\lambda_i t)$. Alternatively, let

$$F(t) = \int^t f(t')\,\mathrm{d}t'. \tag{8.12}$$

The function $F(t)$ has real and imaginary parts $F_r(t)$ and $F_i(t)$ given by

$$\begin{aligned} F_r(t) &= \int^t f_r(t')\,\mathrm{d}t' \\ F_i(t) &= \int^t f_i(t')\,\mathrm{d}t' \end{aligned} \tag{8.13}$$

as shown above. Hence the derivative of $F(t)$ is $F'(t) = F_r'(t) + iF_i'(t) = f_r(t) + if_i(t)$ as one should expect. From this we can deduce that

$$\int^t e^{\lambda t'}\,\mathrm{d}t' = \frac{e^{\lambda t}}{\lambda} \tag{8.14}$$

because

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{e^{\lambda t}}{\lambda} = \frac{1}{\lambda}\frac{\mathrm{d}}{\mathrm{d}t}e^{\lambda t} = e^{\lambda t}. \tag{8.15}$$

8.2 Power series approach

There is an alternative approach using power series. For real-valued x you have seen that the exponential function e^x has the power-series expansion

$$\sum_{n=0}^{\infty}\frac{x^n}{n!} \tag{8.16}$$

which converges for all x. The function e^x can be extended to the complex plane using this power series:

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots = \sum_{n=1}^{\infty}\frac{z^n}{n!} \tag{8.17}$$

for complex-valued z. This works because for any integers $M > N$,

$$\Big|\sum_{n=N}^{M}\frac{z^n}{n!}\Big| \le |\sum_{n=N}^{M}\frac{|z|^n}{n!} \tag{8.18}$$

which goes to zero as $N \to \infty$ because $x = |z|$ is real and the power-series for e^x converges absolutely.

For the function $f(t) = e^{\lambda t}$ with t real we have

$$e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2 t^2}{2} + \frac{\lambda^3 t^3}{3!} + \cdots, \tag{8.19}$$

and differentiating term-by-term we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} e^{\lambda t} &= \lambda + \frac{\lambda^2 2t}{2} + \frac{\lambda^3 3t^2}{3!} + \cdots \\
&= \lambda\Big(1 + \lambda t + \frac{\lambda^2 t^2}{2} + \cdots\Big) \\
&= \lambda e^{\lambda t}.
\end{aligned} \tag{8.20}$$

Similarly we can integrate term-by-term

$$\begin{aligned}
\int^t e^{\lambda t'}\,\mathrm{d}t' &= t + \frac{\lambda t^2}{2} + \frac{\lambda^2 t^3}{3!} + \frac{\lambda^3 t^4}{4!} + \cdots \\
&= \frac{1}{\lambda}\Big(\lambda t + \frac{\lambda^2 t^2}{2} + \frac{\lambda^3 t^3}{3!} + \cdots\Big) \\
&= \frac{1}{\lambda}\Big(e^{\lambda t} - 1\Big).
\end{aligned} \tag{8.21}$$

Since an indefinite integral is defined up to an arbitrary constant we can take

$$\int^t e^{\lambda t'}\,\mathrm{d}t' = \frac{e^{\lambda t}}{\lambda}. \tag{8.22}$$

We could have used a definite integral to integrate:

$$\begin{aligned}
\int_{t_0}^t e^{\lambda t'}\,\mathrm{d}t' &= (t - t_0) + \frac{\lambda t^2}{2} - \frac{\lambda t_0^2}{2} + \frac{\lambda^2 t^3}{3!} - \frac{\lambda^2 t_0^3}{3!} + \cdots \\
&= \frac{1}{\lambda}\Big(\lambda t + \frac{\lambda^2 t^2}{2} + \frac{\lambda^3 t^3}{3!} + \cdots\Big) \\
&\qquad - \frac{1}{\lambda}\Big(\lambda t_0 + \frac{\lambda^2 t_0^2}{2} + \frac{\lambda^3 t_0^3}{3!} + \cdots\Big) \\
&= \frac{1}{\lambda}\Big(e^{\lambda t} - 1\Big) - \frac{1}{\lambda}\Big(e^{\lambda t_0} - 1\Big) \\
&= \frac{1}{\lambda} e^{\lambda t} - \frac{1}{\lambda} e^{\lambda t_0}.
\end{aligned} \tag{8.23}$$

Chapter 9

APPENDIX B: Equality of mixed partial derivatives

Here we prove the following theorem:

Theorem IV: *Let $f(x, y)$ be a scalar field such that the partial derivatives f_x, f_y, and f_{yx} exist on an open rectangle R. Then if f_{yx} is continuous on R the derivative f_{xy} exists and $f_{xy} = f_{yx}$ on R.*

The proof is based on [1] where you can find a more concise treatment.

We start with a weaker version in which we first assume that both f_{yx} and f_{xy} exist and are continuous on R and show that they are then equal.

Theorem IVa: *Let $f(x, y)$ be a scalar field such that the partial derivatives f_x, f_y, and f_{yx} exist on an open rectangle R. Then if f_{yx} and f_{xy} are both continuous on R then $f_{xy} = f_{yx}$ on R.*

Proof: We show that $f_{xy}(a, b) = f_{yx}(a, b)$ at an arbitrary point (a, b) in R. Since R is open we can choose $h > 0$ and $k > 0$ such that the points $(a + h, b)$, $(a, b + k)$ and $(a + h, b + k)$ are also in R. These points are the corners of a rectangle R_{hk} in R. Consider

$$\Delta(h, k) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b) \tag{9.1}$$

which we can write as

$$\Delta(h, k) = [f(a + h, b + k) - f(a + h, b)] - [f(a, b + k) - f(a, b)] \tag{9.2}$$

or

$$\Delta(h, k) = [f(a + h, b + k) - f(a, b + k)] - [f(a + h, b) - f(a, b)] \tag{9.3}$$

We introduce two new functions:

$$G(x) = f(x, b + k) - f(x, b) \tag{9.4}$$

and

$$H(y) = f(a + h, y) - f(a, y). \tag{9.5}$$

$G(x)$ is the change in f between the top and bottom of the rectangle R_{hk} (see figure 9.1) and $H(y)$ is the change between the right and left sides of R_{hk}. In terms of these functions

$$\Delta(h, k) = G(a + h) - G(a) \tag{9.6}$$

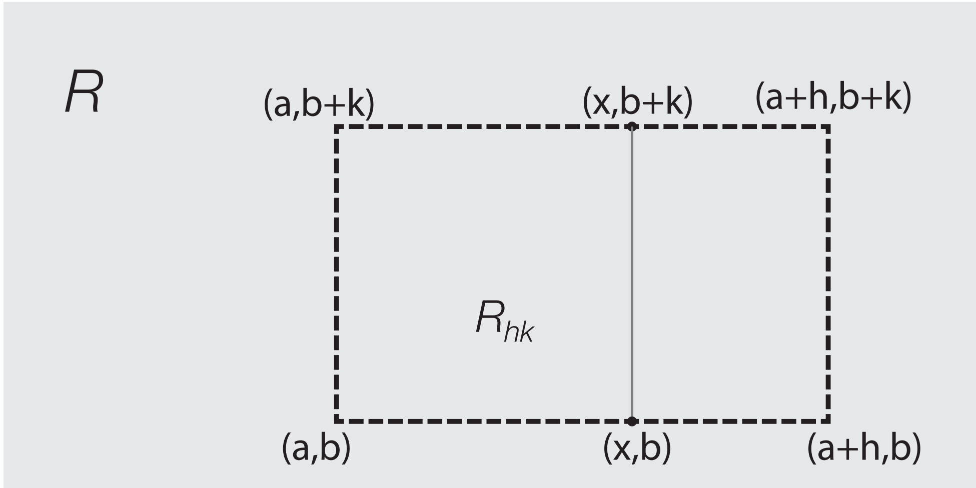

Figure 9.1: The open rectangle R (grey) containing the point (a, b) and the rectangle R_{hk}. The function $G(x)$ is the difference of f between the top and bottom of the rectangle along a vertical line at x.

and

$$\Delta(h, k) = H(b + k) - H(b). \tag{9.7}$$

Since f_x is continuous on R we know that $G'(x)$ is continuous on $[a, a + h]$. Hence by the mean value theorem from (9.6) we have

$$\Delta(h, k) = hG'(x_1) \tag{9.8}$$

for some x_1 in $[a+h, a]$. Note that x_1 depends not only h, but also on k as our function G defined in (9.4) does (they also depend on a and b but we are treating them as fixed constants so ignore this dependence). Using (9.4) we can expand this as

$$\Delta(h, k) = h\Big[f_x(x_1, b + k) - f_x(x_1, b)\Big]. \tag{9.9}$$

Now since $f_{xy}(x_1, y)$ is continuous on $y \in [b, b + k]$ the mean value theorem also gives us

$$f_x(x_1, b + k) - f_x(x_1, b) = kf_{xy}(x_1, y_1) \tag{9.10}$$

for some y_1 in $[b, b + k]$. Thus we have

$$\Delta(h, k) = hkf_{xy}(x_1, y_1). \tag{9.11}$$

Note that y_1 depends on h and k too.

In a similar manner

$$\Delta(h, k) = kH'(y_2) = k\Big[f_y(a + h, y_2) - f_y(a, y_2)\Big] = hkf_{yx}(x_2, y_2) \tag{9.12}$$

for some $x_2 \in [a, a+h]$ and $y_2 \in [b, b+k]$. We now let $(h, k) \to (0, 0)$. Both (x_1, y_1) and (x_2, y_2) go to (a, b) and from (9.11) and (9.12) we get

$$f_{xy}(a, b) = f_{yx}(a, b). \tag{9.13}$$

This step uses continuity of f_{xy} and f_{yx}.

We now turn to the stronger theorem which does not assume the existence of f_{xy}.

Theorem IV: *Let $f(x, y)$ be a scalar field such that the partial derivatives f_x, f_y, and f_{yx} exist on an open rectangle R. Then if f_{yx} is continuous on R the derivative f_{xy} exists and $f_{xy} = f_{yx}$ on R.*

Proof: Since f_{yx} is continuous (9.12) still holds. Equation (9.8) holds as well but because we don't know that f_{xy} exists we no longer have (9.11). We need to prove that f_{xy} exists and is continuous. Then the rest of the proof follows. We prove $f_{xy}(a, b)$ exists which proves it exists in R as (a, b) is an arbitrary point in R. From the definition of derivatives

$$f_{xy}(a, b) = \lim_{k\to 0} \frac{f_x(a, b + k) - f_x(a, b)}{k} \tag{9.14}$$

if the limit exists. Hence, we need to prove that this limit exists and that it is equal to $f_{yx}(a, b)$. Now

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h} \tag{9.15}$$

and

$$f_x(a, b + k) = \lim_{h\to 0} \frac{f(a + h, b + k) - f(a, b + k)}{h} \tag{9.16}$$

so

$$\begin{aligned} \frac{f_x(a, b + k) - f_x(a, b)}{k} &= \lim_{h\to 0} \frac{[f(a + h, b + k) - f(a, b + k)] - [f(a + h, b) - f(a, b)]}{hk} \\ &= \lim_{h\to 0} \frac{\Delta(h, k)}{hk}. \end{aligned} \tag{9.17}$$

This limit exists because we know that f_x exists.

Now recall (9.12) which still holds. Writing it as

$$f_{yx}(x_2, y_2) = \frac{\Delta(h, k)}{hk} \tag{9.18}$$

and using it in (9.17) gives

$$\frac{f_x(a, b + k) - f_x(a, b)}{k} = \lim_{h\to 0} f_{yx}(x_2, y_2). \tag{9.19}$$

Taking the limit as $k \to 0$ gives

$$\lim_{k\to 0} \frac{f_x(a, b + k) - f_x(a, b)}{k} = \lim_{k\to 0}\bigl\{\lim_{h\to 0} f_{yx}(x_2, y_2)\bigr\}. \tag{9.20}$$

If the limit on the right hand side exists and is equal to $f_{yx}(a, b)$ then the limit on left hand side exists and, from (9.14), we have $f_{xy}(a, b) = f_{yx}(a, b)$ and we are done.

From (9.19) we know that

$$\lim_{h\to 0} f_{yx}(x_2, y_2) \tag{9.21}$$

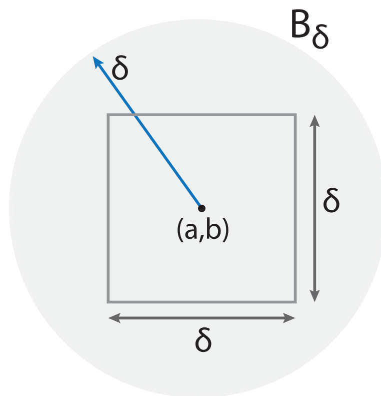

Figure 9.2: B_δ, the ball of radius δ centred at the point (a,b). The point $(a+h, b+k)$ is inside the square with sides of length δ centred at (a,b).

exits. The limiting value depends on k. Call it $V(k)$.

We need to show that $\lim_{k\to 0} V(k) = f_{yx}(a,b)$. For this we use the assumed continuity of f_{yx} on R. Let B_δ be a disk of radius δ centred on (a,b). For any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f_{yx}(x,y) - f_{yx}(a,b)| < \frac{\epsilon}{2} \text{ for } (x,y) \in B_\delta \tag{9.22}$$

Now assume $|h| < \delta/2$ and $|k| < \delta/2$, in which case the whole rectangle R_{kh} is in B_δ (Figure 9.2). In particular (x_2, y_2) is in B_δ hence

$$|f_{yx}(x_2,y_2) - f_{yx}(a,b)| < \frac{\epsilon}{2}. \tag{9.23}$$

Now let $h \to 0$ giving

$$|V(k) - f_{yx}(a,b)| \le \frac{\epsilon}{2} < \epsilon \tag{9.24}$$

provided $|k| \le \delta/2$. Hence we have shown that for any $\epsilon > 0$ there is a $\delta > 0$ such that $|V(k) - f_{yx}(a,b)| < \epsilon$ if $|k| < \delta/2$ which means that $\lim_{k\to 0} V(k)$ exists and is equal to $f_{yx}(a,b)$. This completes the proof.

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