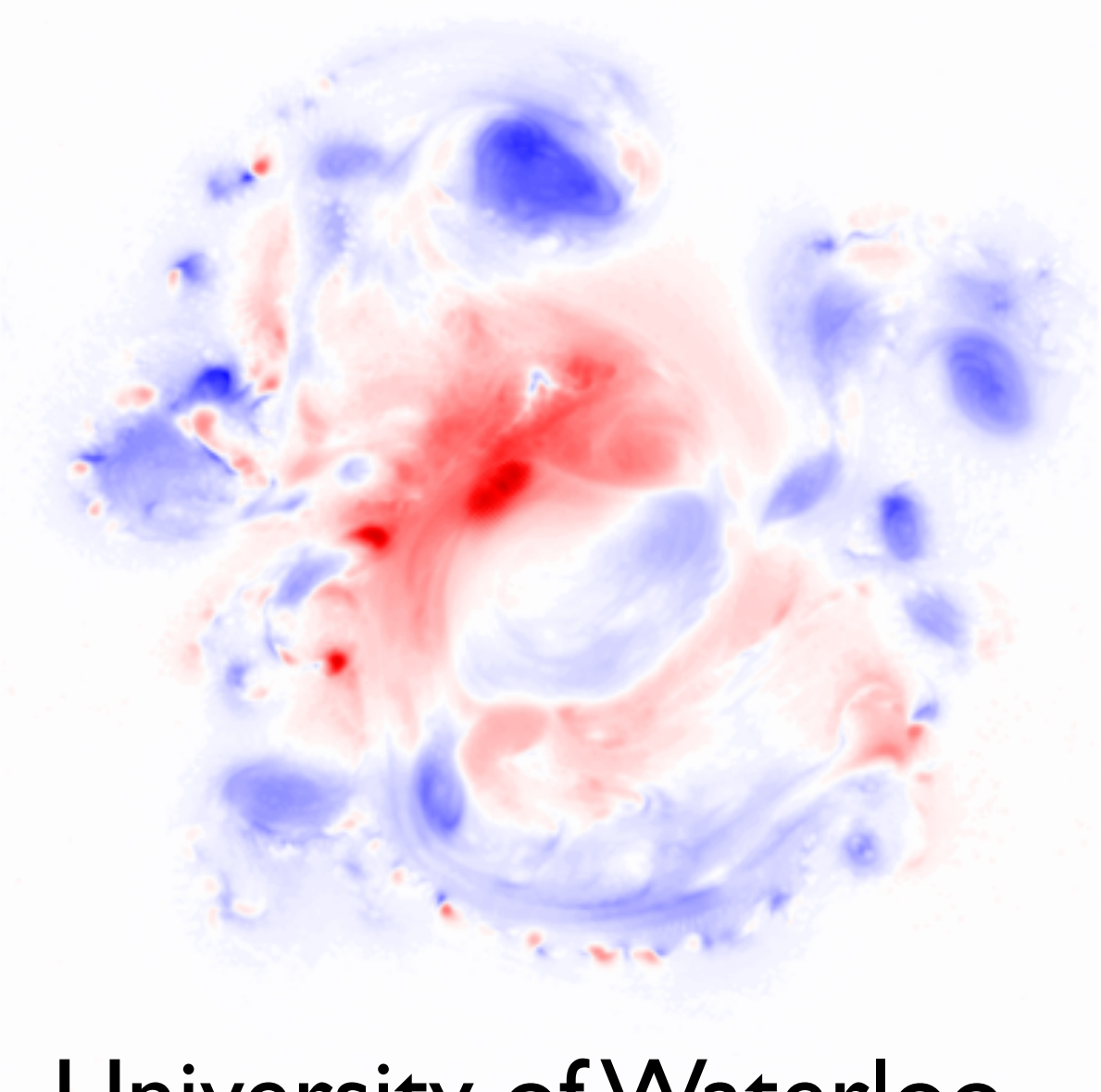
AMATH 353 S2019 Course Notes

# Partial Differential Equations I

## Methods in Mathematical Modelling



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*Everything Should Be Made as Simple as Possible, But Not Simpler.*
Albert Einstein.

# Preface

THESE COURSE NOTES were developed for AMATH 353 : Partial Differential Equations 1 at the University of Waterloo. They were first written in the Spring of 2012 Semester but have been modified much since then. I would like to thank Jenn Hernes and Ghazal Geshnizjani for their helpful comments in fixing up the manuscript. Also, I would like to thank Prof. Geoff Vallis for sharing this latex template with me.

# Part I

# MODELLING

CHAPTER
# ONE

# Modelling with PDEs

## 1.1 INTRODUCTION

A *Partial Differential Equation*, or PDE, is an equation that relates a function, say $u(x,t)$ with its partial derivatives,

$$F(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0.$$

This arises because often it is easier to use basic principles to build a model that describes a process using partial derivatives and relates them in some way. One example of a principle we will discuss shortly is a *conservation law*. Then, using various analytical (or numerical) techniques we can obtain exact or approximate solutions.

In the above equation the independent variables are $x$, for space, and $t$, for time. In general there can be more than two independent variables if we are in a higher dimension and we could have two or more dependent variables, say $u(x,t)$ and $v(x,t)$. If there is any hope of finding a unique solution we need as many equations as we have dependent variables, otherwise the system is said to be underdetermined. Also, to find a solution certain conditions need to be imposed like at the boundary (if there is one) and/or at the initial time (if we are looking at solutions that change in time). It is for this reason that these equations are sometimes known as Initial Boundary Value Problems (IBVPs for short). Note that we do not need a time like variable and will sometimes have only spatial variables. In that case the PDE is more properly described as a Boundary Value Problem (BVP). This is to be contrasted with Initial Value Problems (IVPs) that you studied extensively in AMATH 250/251.

The PDE or system of PDEs is said to be linear if $F(u(x,t))$ has a linear dependency on $u(x,t)$ and its partial derivatives. If it depends nonlinearly on any one of terms involving $u(x,t)$, then it is said to be nonlinear. The vast majority of PDEs that are derived to describe some aspect of nature are in fact nonlinear and are incredibly difficult, if not impossible, to solve exactly. Under certain assumptions we can find linear equations that approximate the true behaviour,

and these are more easily solved. Better to find an approximate solution than no solution at all! The reason why there is such a vast difference between linear and nonlinear equations is that if we have two linear solutions to a homogeneous PDE, then their superposition is also a solution. This is not the case with nonlinear equations. In this course we present mostly the linear theory because it is much more well established, but we do delve into nonlinear equations where we can, since they are of universal importance.

## 1.2 CONSERVATION LAWS

*[This section is based primarily on section 1.2 of Logan (2008)]*

Hundreds of years ago it was believed that matter could spontaneously generate itself. Today it is well accepted that you don't get something from nothing. That is to say that if you are measuring a field, say heat, mass of a chemical or pretty much anything else, that the the total amount in a given region can only change by the "stuff" moving out and/or in. In this section we derive a mathematical equation that says exactly this in one spatial dimension since it begins to illustrate how we use mathematics to describe the world. Then we show how we can derive a similar equation in higher dimension. This is more complicated in that it involves using our knowledge of vector calculus. If you are feeling uncomfortable with this then I suggest you brush up on the material from AMATH 231 (Vector Calculus) right away.

### 1.2.1 One-Dimension

A *conservation law* is a physical principle that when written mathematically can be expressed either as an *integral (global) or differential (local) equation*. This is also called a *governing equation* since it forces a constraint on the field that perhaps lets us determine how the solution changes in time and space. That is if we can solve the equation, which is not always the case. In this course I want you to learn not only how to solve PDEs but how to derive them since equations don't simply fall from the sky. Often we must model the phenomenon that we are interested in before we can begin to solve the equations.

**Fig. 1.1** A cylindrical domain.

To begin consider a long but very thin cylindrical container that is pointed in the $x$-direction. The thinness of the rod is necessary to guarantee that the properties of the rod do not vary in the cross section but can vary along the rod. Suppose there is a quantity (or field) $u(x,t)$ that can change along the rod or with time but does not change with distance from the centre. For concreteness let us say that $u(x,t)$ is the density or concentration that measures the amount of something per unit volume. If we are talking about a chemical than the units would be mass/length$^3$ for example. If the cross section of the tube is uniform, say $A$ (units of length$^2$), then the amount of the substance in the interval $I$, between $x = a$ and $x = b$, can be written as the following integral,

$$\text{Total amount of quantity in } I \equiv \int_a^b u(x,t)A\,dx.$$

Note that the integrand is a *global property* because it sums up a property of the material over a length. We assume that $u(x,t)A$ that has units of $(\text{mass}/\text{length}^3)\cdot\text{length}^2 = \text{mass}/\text{length}$. More generally this can be written as,

$$[uA] = \frac{\text{amount}}{\text{volume}}\cdot\text{area} = \frac{\text{amount}}{\text{length}}.$$

If we integrate with respect to $x$ then we are multiplying by a distance and therefore the final units is the amount of $u$.

In the absence of any flux in or out of the domain the total mass (or amount in general) over the entire domain must necessarily be conserved with respect to time. However, if there is a connection between the rod and the exterior world then the amount of the substance will increase if there is some material coming in from the left or the right ends of the interval. Conversely, the amount of the substance will decrease if the substance is moving out from either the left or right ends of the interval. In these two cases we say there is a *flux* in and out of the system, respectively.

To write this down mathematically we need to define *the flux*, which we denote with $\phi(x,t)$, of $u$ at position $x$ and time $t$. The dimensions of the flux are equal to the dimensions of $u$ times length per time, which is $\text{mass}/(\text{length}^2\cdot\text{time})$ in our example but in general using the standard convention,

$$[\phi] = \frac{\text{amount}}{\text{volume}}\cdot\frac{\text{length}}{\text{time}} = \frac{\text{amount}}{\text{area}\cdot\text{time}}.$$

That is to say the flux is the amount of the substance per unit area and per time. We assume the convention that $\phi(x,t)$ is positive (negative) means that the flow at $x$ is in the positive (negative) $x$-direction. Therefore, at a particular time $t$, the rate at which the quantity is flowing into the interval is,

$$\text{Net rate that the quantity flows into } I \equiv A\phi(a,t) - A\phi(b,t).$$

This quantity has units of amount/time.

In addition to the quantity changing because of transport there is also the possibility that it is being added (there is a source) or it is being removed (there is a sink) at a given location at a given time. To describe this mathematically we define $f(x,t,u)$ to be the *source function* that describes how much of $u$ is generated per volume per time,

$$[f] = \frac{\text{amount}}{\text{volume} \cdot \text{time}}.$$

One can also imagine a case where the source depends on the derivatives but we assume the simpler case here. Therefore, the rate at which $u$ is created or destroyed at $x$ and time $t$ can be written as,

$$\text{Rate that the quantity is produced in } I \equiv \int_a^b f(x,t,u(x,t))A\,dx,$$

which also must have units of amount per time.
In words, the fundamental conservation law states that

$$\boxed{\text{Net rate of change of the amount in } I} = \boxed{\text{Net increase by transport into } I} + \boxed{\text{Rate at which the amount is produced in } I}.$$

Intuitively this should be easy to understand. Given all the work we have done we are now able to state this rather succinctly in the language of mathematics as,

$$\boxed{\frac{d}{dt}\int_a^b u(x,t)\,dx = \phi(a,t) - \phi(b,t) + \int_a^b f(x,t,u(x,t))\,dx}, \tag{1.1}$$

after dividing by $A$ since it is constant throughout. This equation says that the rate of change of the total amount of $u$ in the interval $I$ in time is equal to the sum of the rate at which $u$ is produced by transport and sources. This equation is an example of a *global conservation law* in one spatial dimension. Note that this equation does not require that $u(x,t)$ is differentiable in space since we only integrate it with respect to $x$.

Global properties are very helpful to learn about the dynamics of the whole system. However, if we want to know what is going on at every position we need to obtain a local or differential version. If we make certain assumptions on $u(x,t)$ then we can take this integral equation into a differential equation. In particular, we require that its first order partial derivatives are continuous. Also, we recall the two parts of the Fundamental Theorem of Calculus (see MATH 127/137/147 for details) which state,

$$\int_a^b \frac{\partial \phi}{\partial x}\, dx = \phi(b,t) - \phi(a,t), \tag{1.2}$$

$$\frac{d}{dt}\int_a^b u(x,t)\, dx = \int_a^b \frac{\partial u}{\partial t}\, dx. \tag{1.3}$$

If we use the second part of the theorem on the left of equation (1.1) and the first part of the theorem on the right hand side of the equation, we get after moving everything to the same side

$$\int_a^b \frac{\partial u}{\partial t}(x,t)\, dx = -\int_a^b \frac{\partial \phi}{\partial x}(x,t)\, dx + \int_a^b f(x,t,u(x,t))\, dx, \quad \text{or}$$
$$\int_a^b \left[\frac{\partial u}{\partial t}(x,t) + \frac{\partial \phi}{\partial x}(x,t) - f(x,t,u(x,t))\right] dx = 0.$$

But if this is true for all intervals $I = [a,b]$ then we require that the integrand is exactly zero everywhere and for all time.

To see why this is true you can argue this by contradiction. If it is non-zero somewhere then we can pick an interval that would violate the above equation. Therefore, using our *global conservation law* we deduce a *local conservation law* which is in the form of a differential equation,

$$\boxed{\frac{\partial u}{\partial t}(x,t) + \frac{\partial \phi}{\partial x}(x,t) = f(x,t,u(x,t))}, \tag{1.4}$$

for all $x$ and $t$ in our domain and interval of time in question. This equation shows that the local flux of $u$ in time must be balanced by the rate of change of the flux in space and the source term. Often $\partial\phi/\partial x$ is referred to as the flux in the equation since it is due to the flux across the domain. Also, the source term $f$ is called a *reaction term* (in chemical contexts) or growth or interaction (in biological contexts).

Observe that this is an example of a PDE but it is not closed in the sense that we do not know the relationship between the flux, $\phi$, and the dependent variable, $u$. This relationship is referred to as a *constitutive relation* and when specified it then yields a closed system that perhaps we can solve.

If our solution is smooth then equations (1.1) and (1.4) are equivalent. However, if there is a discontinuity in the solution (as we will see can occur in some problems), the second equation is no longer valid but the first is. Therefore, the global equation is more general but it is harder

to solve and that is why we usually prefer the local expression. Regardless, the latter are the ones that are almost always considered in courses and research.

### 1.2.2 Higher-Dimensions

If we want to consider a higher-dimensional problem we can still begin with equation (1.2.1) but now we must figure out how to describe each term mathematically. To do this in say three-dimensions let us suppose that any point in space is described as $(x, y, z)$. Then, the density must be written as $u(x, y, z, t)$, or $u(\vec{x}, t)$ for brevity. Let the volume in question be denoted by $V$ that has a smooth boundary $\partial V$. Using our knowledge of multivariable calculus we can write the total amount of the substance as,

$$\text{Total amount in } V \equiv \iiint_V u(x, y, z, t)\, dV. \tag{1.5}$$

Note that $dV = dxdydz$ is the volume element in three dimensions.

A second term is the source function, $f(x, y, z, t, u)$. It is basically the same as in the one-dimensional example but here we must also integrate over the entire volume,

$$\text{Rate that } u \text{ is produced by source in } V \equiv \iiint_V f(x, y, z, t, u)\, dV. \tag{1.6}$$

The third and final term that we must deal with is the flux. This is more complicated because now the flux must have a magnitude and a direction, $\vec{\phi}(\vec{x}, t)$. Following convention we use $\hat{n}(\vec{x})$ to denote the unit outwards normal of the boundary at the position $\vec{x}$ and time $t$. Therefore, the outward flux through the boundary is,

$$\text{Net outward flux through } \partial V \equiv \iint_{\partial V} \vec{\phi}(\vec{x}, t) \cdot \hat{n}(\vec{x}, t)\, dA. \tag{1.7}$$

If we combine these three equations we can rewrite equation (1.2.1) as,

$$\frac{d}{dt} \iiint_V u(x, y, z, t)\, dV = -\iint_{\partial V} \vec{\phi}(\vec{x}, t) \cdot \hat{n}(\vec{x}, t)\, dA + \iiint_V f(x, y, z, t, u)\, dV. \tag{1.8}$$

This is the *global conservation* law that governs our system and again does not require that the solution or the flux are smooth. If we furthermore assume that the solution is smooth then we can derive a PDE. To do this in a useful manner we must first recall the *Divergence theorem* which allows us to rewrite the flux as,

$$\iiint_V \vec{\nabla} \cdot \vec{\phi}\, dV = \iint_{\partial V} \vec{\phi}(\vec{x}, t) \cdot \hat{n}(\vec{x}, t)\, dA.$$

Using this we can rewrite the global conservation law as

$$\frac{d}{dt} \iiint_V u(x, y, z, t)\, dV = -\iiint_V \vec{\nabla} \cdot \vec{\phi}\, dV + \iiint_V f(x, y, z, t, u)\, dV. \tag{1.9}$$

These integrals can be combined to yield,

$$\iiint_V \left[ \frac{\partial u}{\partial t}(x, y, z, t) + \vec{\nabla} \cdot \vec{\phi} - f(x, y, z, t, u) \right] dV = 0. \tag{1.10}$$

In this form we can apply the same argument as before that for this to be true for any arbitrary volume we need that the integrand is identically equal to zero. This yields our local or differential form of the conservation law,

$$\boxed{\frac{\partial u}{\partial t}(x, y, z, t) + \vec{\nabla} \cdot \vec{\phi} = f(x, y, z, t, u)}. \tag{1.11}$$

If we compare this with equation (1.4) then we see how this is the natural higher-dimensional extension of what we saw previously.

## 1.3 CONSTITUTIVE RELATIONS

*[This section is based primarily on section 1.3 of Logan (2008)]*

Equations (1.4) and (1.11) are each examples of PDEs where the unknowns are the state variable $u$ and the flux $\phi$, which must be expressed in terms of $u$ and possibly space and time. The *constitutive relation* or *equation of state* is a relationship that specifies this precisely. The most general way to write it for our purposes would be,

$$\phi(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}) = 0.$$

In some sense it is the backbone of the model in that it gives the equation its personality and its form. The fundamental law of conservation is true for many cases and is correct given the assumptions that we made. The constitutive relation is something obtained by empirical observations as to how things tend to flow for a particular problem. In this section we discuss several classical choices that gives rise to famous equations that we will solve throughout the course.

### 1.3.1 Diffusion Equation

In the one-dimensional case, in the absence of any sources the conservation law becomes,

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = 0,$$

for all $x$ in the domain and $t > 0$. We observe that if we have a concentration of a chemical, molecular motion tends to redistribute the chemical and eventually yield a uniform distribution of the substance. The rate at which things are transported depends on the gradient of the density and it always goes from high density to low density. This idea is well encompassed in *Fick's law* which assumes that,

$$\phi(x, t) = -D\frac{\partial u}{\partial x}(x, t), \tag{1.12}$$

where $D$ is the diffusion constant that must have units of,

$$[D] = \frac{\text{length}^2}{\text{time}}.$$

This law is not something that we can derive but is something that is observed to be true in many physical, biological and chemical processes.

If we substitute the above constitutive law into the conservation principle than we get a PDE that is only in terms of one unknown variable, namely $u(x, t)$,

$$\boxed{\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}}. \tag{1.13}$$

This shows that for the solution to evolve it is necessary that the concavity in the solution is non-zero. For example if we have a minimum (maximum) the concavity is positive (negative) and locally the solution will tend to increase (decrease). It does so until $u$ is a linear function of space everywhere. When this occurs we have realized a steady solution.

**Question:** What would the equation look like if diffusion varied in space, i.e. $D(x)$?

Even though we don't know how the solution is going to change by looking at the equation we can learn about the behaviour of the solution. If the solution changes in time then it is the diffusion constant and a typical length scale that will determine the rate at which things change. This can be obtained by asking a rather simple question: how do we get the units of time by combining diffusion, $D$, and length, $L$,? The only way to do this is as follows,

$$\boxed{T = \frac{L^2}{D}}.$$

Thus, we expect the time scale of the solution to be the length scale squared divided by the diffusion rate. The larger the length the larger the time scale. The larger the diffusion, the shorter the time scale. When we find an analytical solution to the diffusion equation we will verify what we have established here through a heuristic argument. Furthermore, we will explain why diffusion acts different on different length scales.

### 1.3.2 Heat Equation

Suppose that $u = u(x,t)$ is the thermal energy density (units of energy per volume) in a heat-conducting object. From thermodynamics we have that,

$$u = \rho C T,$$

where $\rho$ is the mass density (mass per unit volume), $C$ is the specific heat of the object (energy per unit mass per degree) and $T = T(x,t)$ is the temperature in degrees. Again, we neglect sources and then find that the conservation law in local form is,

$$\frac{\partial}{\partial t}(\rho C T) + \frac{\partial \phi}{\partial x} = 0,$$

where $\phi(x,t)$ is the flux of energy. For heat it is typical to use something similar to Fick's law that is called *Fourier's law of heat conduction* that supposes,

$$\phi = -K\frac{\partial T}{\partial x}(x,t),$$

where $K$ is a the *thermal conductivity*. Mathematically, this is the same as Fick's law but we give it a different name because it is a different physical principle and it does give credit to Fourier who stated this for heat.

We substitute Fourier's law into the conservation principle and get,

$$\frac{\partial}{\partial t}(\rho C T) = K\frac{\partial^2 T}{\partial x^2}.$$

If we assume that $\rho$ and $C$ are constant then we can define $k = K/(\rho C)$ and get the famous *heat equation*

$$\boxed{\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}}.$$

The only difference between this equation and equation (1.13) is the name of the variable and the diffusion coefficient. It is for this reason that we often refer to this as the diffusion equation.

Observe that in this case the time scale is,

$$T = \frac{L^2 \rho C}{K}.$$

### 1.3.3 Reaction-Diffusion Equation

To include sources in the context of Fick's law is relatively easy. The conservation principle is

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = f(x,t,u),$$

which when substituted in the constitutive law yields

$$\boxed{\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = f(x,t,u)}. \tag{1.14}$$

This is called a *reaction-diffusion equation* since it has both diffusion, set by $D$, and reactions that can occur because of the sources on the right-hand-side (RHS). This equation is only nonlinear in the case where the RHS is nonlinear. In the next subsection we will look at a particular example of a biological system that is a nonlinear reaction-diffusion equation.

### 1.3.4 Fisher's Equation

In early studies of DE the first equation that is typically studied is the equation for exponential growth,

$$\frac{du}{dt} = ru,$$

where $r$ is the growth (or death) rate of the population. We know that the general family of solutions to this equation is the exponential, $u = ae^{rt}$, where the constant $a$ set by the initial conditions. If the growth rate is positive the solution grows exponentially, which might be realistic in the case of small population sizes but ceases to be when the population gets large enough that its food supply can no longer sustain the population.

This limitation motivates considering a slightly more realistic model, namely the *logistic equation*,

$$\frac{du}{dt} = ru(1 - \frac{u}{K}),$$

where $K$ is not the thermal constant but the *carrying capacity* of the population. The right-hand-side is positive for $0 < u < K$ and therefore is the population size is relatively small the solution will grow. However, if the population exceeds the carrying capacity, i.e. $u > K$ the right-hand-side is negative and the solution will decrease. This is because the environment cannot sustain a population size that is too large.

If we make the logistic assumption for the source in the reaction-diffusion equation we get that the conservation principle is

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = ru\left(1 - \frac{u}{K}\right).$$

After we insert Fick's law then we get what is known as *Fisher's equation*,

$$\boxed{\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = ru\left(1 - \frac{u}{K}\right)}. \tag{1.15}$$

This was first derived by R.A. Fisher in the study of genetics.

### 1.3.5 Advection Equation

The simple assumption that the flux depends linearly on the solution, $u$, has a constitutive law of,

$$\phi = cu,$$

where $c$ has units of speed. With this the conservation law is,

$$\boxed{\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0}. \tag{1.16}$$

This is the *linear advection equation*. It is a simplified version of the inviscid Burger's equation, which you will see next.

### 1.3.6 Burger's Equation

We again neglect source terms and consider a constitutive relation that is slightly more general than Fick's law,

$$\phi = -D\frac{\partial u}{\partial x} + Q(u).$$

The first part on the RHS is exactly Fick's law but in addition we allow for the possibility of the flux depending on the magnitude of the solution through $Q(u)$.

If we substitute this into the conservation principle we get

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} + \frac{\partial Q}{\partial x}(u) = 0. \tag{1.17}$$

This shows that the evolution of $u(x,t)$ is set by two terms: 1) diffusion by Fick's law and 2) an extra term that is due to the transport that we will call *advection*. There are many choices for $Q(u)$ but perhaps the most famous choice is $Q(u) = u^2/2$. In this case the above equation becomes,

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2}}. \tag{1.18}$$

This is *Burger's equation* and is pivotal in the study of fluid mechanics. It includes both diffusion, RHS, and nonlinear advection, LHS and consequently is known as a *nonlinear-advection-diffusion* equation. Note that advection is just a fancy word for transport. In the special case with $D = 0$ we recover what is called the *inviscid Burger's equation*,

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0}. \tag{1.19}$$

This could be described as a *nonlinear advection* equation.

**Question:** How do you construct a nonlinear-advection-diffusion-reaction equation?

### 1.3.7 Diffusion in Three-Dimensions

The higher-dimension version of the conservation principle is

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{\phi} = f(x, y, z, t, u). \tag{1.20}$$

The higher-dimension version of Fick's law is,

$$\vec{\phi} = -D\vec{\nabla}u, \tag{1.21}$$

that is to say the flux is in the opposite direction of where $u$ increases most rapidly. By convention we define the Laplacian operator to be the divergence of the gradient,

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Therefore, when we substitute Fick's law into the conservation principle above we get,

$$\frac{\partial u}{\partial t} - D\nabla^2 u = f(x, y, z, t, u). \tag{1.22}$$

This is the three-dimensional version of the reaction-diffusion equation. If there are no sources, $f = 0$, then we get the diffusion equation,

$$\frac{\partial u}{\partial t} = D\nabla^2 u. \tag{1.23}$$

Note that steady solutions of this equation are governed by the equation,

$$\boxed{\nabla^2 u = 0}, \tag{1.24}$$

which is referred to as *Laplace's equation*.

## 1.4 WAVE EQUATION

*[This section is based primarily on section 4.2 of Haberman (2004)]*

An alternative way to derive PDEs is through using Newton's law which says that the mass times the acceleration of a particle is equal to the sum of the forces. Mathematically it can be written as,

$$F = ma,$$

where $m$ is a mass, $a$ is the acceleration of a particle and $F$ is the sum of the forces.
To make things concrete consider a string that is stretched tightly in the horizontal. Define,

$$y = u(x, t),$$

to be the displacement of the string at position $x$ and time $t$ away from the equilibrium position. Consider a subset of the string that has end points at $x_0 = x - \Delta x$ and $x_1 = x + \Delta x$ and centre of $x$. The parameters that describe the string are listed below:

$$\begin{aligned}
\rho(x) &= \text{density of the string with units of mass/length}^3, \\
A &= \text{cross-sectional area of the string with units of length}^2, \\
T &= \text{string tension } \left(\frac{F}{A}\right) \text{ with units of Newton/length}^2, \\
u(x,t) &= \text{ vertical deflection about the state of no tension with units of length.}
\end{aligned}$$

Therefore, the mass of the string is,

$$\text{Mass} \equiv 2\rho(x)A\Delta x.$$

For simplicity we assume that $A$ is constant.
The vertical component of the acceleration can be written as,

$$\text{Acceleration} \equiv \frac{\partial^2 u}{\partial t^2}.$$

The tension at the left point of the interval is parallel to the string and acts outwards, which we denote with $T(x - \Delta x, t)$. Similarly, at the right end of the interval the tension acts parallel to the end of string with a magnitude of $T(x + \Delta x, t)$. As well, we define the angle that the string makes with the horizontal at these two endpoints is $\theta(x - \Delta x, t)$ and $\theta(x + \Delta x, t)$, respectively. To find the components of tension that are in the vertical we use trigonometry to find that at the left and right ends of the interval we have,

$$-T(x-\Delta x,t)A\sin\theta(x-\Delta x,t),$$
$$T(x+\Delta x,t)A\sin\theta(x+\Delta x,t).$$

Notice that the first is negative because it acts to the left.

For the sake of simplicity, not because it is realistic, we assume that the string is perfectly flexible so that there is no resistance or friction present. If the only two forces that act are gravity and tension, they must balance in both the horizontal and vertical directions. Newton's second law can be written as

$$ma = \text{Force of Tension} + \text{Force of Gravity}.$$

If we substitute into this equation we get

$$2\rho(x)A\Delta x\frac{\partial^2 u}{\partial t^2} = AT(x+\Delta x,t)\sin\theta(x+\Delta x,t) - AT(x-\Delta x,t)\sin\theta(x-\Delta x,t) - 2\rho(x)A\Delta xQ(x,t).$$

We used $Q(x,t)$ to represent the force of gravity per unit mass. In general it can be any body force.

If we divide by $A\Delta x$ and then take the limit as $2\Delta x \to 0$ we get,

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \lim_{\Delta x\to 0}\frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x-\Delta x,t)\sin\theta(x-\Delta x,t)}{2\Delta x} - \rho(x)Q(x,t),$$
$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[T(x,t)\sin\theta(x,t)\right] - \rho(x)Q(x,t).$$

Note that we used the definition of the derivative to rewrite the RHS.

The above equation is not closed because it involves the angle and the displacement. To rewrite everything in terms of one variable, preferably $u$, we need to relate these two. For any displacement we can say that,

$$\frac{\partial u}{\partial x} = \tan\theta = \frac{\sin\theta}{\cos\theta}$$

From this we deduce that,

$$\sin\theta = \frac{\frac{\partial u}{\partial x}}{\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}}.$$

Therefore, Newton's second law yields the following equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[T(x,t)\frac{\frac{\partial u}{\partial x}}{\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}}\right] - \rho(x)Q(x,t).$$

This is a rather complicated and nonlinear PDE that is both second order in space and time. If we assume that the tension is constant and expand out the derivative

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+u_x^2}}\right) &= \frac{\sqrt{1+u_x^2}u_{xx} - 2u_{xx}u_x^2(1+u_x^2)^{-1/2}}{1+u_x^2},\\ &= \frac{u_{xx}}{(1+u_x^2)^{3/2}}.\end{aligned}$$

which means that the PDE can be written as,

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{T}{\left(1+\left(\frac{\partial u}{\partial x}\right)^2\right)^{3/2}}\frac{\partial^2 u}{\partial x^2} - \rho(x)Q(x,t).$$

If the displacement is small, then $\theta$ is also small and we can use our Taylor expansions for trigonometric functions to approximate the above to get $\tan\theta \approx \theta$. Therefore, in the case of small amplitude we get that Newton's law yields the following governing equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho(x)}\frac{\partial^2 u}{\partial x^2} - Q(x,t).$$

Finally, if we assume that the density is constant and that gravity is much smaller than the tension we get the *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where $c^2 = T/\rho$ has the units of speed squared.

If we assume that the density is constant in the nonlinear case we get a similar equation but with a nonlinear coefficient,

$$c^2 \left( \frac{\partial u}{\partial x} \right) = \frac{T}{\rho \left( 1 + \left( \frac{\partial u}{\partial x} \right)^2 \right)^{3/2}}$$

It is important to note that this is a linear equation because we assumed that the amplitude of the displacement is small In general this is not the case and you would get a nonlinear evolution equation. Also, you can easily include resistance to get an equation that is dissipative. This is something that you will explore on the first assignment.

### 1.4.1 Boundary Conditions

To find a unique solution we must impose boundary and initial conditions. Here we discuss four different types that are popular choices.

1) If the string has ends that are clamped down then the appropriate boundary conditions are of the *Dirichlet type*,

$$u(0,t) = 0, \quad \text{and} \quad u(L,t) = 0,$$

if the ends are located at $x = 0, L$. If the left-end is moving up and down at a prescribed rate this can be prescribed as,

$$u(0,t) = f(t),$$

where $f(t)$ is the height of the left end-point at time $t$. This is a non-homogeneous Dirichlet condition

2) It is possible the left-end is free to move but always stays horizontal,

$$\frac{\partial u}{\partial x}(0,t) = 0.$$

This is a *Neumann boundary condition*. In the context of the diffusion or heat equation this corresponds to no-flux at the boundary. So instead of losing the stuff at the boundary it would get reflected, or possibly accumulate at the boundary.

3) A mixed or Robin boundary condition is of the form,

$$a\frac{\partial u}{\partial x}(0,t) + bu(0,t) = 0.$$

4) Periodic boundary conditions. That is to say the solution is equal at the left and right ends of the domain.

Each of these boundary conditions gives rise to very different motion and therefore we should expect that the solution depends inherently on what condition is imposed.

## 1.5 VIBRATING MEMBRANE

Consider a circular membrane of a drum say with a vertical displacement of $z = u(x, y, t)$ in two spatial dimensions and time. We make the following assumptions:

- Displacements are small such that $\partial u/\partial x, \partial u/\partial y \ll 1$
- Vibrations are purely in the vertical
- $\rho(x, y)$ is the mass density. [Units $kg/m^2$]
- $\vec{F_T}$: tensile (line) force tangent to the line membrane and acts along the edge. [Units $N/m$]
- Assume that the force is constant in magnitude $|\vec{F_T}| = T_0$.
- $\hat{t}$: unit tangent vector at the boundary
- $\hat{n}$: outward unit normal vector to the surface
- Using some physics it can be shown that the tensile force is (downwards)

$$\vec{F_T} = T_0\hat{t} \times \hat{n}$$

- Vertical component is

$$\vec{F_T} \cdot \hat{k} = T_0(\hat{t} \times \hat{n}) \cdot \hat{k}$$

With all of these assumptions and definitions we can use Newton's second law to find an evolution equation by balancing the forces.

- The mass times the acceleration at a particular position is

$$ma = \rho_0 \frac{\partial^2 u}{\partial t^2} dA$$

- The total over the surface is therefore,

$$\text{Total} = \iint_A \rho_0 \frac{\partial^2 u}{\partial t^2} dA$$

- Tensile force at a position on the boundary is

$$T_0(\hat{t} \times \hat{n}) \cdot \hat{k}\, ds$$

- If gravity is negligible total force is the surface tension integrated over boundary

$$\oint_{\partial A} T_0(\hat{t} \times \hat{n}) \cdot \hat{k}\, ds = \oint_{\partial A} T_0(\hat{n} \times \hat{k}) \cdot \hat{t}\, ds$$

- Newton's second law then yields

$$\iint_A \rho_0 \frac{\partial^2 u}{\partial t^2} dA = \oint_{\partial A} T_0(\hat{n} \times \hat{k}) \cdot \hat{t}\, ds.$$

- Use Stoke's theorem from Vector Calculus

$$\iint_A \rho_0 \frac{\partial^2 u}{\partial t^2} dA = \iint_A T_0 \left[\vec{\nabla} \times (\hat{n} \times \hat{k})\right] \cdot \hat{n}\, ds.$$

- True for all areas $A$, therefore we can apply our trusty Lemma and get a local eqn

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \left[\vec{\nabla} \times (\hat{n} \times \hat{k})\right] \cdot \hat{n}.$$

- Given that the surface is $z = u(x, y)$ we can find a unit normal easily,

$$\hat{n} = \frac{\left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right)}{|| \left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right) ||} = \frac{\left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right)}{\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}} \approx \left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right)$$

- Above we used the fact that the slopes are very small to make the approximation
- Can compute the inner product between normal and vertical vectors

$$\hat{n} \times \hat{k} \approx \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}, 0\right)$$

- Finally, compute the curl and get,

$$\vec{\nabla} \times \left(\hat{n} \times \hat{k}\right) \approx \left(0, 0, \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

- In summary we have, (where subscript $H$ denotes horizontal)

$$\left[\vec{\nabla} \times (\hat{n} \times \hat{k})\right] \cdot \hat{n} \approx \nabla_H^2 u$$

- This yields that our governing equation is a two-dimensional wave eqn,

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho_0}\nabla_H^2 u \equiv c^2 \nabla_H^2 u.$$

CHAPTER
# TWO

# Classification of 2nd-order PDEs

In one and two-dimensions, three second-order equations that have very different behaviours are the diffusion, wave and Laplace's equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0. \end{aligned}$$

In this chapter we will explain why these three equations are called *parabolic*, *hyperbolic* and *elliptic*, respectively. But we will also see that there are many other equations in each of these three categories and that all second order equations can be put in one of these three areas. For the moment let's observe that the first is first-order in time, the second is second-order in time and the third has no time dependency at all.

## 2.1 GENERAL 2ND-ORDER LINEAR PDES IN 2 VARIABLES

Suppose our two variables are $x$ and $y$, where they do not both need to be space. Indeed, we will see that there is very different behaviour depending on whether both are space-like or whether we have one that is time-like. The most general 2nd-order linear PDE that we can write down is,

$$A\frac{\partial^2 U}{\partial x^2} + 2B\frac{\partial^2 U}{\partial x \partial y} + C\frac{\partial^2 U}{\partial y^2} + D\frac{\partial U}{\partial x} + E\frac{\partial U}{\partial y} + FU = G.$$

The functions that define the equation are,

$$A(x,y), B(x,y), C(x,y), D(x,y), E(x,y), F(x,y), \quad \text{and} \quad G(x,y),$$

and they are all *continuously differentiable functions*, i.e. are in $C^{(1)}(x,y)$.

If we assume that $A$ is not equal to zero then we divide through by it to rewrite the terms that are second-order as

$$\frac{\partial^2 U}{\partial x^2} + \frac{2B}{A}\frac{\partial^2 U}{\partial x \partial y} + \frac{C}{A}\frac{\partial^2 U}{\partial y^2} = \left(\frac{\partial}{\partial x} - \omega^+\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \omega^-\frac{\partial}{\partial y}\right)U + \left(\frac{\partial \omega^-}{\partial x} - \omega^+\frac{\partial \omega^-}{\partial y}\right)\frac{\partial U}{\partial y}$$

if we define $\omega^+$ and $\omega^-$ as,

$$\omega^+ + \omega^- = -\frac{2B}{A}, \quad \text{and} \quad \omega^+\omega^- = \frac{C}{A}.$$

This can be verified by expanding out the product of two differential operators.

At this early stage it is not evident why we would want to write it in this form but for the moment try and follow the steps, and later on it will become more apparent. To determine how to pick these two parameters we must solve these two nonlinear algebraic equations. This can be done by first solving for $\omega^-$ in the first equation and substituting this into the second,

$$\begin{aligned}
\omega^+\omega^- &= \frac{C}{A}, \\
\omega^+\left(-\frac{2B}{A} - \omega^+\right) &= \frac{C}{A}, \\
{\omega^+}^2 + \frac{2B}{A}\omega^+ + \frac{C}{A} &= 0.
\end{aligned}$$

This we can solve using the quadratic equation and taking the positive root (you could also take the negative if you wanted to)

$$\omega^+ = \frac{-B + \sqrt{B^2 - AC}}{A}.$$

If we substitute this into the first equation relating the two variables we recover,

$$\begin{aligned}\omega^- &= -\frac{2B}{A} - \frac{-B + \sqrt{B^2 - AC}}{A}, \\ &= \frac{-B - \sqrt{B^2 - AC}}{A}.\end{aligned}$$

Therefore, we can write the solutions compactly as,

$$\omega^\pm = \frac{-B \pm \sqrt{B^2 - AC}}{A}. \tag{2.1}$$

If we compare this with the three terms that are second order we appreciate that these are simply the factors of the quadratic equation,

$$A\omega^2 + 2B\omega + C = 0.$$

When we first introduced $\omega^\pm$ we also added on linear terms. These are only non-zero if the coefficients are dependent on the independent variables $(x, y)$. Most of the work we will do is for the case of constant coefficients so these terms do not contribute.

From equation 2.1 we know that there are three distinct scenarios depending on the discriminant. This defines the three different types of equations:

$$\begin{aligned}B^2 - AC > 0, &\quad \text{hyperbolic ,} \\ B^2 - AC = 0, &\quad \text{parabolic ,} \\ B^2 - AC < 0, &\quad \text{elliptic.}\end{aligned}$$

With this established we can classify the three equations we stated at the beginning.

1) For the wave equation if we take $y = t$ then we have that $A = 1, B = 0, C = -1$. This implies that $B^2 - AC = -AC = 1 > 0$, and therefore the wave equation is *hyperbolic*. In this case the quadratic equation has two real roots.

2) For the diffusion equation we have $A = 1, B = 0, C = 0$ and therefore $B^2 - AC = 0$, which is *parabolic*. This implies that there is only one real root.

3) For Laplace's equation where $A = 1, B = 0, C = 1$ we have the discriminant is $B^2 - AC = -1 < 0$, which is *elliptic*. In this case there are two imaginary roots.

Students that go on and take AMATH 453 will learn how to reduce equations of each type into a canonical (or standard) form. Here we simply show the canonical forms for the wave, diffusion and Laplace's equation.

## 2.2 REDUCTION TO STANDARD FORM

It turns out that each class of equations can be reduced to a general or canonical form. In this section we present the transformation for the wave equation in detail. The idea generalizes to any hyperbolic system but it does get more complicated. Then we simply discuss the importance of this in the context of elliptic and parabolic equations.

### 2.2.1 Wave Equation

We saw previously that the wave-equation in one-dimension can be factored as,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left( \frac{\partial}{\partial t} - c\frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} \right) u.$$

When we solve this equation using the method of characteristics we will determine that the two characteristic variables are $x - ct$ and $x + ct$, the first and second propagate information to the right and left, respectively. This suggests making the following transformation to characteristic variables,

$$\begin{aligned} \zeta &= x - ct, \\ \eta &= x + ct. \end{aligned}$$

By adding and subtracting these equations we can invert them to obtain,

$$\begin{aligned} x &= \frac{\eta + \zeta}{2}, \\ t &= \frac{\eta - \zeta}{2c}. \end{aligned}$$

Therefore, we can change variables using the chain rule, applied repeatedly,

$$\frac{\partial u}{\partial \zeta} = \frac{\partial x}{\partial \zeta}\frac{\partial u}{\partial x} + \frac{\partial t}{\partial \zeta}\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2c}\frac{\partial u}{\partial t} = -\frac{1}{2c}\left[\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right]$$

and

$$\frac{\partial u}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial u}{\partial x} + \frac{\partial t}{\partial \eta}\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2c}\frac{\partial u}{\partial t} = \frac{1}{2c}\left[\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right]$$

From this we deduce that,

$$\frac{\partial}{\partial \zeta} = -\frac{1}{2c}\left[\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right] \quad \text{and} \quad \frac{\partial}{\partial \eta} = \frac{1}{2c}\left[\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right].$$

This then allows us to rewrite the wave equation as the much simpler, but completely equivalent, PDE,

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = -(2c)^2\frac{\partial^2 u}{\partial \zeta \partial \eta} = 0,$$

or upon diving by the constant,

$$\frac{\partial^2 u}{\partial \zeta \partial \eta} = 0.$$

What's attractive about this equation is that it is so easy to solve, after having set it up in this form. If we first integrate with respect to $\eta$ we get,

$$\frac{\partial u}{\partial \zeta} = \alpha'(\zeta).$$

Note that instead of introducing any function of $\zeta$ we choose the derivative of a function. The reason for this choice becomes apparent now that we integrate with respect to $\zeta$,

$$u = \alpha(\zeta) + \beta(\eta).$$

This solution is in terms of the characteristic variables. If we want to rewrite it in terms of physical quantities we substitute in and find,

$$u = \alpha(x - ct) + \beta(x + ct),$$

which will remind you of d'Alembert's solution. Indeed, we could go back and compare our two different solutions and see that when we impose the initial conditions that $u(x, 0) = f(x)$ and $\partial u/\partial t = g(x)$ we obtain that,

$$\alpha(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_{x-ct}^{0} g(s)\, ds,$$

and

$$\beta(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_{0}^{x+ct} g(s)\, ds.$$

### 2.2.2 Diffusion Equation

If we wanted to factor the principle part of the diffusion equation, by that I mean the second order term(s), into the same form as we did with the wave equation we could do that and we would find that

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} = \left( \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial x} \right) u.$$

This is of the same form as before but now we have that $\omega^+ = \omega^- = 0$, that is to say that $0$ is a root of the differential operator of multiplicity two. Using our previous solution we could define the characteristic equations as,

$$\zeta = x, \quad \text{and} \quad \eta = x.$$

So the characteristics are vertical lines in $t, x$-space because the information does not propagate along characteristics because the solution is not wave-like. Indeed, information actually propagates very differently as we will see later when we solve this equation exactly.

### 2.2.3 Laplace's Equation

As we saw earlier, we can factor Laplace's equation into two first order differential operators but they have complex coefficients. Formally this is true but this is not so useful. That is why we do not think about this anymore. A different perspective is that Laplace's equation solves for steady state solutions that do not propagate information so the idea of characteristics loses it's meaning.

## 2.3 FORMULATION OF INITIAL AND BOUNDARY VALUE PROBLEMS

When we solve for the exact solution of the wave equation in the next chapter we will solve an *Initial Value Problem (IVP)* because we only imposed initial conditions. This is also referred to as a *Cauchy Problem*.

If we were to solve Laplace's equation on a rectangle that would be a *Boundary Value Problem (BVP)* since we must impose conditions on each side of the rectangle.

If we were to solve the wave or diffusion equations on a closed domain, as we will do in the next chapter, that will be an *Initial Boundary Value Problem (IBVP)*.

In all of these cases the number of conditions that must be imposed in order for us to find a unique solution is set by the order of the derivative.

It is easy to impose conditions but not all choices yield reasonable solutions. This idea was expressed by Hadamard who required the following three criteria for a problem to be *well-posed*.

a) A solution must exist.

b) The solution must be unique.

c) The solution must depend continuously on the initial/boundary value data.

To see an example of a problem that at first looks fine but is actually *ill-posed*, that is to say it is not well-posed, consider Laplace's equation in the half-plane,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{on} \quad y > 0 \quad -\infty < x < \infty.$$

Because the equation is second-order we must impose two boundary conditions. One particular choice is,

$$u(x,0) = 0, \quad \text{and} \quad \frac{\partial u}{\partial y}(x,0) = \frac{\sin(nx)}{n}.$$

Obtaining the solution is beyond what we know right now (we will learn how to do this by the end of the course) but it can be verified that the unique solution to the above boundary value problem is,

$$u(x,y) = \frac{\sinh(ny)\sin(nx)}{n^2}.$$

Note that $\sinh(x)$ is the hyperbolic sine function that is defined as $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$. For any finite value of $n$ this solution is finite but how does the solution behave as $n$ increases? Well, the limit of the boundary condition is easy to determine,

$$\lim_{n\to\infty} \frac{\partial u}{\partial y}(x,0) = \lim_{n\to\infty} \frac{\sin(nx)}{n} = 0.$$

You might not be too surprised to find that the unique solution to Laplace's equation that satisfies this boundary condition and the previous one that was stated is $u(x,y) = 0$. So in the limit of $n$ becoming large the boundary condition converges to something that should yield the trivial to Laplace's equation. However, when we take the limit of the solution we find that,

$$\lim_{n\to\infty} u(x,y) = \lim_{n\to\infty} \frac{\sinh(ny)\sin(nx)}{n^2} = \lim_{n\to\infty} \frac{(e^{nx} - e^{-nx})\sin(nx)}{2n^2} \to \infty.$$

Therefore the actual solution grows without bound. Therefore, *arbitrarily small boundary conditions can lead to arbitrarily large solutions and therefore we say the solution is unstable*. By Hadamard's three criteria we deduce that Laplace's equation with these boundary conditions is ill-posed. Laplace's equations is often ill-posed but one should take care to consider the boundary conditions before you decide this.

## 2.4 STABILITY THEORY AND DISPERSION

In this section we look at wave-like solutions to linear PDEs with constant coefficients. Even though these are very specialized solutions you are asked to recall from Fourier Analysis that any well behaved function can be decomposed into sinusoidal components. Therefore, if we can understand how all sinusoidal solutions behave, we can combine them in some way to see how general solutions behave. First, we focus on stability and what it means for a wave or a PDE to be unstable. Second, we look at propagating waves and how to determine their speed of propagation.

### 2.4.1 Stability Theory

We begin by considering a general, scalar, second-order PDE with constant coefficients,

$$Au_{tt} + 2Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = 0.$$

Solving this is general is difficult and, as we have previously stated, you must impose initial and/or boundary conditions in order to find an exact solution. Here we take a different approach and instead consider a particular the of solution of the form,

$$u(x,t) = \exp(ikx + \lambda(k)t).$$

This solution has an amplitude of one because the modulus of anything of the form $\exp(i\theta) = 1$. It has a Fourier behaviour in the $x$-direction and an exponential behaviour in time. If you recall from Fourier analysis in AMATH 231, we know that $k$ is inversely proportional to the wavelength. We refer to $k$ as the *wavenumber* and the wavelength is then $2\pi/k$. Based on this reasoning so far we see that the solution is a sinusoidal function of a given wavelength, we refer to this as a *wave*. This is sometimes called a *normal-mode solution*. The idea here is to determine how does a wave of a particular wavelength evolve in time.

The remaining part is $\lambda(k)$ which can be positive, negative or zero. Even though it looks to be general in this form, if we substitute this equation into the PDE we will find that there is an algebraic relation that arises for $\lambda(k)$ in terms of the governing coefficients. That is to say that if we have a particular wave that evolves in a particular way based on the coefficients, $A, B, C, D, E, F$, which is the nature of the PDE. Now we consider a few examples to better understand what waves can do.

**Example 2.4.1** To understand how waves evolve in the context of the diffusion equation, recall that the equation can be written as,

$$u_t - Du_{xx} = 0.$$

If we substitute our wave solution we get,

$$\begin{aligned} \lambda \exp(ikx + \lambda t) - D(-k^2)\exp(ikx + \lambda(k)t) = 0, \\ \left[\lambda + Dk^2\right]\exp(ikx + \lambda(k)t) = 0. \end{aligned}$$

For this to be true everywhere for all time we need that the coefficient is identically zero, i.e.

$$\lambda = -Dk^2.$$

That means that the temporal behaviour of the solution is decaying exponentially in time. We see that as $D$ changes, it has the same effect on each wave. However, the larger the wavenumber (the smaller the wavelength) the faster the waves decay. This is consistent with the scaling argument that we saw in Chapter 1 when we first discovered the diffusion equation.

**Example 2.4.2** The Helmholtz equation can be written as,

$$u_{xx} + u_{tt} + \rho u = 0,$$

for some constant $\rho$. To see how the waves behave we substitute in the trial solution, as before,

$$\begin{aligned}(-k^2)\exp(ikx + \lambda(k)t) + \lambda^2 \exp(ikx + \lambda t) + \rho \exp(ikx + \lambda t) = 0,\\ \left[-k^2 + \lambda^2 + \rho\right]\exp(ikx + \lambda(k)t) = 0,\\ \implies \quad \lambda = \pm\sqrt{k^2 - \rho}.\end{aligned}$$

This shows that for small enough waves, $k^2 > \rho$ we have that there is a positive root that will grow exponentially, what we describe as unstable. If $k^2 < \rho$ we have that the roots are imaginary. This means that the solutions are not exponentially in time but actually sinusoidal as well. We will study this later in this section.

These examples motivate the following definition.

**Definition 2.4.1.** *If we look for a wave (normal-mode) solution to a linear PDE with constant coefficients than the solution is* **unstable** *for wavenumber $k$ if there exists a root with $\lambda(k) > 0$. If all the roots have $\lambda(k) < 0$ then it is asymptotically stable. If there exists at least one wave that is unstable the PDE is said to be unstable as well.*

The solution we assumed has a modulus (amplitude) of one. To better understand whether the previous PDE is well posed consider a slight modification with,

$$u(x,t) = \frac{1}{\lambda^2(k)}\exp(ikx + \lambda(k)t).$$

This is the same as before but the amplitude is $1/\lambda^2(k)$. Note that the initial conditions are then,

$$u(x,0) = \frac{1}{\lambda^2(k)}\exp(ikx), \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = \frac{1}{\lambda(k)}\exp(ikx).$$

In the previous example where $\lambda(k) \to \infty$ as $k \to \infty$, both the initial solution and its time derivative decay algebraically as $k \to \infty$. This means that waves with large $k$, small wavelengths, even though initially they can be infinitesimal, given enough time they will grow exponentially at an incredible rate. This property is considered to be *well-posed*. Below we make this a bit more precise.

**Definition 2.4.2.** *If we look for a normal-mode (wave) solution to a PDE with $\lambda(k) \to \infty$ as $k \to \infty$, of the form $\frac{1}{\lambda^2(k)} \exp(ikx + \lambda(k)t)$ then the equation is* **not well-posed***, or* **ill-posed** *since very small initial conditions yield huge magnitudes in the solution in very short times.*

### 2.4.2 Dispersion

We continue studying the same linear, second-order PDE with constant coefficients that we introduced in this section but with a slightly different solution. In particular we look for solutions of the form,

$$u(x,t) = \exp(i(kx - \omega t)).$$

As before, $k$ is the wavenumber (inverse of the wavelength) but now we have chosen to write the temporal parts in imaginary form. That is to say we picked $\lambda(k) = -i\omega(k)$. Before $\lambda$ was the rate at which the solution grows exponentially, what we call the *growth rate*. Here we use $\omega$ that is a frequency of the wave. Analogously to the wavenumber, the period of the wave is $2\pi/\omega$. The functional part in the exponential is referred to as the phase and we can write it as,

$$\theta = kx - \omega t.$$

If we want to follow a crest (or trough) we need to keep phase constant. If $k > 0$ and $\omega > 0$ then we can determine the speed at which a wave propagates by differentiating the phase with respect to time,

$$0 = k\frac{dx}{dt} - \omega, \quad \text{or} \quad \frac{dx}{dt} = \frac{\omega}{k}.$$

This is what we define as the *phase speed*, that is to say the speed at which the phases propagate, is $\omega/k$, the frequency divided by the wavenumber.

In some sense we are free to consider any wavelength we like, but it is the PDE that determines how this wave propagates, if it propagates. To determine this we substitute our solution into the PDE, cancel the exponential terms, and the resulting equation, called the *dispersion relation*, determines $\omega(k)$, which determines the frequency and the speed of the waves.

**Example 2.4.3** Consider the Telegraphers equation,

$$u_{tt} - c^2 u_{xx} + 2\gamma u_t = 0.$$

If we substitute our wave solution we get the following dispersion relation,

$$\begin{aligned}(-i\omega)^2 - c^2(ik)^2 + 2\gamma(-i\omega) &= 0,\\ \omega^2 + 2i\gamma\omega - c^2k^2 &= 0.\end{aligned}$$

The solutions to this equation are,

$$\omega = \frac{-2i\gamma \pm \sqrt{-4\gamma^2 + 4c^2}}{2} = -i\gamma \pm \sqrt{c^2 - \gamma^2}.$$

This shows that if $c^2 > \gamma^2$ then there is a real part of the frequency, that indicates that the wave is propagating. But in addition there is also an imaginary part. If we decompose the frequency, $\omega = \omega_R + i\omega_I$ then the temporal behaviour of the solution is,

$$\exp(-i\omega t) = \exp(-i\omega_R t)\exp(\omega_I t).$$

This shows that the imaginary part of the frequency can cause exponential growth. Indeed if $\omega_I > 0$ then we have growth but if it is negative it they decays. For this example we have that $\omega_I = -\gamma$ and therefore the solution is both propagating and decaying. This is not surprising because in the special cases where $\gamma = 0$ or $c = 0$ we recover the wave and diffusion equations, respectively. In general we have a superposition of the two solutions. This is why we have propagating and decaying solutions. Because this is true for every wave we can surmise this will also be true for general solutions. This is why the equation is called a *damped wave equation*

**Example 2.4.4** The Klein-Gordon equation is,

$$u_{tt} - c^2u_{xx} + \alpha^2 u = 0.$$

Again, we substitute in our wave (normal-mode) solution and obtain the associated dispersion relation,

$$\begin{aligned}(-i\omega)^2 - c^2(ik)^2 + \alpha^2 &= 0,\\ -\omega^2 + c^2k^2 + \alpha^2 &= 0,\\ \omega &= \pm\sqrt{c^2k^2 + \alpha^2}.\end{aligned}$$

In the special case where $\alpha = 0$ we have the wave equation and the frequency is $\omega = \pm ck$. This says what we already know that the waves in this equation propagate to the left or the right at a speed of $c$. We have that if $\alpha > 0$ it adds to the magnitude of $c^2k^2$ and therefore the

propagating waves are faster. This is symmetry in the sense that the right and left propagating waves always have the same speed. But there is asymmetry in that not all the waves propagate at the same speeds. The phase speed of these waves is,

$$\frac{\omega}{k} = \pm\sqrt{c^2 + \frac{\alpha^2}{k^2}}.$$

The short waves have $k$ very large and therefore have a phase speed that is close to $c$. However, the long waves can be a lot faster. In fact, as $k \to 0$ we see that the wave speed is unbounded.

## 2.5 ADJOINT AND SELF-ADJOINT OPERATORS

A useful identity that can be derived for the Laplacian operator is what's called Green's identity. If we define $L \equiv \nabla^2$ then we can write,

$$\begin{aligned} wL(u) - uL(w) &= w\nabla^2 u - u\nabla^2 w, \\ &= \nabla \cdot [w\nabla u] - \nabla w \cdot \nabla u - \nabla \cdot [u\nabla w] + \nabla u \cdot \nabla w, \\ &= \nabla \cdot [w\nabla u - u\nabla w]\,. \end{aligned}$$

This rewrites the initial function in terms of the divergence of a difference and is referred to as a *divergence expression*. This becomes of importance when we integrate this equation over a given volume, say $V$ that has boundary $\partial V$, and apply the divergence theorem,

$$\begin{aligned} \iiint_V [wL(u) - uL(w)]\ dV &= \iiint_V \nabla \cdot [w\nabla u - u\nabla w]\ dV, \\ &= \iint_{\partial V} [w\nabla u - u\nabla w] \cdot \hat{n}\ dA, \\ &= \iint_{\partial V} \left[ w\frac{\partial u}{\partial n} - u\frac{\partial w}{\partial n} \right]\ dA. \end{aligned}$$

Above we used the notation $\hat{n}$ for the unit outward normal and then introduced the directional derivative along this normal. The above equation relates the volume integral of $wL(u) - uL(w)$ in terms of the surface integral through the boundary.

If we want to generalize this idea to other differential operators, say $L \equiv \frac{\partial}{\partial t} - \nabla^2$, we need to modify our method. For starters we must define a related operator $L^* \equiv -\frac{\partial}{\partial t} - \nabla^2$. Note that we change the sign on the time derivative because it is first order but we keep the Laplacian as before. Now consider the following difference,

$$
\begin{aligned}
wL(u) - uL^*(w) &= w\left[\frac{\partial u}{\partial t} - \nabla^2 u\right] - u\left[-\frac{\partial w}{\partial t} - \nabla^2 w\right], \\
&= \frac{\partial}{\partial t}(wu) - \nabla\cdot[w\nabla u] + \nabla w\cdot\nabla u + \nabla\cdot[u\nabla w] - \nabla w\cdot\nabla u, \\
&= \frac{\partial}{\partial t}(wu) + \nabla\cdot[u\nabla w - w\nabla u], \\
&= \left[\frac{\partial}{\partial t}, \nabla\right]\cdot(uw, -w\nabla u + u\nabla w).
\end{aligned}
$$

This is the corresponding divergence relation. Note that because we are including time we now have a divergence operator that has both a temporal and spatial component.

**Definition 2.5.1.** *The operator $L^*$ is said to be the adjoint to $L$ because*

$$wL(u) - uL^*(w)$$

*is a divergence relation. If $L^* = L$ then the operator $L$ is said to be self-adjoint.*

We will revisit this when we discuss Sturm-Liouville theory of boundary value problems, in the next chapter.

# Part II

# METHODS OF SOLUTION

CHAPTER
# THREE

# Method of Characteristics

In the previous chapter we saw two important examples of first order PDEs: 1) the inviscid Burger's equation and 2) linear advection equation. In this chapter we focus on developing mathematical techniques to find analytical solutions to these types of equations. We do this using the Method of Characteristics. Before we do that we first show how the wave equation can be written as two first-order PDEs.

## 3.1 REDUCTION OF HIGHER-ORDER EQUATIONS TO SYSTEMS

*[This section is based primarily on section 2.1 of Zauderer (2006)]*

We begin by showing how the wave equation, which is a single PDE that is second-order in space and time can be written as two PDEs that are each first order in space and time. To do this we first rewrite the PDE in terms of a differential operator acting on the variable we wish to determine,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial u}{\partial x^2} = \left(\partial_{tt} - c^2 \partial_{xx}\right) u = 0.$$

It turns out that because the coefficients in this operator are constants we can factor the operator as,

$$u_{tt} - c^2 u_{xx} = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0.$$

We use the notation where $\partial_t$ and $\partial_x$ denote the partial derivatives with respect to time and space respectively.

It turns out that we can factor the differential operator using difference of squares, as we do with algebraic equations. If we introduce a new variable $v(x,t)$ we can rewrite the above single equation as two first order equations,

$$(\partial_t - c\partial_x)u = v, \tag{3.1}$$
$$(\partial_t + c\partial_x)v = 0. \tag{3.2}$$

Even though these equations are coupled we see that if we can solve the second equation first for $v(x,t)$ then we can substitute this into the first equation and then solve for $u(x,t)$, the variable we need to solve for. This means that we can solve a second-order PDE by solving two first-order PDEs.

A similar sort of decomposition can be done for Laplace's equation (in two-dimensions) if we use complex numbers,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (\partial_{xx} + \partial_{yy})\, u = (\partial_x + i\partial_y)\,(\partial_x - i\partial_y)\, u = 0.$$

Therefore, the corresponding two first-order equations also decouple to yield

$$(\partial_t - i\partial_x)u = v, \tag{3.3}$$
$$(\partial_t + i\partial_x)v = 0. \tag{3.4}$$

This yields a similar system but one that is complex, which is problematic since we want to find real solutions to the given equation. That is why an alternative method is usually used. To explain this first begin with Laplace's equation,

$$u_{xx} + u_{yy} = 0.$$

We define,

$$p = u_x, \quad \text{and} \quad q = u_y.$$

We then have two constraints on the new variables. First, that they solve the above PDE. Second we need that $u_{xy} = u_{yx}$. If we translate these two conditions in terms of the new variables we get,

$$p_x + q_y = 0,$$
$$p_y - q_x = 0.$$

These conditions also arise naturally in Complex Variables and they are called the *Cauchy Riemann equations*. If they are satisfied than it guarantees that the complex function,

$$f(z) = q(x, y) + ip(x, y),$$

is analytic, where $z = x + iy$.

For the diffusion equation in one-dimension we can obtain the following system,

$$u_x = v, \tag{3.5}$$
$$u_t = Dv_x, \tag{3.6}$$

which must also be solved simultaneously.

## 3.2 LINEAR FIRST-ORDER PDES

### 3.2.1 From ODEs to PDEs

**An example from ODEs:**

Suppose we have a thin long tube filled with fluid that flows with a prescribed velocity that can vary in the along pipe direction, $x$, and time $t$. We represent the velocity with $u(x, t)$. If at time $t_0$ we insert a particle (which is small and light so that it does not alter the velocity of the fluid) into the pipe at position $x_0$. The thin pipe scenario is illustrated in Figure 3.1. An interesting physical question to ask is, what path does the particle follow? Since the motion is unidirectional the solution for the position is a function of time $x(t)$.

This problem can be formulated as an Ordinary Differential Equation (ODE) where the equation and initial condition (IC) are

$$\frac{dx}{dt} = u(x, t) \qquad \text{with} \qquad x(t_0) = x_0$$


**Fig. 3.1** A thin long pipe filled with a fluid that flows at a given velocity that varies in the along pipe direciton $x$ and in time $t$. The arrows depict the velocity at certain positions. The arrows are of different lengths to illustrate the point that the velocity need not be uniform.

The ODE uses two ideas: 1) the rate of change of position of the particle is equal to the velocity of the particle and 2) it assumes that the velocity of the particle is equal to the velocity of the fluid where it is located. The path that the particle follows is the solution to the above ODE with the IC. Even though we cannot solve this in general, we have learned techniques to solve this for particular forms of the velocity field.

**An example of a PDE:**

Let us say we still have the same geometric set up as before with the thin long fluid filled tube. However, instead of having a particle let us say we add some type of chemical solution (say dye or salt) into the thin tube. This means that we have a concentration of the chemical that can vary with space and time, $\rho(x,t)$. If the velocity is constant in space and time the profile will simply get advected in the direction of the flow. In general, for nonuniform velocities, the concentration profile will get deformed as time evolves. It is then natural to ask the following question: given a velocity profile that can vary in space, $u(x,t)$, what is the concentration of the chemical as a function of time and space, $\rho(x,t)$?

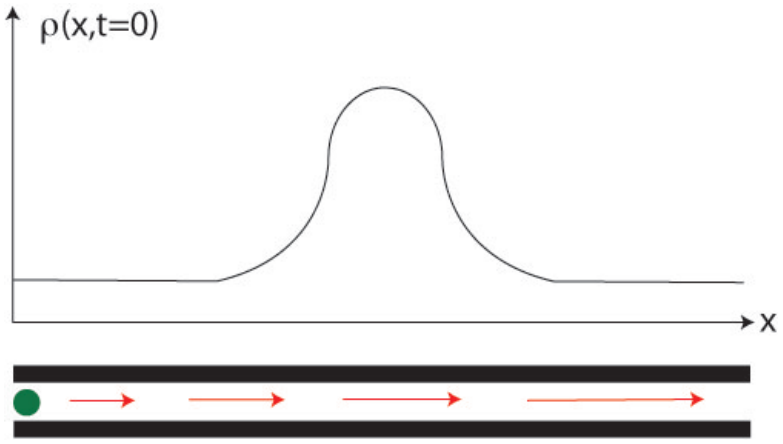


**Fig. 3.2** This illustrates the geometry of a thin pipe but instead of having a particle travelling along the pipe we have a concentration defined as a function of space and time $\rho(x,t)$. The plot above the geometry depicts an initial concentration that one could consider.

There are two possible scenarios. First, the concentration of the chemical is simply advected along with the flow and does not change as it moves around (this is equivalent to saying there is no mixing or diffusion). In this case we simply need to determine the position of a particular

concentration of chemical as it is advected by the fluid. As before, this equation is given by an ODE

$$\frac{dx}{dt} = u(x,t)$$

where the concentration is constant along this path. A second possibility is that the concentration evolves as it moves along with the flow. To determine precisely how the concentration evolves it is necessary to specify how the density changes (say by diffusion) as a function of space, time and perhaps even the concentration itself. This introduces another equation into the problem. The functional dependency, $f$, needs to be determined a priori.

$$\frac{d\rho}{dt} = f(x,t,\rho(x,t))$$

The first case is when $f = 0$. If we solve these two equations simultaneously, then we can determine $\rho(x,t)$ which is a solution to the problem. One equation determines the path of the motion, what we will call the characteristics, and the other how the concentration changes along that path.

If we assume that the for an initial position $x(t_0) = x_0$ the concentration follows a particular path given by $x(t)$ then the concentration following that path is $\rho(x(t),t)$. If we take the total derivative of this with respect to time, and use the above equation which determines how the concentration varies in time, we get with help from the Chain Rule

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{dx}{dt}\frac{\partial\rho}{\partial x} = \frac{\partial\rho}{\partial t} + u(x,t)\frac{\partial\rho}{\partial x} = f(x,t,\rho(x,t))$$

On the left hand side of the above equation $\rho(x(t),t)$ is a composite function of time whereas on the right hand side it is taken to be a function of $x$ and $t$. The last equality above is a first order PDE where the independent variables are space, $x$, and time, $t$. Observe that we have essentially reduced the PDE to two coupled ODEs, those being the two proceeding equations.

### 3.2.2 Basis for the Method of Characteristics

In the previous chapter we saw several examples of first order PDEs. The *general form* for a linear 1-st order PDE is,

$$a(x,t)v_x(x,t) + b(x,t)v_t(x,t) = c(x,t)v(x,t) + d(x,t). \tag{3.7}$$

Recall that $(x,t)$ are the independent variables, $v(x,t)$ is the dependent variable to be determined and $a(x,t), b(x,t), c(x,t), d(x,t)$ are given functions that determine the behaviour of the equation. This equation is linear because these functions only depend on $(x,t)$ NOT on $v(x,t)$ or any of its derivatives. We will consider this more complicated problem in the following subsequently.

Before we explain how to construct a solution suppose we already know the solution $v(x,t)$. If at some initial time, $t_0$, we know the initial solution, $v_0(x) = v(x,t_0)$, then the complete solution must tell us what the solution is at any time $t$ and location $x$. In particular, we can determine how each initial point propagates forward in time.

To do this suppose we can parameterize a solution curve starting on our initial curve using the variable $s$. Then the solution could be written as,

$$v = v(x(s), t(s)).$$

Given this functional form, the chain rule dictates that,

$$\frac{dv}{ds} = \frac{dv(x(s),t(s))}{ds} = \frac{\partial v}{\partial x}\frac{dx}{ds} + \frac{\partial v}{\partial t}\frac{dt}{ds}.$$

If we compare this equation with equation (3.7) we see that the RHS of this equation is very similar to the LHS of equation (3.7). In particular, if we pick the curve such that,

$$\frac{dx}{ds} = a(x,t), \tag{3.8}$$

$$\frac{dt}{ds} = b(x,t), \tag{3.9}$$

then these are forced to be exactly the same. Following this argument along we then get, after combining these two equations,

$$\frac{dv}{ds} = c(x,t)v(x,t) + d(x,t). \tag{3.10}$$

The equation directly above shows how the field $v$ changes as the solution advances. In contrast, equations (3.8) and (3.9) determine the curve that the solution evolves on. These curves are known as *characteristic curves* of the PDE. These equations can be solved independently and then substituted into equation (3.10) to determine how the solution changes along these curves.

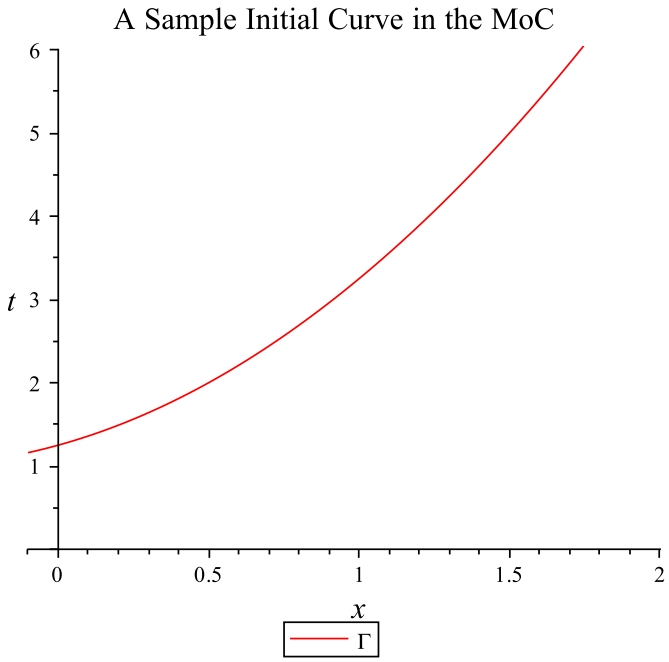


**Fig. 3.3** A figure that depicts the basic idea of an initial curve in the method of characteristics. Each solution curve must intersect this curve at one point and then propagates along.

This is the basis of the *method of characteristics* and is surprising because we can reduce a 1st-order PDE into three coupled ODEs. Since we already know how to solve ODEs this is a much simpler task. If $a, b, c, d$ are all smooth functions than theorems from ODEs ensure that the solution exists and is unique that passes through some initial point $(x_0, t_0, v_0)$. This is known as an *initial value problem*. Given the above arguments we can use the following approach to solve the PDE in question.

### 3.2.3 Method of Characteristics

**Theorem 3.2.1.** *Suppose that $a(x,t), b(x,t), c(x,t), d(x,t)$ are smooth functions and that the initial data is given on a curve in the $(x,t)$-plane, denoted with $\Gamma$, that can be parameterized with,*

$$\Gamma = \left\{(x,t) \mid x = \hat{x}(\tau), t = \hat{t}(\tau)\right\}, \quad \textit{with} \quad v = \hat{v}(\tau) \textit{ on } \Gamma.$$

*Then there is a* unique *solution to the linear first-order PDE*

$$a(x,t)\frac{\partial v}{\partial x}(x,t) + b(x,t)\frac{\partial v}{\partial t}(x,t) = c(x,t)v(x,t) + d(x,t). \tag{3.11}$$

Below, we explain in detail the steps that allow us to construct the solution for a given initial curve. There is some flexibility is to when you solve certain equations but for the moment let's say there are four parts.

1) If we are given the solution along an initial curve $\Gamma$ we parameterize this curve using $\tau$ and get,

$$x = \hat{x}(\tau), \quad t = \hat{t}(\tau), \quad v = \hat{v}(\tau).$$

2) Then, solve the characteristic equations subject to the initial conditions specified in part $1)$ that at $s = 0$ we have the correct information along the initial curve,

$$\begin{aligned} \frac{dx}{ds} &= a(x,t), \quad \text{with} \quad x|_{s=0} = \hat{x}(\tau), \\ \frac{dt}{ds} &= b(x,t), \quad \text{with} \quad t|_{s=0} = \hat{t}(\tau). \end{aligned}$$

This then yields solutions of the form,

$$x = x(s,\tau), \quad t = t(s,\tau).$$

3) Substitute these functions into

$$\frac{dv}{ds} = c(x(s,\tau), t(s,\tau))v + d(x(s,\tau), t(s,\tau)),$$

and solve the ODE subject to the initial condition,

$$v|_{s=0} = \hat{v}(\tau).$$

This yields solutions in terms of the two parameters,

$$v = v(s,\tau), \quad x = x(s,\tau), \quad t = t(s,\tau).$$

Theory of ODEs ensures that there is a unique solution by the smoothness of the coefficients.

4) Because we want $v(x,t)$ and not $v(s,\tau)$ we must invert the characteristic curves to find

$$s = s(x,t), \quad \tau = \tau(x,t).$$

Then we substitute into our solution to obtain,

$$v(x,t) = v(s(x,t),\tau(x,t)).$$

### 3.2.4 Example where the Initial Curve is t=0

**Example 3.2.1** The advection equation derived in Chapter 1 on the infinite interval with an initial profile given at $t = 0$ can be written as,

$$\begin{aligned} &\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0, \\ &v(x,0) = f(x), \qquad -\infty < x < \infty. \end{aligned}$$

**Solution:**

1) The initial curve $\Gamma$ is the $x$-axis and the parameterization is,

$$x = \tau, \quad t = 0, \quad , v = f(\tau).$$

2) The characteristic equations are,

$$\frac{dt}{ds} = 1, \quad t|_{s=0} = 0,$$
$$\frac{dx}{ds} = c, \quad x|_{s=0} = \tau.$$

These equations are decoupled and therefore we can integrate each and then and impose the initial condition to find

$$t = s, \quad x = cs + \tau.$$

Note that when we integrate these functions we have constants of integration that can depend on $\tau$ since $s$ and $\tau$ are independent.

3) We substitute our result into the ODE,

$$\frac{dv}{ds} = 0, \quad v|_{s=0} = f(\tau).$$

We integrate the equation and get $v = \phi(\tau)$ and impose the initial condition to get,

$$v = f(\tau).$$

4) Finally, we must invert our characteristic equations. This is easy since $s = t$ and $\tau = x - ct$. Therefore, our solution is,

$$v = f(x - ct).$$

The characteristic curves in space-time for a constant $\tau$ are,

$$\tau = x - ct,$$

which are lines in the $x - t$ plane that have a slope of $1/c$. The solution initially starts along the $x$-axis and the initial data propagates along the characteristics. In this example it is constant along each characteristics.

If we pick $f(x) = e^{-x^2}$ than the solution is

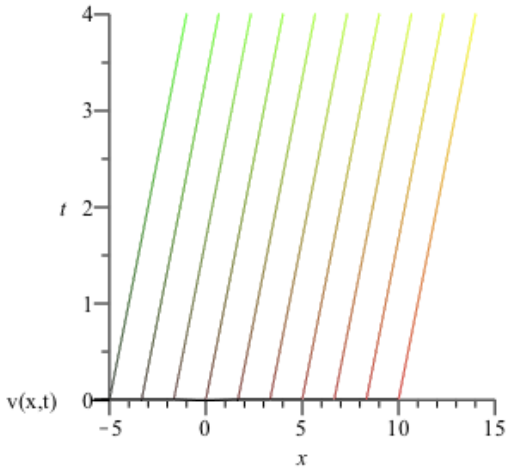


**Fig. 3.4** This figure depicts the characteristics for the linear advection equation discussed in this section for the case of $c = 1$. Each characteristic is described by the equation $\tau = x - t$. The one on the far left is $-5 = x - t$ and the one on the right right is $5 = x - t$.

$$v(x,t) = e^{-(x-ct)^2}.$$

Initially the solution is a Gaussian centred at $x = 0$ and it simply probates along to the right and at time $t = T$ it is centred along $cT$.

Note: when you derive a solution it is always possible to check whether it satisfies the PDE and the IC. In this case we can verify that this is indeed the unique solution we were looking for.

### 3.2.5 Example with conditions at all time

**Example 3.2.2** A slightly different problem is

$$v_t + 2v_x = -v, \quad \text{with} \quad v(x,t) = \frac{1}{1+x^2}, \quad \text{on} \quad x + t = 0.$$

**Solution:**

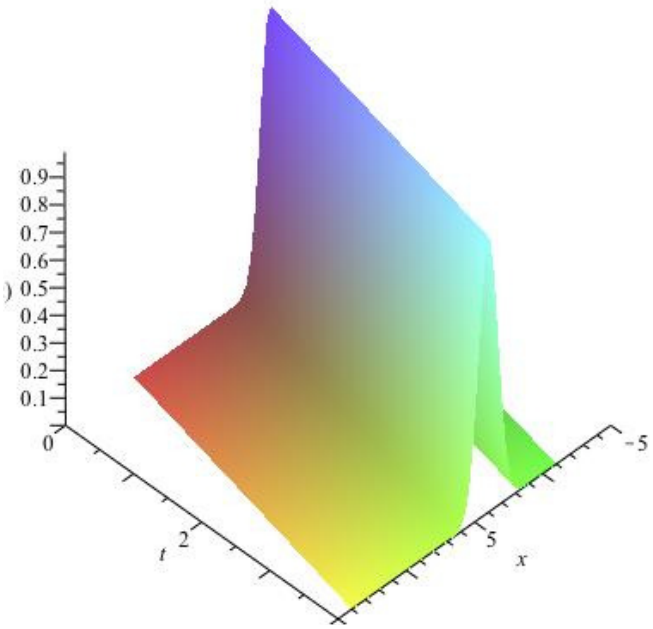


**Fig. 3.5** This figure depicts the solution for the linear advection equation discussed in this section for the case of $c = 1$ using a surface plot. As we saw previously, the solution is simply transported along never changing it's shape.

Note that this question is different in several ways: 1) the speed is set to 2, 2) the RHS of the equation is non-zero, 3) the initial condition is specified and most importantly 4) the "initial curve" is not at $t = 0$ but at t=-x. That means that the information is given for all time but at a particular location in each time. We still use the same strategy as before to solve this system.

1) The initial curve $\Gamma$ is given by $x = -t$ and therefore we can parameterization it as,

$$x = \tau, \quad t = -\tau, \quad , v = \frac{1}{1+\tau^2}.$$

2) The characteristic equations are,

$$\frac{dt}{ds} = 1, \quad t|_{s=0} = -\tau,$$
$$\frac{dx}{ds} = 2, \quad x|_{s=0} = \tau.$$

We integrate the first equation to get,

$$t = s + \alpha(\tau),$$

and then evaluate this at $s = 0$ to obtain,

$$\boxed{t = s - \tau}.$$

Similarly, we integrate the $x$ equation and obtain,

$$x = 2s + \beta(\tau),$$

which after applying the initial condition becomes,

$$\boxed{x = 2s + \tau}.$$

3) We substitute our result into the ODE,

$$\frac{dv}{ds} = -v, \quad v|_{s=0} = \frac{1}{1+\tau^2}.$$

This equation is linear and separable so we have the choice of techniques. If we use the former we get,

$$v = \delta(\tau)e^{-s}.$$

Then, apply the IC to find that,

$$\boxed{v = \frac{e^{-s}}{1+\tau^2}}.$$

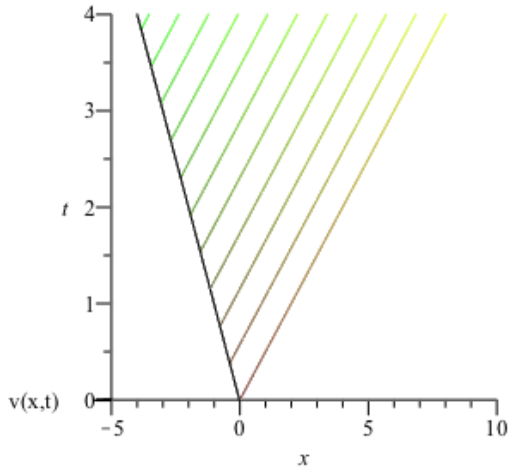


**Fig. 3.6** This figure depicts the characteristics for the solution to example 2 in the course notes.

4) We must invert the equations $t = s - \tau$ and $x = 2s + \tau$ to find $s(x,t)$ and $\tau(x,t)$. If we add the equations we get

$$s = \frac{x+t}{3}.$$

Similarly, if you subtract two times the first equation from the second equations you get,

$$\tau = \frac{x-2t}{3}.$$

Therefore, our complete solution is,

$$\boxed{v = \frac{e^{-\left(\frac{x+t}{3}\right)}}{1+\left(\frac{x-2t}{3}\right)^2}}.$$

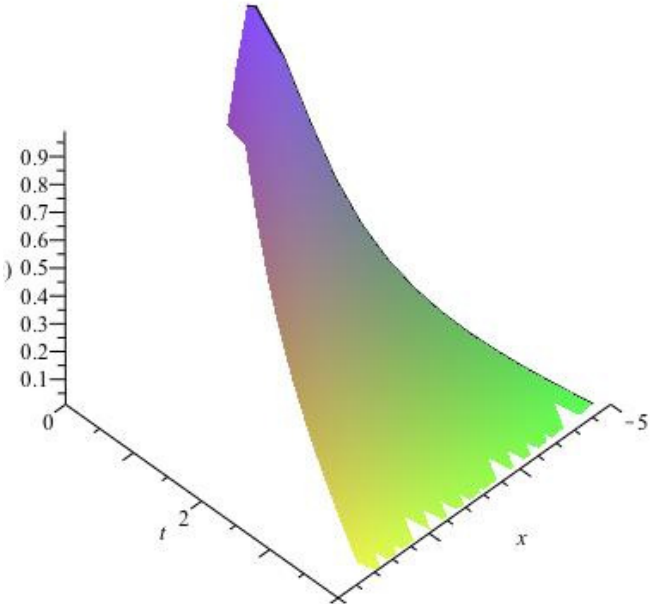


**Fig. 3.7** This figure depicts the surface plot of the solution to example 2 in the course notes. Note that Maple starts time at the top.

### 3.2.6 Boundary Value Problem

Consider the Boundary Value Problem,

$$\begin{aligned} u_t + u_x &= 0, \quad x > 0, \quad t > 0, \\ u(x,0) &= 0, \quad x > 0, \\ u(0,t) &= te^{-t}. \end{aligned}$$

We know that the information of this equation propagates to the right at a speed of 1. Therefore, there is information propagating from the boundary and also information propagating from the initial conditions. It is therefore convenient to separate the first quadrant into two regions divided by the curve $x = t$.

We could go through the details for the information propagating form the initial conditions but since at the initial time $u = 0$ we know that the solution must be zero everywhere below the curve $t = x$. Therefore, we only solve for the solution that is due to the boundary.

1) The “initial" curve $\Gamma$ is given by $x = 0$ and therefore we can parameterization it as,

$$x = 0, \quad t = \tau, \quad , v = \tau e^{-\tau}.$$

2) The characteristic equations are,

$$\frac{dt}{ds} = 1, \quad t|_{s=0} = \tau,$$
$$\frac{dx}{ds} = 1, \quad x|_{s=0} = 0.$$

We integrate the first equation to get,

$$t = s + \tau.$$

Similarly, we integrate the $x$ equation and obtain,

$$x = s.$$

3) We substitute our result into the ODE,

$$\frac{dv}{ds} = 0, \quad v|_{s=0} = \tau e^{-\tau}.$$

We integrate this and find,

$$v = \tau e^{-\tau}.$$

4) We must invert the equations $t = s + \tau$ and $x = s$. But that's easy since we can find that $\tau = t - x$. When we substitute this into our solution we get,

$$v(x,t) = (t-x)e^{x-t}.$$

### 3.2.7 d'Alembert's solution for the linear wave equation

The wave equation on an infinite domain with an initial position, $f(x)$, and velocity, $g(x)$, is

$$v_{tt} - c^2 v_{xx} = 0, \qquad -\infty < x < \infty, \quad t > 0,$$
$$v(x,0) = f(x), \quad v_t(x,0) = g(x), \quad -\infty < x < \infty.$$

As we have previously shown, this can be rewritten as two first-order systems of equations.

$$u_t + c u_x = 0,$$
$$v_t - c v_x = u$$

We use our four steps to find a general solution to the system that then yields a solution to the original equation. First we solve for $u$ and then we use this to solve for $v$.

1) The initial curve $\Gamma$ is given at $t = 0$ and therefore we can parameterization it as,

$$x = \tau, \quad t = 0,$$
$$v|_{s=0} = f(\tau),$$
$$u|_{s=0} = (v_t - c v_x)\,|_{s=0} = g(\tau) - c f'(\tau).$$

Note that the prime means the derivative with respect to the argument.

2) and 3) The characteristic equations for $u$ are,

$$\frac{dt}{ds} = 1, \quad t|_{s=0} = 0,$$
$$\frac{dx}{ds} = c, \quad x|_{s=0} = \tau,$$
$$\frac{du}{ds} = 0, \quad u|_{s=0} = g(\tau) - c f'(\tau).$$

The solutions to the three equations are,

$$
\begin{aligned}
t &= s, \\
x &= cs + \tau, \\
u &= g(\tau) - cf'(\tau).
\end{aligned}
$$

4) The inversion of the characteristics for the first equation is rather easy and we get $s = t$ and $\tau = x - ct$ and therefore the solution for $u$ is,

$$u = g(x - ct) - cf'(x - ct).$$

Now we can solve for $v(x, t)$. But first we substitute our solution for $u$ into the equation to find what we need to solve,

$$v_t - cv_x = g(x - ct) - cf'(x - ct).$$

Now we proceed using our 4 step method.

1) The initial curve $\Gamma$ is given at $t = 0$ and therefore we can parameterization it as,

$$x = \tau, \quad t = 0, \quad v|_{s=0} = f(\tau).$$

Note that the prime means the derivative with respect to the argument.

2) The characteristic equations for $v$ are,

$$
\begin{aligned}
\frac{dt}{ds} &= 1, \quad t|_{s=0} = 0, \\
\frac{dx}{ds} &= -c, \quad x|_{s=0} = \tau,
\end{aligned}
$$

The solutions to the three equations are,

$$\begin{aligned} t &= s, \\ x &= -cs + \tau. \end{aligned}$$

4) The inversion is $s = t$ and $\tau = x + ct$.

3) The equations for $v$ are,

$$\begin{aligned} \frac{dv}{ds} &= g(-cs + \tau - cs) - cf'(-cs + \tau - cs), \\ &= g(\tau - 2cs) - cf'(\tau - 2cs). \end{aligned}$$

The initial condition is

$$v|_{s=0} = f(\tau).$$

The solutions to the three equations are,

$$v = \int_0^s \left[g(\tau - 2c\zeta) - cf'(\tau - 2c\zeta)\right] d\zeta + f(\tau).$$

To integrate this we substitute $\beta = \tau - 2c\zeta$ that implies $d\beta = -2cd\zeta$ and get,

$$\begin{aligned} v &= f(\tau) - \frac{1}{2c}\int_\tau^{\tau-2cs} \left[g(\beta) - c\frac{df}{d\beta}\right] d\beta, \\ v &= f(\tau) + \frac{1}{2}\left[f(\tau - 2cs) - f(\tau)\right] - \frac{1}{2c}\int_\tau^{\tau-2cs} g(\beta)d\beta, \\ v &= \frac{1}{2}\left[f(x - ct) + f(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\beta)d\beta, \end{aligned}$$

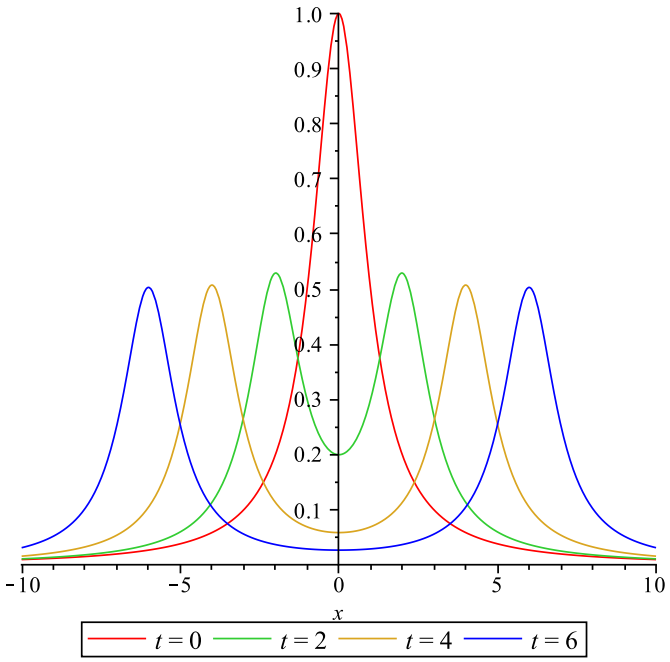


**Fig. 3.8** Plots of the exact solution to the wave equation at four different times. It shows how there are components that are propagating to the left and the right that are symmetric.

where in the last equation we substituted for $\tau$ and $s$ interns of $x$ and $t$ and also exchanged the order of integration.

Even though we changed the order a little bit we still followed out method and obtained a solution, which is indeed the general solution to the wave equation. Notice that there are two components: the first is due to the initial displacement and the second is because of the initial velocity. If initially both are zero then the solution stays at rest. However, if even one is non-zero then there will tend to be some oscillatory like motion.

The structure of the solution do to the initial condition shouldn't be too surprising. Previously we solved the advection equation (with constant speed) and found the solution is $f(x - ct)$ so that the initial disturbance travels to the right (if $c$ is positive) with a speed of $c$. As our factoring of the wave equation showed, it is simply a product of a right-ward advection equation and a left-ward advection equation. So half of the initial disturbance propagates to the right at a speed of $c$ and the other half propagates to the left at a speed of $c$.

### 3.2.8 Interpretation of d'Alembert's Solution

Recall that the general solution to the wave equation is,

$$v = \frac{1}{2}\left[f(x - ct) + f(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\beta)d\beta.$$

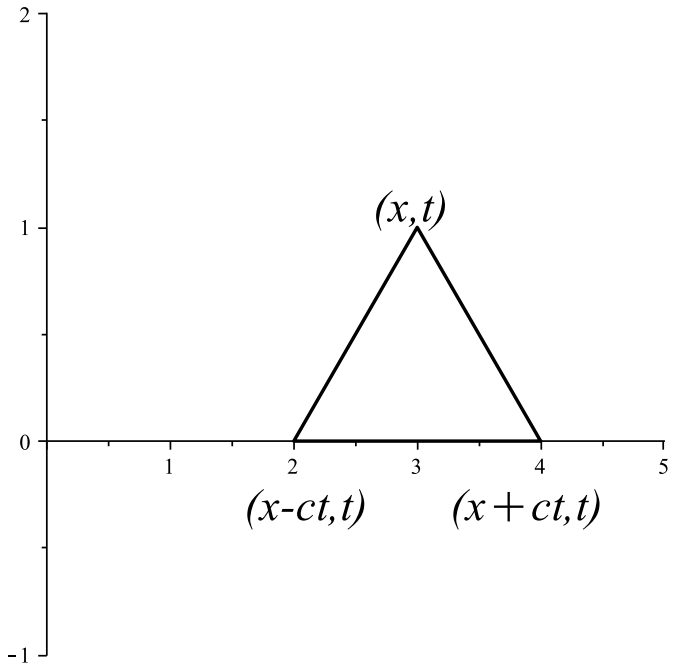


**Fig. 3.9** A plot of the two characteristic curves that transport information from the initial curve to any position $(x, t)$. This is appropriate for the initial displacement $v(x, 0)$.

The first term is only present if $f(x) \neq 0$ and the second is present if $g(x) \neq 0$. Below we consider two separate cases that help to illustrate the nature of the different components of the solution. Figure 3.8 shows the solution to the linear wave equation at several different times to illustrate the behaviour. This picked the initial velocity to be zero and the initial position to be $1/(1 + x^2)$.

First, suppose that $g(x) = 0$ so that we only have an initial displacement from rest. In this case the solution reduces to

$$v = \frac{1}{2}\left[f(x - ct) + f(x + ct)\right].$$

This means that the solution at time $t$ and position $x$ is equal to the average of the initial solution at $x - ct$ and $x + ct$. This is because information only propagates at a speed of $c$ either to the left or the right. In the case of a plucked string (as it's sometimes called) we need to look back along these two lines and if we know the initial conditions we know the solution. The interval between $x - ct$ and $x + ct$ is sometimes called the *domain of dependence*. Figure 3.9 shows this in $x - t$ space.

Second, assume that $f(x) = 0$ so the string is at rest but it has an initially velocity. Then the solution is

$$v = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\beta)d\beta,$$
$$v = \frac{1}{2c}\int_{x-ct}^{x+ct} v_t(\beta,0)d\beta.$$

This shows that the solution at time $t$ and position $x$ is determine by the initial velocity between $x-ct$ and $x+ct$. It is the integral over an interval of length $2c$, divided by $2ct$ and therefore we can think of it as the average velocity of all the initial velocities in the *domain of dependence*. Figure 3.10 shows the domain of dependence in $x-t$ space.

To understand why this is true let us first appreciate that the fundamental theorem of calculus states that,

$$\int_0^t v_t(x,\beta)d\beta,$$

since $v(x,0)=0$ as we have assumed. This says that *the position is equal to the integral of the velocity with respect to time*. This is true in particular for mechanics and for a continuum (i.e. the string) as well. That means if you integrate the velocity at position $x$ from the initial times to $t$ then we find the position at that time.

Notice that if we differentiate the solution with respect to time we get, after using the other part of the Fundamental Theorem of Calculus,

$$\begin{aligned} v(x,t)-v(x,0) &= \frac{d}{dt}\frac{1}{2c}\int_{x-ct}^{x+ct} v_t(\beta,0)d\beta, \\ &= \frac{d}{dt}\left[\frac{1}{2c}\int_0^{x+ct} v_t(\beta,0)d\beta - \frac{1}{2c}\int_0^{x-ct} v_t(\beta,0)d\beta\right], \\ &= \left[\frac{c}{2c}v_t(x+ct,0) - \frac{c}{2c}v_t(x-ct,0)\right], \\ &= \frac{1}{2}\left[g(x+ct)-g(x-ct)\right]. \end{aligned}$$

This shows that the velocity solves the same time of equation as in the first case. This says that the velocity at $(x,t)$ is an average of the initial velocity at positions $x-ct$ and $x+ct$.

### 3.2.9 Existence of Solutions

We have a method for finding solutions but unfortunately it does not guarantee that a unique solution always exists. Imagine for example that the *initial curve* happens to coincide with a

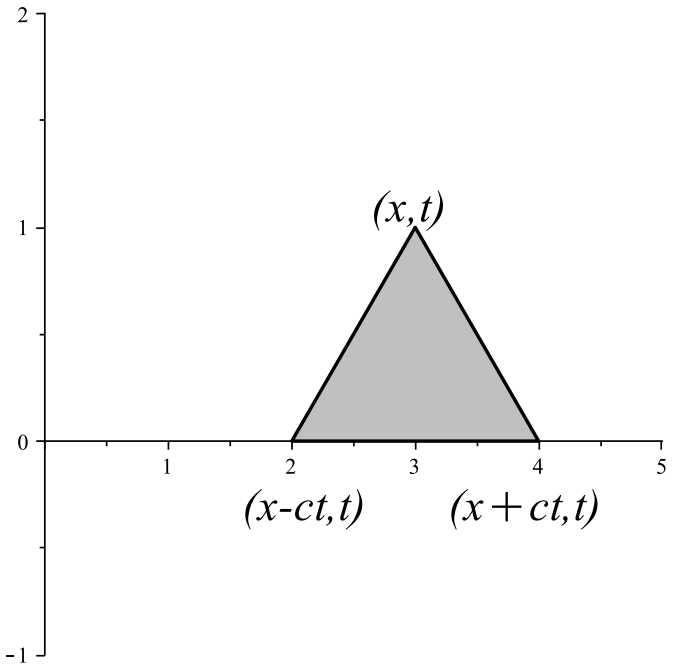


**Fig. 3.10** A plot showing the area that determines the solution at the point $(x,t)$ due to the non-zero initial velocity. Unlike the previous case this is the interior of the triangle, not just the boundary.

characteristic. Then, since the characteristics never intersect (for the linear case) we cannot expect information to propagate from the initial curve to all the other characteristics.

**Theorem 3.2.2.** *This somewhat intuitive result can be stated mathematically in terms of the Jacobian of the transformation. If the Jacobian,*

$$\left|\frac{\partial(x,t)}{\partial(s,\tau)}\right| = \begin{vmatrix} x_s & x_\tau \\ t_s & t_\tau \end{vmatrix} = \begin{vmatrix} b & x_\tau \\ a & t_\tau \end{vmatrix} = bt_\tau - ax_\tau \neq 0.$$

*then,*

$$x = x(s,\tau), \quad \textit{and} \quad t = t(s,\tau),$$

*can be inverted to yield,*

$$s = s(x,t), \quad \textit{and} \quad \tau = \tau(x,t).$$

Note the Jacobian not equal to zero is equivalent to saying that the initial curve and the characteristics are not linearly dependent, i.e. parallel.

This is not very detailed but it can be shown that with this criteria satisfied, and additional criteria for smoothness, that you do have that a solution exists.

### 3.2.10 Uniqueness of the Solution

**Theorem 3.2.3.** *The previous solution to the wave equation is unique.*

**Solution:**

Suppose that there exist two solutions, $v_1$ and $v_2$. Each satisfies the same wave equation and initial conditions,

$$\begin{aligned}\left(\partial_{tt} - c^2\partial_{xx}\right) v_1 = 0, \qquad -\infty < x < \infty, \quad t > 0,\\ v_1(x,0) = f(x), \quad \partial_t v_1(x,0) = g(x), \quad -\infty < x < \infty,\end{aligned}$$

and

$$\begin{aligned}\left(\partial_{tt} - c^2\partial_{xx}\right) v_2 = 0, \qquad -\infty < x < \infty, \quad t > 0,\\ v_2(x,0) = f(x), \quad \partial_t v_2(x,0) = g(x), \quad -\infty < x < \infty.\end{aligned}$$

We define $v = v_1 - v_2$. If we subtract the above PDEs and ICs we find that the variable $v$ satisfies the same PDE but with zero ICs,

$$\begin{aligned}\left(\partial_{tt} - c^2\partial_{tt}\right) v = 0, \qquad -\infty < x < \infty, \quad t > 0,\\ v(x,0) = 0, \quad \partial_t v(x,0) = 0, \quad -\infty < x < \infty.\end{aligned}$$

At this point if we use l'Alembert's solution we could find that $v = 0$ is a solution. However, maybe there are other solutions? To put this troublesome thought aside we need to use a different argument. What we do is we form an energy equation. To do this we multiply the above PDE by $v_t$ and then simplify,

$$\begin{aligned}v_t v_{tt} &= c^2 v_t v_{xx},\\ \frac{1}{2}\partial_t(v_t^2) &= c^2\partial_x(v_t v_x) - c^2 v_{tx} v_x,\end{aligned}$$

$$\frac{1}{2}\partial_t(v_t^2) = c^2\partial_x(v_t v_x) - c^2\partial_t(v_x^2),$$
$$\frac{1}{2}\partial_t\left(v_t^2 + c^2 v_x^2\right) - \partial_x(c^2 v_t v_x) = 0.$$

This equation requires some thought to appreciate what it says. We know that $v$ is the displacement, $v_t$ is the speed and therefore $\frac{1}{2}v_t^2$ is the kinetic energy at a given position and time. Also, $v_x$ is non-zero only if the string is not flat and therefore $c^2 v_x^2$ is proportional to the potential energy. The sum of these to is the *energy density*, really the total energy per unit length. The second term is what we call the energy flux. Note that it is similar to the flux equation that we saw in the first chapter.

If we integrate this equation over the entire real line and assume that the displacement vanishes at infinity, because the displacements come from the interior we get,

$$\int_{-\infty}^{\infty} \frac{1}{2}\partial_t\left(v_t^2 + c^2 v_x^2\right) dx = \int_{-\infty}^{\infty} \partial_x(c^2 v_t v_x) dx,$$
$$\frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}\left(v_t^2 + c^2 v_x^2\right) dx = \left[(c^2 v_t v_x)\right]_{-\infty}^{\infty},$$
$$\frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}\left(v_t^2 + c^2 v_x^2\right) dx = 0.$$

That is to say the total energy over the string is constant, or conserved, in time. Initially we have that $v = 0$ and $v_t = 0$ but the first implies that $v_x = 0$ at that time. Since the energy density is zero initially it must be zero for all time. That means that the energy density is zero at every point and therefore,

$$v_t = 0, \quad \text{and } v_x = 0.$$

This is because we have a sum of squares, each of which is non-negative. Since $v_t = 0$ and initially $v = 0$ that means that $v = 0$ for all time. Therefore, $v_1 = v_2$ and so the solution to the wave equation is unique.

## 3.3 QUASI-LINEAR EQUATIONS

The definition for a *Quasi-linear PDE* is like the linear PDE but now the coefficients can depend on the solution, $v$. The equation can be written as,

$$a(x,t,v)\frac{\partial v}{\partial x} + b(x,t,v)\frac{\partial v}{\partial t} = c(x,t,v).$$

The second-order analogue of this in the context of the wave equation is,

$$v_{tt} - c^2(x,t,v,v_t,v_x)v_{xx} = f(x,t,v,v_x,v_t).$$

That is to say that the coefficients can depend on the solution but only on derivatives that are one order less than the highest order that appears in the equation.

As we saw previously, the nonlinear wave equation with constant tension is,

$$v_{tt} - \frac{1}{(1+v_x^2)^{3/2}}v_{xx} = 0,$$

which is to say we have a nonlinear wave equation with,

$$c^2 = \frac{1}{(1+v_x^2)^{3/2}}.$$

This says that the speed at which the signal or waves propagate depends on the slope of the string. The stronger the slope the smaller the speed. In the case of weak slopes we can approximate the speed as a constant, in the case one.

If you consider a Gaussian shape we see that at the maximum where the slope is zero but to the left and right we have positions where the slope is strongest. In these locations the speed will be smaller. That means that the maximum travels faster than the string to the left and right. Mathematically, we could write this as,

$$c^2|_{v_x \approx 0} > c^2|_{v_x \neq 0}$$

That shows that the top or crest will start to overtake the string on the right and this creates what is called *amplitude steepening*. Given enough time this can create a singularity where the string is vertical. This is said to be singular because $v_x$ becomes infinite and the PDE starts to break down. The occurrence of a vertical string is called a *shock* and the process by which it forms is known as *shock formation*. This is the basic mechanism that explains why waves break at the beach and why people can surf!

### 3.3.1 Method of Characteristics for Quasi-linear 1st order PDEs

We will show that for a certain class of problems the method of characteristics that we previously established for linear equations extends naturally to quasi-linear equations. To begin suppose that we can parameterize a solution that passes through the initial curve in terms of $s$ as,

$$v = v(x(s), t(s)).$$

Then the chain rules implies that,

$$\frac{dv}{ds} = v_x \frac{dx}{ds} + v_t \frac{dt}{ds}.$$

As before, if we pick,

$$\begin{aligned} \frac{dx}{ds} &= a(x,t,v), \\ \frac{dt}{ds} &= b(x,t,v), \end{aligned}$$

then the PDE necessarily implies that,

$$\frac{dv}{ds} = c(x,t,v),$$

Note that unlike before the characteristic curves depend on the solution $v$. This is unlike before in that previously we could solve for the characteristics first and then determine after the fact how the solution changed along each curve. Now we must solve a system of three first order equations simultaneously. This is harder but can in fact be done in some cases.

**Theorem 3.3.1.** *Suppose the following. First, that $a(x,t,v), b(x,t,v)$ and $c(x,t,v)$ are smooth functions of their arguments. Second, that the initial data, say $\Gamma$, can be written parametrically in the form,*

$$x = \hat{x}(\tau), \quad t = \hat{t}(\tau), \quad \textit{and} \quad v = \hat{v}(\tau).$$

*Third, and finally, that the Jacobian is non-zero,*

$$\Delta(\tau) \equiv \frac{d\hat{t}}{d\tau} a(\hat{x}, \hat{t}, \hat{v}) - \frac{d\hat{x}}{d\tau} b(\hat{x}, \hat{t}, \hat{v}) \neq 0,$$

*then the PDE*

$$a(x,t,v)\frac{\partial v}{\partial x} + b(x,t,v)\frac{\partial v}{\partial t} = c(x,t,v).$$

*has a unique solution in at least some neighbourhood* $\Gamma$ *in space-time determined by,*

$$\begin{aligned}
\frac{dx}{ds} &= a(x,t,v), \quad x|_{s=0} = \hat{x}(\tau), \\
\frac{dt}{ds} &= b(x,t,v), \quad t|_{s=0} = \hat{t}(\tau), \\
\frac{dv}{ds} &= c(x,t,v), \quad v|_{s=0} = \hat{v}(\tau).
\end{aligned}$$

Note that in addition to having to work harder we are not guaranteed that if a solution exists it exists for all time. Instead, we are only ensured that it exists for some span of time. If you recall the example about shock formation, that solution breaks down exactly where the shock forms. After that more work needs to be done.

To show that a solution exists we need that $a, b, c$ are smooth and then by applying ODE theory we know that there is a unique solution that is continuously differentiable for $0 < s \leq s_*$ of the form,

$$x = x(\tau, s), \quad t = t(\tau, s), \quad \text{and} \quad v = v(\tau, s),$$

that satisfy the initial conditions,

$$x(\tau, 0) = \hat{x}(\tau), \quad t(\tau, 0) = \hat{t}(\tau), \quad v(\tau, 0) = \hat{v}(\tau).$$

The condition that $\Delta(\tau) \neq 0$ is equivalent to,

$$\left| \frac{\partial(x,t)}{\partial(s,\tau)} \right|_{s=0} \neq 0.$$

Hence, by the *Implicit Function Theorem*, there is a neighbourhood of $s = 0$, why we introduce $0 < s \leq s_* < \infty$ for which we can invert,

$$x = x(\tau, s), \quad t = t(\tau, s),$$

to obtain,

$$\tau = \tau(x, t), \quad s = s(x, t),$$

where $\tau(x, t) =$constant define the characteristic curves and the solution is,

$$v = v(\tau(x, t), s(x, t)) = V(x, t).$$

### 3.3.2 Inviscid Burger's Equation

We apply this method to the inviscid Burger's equation,

$$\begin{aligned} &v_t + vv_x = 0, \quad -\infty < x < \infty, t > 0, \\ &v(x, 0) = f(x), -\infty < x < \infty, \quad \text{and } f(x) \text{ is continuously differentiable.} \end{aligned}$$

The method of characteristics can be divided into several important steps.

1) Parameterize the curve $\Gamma$ from the ICs,

$$x = \tau, \quad t = 0, \quad v = f(\tau).$$

2) The characteristic equations that must be solved are,

$$\begin{aligned} \frac{dx}{ds} &= v, \quad x|_{s=0} = \tau, \\ \frac{dt}{ds} &= 1, \quad t|_{s=0} = 0, \\ \frac{dv}{ds} &= 0, \quad v|_{s=0} = f(\tau). \end{aligned}$$

We check that the Jacobian is non-zero,

$$\Delta(\tau) = (x_s \hat{t}_\tau - \hat{x}_\tau t_s)|_{s=0} = f(\tau)0 - (1)(1) = -1 \neq 0,$$

and therefore we are insured that a solution does exist.

First, we can solve for time,

$$t = s + \alpha(\tau) \qquad \Longrightarrow \qquad t = s,$$

because of the initial conditions. Second, because the RHS of the PDE is zero we can solve for $v$ and find

$$v = f(\tau).$$

Third, we substitute this into the equation for $x$,

$$\frac{dx}{ds} = v = f(\tau), \quad x|_{s=0} = \tau.$$

We integrate this,

$$x = sf(\tau) + \beta(\tau),$$

and apply the ICs to get,

$$x = sf(\tau) + \tau.$$

3) Therefore, our solution can be written implicitly as,

$$\begin{aligned} v(x,t) &= f(\tau), \\ x &= tf(\tau) + \tau. \end{aligned}$$

For a particular value of $(x, t)$ we solve the algebraic equation $x = tf(\tau) + \tau$ for $\tau$ and then substitute this into the solution $v = f(\tau)$ to obtain an explicit form for $v$.

Since we know that $v = f(\tau)$ we can also write the solution as,

$$v = f(x - tv).$$

This is attractive because it has the same form as the linear advection equation. We see that the initial condition is propagating forward but because $v$ is in the argument, it can get compressed or extended depending on the magnitude of the solution.

The characteristic curves in $(x, t)$-space occur for constant values of $\tau$ and are set by the equation,

$$x = f(\tau)t + \tau.$$

The slope of these lines is $f(\tau)$ which depends on the amplitude of the initial signal. But after we determine the characteristics, in this problem the solution $u = f(\tau)$ is constant along each characteristic.

### 3.3.3 Shock Formation

Recall that the solution can be written as

$$\begin{aligned} v(x, t) &= f(\tau), \\ x &= tf(\tau) + \tau. \end{aligned}$$

Suppose that the initial conditions at $t = 0$ are $v = f(x)$ that is a Gaussian like bump.

A point to the right of the maximum is $x_0$ and further along another is $x_1$. The IC at these positions are $f(x_0)$ and $f(x_1)$, respectively. We will always have $f(x_0) > f(x_1)$. Consider the two characteristics that pass through $(x, t) = (x_0, 0)$ and $(x, t) = (x_1, 0)$, respectively,

The characteristic coming from $x_0$ is determined by the equation $x = tf(x_0) + x_0$ and the solution along the curve is $v = f(x_0)$. Similarly, the other characteristic is set by $x = tf(x_1)+x_1$ and the solution is $v = f(x_1)$ along that curve. Mathematically, we can write these conditions as,

$$\begin{aligned} \frac{dx}{dt}|_{\tau=x_0} &= f(x_0), \\ \frac{dx}{dt}|_{\tau=x_1} &= f(x_1), \end{aligned}$$

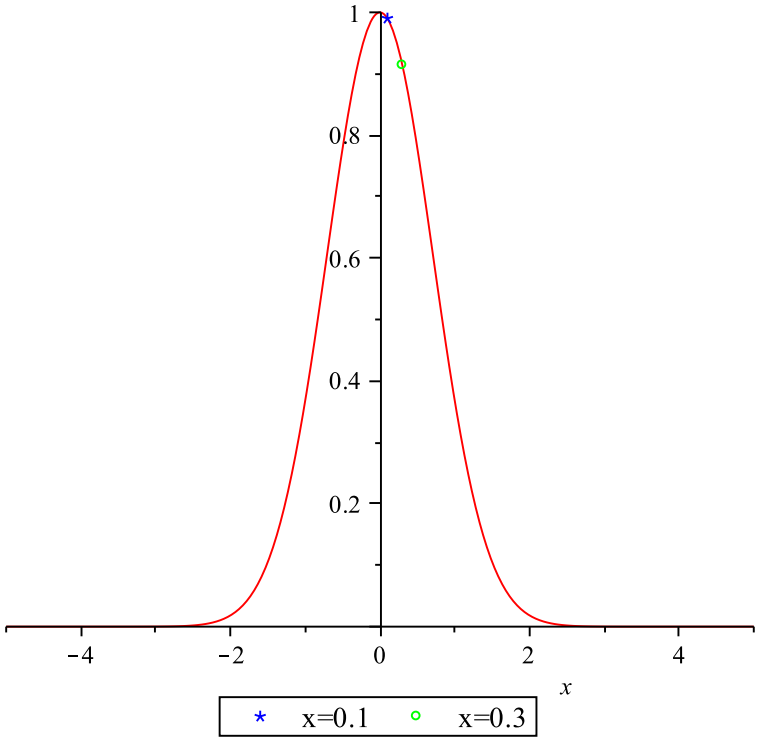


**Fig. 3.11** A plot of a Gaussian curve and two points to the right of centre.

with $f(x_0) > f(x_1)$. The following properties can be observed:

1) Since $f(x_0) > f(x_1)$ the slopes of the characteristics are not equal and also must intersect somewhere.

2) But along each characteristic, $v$ is constant and almost always have different values.

3) At the point of intersection, the theory suggests that $v$ becomes multivalued.

How is this possible? This occurs if there exists an $(x^*, t^*)$ such that $|v_x(x^*, t^*)| \to \infty$. This determines the location in space and time of the shock.

### 3.3.4 Determining the location and time of the shock

We begin with the PDE and the IC,

$$\begin{aligned} v_t + vv_x &= 0, \\ v(x, 0) &= f(x), \end{aligned}$$

that we know have a solution of

$$\begin{aligned} v(x,t) &= f(\tau), \\ x &= tf(\tau) + \tau. \end{aligned}$$

To determine where the slope becomes infinite we differentiate the solution and the characteristic equation with respect to $x$,

$$\begin{aligned} v_x &= f'(\tau)\tau_x, \\ 1 &= tf'(\tau)\tau_x + \tau_x. \end{aligned}$$

If we solve the second equation for $\tau_x$ we obtain,

$$\tau_x = \frac{1}{tf'(\tau)+1}.$$

We can substitute this into our equation for the slope and get,

$$v_x = f'(\tau)\tau_x = \frac{f'(\tau)}{tf'(\tau)+1}.$$

If we assume that initially the solution has finite slope, $|f'| < \infty$, that is to say there is no shock in the initial data, then the only way for a shock to occur is if the denominator is equal to zero. This can only happen if,

$$tf'(\tau) + 1 = 0, \quad \Longrightarrow \quad t = -\frac{1}{f'(\tau)}.$$

**Lemma 3.3.2.** *Since $t > 0$ a necessary and sufficient condition for a shock to occur in the example is that there exists an $x^*$ such that $f'(x^*) < 0$. That is to say if the initial data is decreasing anywhere, then a shock will occur in the inviscid Burger's equation.*

**Theorem 3.3.3.** *Suppose there exists $x^*$ such that $f'(x^*) < 0$, then the* first time *a shock will occur is determined by,*

$$t_s = \min_{\tau} \left( \frac{-1}{f'(\tau)} \right) > 0,$$

*and if the minimum of $-1/f'(\tau)$ is denoted by $\tau_{\min}$, then the first shock will occur at $x = x_s$ given by,*

$$x_s = t_s f(\tau_{\min}) + \tau_{\min},$$

*in the example.*

**Example 3.3.1** We look at finding solutions to the inviscid Burger's equation with particle initial conditions. The given problem is,

$$u_t + uu_x = 0, \quad \text{with} \quad u(x,0) = \frac{1}{1+x^2}.$$

We know that the solution can be written formally as,

$$u = \frac{1}{1+\tau^2} = f(\tau), \quad \text{where} \quad x = tf(\tau) + \tau = \frac{t}{1+\tau^2} + \tau.$$

With this we determine the time and location of shock formation.
We begin by differentiating the initial condition,

$$f'(\tau) = -\frac{2\tau}{(1+\tau^2)^2}.$$

Using our formula we can determine the time of the first shock,

$$t_s = \min_{\tau} \left( -\frac{1}{f(\tau)} \right) = \min_{\tau} \left( \frac{(1+\tau^2)^2}{2\tau} \right).$$

We look for the extrema of the function by computing the derivative,

$$\frac{d}{d\tau}\left(\frac{(1+\tau^2)^2}{2\tau}\right) = \left(\frac{3\tau^4+2\tau^2-1}{2\tau^2}\right).$$

This has its zeros at the roots of the polynomial

$$3\tau^4+2\tau^2-1 = (3\tau^2-1)(\tau^2+1),$$

which means that the only real roots are $\tau = 1/\sqrt{3}$. Therefore the minimal time is,

$$t_s = \frac{8}{3\sqrt{3}} \approx 1.54.$$

Figure 3.11 plots the function we are minimizing and it does show that the shock is close to $1.73$, as we have computed.

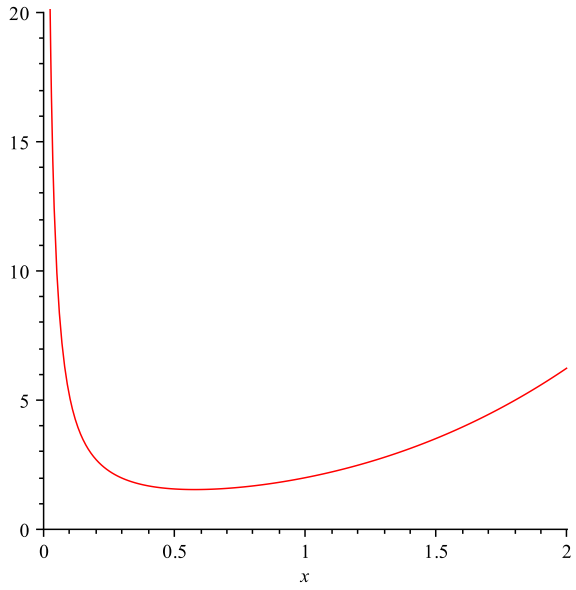


**Fig. 3.12** Figure of the function that we must minimize to determine the initial shock time and location.

Now that we know the shock time we can compute the shock location,

$$x_s = t_s f(\tau_s) + \tau_s = \frac{8}{9\sqrt{3}}\frac{1}{1+\frac{1}{3}} + \frac{1}{\sqrt{3}} = \frac{5}{3\sqrt{3}} \approx 0.96.$$

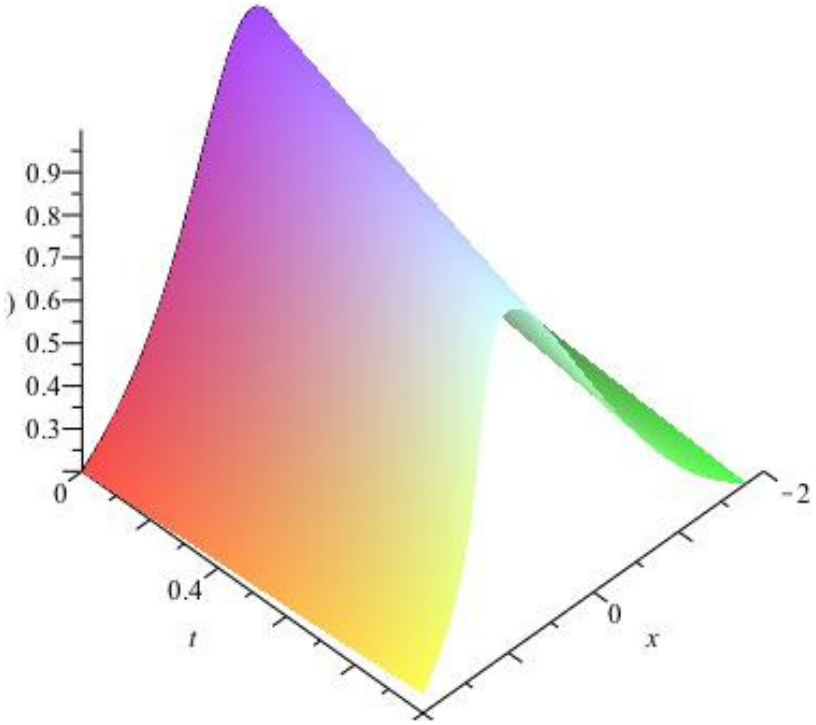


**Fig. 3.13** Surface plot of the solution to the nonlinear advection equation.

### 3.3.5 Expansion Fans

The final example we consider in the lectures is a case where a shock never occurs. Instead, something entirely different does, what we call an *expansion fan*. To understand what this is begin with the nonlinear Burger's equation where the initial condition is something like a Heaviside step function,

$$u_t + uu_x = 0, \quad \text{with} \quad u(x,0) = \begin{cases} B & x \leq a \\ A & x > a \end{cases},$$

with $B < A$. Figure 3.15 shows the plot of the initial conditions for the special case where $B = 1, A = 2, a = 0$. The formal solution can be written as before,

$$u = f(\tau), \quad \text{where} \quad \tau = x - tf(\tau).$$

For this initial condition we have that for $\tau > 0 \implies f(\tau) = A$, which yields,

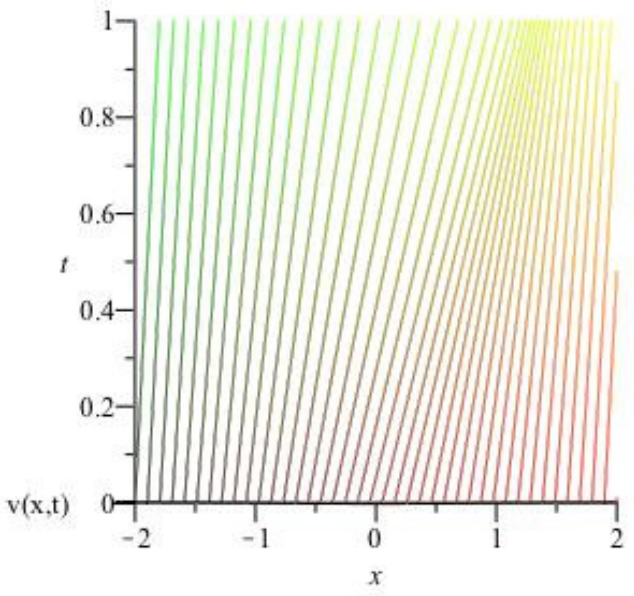


**Fig. 3.14** Characteristics of the solution to the nonlinear advection equation.

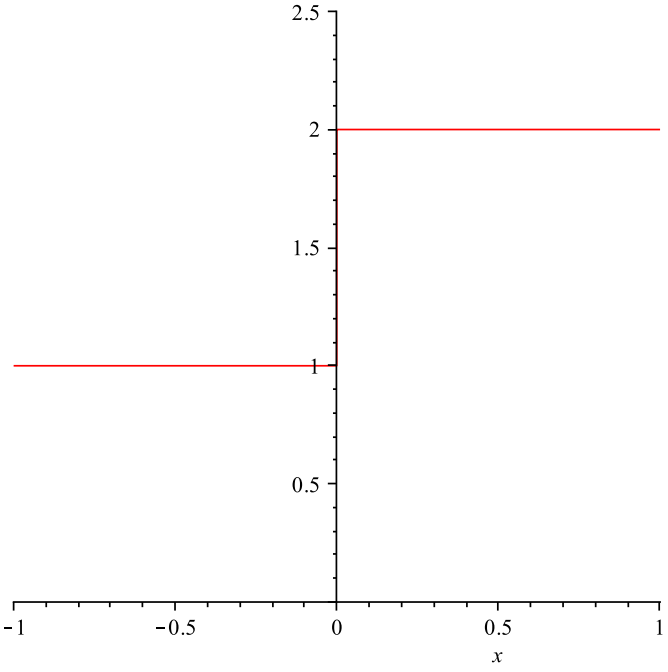


**Fig. 3.15** Characteristics of the solution to the nonlinear advection equation.

$$\tau = x - tA, \quad \Longrightarrow \quad x - tA > a \quad \Longrightarrow \quad x > a + tA.$$

This shows that the solution that begins to the right of $x = a$ moves to the right at a speed of $A$. This is not surprising since initially $u = A$, which is the propagation speed.

Similarly, if $\tau \leq a$ then $f(\tau) = B$ and

$$\tau = x - tB, \quad \Longrightarrow \quad x - tB \leq a \quad \Longrightarrow \quad x \leq a + tB.$$

This means that the boundary of the region $x \leq a$ is also propagating to the right but more slowly.

Combining the above two results yields the solution,

$$u(x,t) = \begin{cases} B & x \leq a + tB \\ A & x > a + tA \end{cases},$$

This raises the question, what happens for the domain,

$$a + tB < x < a + tA?$$

To fill in the void we insert an *expansion fan*. We define a function,

$$\phi(x,t) = \frac{x - x_0}{t - t_0},$$

which we note solves the PDE. This is easy to show because

$$\phi_t = -\frac{(x - x_0)}{(t - t_0)^2}, \quad \text{and} \quad \phi_x = \frac{1}{t - t_0},$$

and therefore,

$$\phi_t + \phi\phi_x = -\frac{(x - x_0)}{(t - t_0)^2} + \frac{x - x_0}{t - t_0}\frac{1}{t - t_0} = 0.$$

We must choose $x_0$ and $t_0$ to be consistent with our solution. By that I mean that,

$$\begin{aligned} \phi(a + tA, t) &= A, \quad \text{for all } t, \\ \phi(a + tB, t) &= B, \quad \text{for all } t. \end{aligned}$$

This translates to saying,

$$\begin{aligned} \frac{a + tA - x_0}{t - t_0} &= A, \quad \text{or} \quad a + t_0 A - x_0 = 0, \\ \frac{a + tB - x_0}{t - t_0} &= B, \quad \text{or} \quad a + t_0 B - x_0 = 0. \end{aligned}$$

If we subtract these two equations we deduce,

$$t_0(A - B) = 0, \quad \Longrightarrow \quad t_0 = 0.$$

If we substitute this back into either of the equations we further learn that,

$$x_0 = a.$$

Therefore, we have surmised that a solution to the PDE of this form that is consistent with our initial conditions and analytical solution is,

$$\phi(x, t) = \frac{x - a}{t}.$$

This implies that the complete solution can be written as

$$u(x, t) = \begin{cases} B & x \leq a + tB \\ \frac{x-a}{t} & a + tB < x < a + tA \\ A & x > a + tB \end{cases}.$$

This is known as an expansion fan. Figure 3.16 shows the characteristics for the solution with an expansion fan.

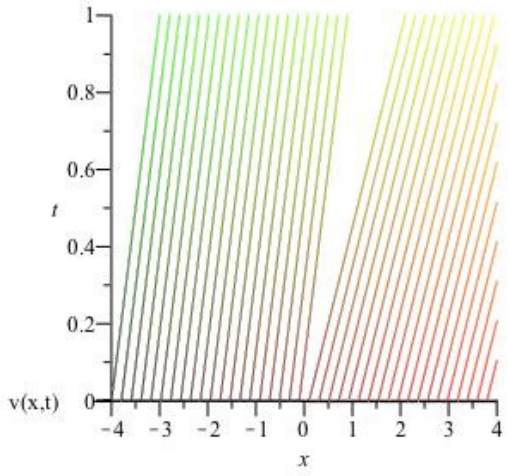


**Fig. 3.16** Characteristics of the solution to the nonlinear advection equation with an expansion fan.

CHAPTER
# FOUR

# Initial Boundary Value Problems in Bounded Domains

In this chapter we learn to solve initial boundary value problems in bounded domains using the *Method of Separation of Variables*. To do so will require solving *eigenvalue problems* of differential equations and what is called *Sturm-Liouville Theory*. This theory will be the foundation upon which we build our solutions.

## 4.1 INTRODUCTION

Here we present the governing equations and boundary conditions for the problems we will solve in this section.

### 4.1.1 Governing Equations

Before we begin recall that in higher-dimensions we can write the local version of the conservation law as

$$\rho \frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{\phi} = f.$$

We may pick the flux to be defined by Fick's or Fourier's law,

$$\vec{\phi} = -p(\vec{x})\vec{\nabla}u,$$

where the diffusion parameter is allowed to vary in space. The source is set by *Newton's law of cooling* so that it is a linear function of the solution,

$$f = -q(\vec{x})u + \rho F.$$

Physically, this states that the heat flow through the lateral boundary is proportional to the difference between the temperature of the rod and the external temperature. The function $q(\vec{x})$ determines how the cooling takes place at different locations in our domain.

Then the above equation can be written as,

$$\rho\frac{\partial u}{\partial t} - \vec{\nabla}\cdot\left(p\vec{\nabla}u\right) + qu = \rho F.$$

The one-dimensional analogue is,

$$\rho\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) + qu = \rho F.$$

With this very general equation we can define the following differential operator that only depends on space,

$$Lu \equiv -\vec{\nabla}\cdot\left(p\vec{\nabla}u\right) + qu.$$

Again, in one spatial dimension this becomes,

$$Lu \equiv -\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) + qu.$$

With this notation established, we can rewrite out equation as,

$$\boxed{\rho\frac{\partial u}{\partial t} + Lu = \rho F}.$$

This is a parabolic equation and we can recover the diffusion equation by picking $\rho = 1, p = 1, q = 0, F = 0$.

The associated elliptic equation is,

$$\boxed{Lu = \rho F}.$$

We can recover Laplace's equation by picking $p = 1, q = 0, F = 0$.

The related hyperbolic equation is,

$$\boxed{\rho \frac{\partial^2 u}{\partial t^2} + Lu = \rho F}.$$

The wave equation occurs for $\rho = 1, p = 1, q = 0, F = 0$.

*In general we assume that $\rho, p$ are strictly positive functions and that $q$ is non-negative. The forcing on the right-hand side can be of any sign depending on the nature of the forcing.*

### 4.1.2 Boundary Conditions

The most general boundary conditions that we consider in higher-dimensions is,

$$\alpha(\vec{x})u + \beta(\vec{x}) \left. \frac{\partial u}{\partial n} \right|_{\partial V} = B(\vec{x}, t).$$

where $\alpha(\vec{x}), \beta(\vec{x})$ and $B(\vec{x}, t)$ are prescribed functions on the boundary of the volume, again denoted by $\partial V$. If the problem is elliptic, so there is no time variation, then we need $B(\vec{x})$. Because $\partial u/\partial n$ is the exterior normal derivative on the boundary this boundary condition relates the values of $u$ with its flux through the given boundary. Notice that in the case where $\beta$ and $B$ are zero we have a homogeneous*Dirichlet* boundary condition. Also, if $\alpha$ and $B$ are zero we have a homogeneous*Neumann* boundary condition. If $B$ is not exactly zero than it is a non-homogeneous boundary condition, which is still okay.

In the one-dimensional case if the ends of the domain are at $x = 0$ and $x = L$, then this boundary reduces to,

$$\alpha_1 u(0, t) - \beta_1 \frac{\partial u}{\partial x}(0, t) = B_1(t),$$
$$\alpha_2 u(L, t) + \beta_2 \frac{\partial u}{\partial x}(L, t) = B_2(t).$$

Note that there is a sign difference between the boundary conditions at the left and the right. The reason is that on the real line the outward normal on the left points in the negative $x$-direction, and thus $\partial u/\partial n(0,t) = -\partial u/\partial x(0,t)$.

## 4.2 SEPARATION OF VARIABLES

In this section we show how the method of separation of variables can be applied to linear PDEs to find analytical solutions in terms of an infinite series. Even though you need an infinite number to get an exact solution we can usually truncate the series to get a good approximation to the solution.

### 4.2.1 Hyperbolic Case

If we consider a homogeneous PDE with boundary conditions, then the governing model is,

$$\rho(x)\frac{\partial^2 u}{\partial t^2} + Lu = \rho F,$$

with

$$\alpha(\vec{x})u + \beta(\vec{x})\left.\frac{\partial u}{\partial n}\right|_{\partial V} = 0,$$

in higher dimensions and

$$\alpha_1 u(0,t) - \beta_1 \frac{\partial u}{\partial x}(0,t) = 0,$$
$$\alpha_2 u(L,t) + \beta_2 \frac{\partial u}{\partial x}(L,t) = 0,$$

in one dimension. But to find a unique solution we need to also impose two initial conditions,

$$u(x,0) = f(x), \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

The conditions aboves are stated in 1D but in higher dimensions, simply replace $x$ with $\vec{x}$ defined in the domain.

### 4.2.2 Parabolic Case

The homogeneous parabolic problem is very similar to before but now there is a first order partial derivative with respect to time, and consequently, one less initial condition to impose. The PDE is

$$\rho(x)\frac{\partial u}{\partial t} + Lu = \rho F,$$

with

$$\alpha(\vec{x})u + \beta(\vec{x})\left.\frac{\partial u}{\partial n}\right|_{\partial V} = 0,$$

in higher dimensions and

$$\alpha_1 u(0,t) - \beta_1 \frac{\partial u}{\partial x}(0,t) = 0,$$
$$\alpha_2 u(L,t) + \beta_2 \frac{\partial u}{\partial x}(L,t) = 0,$$

in one dimension. To find a unique solution, we also need to also impose one initial condition,

$$u(x,0) = f(x).$$

The condition above is stated in 1D but in higher dimensions simply replace $x$ with $\vec{x}$ defined in the domain.

### 4.2.3 Elliptic Case

The homogeneous elliptic problem has no time-derivatives and the PDE is

$$\rho(x)\frac{\partial^2 u}{\partial y^2} - Lu = 0,$$

with

$$\alpha(\vec{x})u + \beta(\vec{x})\left.\frac{\partial u}{\partial n}\right|_{\partial V} = 0,$$

and no initial conditions.

**Example 4.2.1** We begin by looking at one-dimensional problems with constant coefficients: i.e. $\rho = p = 1$ and $q = 0$. In the three different cases the equations reduce down to the following.

1) Hyperbolic: **Initial Boundary Value Problem**

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L, \quad t > 0 \\
\alpha_1 u(0,t) - \beta_1 \frac{\partial u}{\partial x}(0,t) &= 0, \quad t > 0 \\
\alpha_2 u(L,t) + \beta_2 \frac{\partial u}{\partial x}(L,t) &= 0, \quad t > 0, \\
u(x,0) &= f(x), \quad 0 \le x \le L \\
\frac{\partial u}{\partial t}(x,0) &= g(x) \quad 0 \le x \le L.
\end{aligned}$$

2) Parabolic: **Initial Boundary Value Problem**

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L, \quad t > 0 \\
\alpha_1 u(0,t) - \beta_1 \frac{\partial u}{\partial x}(0,t) &= 0, \quad t > 0 \\
\alpha_2 u(L,t) + \beta_2 \frac{\partial u}{\partial x}(L,t) &= 0, \quad t > 0 \\
u(x,0) &= f(x), \quad 0 \le x \le L.
\end{aligned}$$

3) Elliptic: **Boundary Value Problem**

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \quad 0 \le x \le L, \quad 0 \le y \le L_y \\
\alpha_1 u(0,t) - \beta_1 \frac{\partial u}{\partial x}(0,t) &= 0, \quad 0 \le y \le L_y
\end{aligned}$$

$$\begin{aligned}\alpha_2 u(L,t) + \beta_2 \frac{\partial u}{\partial x}(L,t) = 0, \quad 0 \le y \le L_y \\ u(x,0) = f(x), \quad 0 \le x \le L \\ u(x,L_y) = g(x) \quad 0 \le x \le L.\end{aligned}$$

### 4.2.4 Separating the solution

We assume that it is possible to find a solution that consists of a product of a function of $x$ and a function of $t$, for the wave and diffusion equations, and a function of $x$ and a function of $y$ for Laplace's equation,

$$u(x,t) = M(x)N(t), \quad \text{and} \quad u(x,y) = M(x)N(y).$$

We substitute these solutions into the three different homogeneous equations and then divide by $\rho MN$. Because we are looking for nontrivial solutions we can assume that the functions are not always zero.

We begin with the hyperbolic equation,

$$\begin{aligned}\rho(x)\frac{\partial^2 u}{\partial t^2} &= -Lu, \\ \rho(x)M(x)N''(t) &= -N(t)LM(x), \\ \frac{N''(t)}{N(t)} &= -\frac{LM(x)}{\rho(x)M(x)}.\end{aligned}$$

Note that in the above we use primes to denote the derivative of the function with respect to its argument. As long as we know what the independent variable is for each function there is no ambiguity. Similarly for the parabolic equation,

$$\begin{aligned}\rho(x)\frac{\partial u}{\partial t} &= -Lu, \\ \rho(x)M(x)N'(t) &= -N(t)LM(x), \\ \frac{N'(t)}{N(t)} &= -\frac{LM(x)}{\rho(x)M(x)},\end{aligned}$$

and the elliptic equation,

$$
\begin{aligned}
-\rho(x)\frac{\partial^2 u}{\partial y^2} &= -Lu, \\
-\rho(x)M(x)N''(y) &= -N(t)LM(x), \\
-\frac{N''(y)}{N(y)} &= -\frac{LM(x)}{\rho(x)M(x)},
\end{aligned}
$$

In each of the three equations the right hand side depends only on $x$ and the left hand side only depends on the other variable, whether it be $t$ or $y$. Because $x$ is an independent variable, this is only possible if the functional dependency cancels out and we have that each equation is a constant. This is the only consistent choice we can make.

If we assume the constant is $-\lambda$, then the right hand side of each equation yields,

$$
LM(x) = \lambda\rho(x)M(x).
$$

The left hand sides of the equations each yields a corresponding equation in terms of this constant,

$$
\begin{aligned}
N''(t) + \lambda N(t) &= 0, \quad \text{hyperbolic}, \\
N'(t) + \lambda N(t) &= 0, \quad \text{parabolic}, \\
N''(y) - \lambda N(y) &= 0, \quad \text{elliptic}.
\end{aligned}
$$

The three problems each have the same mixed homogeneous boundary conditions in the $x$-direction. To impose these on our separable solution we require that $M(x)$ satisfies these boundary conditions. Therefore, if we combine our ODE and boundary conditions we get,

$$
\begin{aligned}
&\frac{1}{\rho}LM = \lambda M, && && (4.1)\\
&\alpha_1 M - \beta_1 M' = 0, && \text{at} \quad x = 0, && (4.2)\\
&\alpha_2 M + \beta_2 M' = 0, && \text{at} \quad x = L. && (4.3)
\end{aligned}
$$

Note that if we multiply the second and third equations above by $N$ and combine $N$ and $M$ to get $u$ we do recover the same boundary conditions on $u$ as were originally specified in the problem.

The idea of eigenvalues is not new to you since you would have seen eigenvalues of matrices in Linear Algebra. This is the same idea but generalized to linear functions. In the context of matrices we ask, for a particular matrix, what vectors do we act the matrix on to get a scalar multiple of the vector. The vector in question is the eigenvector and the scalar multiple is the eigenvalue. In the context of differential operators it is very similar. What functions do we act a linear operator on to get a scalar multiple of that function. It turns out to work for a special class of functions, called eigenfunctions, and the multiple is again the eigenvalue.

Solving this second order ODE and two boundary conditions for $M(x)$ is what is called an *eigenvalue problem*. The possible values for $\lambda$ that are permitted are called the *eigenvalues* and the associated functions that match it are the *eigenfunctions*. Together these form what are called a *Sturm-Liouville problem*. The differential operator that we are studying are $1/\rho L$. It turns out that for our particular choice of $L$ we have that the operator is *self-adjoint* and also positive, i.e. the eigenvalues are strictly positive. It turns out that in general there is not only one solution but a countably infinite number of solutions, something that we will investigate further in the next section.

### *Self-Adjoint and Positivity*

Next we show that this operator that arises naturally from our general PDE through separation of variables is self-adjoint and positive. The former is shown using the the same techniques that we used before in the context of the Laplacian.

Before we show this we introduce some notation that originates from Linear Algebra. Before we do realize that in equation 4.1 there is a function, $\rho(x) > 0$ that we assume is strictly positive.

**Definition 4.2.1.** *We define* **the inner product** *of two functions $f$ and $g$ with* **weight** $\rho(x) > 0$ *as,*

$$(f, g) = \int_0^\ell \rho(x) f(x) g(x)\, dx, \tag{4.4}$$

*where $\ell$ is the length of the domain.*

*The equation above is stated for the case of 1D but in higher dimensions the equation looks almost identical except for more integrals,*

$$(f, g) = \iiint_V \rho(\vec{x}) f(\vec{x}) g(\vec{x})\, dV. \tag{4.5}$$

*This inner product is for real functions and is clearly symmetric since it doesn't matter what order we integrate the product of $\rho$, $f$ and $g$.*

**Definition 4.2.2.** *Two functions $f$ and $g$ are said to be* **orthogonal** *with respect to the weight $\rho$ if,*

$$(f, g) = 0.$$

**Definition 4.2.3.** *With an inner product we define the* **norm** *of a function $f(x)$ to be,*

$$||f|| = \sqrt{(f, f)} = \sqrt{\int_0^L \rho(x) f^2(x)\, dx}$$

*Note that the last equation above can easily be modified in the higher-dimensional case by putting in the correct inner product.*

*The norm is non-negative for real valued functions $f$ and measures the magnitude of the function. If $f$ is a continuous function then $||f|| = 0$ if and only if $f = 0$. Usually people define the norm with $\rho(x) = 1$ but for our purposes we need this generality.*

**Definition 4.2.4.** *We say a differential operator $L$ is* **self-adjoint** *if*

$$(w, Lu) = (Lw, u),$$

*for any two functions $u$ and $w$. We will show that this property requires knowing the differential operator and certain boundary conditions.*

With the foundations well defined now we can go through the process of showing that the linear differential operator that arises in our eigenvalue problem is in fact self adjoint. To do this we begin by looking at the difference between $wLu - uLw$:

$$\begin{aligned} wL(u) - uL(w) &= w\left[-\vec{\nabla}\cdot\left(p\vec{\nabla}u\right) + qu\right] - u\left[-\vec{\nabla}\cdot\left(p\vec{\nabla}w\right) + qw\right], \\ &= -w\vec{\nabla}\cdot\left(p\vec{\nabla}u\right) + u\vec{\nabla}\cdot\left(p\vec{\nabla}w\right), \\ &= -\vec{\nabla}\cdot\left(pw\vec{\nabla}u\right) + \vec{\nabla}w\cdot\left(p\vec{\nabla}u\right) + \vec{\nabla}\cdot\left(pu\vec{\nabla}w\right) - \vec{\nabla}u\cdot\left(p\vec{\nabla}w\right), \\ &= -\vec{\nabla}\cdot\left(p\left[w\vec{\nabla}u - u\vec{\nabla}w\right]\right). \end{aligned}$$

With this *divergence relation* established we integrate over the volume $V$ and then use the divergence theorem to transform the integral on the right hand side to a surface integral over the boundary $\partial V$

$$\begin{aligned}\iiint_V \left[wL(u) - uL(w)\right]\, dV &= -\iiint_V \vec{\nabla}\cdot\left(p\left[w\vec{\nabla}u - u\vec{\nabla}w\right]\right)\, dV,\\ &= -\iint_{\partial V}\left(p\left[w\vec{\nabla}u - u\vec{\nabla}w\right]\right)\cdot\vec{n}\, dA,\\ &= -\iint_{\partial V} p\left[w\frac{\partial u}{\partial n} - u\frac{\partial w}{\partial n}\right]\, dA.\end{aligned}$$

But in addition to the PDE we also have homogeneous boundary conditions,

$$\alpha(\vec{x})u + \beta(\vec{x})\left.\frac{\partial u}{\partial n}\right|_{\partial V} = 0,$$

that we can use to rewrite the above equation to see that the right-hand side becomes,

$$-\iint_{\partial V} p\left[w\frac{\partial u}{\partial n} - u\frac{\partial w}{\partial n}\right]\, dA = -\iint_{\partial V} p\left[-\frac{\alpha}{\beta}wu + \frac{\alpha}{\beta}uw\right]\, dA = 0.$$

Therefore, solutions to the eigenvalue problem, both PDE and boundary conditions, satisfies the above equation that we can rewrite in terms of the inner product,

$$\left(w, \frac{1}{\rho}Lu\right) = \left(u, \frac{1}{\rho}Lw\right),$$

which does show that the operator is self-adjoint.
To show that the operator is positive we consider the following,

$$\begin{aligned}\left(u, \frac{1}{\rho}Lu\right) &= \iiint_V \left[uLu\right],\\ &= \iiint_V \left[-u\vec{\nabla}\cdot\left(p\vec{\nabla}u\right) + qu^2\right].\end{aligned}$$

We use a product rule and the divergence theorem to simply this to,

$$\begin{aligned}\left(u, \frac{1}{\rho}Lu\right) &= \iiint_V \left[-\vec{\nabla}\cdot\left(pu\vec{\nabla}u\right) + p(\vec{\nabla}u)^2 + qu^2\right]\, dV, \\ &= \iiint_V \left[p(\vec{\nabla}u)^2 + qu^2\right] dV - \iint_{\partial V} pu\frac{\partial u}{\partial n}\, dA.\end{aligned}$$

Note, the product rule that we use is

$$f\vec{\nabla}\cdot\vec{v} = \vec{\nabla}\cdot(f\vec{v}) - \vec{\nabla}f\cdot\vec{v}.$$

Again, if we substitute the boundary condition we get,

$$\left(u, \frac{1}{\rho}Lu\right) = \iiint_V \left[p(\vec{\nabla}u)^2 + qu^2\right] dV + \iint_{\partial V} p\frac{\alpha}{\beta}u^2\, dA.$$

Because $p, q, \alpha, \beta$ are all non-negative and $u$ only appears in quadratic terms, we see that the right hand side cannot be negative. At worse it can be zero. But for it to be zero it requires that each term is zero. That means that $u = 0$, which is the trivial solution. It is for this reason that we deduce that,

$$\iiint_V uLu\, dV = \left(u, \frac{1}{\rho}Lu\right) \geq 0.$$

This means that $1/\rho L$ is a positive operator and if $\rho = 1$ then that implies that $L$ is a positive operator.

### *Eigenfunction Expansions*

Suppose for the moment we have found solutions to the eigenvalue problem, equations (4.1) to (4.3), and that we denote each eigenvalue and eigenfunction pair $\lambda_k$ and $M_k$, for $k = 1, 2, 3\ldots$. Later we will show a variety of properties that these solutions must posses but for now we will simply show a couple before we start building our general solution to the original PDE.

First, if we recall the self-adjointness property and pick $u = M_k$ and $w = M_j$ to be two solutions of the eigenvalue problem then the equation becomes,

$$(M_k, \frac{1}{\rho}L[M_j]) - (M_j, \frac{1}{\rho}L[M_k]) = 0.$$

Next, using the fact that each solution satisfies the eigenvalue problem we know that $\frac{1}{\rho}LM_k = \lambda_k M_k$ and a similar equation for $M_j$, we obtain,

$$(M_k, \lambda_j M_j) - (M_j, \lambda_k M_k) = (\lambda_j - \lambda_k)(M_k, M_j) = 0,$$

because the inner product is independent of the order, as we previously stated. The last identity above shows that if if two eigenfunctions have distinct eigenvalues, then they are orthogonal, i.e. their inner product is zero.

Later we will show that the eigenvalues are necessarily real.

The next property that we concern ourselves with is the sign of the eigenvalues Previously we showed that the differential operator $\frac{1}{\rho}L$ is positive. If we pick $u = v = M_k$, this implies that

$$\begin{aligned} \left(M_k, \frac{1}{\rho}L[M_k]\right) &\geq 0, \\ \lambda_k \left(M_k, M_k\right) &\geq 0, \\ \lambda_k ||M_k||^2 &\geq 0, \quad \text{which yields ,} \\ \lambda_k &\geq 0, \end{aligned}$$

by using the eigenvalue equation and the fact that the norm is strictly non-negative. This result, that the eigenvalues are non-negative is very important and using this result later can save us a lot of time rather than having to verify this in the context of each equation. In the next section we will show there are a countably infinite number of eigenvalues and corresponding eigenfunctions.

Consider a particular eigenvalue that for simplicity we assume to be positive, $\lambda_k > 0$; later we will worry about when it can be zero or not. For the three different classes of equations we have that the temporal component of the problem is governed by the following equations,

$$\begin{aligned} N_k''(t) + \lambda_k N_k(t) &= 0, \quad \text{hyperbolic,} \\ N_k'(t) + \lambda_k N_k(t) &= 0, \quad \text{parabolic,} \\ N_k''(y) - \lambda_k N_k(y) &= 0, \quad \text{elliptic.} \end{aligned}$$

We inserted the index $k$ to emphasize the fact that these equations do change for each eigenvalue. Given that we are assuming the eigenvalues are positive we know how to solve each of these ODEs using our knowledge of AMATH 250. The solutions can be written as,

$$\begin{aligned} N_k &= a_k \cos(\sqrt{\lambda_k} t) + b_k \sin(\sqrt{\lambda_k} t), \quad \text{hyperbolic}, \\ N_k &= a_k \exp(-\lambda_k t), \qquad\quad \text{parabolic}, \\ N_k &= a_k \cosh(\sqrt{\lambda_k} y) + b_k \sinh(\sqrt{\lambda_k} y), \quad \text{elliptic}. \end{aligned}$$

In each case the constants $a_k$ and possibly $b_k$ must be determined for a given problem.

Note that previously, you would probably have written the last equation in terms of $\exp(\pm\sqrt{\lambda_a} t)$ as the basis of the solution space, and that is completely valid. However, because the hyperbolic trigonometric functions are linear superpositions of these functions they yield the same solution space. Furthermore, it becomes easier to impose boundary conditions when we use the hyperbolic trigonometric functions.

Whichever case we are considering we can write a solution to the PDE as,

$$u_k = M_k(x) N_k(t)$$

This solution for a given $\lambda_k$ satisfies the homogeneous PDE as well as the homogeneous boundary conditions. This solution will not in general solve the initial conditions, which is true for every eigenfunction. If we want to find a solution that can reproduce any initial condition we need to use all of the eigenfunctions that we found. We can do this since the PDE and BCs are linear and therefore we can use the superposition principle to sum over all possible values of $k$. By that I mean we have that the following is a solution to everything except the initial conditions,

$$u = \sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} M_k(x) N_k(t).$$

Because the function $N_k(t)$ has constants we still have some freedom to specify our solution. For the hyperbolic case we have two initial conditions that we must satisfy and they become,

$$\begin{aligned} u(x,0) &= \sum_{k=1}^{\infty} M_k(x) N_k(0) = \sum_{k=1}^{\infty} a_k M_k(x) = f(x), \\ \frac{\partial u}{\partial t}(x,0) &= \sum_{k=1}^{\infty} M_k(x) N_k'(0) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} b_k M_k(x) = g(x). \end{aligned}$$

If we look at the last identity in each line we see that we have what are called *eigenfunction expansions* of the functions $f(x)$ and $g(x)$ in terms of our eigenfunctions. To be able to do this in general we need that the eigenfunctions form a *complete basis*. This is analogous to linear algebra where we need a complete set of eigenvectors to reproduce an arbitrary vector in our vector space. Showing completeness is a bit complicated but one requirement that is necessary but not sufficient is that there is an infinite number of eigenfunctions.

Assuming that the eigenfunctions form a complete set we are able to determine the *Fourier coefficients*, $a_k$ and $b_k$ by using a *Gram-Schmidt orthogonalization*. To do this we essentially project the initial condition onto each basis function, something that we can do using the inner product because the eigenfunctions are orthogonal.

If we multiply the first equation by $M_j$ and then integrate over the domain, this can be restated as taking the inner product of this equation with respect to the function $M_j$ we get,

$$\begin{aligned}
(f(x), M_j) &= \left( \sum_{k=1}^{\infty} a_k M_k, M_j \right), \quad \text{use linearity,} \\
&= \sum_{k=1}^{\infty} a_k \left( M_k, M_j \right), \quad \text{use orthogonality,} \\
&= \sum_{k=1}^{\infty} a_k \delta_{kj} \left( M_k, M_j \right), \quad \text{with the Kronecker delta,} \\
&= a_j \left( M_j, M_j \right).
\end{aligned}$$

In summary we have determined that the Fourier coefficient, say $a_k$, is,

$$a_k = \frac{(f(x), M_k)}{(M_k, M_k)}.$$

This equation shows that the $k$-th Fourier coefficient is equal to the projection of the initial solution $f(x)$ onto $M_k$, divided by the norm of the eigenfunction in question. We have to divide by the norm because our eigenfunctions are only orthogonal, but not *orthonormal*, that is to say they have norm one. If we do a little bit more work at the beginning by defining an orthonormal basis,

$$\hat{M}_k = \frac{M_k}{||M_k||}$$

then the above equation is replaced by,

$$a_k = \left( f(x), \hat{M}_k \right).$$

Using a similar argument for the initial velocity we can show that,

$$\begin{aligned}
(g(x), M_j) &= \left( \sum_{k=1}^{\infty} \sqrt{\lambda_k} b_k M_k, M_j \right), \quad \text{use linearity,} \\
&= \sum_{k=1}^{\infty} \sqrt{\lambda_k} b_k \left( M_k, M_j \right), \quad \text{use orthogonality,} \\
&= \sum_{k=1}^{\infty} \sqrt{\lambda_k} b_k \delta_{kj} \left( M_k, M_j \right), \quad \text{with the Kronecker delta,} \\
&= \sqrt{\lambda_k} b_k \left( M_j, M_j \right).
\end{aligned}$$

Form this we recover,

$$b_k = \frac{(g(x), M_k)}{\sqrt{\lambda_k} \left( M_k, M_k \right)},$$

or

$$b_k = \frac{1}{\sqrt{\lambda_k}} \left( g(x), \hat{M}_k \right).$$

After we have computed the Fourier coefficients we have our complete solution

$$u = \sum_{k=1}^{\infty} \left[ a_k \cos(\sqrt{\lambda_k} t) + b_k \sin(\sqrt{\lambda_k} t) \right] M_k(x).$$

For the parabolic equation we only have one initial condition to impose,

$$u(x, 0) = \sum_{k=1}^{\infty} M_k(x) N_k(0) = \sum_{k=1}^{\infty} a_k M_k(x) = f(x).$$

If we project onto the $k$-th eigenfuction we get the same solution as before,

$$a_k = \frac{(f(x), M_k)}{(M_k, M_k)}.$$

This yields the following solution,

$$u = \sum_{k=1}^{\infty} a_k \exp(-\lambda_k t) M_k(x).$$

For the elliptic case we have two boundary conditions to impose instead of initial conditions, and this yields the following two eigenfunction expansions,

$$\begin{aligned} u(x,0) &= \sum_{k=1}^{\infty} a_k M_k(x) = f(x), \\ u(x,L_y) &= \sum_{k=1}^{\infty} \sqrt{\lambda_k} \left[ a_k \cosh(\sqrt{\lambda_k} L_y) + b_k \sinh(\sqrt{\lambda_k} L_y) \right] M_k(x) = g(x). \end{aligned}$$

We project the boundary condition at $y = 0$ and recover the same solution for $a_k$ as before,

$$a_k = \frac{(f(x), M_k)}{(M_k, M_k)}.$$

For the second coefficient we need to do a bit more work but the method of projecting still works,

$$\begin{aligned} (g(x), M_j) &= \left( \sum_{k=1}^{\infty} \sqrt{\lambda_k} \left[ a_k \cosh(\sqrt{\lambda_k} L_y) + b_k \sinh(\sqrt{\lambda_k} L_y) \right] M_k, M_j \right), \\ &= \sum_{k=1}^{\infty} \sqrt{\lambda_k} \left[ a_k \cosh(\sqrt{\lambda_k} L_y) + b_k \sinh(\sqrt{\lambda_k} L_y) \right] (M_k, M_j), \\ &= \sqrt{\lambda_j} \left[ a_j \cosh(\sqrt{\lambda_j} L_y) + b_j \sinh(\sqrt{\lambda_j} L_y) \right] (M_j, M_j). \end{aligned}$$

With this equation we can solve for $b_j$ in terms of the $a_j$ that we already know,

$$b_k = \frac{1}{\sinh(\sqrt{\lambda_k} L_y)} \left[ \frac{(g(x), M_k)}{\sqrt{\lambda_k}\,(M_k, M_k)} - a_k \cosh(\sqrt{\lambda_k} L_y) \right].$$

Then, the general solution is

$$u = \sum_{k=1}^{\infty} \left[ a_k \cosh(\sqrt{\lambda_k} y) + b_k \sinh(\sqrt{\lambda_k} y) \right] M_k(x).$$

In this subsection we have derived formal solutions in that we know the solutions for the hyperbolic, parabolic and elliptic problems that solve the PDE and extra conditions, if we know the eigenvalues and eigenfunctions. In the next section we focus on the properties of the solutions without actually finding the solutions.

## 4.3 STURM-LIOUVILLE PROBLEM AND FOURIER SERIES

Instead of proving the properties in general we focus on the one-dimensional eigenvalue problem,

$$L[v(x)] = -\frac{d}{dx}\left[ p(x) \frac{dv}{dx} \right] + q(x) v(x) = \lambda \rho(x) v(x),$$

on $0 < x < \ell$ with the mixed boundary conditions,

$$\alpha_1 v(0) - \beta_1 \frac{dv}{dx}(0) = 0,$$
$$\alpha_2 v(\ell) + \beta_2 \frac{dv}{dx}(\ell) = 0.$$

Collectively, the ODE and two BCs are known as the *Sturm-Liouville problem* if $p(x) > 0, \rho(x) > 0$ and $q(x) \geq 0$. Furthermore, we also require that $p(x), \rho(x), q(x)$ and $p'(x)$ are continuous on the closed interval $0 \leq x \leq \ell$.

If furthermore, for the boundary conditions we require $\alpha_i \geq 0, \beta_i \geq 0$ and $\alpha_i + \beta_i > 0$ for $i = 1, 2$ then we have a *Regular Sturm-Liouville Problem*. If one of these conditions related to the boundary is not satisfied then we have a *Singular Sturm-Liouville Problem*.

### 4.3.1 Definitions

Next, we restate some definitions in lower-dimensions, where we allow for complex functions. Below is a list of definitions that we will need.

1) The **(Hermitian) inner product** for two functions, possibly complex, $\varphi(x)$ and $\psi(x)$ is

$$(\varphi, \psi) = \int_0^L \rho(x)\varphi(x)\overline{\psi(x)}\, dx,$$

with respect to the weighting function $\rho(x) > 0$. In the case where the two functions are real it reduces to what we saw previously. The symmetry property for the complex case is,

$$(\varphi, \psi) = \overline{(\psi, \varphi)}.$$

2) The **norm** of a function $\varphi(x)$ is

$$||\varphi(x)||^2 = (\varphi, \varphi) = \int_0^L \rho(x)\varphi(x)\overline{\varphi(x)}\, dx, = \int_0^L \rho(x)|\varphi(x)|^2\, dx,$$

which is necessarily non-negative. This is sometimes called the two-norm.

3) If the $2-$norm of a function is finite then the function is said to be **square integrable**.

4) Given a function $\varphi(x)$ that is square integrable we can redefine it so that it has a norm of one. When this is done we say that the function is **normalized**.

5) Two functions are said to be orthogonal if,

$$(\varphi, \psi) = 0.$$

6) A set of functions $\{\varphi_k(x)\}$ for $k = 1, 2, 3, \cdots$ is said to be an **orthogonal set** if

$$(\varphi_k, \varphi_j) = 0, \quad \text{if } j \neq k.$$

7) An orthogonal set that is normalized is said to be an **orthonormal set**.

8) Given an orthonormal set of square integrable functions, $\{\varphi_k(x)\}$ for $k = 1, 2, 3, \cdots$, we define the **Fourier Coefficients** of a square integrable function $\varphi(x)$ to be,

$$(\varphi, \varphi_k)\,.$$

9) The **Fourier Series** of $\varphi(x)$ is,

$$\varphi(x) = \sum_{k=1}^{\infty} (\varphi, \varphi_k)\, \varphi_k(x).$$

When introduced to power series in MATH 138 you determined the formula for the Taylor coefficients of a Taylor series for a particular function. This shows that if the power series of a function exists the coefficients must have this particular form. Then, it was necessary to show that the power series actually converged to the given function. Otherwise the coefficients become rather meaningless. To do that we define a sequence of partial sums and then determine whether that sequence converges or not.

### 4.3.2 Convergence

To better understand convergence, consider the $N$-th partial sum of a Fourier Series,

$$\sum_{k=1}^{N} (\varphi, \varphi_k)\, \varphi_k(x).$$

The difference between the actual function and the $N$-th partial sum can be computed as follows,

$$\begin{aligned}
\|\varphi - \sum_{k=1}^{N} (\varphi, \varphi_k)\, \varphi_k(x)\|^2 &= \left( \varphi - \sum_{k=1}^{N} (\varphi, \varphi_k)\, \varphi_k(x), \varphi - \sum_{k=1}^{N} (\varphi, \varphi_k)\, \varphi_k(x) \right), \\
&= (\varphi, \varphi) - 2\sum_{k=1}^{N} (\varphi, \varphi_k)^2 + \sum_{k=1}^{N} (\varphi, \varphi_k)^2\,, \\
&= \|\varphi\|^2 - \sum_{k=1}^{N} (\varphi, \varphi_k)^2 \geq 0.
\end{aligned}$$

The last inequality holds because the norm is a non-negative function by definition. Previously we used the property that the inner product is linear and furthermore that the set is orthonormal. This implies that,

$$\sum_{k=1}^{N} (\varphi, \varphi_k)^2 \leq \|\varphi\|^2,$$

for any value of $N$ and therefore this must hold as $N \to \infty$. Therefore, we get in this limit that,

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \leq \|\varphi\|^2,$$

what is known as *Bessel's Inequality*. This shows that the sum of the square of the Fourier coefficients of any square integrable function $\varphi(x)$ converges. This requires that,

$$(\varphi, \varphi_k) \to 0, \quad \text{as} \quad k \to \infty.$$

**Definition 4.3.1.** *A sequence of square integrable functions* $\{\psi_N(x)\}, N = 1, 2, \cdots$, *is said to* **converge to a function** $\varphi(x)$ **in the mean** *if,*

$$\lim_{N\to\infty} \|\varphi(x) - \psi_N(x)\| = 0.$$

*This is referred to as* **mean square convergence**.

If we define the $N$-th partial sums to be,

$$\psi_N(x) = \sum_{k=1}^{N} (\varphi, \varphi_k)\, \varphi_k(x),$$

and if *Parseval's Equality*,

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = \|\varphi\|^2,$$

holds then this implies that,

$$\lim_{N\to\infty} \|\varphi(x) - \psi_N(x)\|^2 = \lim_{N\to\infty} \|\varphi(x) - \sum_{k=1}^{N} (\varphi, \varphi_k)\, \varphi_k(x)\|^2 = 0.$$

This follows from the previous expansion that we presented. Therefore, if Parseval's equality is satisfied then the Fourier series converges in the mean square sense.

**Definition 4.3.2.** *A set of square integrable functions is said to be complete if for any square integrable function $\varphi(x)$, its Fourier series converges to it in the mean.*

Showing completeness is an advanced topic that is beyond the scope of this course.

### 4.3.3 Properties

P1) *Eigenfunctions corresponding to different eigenvalues are orthogonal.*

To prove this suppose that $\lambda_i$ and $\lambda_j$ are two distinct eigenvalues with $v_i(x)$ and $v_j(x)$ as their corresponding eigenfunctions. Using Green's identity we have,

$$\begin{aligned}
\int_0^L \left[v_i \overline{Lv_j} - \overline{v_j} L v_i\right]\, dx &= \int_0^l \left[\overline{v_j}(pv_i')' - v_i(p\overline{v_j}')'\right]\, dx, \\
&= \int_0^l \frac{d}{dx}\left[p\overline{v_j}v_i' - pv_i\overline{v_j'}\right]\, dx, \\
&= 0,
\end{aligned}$$

because of the boundary conditions that we are imposing on both $v_i$ and $v_j$.

Alternatively, we can also show that

$$\int_0^l \left[v_i L v_j - v_j L v_i\right]\, dx = (\lambda_j - \lambda_i) \int_0^l \rho v_i \overline{v_j}\, dx = (\lambda_j - \lambda_i)(v_i, v_j).$$

Because the eigenvalues are distinct we then conclude that the inner product of the two associated eigenfunctions is zero,

$$(v_i, v_j) = 0,$$

which is to say that the two eigenfunctions are orthogonal.

P2) *The eigenvalues are real and non-negative and the eigenfunctions may be chosen to be real valued.*

To prove this result suppose that $\lambda_i$ is complex valued with eigenfunction $v_i$. Since $p(x), q(x), \rho(x)$ are all real functions, we can compute the complex conjugate of the eigenvalue relation and find that $\overline{\lambda_i}$ and $\overline{v_i}$ are also an eigenpair,

$$Lv_i = \rho\lambda_i v_i, \qquad \text{and} \qquad L\overline{v_i} = \rho\overline{\lambda_i}\overline{v_i}.$$

Since we know the operator is self-adjoint, we can substitute in $v_i$ and obtain

$$\begin{aligned}
\left(v_i, \frac{1}{\rho}L[v_i]\right) - \left(\frac{1}{\rho}L[v_i], v_i\right) &= 0, \\
(v_i, \lambda_i v_i) - (\lambda_i v_i, v_i) &= 0, \\
\overline{\lambda_i}\,(v_i, v_i) - \lambda_i\,(v_i, v_i) &= 0, \\
\left(\overline{\lambda_i} - \lambda_i\right)\|v_i\|^2 &= 0.
\end{aligned}$$

Since the eigenvalue is non-trivial it must have a non-zero norm and therefor $\overline{\lambda_i} - \lambda_i = 0$, which means that the eigenvalue is necessarily real.

Next, we show non-negativity,

$$\begin{aligned}
\left(v, \frac{1}{\rho}Lv\right) &= -\int_0^l v\frac{d}{dx}\left(pv'\right)'\,dx + \int_0^l qv^2\,dx, \\
&= -pvv'|_0^l + \int_0^l pv'^2\,dx + \int_0^l qv^2\,dx.
\end{aligned}$$

But the boundary conditions imply that,

$$-p(l)v(l)v'(l) = \begin{cases} \frac{\alpha_2}{\beta_2}p(l)v(l)^2 \geq 0, & \beta_2 > 0 \\ 0, & \beta_2 = 0, \end{cases}$$

and similarly,

$$p(0)v(0)v'(0) = \begin{cases} \frac{\alpha_2}{\beta_2} p(0)v(0)^2 \geq 0, & \beta_2 > 0 \\ 0, & \beta_2 = 0, \end{cases}$$

Therefore, when we include the boundary conditions we conclude that,

$$\left(v, \frac{1}{\rho} Lv\right) \geq 0,$$

because each term on the right-hand side is non-negative. As an aside, the right-hand side will only be zero for trivial solutions.

If we substitute in the DE we get that,

$$\left(v, \frac{1}{\rho} Lv\right) = \lambda \left(v, v\right) = \lambda \|v\|^2,$$

which when combined with our inequality yields that the eigenvalues are non-negative,

$$\lambda = \frac{\left(v, \frac{1}{\rho} Lv\right)}{\|v\|^2} \geq 0.$$

We should appreciate that it is the positivity of the operator that then implies the positivity of the eigenvalues in the above equation.

Given that the eigenvalue is real it is conceivable that the eigenfunction is complex, say $v_i = v_R + iv_I$. If this were the case then the eigenvalue relation would yield

$$\begin{aligned} L[v_i] &= \rho \lambda_i v_i, \\ L[v_R + iv_I] &= \rho \lambda_i (v_R + iv_I), \\ L[v_R] + iL[v_I] &= \rho \lambda_i v_R + i\rho \lambda_i v_I. \end{aligned}$$

Since the real and imaginary parts must each vanish separately, we deduce that both the real and imaginary parties of the eigenfunctions, which must necessarily be real, satisfy the same eigenvalue relation. Therefore, if we solve this relation for all the eigenvalues and eigenfunctions, we have all of the solutions.

P3) *Each eigenvalue is simple*: that is to say each eigenvalue has a multiplicity of one. For each eigenvalue $\lambda$ there are two linearly independent solutions to the DE because it is second order in space. Since each solution solves the same DE and boundary conditions, they must be equal by uniqueness. This is not shown explicitly here but one can show that the Wronskian of the two functions is zero and therefore one is simply a scalar multiple of each other and, if then they are normalized, they are equal.

P4) *There is a countable infinity of eigenvalues having a limit point at infinity*: That is to say the set of eigenvalues can be ordered,

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots,$$

with $\lambda_k \to \infty$ as $k \to \infty$.

**Definition 4.3.3.** *The set of eigenvalues is called the* **spectrum** *of the differential operator.*

For the Sturm-Liouville problem the spectrum of the operator $L$ is discrete, non-negative and is unbounded.

P5) *The set of eigenfunctions* $\{v_k(x)\}, k = 1, 2, \cdots$ *forms a complete orthonormal set of square integrable functions over the interval* $0 < x < l$. As a consequence of this property we have that the eigenfunction expansion (or Fourier Series) converges to the function in the mean,

$$v(x) = \sum_{k=1}^{\infty} (v, v_k) v_k(x).$$

If $v(x)$ is continuous then it can be shown that the series converges uniformly to $v(x)$ in the interval in question.

### 4.3.4 Examples

It is possible to express our eigenfunctions, i.e. solutions to the eigenvalue problem, in terms of two particular solutions. We define $V(x;\lambda)$ and $W(x;\lambda)$ to be solutions to the BVP with conditions,

$$V(0;\lambda) = 1, \qquad V'(0;\lambda) = 0, \tag{4.6}$$
$$W(0;\lambda) = 0, \qquad W'(0;\lambda) = 1. \tag{4.7}$$

Note that there are two conditions specified at $x = 0$ and therefore $V$ and $W$ solve initial value problems. We will show how it is possible to write the solution to our BVP in terms of these two IVPs. In particular, note that,

$$v(x;\lambda) = \beta_1 V(x;\lambda) + \alpha_1 W(x;\lambda),$$

solves the two boundary conditions at $x = 0$.

To enforce that it also solves the boundary conditions at $x = l$ we require,

$$\begin{aligned}\alpha_2 \left[\beta_1 V(l;\lambda) + \alpha_1 W(l;\lambda)\right] + \beta_2 \left[\beta_1 V'(l;\lambda) + \alpha_1 W'(l;\lambda)\right] = 0,\\ \alpha_2\beta_1 V(l;\lambda) + \alpha_2\alpha_1 W(l;\lambda) + \beta_2\beta_1 V'(l;\lambda) + \beta_2\alpha_1 W'(l;\lambda) = 0.\end{aligned}$$

This is an algebraic equation that determines the eigenvalues of the BVP, $\lambda = \lambda_k$ for $k = 1, 2, 3, \cdots$. After solving for these eigenvalues the eigenfunctions are,

$$v_k(x) = v(x;\lambda_k) = \beta_1 V(x;\lambda_k) + \alpha_1 W(x;\lambda_k).$$

Formally, this is all true but in practice it's usually easier to complete them directly, when we can.

*Trigonometric Eigenfunctions*

We set $p(x) = 1$, $\rho(x) = 1$ and $q(x) = 0$ in our boundary value problem. The equation simplifies to,

$$-v''(x) = \lambda v(x).$$

The solutions to this equation that satisfy equations (4.6) and (4.7) are,

$$V(x;\lambda) = \cos(\sqrt{\lambda}x), \quad \text{and} \quad W(x;\lambda) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}},$$

for $\lambda > 0$. Below we consider the case if $\lambda = 0$.

Then from the previous discussion we know that we can rewrite the transcendental equation that determines the eigenvalues as,

$$\alpha_2\beta_1 V(l;\lambda) + \alpha_2\alpha_1 W(l;\lambda) + \beta_2\beta_1 V'(l;\lambda) + \beta_2\alpha_1 W'(l;\lambda) = 0,$$
$$\alpha_2\beta_1 \cos(\sqrt{\lambda}l) + \alpha_2\alpha_1 \frac{\sin(\sqrt{\lambda}l)}{\sqrt{\lambda}} - \sqrt{\lambda}\beta_2\beta_1 \sin(\sqrt{\lambda}l) + \beta_2\alpha_1 \cos(\sqrt{\lambda}l) = 0,$$
$$\sqrt{\lambda}\,(\alpha_1\beta_2 + \alpha_2\beta_1)\cos(\sqrt{\lambda}l) + (\alpha_1\alpha_2 - \lambda\beta_1\beta_2)\sin(\sqrt{\lambda}l) = 0.$$

For given parameters $\alpha_i, \beta_i$ with $i = 1, 2$ we can find solutions to this equation that specify the spectrum of the eigenvalue problem. Once this is done we can write down the corresponding eigenfunctions as,

$$v_k(x) = v(x;\lambda_k) = \beta_1 \cos(\sqrt{\lambda_k}x) + \alpha_1 \frac{\sin(\sqrt{\lambda_k}x)}{\sqrt{\lambda_k}}, \quad \text{for } k = 1, 2, 3, \cdots$$

If $\lambda = 0$ then the general solution to the ODE is $v = Ax + B$ and we find that $V(x;\lambda) = 1$ and $W(x;\lambda) = x$. The eigenvalue equation then becomes,

$$\alpha_2\beta_1 V(l;\lambda) + \alpha_2\alpha_1 W(l;\lambda) + \beta_2\beta_1 V'(l;\lambda) + \beta_2\alpha_1 W'(l;\lambda) = 0,$$
$$\alpha_2\beta_1 + \alpha_2\alpha_1 l + +\beta_2\alpha_1 = 0.$$

This can only be true if $\alpha_1 = \alpha_2 = 0$. Therefore, the eigenfunction for the zero eigenvalue is,

$$v_0(x) = v(x;0) = \beta_1.$$

*Fourier Sine Series*

Let's consider a special case of the trigonometric eigenfunctions with,

$$\alpha_1 = \alpha_2 = 1, \quad \text{and} \quad \beta_1 = \beta_2 = 0.$$

Note that $\lambda = 0$ is not an eigenvalue in this case.
The eigenvalue equation is

$$\sin(\sqrt{\lambda}l) = 0, \quad \Longrightarrow \quad \lambda_k = \left(\frac{\pi k}{l}\right)^2.$$

Furthermore, the orthonormalized eigenfunctions are,

$$v_k(x) = v(x;\lambda_k) = \sqrt{\frac{2}{l}}\sin\left(\frac{\pi k x}{l}\right), \quad \text{for } k = 1, 2, 3, \cdots$$

This choice is known as a *Fourier Sine series*.

*Fourier Cosine Series*

Let's consider a special case of the trigonometric eigenfunctions with,

$$\alpha_1 = \alpha_2 = 0, \quad \text{and} \quad \beta_1 = \beta_2 = 1.$$

The eigenvalue equation with $\lambda > 0$ is

$$\sin(\sqrt{\lambda}l) = 0, \quad \Longrightarrow \quad \lambda_k = \left(\frac{\pi k}{l}\right)^2,$$

for $k = 1, 2, 3, \cdots$. But $\lambda = 0$ is also an eigenvalue so we pick $k = 0, 1, 2, 3, \cdots$. Furthermore, the orthonormalized eigenfunctions are,

$$\begin{aligned} v_0(x) &= \frac{1}{\sqrt{l}}, \\ v_k(x) &= \sqrt{\frac{2}{l}}\cos\left(\frac{\pi k x}{l}\right), \quad \text{for } k = 1, 2, 3, \cdots \end{aligned}$$

This choice is known as a *Fourier Cosine series*.

*Complete Fourier Series*

Let's consider a special case of the trigonometric eigenfunctions but now with periodic boundary conditions, which are not of the Sturm-Liouville type,

$$-v''(x) = \lambda v(x), \qquad -l < x < l,$$

with end conditions that,

$$v(-l) = v(l), \quad \text{and} \quad v'(-l) = v'(l).$$

It turns out that even though this is not a strict Sturm-Liouville eigenvalue problem, many of the properties still hold.

The general solution can be written as,

$$v(x;\lambda) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x).$$

for $k = 1, 2, 3, \cdots$. But $\lambda = 0$ is also an eigenvalue so we pick $k = 0, 1, 2, 3, \cdots$. For this solution to be periodic we require that each component is periodic and in particular that,

$$\lambda_k = \left(\frac{\pi k}{l}\right)^2, \quad \text{for} \quad k = 0, 1, 2, 3, \cdots$$

Note that for each $\lambda_k$ there are two eigenfunctions, the sine and the cosine, and therefore we do not have simple roots but instead have double roots. The eigenfunctions that we get is a combination of the Fourier Sine and Cosines series that we just presented, which are,

$$\begin{aligned}
\hat{v}_0(x) &= \frac{1}{\sqrt{2l}}, \\
\hat{v}_k(x) &= \sqrt{\frac{1}{l}}\cos\left(\frac{\pi k x}{l}\right), \\
v_k(x) &= \sqrt{\frac{1}{l}}\sin\left(\frac{\pi k x}{l}\right),
\end{aligned}$$

Orthogonality of the eigenfunctions is something that can be readily tested and we get that,

$$\begin{aligned}(\hat{v}_k, \hat{v}_\ell) &= (v_k, v_\ell) = 0, \quad k \neq l,\\ (\hat{v}_k, v_l) &= 0, \qquad k, l = 0, 1, 2, \cdots .\end{aligned}$$

This choice is known as a *Fourier series*. It turns out that any continuous function (even some that are not continuous) $v(x)$ can be decomposed as follows,

$$v(x) = (v, \hat{v}_0)\hat{v}_0 + \sum_{k=1}^{\infty} \left[(v, \hat{v}_k)\hat{v}_k(x) + (v, v_k)v_k(x)\right],$$

which converges to $v(x)$ in the mean. If $v(x)$ is continuous and has piecewise continuous first derivatives on $-l \leq x \leq l$ then the series converges uniformly to $v(x)$.

*Bessel Function Expansions*

If you look at the normal-modes on a circular drum it can be shown that the resulting equation for the radial direction is Bessel's equation. It is usually written as,

$$x^2 v'' + xv' + (x^2 - n^2)v = 0,$$

where $n$ determines the order of the DE, which we assume is a positive integer. If we rescale space $x = \sqrt{\lambda}x$ then the equation can be written as,

$$-x^2 v'' - xv' + n^2 v = \lambda x^2 v.$$

Next, to write this in Sturm-Liouville form we divide by $x$ and then combine the first two terms to get,

$$L[v(x)] = -\frac{d}{dx}\left(x\frac{dv}{dx}\right) + \left(\frac{n^2}{x}\right)v = \lambda x v,$$

on $0 < x < l$. This is a special case of our differential operator $L$ with $p(x) = x, \rho(x) = x$ and $q(x) = n^2/x$. These functions are positive on the interior but the first two vanish at $x$ and the third is actually singular at $x = 0$. But using some knowledge of ODES (see AMATH 351), $x = 0$ is a *regular singular point* of the ODE and it is possible to find bounded solutions in the form of a Frobenius power series.

The boundary conditions that we impose are,

$$v(x) \quad \text{is bounded at } x = 0, \quad \text{and} \quad v(l) = 0.$$

That is to say there is a Dirichlet BC at the right end of the interval, that can occur when the outer edge of the drum is clamped down. The first BC seems odd but can be interpreted to say that the interior of the drum never goes off to infinity. It doesn't seem to be asking for much and in fact it does allow us to find a complete set of eigenfunctions for this differential operator.

In the context of a regular Sturm-Liouville problem we have that the right hand side of the lower equation is zero, due to the mixed boundary condition, and therefore the eigenfunctions are orthogonal,

$$\int_0^l \left[v_i L v_j - v_j L v_i\right]\, dx = (\lambda_j - \lambda_i) \int_0^l v_i v_j \, dx = \left[p v_j v_i' - p v_i v_j'\right]_0^l .$$

For Bessel's equation we have that $v(l) = 0$ and therefore the only potentially non-zero contribution is from the contribution at $x = 0$. The boundary condition that we impose is that $v(0)$ is finite and therefore, for the eigenfunctions to be orthogonal we require that,

$$\lim_{x \to 0} p(x) v'(x) = 0.$$

Recall that $p(x) = x$, and therefore, as long as the limit of $v'(x)$ as $x \to 0$ is finite, then we have this above limit is true and the eigenfunctions to the Bessel's equation are indeed orthogonal.

For the linear harmonic oscillator, $y'' + y = 0$, two linearly independent solutions are $\sin(x)$ and $\cos(x)$ and they form a general solution to the DE. In a similar way we can find two linearly independent solutions to the Bessel's equation. The functions are not so simple however. To find these functions you can use a power-series expansion about $x = 0$, as is done in AMATH 351, and we can define the two solutions to be $J_n(x)$, the Bessel function of order $n$ of the first kind and $Y_n(x)$, the Bessel function of order $n$ of the second kind. The general solution can be written as,

$$v(x) = c_1 J_n(x) + c_2 Y_n(x).$$

By looking at the power series for each function it is easy to show that $Y_n(x)$ is singular as $x \to 0$, by that we mean that it is unbounded. Therefore, if we want solutions to the Bessel's equation that satisfy the bounded condition at the centre, then the solution can be written as,

$$v_n(x) = J_n(\sqrt{\lambda}x).$$

We have neglected the constant since eigenfunctions are defined within a constant.
To satisfy the second boundary condition we need to determine the values of $\lambda$ such that,

$$J_n(\sqrt{\lambda}l) = 0.$$

That is to say that the lambdas are determined by the zeros of the Bessel function. This is similar to how in the case of the simple harmonic oscillator the eigenvalues are set by the zeros of the trigonometric functions. The major difference is that the zeros of trig functions are evenly spaced and are well known. Bessel functions are less regular and zeros must be found numerically, or using a table if you want to do it the old fashioned way. But by the Sturm-Liouville theory, we can deduce that there are an infinite number of zeros to this equation.
Formally we can solve the above equation by saying that,

$$\lambda_{kn} = \left(\frac{\alpha_{kn}}{l}\right)^2, \quad \text{for} \quad k = 1, 2, 3, \cdots.$$

Note that $\alpha_{kn}$ is defined to be the $k$-th zero of the $n$-th order Bessel function of the first kind. With this established, we can write the different eigenfunctions that solve Bessel's equation along with the two boundary conditions as,

$$v_{kn}(x) = J_n\left(\frac{\alpha_{kn}x}{l}\right).$$

This is an orthogonal set of eigenfunctions but their norms are not one. Using the fact that,

$$\int_0^l x^2 J_n^2(\sqrt{\lambda_{kn}}x)\, dx = J_{n+1}^2(\sqrt{\lambda_{kn}}l),$$

we could define the orthonormal basis as,

$$\hat{v}_{kn}(x) = \frac{\sqrt{2}}{l}\frac{J_n\left(\alpha_{kn}\frac{x}{l}\right)}{|J_{n+1}\left(\alpha_{kn}\right)|}, \quad \text{with} \quad k = 1, 2, \cdots.$$

It can be shown that this set of eigenfunctions is complete and therefore we can reproduce any continuous function in terms of this orthonormal basis on the interval $0 \leq x \leq l$. The expansion takes the form,

$$v(x) = \sum_{k=1}^{\infty} (v(x), \hat{v}_{kn}(x))\, \hat{v}_{kn}(x).$$

## 4.4 SERIES SOLUTIONS OF BVPS AND IBVPS

In this section we show how the previous theory can be straight forwardly applied to PDEs to find solutions. We consider one equation in each class to see how the solutions differ. In particular we look at the wave equation, diffusion equation and Laplace's equation.

### 4.4.1 The Wave Equation

Recall that we derived the wave equation as a linearized model to describe the evolution of a tight string,

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0, \quad 0 < x < l, t > 0,$$

with $c = \sqrt{T/\rho}$ where $T$ is the tension and $\rho$ is the density, both of which we assume to be constant. To impose the constraint that the boundaries are clamped down we enforce homogeneous Dirichlet boundary conditions,

$$u(0,t) = 0, \quad \text{and} \quad u(l,t) = 0.$$

Thirdly, we must impose two initial conditions because we have a second order time-derivative in the wave equation. We impose the initial position and velocities,

$$u(x,0) = f(x), \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 < x < l.$$

We use the method of separation of variables, and substitute

$$u(x,t) = M(x)N(t)$$

into the PDE. This yields, upon dividing through by $c^2 MN$,

$$MN'' = c^2 M'' N,$$
$$\frac{N''}{c^2 N} = \frac{M''}{M}.$$

Since time and space are independent, we need that this is equal to a constant. Based on the equation that we write down next we know that the constant must be negative, since the eigenvalues are non-negative,

$$M'' + \lambda M = 0, \quad \text{with} \quad M(0) = 0, \quad M(l) = 0.$$

We substituted the separable solution into the boundary conditions to obtain the second and third equation above. The solutions to this equations are sine functions, as we have already seen,

$$\lambda_k = \left(\frac{\pi k}{l}\right)^2, \quad \text{with} \quad M_k(x) = \sqrt{\frac{2}{l}} \sin\left(\frac{\pi k}{l} x\right), \quad \text{for} \quad k = 1, 2, \cdots.$$

But this choice also implies that the temporal equation is,

$$N'' + c^2 \lambda N = 0,$$

whose solution is,

$$N_k(t) = a_k \cos\left(\frac{\pi k c}{l} t\right) + b_k \sin\left(\frac{\pi k c}{l} t\right), \quad k = 1, 2, 3, \cdots.$$

When we combine these we get that each mode, or each part of the solution is,

$$u_k(x,t) = \left[a_k \cos\left(\frac{\pi k c}{l} t\right) + b_k \sin\left(\frac{\pi k c}{l} t\right)\right] \sqrt{\frac{2}{l}} \sin\left(\frac{\pi k}{l} x\right), \quad k = 1, 2, \cdots.$$

These particular solutions solve the PDE and the two boundary conditions.

What is left to impose are the two initial conditions. To be able to solve any initial conditions we must superimpose over all the solutions to get a general solution,

$$u(x,t) = \sum_{k=1}^{\infty} \left[ a_k \cos\left(\frac{\pi k c}{l} t\right) + b_k \sin\left(\frac{\pi k c}{l} t\right) \right] \sqrt{\frac{2}{l}} \sin\left(\frac{\pi k}{l} x\right)$$

We now impose the two initial conditions,

$$\begin{aligned} u(x,0) &= \sum_{k=1}^{\infty} u_k(k,0) = \sum_{k=1}^{\infty} a_k \sqrt{\frac{2}{l}} \sin\left(\frac{\pi k}{l} x\right) = f(x), \\ u_t(x,0) &= \sum_{k=1}^{\infty} \frac{\partial u_k}{\partial t}(x,0) = \sum_{k=1}^{\infty} \left(\frac{\pi k c}{l}\right) b_k \sqrt{\frac{2}{l}} \sin\left(\frac{\pi k}{l} x\right) = g(x). \end{aligned}$$

This shows that we need to determine the Fourier Sine series for $f(x)$ and $g(x)$. Because we have already normalized our eigenfunctions this can be done easily. We project each function onto each eigenfunction,

$$\begin{aligned} a_k &= (f(x), M_k(x)) = \sqrt{\frac{2}{l}} \int_0^l f(x) \sin\left(\frac{\pi k}{l} x\right) \, dx, \quad k = 1, 2, 3, \cdots, \\ b_k &= \left(\frac{g(x)}{\sqrt{\lambda_k}}, M_k(x)\right) = \frac{\sqrt{2l}}{\pi k c} \int_0^l g(x) \sin\left(\frac{\pi k}{l} x\right) \, dx, \quad k = 1, 2, 3, \cdots. \end{aligned}$$

This can be applied for any given initial conditions that have a Fourier Sine series expansion to give us the complete solutions.

Given the solution we can analyze it to learn more about the nature of the solution. Each mode of the solution can be written as,

$$u_k(x,t) = \alpha_k \cos\left(\frac{\pi k c}{l}(t + \delta_k)\right) \sin\left(\frac{\pi k}{l} x\right), \quad k = 1, 2, \cdots,$$

where the amplitude and the phase shift are,

$$\alpha_k = \sqrt{\frac{2}{l}} \sqrt{a_k^2 + b_k^2}, \quad \delta_k = -\frac{l}{\pi k c} \arctan\left(\frac{b_k}{a_k}\right), \quad k = 1, 2, \cdots.$$

These solutions are called *waves* or sometimes *standing waves* since they show the spatial structures that can oscillate in this particular domain. They are also known as the $k$-th harmonic of the system.

Note that the zeros of the modes are called *nodes* and are equal to,

$$x_m = \frac{ml}{k}, \quad \text{for} \quad m = 1, 2, 3, \cdots, k-1.$$

That is to say that,

$$\sin\left(\frac{\pi k}{l} x_m\right) = 0,$$

for all time. The maximum amplitudes occur at *antinodes* and are given by

$$x_n = \frac{(2n+1)l}{2k},$$

for $n = 0, 1, 2, \cdots, k-1$, where it follows that,

$$\sin\left(\frac{\pi k}{l} x_n\right) = \pm 1.$$

The frequencies of vibrations are,

$$\omega_k = \frac{\pi k c}{l} = c\sqrt{\lambda_k}, \quad k = 1, 2, 3, \cdots.$$

The energy of the vibrating string is,

$$E = \frac{1}{2}\int_0^l \left[\rho u_t^2 + T u_x^2\right]\, dx.$$

Given our solution we can compute the Kinetic Energy

$$KE = \frac{1}{2}\int_0^l \left[\sum_{k=0}^{\infty}\left(\frac{\pi kc}{l}\right)\left[-a_k \sin\left(\frac{\pi kc}{l}t\right) + b_k \cos\left(\frac{\pi kc}{l}t\right)\right]\sqrt{\frac{2}{\ell}}\sin\left(\frac{\pi k}{l}x\right)\right]^2 dx,$$
$$= \sum_{k=0}^{\infty}\frac{1}{2}\left(\frac{\pi kc}{l}\right)^2\left[a_k \sin\left(\frac{\pi kc}{l}t\right) - b_k \cos\left(\frac{\pi kc}{l}t\right)\right]^2.$$

Similarly for the potential energy density,

$$PE = \frac{1}{2}\int_0^l \left[\sum_{k=0}^{\infty}\left(\frac{\pi kc}{l}\right)\left[a_k \cos\left(\frac{\pi kc}{l}t\right) + b_k \sin\left(\frac{\pi kc}{l}t\right)\right]\sqrt{\frac{2}{\ell}}\cos\left(\frac{\pi k}{l}x\right)\, dx\right]^2,$$
$$= \sum_{k=0}^{\infty}\frac{1}{2}\left(\frac{\pi kc}{l}\right)^2\left[a_k \cos\left(\frac{\pi kc}{l}t\right) + b_k \sin\left(\frac{\pi kc}{l}t\right)\right]^2.$$

Note that in both of the calculations above we used the fact that the eigenfunctions are orthonormal and therefore the double sums reduce to single sums in each case.

When we add these two we get the total energy density

$$E = \sum_{k=0}^{\infty}\frac{1}{2}\left(\frac{\pi kc}{l}\right)^2\left[a_k^2 + b_k^2\right]^2$$

This shows that the total energy is the sum of the energy over all the wavenumbers.

This is the total energy in the system at time $t$, which is a sum of the kinetic and potential energies. The total energy in each mode is,

$$E_k(t) = \frac{1}{2}\int_0^l \left[\rho\left(\frac{\partial u_k}{\partial t}\right)^2 + T\left(\frac{\partial u_k}{\partial x}\right)^2\right] dx = \frac{\omega_k^2 m(a_k^2 + b_k^2)}{2l},$$

where $m = \rho l$ is the mass of the string. Therefore, the total energy is the sum of the energy of each mode,

$$E(t) = \sum_{k=1}^{\infty} E_k(t) = \frac{m}{2l}\sum_{k=1}^{\infty}\omega_k^2(a_k^2 + b_k^2),$$

which is constant because the energy is a conserved quantity in time.

### 4.4.2 Diffusion Equation

The diffusion equation is,

$$u_t(x,t) - c^2 u_{xx}(x,t) = 0, \quad 0 < x < l, t > 0,$$

where $c^2$ is the diffusion coefficient. We fix the temperature at each end to be zero degrees,

$$u(0,t) = 0, \quad \text{and} \quad u(l,t) = 0.$$

Thirdly, we prescribe an initial temperature distribution,

$$u(x,0) = f(x), \quad 0 < x < l.$$

We use the method of separation of variables, and substitute

$$u(x,t) = M(x)N(t)$$

into the PDE. This yields, upon dividing through by $c^2 MN$,

$$\begin{aligned} MN' &= c^2 M''N, \\ \frac{N'}{c^2 N} &= \frac{M''}{M}. \end{aligned}$$

Since time and space are independent, we need that this is equal to a constant. Based on the equation that we write down next we know that the constant must be negative, since the eigenvalues are non-negative,

$$M'' + \lambda M = 0, \quad \text{with} \quad M(0) = 0, \quad M(l) = 0.$$

We substituted the separable solution into the boundary conditions to obtain the second and third equation above. The solutions to this equations are sine functions, as we have already seen,

$$\lambda_k = \left(\frac{\pi k}{l}\right)^2, \quad \text{with} \quad M_k(x) = \sqrt{\frac{2}{l}} \sin\left(\frac{\pi k}{l} x\right), \quad \text{for} \quad k = 1, 2, \cdots .$$

But this choice also implies that the temporal equation is,

$$N' - c^2 \lambda N = 0,$$

whose solution is,

$$N_k(t) = a_k \exp\left[-\left(\frac{\pi k c}{l}\right)^2 t\right].$$

When we combine these we get that each mode, or each part of the solution is,

$$u_k(x,t) = \sqrt{\frac{2}{l}} a_k \exp\left[-\left(\frac{\pi k c}{l}\right)^2 t\right] \sin\left(\frac{\pi k}{l} x\right), \quad k = 1, 2, \cdots .$$

We superimpose all the solutions to get the general solution,

$$u(x,t) = \sqrt{\frac{2}{l}} \sum_{k=1}^{\infty} a_k \exp\left[-\left(\frac{\pi k c}{l}\right)^2 t\right] \sin\left(\frac{\pi k}{l} x\right).$$

What is left to impose is the initial condition,

$$u(x,0) = \sum_{k=1}^{\infty} u_k(x,0) = \sum_{k=1}^{\infty} a_k \sqrt{\frac{2}{l}} \sin\left(\frac{\pi k}{l} x\right) = f(x).$$

This is exactly the same equation as before and we must therefore have the same solution,

$$a_k = (f(x), M_k(x)) = \sqrt{\frac{2}{l}} \int_0^l f(x) \sin\left(\frac{\pi k}{l} x\right) \, dx, \quad k = 1, 2, 3, \cdots .$$

This can be applied for any initial conditions that have a Fourier Sine series expansion to obtain the complete solution.

### 4.4.3 Laplace's Equation on a Rectangle

Laplace's equation in two dimensions is,

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad 0 < x < l_x, 0 < y < l_y.$$

We assume non homogeneous Dirichlet boundary conditions at the top and bottom and homogeneous Dirichlet boundary conditions to the left and right,

$$u(x,0) = f(x), \quad u(x,l_y) = g(x), \quad u(0,y) = 0, \quad \text{and} \quad u(l,y) = 0.$$

Physically, this describes the steady state distribution inside a rectangle that is heated at the top and bottom.

We use the method of separation of variables, and substitute

$$u(x,y) = M(x)N(y)$$

into the PDE. This yields, upon dividing through by $MN$,

$$\begin{aligned} MN'' &= -M''N, \\ \frac{N''}{N} &= -\frac{M''}{M}. \end{aligned}$$

Because we have zero Dirichlet conditions in the $x$-direction we get the same eigenvalue problem in the $x$-direction,

$$M'' + \lambda M = 0, \quad \text{with} \quad M(0) = 0, \quad M(l) = 0.$$

We substituted the separable solution into the boundary conditions to obtain the second and third equation above. The solutions to this equations are sine functions, as we have already seen,

$$\lambda_k = \left(\frac{\pi k}{l_x}\right)^2, \quad \text{with} \quad M_k(x) = \sqrt{\frac{2}{l_x}} \sin\left(\frac{\pi k}{l_x} x\right), \quad \text{for} \quad k = 1, 2, \cdots .$$

But this choice also implies that the temporal equation is,

$$N'' - \lambda N = 0,$$

The solution to this equation can be written in different forms. Perhaps the first you would think of is

$$N_k(y) = a_k \exp\left(\frac{\pi k y}{l_x}\right) + b_k \exp\left(-\frac{\pi k y}{l_x}\right).$$

One term is growing exponentially and the second is decaying exponentially. However, we have already seen that the hyperbolic trigonometric functions another basis for this solution space. That is to say we could also write this solution as,

$$N_k(y) = a_k \sinh\left(\frac{\pi k y}{l_x}\right) + b_k \cosh\left(\frac{\pi k y}{l_x}\right),$$

where the constants are different from before. Both of these are correct and will yield a correct answer. A more clever option, which is also less intuitive, is instead to pick that

$$N_k(y) = a_k \sinh\left(\frac{\pi k y}{l_x}\right) + b_k \sinh\left(\frac{\pi k (y - l_y)}{l_x}\right),$$

Even though both of the functions are hyperbolic sines, there is a translation (or phase shift) in the second that makes them linearly independent. This is the form that we use because it turns out to be the easiest but if you don't like this you don't have to. If you have any concerns about whether this is really a solution space you can always compute the Wronskian of the two functions and convince yourself that it is zero and therefore this is a general solution.

When we combine these we get that each mode, or each part of the solution is,

$$u_k(x,t) = \sqrt{\frac{2}{l_x}} \left[a_k \sinh\left(\frac{\pi k y}{l_x}\right) + b_k \sinh\left(\frac{\pi k (y - l_y)}{l_x}\right)\right] \sin\left(\frac{\pi k}{l_x} x\right), \quad k = 1, 2, \cdots .$$

We superimpose all the solutions to get the general solution,

$$u(x,t) = \sqrt{\frac{2}{l}} \sum_{k=1}^{\infty} \left[ a_k \sinh\left(\frac{\pi k y}{l_x}\right) + b_k \sinh\left(\frac{\pi k (y - l_y)}{l_x}\right) \right] \sin\left(\frac{\pi k}{l_x} x\right).$$

What is left to impose is the boundary conditions on the top and bottom,

$$u(x,0) = \sqrt{\frac{2}{l_x}} \sum_{k=1}^{\infty} b_k \sinh\left(-\frac{\pi k l_y}{l_x}\right) \sin\left(\frac{\pi k}{l_x} x\right) = f(x),$$
$$u(x,l_y) = \sqrt{\frac{2}{l_x}} \sum_{k=1}^{\infty} a_k \sinh\left(\frac{\pi k l_y}{l_x}\right) \sin\left(\frac{\pi k}{l} x\right) = g(x).$$

Using the inner product we can project onto each and find the appropriate Fourier coefficients,

$$a_k = \frac{(f(x), M_k(x))}{\sinh\left(-\frac{\pi k l_y}{l_x}\right)}, \quad k = 1, 2, 3, \cdots,$$
$$b_k = \frac{(g(x), M_k(x))}{\sinh\left(\frac{\pi k l_y}{l_x}\right)} \quad k = 1, 2, 3, \cdots.$$

This is the general solution to Laplace's equation in two spatial dimensions for the particular boundary conditions that we have imposed.

If instead of imposing non-homogeneous boundary conditions at the top and bottom, they were imposed on the left and right, what would be different? We would then have zero boundary conditions in the $y$ direction and the eigenvalue problem will be defined by the $N(y)$ variable. Then we can use this to reproduce any of the functions imposed on the other boundaries. By symmetry it is easy to argue that the solution would be,

$$u(x,t) = \sqrt{\frac{2}{l_y}} \sum_{k=1}^{\infty} \left[ a_k \sinh\left(\frac{\pi k x}{l_y}\right) + b_k \sinh\left(\frac{\pi k (x - l_x)}{l_y}\right) \right] \sin\left(\frac{\pi k}{l_y} y\right).$$

If we had non-homogeneous boundary conditions our method would fail because we couldn't obtain an eigenvalue problem that can be superimposed to yield a solution to the original

equation. This case can be dealt with by decomposing the solution into two solutions. One that solves the problem above and the second that solves the other problem. Therefore, the linearity of Laplace's equation allows us to find a general solution to simpler problems. We delve into the issue of inhomogeneity in the next section.

### 4.4.4 Laplace's Equation on a Circle

Consider,

$$\nabla^2 u = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right] = 0,$$

on the domain $0 \leq \theta \leq 2\pi$ and $0 < r < a$ with the BCs,

$$u(a, \theta) = f(\theta).$$

In addition to this conditions that are three other,

$$\begin{aligned} u(0, \theta) &\text{ is bounded ,} \\ u(r, 0) &= u(r, 2\pi), \\ \frac{\partial u}{\partial \theta}(r, 0) &= \frac{\partial u}{\partial \theta}(r, 2\pi). \end{aligned}$$

The second and third condition above state that the solution and the derivative of the solution are periodic on the drum.

We solve this using separation of variables, $u(r, \theta) = R(r)\Theta(\theta)$,

$$\begin{aligned} \frac{\Theta}{r}(rR')' + \frac{1}{r^2}R\Theta'' &= 0, \\ \frac{r}{R}(rR')' &= -\frac{\Theta''}{\Theta} = \lambda^2. \end{aligned}$$

Because the two variables, $r, \theta$ are independent, we deduce that the functional dependency above must cancel. This yields

$$\Theta'' + \lambda^2 \Theta = 0,$$

since the $\theta$ dependency must cancel. This means that the equation above becomes,

$$r^2 R'' + rR' - \lambda^2 R = 0.$$

The periodic conditions on $u$ transfer onto $\Theta$ to yield the following BVP,

$$\begin{aligned} \Theta'' + \lambda^2 \Theta &= 0, \\ \Theta(0) &= \Theta(2\pi), \\ \Theta'(0) &= \Theta'(2\pi). \end{aligned}$$

Note that the DE is of Sturm-Liouville type if we pick $p(\theta) = 1, q(\theta) = 0, r(\theta) = 1$ but the BCs are not. However, we know that the general solution is,

$$\Theta(\theta) = A_n \cos(\lambda\theta) + B_n \sin(\lambda\theta).$$

The first and second BC when combined imply that,

$$\begin{aligned} A_n &= A_n \cos(2\pi\lambda) + B_n \sin(2\pi\lambda), \\ \lambda B_n &= \lambda \left( -A_n \sin(2\pi\lambda) + B_n \cos(2\pi\lambda) \right). \end{aligned}$$

It is clear that one way for these conditions to be satisfied is to pick $\lambda = n$ where $n = 0, 1, 2, \cdots$. It turns out that if you solve the above system and look for nontrivial solutions, this is the only possible choice. Therefore, our solution to the first BVP is,

$$\Theta(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta).$$

When we substitute the fact that $\lambda = n$ into the radial equation and the BCs we get

$$r^2R'' + rR' - \lambda^2 R = 0,$$
$$R(0) < \infty.$$

This equation is of the Euler type and therefore can be solved using $R = r^\alpha$ and we find that the characteristic equation is $\alpha^2 = n^2$ or $\alpha = \pm n$.

$$R_n(r) = D_n r^n + E_n r^{-n}.$$

For $n = 0$ we need to have

$$R_0(r) = D_0 + E_0 \ln r.$$

To force the solution to be bounded at $r = 0$ we need that $E_n = 0$ for all $n$. The general solution is then

$$u(r, \theta) = \sum_{n=0}^{\infty} \left[A_{nm} \cos(n\theta) + B_{nm} \sin(n\theta)\right] r^n \tag{4.8}$$

To enforce the BCs at $r = a$ we then require that,

$$\sum_{n=0}^{\infty} a^n \left[A_{nm} \cos(n\theta) + B_{nm} \sin(n\theta)\right] = f(\theta),$$

which is equivalent to finding the Fourier series expansion for $f(\theta)$. This is our final solution.

### 4.4.5 Vibrating Drumhead

Consider,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right],$$

on the domain $0 \leq \theta \leq 2\pi$ and $0 < r < 1$ with the BCs,

$$
\begin{aligned}
u(1,\theta,t) &= 0,\\
u(r,\theta,0) &= 0,\\
\frac{\partial u}{\partial t}(r,\theta,0) &= 0.
\end{aligned}
$$

In addition to these conditions, there are three other,

$$
\begin{aligned}
u(0,\theta,t) &\text{ is bounded },\\
u(r,0,t) &= u(r,2\pi,t),\\
\frac{\partial u}{\partial \theta}(r,0,t) &= \frac{\partial u}{\partial \theta}(r,2\pi,t).
\end{aligned}
$$

The second and third condition above state that the solution and the derivative of the solution are periodic on the drum.

We solve this using separation of variables, $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$,

$$
\begin{aligned}
R\Theta T'' &= c^2\left[\frac{\Theta T}{r}(rR')' + \frac{1}{r^2}R\Theta'' T\right],\\
\frac{T''}{c^2 T} &= \left[\frac{1}{rR}(rR')' + \frac{1}{r^2}\frac{\Theta''}{\Theta}\right] = -\lambda^2.
\end{aligned}
$$

Because the three variables, $r,\theta,t$ are independent, we deduce that the functional dependency above must cancel. This yields

$$
T'' + c^2\lambda^2 T = 0.
$$

Moreover, we must that,

$$
\Theta'' + \mu^2\Theta = 0,
$$

since the $\theta$ dependency must cancel. This means that the equation above becomes,

$$(rR')' - \frac{\mu^2}{r}R + \lambda^2 rR = 0.$$

The periodic conditions on $u$ transfer onto $\Theta$ to yield the following BVP,

$$\begin{aligned} \Theta'' + \mu^2\Theta &= 0, \\ \Theta(0) &= \Theta(2\pi), \\ \Theta'(0) &= \Theta'(2\pi). \end{aligned}$$

Note that the DE is of Sturm-Liouville type if we pick $p(\theta) = 1, q(\theta) = 0, r(\theta) = 1$ but the BCs are not. However, we know that the general solution is,

$$\Theta(\theta) = A_n \cos(\mu\theta) + B_n \sin(\mu\theta).$$

The first and second BC when combined imply that,

$$\begin{aligned} A_n &= A_n \cos(2\pi\mu) + B_n \sin(2\pi\mu), \\ \mu B_n &= \mu \left(-A_n \sin(2\pi\mu) + B_n \cos(2\pi\mu)\right). \end{aligned}$$

It is clear that one way for these conditions to be satisfied is to pick $\mu = n$ where $n = 0, 1, 2, \cdots$. It turns out that if you solve the above system and look for nontrivial solutions, this is the only possible choice. Therefore, our solution to the first BVP is,

$$\Theta(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta).$$

When we substitute the fact that $\mu = n$ into the radial equation and the BCs we get

$$\begin{aligned} (rR')' - \frac{n^2}{r}R + \lambda^2 rR &= 0, \\ R(0) &< \infty, \\ R(1) &= 0. \end{aligned}$$

Observe that the DE is of the Sturm-Liouville type if we pick $p(r) = r, q(r) = n^2/r, r^*(r) = r$, the outer BC is also of the right form but the inner BC is not. If we had a drum on a ring instead of a circle (basically cut a hole in the centre) then we would have a boundary condition of the right type and we could apply Sturm-Liouville theory directly. As it stands, we must be a bit careful with the inner BC.

The general solution to this Bessel equation of order $n$ is,

$$R(r) = D_n J_n(\lambda r) + E_n Y_n(\lambda r),$$

where $J_n(\lambda r)$ and $Y_n(\lambda r)$ are the $n$-th order Bessel functions of the first and second kind. We derived the Power Series representation of the first but not the second. We know that the first equation is bounded at $r = 0$ (either 1 if $n = 0$ or 0 if $n > 0$) and it can be shown that $Y_n(r)$ is unbounded at $r = 0$ for all values of $n$. Therefore, to satisfy the bounded condition at the centre point we are forced to look for a solution of the form,

$$R(r) = D_n J_n(\lambda r).$$

The outer BC requires that,

$$J_n(\lambda) = 0.$$

Since Bessel's equation is of the Sturm-Liouville type we know that there are an infinite number of zeros. There are no closed form expression for these zeros but they can be determined numerically. In fact, there are many tables showing you what they in case you don't have a computer. For the $n$-th order Bessel function we define $\lambda_{n,m}$ to be the $m$-th zero.

The solution to the temporal equation is,

$$T(t) = F_n \cos(c\lambda t) + G_n \sin(c\lambda t).$$

Because our initial condition has a zero velocity we have that $T'(0) = 0$ and thus $G_n = 0$ for all $n$. Therefore, the temporal part of the solution is,

$$T(t) = F_n \cos(c\lambda t).$$

This yields that the general solution is a superposition over all the eigenfunctions,

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left[A_{nm} \cos(n\theta) + B_{nm} \sin(n\theta)\right] J_n(\lambda_{n,m} r) \cos(\lambda_{n,m} t). \tag{4.9}$$

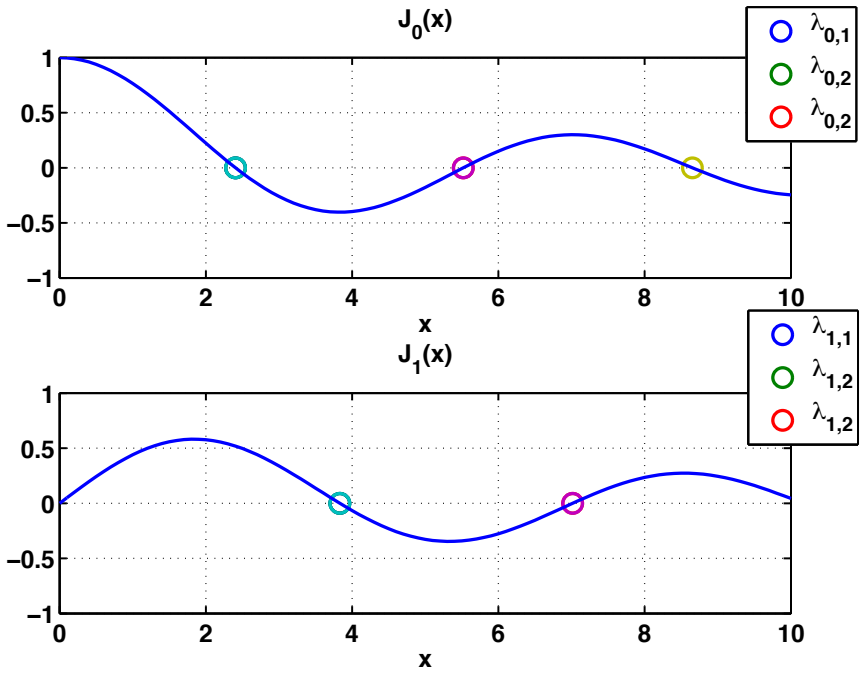


**Fig. 4.1** Zeros of the Bessel function of the first kind of order 0 and 1.

This shows that the modes on the drum are of the form,

$$\cos(n\theta)J_n(\lambda_{n,m}r).$$

Some modes only vary in $\theta$, others only in $r$ and others in both. Figure 4.2 show the first 20 modes that fit in the drum and Figure 4.3 shows the three-dimensional structure of some of these modes. The amplitudes are exaggerated for illustrative purposes. The reader is reminded that an important assumption in deriving the wave equation to describe the vibrating membrane is that the amplitude is small. If the amplitude is not so small then nonlinear effects come into play and the problem is much more complicated.

The general solution in equation (4.9) satisfies the PDE as well as all the BCs and one of the ICs. The only condition we have not yet imposed is $u(r,\theta,0) = 1 - r^2$. To impose this we need the orthogonality conditions for the azimuthal eigenfunctions,

$$\int_0^{2\pi} \cos(n\theta)\cos(m\theta)d\theta = \pi\delta_{nm}, \quad \text{with } n \text{ and/or } m \text{ nonzero}$$
$$\int_0^{2\pi} \sin(n\theta)\sin(m\theta)d\theta = \pi\delta_{nm},$$
$$\int_0^{2\pi} \cos(n\theta)\sin(m\theta)d\theta = 0,$$

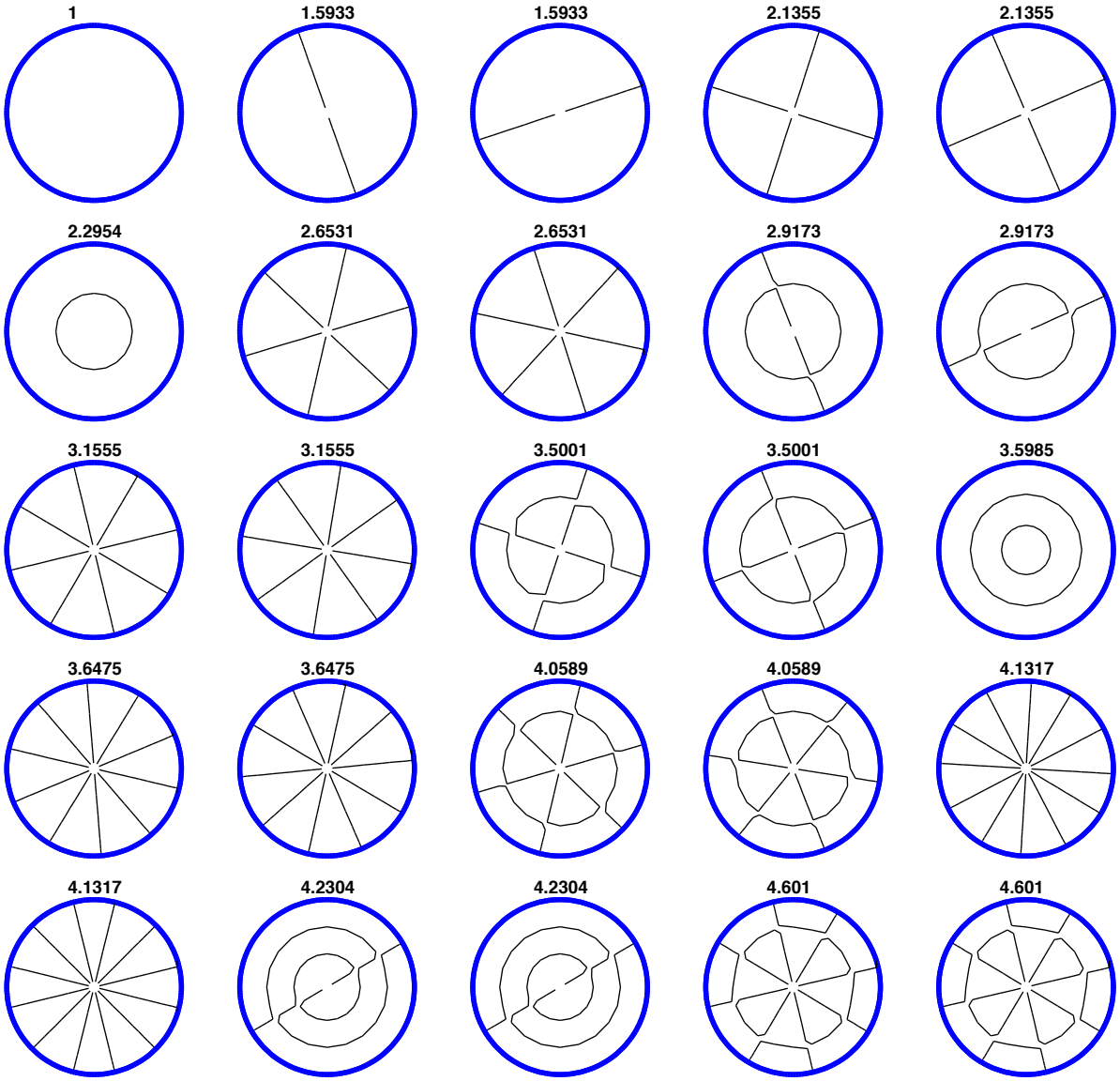


**Fig. 4.2** Nodes of the first 20 modes on the drum. Taken from the textbook Spectral Methods in Matlab.

$$\int_0^{2\pi} 1 d\theta = 2\pi, \quad \text{for } n = m = 0.$$

and the orthogonality condition for the radial eigenfunctions,

$$\int_0^1 J_n(\lambda_{n,m} r) J_n(\lambda_{l,k} r) r dr = \frac{1}{2} J_{n+1}^2(\lambda_{n,m}) \delta_{nl} \delta_{mk}.$$

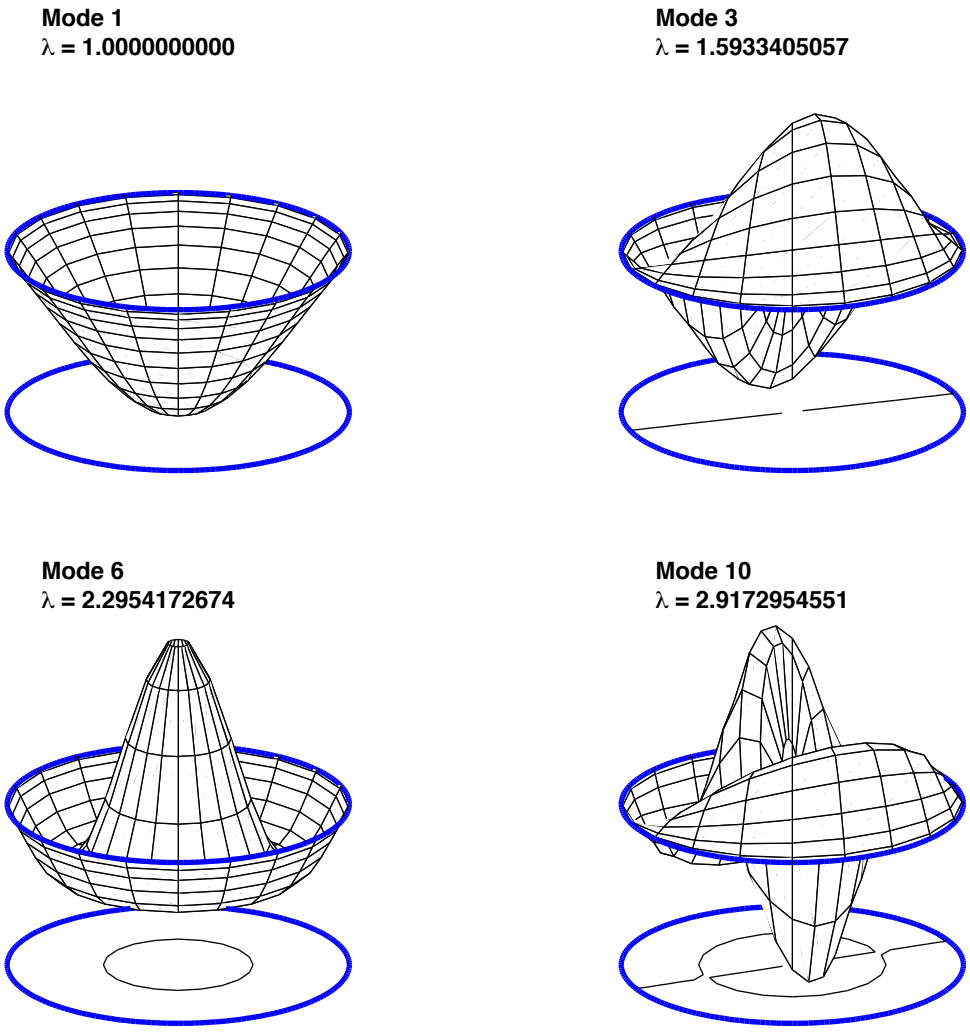


**Fig. 4.3** Three-dimensional plots of several eigenfunctions on the drum. Taken from the textbook Spectral Methods in Matlab.

This is to be proven on the next assignment. Then, we impose the ICs,

$$u(r,\theta,0) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \left[A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)\right] J_n(\lambda_{n,m} r) = 1 - r^2.$$

To begin to determine the generalized Fourier coefficients we can multiply this equation by $\cos(n\theta)$ and $\sin(n\theta)$ and integrate over the full theta domain. Alternatively, we can observe that the right-hand side does not depend on any of the Fourier basis functions except for one, namely $1$. Therefore, we find that $B_{nm} = 0$, for all $n, m$ and $A_{nm} = 0$ for all $n, m$ except

$n = 0$. This leaves us with,

$$\sum_{m=1}^{\infty} A_{0m} J_0(\lambda_{0,m} r) = 1 - r^2.$$

We multiply by $rJ_0(\lambda_{0,p} r)$ and integrate over the full radial domain,

$$\begin{aligned}
\int_0^1 \sum_{m=1}^{\infty} A_{0m} J_0(\lambda_{0,m} r) J_0(\lambda_{0,p} r) r dr &= \int_0^1 (1 - r^2) J_{0,p}(\lambda_{0,p} r) r dr, \\
\sum_{m=1}^{\infty} A_{0m} \int_0^1 J_0(\lambda_{0,m} r) J_0(\lambda_{0,p} r) r dr &= \int_0^1 (1 - r^2) J_{0,p}(\lambda_{0,p} r) r dr, \\
\sum_{m=1}^{\infty} A_{0m} \frac{1}{2} J_1^2(\lambda_{0,m}) \delta_{mp} &= \int_0^1 (1 - r^2) J_{0,p}(\lambda_{0,p} r) r dr, \\
A_{0p} \frac{1}{2} J_1^2(\lambda_{0,p}) &= \int_0^1 (1 - r^2) J_{0,p}(\lambda_{0,p} r) r dr, \\
A_{0p} &= \frac{\int_0^1 (1 - r^2) J_{0,p}(\lambda_{0,p} r) r dr}{\frac{1}{2} J_1^2(\lambda_{0,p})}.
\end{aligned}$$

Because of the relatively simple form of the IC we can evaluate this integral exactly. To do this we need the following identify,

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x).$$

We begin by defining $x = \lambda_{0,p} r$

$$\begin{aligned}
\int_0^1 (1 - r^2) J_0(\lambda_{0,p} r) r dr &= \int_0^{\lambda_{0,p}} \left[1 - \left(\frac{x}{\lambda_{0,p}}\right)^2\right] J_0(x) \frac{x}{\lambda_{0,p}^2} dx, \\
&= \frac{1}{\lambda_{0,p}^4} \int_0^{\lambda_{0,p}} \left[\lambda_{0,p}^2 - x^2\right] x J_0(x) dx, \\
&= \frac{1}{\lambda_{0,p}^4} \int_0^{\lambda_{0,p}} \left[\lambda_{0,p}^2 - x^2\right] \frac{d}{dx}\left(x J_1(x)\right) dx, \\
&= \frac{1}{\lambda_{0,p}^4} \left[ \left[(\lambda_{0,p}^2 - x^2) x J_1(x)\right]_0^{\lambda_{0,p}} + \int_0^{\lambda_{0,p}} 2x^2 J_1(x) dx \right],
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\lambda_{0,p}^4}\left[\int_0^{\lambda_{0,p}} \frac{d}{dx}\left[x^2 J_2(x)\right] dx\right], \\
&= \frac{2}{\lambda_{0,p}^4}\left[x^2 J_2(x)\right]_0^{\lambda_{0,p}}, \\
&= \frac{2}{\lambda_{0,p}^2} J_2(\lambda_{0,p}).
\end{aligned}
$$

Therefore, the solution to the wave equation is

$$u(r,\theta,t) = 4\sum_{m=1}^{\infty} \frac{J_2(\lambda_{0,m})}{\lambda_{0,m}^2 J_1^2(\lambda_{0,m} r)} J_0(\lambda_{0,m} r)\cos(\lambda_{0,m} t).$$

Because the solution does not depend on $\theta$ the vibrations are radially symmetric. To have variations in $\theta$ it is necessary to have an initial disturbance that depends on $\theta$. The reason being is that this is an unforced linear problem. Whatever modes you have initially are the only modes that can exist for all time, in the absence of forcing.

### 4.4.6 Diffusion Equation on a Circle

Consider,

$$\frac{\partial u}{\partial t} = D\nabla^2 u = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right],$$

on the domain $0 \le \theta \le 2\pi$ and $0 < r < 1$ with ICs

$$u(r,\theta,0) = f(r,\theta)$$

and BCs

$$
\begin{aligned}
&u(1,\theta,t) = 0, \\
&u(0,\theta,t) \text{ is bounded },
\end{aligned}
$$

$$u(r,0,t) = u(r,2\pi,t),$$
$$\frac{\partial u}{\partial \theta}(r,0,t) = \frac{\partial u}{\partial \theta}(r,2\pi,t).$$

We solve this using separation of variables, $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$,

$$R\Theta T' = D\left[\frac{\Theta T}{r}(rR')' + \frac{1}{r^2}R\Theta'' T\right],$$
$$\frac{T'}{c^2 T} = \left[\frac{1}{rR}(rR')' + \frac{1}{r^2}\frac{\Theta''}{\Theta}\right] = -\lambda.$$

Temporal equation

$$T' + D\lambda T = 0.$$

Moreover, we also must that,

$$\Theta'' + \mu\Theta = 0,$$

since the $\theta$ dependency must cancel. This means that the equation above becomes,

$$(rR')' - \frac{\mu}{r}R + \lambda r R = 0.$$

The periodic conditions on $u$ transfer onto $\Theta$ to yield the following BVP,

$$\Theta'' + \mu\Theta = 0,$$
$$\Theta(0) = \Theta(2\pi),$$
$$\Theta'(0) = \Theta'(2\pi).$$

Our solution to the first BVP is,

$$\Theta(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta).$$

When we substitute the fact that $\mu = n^2$ into the radial equation and the BCs we get

$$\begin{aligned} (rR')' - \frac{n^2}{r}R + \lambda r R = 0, \\ R(0) < \infty, \\ R(1) = 0. \end{aligned}$$

The general solution is

$$R_n(r) = D_n J_n(\sqrt{\lambda} r) + E_n Y_n(\sqrt{\lambda} r).$$

But for the solution to be bounded at $r = 0$ we again require $E_n = 0$.

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left[A_{nm} \cos(n\theta) + B_{nm} \sin(n\theta)\right] J_n(\lambda_{n,m} \frac{r}{L}) \exp(-\sqrt{\lambda_{n,m}} D t).$$

In the limit of large but finite time we have that the term that is going survive the longest is the one with the smallest eigenvalue, namely $lambda_{0,1}$. That is why the asymptotic behaviour is

$$\lim_{t \to \infty} u(r, \theta, t) = A_{01} J_0(\sqrt{\lambda_{0,1}} \frac{r}{L})$$

Since the zero eigenvalue is not allowed in this case since the first non-zero root of the $n - th$ order Bessel function of the first kind is non-zero, we have that the solution tends to zero as $t \to \infty$. Physically, this is easy enough to see since we are cooling the perimieter of the circle and do not have insulating boundary conditions. If one were to solve the same problem with Neumann boundary conditions, then we would get that the first term would be non-zero. An interesting question is would the shape be constant or not?

### 4.4.7 Schrödinger Equation

In classical mechanics we can use Newton's 2nd law,

$$m\frac{d^2x}{dt^2} = F = -\frac{\partial V}{\partial x}$$

to determine the exact position of an object if we know the initial conditions and all of the forces, $F$. In the above we assume that the forces are conservative and therefore write the force as the gradient of a potential function.

In quantum mechanics we instead determine the probability that a particle is at a given location at a given time. The probability is specified by the wave function $|\Psi(x,t)|^2$, and it's evolution is set by the Schrödinger equation,

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi.$$

Note that the wave function is complex valued and

$$|\Psi(x,t)|^2 dx,$$

denotes the probability of finding the particle between $x$ and $x + dx$ at time $t$.

*Derivation*

Any mathematical model based on the real world must be based on observations. Doing this in Quantum Mechanics is clearly challenging since observing the quantum scales is incredibly difficult, maybe impossible.

We suppose that the wavefunction is sinusoidal with wavenumber $k$ and frequency $\omega$,

$$\Psi = Ae^{i(kx-\omega t)}.$$

With this form it is clear that the partial derivatives yield

$$\frac{\partial \Psi}{\partial x} = ik\Psi, \quad \text{and} \quad \frac{\partial \Psi}{\partial t} = -i\omega\Psi.$$

But it is hypothesized that the momentum and energy can be written in terms of the wavenumber and frequency as

$$k = \frac{p}{\hbar}, \quad \text{and} \quad \omega = \frac{E}{\hbar}.$$

Note that $p$ and $E$ are the momentum and energy of an particle. I believe that these assumptions are known as deBroglie relations and are motivated by classical mechanics.

Using these relations we can rewrite the above expressions as

$$-i\hbar\frac{\partial\Psi}{\partial x} = p\Psi, \quad \text{and} \quad i\hbar\frac{\partial\Psi}{\partial t} = E\Psi.$$

But if we believe that energy and monentum in quantum mechanics follow the same relation as in classical mechanics, that is to say

$$E = \frac{1}{2}mv^2 + V(x) = \frac{1}{2}\frac{p^2}{m} + V(x),$$

if we substitute in the above relations we get,

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} &= \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\Psi + V(x)\Psi, \\ i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi. \end{aligned}$$

This is of course Schrödinger's equation. I have read that this equation can be derived using a Path Integral formulation due to Richard Feynmann but I have not seen it and I am pretty sure it is beyond the scope of this course.

*Separated Solutions*

The time-dependent Schrödinger equation is

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi.$$

We look for a separated solution of the form $\Psi(x,t) = M(x)N(t)$.
We substitute our trial solution into the PDE and divided by the solution to obtain

$$\begin{aligned} i\hbar MN' &= -\frac{\hbar^2}{2m}M''N + VMN, \\ i\hbar\frac{N'}{N} &= -\frac{\hbar^2}{2m}\frac{M''}{M} + V = E \end{aligned}$$

Since the variables are separated we know that the above equation must be a constant. Previously we set this equal to a constant, say $\lambda$, but I follow convention and set it equal to $E$, since it will be related to the energy. Therefore the two equations are

$$\begin{aligned} i\hbar N'' &= EN, \\ -\frac{\hbar^2}{2m}M'' + VM &= EM \end{aligned}$$

The solution to the temporal problem is clearly,

$$N = \exp\left(-i\frac{E}{\hbar}t\right).$$

Note that even thought the Schrödinger equation looks like the Diffusion equation, we get oscillatory solutions because of the $i$ that appears in the exponential.

The ODE that yields the eigenvalue problem is referred to as the time-independent Schrödinger equation. Clearly the eigenfunctions $M_n$ and eigenvalues $E_n$ depend on the potential function $V(x)$. This is not something that we can solve in general but needs to be solved on a case by case basis.

Whatever the solutions happen to be the solution for each eigenvalue is

$$\Psi_n(x,t) = M_n(x)\exp\left(-i\frac{E_n}{\hbar}t\right).$$

With this we can determine the probability for each eigenvalue $E_n$,

$$|\Psi_n(x,t)|^2 = M_n^*(x)\exp\left(i\frac{E_n}{\hbar}t\right)M_n(x)\exp\left(-i\frac{E_n}{\hbar}t\right) = M_n^* M_n = |M_n|^2.$$

The general solution is the superposition over all the possible eigenfunctions and can be written as

$$\Psi(x,t) = \sum_{n=1}^{\infty} M_n(x)\exp\left(-i\frac{E_n}{\hbar}t\right).$$

The eigenfunction is of Sturm-Liouville type with $p(x) = -\hbar^2/(2m)$, $q(x) = V(x)$ and $\rho(x) = 1$. As long as the boundary conditions are "nice enough" the eigenfunctions are orthogonal, i.e.

$$(M_n, M_m) = 0,$$

if $n \neq m$. But it should be noted that the inner product is usually written differently in terms of the bra and the ket notation.

*The Infinite Square Well*

We consider the case with

$$V(x) = \begin{cases} 0, & \text{if } \ 0 \leq x \leq L, \\ \infty, & \text{otherwise} \end{cases}$$

If we consider the regime $0 \leq x \leq a$ then the ODE becomes

$$M'' + \frac{2mE}{\hbar^2} M = 0.$$

The general solution is of course

$$M(x) = a \sin\left(\frac{\sqrt{2mE}}{\hbar} x\right) + b \cos\left(\frac{\sqrt{2mE}}{\hbar} x\right).$$

If we impose zero Dirichlet boundary conditions, justified, by the requirements on $\Psi$,

$$M(0) = M(L) = 0,$$

then the eigenvalues and eigenfunctions are

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}, \quad \text{and} \quad M_n = a_n \sin\left(\frac{n\pi x}{L}\right),$$

for $n = 1, 2, 3, \cdots$.

*Position and Momentum*

If we compute the complex conjugate of the PDE we get

$$-i\hbar \frac{\partial \Psi^*}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} + V(x)\Psi^*.$$

If we compute the time derivative of the modulus of the wave function then we get

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi^* \Psi),$$

$$
\begin{aligned}
&= \frac{\partial \Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial \Psi}{\partial t}, \\
&= -\frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right]\Psi + \frac{1}{i\hbar}\Psi^*\left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right], \\
&= \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \Psi\frac{\partial^2\Psi}{\partial x^2}\right)
\end{aligned}
$$

Integrating this equation over the real line then yields

$$
\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty}|\Psi(x,t)|^2dx &= \int_{-\infty}^{\infty}\frac{\partial}{\partial t}|\Psi(x,t)|^2dx, \\
&= \int_{-\infty}^{\infty}\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \Psi\frac{\partial^2\Psi}{\partial x^2}\right)dx, \\
&= \frac{i\hbar}{2m}\left[\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi}{\partial x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{i\hbar}{2m}\left(\frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial\Psi}{\partial x}\right)dx, \\
&= 0.
\end{aligned}
$$

The first term vanishes because we assume that the wave function and its spatial derivative vanish as $x$ tends to plus or minus infinity. Otherwise, its integral over the real line will in general not converge. The second term vanishes trivially. Therefore the integral of the probability over the real line is conserved in time. Without loss of generality, we can assume that initially the total probability is unity.

Therefore, the wave function must satisfy the following

$$
\int_{-\infty}^{\infty}|\Psi(x,t)|^2dx = 1.
$$

The expected, or mean, value of the position is

$$
\langle x\rangle = \int_{-\infty}^{\infty}x|\Psi(x,t)|^2dx.
$$

This is often written as

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^*(x)\Psi dx,$$

where $(x)$ is thought of as an operator.

If we define the momentum to be $p = mv = m\frac{dx}{dt}$, we can use the above to determine the expected momentum,

$$\begin{aligned}
\langle p \rangle &= m\frac{d}{dt}\langle x \rangle, \\
&= m\frac{d}{dt}\int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx, \\
&= m\int_{-\infty}^{\infty} x\frac{\partial}{\partial t}(\Psi^*\Psi)dx, \\
&= \frac{i\hbar}{2}\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial \Psi}{\partial x} - \Psi\frac{\partial \Psi}{\partial x}\right)dx, \\
&= -\frac{i\hbar}{2}\int_{-\infty}^{\infty} \left(\Psi^*\frac{\partial \Psi}{\partial x} - \Psi\frac{\partial \Psi}{\partial x}\right)dx, \\
&= -i\hbar\int_{-\infty}^{\infty} \left(\Psi^*\frac{\partial \Psi}{\partial x}\right)dx.
\end{aligned}$$

Note that in the above we use integration by parts in two different eqations to obtain the final result.

The above two results can be summarized as

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} \Psi^*(x)\Psi dx, \\
\langle p \rangle &= \int_{-\infty}^{\infty} \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi dx.
\end{aligned}$$

*Notation for Inner Product*

If we use our complex inner product that we defined previously we could write the expectation of the position and momentum as

$$\langle x \rangle = (\Psi, x\Psi),$$
$$\langle p \rangle = (\Psi, \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi).$$

This would be correct and precise, but this is not the convention that people have used.

Dirac introduced what is referred to as the "bra" and "ket" notation, which is because when you put them together you get a bracket. To make things precise, consider the example where we want to find the inner product of two functions, $\alpha$ and $\beta$. The idea of Dirac is to separate the inner product, which is often used using angular brackets, into two pieces,

$$\langle \alpha, \beta \rangle = \langle \alpha | \beta \rangle.$$

If one accepts this notation, as many people do, then we can go a little further, and introduce functions or even differential operators inbetween the bra and the ket. Therefore, the expection of the position and monentum can be written using the following notation,

$$\langle x \rangle = \langle \Psi | x | \Psi \rangle,$$
$$\langle p \rangle = \langle \Psi | \frac{\hbar}{i} \frac{\partial}{\partial x} | \Psi \rangle.$$

### 4.4.8 Spherical Wave Equation

The wave equation in 3D can be written as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u,$$

where we write the Laplacian operator in a coordinate system that is convenient given the boundary conditions.

We have previously solved this in 1D, on a string, and in 2D, on a circle in lecture and on a rectangle in an assignment. Next, we consider the wave equation on a sphere.

*Mathematical Model*

In spherical coordinates we typically describe any position in terms of three variables, $(r, \theta, \phi)$, which are the radial distance from the origin, the angle in the horizontal $(x, y)$ plane and the angle with respect to the North pole.

The Laplacian has a particular representation in spherical coordiantes and allows us to rewrite the wave equation as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin\phi} \frac{\partial}{\partial \phi} \left( \sin\phi \frac{\partial u}{\partial \phi} \right) + \frac{1}{r^2 \sin^2\phi} \frac{\partial^2 u}{\partial \theta^2} \right],$$

The two initial conditions are

$$\begin{aligned} u(r, \theta, \phi, 0) &= f(r, \theta, \phi), \\ \frac{\partial u}{\partial t}(r, \theta, \phi, 0) &= g(r, \theta, \phi). \end{aligned}$$

The boundary conditions must of course be specified but let's save this for later.

*Separation of Variables*

First, we assume that we can separate space and time using

$$u(\vec{x}, t) = M(\vec{x}) N(t).$$

We subtitute this into the PDE and divide through by $c^2 u$ to obtain

$$\begin{aligned} MN'' &= c^2 N \nabla^2 M, \\ \frac{N''}{c^2 N} &= \frac{1}{M} \nabla^2 M = -\lambda. \end{aligned}$$

In the last line above we used the fact that the variables are separated and introduced an eigenvalue, $\lambda$.

One might not be sure as to what sign to pick for the RHS and in some sense it should not matter. But if we expect oscillatory solutions then that means the RHS is negative, otherwise the solution will not oscillate in time. But this must be justified, and it will be in due time.

From the above we have the same temporal equation as before,

$$N'' + c^2 \lambda N = 0.$$

Well, it looks almost the same. One slight difference we will find is that the eigenvalues differ depending on the geometry.

The eigenvalue problem takes the form,

$$\nabla^2 M + \lambda M = 0.$$

or

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial M}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial M}{\partial \phi}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 M}{\partial \theta^2} + \lambda M = 0.$$

This is a 3D eigenvalue problem where the eigenvalue is $\lambda$. If we had started off with the wave equation on a rectangular box we would have found something very similar. All that would be different is we would rewrite the Laplacian in Cartesian coordinates,

$$\frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} + \frac{\partial^2 M}{\partial z^2} + \lambda M = 0.$$

We pick one of the two representations for the Laplacians above based on whether we are solving the wave equation on a sphere or on a cube.

Compare the 1D, 2D and 3D cases in Cartesian coordinates,

$$\begin{aligned} \frac{d^2 M}{dx^2} + \lambda M &= 0, \\ \frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} + \lambda M &= 0, \\ \frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} + \frac{\partial^2 M}{\partial z^2} + \lambda M &= 0. \end{aligned}$$

The vector form nicely represents all 3 cases.
But if we write it in vector form the values of $\lambda$'s will change in different dimension.
Also, they change because of different coordinates systems.

*Spherical Laplacian*

Next, we assume that our eigenfunction, $M$, can be separated into the product of 3 functions:

$$M(r,\phi,\theta) = R(r)\Phi(\phi)\Theta(\theta).$$

We plug this into the Laplacian and then divide by $R\Phi\theta/r^2$,

$$\begin{aligned}
&\frac{\Phi\Theta}{r^2}(r^2R')' + \frac{R\Theta}{r^2\sin\phi}(\sin\phi\Phi')' + \frac{R\Phi}{r^2\sin^2\phi}\Theta'' + \lambda M = 0,\\
&\frac{1}{R}(r^2R')' + \frac{1}{\Phi\sin\phi}(\sin\phi\Phi')' + \frac{1}{\Theta\sin^2\phi}\Theta'' + \lambda r^2 M = 0,\\
&\qquad\qquad \frac{1}{R}(r^2R')' + \lambda r^2 = -\frac{1}{\Phi\sin\phi}(\sin\phi\Phi')' - \frac{1}{\sin^2\phi}\frac{\Theta''}{\Theta} = \gamma
\end{aligned}$$

This equation has 3 variables, and they must be independent.
First, we deduce that the $\theta$ dependency must cancel and we do this by defining

$$\frac{\Theta''}{\Theta} = -\mu$$

If the solution is periodic in $\theta$, which it must be, then we get the same solution as in the case of a circular disk,

$$\Theta = a_m\cos m\theta + b_m\sin m\theta, \quad \text{with} \quad \mu = m^2,$$

where $m$ is a non-negative integer.
This allows us to rewrite the above equation as

$$\frac{1}{R}(r^2R')' + \lambda r^2 = -\frac{1}{\Phi \sin\phi}(\sin\phi\Phi')' + \frac{m^2}{\sin^2\phi} = \gamma.$$

This then yields two different ODEs for the $r$ and $\phi$ directions

$$(r^2R')' + (\lambda r^2 - \gamma)R = 0,$$

and

$$(\sin\phi\Phi')' + (\gamma\sin\phi - \frac{m^2}{\sin\phi})\Phi = 0.$$

*Associated Legendre Functions*

The DE for $\Phi$ can be written as

$$-(\sin\phi\Phi')' + \frac{m^2}{\sin\phi}\Phi = \gamma\sin\phi\Phi.$$

This looks like it is in a Sturm Liouville form with

$$p(\phi) = \sin\phi, \quad q(\phi) = \frac{m^2}{\sin\phi}, \quad \text{and} \quad \rho(\phi) = \sin\phi,$$

and the eigenvalue is $\gamma$. Recall that $-\pi \leq \phi \leq \pi$.

This is not a regular Sturm-Liouville problem because both $p$ and $\rho$ vanish in the domain, aleit the ends of the domain. This is similar to what we saw in the context of the Bessel equation.

To rewrite this equation in terms of a well known equation we make the definition

$$x = \cos\phi, \quad \text{and} \quad \frac{d}{d\phi} = \frac{dx}{d\phi}\frac{d}{dx} = -\sin\phi\frac{d}{dx} = -\sqrt{1-x^2}\frac{d}{dx}$$

We can then rewrite the equation as

$$
\begin{aligned}
-\frac{d}{d\phi}(\sin\phi\frac{d\Phi}{d\phi}) + \frac{m^2}{\sin\phi}\Phi &= \gamma\sin\phi,\\
-\sqrt{1-x^2}\frac{d}{dx}\left((1-x^2)\frac{d\Phi}{dx}\right) + \frac{m^2}{\sqrt{1-x^2}}\Phi &= \gamma\sqrt{1-x^2}\Phi,\\
-\frac{d}{dx}\left((1-x^2)\frac{d\Phi}{dx}\right) + \frac{m^2}{1-x^2}\Phi &= \gamma\Phi,\\
\frac{d}{dx}\left((1-x^2)\frac{d\Phi}{dx}\right) + \left(\gamma - \frac{m^2}{1-x^2}\right)\Phi &= 0.
\end{aligned}
$$

It is beyond the scope of this course but it can be shown that if we want a solution to be bounded at $x = \pm 1$, which corresponds to the North and South poles, then we need

$$\gamma = n(n+1),$$

where $n$ is an integer. This sets the eigenvalue for the latitudinal direction.

With this choice it can be shown that the solutions are the associated Legendre functions of the first and second kind, denoted by

$$P_n^m(x), \quad \text{and} \quad Q_n^m(x).$$

These are the eigenfunctions in latitude. Since $Q_n^m(x)$ are not bounded at $x = \pm 1$, we need to take the functions of the first kind,

$$\Phi = P_n^m(x) = P_n^m(\cos\phi).$$

If $m = 0$ then we have Legendre Polynomials. The first few look like

$$
\begin{aligned}
n = 0, &\quad P_0^0(x) = 1,\\
n = 1, &\quad P_1^0(x) = x = \cos\phi,
\end{aligned}
$$

$$n = 2, \quad P_2^0(x) = \frac{1}{2}(3x^2 - 1) = \frac{1}{4}(3\cos 2\phi + 1).$$

For $m > 0$ we have associated Legendre functions that can be written in terms of power series.

*Radial Equation*

The final problem we need to solve is the Radial Eigenvalue problem,

$$(r^2 R')' + (\lambda r^2 - n(n+1))R = 0.$$

The solutions to this equation are the spherical Bessel functions. The ones that are bounded on the whole domain are

$$R(r) = r^{-1/2} J_{n+1/2}(\sqrt{\lambda} r).$$

If we have a homogeneous boundary condition at $r = a$ then the eigenvalues are set by

$$J_{n+1/2}(\sqrt{\lambda} a) = 0.$$

Therefore, the eigenvalues are determined by the roots of the Bessel function of the first kind of order $n + 1/2$. Using our previous definition, we can denote the $p$-th root as $\alpha_{p(n+1/2)}$, therefore

$$\sqrt{\lambda} = \frac{\alpha_{p(n+1/2)}}{a}$$

*Complete Solution*

Since we have assumed a separated solution we have that each eigenfunction will be of the form

$$u(r,\theta,\phi,t) = \left\{ \begin{array}{c} \cos c\sqrt{\lambda}t \\ \sin c\sqrt{\lambda}t \end{array} \right\} r^{-1/2} J_{n+1/2}(\sqrt{\lambda}r) \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} P_n^m(\cos\phi).$$

The frequencies of vibration are set by solving

$$J_{n+1/2}(\sqrt{\lambda}a) = 0.$$

When we put this together we get that the most general solution is

$$u = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} r^{-1/2} J_{n+1/2}(\sqrt{\lambda}r) \left(A_{nmp}\cos m\theta + B_{nmp}\sin m\theta\right) P_n^m(\cos\phi) \left(C_{nmp}\cos c\sqrt{\lambda}t + D_{nmp}\sin c\sqrt{\lambda}t\right)$$

If we have initial conditions then we evaluate $u$ or its time derivative at $t = 0$ and determine the generalized Fourier series representation of those initial conditions.

Spherical harmonics arise in many contexts. In physics, forces usually have a radial dependency on the distance from the source. In meteorology, we often want to determine the temperature, pressure or velocity on the surface of the Earth. In that case things get a little easier because $r$ does not change very much. Indeed, if we want to solve the wave equation on the surface of a sphere it will look very similar to the above but it will not depend on $r$ and therefore we have a double sum,

$$u(\theta,\phi,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \left(A_{nmp}\cos m\theta + B_{nmp}\sin m\theta\right) P_n^m(\cos\phi) \left(C_{nmp}\cos c\sqrt{\lambda}t + D_{nmp}\sin c\sqrt{\lambda}t\right).$$

## 4.5 INHOMOGENEOUS EQUATIONS: DUHAMEL'S PRINCIPLE

In this section we show how we can solve inhomogeneous PDEs by relating them to ones that are homogeneous, where our standard methods apply. This is based on what we refer to as *Duhamel's Principle*. To begin the discussion consider the general hyperbolic and parabolic equations:

$$\begin{aligned} \rho u_{tt} + L[u] &= g(x,t), \quad \text{hyperbolic,} \\ \rho u_t + L[u] &= g(x,t), \quad \text{parabolic,} \end{aligned}$$

where $L$ is the same differential operator that we defined previously. As usual, we require positivity in the weighting function, $\rho > 0$. This differs from before because the right hand side is non-zero and $g(x,t)$ is the forcing or source term. The above equations are written in the scalar form but the idea of Duhamel's principle works in any dimension. To find unique solutions we must impose boundary conditions as well as initial conditions. These take the form,

$$\begin{aligned} u(x,0) = u_t(x,0) &= 0, \quad \text{hyperbolic,} \\ u(x,0) &= 0, \quad \text{parabolic.} \end{aligned}$$

Duhamel's principle asks that we consider the associated homogeneous equation in terms of the variable $v(x,t;\tau)$,

$$\begin{aligned} \rho v_{tt} + L[v] &= 0, \quad t > \tau \quad \text{hyperbolic,} \\ \rho v_t + L[v] &= 0, \quad t > \tau \quad \text{parabolic,} \end{aligned}$$

that satisfies exactly the same boundary conditions as $u(x,t)$ but instead has different initial conditions for $t = \tau$, defined to be,

$$\begin{aligned} v(x,\tau;\tau) = 0, \quad v_t(x,\tau;\tau) &= \frac{g(x,\tau)}{\rho(x)}, \quad \text{hyperbolic,} \\ v(x,\tau;\tau) &= \frac{g(x,\tau)}{\rho(x)}, \quad \text{parabolic,} \end{aligned}$$

where $\rho(x)$ is the weighting function and $g(x,t)$ is the forcing in the given PDE.

If the boundary conditions are of the right form, as what we previously considered, we can apply our method to solve for this new variable that we now denote as $v(x,t;\tau)$. Duhamel's principle states that the solution of the inhomogeneous problem is obtained by the following integral,

$$u(x,t) = \int_0^t v(x,t;\tau)\, d\tau.$$

To verify that this does in fact yield a function that solves the correct PDE we can substitute it into the given equation. Before we do that note that **Leibniz's principle** states,

$$\boxed{\frac{d}{dt}\int_{g(t)}^{f(t)} h(s,t)\, ds = \frac{df}{dt}h(f(t),t) - \frac{dg}{dt}h(g(t),t) + \int_{g(t)}^{f(t)} \frac{\partial h}{\partial t}(s,t)\, ds}\,.$$

If we apply this to the above expression for $u(x,t)$ we obtain,

$$\begin{aligned} u_t(x,t) &= v(x,t;t) + \int_0^t v_t(x,t;\tau)\, d\tau, \quad \text{and} \\ u_{tt}(x,t) &= \frac{\partial}{\partial t}\left[v(x,t;t)\right] + v_t(x,t;t) + \int_0^t v_{tt}(x,t;\tau)\, d\tau. \end{aligned}$$

The initial conditions on $v(x,t;\tau)$ are for the parabolic case,

$$v(x,\tau;\tau) = \frac{g(x,\tau)}{\rho(x)},$$

and similarly for the hyperbolic case we get,

$$v(x,\tau;\tau) = 0, \quad v_t(x,\tau;\tau) = \frac{g(x,\tau)}{\rho(x)}$$

Therefore, for the hyperbolic case we substitute this and obtain,

$$\begin{aligned} \rho u_{tt} + L[u] &= \rho\left[\frac{\partial}{\partial t}\left[v(x,t;t)\right] + v_t(x,t;\tau) + \int_0^t v_{tt}(x,t;\tau)\, d\tau\right] + \int_0^t L[v(x,t;\tau)]\, d\tau, \\ &= \rho\left[\frac{g(x,t)}{\rho(x)}\right] + \int_0^t \left[\rho v_{tt}(x,t;\tau) + L[v(x,t;\tau)]\right]\, d\tau, \\ &= g(x,t). \end{aligned}$$

Above we used the fact that $v$ satisfies the homogeneous PDE. Similarly for the parabolic equation

$$\begin{aligned}\rho u_t + L[u] &= \rho\left[v(x,t;\tau) + \int_0^t v_t(x,t;\tau)\,d\tau\right] + \int_0^t L[v(x,t;\tau)]\,d\tau,\\ &= \rho\left[\frac{g(x,t)}{\rho(x)}\right] + \int_0^t \left[\rho v_t(x,t;\tau) + L[v(x,t;\tau)]\right]\,d\tau,\\ &= g(x,t).\end{aligned}$$

This proves that the equations for $v(x,t;\tau)$ allow us to solve the correct inhomogeneous PDE. Next, we apply this to find general solutions to the inhomogeneous wave equation.

### 4.5.1 Inhomogeneous Wave Equation

The PDE on an infinite domain is,

$$u_{tt} - c^2 u_{xx} = g(x,t), \quad -\infty < x < \infty, \quad t > 0,$$

with homogeneous initial conditions,

$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad -\infty < x < \infty.$$

We use Duhamel's principle and define the new problem for $v(x,t;\tau)$,

$$v_{tt} - c^2 v_{xx} = 0, \quad -\infty < x < \infty, \quad t > \tau,$$

with non-homogeneous initial conditions,

$$v(x,\tau;\tau) = 0, \quad v_t(x,\tau;\tau) = g(x,\tau), \quad -\infty < x < \infty.$$

This is a problem that can be solved using d'Alembert's solution that we previously obtained using the method of characteristics. It implies,

$$v(x,t;\tau) = \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} g(s,\tau)ds.$$

With this we can compute the solution we are looking for,

$$u(x,t) = \int_0^t v(x,t;\tau)\,d\tau = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} g(s,\tau)\,ds\,d\tau.$$

If we want the general solution to the wave equation with any initial conditions we can define the solution to be a sum of two terms: one that results from the non-zero initial conditions and the second that is due to the forcing. That is why the general solution with,

$$u(x,0) = F(x), \quad \text{and} \quad u_t(x,0) = G(x),$$

with the forcing we described above is,

$$u(x,t) = \frac{1}{2}\left[F(x+ct) + F(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} g(s,\tau)\,ds\,d\tau.$$

Let's consider the domain of dependence of the third term due to the forcing in the wave equation. At $t = 0$ there is no effect because the forcing has not had time to do anything. Then for $t > 0$ we see that the first integral integrates between $x - c(t-\tau)$ and $x + c(t-\tau)$ where $\tau$ ranges from 0 to $t$. This interval is a point when $\tau = t$ and is largest for $\tau = 0$. Therefore, we see that the domain of dependence is still the triangle between $0 \le t$ with base between $x - ct$ and $x + ct$.

### 4.5.2 Inhomogeneous Heat Equation

The non homogeneous equation is,

$$u_t - c^2 u_{xx} = g(x,t), \quad 0 < x < l, \quad t > 0.$$

For simplicity we assume zero initial temperature,

$$u(x,0) = 0,$$

and zero Dirichlet boundary conditions,

$$u(0,t) = 0, \quad u(l,t) = 0.$$

To find a general solution to the inhomogeneous equation we apply Duhamel's principle. We define a new variable $v(x,t;\tau)$ that is determined by the following system of equations,

$$v_t - c^2 v_{xx} = 0, \quad 0 < x < l, \quad t > \tau.$$

with initial temperature,

$$v(x,\tau;\tau) = g(x,\tau),$$

and of course zero Dirichlet boundary conditions. We have already solved this problem exactly in the previous section. We apply that solution but now replace $t$ with $t-\tau$ since the starting time is $t = \tau$,

$$v(x,t;\tau) = \sqrt{\frac{2}{l}} \sum_{k=1}^{\infty} a_k(\tau) \exp\left[ -\left(\frac{\pi k c}{l}\right)^2 (t-\tau) \right] \sin\left(\frac{\pi k}{l} x\right).$$

Note that $a_k(\tau)$ are determined from the initial conditions,

$$v(x,\tau;\tau) = g(x;\tau) = \sqrt{\frac{2}{l}} \sum_{k=1}^{\infty} a_k(\tau) \sin\left(\frac{\pi k}{l} x\right)$$

What is different about this is that the Fourier coefficient depends on the parameter $\tau$ and therefore can change at different times. Finally, we recover the actual solution by integrating over all the possible $\tau$'s,

$$u(x,t) = \sqrt{\frac{2}{l}} \sum_{k=1}^{\infty} \left\{ \int_0^t a_k(\tau) \exp\left[ -\left(\frac{\pi k c}{l}\right)^2 (t-\tau) \right] \, d\tau \right\} \sin\left(\frac{\pi k}{l} x\right).$$

Consider the special case where $g(x,t) = \sin(\pi x/l)$, so the forcing is constant but sinusoidal in space. It is easy to see that the Fourier coefficients are such that $a_1 = \sqrt{l/2}$ and $a_k = 0$ for $k > 1$. For this example the solution becomes,

$$\begin{aligned} u(x,t) &= \left\{ \int_0^t \exp\left[ -\left(\frac{\pi c}{l}\right)^2 (t-\tau) \right] \, d\tau \right\} \sin\left(\frac{\pi}{l}x\right), \\ &= \left(\frac{l}{\pi c}\right)^2 \left\{ 1 - \exp\left[ -\left(\frac{\pi c}{l}\right)^2 t \right] \right\} \sin\left(\frac{\pi}{l}x\right). \end{aligned}$$

This has assumed that initially $u(x,0) = 0$ but this can be relaxed if desired.

In the long time limit we see that the solution tends to,

$$u_0(x) = \lim_{t\to\infty} u(x,t) = \left(\frac{l}{\pi c}\right)^2 \sin\left(\frac{\pi}{l}x\right).$$

Since the solution tends to this solution and it is independent of time, it must be a *steady state solution* to the forced diffusion equation. Indeed it is easy to verify that it satisfies the equation,

$$-c^2 u_0'' = \sin\left(\frac{\pi}{l}x\right), \quad \text{with} \quad u_0(0) = u_0(l) = 0.$$

In the general case of any initial conditions we can still take the limit and find that the solution, which looks like this,

$$u(x,t) = \sqrt{\frac{2}{l}} \sum_{k=1}^{\infty} \left\{ \int_0^t \left(\frac{l}{\pi k c}\right)^2 a_k(\tau) \left( 1 - \exp\left[ -\left(\frac{\pi k c}{l}\right)^2 t \right] \right) \, d\tau \right\} \sin\left(\frac{\pi k}{l}x\right).$$

tends to,

$$u_0(x) = \lim_{t\to\infty} u(x,t) = \sqrt{\frac{2}{l}} \sum_{k=1}^{\infty} \left(\frac{l}{\pi k c}\right)^2 a_k \sin\left(\frac{\pi k}{l}x\right).$$

This shows that the solution tends to a superposition of modes, each of which solves the steady-state diffusion equation.

We have previously seen that any initial condition to the diffusion equation will diffuse and disappear given enough time. One way to describe this is to say that the equation forgets about the initial conditions. However, continual forcing can have a lasting effect. The terms that persist are normal modes and solve the steady state problem.

## 4.6 EIGENFUNCTION EXPANSIONS

When we first used the method of separation of variables we found that for homogeneous PDEs and boundary conditions we could solve for any initial conditions. The solution was always written in terms of eigenfunctions. If this set is complete then we can find any solution. In this section we look at the more general cases of non homogeneous equations but assume that the solution can be found in terms of a superpositon of the eigenfunctions defined by the PDE and boundary conditions. This is called the *Method of Eigenfunction Expansions* or sometimes called *Finite Fourier Transforms*.

The three different problems that we have solved can be written as,

$$\begin{aligned} \rho\frac{\partial^2 u}{\partial t^2} + Lu = \rho F, &\quad \text{hyperbolic,} \\ \rho\frac{\partial u}{\partial t} + Lu = \rho F, &\quad \text{parabolic,} \\ -\rho\frac{\partial^2 u}{\partial y^2} + Lu = \rho F, &\quad \text{elliptic.} \end{aligned}$$

In each case we found that the relevant eigenvalue problem is,

$$LM_k(x) = \lambda_k \rho(x) M_k(x), \quad k = 1, 2, 3, \cdots,$$

where we use the index $k$ because there is an infinite number of these eigenfunctions. Recall that for Sturm-Liouville problems we have an orthogonality relation,

$$(M_k(x), M_j(x)) = \delta_{kj}.$$

Keep in mind that the inner product is the integral of the product of the two functions with the weighting function, $\rho(x)$, over the length of the domain.

Armed with this, we looked for solutions of the form,

$$u = \sum_{k=1}^{\infty} N_k M_k(x).$$

If we multiply this equation by $\rho M_j$ and integrate, we can determine the other function through projecting,

$$N_k \equiv (u, M_k(x)) \,.$$

We begin by solving inhomogeneous equations by assuming a solution of the form written two equations above. But rather than substituting this into the equation directly, which has issues with convergence, instead we project the equation onto each mode and then solve each of those.

To show how this is done for the hyperbolic equation, multiply the equation by $M_k(x)$ and integrate over the domain, i.e. project the equation onto each eigenfunction,

$$\begin{aligned}\left(\frac{\partial^2 u}{\partial t^2}, M_k\right) &= -\left(\frac{1}{\rho}Lu, M_k\right) + (F, M_k)\,,\\ \frac{d^2}{dt^2}(u, M_k) &= -\left(\frac{1}{\rho}Lu, M_k\right) + (F, M_k)\,,\end{aligned}$$

by factoring out the time derivative.

To simplify this equation we use Green's identity that we derived before and pick $u = u$ and $w = M_k$,

$$\left(M_k, \frac{1}{\rho}Lu\right) = \left(\frac{1}{\rho}LM_k, u\right) - \left[p(x)\left[M_k\frac{\partial u}{\partial n} - u\frac{\partial M_k}{\partial n}\right]\right]_{\partial V}.$$

If we assume that we have either homogeneous Dirichlet or Neumann conditions (for simplicity) and substitute in the eigenvalue relation into the RHS we find that the above becomes,

$$\left(M_k, \frac{1}{\rho}Lu\right) = \lambda_k\,(M_k, u) = \lambda_k N_k,$$

where we have defined,

$$N_k \equiv (u, M_k)$$

When we substitute this into the projected equation we get,

$$\boxed{\frac{d^2N_k}{dt^2} + \lambda N_k = F_k, \quad k = 1, 2, \cdots},$$

if we furthermore define the Fourier coefficients of the forcing to be,

$$F_k = (F, M_k)\,.$$

To solve the problem we must also include the initial conditions. This is done by realizing that,

$$u(x,0) = f(x) = \sum_{k=1}^{\infty} N_k(0) M_k(x),$$
$$u_t(x,0) = g(x) = \sum_{k=1}^{\infty} N_k'(0) M_k(x).$$

We can essentially invert these relations by projecting onto each eigenfunction and obtain,

$$N_k(0) = (f(x), M_k(x)), \quad \text{and} \quad N_k'(0) = (g(x), M_k(x)).$$

The above equation is solved for with these initial conditions for each $k$, and together we get the complete solution.

Similarly, we obtain that the parabolic equation becomes

$$\boxed{\frac{dN_k}{dt} + \lambda N_k = F_k, \quad k = 1, 2, \cdots}.$$

The initial conditions that we impose are,

$$N_k(0) = (f(x), M_k(x)).$$

In the elliptic case the governing system of ODEs is

$$\boxed{-\frac{d^2N_k}{dy^2} + \lambda N_k = F_k, \quad k = 1, 2, \cdots}.$$

The boundary conditions are slightly more complicated and not shown here, but they can be decomposed in a similar way as at the other two cases.

We should appreciate that each of these equations has the same form as the equation for $N_k$ in each of the three cases when the PDEs were homogeneous. What we find here is more general though. We must impose initial conditions for each but then we can solve for each mode and then sum up all of the solutions to find the solution to the inhomogeneous equation.

For the parabolic case the solution for each $k$ is,

$$\boxed{N_k(t) = (f(x), M_k)\exp(-\lambda_k t) + \int_0^t F_k(\tau)\exp(-\lambda_k(t-\tau))\, d\tau.}$$

For the hyperbolic case the solution is,

$$\boxed{N_k(t) = (f(x), M_k)\cos(\sqrt{\lambda_k}t) + \frac{1}{\sqrt{\lambda_k}}(g(x), M_k)\sin(\sqrt{\lambda_k}t) + \frac{1}{\sqrt{\lambda_k}}\int_0^t F_k(\tau)\sin(\sqrt{\lambda_k}(t-\tau))\, d\tau.}$$

### 4.6.1 Solving Inhomogeneous ODEs with Laplace Transforms

In this subsection we show how we can solve the inhomogeneous second order equations.

First recall formula's for the Laplace transform:

1) Differentiation:

$$\begin{aligned}\mathcal{L}[y'] &= s\mathcal{L}[y] - y(0),\\ \mathcal{L}[y''] &= s^2\mathcal{L}[y] - sy(0) - y'(0).\end{aligned}$$

2) Convolution:

$$\mathcal{L}^{-1}[\mathcal{L}[f]\mathcal{L}[g]] = f * g,$$

where the convolution is defined as

$$f * g = \int_0^t f(t-\tau)g(\tau)\, d\tau.$$

3) Transform of exponentials:

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}.$$

4) Transform of Trigonometric functions:

$$\mathcal{L}[\cos \omega t] = \frac{s}{s^2+\omega^2}, \quad \text{and} \quad \mathcal{L}[\sin \omega t] = \frac{\omega}{s^2+\omega^2},$$

5) Transform of Hypertrigonometric functions:

$$\mathcal{L}[\cosh \omega t] = \frac{s}{s^2-\omega^2}, \quad \text{and} \quad \mathcal{L}[\sinh \omega t] = \frac{\omega}{s^2-\omega^2},$$

We begin with the hyperbolic equation and compute the Laplace transform

$$\begin{aligned}
\mathcal{L}[\frac{d^2N_k}{dt^2}] + \lambda_k \mathcal{L}[N_k] &= \mathcal{L}[F_k], \\
s^2\mathcal{L}[N_k] - sN_k(0) - N_k'(0) + \lambda_k \mathcal{L}[N_k] &= \mathcal{L}[F_k], \\
(s^2+\lambda_k)\mathcal{L}[N_k] &= sN_k(0) + N_k'(0) + \mathcal{L}[F_k], \\
\mathcal{L}[N_k] &= \frac{s}{(s^2+\lambda_k)}N_k(0) + \frac{1}{(s^2+\lambda_k)}N_k'(0) + \frac{\mathcal{L}[F_k]}{s^2+\lambda_k}.
\end{aligned}$$

When we take the inverse transform of this equation we find,

$$N_k(t) = \cos(\sqrt{\lambda_k}t)N_k(0) + \frac{\sin(\sqrt{\lambda_k}t)}{\sqrt{\lambda_k}}N_k'(0) + (F_k(t) * \frac{\sin(\sqrt{\lambda_k}t)}{\sqrt{\lambda_k}})(t),$$
$$N_k(t) = \cos(\sqrt{\lambda_k}t)N_k(0) + \frac{\sin(\sqrt{\lambda_k}t)}{\sqrt{\lambda_k}}N_k'(0) + \int_0^t F_k(\tau)\frac{\sin(\sqrt{\lambda_k}(t-\tau))}{\sqrt{\lambda_k}}d\tau.$$

Similarly, for the elliptic equation,

$$\begin{aligned}
-\mathcal{L}[\frac{d^2N_k}{dy^2}] + \lambda_k\mathcal{L}[N_k] &= \mathcal{L}[F_k], \\
-s^2\mathcal{L}[N_k] + sN_k(0) + N_k'(0) + \lambda_k\mathcal{L}[N_k] &= \mathcal{L}[F_k], \\
(-s^2 + \lambda_k)\mathcal{L}[N_k] &= -sN_k(0) - N_k'(0) + \mathcal{L}[F_k], \\
\mathcal{L}[N_k] &= \frac{s}{(s^2-\lambda_k)}N_k(0) + \frac{1}{(s^2-\lambda_k)}N_k'(0) + \frac{\mathcal{L}[F_k]}{s^2-\lambda_k}.
\end{aligned}$$

When we take the inverse transform of this equation we find,

$$N_k(t) = \cosh(\sqrt{\lambda_k}t)N_k(0) + \frac{\sinh(\sqrt{\lambda_k}t)}{\sqrt{\lambda_k}}N_k'(0) + (F_k(t) * \frac{\sinh(\sqrt{\lambda_k}t)}{\sqrt{\lambda_k}})(t),$$
$$N_k(t) = \cosh(\sqrt{\lambda_k}t)N_k(0) + \frac{\sinh(\sqrt{\lambda_k}t)}{\sqrt{\lambda_k}}N_k'(0) + \int_0^t F_k(\tau)\frac{\sinh(\sqrt{\lambda_k}(t-\tau))}{\sqrt{\lambda_k}}\,d\tau.$$

### 4.6.2 Hyperbolic Equations: Resonance

Suppose we want to solve a forced hyperbolic equation of the form,

$$\rho u_{tt} + Lu = \rho M_i \sin(\omega t),$$

where $M_i(x)$ is one of the eigenfunctions. That is to say we are forcing the equation with the spatial structure of one of the eigenfunctions and a temporal part that is sinusoidal of frequency $\omega$. For simplicity we assume the initial conditions are zero, $f(x) = g(x) = 0$.

In this form it is easy to see that,

$$F_k(t) = (F(x,t), M_k(x)) = \sin(\omega t)(M_i, M_k) = \delta_{ik}\sin(\omega t).$$

Because of the zero initial conditions the solution reduces greatly to,

$$\begin{aligned} N_k(t) &= \frac{1}{\sqrt{\lambda_k}}\int_0^t F_k(\tau)\sin(\sqrt{\lambda_k}(t-\tau))\,d\tau, \\ &= \frac{1}{\sqrt{\lambda_k}}\int_0^t \delta_{ik}\sin(\omega\tau)\sin(\sqrt{\lambda_k}(t-\tau))\,d\tau, \\ &= \frac{\omega\sqrt{\lambda_i}\sin(\sqrt{\lambda_i}t) - \sin(\omega_i t)}{\sqrt{\lambda_i}(\omega^2 - \lambda_i)}. \end{aligned}$$

and $N_k(t) = 0$ for $k \neq i$. Therefore, our series solution reduces to,

$$u(x,t) = \frac{1}{\omega^2 - \lambda_i}\left[\frac{\omega}{\sqrt{\lambda_i}}\sin(\sqrt{\lambda_i}t) - \sin(\omega_i t)\right] M_i(x).$$

This is an exact solution.

The solution is not valid if $\omega = \sqrt{\lambda_i}$ but we can recover the solution by taking the limit and using l'hopital's rule,

$$u(x,t) = \frac{1}{2\sqrt{\lambda_i}}\left[\frac{\sin(\omega t)}{\omega} - t\cos(\omega t)\right] M_i(x).$$

This shows that whenever we force the hyperbolic equation with the structure of an eigenfunction and exactly the right frequency for this eigenfunction, the eigenvalue, then we get a solution that has the same spatial structure and grows linearly in time.

### 4.6.3 Inhomogeneous Boundary Conditions

Suppose we have a mixed boundary condition that is inhomogeneous,

$$\alpha(\vec{x})V(\vec{x},t) + \beta(\vec{x})\frac{\partial V(\vec{x},t)}{\partial n}\bigg|_{\partial V} = B(\vec{x},t),$$

where we have stated it in three-dimensions with a boundary denoted with $\partial V$. In the one-dimensional case we would have the two equations at the ends of the domain,

$$\alpha_1 V(0,t) - \beta_1 \frac{\partial V(0,t)}{\partial x} = g_1(t),$$
$$\alpha_2 V(l,t) + \beta_2 \frac{\partial V(l,t)}{\partial x} = g_2(t).$$

To solve a PDE with inhomogeneous boundary conditions we decompose the solution into two parts,

$$u(\vec{x},t) = W(\vec{x},t) + V(\vec{x},t).$$

In this decomposition, $V(\vec{x},t)$ is a linear function of space that solves the boundary conditions and the PDE. This is usually obtained by linearly interpolating between the two boundary conditions. Also, $W(\vec{x},t)$ solves the system of equations with homogeneous boundary conditions but extra inhomogeneous terms in the equations to account for the difference.

To illustrate how this method works we consider the diffusion equation with inhomogeneity in both the PDE and the BCs,

$$u_t(x,t) - c^2 u_{xx}(x,t) = g(x,t), \qquad 0 < x < l, \quad t > 0,$$
$$u(x,0) = f(x), 0 < x < l, \quad u(0,t) = g_1(t), \quad u(l,t) = g_2(t), \quad t > 0.$$

As stated above, we decompose the solution into two parts, $W(x,t)$ and $V(x,t)$. The idea is that we want $V(x,t)$ to take care of the boundary conditions. In doing so we have to then modify the equation that $W(x,t)$ satisfies. But, since we know how to solve inhomogeneous equations using Duhamel's principle, we are able to account for this.

Observe that if we define,

$$V(x,t) = \frac{1}{l}\left[x g_2(t) + (l-x) g_1(t)\right].$$

This doesn't satisfy the PDE but it does satisfy the two boundary conditions. Next, when we substitute $u(x,t) = W(x,t) + V(x,t)$ into the PDE we find that $W(x,t)$ satisfies,

$$u_t(x,t) - c^2 u_{xx}(x,t) = g(x,t),$$
$$W_t(x,t) + V_t - c^2 W_{xx}(x,t) = g(x,t),$$

$$W_t(x,t) - c^2 W_{xx}(x,t) = g(x,t) - V_t,$$
$$W_t(x,t) - c^2 W_{xx}(x,t) = g(x,t) - \frac{1}{l}\left[x g_2'(t) + (l-x) g_1'(t)\right].$$

Note that in the above calculation we used the fact that $V(x,t)$ is linear in $x$ and therefore its second derivative is identically zero.

Furthermore, the initial conditions become,

$$\begin{aligned} u(x,0) &= f(x), \\ W(x,0) + V(x,0) &= f(x), \\ W(x,0) &= f(x) - \frac{1}{l}\left[x g_2(0) + (l-x) g_1(0)\right]. \end{aligned}$$

Finally the boundary conditions are,

$$W(0,t) = 0, \quad \text{and} \quad W(l,t) = 0.$$

Therefore, we have that $W(x,t)$ solves an inhomogeneous diffusion equation that we can solve. This is as far s we go here but the interested reader is welcome to construct a solution for this system using what we have done previously. It may be tedious but it is doable.

## 4.7 NONLINEAR STABILITY THEORY: EIGENFUNCTION EXPANSIONS

In this final section of the chapter we investigate a nonlinear reaction-diffusion equation, very similar to the Fisher equation that we derived in Chapter one,

$$u_t(x,t) - u_{xx}(x,t) = \hat{\lambda} u(x,t)(1 - u^2(x,t)).$$

This differs form Fisher's equation in that the right hand side involves a cubicly nonlinear term. Before we investigate the stability of the PDE let's consider the associated ODE.

### 4.7.1 Reaction Equation

If we wanted to look for a solution to the above equation that did not depend on space it would satisfy the following ODE,

$$\frac{dU}{dt} = \hat{\lambda}U(1-U^2), \quad \text{with} \quad U(0) = U_0.$$

This is a nonlinear ODE but it is possible to solve it because it is separable. Using this property along with partial fractions we find that the solution is of the form,

$$U(t) = \frac{U_0 \exp(\hat{\lambda}t)}{\sqrt{1 + U_0^2(\exp(2\hat{\lambda}t) - 1)}}$$

In Figure 4.4 we plot three different solutions with $\hat{\lambda} = 1$ and $U(0) = 0.2, 1.0, 2.0$. These are to show that the typical behaviour of the solutions is to tend to $U(t) = 1$. Indeed, if we look at the RHS of the ODE we see that, assuming $\hat{\lambda} > 0$, it is positive for $U < 1$ but then becomes negative for $U > 1$ and is exactly zero for $U = 0, U = 1$. That is to say the two equilibrium solutions are $U = 0, 1, -1$. If we restrict the concentrations to be non-negative that leaves us with $U = 0, 1$, which are the only ones we consider. All the solutions between $0 < U < 1$ increase to asymptotically approach $U = 1$. Similarly, for $U > 1$, all the solutions decrease to approach $U = 1$. Without any formal definitions it should be evident that the steady solution $U = 1$ is stable since everything tends to it. This is actually referred as a global attractor. In contrast, $U = 0$ is unstable and is known as a repeller.

To be more formal we can determine the stability of each steady solution using a *linear stability analysis*. We we do is decompose our solution into a mean part plus a small amplitude fluctuation above that state,

$$U(t) = \overline{U} + \epsilon U'(t),$$

where $\overline{U}$ can either be $0$ or $1$. Also, we have an $\epsilon$ in front of $U'(t)$ because we want only to consider small perturbations, i.e. we assume $\epsilon \ll 1$. We substitute this into the ODE we get

$$\begin{aligned}
\frac{d}{dt}(\overline{U} + \epsilon U'(t)) &= \hat{\lambda}\left[\overline{U} + U'(t)\right](1 - \left[\overline{U} + \epsilon U'(t)\right]^2), \\
\epsilon\frac{dU'}{dt} &= \hat{\lambda}\left[\overline{U} + \epsilon U'(t)\right](1 - \left[\overline{U}^2 + 2\epsilon\overline{U}U'(t) + \epsilon^2 U'(t)^2\right]), \\
\epsilon\frac{dU'}{dt} &= \hat{\lambda}\left[\overline{U}(1-\overline{U}^2) + \epsilon U'(t)(1-\overline{U}^2) - 2\epsilon\overline{U}^2 U'(t) + O(\epsilon^2)\right].
\end{aligned}$$

In the above we use the notation $O(\epsilon)$ to denote the fact that all other terms are proportional to $\epsilon^2$ or higher powers of $\epsilon$. Since $\overline{U}$ is either $0$ or $1$ the first term is identically zero. What is left can be divided by $\epsilon$ to get, upon neglecting the higher order terms,

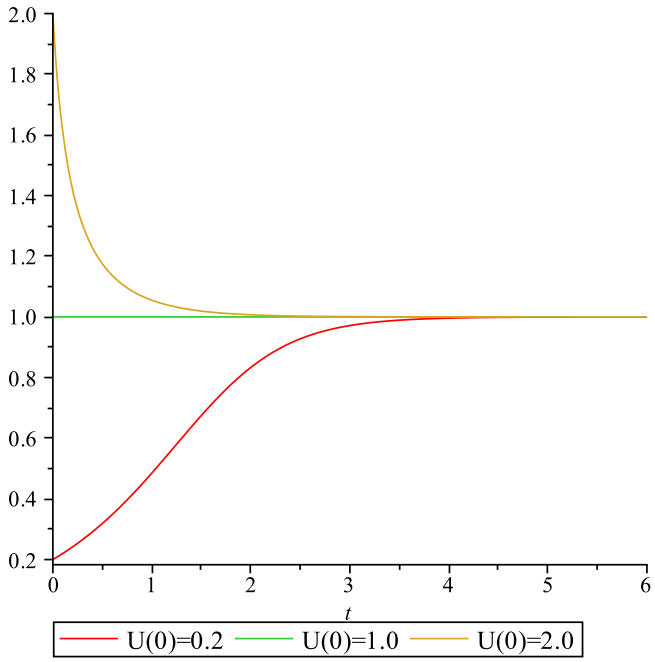


**Fig. 4.4** Three different solutions to the reaction part of the PDE without any spatial variation, that is to say solving the associated ODE.

$$\frac{dU'}{dt} = \hat{\lambda}\left[U'(t)(1-\overline{U}^2) - 2\overline{U}^2 U'(t)\right].$$

With this equation there are two cases to consider. First, if we substitute $\overline{U} = 0$ into the equation we get,

$$\frac{dU'}{dt} = \hat{\lambda}U'(t).$$

We know the solution to this equation is proportional to $\exp(\hat{\lambda}t)$ and since $\hat{\lambda} > 0$ we know that this solution will grow exponentially. This is sufficient for us to describe $\overline{U} = 0$ as an unstable solution. Note that we saw previously that the solution does not grow exponentially for all time. It must necessarily saturate near $U = 1$. That is because we have linearized the above equation assuming that the perturbations are small. When the perturbations because large enough the approximation breaks down and we must retain the nonlinear terms to get an accurate solution.

The second solution is $\overline{U} = 1$. When we substitute this into the linearized equation we get,

$$\frac{dU'}{dt} = -2\hat{\lambda}U'(t).$$

The solution to this approximate equation is proportional to $\exp(-2\hat{\lambda}t)$, which decays exponentially in time. This means that if we start off near $U = 1$, small perturbations will decay. Indeed this is consistent with what we saw previously and therefore why we call this solution stable. To be more precise some people say asymptotically stable.

Now that we understand the stability of the scalar ODE, we are now in a position to study the stability of the PDE using similar methods. There are two different situations we will consider. First, an infinite domain where we look for normal-mode or wave solutions, as we saw previously. Second, a finite domain where we look for solutions based on eigenfunction expansions.

### 4.7.2 Nonlinear Heat Equation: Infinite Domain

To begin let us appreciate that the two solutions that we found in the context of the ODE are still solutions for the PDE. That is because they are both constant with respect to $x$ and therefore $u_{xx}$ vanishes. One can imagine that there are more complicated solutions but it turns out there are in fact the only two steady (equilibrium) solutions. We continue to denote them with $\overline{U}$.

As before, we perturb the steady states but now with one that is variable in both space and time,

$$u(x,t) = \overline{U} + \epsilon u'(x,t).$$

We substitute this into the PDE and get,

$$\begin{aligned}
\frac{\partial}{\partial t}(\overline{U} + \epsilon u'(x,t)) - \frac{\partial^2}{\partial x^2}(\overline{U} + \epsilon u'(x,t)) &= \hat{\lambda}\left[\overline{U} + u'(x,t)\right](1 - \left[\overline{U} + \epsilon u'(x,t)\right]^2), \\
\epsilon\left[\frac{\partial u'}{\partial t} - \frac{\partial^2 u'}{\partial x^2}\right] &= \hat{\lambda}\left[\overline{U} + \epsilon u'\right](1 - \left[\overline{U}^2 + 2\epsilon\overline{U}u' + \epsilon^2 u'^2\right]), \\
\epsilon\left[\frac{\partial u'}{\partial t} - \frac{\partial^2 u'}{\partial x^2}\right] &= \hat{\lambda}\left[\overline{U}(1-\overline{U}^2) + \epsilon u'(1-\overline{U}^2) - 2\epsilon\overline{U}^2 u' + O(\epsilon^2)\right], \\
\epsilon\left[\frac{\partial u'}{\partial t} - \frac{\partial^2 u'}{\partial x^2}\right] &\approx \hat{\lambda}\left[\epsilon u'(1-\overline{U}^2) - 2\epsilon\overline{U}^2 u'\right].
\end{aligned}$$

In the last equation above we used the fact that $\overline{U}$ solves a particular cubic relation and also neglected the smaller terms, and that is why we have an approximation symbol.

First we specialize this for $\overline{U} = 0$ and obtain, after dividing through by $\epsilon$, the linearized equation

$$\frac{\partial u'}{\partial t} - \frac{\partial^2 u'}{\partial x^2} = \hat{\lambda} u'.$$

If we substitute in our normal mode (wave) solution of the form,

$$u'(x,t) = \exp(ikx + \lambda(k)t),$$

we find that the resulting growth/dispersion relation is,

$$\lambda = \hat{\lambda} - k^2.$$

In the ODE case we found that the solution $\overline{U} = 0$ was always unstable. Here, we find that the stability depends on the wavenumber. If $\hat{\lambda} > k^2$, which is true for larger waves, then the zero solution is unstable because the growth rate is positive. However, for any $\hat{\lambda}$ we know that there will be a cut-off value, $k_c^2 = \hat{\lambda}$ where for $k > k_c$, smaller waves, the solution is stable. Here we see that diffusion has the effect of stabilizing the smaller wavelengths. This is perhaps not surprising because we know that diffusion dampens the smaller wavelengths very quickly.

The second solution is for $\overline{U} = 1$ and the linearized equation is

$$\frac{\partial u'}{\partial t} - \frac{\partial^2 u'}{\partial x^2} = -2\hat{\lambda} u'.$$

Again, we look for normal-mode solutions to this equation and get a growth/dispersion relation of,

$$\lambda = -2\hat{\lambda} - k^2.$$

This shows that all the wave numbers are stable but the diffusion causes the decay rates to increase quadratically with the wavenumber.

### 4.7.3 Nonlinear Heat Equation: Bounded Domain

If we assume that we are imposing Dirichlet boundary conditions at the end of our domain,

$$u(0,t) = 0, \quad \text{and} \quad u(l,t) = 0,$$

then we know, from previous calculations, that the eigenfunctions for the Diffusion equation are sinusoidal functions. This is what requires that we look for an eigenfunction expansion solution of the form,

$$u(x,t) = \exp(\lambda t)\sin(n\pi x),$$

for any positive integer $n$. When we substitute this into the linearized PDE about $\overline{U} = 0$ we obtain that $\lambda = \hat{\lambda} - n^2\pi^2$. For the other steady solution we obtain $\lambda = -2\hat{\lambda} - n^2\pi^2$. These are essentially the same growth/dispersion relations as in the infinite case except for that the wavenumber achieves only discrete values, $k_n = n\pi$. The same behaviour occurs as in the previous case.

### 4.7.4 Predator-Prey Systems

In an ODE course some of you might have seen a Lotka-Volterra system that describes the interaction between a predator and a prey. (Will be covered in AMATH 351). Here, we consider an extension of the Lotka-Volterra system where the reaction terms are slightly modified and we include diffusion to see what effect this can have on the evolution of the ecosystem. The material from this subsection is found in Murray (1993).

*Reaction System*

To begin we define $u(t)$ and $v(t)$ to be the concentration of the prey and predator, respectively. The modified Lotka-Volterra system for $u(t)$ and $v(t)$ is

$$\begin{aligned}\frac{du}{dt} &= u(1-u-v),\\ \frac{dv}{dt} &= av(u-b).\end{aligned}$$

In the prey equation we have three terms on the RHS: $u$, $-u^2$ and $-uv$. The first term is a growth term because of the prey eating their food supply, which is assumed to be in infinite abundance. The second is a saturation term to make sure that the prey does not grow too large. The first and second term combined form what is called *Logistic Growth*. The third term is a death rate term because of the predator eating the prey. Similarly, in the predator equation we have that the concentration changes in time due to two factors: $auv$ and $-abv$. The first is a growth term because of predators consuming preys and the second is a natural mortality term or mortality due to other species consuming them.

There are three steady state solutions that are possible, all of which occur when the RHS of the two equations are zero. From the second equation we see that we need either $v = 0$ or $u = b$. Then from the first equation we have either $u = 0$ or $u + v = 1$. From this we find the following possible solutions,

$$(0,0), \quad (1,0), \quad (b,1-b),$$

for $b < 1$, which we assume throughout. The first describes an ecosystem with no residents, there is nothing there. Not very exciting but a possibility nonetheless. The second has prey present but no predators. The third, by far the most interesting one, has both preys and predators present.

Now that we know all the possible steady solutions it is possible to answer the question, which ones are stable and unstable? As before, we define the steady solution to be $(\overline{U}, \overline{V})$ and perturb this basic state with small perturbations,

$$(u(t), v(t)) = (\overline{U} + \epsilon u'(t), \overline{V} + \epsilon v'(t)),$$

where $\epsilon \ll 1$. We substitute this into the system of two first-order ODEs and obtain,

$$\begin{aligned}
\frac{d}{dt}\left[\overline{U} + \epsilon u'(t)\right] &= \left[\overline{U} + \epsilon u'(t)\right](1 - \left[\overline{U} + \epsilon u'(t)\right] - \left[\overline{V} + \epsilon v'(t)\right]), \\
\epsilon \frac{du'}{dt} &= \overline{U}(1 - \overline{U} - \overline{V}) + \epsilon u'(1 - \overline{U} - \overline{V}) - \epsilon \overline{U}(u' + v') + O(\epsilon^2), \\
\frac{du'}{dt} &= u'(1 - 2\overline{U} - \overline{V}) - \overline{U}v' + O(\epsilon),
\end{aligned}$$

if we divide each of the above equations by $\epsilon$ and use the fact that the steady states are roots of the flux functions. A similar calculation for the other equation yields,

$$\begin{aligned}
\frac{d}{dt}\left[\overline{V} + \epsilon v'(t)\right] &= a\left[\overline{V} + \epsilon v'(t)\right](\left[\overline{U} + \epsilon u'(t)\right] - b), \\
\epsilon \frac{dv'}{dt} &= a\overline{V}(\overline{U} - b) + a\epsilon v'(\overline{U} - b) + a\epsilon \overline{V}u' + O(\epsilon^2), \\
\frac{dv'}{dt} &= a\overline{V}u' + av'(\overline{U} - b) + O(\epsilon).
\end{aligned}$$

Therefore, when we approximate (linearized) these equations we get

$$\frac{du'}{dt} = u'(1 - 2\overline{U} - \overline{V}) - \overline{U}v',$$

$$\frac{dv'}{dt} = a\overline{V}u' + av'(\overline{U} - b).$$

The observant reader might notice that the right hand side is the Jacobian of the original RHS of the system of two DEs.

Indeed, given that the reaction functions are $f(u, v)$ and $g(u, v)$ in the two equations, respectively, and $(\bar{U}, \bar{V})$ is an exact solution, we can Taylor expand about this state and get,

$$f(\bar{U} + \epsilon u', \bar{U} + \epsilon v') = f(\bar{U}, \bar{V}) + \epsilon u' \frac{\partial f}{\partial u}(\bar{U}, \bar{V}) + \epsilon v' \frac{\partial f}{\partial v}(\bar{U}, \bar{V}) + O(\epsilon^2),$$
$$g(\bar{U} + \epsilon u', \bar{U} + \epsilon v') = g(\bar{U}, \bar{V}) + \epsilon u' \frac{\partial g}{\partial u}(\bar{U}, \bar{V}) + \epsilon v' \frac{\partial g}{\partial v}(\bar{U}, \bar{V}) + O(\epsilon^2).$$

Since the steady solution is a zero of the reaction terms, the first terms on the right are both zero. So far the equation is exact. If we assume that $\epsilon \ll 1$, then we can neglect the terms that are quadratic and higher order in epsilon, what we denote with $O(\epsilon^2)$. That is why we can substitute this into the original system of equations and get,

$$\frac{d}{dt}\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}$$

The previous system of equations we derived by substituting in and expanding is of this form where we substituted in for this particular reaction equation. If we look for solutions that are proportional to $e^{\lambda t}$ then we get our eigenvalue problem.

We consider each case separately.

a) Solution $(0, 0)$. If we substitute this into the above equation we get,

$$\frac{du'}{dt} = u',$$
$$\frac{dv'}{dt} = -abv'.$$

If we look for solutions of the form, $u = \alpha e^{\lambda t}$ and $v = \beta e^{\lambda t}$ we find that the above equations become an eigenvalue problem,

$$\lambda \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -ab \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

The eigenvalues and eigenvectors are

$$\lambda_1 = 1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = -ab, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

One is positive and one is negative. Because there exists an eigenvalue with a positive real part we know the solution is unstable. In particular, this shows that for small perturbations about the zero solution the prey grows exponentially and the predator dies exponentially. That is because the prey has an infinite resource of food and the predator is starved because of the lack of prey. Because one of the solutions grows exponentially the solution is unstable.

The eigenvalues tell us the nature of the instability. That is to say if we perturb the solution $(0, 0)$ with some prey and no predators then that is going to grow. That is why the eigenvalue corresponding to the vector $(1, 0)^T$ is positive. Similarly, if we introduce some predator but no prey they will die, and therefore why the vector $(0, 1)^T$ has a negative eigenvalue.

b) Solution $(1, 0)$. The linearized equations for this solution are,

$$\begin{aligned} \frac{du'}{dt} &= -u' - v', \\ \frac{dv'}{dt} &= av'(1-b). \end{aligned}$$

The characteristic equation of the above system can be written in vector form as

$$\lambda \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & a(1-b) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Because the matrix is upper tridiagonal we know the eigenvalues are equal to the the elements in the diagonal, $-1 < 0$ and $a(1-b) > 0$. One is positive and the other

negative and so the system is unstable. This corresponds to the fact that if there is a lot of prey around and a little bit of predators are introduced, they will grow in numbers.

Another approach is to compute the eigenvalues and eigenvectors, which are

$$\lambda_1 = a(1-b), \quad \vec{v}_1 = \begin{bmatrix} \frac{-1}{a(1-b)+1} \\ 1 \end{bmatrix}, \quad \lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

For $0 < b < 1$ we have that the first is positive. That means that the nature of the unstable part is to decrease the population of prey (because the first element of the eigenvector is negative) and increase the predator. This is because the steady solution has a lot of prey and no predators. Similarly, if we add only prey but no predators, what is indicated by the second eigenvector, we see that the prey population will decrease because it exceeds the carrying capacity, which in this case is one.

c) Solution $(b, 1-b)$: In this final case we get linearized set of equations that will yield a qualitatively different result.

$$\begin{aligned} \frac{du'}{dt} &= -bu' - bv', \\ \frac{dv'}{dt} &= a(1-b)u'. \end{aligned}$$

The associated eigenvalue problem is,

$$\lambda \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -b & -b \\ a(1-b) & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

This solution is more difficult to obtain (but those of you taking AMATH 351 should be able to do it easily). It can be shown that the stability of the system is determined by the eigenvalues of the matrix of the system of equations. The characteristic equation is,

$$\begin{aligned} (\lambda + b)(\lambda) + ab(1-b) &= 0, \\ \lambda^2 + b\lambda + ab(1-b) &= 0, \\ \lambda = \frac{-b \pm \sqrt{b^2 - 4ab(1-b)}}{2}&. \end{aligned}$$

Because $4ab(1-b) \geq 0$ we have that the quantity in the square root will always have a magnitude less than or equal to $b^2$ and therefore it is impossible to have a positive real part. That is how we know that this steady solution is stable.

The eigenvectors are

$$\vec{v}_{1,2} = \begin{bmatrix} -\frac{2b}{-b \pm \sqrt{b^2 - 4ab(1-b)}} \\ 1 \end{bmatrix},$$

but since both eigenvalues are negative this simply tells us how the two different directions decay slightly differently.

*Reaction-Diffusion System*

If we include diffusion in our reaction system we get the system of equations for $u(x,t)$ and $v(x,t)$,

$$\begin{aligned} \frac{\partial u}{\partial t} &= u(1-u-v) + D\frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial v}{\partial t} &= av(u-b) + \frac{\partial^2 v}{\partial x^2}. \end{aligned}$$

Note that we have set the diffusion rate to be one for the predators. This leaves us with $D$ in the prey equation and therefore we can determine the relative diffusion that occurs between the two species. Because the steady solutions we found in the purely reaction case is a constant in the reaction-diffusion, they are also solutions to this system of equations.

To determine the stability in the presence of diffusion, we begin by perturbing the steady solutions,

$$(u(x,t), v(x,t)) = (\overline{U} + \epsilon u'(x,t), \overline{V} + \epsilon v'(x,t)),$$

where $\epsilon \ll 1$. We substitute this into the system of two first-order ODEs and obtain,

$$\begin{aligned} \left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right)\left[\overline{U} + \epsilon u'(t)\right] &= \left[\overline{U} + \epsilon u'\right](1 - \left[\overline{U} + \epsilon u'\right] - \left[\overline{V} + \epsilon v'\right]), \\ \epsilon\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right)u' &= \overline{U}(1-\overline{U}-\overline{V}) + \epsilon u'(1-\overline{U}-\overline{V}) - \epsilon\overline{U}(u'+v') + O(\epsilon^2), \\ \left(\frac{\partial u'}{\partial t} - D\frac{\partial^2 u'}{\partial x^2}\right) &= u'(1-2\overline{U}-\overline{V}) - \overline{U}v' + O(\epsilon), \end{aligned}$$

if we divide each of the above equations by $\epsilon$ and use the fact that the steady states are roots of the flux functions. A similar calculation for the other equation yields,

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left[\overline{V} + \epsilon v'(t)\right] = a\left[\overline{V} + \epsilon v'(t)\right]\left(\left[\overline{U} + \epsilon u'(t)\right] - b\right),$$
$$\epsilon\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)v' = a\overline{V}(\overline{U} - b) + a\epsilon v'(\overline{U} - b) + a\epsilon\overline{V}u' + O(\epsilon^2),$$
$$\left(\frac{\partial v'}{\partial t} - \frac{\partial^2 v'}{\partial x^2}\right) = a\overline{V}u' + av'(\overline{U} - b) + O(\epsilon).$$

Therefore, when we approximate (linearized) these equations we get

$$\frac{\partial u'}{\partial t} = u'(1 - 2\overline{U} - \overline{V}) - \overline{U}v' + D\frac{\partial^2 u'}{\partial x^2},$$
$$\frac{\partial v'}{\partial t} = a\overline{V}u' + av'(\overline{U} - b) + \frac{\partial^2 v'}{\partial x^2}.$$

The observant reader might notice that the right hand side is the Jacobian of the original RHS of the system of two DEs.

As before, we consider each case separately.

a) Solution $(0, 0)$. If we substitute this into the above equation we get,

$$\frac{\partial u'}{\partial t} = u' + D\frac{\partial^2 u'}{\partial x^2},$$
$$\frac{\partial v'}{\partial t} = -abv' + \frac{\partial^2 v'}{\partial x^2},$$

and so the equations decouple, as before. If we look for normal mode solutions of the form

$$u' = \alpha\exp(ikx + \lambda t),$$
$$v' = \beta\exp(ikx + \lambda t),$$

we can substitute this into our system of equations and get,

$$\lambda \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 - k^2 D & 0 \\ 0 & -(ab + k^2) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

This shows quite nicely how $\lambda$ plays the role of the eigenvalue and $[\alpha, \beta]^T$ is the corresponding eigenvector. We see that diffusion acts to stabilize this solution for small wavelengths, i.e. large wavenumber. The critical value is $k_c^2 = 1/D$. For $k > k_c$ the system is stable but for $k < k_c$ the system is unstable. Overall the system is still unstable for this particular steady solution.

b) Solution $(1, 0)$. The linearized equations for this solution are,

$$\frac{\partial u'}{\partial t} = -u' - v' + D\frac{\partial^2 u'}{\partial x^2},$$
$$\frac{\partial v'}{\partial t} = av'(1 - b) + \frac{\partial^2 v'}{\partial x^2}.$$

We substitute in our normal-mode solutions to obtain,

$$\lambda \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 - k^2 D & -1 \\ 0 & a(1 - b) - k^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

We observe that the eigenvalues of the matrix are $-1 - k^2 D < 0$ and $a(1 - b) - k^2$. The first is negative and therefore stable but now that it decreases with increasing $k$. The second eigenvalue can be positive for large wavelengths $k \ll 1$ but is necessarily stable for $k > k_c$ where the critical wavenumber is defined as $k_c^2 = a(1 + b)$. This steady solution is still stable, but less so than in the purely reaction case.

c) Solution $(b, 1 - b)$: In this final case we get linearized set of equations that will yield a qualitatively different result.

$$\frac{\partial u'}{\partial t} = -bu' - (b + Dk^2)v',$$
$$\frac{\partial v'}{\partial t} = a(1 - b)u' - k^2 v'.$$

The associated eigenvalue problem is,

$$\lambda \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -b & -(b+k^2) \\ a(1-b) & -Dk^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

This solution is more difficult to obtain (but those of you taking AMATH 351 should be able to do it easily). It can be shown that the stability of the system is determined by the eigenvalues of the matrix of the system of equations. The characteristic equation is,

$$\begin{aligned} (\lambda+b)(\lambda+k^2) + a(1-b)(b+Dk^2) &= 0, \\ \lambda^2 + (b+k^2)\lambda + bk^2 + a(1-b)(b+Dk^2) &= 0, \\ \lambda = \frac{-(b+k^2) \pm \sqrt{(b+k^2)^2 - 4bk^2 - 4a(1-b)(b+Dk^2)}}{2}&. \end{aligned}$$

Because $4ab(1-b) \geq 0$ we have that the quantity in the square root will always be less than or equal to $(b+k)^2$ and therefore it is impossible to have a positive real part. That is how we know that this steady solution is stable. If the quantity under the square root is negative we will have a negative real part, which produces an asymptotically stable spiral. If the eigenvalues are real then we have nodes instead.

CHAPTER
# FIVE

# Fourier Transform Methods

In this chapter we focus on solving PDEs on infinite domains. Unlike in the previous chapter where we used essentially what can be described as *Fourier Series Eigenfunction Expansions*, here it is still based on the idea of separation of variables but now with *Fourier Transforms*. These ideas can be generalized to other equations to include Hankel transforms, Laplace transforms and others, but this is beyond the scope of the course. Not because it is harder but because we simply don't have time. The interested reader is directed to Zauderer (2006) for more details.

## 5.1 INTRODUCTION

The fundamental difference between the solutions on infinite and finite domains is in the spectrum. In the infinite domain the spectrum is continuous, since all waves wave in an infinite domain. Therefore, the wavenumber $k$ can be any real number, i.e.

$$-\infty < \lambda < \infty.$$

This is in contrast to a finite domain where there is only a countably infinite number of waves that fit in our domain, and therefore the spectrum is discrete, i.e.

$$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots .$$

The ideas of the eigenfunction expansion still hold however. We look for a solution that is a sum over all the eigenfunctions so we can think of the solution on the infinite domain as,

$$u(x,t) = \int_{-\infty}^{\infty} M_\lambda N_\lambda d\lambda.$$

One issue is that with a continuous spectrum we do not get orthogonality in the same way. This is restrictive in some way but we still find that much progress can still be made in solving these problems.

## 5.2 ONE-DIMENSIONAL FOURIER TRANSFORM

We begin with the same PDE that we considered previously for the hyperbolic, parabolic and elliptic equations. In each case we use the method of separation of variables, $u(x,t) = M(x)N(t)$ or $u(x,y) = M(x)N(y)$ and obtain a type of eigenvalue problem for the $M(x)$:

$$LM(x) = \lambda^2 \rho(x) M(x),$$

where we defined the eigenvalue to be $\lambda^2$ that will facilitate in how we construct the solution. It is not necessary to do this however if you would rather not. The other three equations are,

$$\begin{aligned} N''(t) + \lambda^2 N(t) &= 0, \qquad \text{hyperbolic,} \\ N'(t) + \lambda^2 N(t) &= 0, \qquad \text{parabolic,} \\ N''(t) - \lambda^2 N(t) &= 0, \qquad \text{elliptic.} \end{aligned}$$

For the special choice of $\rho = 1, p = 1, q = 0$ we still use separation of variables and get the same equation we studied before,

$$M''(x) + \lambda^2 M(x) = 0, \qquad -\infty < x < \infty.$$

We know that for any value of $\lambda$ the solution can be written as,

$$M_\lambda(x) = \alpha(\lambda) e^{i\lambda x} + \beta(\lambda) e^{-i\lambda x},$$

where $\alpha(\lambda)$ and $\beta(\lambda)$ are constants that must be solved for. If the spectrum is continuous and infinite then we must solve this for each real number $\lambda$.

Whether we are solving hyperbolic, parabolic or elliptic equations, if we have an initial condition of $f(x)$, or boundary condition for the elliptic case, we know that we need to

decompose this function in terms of our eigenfunctions. That means that we need to determine the coefficients such that,

$$f(x) = \int_{-\infty}^{\infty} \left[\alpha(\lambda)e^{i\lambda x} + \beta(\lambda)e^{-i\lambda x}\right]\, d\lambda.$$

If we make a substitution in the first term of $\lambda = -\lambda$, we can define $\gamma(\lambda) = \alpha(-\lambda) + \beta(\lambda)$ and rewrite the above equation as,

$$f(x) = \int_{-\infty}^{\infty} \gamma(\lambda)e^{-i\lambda x}\, d\lambda. \tag{5.1}$$

Given this equation we need to invert it to solve for $\gamma(\lambda)$ in terms of the initial condition $f(x)$. To do this we recall the definition of the Fourier Transform from AMATH 231. For a well behaved function $f(x)$, its *Fourier Transform* is,

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} f(x)\, dx,$$

and we can recover the function through the *Inverse Fourier Transform*,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} F(\lambda)\, d\lambda.$$

Note that if we substitute the former into the latter we get the *Fourier Integral Formula*,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\lambda(x-s)} f(s)\, ds\, d\lambda.$$

For this to be true we can introduce the *Dirac delta function* that is a generalized function that satisfies the following equality,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda(x-s)}\, d\lambda = \delta(x-s).$$

Given this, we can substitute into the RHS of the previous equation and get,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\lambda(x-s)}f(s)\,ds\,d\lambda &= \int_{-\infty}^{\infty}\delta(x-s)f(s)\,ds,\\ &= f(x),\end{aligned}$$

as desired. If you make a change of variables it is easy to see that we get a similar equation,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix(\lambda-\hat{\lambda})}\,dx = \delta(\lambda-\hat{\lambda}).$$

Those of you that have see this delta function might understand this, those of you that haven't will see this derived in detail later on. If we take this relation to be true for the moment, then we can multiply equation (5.1) by $(1/(2\pi))e^{i\hat{\lambda}x}$ and integrate with respect to $x$ over the real line, exchange the order of integration and final recover,

$$\begin{aligned}f(x) &= \int_{-\infty}^{\infty}\gamma(\lambda)e^{-i\lambda x}\,d\lambda,\\ \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\hat{\lambda}x}\,dx &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\gamma(\lambda)e^{-i(\lambda-\hat{\lambda})x}\,d\lambda dx,\\ \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\hat{\lambda}x}\,dx &= \int_{-\infty}^{\infty}\gamma(\lambda)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\lambda-\hat{\lambda})x}\,dx d\lambda,\\ \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\hat{\lambda}x}\,dx &= \int_{-\infty}^{\infty}\gamma(\lambda)\delta(\lambda-\hat{\lambda})\,d\lambda,\\ \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\hat{\lambda}x}\,dx &= \gamma(\hat{\lambda}).\end{aligned}$$

Besides the fact that we have hats over the $\lambda$ and a constant factor, this is precisely what we have in the Fourier Transform.

### 5.2.1 Dirac Delta Function

If you take the Fourier transform of $\delta(x-x_0)$ then you get,

$$\mathcal{F}[\delta(x-x_0)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x}\delta(x-x_0)\,dx = \frac{1}{\sqrt{2\pi}}e^{i\lambda x_0}.$$

From this we can use the inverse Fourier transform equation to determine that,

$$\delta(x - x_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda(x-x_0)}\, d\lambda.$$

This is precisely what we have already stated.

To prove this above result you can consider the Dirac delta function to be the limit of a rectangles,

$$\delta(x) = \lim_{N\to\infty} \delta_N(x),$$

where

$$\delta_N(x) = \sqrt{2\pi} \begin{cases} N & |x| \leq \frac{1}{2N} \\ 0 & |x| > \frac{1}{2N} \end{cases}$$

Note that the area within the rectangle is always equal to $\sqrt{2\pi}$. This is done to normalize the Dirac delta function in a particular way. Other choices are possible.

We begin by showing that the Fourier transform of $\delta_N(x)$ is in fact one,

$$\begin{aligned} \mathcal{F}[\delta_N(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} \delta_N(x)\, dx, \\ &= N \int_{-\frac{1}{2N}}^{\frac{1}{2N}} e^{i\lambda x}\, dx, \\ &= N \left[ \frac{1}{i\lambda} e^{i\lambda x} \right]_{-\frac{1}{2N}}^{\frac{1}{2N}}, \\ &= N \frac{1}{i\lambda} \left[ e^{i\frac{\lambda}{2N}} - e^{i\frac{\lambda}{2N}} \right], \\ &= N \frac{1}{i\lambda} \left[ e^{i\frac{\lambda}{2N}} - e^{i\frac{\lambda}{2N}} \right], \\ &= \frac{\sin\frac{\lambda}{2N}}{\frac{\lambda}{2N}}. \end{aligned}$$

The Fourier Transform pair then shows,

$$\delta_N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} \frac{\sin \frac{\lambda}{2N}}{\frac{\lambda}{2N}} \, d\lambda$$

To obtain the continuous limit we take the limit as $N$ approaches infinity,

$$\begin{aligned} \mathcal{F}[\delta(x)] &= \lim_{N\to\infty} \mathcal{F}[\delta_N(x)], \\ &= \lim_{N\to\infty} \frac{\sin \frac{\lambda}{2N}}{\frac{\lambda}{2N}}, \\ &= 1. \end{aligned}$$

and

$$\begin{aligned} \delta(x) &= \lim_{N\to\infty} \delta_N(x), \\ &= \frac{1}{\sqrt{2\pi}} \lim_{N\to\infty} \int_{-\infty}^{\infty} e^{-i\lambda x} \frac{\sin \frac{\lambda}{2N}}{\frac{\lambda}{2N}} \, d\lambda, \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} \, d\lambda. \end{aligned}$$

This is precisely the relation we defined above with $x_0 = 0$.

### 5.2.2 General Properties

1) If $f(x)$ is piecewise continuously differentiable and,

$$\int_{-\infty}^{\infty} |f(x)| \, dx < \infty,$$

the Fourier Integral Formula converges pointwise.

2) If $f(x)$ is discontinuous at $x_0$ then the integral converges to,

$$\frac{1}{2} \left[ f(x_0-) + f(x_0+) \right].$$

Note that the minus and plus mean the limit from the left and right, respectively.

3) If $F(\lambda) = \mathcal{F}(f(x))$ and $G(\lambda) = \mathcal{F}(g(x))$ and $H(\lambda) = F(\lambda)G(\lambda)$ then

$$\begin{aligned}
\mathcal{F}^{-1}(H(\lambda)) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} F(\lambda)G(\lambda)\, d\lambda, \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\lambda x} e^{i\lambda s} f(s)G(\lambda)\, ds\, d\lambda, \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \int_{-\infty}^{\infty} e^{-i\lambda(x-s)} G(\lambda)\, d\lambda\, ds, \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)g(x-s)\, ds, \\
&= \frac{1}{\sqrt{2\pi}} (f * g)(x).
\end{aligned}$$

In the last line above we defined the *convolution* of two functions.

4) If we pick $G(\lambda) = \overline{F(\lambda)}$ in the above equation we obtain *Parseval's equation* for Fourier Transforms. To derive this we begin by integrating the produce of the two Fourier Transforms without the exponential term,

$$\begin{aligned}
\int_{-\infty}^{\infty} |F(\lambda)|^2\, d\lambda &= \int_{-\infty}^{\infty} F(\lambda)\overline{F(\lambda)}\, d\lambda, \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\lambda(x-s)} f(x)\overline{f(s)}\, dx\, ds\, d\lambda, \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\overline{f(s)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda(x-s)}\, d\lambda\, dx\, ds, \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\overline{f(s)}\delta(x-s)\, dx\, ds, \\
&= \int_{-\infty}^{\infty} |f(x)|^2\, dx.
\end{aligned}$$

In summary, we have

$$\int_{-\infty}^{\infty} |F(\lambda)|^2\, d\lambda = \int_{-\infty}^{\infty} |f(x)|^2\, dx.$$

5) The Fourier Transform of the derivative of a function can be simplified by integrating by parts,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x}\frac{df}{dx}(x)\,dx &= \frac{1}{\sqrt{2\pi}}\left[e^{i\lambda x}f(x)\right]_{-\infty}^{\infty} - i\lambda\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x}f(x)\,dx,\\ &= -i\lambda\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x}f(x)\,dx,\end{aligned}$$

if the function $f(x)$ vanishes as $x \to \pm\infty$. Alternatively, this can be written as,

$$\mathcal{F}[\frac{df}{dx}] = -i\lambda\mathcal{F}[f].$$

Using induction, this can be extended to the following more general result,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x}f^{(n)}(x)\,dx = (-i\lambda)^n F(\lambda), \quad \text{for} \quad n = 1, 2, 3, \cdots.$$

### 5.2.3 Applications to ODEs

Consider the ODE,

$$y''(x) - k^2 y(x) = -f(x), \quad -\infty < x < \infty,$$

where we assume that $k$ is constant and $f(x)$ is a given forcing function. Technically, because there are no boundaries we don't have boundary conditions to apply. However, we do require that the solution vanishes as $x \to \pm\infty$, what is often described as *the far field*,

$$y(x), y'(x) \to 0, \quad \text{as} \quad |x| \to \infty.$$

Also, we define $\mathcal{F}[y(x)] = Y(\lambda)$ and $\mathcal{F}[f(x)] = F(\lambda)$.

We take the Fourier Transform of the ODE. This requires using the differentiation formula $\mathcal{F}[y''] = -\lambda^2 Y$ but then we get,

$$\begin{aligned} -(\lambda^2 + k^2)Y(\lambda) &= -F(\lambda), \\ Y(\lambda) &= \frac{F(\lambda)}{\lambda^2 + k^2}, \\ Y(\lambda) &= F(\lambda)G(\lambda), \end{aligned}$$

where $G(\lambda) = 1/(\lambda^2 + k^2)$. If we can invert this function then we can use the convolution theorem to obtain our solution. The inverse is obtained by evaluating the integral directly,

$$\begin{aligned} \mathcal{F}^{-1}[G] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} G(\lambda)\, d\lambda, \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{\lambda^2 + k^2}\, d\lambda, \\ &= \frac{\sqrt{2\pi}}{2k} e^{-k|x|}. \end{aligned}$$

To evaluate the above integral we used a table.
Finally, we can obtain our complete solution using the convolution theorem,

$$y(x) = (f * \frac{\sqrt{2\pi}}{2k} e^{-k|x|})(x) = \frac{1}{2k} \int_{-\infty}^{\infty} e^{-k|x-t|} f(t)\, dt.$$

We consider two special cases. First, $f(x) = 1$, implies that the above becomes,

$$\begin{aligned} y(x) &= \frac{1}{2k} \left[ \int_{-\infty}^{x} e^{-k(x-t)}\, dt + \int_{x}^{\infty} e^{k(x-t)}\, dt \right], \\ &= \frac{1}{2k^2} \left[ \left( e^{-k(x-t)} \right)_{-\infty}^{x} - \left( e^{k(x-t)} \right)_{x}^{\infty} \right], \\ &= \frac{1}{k^2}. \end{aligned}$$

Second, we take $f(x) = \delta(x - \zeta)$,

$$y(x) = \frac{1}{2k}\int_{-\infty}^{\infty} e^{-k|x-t|}\delta(t-\zeta)\,dt = \frac{1}{2k}e^{-k|x-\zeta|}$$

Each of these solutions has their own problems though. The first does not vanish as $x \to \pm\infty$, the far field, as we required but does satisfy the equation. The second satisfies the far field conditions but it is not differentiable at $x = \zeta$ and so it does not strictly satisfy the PDE. This is because we are required to deal with what are called *generalized functions*, something we will deal with later in this Chapter.

### 5.2.4 Applications to PDEs

*Diffusion Equation*

We begin with the diffusion equation,

$$u_t - c^2 u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

for a constant $c$ and initial conditions,

$$u(x,0) = f(x), \quad -\infty < x < \infty.$$

This is an equation of two variables, $x$ is infinite and $t$ is semi-infinite. We will compute the Fourier transform in $x$, which will convert the PDE to an ODE that we can solve easily. An alternative approach is to take the Fourier transform in $x$ and then a Laplace transform in $t$. Those of you that have taken AMATH 351 should be able to do this approach. If you try and have problems please let me know.

We define $\mathcal{F}[u(x,t)] = U(\lambda,t)$ and then get after transforming the equation,

$$\begin{aligned}
\mathcal{F}[u_t] - \mathcal{F}[c^2 u_{xx}] &= \mathcal{F}[0], \\
\frac{\partial U}{\partial t} - c^2(-\lambda^2)U &= 0, \\
\frac{\partial U}{\partial t} + c^2\lambda^2 U &= 0.
\end{aligned}$$

We must also transform the initial conditions,

$$U(\lambda, 0) = F(\lambda),$$

where we define $F(\lambda) = \mathcal{F}[f]$. The solution to the DE and IC is,

$$U(\lambda, t) = F(\lambda) e^{-c^2\lambda^2 t}.$$

This is the solution in transform space but to recover the solution in physical space we must invert this equation,

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}[U], \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} U(\lambda, t)\, d\lambda, \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x - c^2\lambda^2 t} F(\lambda)\, d\lambda, \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\lambda(x-s) - c^2\lambda^2 t} f(s)\, ds\, d\lambda, \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \int_{-\infty}^{\infty} e^{-i\lambda(x-s) - c^2\lambda^2 t}\, d\lambda\, ds. \end{aligned}$$

To derive this result we substituted in the definition for $U(\lambda, t)$, then $F(\lambda)$ and finally exchanged the order of integration.

Next we evaluate the inner integral,

$$\int_{-\infty}^{\infty} e^{-i\lambda(x-s) - c^2\lambda^2 t}\, d\lambda = \int_{-\infty}^{\infty} \left[\cos\lambda(x-s) - i\sin\lambda(x-s)\right] e^{-c^2\lambda^2 t}\, d\lambda.$$

Since $\sin$ is odd and $\cos$ is even we can deduce that the imaginary part of the integral is identically zero and furthermore,

$$\int_{-\infty}^{\infty} e^{-i\lambda(x-s) - c^2\lambda^2 t}\, d\lambda = 2\int_{0}^{\infty} \cos\lambda(x-s) e^{-c^2\lambda^2 t}\, d\lambda.$$

The above integral is of the form,

$$I(\alpha) = 2\int_0^\infty e^{-c^2\lambda^2 t}\cos(\lambda\alpha)\,d\lambda.$$

One way to evaluate this integral is to first differentiate it and then use integration by parts,

$$\begin{aligned}\frac{dI}{d\alpha}(\alpha) &= -2\int_0^\infty \lambda e^{-c^2\lambda^2 t}\sin(\lambda\alpha)\,d\lambda,\\ &= \frac{1}{c^2 t}\left[e^{-c^2\lambda^2 t}\sin(\lambda\alpha)\right]_0^\infty - \frac{\alpha}{c^2 t}\int_0^\infty \lambda e^{-c^2\lambda^2 t}\cos(\lambda\alpha)\,d\lambda,\\ &= -\frac{\alpha}{2c^2 t}I(\alpha).\end{aligned}$$

Also, the initial value is,

$$I(0) = 2\int_0^\infty e^{-c^2\lambda^2 t}\,d\lambda = \sqrt{\frac{\pi}{c^2 t}}.$$

To show this we used the identity $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$ along with a change of variables. Since we know the DE that $I(\alpha)$ satisfies along with the IC we can deduce it's form by solving the DE directly,

$$I(\alpha) = \sqrt{\frac{\pi}{c^2 t}}\exp\left[-\frac{\alpha^2}{4c^2 t}\right].$$

With this established we can determine the solution to our original problem,

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}}\int_{-\infty}^\infty \exp\left[-\frac{(x-s)^2}{4c^2 t}\right] f(s)\,ds.$$

This is the general solution to the heat equation for any initial condition $f(x)$ on the real line.

The function,

$$G(x-\zeta,t)=\frac{1}{\sqrt{4\pi c^2 t}}\exp\left[-\frac{(x-\zeta)^2}{4c^2t}\right],$$

that appears in the integral is known as the *fundamental solution of the diffusion or heat equation*. It is sometimes referred to as the *heat kernel*. Using this definition we can write the general solution as,

$$u(x,t)=\int_{-\infty}^{\infty}G(x-s,t)f(s)\,ds.$$

If we pick $f(s)=\delta(s-\zeta)$ we find that the solution becomes $u(x,t)=G(x-\zeta,t)$. That is to say if the heat is initially located at $\zeta$, $G(x-\zeta,t)$ is the solution at position $x$ and time $t$. From this we gather that the fundamental solution, also related to the Green's function, determine how the point source at position $x=\zeta$ with magnitude $f(\zeta)$ evolves in time and space. In general the solution is a superposition over all of these initial positions.

Note that it is possible to evaluate the integral,

$$I=\int_{-\infty}^{\infty}e^{-x^2}\,dx,$$

by multiplying this by itself and then changing to polar co-ordinates:

$$\begin{aligned}
I^2&=\int_{-\infty}^{\infty}e^{-x^2}\,dx\int_{-\infty}^{\infty}e^{-y^2}\,dy,\\
I^2&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(x^2+y^2)}\,dx\,dy,\\
I^2&=\int_{0}^{\infty}\int_{0}^{2\pi}e^{-r^2}\,r d\theta\,dr,\\
I^2&=2\pi\int_{0}^{\infty}re^{-r^2}\,dr,\\
I^2&=\pi\left[-e^{-r^2}\right]_0^{\infty},\\
I^2&=\pi.
\end{aligned}$$

Therefore, we deduce,

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

*Wave Equation*

The wave equation is,

$$u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

for a constant $c$ and two initial conditions,

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad -\infty < x < \infty.$$

We take the Fourier transform with respect to $x$ and get,

$$\begin{aligned} \mathcal{F}[u_{tt}] - \mathcal{F}[c^2 u_{xx}] &= \mathcal{F}[0], \\ \frac{\partial^2 U}{\partial t^2} + c^2 \lambda^2 U &= 0. \end{aligned}$$

We must also transform the initial conditions,

$$U(\lambda, 0) = F(\lambda), \quad \text{and} \quad U_t(\lambda, 0) = G(\lambda).$$

The solution to the DE and ICs is,

$$U(\lambda, t) = \alpha(\lambda) e^{ic\lambda t} + \beta(\lambda) e^{-ic\lambda t}.$$

Note that we do not necessarily want to use trigonometric functions because in the solution in the transform space is complex and so in general the solution will be complex.

To satisfy the ICs we need,

$$\begin{aligned} \alpha(\lambda) + \beta(\lambda) &= F(\lambda), \\ ic\lambda\alpha(\lambda) - ic\lambda\beta(\lambda) &= G(\lambda). \end{aligned}$$

This is a two by two system that is easily solved to yield a general solution of,

$$U(\lambda,t)=\left[\frac{1}{2}F(\lambda)+\frac{1}{2i\lambda c}G(\lambda)\right]e^{ic\lambda t}+\left[\frac{1}{2}F(\lambda)-\frac{1}{2i\lambda c}G(\lambda)\right]e^{-ic\lambda t}.$$

The last step is to invert this to discover the solution in the physical variables $(x,t)$,

$$\begin{aligned}u(x,t)&=\mathcal{F}^{-1}[\left[F(\lambda)+\frac{1}{2i\lambda c}G(\lambda)\right]e^{ic\lambda t}+\left[F(\lambda)-\frac{1}{2i\lambda c}G(\lambda)\right]e^{-ic\lambda t}],\\&=\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\lambda(x-ct)}F(\lambda)\,d\lambda+\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\lambda(x+ct)}F(\lambda)\,d\lambda\\&+\frac{1}{2c\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\lambda(x-ct)}\frac{G(\lambda)}{i\lambda}\,d\lambda-\frac{1}{2c\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\lambda(x+ct)}\frac{G(\lambda)}{i\lambda}\,d\lambda.\end{aligned}$$

If we express $g(x)$ in terms of it's Fourier transform,

$$g(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\lambda x}G(\lambda)\,d\lambda,$$

then it's definite integral is

$$\int^{x}g(s)\,ds=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\lambda x}\frac{G(\lambda)}{i\lambda}\,d\lambda.$$

From this we can invert the transforms above and obtain,

$$u(x,t)=\frac{1}{2}\left[f(x-ct)+f(x+ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)\,ds.$$

This is of course d'Alembert's solution.

*Laplace's Equation*

Laplace's equation on the half-plane is,

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, y > 0.$$

The boundary conditions that we impose are,

$$\begin{aligned} &u(x,0) = f(x), \\ &u(x,y) \to 0 \quad \text{as} \quad |x| \to \infty, \\ &u(x,y) \text{ bounded as } y \to \infty. \end{aligned}$$

We define the Fourier Transform as,

$$U(\lambda, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} u(x,y)\, dx,$$

and then transform the equation with respect to $x$,

$$\begin{aligned} \mathcal{F}[u_{xx}] + \mathcal{F}[u_{yy}] &= \mathcal{F}[0], \\ \frac{\partial^2 U}{\partial y^2} - \lambda^2 U &= 0. \end{aligned}$$

We must also transform the boundary conditions,

$$\begin{aligned} &U(\lambda, 0) = F(\lambda), \\ &U(\lambda, y) \quad \text{bounded as} \quad y \to \infty. \end{aligned}$$

Therefore, the solution to the DE and ICs is,

$$U(\lambda, y) = \alpha(\lambda) e^{-|\lambda| y} + \beta(\lambda) e^{|\lambda| y}.$$

For this to be bounded as $y \to \infty$ we need that $\beta(\lambda) = 0$. Next, we evaluate this at $y = 0$ and get

$$U(\lambda, 0) = \alpha(\lambda) = F(\lambda).$$

Therefore, the solution in transform space is,

$$U(\lambda, y) = F(\lambda)e^{-|\lambda|y}.$$

Finally, we must invert the transformed solution into the physical variables $(x, y)$,

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left[F(\lambda)e^{-|\lambda|y}\right], \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda x-|\lambda|y}F(\lambda)\,d\lambda, \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\lambda(x-s)-|\lambda|y}f(s)\,ds\,d\lambda, \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\int_{-\infty}^{\infty} e^{-i\lambda(x-s)-|\lambda|y}\,d\lambda\,ds. \end{aligned}$$

The integral that we need to evaluate is equal to the following,

$$\int_{-\infty}^{\infty} e^{-i\lambda(x-s)-|\lambda|y}\,d\lambda = \frac{2y}{(x-s)^2+y^2}.$$

Therefore, the complete solution is,

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{(x-s)^2+y^2}\,ds.$$

It is easy to verify that this function does satisfy Laplace's equation. Verifying the BC at $y = 0$ is harder to show. At first glance you might guess that $u(x, 0) = 0$ because of the $y$ in front of the integral. However, the integral is singular. It turns out that it does satisfy the BC but it is difficult to show in general.

Instead we consider the special case where $f(x) = H(x)$. In this case the integral becomes,

$$\begin{aligned} u(x,y) &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{H(s)}{(x-s)^2+y^2}\,ds, \\ &= \frac{1}{\pi}\int_0^{\infty} \frac{1}{(\frac{s-x}{y})^2+1}\,ds, \quad \text{sub} \quad u = \frac{s-x}{y} \\ &= \frac{1}{\pi}\int_0^{\infty} \frac{1}{u^2+1}du, \\ &= \frac{1}{\pi}\arctan\left(\frac{s-x}{y}\right)\Big|_0^{\infty}, \\ &= \left[\frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{x}{y}\right)\right]. \end{aligned}$$

When we evaluate this at $x > 0$ as $y$ tends to zero we notice that $x/y$ tends to $+\infty$ and $\arctan(x/y)$ tends to $\pi/2$ and therefore,

$$\lim_{y\to 0} u(x,y) = 1, \quad \text{for} \quad x > 0.$$

Similarly, for $x < 0$ we have that $x/y$ tends to $-\infty$ and so the arctan approaches $-\pi/2$, and so,

$$\lim_{y\to 0} u(x,y) = 0, \quad \text{for} \quad x < 0.$$

Both of these are consistent with the boundary conditions. At the origin we notice that for any $y > 0$ if $x = 0$ we have that,

$$u(0,y) = \frac{1}{2},$$

which is the average of the value on the left and the right. Therefore, this example shows that indeed the solution to Laplace's equation on the half plane does yield the correct boundary conditions, at least in this particular example. It is true in general but harder to show.

CHAPTER
# SIX

# Basic Wave Theory featuring KdV

## 6.1 INTRODUCTION

Recall that if a function $f(x)$ is defined on a finite interval of length $\ell$, its periodic extension has a Fourier series which can be written as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right]$$

where the Fourier coefficients are computed with the necessary formulas. This equation projects the function $f(x)$ onto the complete basis of trigonometric functions. This is analogous to linear algebra where you decompose a vector in terms of the components of a basis but now the basis consists of functions instead of vectors.

If instead the function is defined on an infinite region it is necessary to consider the Fourier transform instead of the Fourier series. The Fourier transform of $f(x)$ is

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} f(x)dx$$

where the inversion formula is

$$f(x) = \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) dk$$

In this equation $f(x)$ is written as a sum (or integal) of the basis $e^{ikx}$, which is both orthogonal and complete. With the use of Euler's formula, $\exp i\theta = \cos\theta + i\sin\theta$, we see that the Fourier Series also has a similar basis $e^{i(n\pi/\ell)x}$. Both the Fourier Series and the Transform are summing over all of the possible modes where the "amount" of the $k$-th or $n$-th mode is given by the coefficient. In the case of a finite domain the modes are discrete since we have a sum over all the interger values of $n$. In contrast, for the case of an infinite domain, $k$ is any real number which is why we say that the modes are continuous. See "Spectral methods in Matlab" by L.N. Trefethen for more details.

## 6.2 LINEAR DISSIPATION

Suppose we have a PDE that is defined on an infinite domain. Even though all physical domains are necessarily bounded, an infinite domain is a good approximation if we consider time scales that are small enough so that the boundaries do not effect the dynamics. Consider the linearized version of the Burger's equation (with viscosity).

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}.$$

If $u$ is the velocity then $\partial u/\partial t$ is the acceleration which changes due to advection (or transport or even convection as its sometimes called) $c\frac{\partial u}{\partial x}$ and diffusion $\nu\frac{\partial^2 u}{\partial x^2}$.

As you saw in assignment 4, we can find traveling wave solutions of the form $u(x,t) = f(x-Ut)$ where $U$ is the speed of the traveling wave. If $f$ has a Fourier Transform then the traveling wave can be written as

$$f(x-Ut) = \int_{-\infty}^{\infty} e^{ik(x-Ut)} \hat{f}(k) dk$$

Since the PDE we are solving is linear we can consider each mode separately. If you wanted to solve an initial value problem we would need to determine the coefficients $\hat{f}(k)$ of each mode and the solution would be the superposition of all the modes. However, that is not our intent. We will instead look for plane wave (or free wave) solutions that solve this PDE, and by that we mean solutions with no constraints other than the PDE itself. Our trial solution is of the form

$$u(x,t) = f(x-Ut) = e^{ik(x-Ut)}$$

We substitute this into the PDE and obtain, after differentiation, the following equation

$$-ikUf(\xi) + cikf(\xi) = \nu(ik)^2 f(\xi)$$

where $\xi = x - Ut$. Since the exponential is not zero we can divide by $-ikf(\xi)$ and obtain what is refered to as a dispersion relation

$$U = c - i\nu k$$

This indicates that the wave speed has a real component, $c$, but it also has an imaginary component. To better understand what this means let us substitute this back into our solution.

$$u(x,t) = f(x - Ut) = e^{ik(x-(c-i\nu k)t)} = e^{ik(x-ct)}e^{-\nu k^2 t}$$

The advection term in the equation causes the wave to travel in the positive $x$ direction at a speed of $c$. This is clear since for $x - ct$ to remain constant as time passes, $x$ must increase. The effect of the diffusion term on each mode, or what I will now call a wave, is not to transport the solution but instead to cause exponential decay (this is assuming that $\nu$ is positive). This process is known physically as dissipation since it dissipates energy from the system. If we define $\lambda$ to be the wavelength of each wave then, from the expression above, it is readily obtained that the wavelength is $\lambda = \frac{2\pi}{k}$. The parameter $k$ is referred to as the wavenumber. The rate at which each wave is diffused is proportional to $\nu k^2$ which signifies that small waves are dissipated faster than large ones.

The frequency of the wave is a product of wave speed and the wavenumber, $\omega = kU$. In general the dispersion relation relates the wave speed or frequency with the wavenumber. It could equivalently be written as,

$$\omega = kc - i\nu k^2$$

We summarize the parameters in the problem for the Burger's equation.

$c$ : wave speed
$\omega$ : wave frequency
$\lambda$ : wavelength
$k$ : wavenumber $k = \frac{2\pi}{\lambda}$
$\nu$ : friction parameter

## 6.3 LINEAR DISPERSION

Let us consider a slight modification of the Burger's equation where the right hand side is not the second partial derivative of $u$ with respect to $x$ but instead the third derivative. Even though this is only a slight mathematical variation, it will lead to a very different physical phenomenon.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \beta\frac{\partial^3 u}{\partial x^3}.$$

We look for a wave solution of the same form as before.

$$u(x,t) = f(x - Ut) = e^{ik(x-Ut)}$$

When we substitute this into the PDE and obtain, after differentiation, the following equation

$$-ikUf(\xi) + cikf(\xi) = \beta(-ik^3)f(\xi)$$

Again, since the exponential is not zero we can divide by $-ikf(\xi)$ and obtain the dispersion relation for this problem

$$U = c + \beta k^2$$

Since the wave speed is real we will not get any exponential growth or decay. Instead, what we find is that the wave speed is dependent on the wavenumber. In particular, the longer waves have a speed near $c$ whereas the shorter waves have speeds that are higher than $c$. This means that if we begin with a collection of waves at the same location each wave propagates at a different speed. Therefore, the collection of waves will separate, or disperse, where the shortest ones are at the front and the largest ones are at the rear. The previous example had no dispersion. The parameter $\beta$ determines at what rate the waves disperse since the larger it is the faster the shorter waves move.

Think of the scenario when you drop a pebble into a pond. Since water waves are dispersive, if you follow the ripples you will notice that there is a separation of waves of different lengths.

## 6.4 NONLINEAR DISPERSION

A nonlinear version of the linear dispersion PDE is the Korteweg-deVries Equation, commonly referred to as the KdV equation. It is

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

We saw in the treatment of shocks that nonlinear advection can lead to the formation of shocks since it can steepen waves. Dispersion on the other hand causes waves to spread out. These two forces act in opposition to one another. The KdV equation is an interesting example where there can be a balance between these two forces which can create a very special type of isolated (or solitary) wave that is referred to as a soliton.

Since the equation is nonlinear we cannot simply consider each wave by itself since the nonlineariety creates interactions between all the waves. Instead we look for a traveling wave solution of the form

$$u(x,t) = f(x - Ut)$$

We will focus our attention on waves that vanishes both up and down stream.

We substitute this into the KdV equation and obtain

$$-Uf' + 6ff' + f''' = 0$$

Since $ff' = 1/2(f^2)'$, we can integrate this equation once and obtain

$$-Uf + 3f^2 + f'' = A$$

If we impose the condition that $f$, $f'$ and $f''$ vanish as $x \to \pm\infty$ then $A = 0$. These conditions are called far field conditions and follow from the fact that we are looking for isolated waves. Next, we multiply by $f'$, which serves as an integrating factor,

$$-Uff' + 3f^2 f' + f'f'' = 0$$

We can integrate the equation once more to obtain

$$-\frac{1}{2}Uf^2 + f^3 + \frac{1}{2}f'^2 = B$$

If we again apply the far field conditions we deduce that $B = 0$. Therefore, the equation can be rewritten as

$$f'^2 = Uf^2 - 2f^3 \text{ or } f' = f\sqrt{U-2f}$$

where for a solution to exist we require $\sqrt{U-2f} \geq 0$. Since the equation is separable we rewrite it as

$$\frac{df}{f\sqrt{U-2f}} = \pm d\xi$$

which we will integrate. To do so we make the substitution of

$$f = \frac{U}{2}\text{sech}^2\theta \quad \text{which implies} \quad df = U\text{sech}^2\theta\tanh\theta d\theta$$

We substitute these identities into the integral of the third equation above and recall that since $\sinh^2\theta = \cosh^2\theta - 1$ we deduce that $\sqrt{U-2f} = \sqrt{U}\tanh\theta$.

$$\frac{df}{f\sqrt{U-2f}} = \frac{U\text{sech}^2\theta\tanh\theta d\theta}{\frac{U}{2}\text{sech}^2\theta\sqrt{U}\tanh\theta} = \frac{d\theta}{\sqrt{U}}$$

Therefore, the integral yields

$$\theta = \sqrt{U}(\pm\xi + x_0)$$

If we substitute this result into our formula for $f$ we get

$$f(x-Ut) = \frac{U}{2}\text{sech}^2\theta = \frac{U}{2}\text{sech}^2\left(\sqrt{c}(\xi+x_0)\right) = \frac{U}{2}\text{sech}^2\left(\sqrt{U}(x-Ut+x_0)\right)$$

where $x_0$ is an arbitrary constant of integration. We have dropped the $\pm$ since $\text{sech}$ is an even function so it does not make any difference in the solution. Observe that this solution is valid for any positive wave speed $U$. Also, large values of $U$ correspond to fast waves that are tall but narrow since there is a factor of $\sqrt{U}$ within the $\text{sech}^2$.

These solutions to the nonlinear KdV equation are called solitons and are a type of solitary wave. What is rather interesting about them is that they interact with each other in a linear

fashion. If you consider two solitons that interact you will find that the shape of each wave is precisely the same before and after interaction. The only difference is that there is a slight phase difference after interaction. These nonlinear waves have appeared in many fields of research such as water waves, optics, electromagnetism, general relativity and quantum field theory, to name but a few.

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