MATH 148: Analysis 2

Instructor: Stephen New

Winter 2021

Table of Contents

MATH 148 Calculus 2 (Advanced Level), Course Outline, Winter 2021

Lectures: This is a special online offering of the course and there are no in-class lectures. Course materials will be available on LEARN and at www.math.uwaterloo.ca\~snew

Instructor: The course instructor is Stephen New. He can be contacted by email at snew@uwaterloo.ca

Text: There is no required textbook. Lecture notes will be provided.

Course Outline: We will cover all of the material in the lecture notes:

Chapter 1. The Riemann Integral
Chapter 2. Methods of Integration
Chapter 3. Applications of the Definite Integral
Chapter 4. Parametric and Polar Curves
Chapter 5. Differential Equations
Chapter 6. Sequences and Series
Chapter 7. Sequences and Series of Functions

Assignments: there will be 5 assignments. You must complete all 5 assignments and each will count for 20% of your final grade. The assignments must be submitted using Crowdmark. Each assignment must be submitted before 5:00 pm on the due date (it is recommended that you submit them well in advance in case technical difficulties arise). You should try to complete Assignment 2 by Tues June 16 (for sensible budgeting of time) but it is due on June 23 because June 16 falls during Reading Week. Here is the schedule.

Assignment 1 covers Chapter 1 and is due on Tues Feb 2,
Assignment 2 covers Chapter 2 and is due on Tues June 23,
Assignment 3 covers Chapters 3 and 4 and is due on Tues Mar 9,
Assignment 4 covers Chapters 5 and 6 and is due on Tues Mar 30, and
Assignment 5 covers Chapter 7 and is due on Tues Apr 13.

Tests: There will be no midterm test and no final examination.

Course Mark: The final course grade will be entirely based on your assignment marks, and all 5 assignments will be given equal weight.

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For typical penalties check Guidelines for the Assessment of Penalties, http://www.adm.uwaterloo.ca/infosec/guidelines/penaltyguidelines.htm.

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Chapter 1. The Riemann Integral

The Riemann Integral

1.1 Definition: A **partition** of the closed interval $[a,b]$ is a set $X = \{x_0, x_1, \cdots, x_n\}$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b\,.$$

The intervals $[x_{k-1}, x_k]$ are called the **subintervals** of $[a,b]$, and we write

$$\Delta_k x = x_k - x_{k-1}$$

for the size of the k^{th} subinterval. Note that

$$\sum_{k=1}^{n} \Delta_k x = b - a\,.$$

The **size** of the partition X, denoted by $|X|$ is

$$|X| = \max\left\{\Delta_k x \middle| 1 \le k \le n\right\}.$$

1.2 Definition: Let X be a partition of $[a,b]$, and let $f : [a,b] \to \mathbf{R}$ be bounded. A **Riemann sum** for f on X is a sum of the form

$$S = \sum_{k=1}^{n} f(t_k)\Delta_k x \quad \text{ for some } t_k \in [x_{k-1}, x_k]\,.$$

The points t_k are called **sample points**.

1.3 Definition: Let $f : [a,b] \to \mathbf{R}$ be bounded. We say that f is **(Riemann) integrable** on $[a,b]$ when there exists a number I with the property that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every partition X of $[a,b]$ with $|X| < \delta$ we have $|S - I| < \epsilon$ for every Riemann sum for f on X, that is

$$\left| \sum_{k=1}^{n} f(t_k)\Delta_k x - I \right| < \epsilon\,.$$

for every choice of $t_k \in [x_{k-1}, x_k]$ This number I is unique (as we prove below); it is called the **(Riemann) integral** of f on $[a,b]$, and we write

$$I = \int_a^b f \text{ , or } I = \int_a^b f(x)\,dx\,.$$

Proof: Suppose that I and J are two such numbers. Let $\epsilon > 0$ be arbitrary. Choose δ_1 so that for every partition X with $|X| < \delta_1$ we have $|S - I| < \frac{\epsilon}{2}$ for every Riemann sum S on X, and choose $\delta_2 > 0$ so that for every partition X with $|X| < \delta_2$ we have $|S - J| < \frac{\epsilon}{2}$ for every Riemann sum S on X. Let $\delta = \min\{\delta_1, \delta_2\}$. Let X be any partition of $[a,b]$ with $|X| < \delta$. Choose $t_k \in [x_{k-1}, x_k]$ and let $S = \sum\limits_{k=1}^{n} f(t_k)\Delta_k x$. Then we have $|I - J| \le |I - S| + |S - J| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since ϵ was arbitrary, we must have $I = J$.

1.4 Example: Let $f(x)=\begin{cases}1 & \text{if} \quad x\in\mathbf{Q}\\ 0 & \text{if} \quad x\notin\mathbf{Q}\,.\end{cases}$ Show that f is not integrable on $[0,1]$.

Solution: Suppose, for a contradiction, that f is integrable on $[0,1]$, and write $I=\int_0^1 f$. Let $\epsilon=\frac{1}{2}$. Choose δ so that for every partition X with $|X|<\delta$ we have $|S-I|<\frac{1}{2}$ for every Riemann sum S for f on X. Choose a partition X with $|X|<\delta$. Let $S_1=\sum\limits_{k=1}^n f(t_k)\Delta_i kx$ where each $t_k\in[x_{k-1},x_k]$ is chosen with $t_k\in\mathbf{Q}$, and let $S_2=\sum\limits_{k=1}^n f(s_k)\Delta_k x$ where each $s_k\in[x_{k-1},x_k]$ is chosen with $s_k\notin\mathbf{Q}$. Note that we have $|S_1-I|<\frac{1}{2}$ and $|S_2-I|<\frac{1}{2}$. Since each $t_k\in\mathbf{Q}$ we have $f(t_k)=1$ and so $S_1=\sum\limits_{k=1}^n f(t_k)\Delta_k x=\sum\limits_{k=1}^n \Delta_k x=1-0=1$, and since each $s_k\notin\mathbf{Q}$ we have $f(s_k)=0$ and so $S_2=\sum\limits_{k=1}^n f(s_k)\Delta_k x=0$. Since $|S_1-I|<\frac{1}{2}$ we have $|1-I|<\frac{1}{2}$ and so $\frac{1}{2}<I<\frac{3}{2}$, and since $|S_2-I|<\frac{1}{2}$ we have $|0-I|<\frac{1}{2}$ and so $-\frac{1}{2}<I<\frac{1}{2}$, giving a contradiction.

1.5 Example: Show that the constant function $f(x)=c$ is integrable on any interval $[a,b]$ and we have $\displaystyle\int_a^b c\,dx=c(b-a)$.

Solution: The solution is left as an exercise.

1.6 Example: Show that the identity function $f(x)=x$ is integrable on any interval $[a,b]$, and we have $\displaystyle\int_a^b x\,dx=\tfrac{1}{2}(b^2-a^2)$.

Solution: Let $\epsilon>0$. Choose $\delta=\frac{2\epsilon}{b-a}$. Let X be any partition of $[a,b]$ with $|X|<\delta$. Let $t_k\in[x_{k-1},x_k]$ and set $S=\sum\limits_{k=1}^n f(t_k)\Delta_k x=\sum\limits_{k=1}^n t_i\Delta_k x$. We must show that $|S-\frac{1}{2}(b^2-a^2)|<\epsilon$. Notice that

$$\begin{aligned}\sum_{k=1}^n (x_k+x_{k-1})\Delta_k x&=\sum_{k=1}^n (x_k+x_{k-1})(x_k-x_{k-1})=\sum_{k=1}^n {x_k}^2-{x_{k-1}}^2\\ &=({x_1}^2-{x_0}^2)+({x_2}^2-{x_1}^2)+\cdots+({x_{n-1}}^2-{x_{n-2}}^2)+({x_n}^2-{x_{n-1}}^2)\\ &=-{x_0}^2+({x_1}^2-{x_1}^2)+\cdots+({x_{n-1}}^2-{x_{n-1}}^2)+{x_n}^2\\ &={x_n}^2-{x_0}^2=b^2-a^2\end{aligned}$$

and that when $t_k\in[x_{k-1},x_k]$ we have $\left|t_k-\frac{1}{2}(x_k+x_{k-1})\right|\le\frac{1}{2}(x_k-x_{k-1})=\frac{1}{2}\Delta_k x$, and so

$$\begin{aligned}\left|S-\tfrac{1}{2}(b^2-a^2)\right|&=\left|\sum_{k=1}^n t_k\Delta_k x-\tfrac{1}{2}\sum_{k=1}^n (x_k+x_{k-1})\Delta_k x\right|\\ &=\left|\sum_{k=1}^n \left(t_k-\tfrac{1}{2}(x_k+x_{k-1})\right)\Delta_k x\right|\\ &\le\sum_{k=1}^n \left|t_k-\tfrac{1}{2}(x_k+x_{k-1})\right|\Delta_k x\\ &\le\sum_{k=1}^n \tfrac{1}{2}\Delta_k x\,\Delta_k x\le\sum_{k=1}^n \tfrac{1}{2}\delta\,\Delta_k x\\ &=\tfrac{1}{2}\delta(b-a)=\epsilon\,.\end{aligned}$$

Upper and Lower Riemann Sums

1.7 Definition: Let X be a partition for $[a,b]$ and let $f:[a,b]\to\mathbf{R}$ be bounded. The **upper Riemann sum** for f on X, denoted by $U(f,X)$, is

$$U(f,X)=\sum_{k=1}^{n} M_k\,\Delta_k x \quad \text{where } M_k=\sup\left\{f(t)\middle|t\in[x_{k-1},x_k]\right\}$$

and the **lower Riemann sum** for f on X, denoted by $L(f,X)$ is

$$L(f,X)=\sum_{k=1}^{n} m_k\,\Delta_k x \quad \text{where } m_k=\inf\left\{f(t)\middle|t\in[x_{k-1},x_k]\right\}.$$

1.8 Remark: The upper and lower Riemann sums $U(f,X)$ and $L(f,X)$ are not, in general, Riemann sums at all, since we do not always have $M_k=f(t_k)$ or $m_k=f(s_k)$ for any $t_k,s_k\in[x_{k-1},x_k]$. If f is increasing, then $M_k=f(x_k)$ and $m_k=f(x_{k-1})$, and so in this case $U(f,X)$ and $L(f,X)$ are indeed Riemann sums. Similarly, if f is decreasing then $U(f,X)$ and $L(f,X)$ are Riemann sums. Also, if f is continuous then, by the Extreme Value Theorem, we have $M_k=f(t_k)$ and $m_k=f(s_k)$ for some $t_k,s_k\in[x_{k-1},x_k]$, and so in this case $U(f,X)$ and $L(f,X)$ are again Riemann sums.

1.9 Note: Let X be a partition of $[a,b]$, and let $f:[a,b]\to\mathbf{R}$. be bounded. Then

$$U(f,X)=\sup\left\{S\middle|S \text{ is a Riemann sum for } f \text{ on } X\right\}\text{ , and}$$
$$L(f,X)=\inf\left\{S\middle|S \text{ is a Riemann sum for } f \text{ on } X\right\}.$$

In particular, for every Riemann sum S for f on X we have

$$L(f,X)\le S\le U(f,X)$$

Proof: We show that $U(f,X)=\sup\left\{S\middle|S \text{ is a Riemann sum for } f \text{ on } X\right\}$ (the other statement is proved similarly). Let $\mathcal{T}=\left\{S\middle|S \text{ is a Riemann sum for } f \text{ on } X\right\}$. For $S\in\mathcal{T}$, say $S=\sum\limits_{k=1}^{n} f(t_k)\Delta_k x$ where $t_k\in[x_{k-1},x_k]$, we have

$$S=\sum_{k=1}^{n} f(t_k)\Delta_k x\le\sum_{k=1}^{n} M_k\Delta_k x=U(f,X).$$

Thus $U(f,X)$ is an upper bound for $\mathcal{T}$ so we have $U(f,X)\ge\sup\mathcal{T}$. It remains to show that given any $\epsilon>0$ we can find $S\in\mathcal{T}$ with $U(f,X)-S<\epsilon$. Let $\epsilon>0$ be arbitrary. Since $M_k=\sup\left\{f(t)\middle|t\in[x_{k-1},x_k]\right\}$, we can choose $t_k\in[x_{k-1},x_k]$ with $M_k-f(t_k)<\frac{\epsilon}{b-a}$. Then we have

$$U(f,X)-S=\sum_{k=1}^{n} M_k\Delta_k x-\sum_{k=1}^{n} f(t_k)\Delta_k x=\sum_{k=1}^{n}\big(M_k-f(t_k)\big)\Delta_k x<\sum_{k=1}^{n}\tfrac{\epsilon}{b-a}\,\Delta_k x=\epsilon$$

1.10 Lemma: *Let $f : [a,b] \to \mathbf{R}$ be bounded with upper and lower bounds M and m. Let X and Y be partitions of $[a,b]$ such that $Y = X \cup \{c\}$ for some $c \notin X$. Then*

$$0 \leq L(f,Y) - L(f,X) \leq (M-m)|X| \text{ , and}$$
$$0 \leq U(f,X) - U(f,Y) \leq (M-m)|X| \,.$$

Proof: We shall prove that $0 \leq L(f,Y) - L(f,X) \leq (M-m)|X|$ (the proof that $0 \leq U(f,X) - U(f,Y) \leq (M-m)|X|$ is similar). Say $X = \{x_0, x_1, \cdots, x_n\}$ and $c \in [x_{k-1}, x_k]$ so $Y = \{x_0, x_1, \cdots, x_{k-1}, c, x_k, \cdots, x_n\}$. Then

$$L(f,Y) - L(f,X) = r\,(c - x_{k-1}) + s\,(x_k - c) - m_k(x_k - x_{k-1})$$

where

$$r = \inf\left\{f(t)\big|t \in [x_{k-1}, c]\right\},\ s = \inf\left\{f(t)\big|t \in [c, x_k]\right\}\ ,\ m_k = \inf\left\{f(t)\big|t \in [x_{k-1}, x_k]\right\}.$$

Since $m_k = \min\{r, s\}$ we have $r \geq m_k$ and $s \geq m_k$, so

$$L(f,Y) - L(f,X) \geq m_k(c - x_{k-1}) + m_k(x_k - c) - m_k(x_k - x_{k-1}) = 0\,.$$

Since $r \leq M$ and $s \leq M$ and $m_k \geq m$ we have

$$\begin{aligned} L(f,Y) - L(f,X) &\leq M(c - x_{k-1}) + M(x_k - c) - m(x_k - x_{k-1}) \\ &= (M-m)(x_k - x_{k-1}) \leq (M-m)|X|\,. \end{aligned}$$

1.11 Note: Let X and Y be partitions of $[a,b]$ with $X \subseteq Y$. Then

$$L(f,X) \leq L(f,Y) \leq U(f,Y) \leq U(f,X)\,.$$

Proof: If Y is obtained by adding one point to X then this follows from the above lemma. In general, Y can be obtained by adding finitely many points to X, one point at a time.

1.12 Note: Let X and Y be any partitions of $[a,b]$. Then $L(f,X) \leq U(f,Y)$.

Proof: Let $Z = X \cup Y$. Then by the above note,

$$L(f,X) \leq L(f,Z) \leq U(f,Z) \leq U(f,Y)\,.$$

1.13 Definition: Let $f : [a,b] \to \mathbf{R}$ be bounded. The **upper integral** of f on $[a,b]$, denoted by $U(f)$, is given by

$$U(f) = \inf\left\{U(f,X)\big|X \text{ is a partition of } [a,b]\right\}$$

and the **lower integral** of f on $[a,b]$, denoted by $L(f)$, is given by

$$L(f) = \sup\left\{L(f,X)\big|X \text{ is a partition of } [a,b]\right\}.$$

1.14 Note: The upper and lower integrals of f both exist even when f is not integrable.

1.15 Note: We always have $L(f) \leq U(f)$.

Proof: Let $\epsilon > 0$ be arbitrary. Choose a partition X_1 so that $L(f) - L(f,X_1) < \frac{\epsilon}{2}$ and choose a partition X_2 so that $U(f,X_2) - U(f) < \frac{\epsilon}{2}$. Then

$$\begin{aligned} U(f) - L(f) &= \big(U(f) - U(f,X_2)\big) + \big(U(f,X_2) - L(f,X_1)\big) + \big(L(f,X_1) - L(f)\big) \\ &> -\tfrac{\epsilon}{2} + 0 - \tfrac{\epsilon}{2} = -\epsilon\,. \end{aligned}$$

Since ϵ was arbitrary, this implies that $U(f) - L(f) \geq 0$.

1.16 Theorem: *(Equivalent Definitions of Integrability) Let $f : [a, b] \to \mathbf{R}$ be bounded. Then the following are equivalent.*
(1) $L(f) = U(f)$.
(2) For all $\epsilon > 0$ there exists a partition X such that $U(f, X) - L(f, X) < \epsilon$.
(3) f is integrable on $[a, b]$.

Proof: (1) $\implies$ (2). Suppose that $L(f) = U(f)$. Let $\epsilon > 0$. Choose a partition X_1 so that $L(f) - L(f, X_1) < \frac{\epsilon}{2}$ and choose a partition X_2 so that $U(f, X_2) - U(f) < \frac{\epsilon}{2}$. Let $X = X_1 \cup X_2$. Then $L(f, X_1) \le L(f, X) \le L(f)$ so $L(f) - L(f, X) \le L(f) - L(f, X_1) < \frac{\epsilon}{2}$, and $U(f) \le U(f, X) \le U(f, X_2)$ so $U(f, X) - U(f) < \frac{\epsilon}{2}$. Thus

$$\begin{aligned} U(f, X) - L(f, X) &= \big(U(f, X) - U(f)\big) + \big(U(f) - L(f)\big) + \big(L(f) - L(f, X)\big) \\ &< \tfrac{\epsilon}{2} + 0 + \tfrac{\epsilon}{2} = \epsilon\,. \end{aligned}$$

(2) $\implies$ (1). Suppose that for all $\epsilon > 0$ there is a partition X such that $U(f, X) - L(f, X) < \epsilon$. Let $\epsilon > 0$. Choose X so that $U(f, X) - L(f, X) < \epsilon$. Then

$$\begin{aligned} U(f) - L(f) &= \big(U(f) - U(f, X)\big) + \big(U(f, X) - L(f, X)\big) + \big(L(f, X) - L(f)\big) \\ &< 0 + \epsilon + 0 = \epsilon\,. \end{aligned}$$

Since $0 \le U(f) - L(f) < \epsilon$ for every $\epsilon > 0$, we have $U(f) = L(f)$.

(3) $\implies$ (2). Suppose that f is integrable on $[a, b]$ with $I = \int_a^b f$. Let $\epsilon > 0$. Choose $\delta > 0$ so that for every partition X with $|X| < \delta$ we have $|S - I| < \frac{\epsilon}{4}$ for every Riemann sum S on X. Let X be a partition with $|X| < \delta$. Let S_1 be a Riemann sum for f on X with $|U(f, X) - S_1| < \frac{\epsilon}{4}$, and let S_2 be a Riemann sum for f on X with $|S_2 - L(f, X)| < \frac{\epsilon}{4}$. Then

$$\begin{aligned} |U(f, X) - L(f, X)| &\le |U(f, X) - S_1| + |S_1 - I| + |I - S_2| + |S_2 - L(f, X)| \\ &< \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} = \epsilon\,. \end{aligned}$$

(1) $\implies$ (3). Suppose that $L(f) = U(f)$ and let $I = L(f) = U(f)$. Let $\epsilon > 0$. Choose a partition X_0 of $[a, b]$ so that $L(f) - L(f, X_0) < \frac{\epsilon}{2}$ and $U(f, X_0) - U(f) < \frac{\epsilon}{2}$. Say $X_0 = \{x_0, x_1, \cdots, x_n\}$ and set $\delta = \frac{\epsilon}{2(n-1)(M-m)}$, where M and m are upper and lower bounds for f on $[a, b]$. Let X be any partition of $[a, b]$ with $|X| < \delta$. Let $Y = X_0 \cup X$. Note that Y is obtained from X by adding at most $n - 1$ points, and each time we add a point, the size of the new partition is at most $|X| < \delta$. By lemma 1.10, applied $n - 1$ times, we have

$$\begin{aligned} &0 \le U(f, X) - U(f, Y) \le (n-1)(M-m)|X| < (n-1)(M-m)\delta = \tfrac{\epsilon}{2}\ , \text{ and} \\ &0 \le L(f, Y) - L(f, X) \le (n-1)(M-m)|X| < (n-1)(M-m)\delta = \tfrac{\epsilon}{2}\,. \end{aligned}$$

Now let S be any Riemann sum for f on X. Note that $L(f, X_0) \le L(f, Y) \le L(f) = U(f) \le U(f, Y) \le U(f, X_0)$ and $L(f, X) \le S \le U(f, X)$, so we have

$$\begin{aligned} S - I \le U(f, X) - I = U(f, X) - U(f) &= \big(U(f, X) - U(f, Y)\big) + \big(U(f, Y) - U(f)\big) \\ &\le \big(U(f, X) - U(f, Y)\big) + \big(U(f, X_0) - U(f)\big) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \end{aligned}$$

and

$$\begin{aligned} I - S = I - L(f, X) = L(f) - L(f, X) &= \big(L(f) - L(f, Y)\big) + \big(L(f, Y) - L(f, X)\big) \\ &\le \big(L(f) - L(f, X_0)\big) + \big(L(f, Y) - L(f, X)\big) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,. \end{aligned}$$

Evaluating Integrals of Continuous Functions

1.17 Theorem: *(Continuous Functions are Integrable) Let $f:[a,b]\to\mathbf{R}$ be continuous. Then f is integrable on $[a,b]$.*

Proof: Let $\epsilon>0$. Since f is uniformly continuous on $[a,b]$, we can choose $\delta>0$ such that for all $x,y\in[a,b]$ we have $|x-y|<\delta \implies |f(x)-f(y)|<\frac{\epsilon}{b-a}$. Let X be any partition of $[a,b]$ with $|X|<\delta$. By the Extreme Value Theorem we have $M_k=f(t_k)$ and $m_k=f(s_k)$ for some $t_k,s_k\in[x_{k-1},x_k]$. Since $|t_k-s_k|\le|x_k-x_{k-1}|\le|X|=\delta$, we have $|M_k-m_k|=|f(t_k)-f(s_k)|<\frac{\epsilon}{b-a}$. Thus

$$U(f,X)-L(f,X)=\sum_{k=1}^{n}(M_k-m_k)\Delta_k x<\tfrac{\epsilon}{b-a}\sum_{k=1}^{n}\Delta_k x=\epsilon\,.$$

1.18 Note: Let f be integrable on $[a,b]$. Let X_n be any sequence of partitions of $[a,b]$ with $\lim\limits_{n\to\infty}|X_n|=0$. Let S_n be any Riemann sum for f on X_n. Then $\{S_n\}$ converges with

$$\lim_{n\to\infty}S_n=\int_a^b f(x)\,dx\,.$$

Proof: Write $I=\int_a^b f$. Given $\epsilon>0$, choose $\delta>0$ so that for every partition X of $[a,b]$ with $|X|<\delta$ we have $|S-I|<\epsilon$ for every Riemann sum S for f on X, and then choose N so that $n>N\implies|X_n|<\delta$. Then we have $n>N\implies|S_n-I|<\epsilon$.

1.19 Note: Let f be integrable on $[a,b]$. If we let X_n be the partition of $[a,b]$ into n equal-sized subintervals, and we let S_n be the Riemann sum on X_n using right-endpoints, then by the above note we obtain the formula

$$\int_a^b f(x)\,dx=\lim_{n\to\infty}\sum_{k=1}^{n}f(x_{n,k})\Delta_{n,k}x=\lim_{n\to\infty}\sum_{k=1}^{n}f\big(a+\tfrac{b-a}{n}\,k\big)\,\tfrac{b-a}{n}.$$

1.20 Example: Find $\displaystyle\int_0^2 2^x\,dx$.

Solution: Let $f(x)=2^x$. Note that f is continuous and hence integrable, so we have

$$\int_0^2 2^x\,dx=\lim_{n\to\infty}\sum_{k=1}^{n}f(x_{n,k})\Delta_{n,k}x=\lim_{n\to\infty}\sum_{k=1}^{n}f\left(\tfrac{2k}{n}\right)\left(\tfrac{2}{n}\right)=\lim_{n\to\infty}\sum_{k=1}^{n}2^{2k/n}\left(\tfrac{2}{n}\right)$$

$$=\lim_{n\to\infty}\frac{2\cdot 4^{1/n}}{n}\cdot\frac{4-1}{4^{1/n}-1}\ \text{, by the formula for the sum of a geometric sequence}$$

$$=\Big(\lim_{n\to\infty}6\cdot 4^{1/n}\Big)\Big(\lim_{n\to\infty}\frac{1}{n\left(4^{1/n}-1\right)}\Big)=6\lim_{n\to\infty}\frac{\frac{1}{n}}{4^{1/n}-1}=6\lim_{x\to 0}\frac{x}{4^x-1}$$

$$=6\lim_{x\to 0}\frac{1}{\ln 4\cdot 4^x}\ \text{, by l'Hôpital's Rule}$$

$$=\tfrac{6}{\ln 4}=\tfrac{3}{\ln 2}\,.$$

1.21 Lemma: *(Summation Formulas) We have*

$$\sum_{i=1}^{n} 1 = n\ ,\ \sum_{i=1}^{n} i = \frac{n(n+1)}{2}\ ,\ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\ ,\ \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Proof: These formulas could be proven by induction, but we give a more constructive proof. It is obvious that $\sum\limits_{k=1}^{n} 1 = 1+1+\cdots 1 = n$. To find $\sum\limits_{k=1}^{n} k$, consider $\sum\limits_{k=1}^{n} \big(k^2-(k-1)^2\big)$. On the one hand, we have

$$\begin{aligned}\sum_{k=1}^{n} \big(k^2-(k-1)^2\big) &= (1^2-0^2)+(2^2-1^2)+\cdots+((n-1)^2-(n-2)^2)+(n^2-(n-1)^2) \\ &= -0^2+(1^2-1^2)+(2^2-2^2)+\cdots+((n-1)^2-(n-1)^2)+n^2 \\ &= n^2\end{aligned}$$

and on the other hand,

$$\sum_{k=1}^{n} \big(k^2-(k-1)^2\big) = \sum_{k=1}^{n} \big(k^2-(k^2-2k+1)\big) = \sum_{k=1}^{n} (2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

Equating these gives $n^2 = 2\sum\limits_{k=1}^{n} k - \sum\limits_{k=1}^{n} 1$ and so

$$2\sum_{k=1}^{n} k = n^2 + \sum_{k=1}^{n} 1 = n^2+n = n(n+1)\,,$$

as required. Next, to find $\sum\limits_{k=1}^{n} k^2$, consider $\sum\limits_{k=1}^{n} \big(k^3-(k-1)^3\big)$. On the one hand we have

$$\begin{aligned}\sum_{k=1}^{n} \big(k^3-(k-1)^3\big) &= (1^3-0^3)+(2^3-1^3)+(3^3-2^3)+\cdots+(n^3-(n-1)^3) \\ &= -0^3+(1^3-1^3)+(2^3-2^3)+\cdots+((n-1)^3-(n-1)^3)+n^3 \\ &= n^3\end{aligned}$$

and on the other hand,

$$\begin{aligned}\sum_{k=1}^{n} \big(k^3-(k-1)^3\big) &= \sum_{k=1}^{n} \big(k^3-(k^3-3k^2+3k-1)\big) \\ &= \sum_{k=1}^{n} (3k^2-3k+1) = 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\,.\end{aligned}$$

Equating these gives $n^3 = 3\sum\limits_{k=1}^{n} k^2 - 3\sum\limits_{k=1}^{n} k + \sum\limits_{k=1}^{n} 1$ and so

$$6\sum_{k=1}^{n} k^2 = 2n^3 + 6\sum_{k=1}^{n} k - 2\sum_{k=1}^{n} 1 = 2n^3+3n(n+1)-2n = n(n+1)(2n+1)$$

as required. Finally, to find $\sum\limits_{k=1}^{n} k^3$, consider $\sum\limits_{k=1}^{n} \big(k^4-(k-1)^4\big)$. On the one hand we have

$$\sum_{k=1}^{n} \big(k^4-(k-1)^4\big) = n^4\,,$$

(as above) and on the other hand we have

$$\sum_{k=1}^{n} \big(k^4-(k-1)^4\big) = \sum_{k=1}^{n} (4k^3-6k^2+4k-1) = 4\sum_{k=1}^{n} k^3 - 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1\,.$$

Equating these gives $n^4 = 4\sum\limits_{k=1}^{n} k^3 - 6\sum\limits_{k=1}^{n} k^2 + 4\sum\limits_{k=1}^{n} k - \sum\limits_{k=1}^{n} 1$ and so

$$\begin{aligned}
4\sum_{k=1}^{n} k^3 &= n^4 + 6\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
&= n^4 + n(n+1)(2n+1) - 2n(n+1) + n \\
&= n^4 + 2n^3 + n^2 = n^2(n+1)^2\,,
\end{aligned}$$

as required.

1.22 Example: Find $\displaystyle\int_1^3 x + 2x^3\ dx$.

Solution: Let $f(x) = x + 2x^3$. Then

$$\begin{aligned}
\int_1^3 x + 2x^3\ dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_{n,k})\Delta_{n,k}x = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(1 + \tfrac{2}{n}\,k\right)\left(\tfrac{2}{n}\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(1 + \tfrac{2}{n}\,k\right) + 2\left(1 + \tfrac{2}{n}\,k\right)^3\right)\left(\tfrac{2}{n}\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(1 + \tfrac{2}{n}\,k + 2\left(1 + \tfrac{6}{n}\,k + \tfrac{12}{n^2}\,k^2 + \tfrac{8}{n^3}\,k^3\right)\right)\left(\tfrac{2}{n}\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\tfrac{6}{n} + \tfrac{28}{n^2}\,k + \tfrac{48}{n^3}\,k^2 + \tfrac{32}{n^4}\,k^3\right) \\
&= \lim_{n\to\infty}\left(\tfrac{6}{n}\sum_{k=1}^{n} 1 + \tfrac{28}{n^2}\sum_{k=1}^{n} k + \tfrac{48}{n^3}\sum_{k=1}^{n} k^2 + \tfrac{32}{n^4}\sum_{k=1}^{n} k^3\right) \\
&= \lim_{n\to\infty}\left(\tfrac{6}{n}\cdot n + \tfrac{28}{n^2}\cdot\tfrac{n(n+1)}{2} + \tfrac{48}{n^3}\cdot\tfrac{n(n+1)(2n+1)}{6} + \tfrac{32}{n^4}\cdot\tfrac{n^2(n+1)^2}{4}\right) \\
&= 6 + \tfrac{28}{2} + \tfrac{48\cdot 2}{6} + \tfrac{32}{4} = 44\,.
\end{aligned}$$

Basic Properties of Integrals

1.23 Theorem: *(Linearity) Let f and g be integrable on $[a,b]$ and let $c\in\mathbf{R}$. Then $f+g$ and cf are both integrable on $[a,b]$ and*
$$\int_a^b (f+g)=\int_a^b f+\int_a^b g$$
and
$$\int_a^b cf = c\int_a^b f\,.$$
Proof: The proof is left as an exercise.

1.24 Theorem: *(Comparison) Let f and g be integrable on $[a,b]$. If $f(x)\le g(x)$ for all $x\in[a,b]$ then*
$$\int_a^b f\le\int_a^b g\,.$$
Proof: The proof is left as an exercise.

1.25 Theorem: *(Additivity) Let $a<b<c$ and let $f:[a,c]\to\mathbf{R}$ be bounded. Then f is integrable on $[a,c]$ if and only if f is integrable both on $[a,b]$ and on $[b,c]$, and in this case*
$$\int_a^b f+\int_b^c f=\int_a^c f\,.$$
Proof: Suppose that f is integrable on $[a,c]$. Choose a partition X of $[a,c]$ such that $U(f,X)-L(f,X)<\epsilon$. Say that $b\in[x_{k-1},x_k]$ and let $Y=\{x_0,x_1,\cdots,x_{k-1},b\}$ and $Z=\{b,x_k,x_{k+1},\cdots,x_n\}$ so that Y and Z are partitions of $[a,b]$ and of $[b,c]$. Then we have $U(f,Y)-L(f,Y)\le U(f,X\cup\{b\})-L(f,X\cup\{b\})\le U(f,X)-L(f,X)<\epsilon$ and also $U(f,Z)-L(f,Z)\le U(f,X\cup\{b\})-L(f,X\cup\{b\})\le U(f,X)-L(f,X)<\epsilon$ and so f is integrable both on $[a,b]$ and on $[b,c]$.

Conversely, suppose that f is integrable both on $[a,b]$ and on $[b,c]$. Choose a partition Y of $[a,b]$ so that $U(f,Y)-L(f,Y)<\frac{\epsilon}{2}$ and choose a partition Z of $[b,c]$ such that $U(f,Z)-L(f,Z)<\frac{\epsilon}{2}$. Let $X=Y\cup Z$. Then X is a partition of $[a,c]$ and we have $U(f,X)-L(f,X)=\big(U(f,Y)+U(f,Z)\big)-\big(L(f,Y)+L(f,Z)\big)<\epsilon$.

Now suppose that f is integrable on $[a,c]$ (hence also on $[a,b]$ and on $[b,c]$) with $I_1=\displaystyle\int_a^b f$, $I_2=\displaystyle\int_b^c f$ and $I=\displaystyle\int_a^c f$. Let $\epsilon>0$. Choose $\delta>0$ so that for all partitions X_1, X_2 and X of $[a,b]$, $[b,c]$ and $[a,c]$ respectively with $|X_1|<\delta$, $|X_2|<\delta$ and $|X|<\delta$, we have $|S_1-I_1|<\frac{\epsilon}{3}$, $|S_2-I_2|<\frac{\epsilon}{3}$ and $|S-I|<\frac{\epsilon}{3}$ for all Riemann sums S_1, S_2 and S for f on X_1, X_2 and X respectively. Choose partitions X_1 and X_2 of $[a,b]$ and $[b,c]$ with $|X_1|<\delta$ and $|X_2|<\delta$. Choose Riemann sums S_1 and S_2 for f on X_1 and X_2. Let $X=X_1\cup X_2$ and note that $|X|<\delta$ and that $S=S_1+S_2$ is a Riemann sum for f on X. Then we have
$$\big|I-(I_1+I_2)\big|=\big|(I-S)+(S_1-I_1)+(S_2-I_2)\big|\le\big|I-S|+|S_1-I_1|+|S_2-I_2|\le\tfrac{\epsilon}{3}+\tfrac{\epsilon}{3}+\tfrac{\epsilon}{3}=\epsilon.$$

1.26 Corollary: *(Piecewise Continuous Functions are Integrable) Let $X=(x_0,x_1,\cdots,x_n)$ be a partition of $[a,b]$. Let $g_k:[x_{k-1},x_k]\to\mathbf{R}$ be continuous for $1\le k\le n$. Let $f:[a,b]\to\mathbf{R}$ be a function with $f(t)=g_k(t)$ for all $t\in(x_{k-1},x_k)$. Then f is integrable on $[a,b]$ with $\displaystyle\int_a^b f(x)\,dx=\sum_{k=1}^n\int_{x_{k-1}}^{x_k}g_k(x)\,dx$.*

1.27 Definition: We define $\displaystyle\int_a^a f = 0$ and for $a < b$ we define $\displaystyle\int_b^a f = -\int_a^b f$.

1.28 Note: Using the above definition, the Additivity Theorem extends to the case that $a, b, c \in \mathbf{R}$ are not in increasing order: for any $a, b, c \in \mathbf{R}$, if f is integrable on $\big[\min\{a,b,c\}, \max\{a,b,c\}\big]$ then

$$\int_a^b f + \int_b^c f = \int_a^c f\,.$$

1.29 Theorem: *(Estimation) Let f be integrable on $[a,b]$. Then $|f|$ is integrable on $[a,b]$ and*

$$\left|\int_a^b f\right| \leq \int_a^b |f|\,.$$

Proof: Let $\epsilon > 0$. Choose a partition X of $[a,b]$ such that $U(f,X) - L(f,X) < \epsilon$. Write $M_k(f) = \sup\big\{f(t)\big|t \in [x_{k-1}, x_k]\big\}$ and $M_k(|f|) = \sup\big\{|f(t)|\big|t \in [x_{k-1}, x_k]\big\}$, and similarly for $m_k(f)$ and $m_k(|f|)$.

When $0 \leq m_k(f) \leq M_k(f)$ we have $M_k(|f|) = M_k(f)$ and $m_k(|f|) = m_k(f)$. When $m_k(f) \leq 0 \leq M_k(f)$ we have $M_k(|f|) = \max\{M_k(f), -m_k(f)\}$ and $m_k(|f|) \geq 0$, and so $M_k(|f|) - m_k(|f|) \leq \max\big\{M_k(f), -m_k(f)\big\} \leq M_k(f) - m_k(f)$. When $m_k(f) \leq M_k(f) \leq 0$, $M_k(|f|) = -m_k(f)$ and $m_k(|f|) = -M_k(f)$, and so $M_k(|f|) - m_k(|f|) = M_k(f) - m_k(f)$. Thus in all three cases we have

$$M_k(|f|) - m_k(|f|) \leq M_k(f) - m_k(f)$$

and so

$$\begin{aligned} U(|f|,X) - L(|f|,X) &= \sum_{k=1}^n \big(M_k(|f|) - m_k(|f|)\big)\Delta_k x \leq \sum_{k=1}^n \big(M_k(f) - m_k(f)\big)\Delta_k x \\ &= U(f,X) - L(f,X) < \epsilon\,. \end{aligned}$$

Thus $|f|$ is integrable on $[a,b]$.

Again, let $\epsilon > 0$. Choose a partition X on $[a,b]$ and choose values $t_k \in [x_{k-1}, x_k]$ so that

$$\left|\sum_{k=1}^n f(t_k)\Delta_k x - \int_a^b f\right| < \tfrac{\epsilon}{2} \text{ and } \left|\sum_{k=1}^n |f(t_k)|\Delta_k x - \int_a^b |f|\right| < \tfrac{\epsilon}{2}\,.$$

Note that by the triangle inequality we have $\left|\sum\limits_{k=1}^n f(t_k)\Delta_k x\right| \leq \sum\limits_{k=1}^n |f(t_k)|\Delta_k x$, and so

$$\begin{aligned} \left|\int_a^b f\right| - \int_a^b |f| &= \left(\left|\int_a^b f\right| - \left|\sum_{k=1}^n f(t_k)\Delta_k x\right|\right) + \left(\left|\sum_{k=1}^n f(t_k)\Delta_k x\right| - \sum_{k=1}^n \big|f(t_k)\big|\Delta_k x\right) \\ &\quad + \left(\sum_{k=1}^n \big|f(t_k)\big|\Delta_k x - \int_a^b |f|\right) \\ &< \tfrac{\epsilon}{2} + 0 + \tfrac{\epsilon}{2} = \epsilon \end{aligned}$$

Since $\left|\displaystyle\int_a^b f\right| - \displaystyle\int_a^b |f| < \epsilon$ for every $\epsilon > 0$, we have $\left|\displaystyle\int_a^b f\right| - \displaystyle\int_a^b |f| \leq 0$, as required.

The Fundamental Theorem of Calculus

1.30 Notation: For a function F, defined on an interval containing $[a,b]$, we write

$$\Big[F(x)\Big]_a^b = F(b) - F(a)\,.$$

1.31 Theorem: *(The Fundamental Theorem of Calculus)*
(1) Let f be integrable on $[a,b]$. Define $F:[a,b]\to\mathbf{R}$ by

$$F(x) = \int_a^x f = \int_a^x f(t)\,dt\,.$$

Then F is continuous on $[a,b]$. Moreover, if f is continuous at a point $x\in[a,b]$ then F is differentiable at x and

$$F'(x) = f(x)\,.$$

(2) Let f be integrable on $[a,b]$. Let F be differentiable on $[a,b]$ with $F'=f$. Then

$$\int_a^b f = \Big[F(x)\Big]_a^b = F(b) - F(a)\,.$$

Proof: (1) Let M be an upper bound for $|f|$ on $[a,b]$. For $a\le x,y\le b$ we have

$$\big|F(y)-F(x)\big| = \left|\int_a^y f - \int_a^x f\right| = \left|\int_x^y f\right| \le \left|\int_x^y |f|\right| \le \left|\int_x^y M\right| = M|y-x|$$

so given $\epsilon>0$ we can choose $\delta=\frac{\epsilon}{M}$ to get

$$|y-x|<\delta \implies \big|F(y)-F(x)\big| \le M|y-x| < M\delta = \epsilon\,.$$

Thus F is continuous (indeed uniformly continuous) on $[a,b]$. Now suppose that f is continuous at the point $x\in[a,b]$. Note that for $a\le x,y\le b$ with $x\ne y$ we have

$$\begin{aligned}
\left|\frac{F(y)-F(x)}{y-x} - f(x)\right| &= \left|\frac{\int_a^y f - \int_a^x f}{y-x} - f(x)\right| \\
&= \left|\frac{\int_x^y f}{y-x} - \frac{\int_x^y f(x)}{y-x}\right| \\
&= \frac{1}{|y-x|}\left|\int_x^y \big(f(t)-f(x)\big)\,dt\right| \\
&\le \frac{1}{|y-x|}\left|\int_x^y \big|f(t)-f(x)\big|\,dt\right|\,.
\end{aligned}$$

Given $\epsilon>0$, since f is continuous at x we can choose $\delta>0$ so that

$$|y-x|<\delta \implies \big|f(y)-f(x)\big|<\epsilon$$

and then for $0<|y-x|<\delta$ we have

$$\begin{aligned}
\left|\frac{F(y)-F(x)}{y-x} - f(x)\right| &\le \frac{1}{|y-x|}\left|\int_x^y \big|f(t)-f(x)\big|\,dt\right| \\
&\le \frac{1}{|y-x|}\left|\int_x^y \epsilon\,dt\right| = \frac{1}{|y-x|}\,\epsilon|y-x| = \epsilon\,.
\end{aligned}$$

and thus we have $F'(x)=f(x)$ as required.

(2) Let f be integrable on $[a,b]$. Suppose that F is differentiable on $[a,b]$ with $F'=f$. Let $\epsilon>0$ be arbitrary. Choose $\delta>0$ so that for every partition X of $[a,b]$ with $|X|<\delta$ we have $\left|\int_a^b f-\sum_{k=1}^n f(t_k)\Delta_k x\right|<\epsilon$ for every choice of sample points $t_k\in[x_{k-1},x_k]$. Choose sample points $t_k\in[x_{k-1},x_k]$ as in the Mean Value Theorem so that

$$F'(t_k)=\frac{F(x_k)-F(x_{k-1})}{x_k-x_{k-1}}\,,$$

that is $f(t_k)\Delta_k x=F(x_k)-F(x_{k-1})$. Then $\left|\int_a^b f-\sum_{k=1}^n f(t_k)\Delta_k x\right|<\epsilon$, and

$$\begin{aligned}\sum_{k=1}^n f(t_k)\Delta_k x&=\sum_{k=1}^n\big(F(x_k)-F(x_{k-1})\big)\\&=\big(F(x_1)-F(x_0)\big)+\big(F(x_2)-F(x_1)\big)+\cdots+\big(F(x_{n-1})-F(x_n)\big)\\&=-F(x_0)+\big(F(x_1)-F(x_1)\big)+\cdots+\big(F(x_{n-1})-F(x_{n-1})\big)+F(x_n)\\&=F(x_n)-F(x_0)=F(b)-F(a)\,.\end{aligned}$$

and so $\left|\int_a^b f-\big(F(b)-F(a)\big)\right|<\epsilon$. Since ϵ was arbitrary, $\left|\int_a^b f-\big(F(b)-F(a)\big)\right|=0$.

1.32 Definition: A function F such that $F'=f$ on an interval is called an **antiderivative** of f on the interval.

1.33 Note: If $G'=F'=f$ on an interval, then $(G-F)'=0$, and so $G-F$ is constant on the interval, that is $G=F+c$ for some constant c.

1.34 Notation: We write

$$\int f=F\ ,\text{ or }\int f=F+c\ ,\text{ or }\int f(x)=F(x)\ ,\text{ or }\int f(x)\,dx=F(x)+c$$

to indicate that F is an antiderivative of f on an interval, so that the antiderivatives of f on the interval are the functions of the form $G=F+c$ for some constant c.

1.35 Example: Find $\displaystyle\int_0^{\sqrt{3}}\frac{dx}{1+x^2}$.

Solution: Since $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$ or, equivalently, since $\displaystyle\int\frac{dx}{1+x^2}=\tan^{-1}x$, it follows from Part 2 of the Fundamental Theorem of Calculus that

$$\int_0^{\sqrt{3}}\frac{dx}{1+x^2}=\Big[\tan^{-1}x\Big]_0^{\sqrt{3}}=\tan^{-1}\sqrt{3}-\tan^{-1}0=\tfrac{\pi}{3}\,.$$

Chapter 2. Methods of Integration

Basic Integrals

2.1 Note: We have the following list of **Basic Integrals**

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + c \text{ , for } p \neq -1 \qquad \int \sec^2 x\; dx = \tan x + c$$
$$\int \frac{dx}{x} = \ln|x| + c \qquad \int \sec x \tan x\; dx = \sec x + c$$
$$\int e^x\,dx = e^x + c \qquad \int \tan x\; dx = \ln|\sec x| + c$$
$$\int a^x\,dx = \frac{a^x}{\ln a} + c \qquad \int \sec x\; dx = \ln|\sec x + \tan x| + c$$
$$\int \ln x\; dx = x\ln x - x + c \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x + c$$
$$\int \sin x\; dx = -\cos x + c \qquad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$$
$$\int \cos x\; dx = \sin x + c \qquad \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$$

Proof: Each of these equalities is easy to verify by taking the derivative of the right side. For example, we have $\int \ln x\; dx = x\ln x - x + c$ since $\frac{d}{dx}(x\ln x - x) = 1\cdot\ln x + x\cdot\frac{1}{x} - 1 = \ln x$, and we have $\int \tan x\; dx = \ln|\sec x| + c$ since $\frac{d}{dx}(\ln|\sec x|) = \frac{\sec x\tan x}{\sec x} = \tan x$, and we have $\int \sec x\; dx = \ln|\sec x + \tan x| + c$ since $\frac{d}{dx}(\ln|\sec x + \tan x|) = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x} = \sec x$.

2.2 Example: Find $\int_1^4 \frac{x^2-5}{\sqrt{x}}\,dx$.

Solution: By the Fundamental Theorem of Calculus and Linearity, we have

$$\int_1^4 \frac{x^2-5}{\sqrt{x}}\,dx = \int_1^4 x^{3/2} - 5x^{-1/2}\,dx = \Big[\tfrac{2}{5}x^{5/2} - 10x^{1/2}\Big]_1^4 = \big(\tfrac{64}{5} - 20\big) - \big(\tfrac{2}{5} - 10\big) = \tfrac{12}{5}\,.$$

2.3 Example: Find $\int_{\pi/6}^{\sqrt{\pi}/3} \sin 2x + \cos 3x\; dx$.

Solution: We find antiderivatives for $\sin 2x$ and $\cos 3x$. Since $\frac{d}{dx}\cos 2x = -2\sin 2x$ we have $\frac{d}{dx}\big(-\frac{1}{2}\cos 2x\big) = \sin 2x$ and since $\frac{d}{dx}\sin 3x = 3\cos 3x$ we have $\frac{d}{dx}\big(\frac{1}{3}\sin 3x\big)\cos 3x$, and so

$$\int_{\pi/6}^{\pi/3} \sin 2x + \cos 3x\; dx = \Big[-\tfrac{1}{2}\cos 2x + \tfrac{1}{3}\sin 3x\Big]_{\pi/6}^{\pi/3} = \big(\tfrac{1}{4} + 0\big) - \big(-\tfrac{1}{4} + \tfrac{1}{3}\big) = \tfrac{1}{6}\,.$$

Substitution

2.4 Theorem: *(Substitution, or Change of Variables) Let $u = g(x)$ be differentiable on an interval and let $f(u)$ be continuous on the range of $g(x)$. Then*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

and

$$\int_{x=a}^{b} f(g(x))g'(x)\,dx = \int_{u=g(a)}^{g(b)} f(u)\,du\,.$$

Proof: Let $F(u)$ be an antiderivative of $f(u)$ so $F'(u) = f(u)$ and $\int f(u)\,du = F(u) + c$. Then from the Chain Rule, we have $\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$, and so

$$\int f(g(x))g'(x)\,dx = F(g(x)) + c = F(u) + c = \int f(u)\,du$$

and

$$\begin{aligned}\int_{x=a}^{b} f(g(x))g'(x)\,dx &= \Big[F(g(x))\Big]_{x=a}^{b} = F(g(b)) - F(g(a)) \\ &= \Big[F(u)\Big]_{u=g(a)}^{g(b)} = \int_{u=g(a)}^{g(b)} f(u)\,du\,.\end{aligned}$$

2.5 Notation: For $u = g(x)$ we write $du = g'(x)\,dx$. More generally, for $f(u) = g(x)$ we write $f'(u)\,du = g'(x)\,dx$. This notation makes the above theorem easy to remember and to apply.

2.6 Example: Find $\int \sqrt{2x+3}\,dx$.

Solution: Make the substitution $u = 2x + 3$ so $du = 2dx$. Then

$$\int \sqrt{2x+3}\,dx = \int \tfrac{1}{2}u^{1/2}\,du = \tfrac{1}{3}\,u^{3/2} + c = \tfrac{1}{3}(2x+3)^{3/2} + c\,.$$

(In applying the Substitution Rule, we used $u = g(x) = 2x + 3$ and $f(u) = \sqrt{u} = u^{1/2}$, but the notation $du = g'(x)\,dx$ allows us to avoid explicit mention of the function $f(u)$ in our solution).

2.7 Example: Find $\int x\,e^{x^2}\,dx$.

Solution: Make the substitution $u = x^2$ so $du = 2x\,dx$. Then

$$\int x\,e^{x^2}\,dx = \int \tfrac{1}{2}e^u\,du = \tfrac{1}{2}e^u + c = \tfrac{1}{2}e^{x^2} + c\,.$$

2.8 Example: Find $\displaystyle\int \frac{\ln x}{x}\,dx$.

Solution: Let $u = \ln x$ so $du = \dfrac{1}{x}\,dx$. Then

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \tfrac{1}{2}u^2 + c = \tfrac{1}{2}(\ln x)^2 + c\,.$$

2.9 Example: Find $\displaystyle\int \tan x\; dx$.

Solution: We have $\tan x = \dfrac{\sin x}{\cos x}$. Let $u = \cos x$ so $du = -\sin x\; dx$. Then

$$\int \tan x\; dx = \int \frac{\sin x\; dx}{\cos x} = \int \frac{-du}{u} = -\ln|u| + c = -\ln|\cos x| + c = \ln|\sec x| + c\,.$$

2.10 Example: Find $\displaystyle\int \frac{dx}{x+\sqrt{x}}$.

Solution: Let $u = \sqrt{x}$ so $u^2 = x$ and $2u\,du = dx$. Then

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2u\,du}{u^2+u} = \int \frac{2\,du}{u+1}\,.$$

Now let $v = u+1$ do $dv = du$. Then

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2\,du}{u+1} = \int \frac{2}{v}\,dv = 2\ln|v| + c = 2\ln|u+1| + c = 2\ln(\sqrt{x}+1) + c\,.$$

2.11 Example: Find $\displaystyle\int_0^2 \frac{x\,dx}{\sqrt{2x^2+1}}$.

Solution: Let $u = 2x^2+1$ so $du = 4x\,dx$. Then

$$\int_{x=0}^2 \frac{x\,dx}{\sqrt{2x^2+1}} = \int_{u=1}^9 \frac{\frac{1}{4}\,du}{\sqrt{u}} = \int_1^9 \tfrac{1}{4}\,u^{-1/2}\,du = \Big[\tfrac{1}{2}\,u^{1/2}\Big]_1^9 = \tfrac{3}{2} - \tfrac{1}{2} = 1\,.$$

2.12 Example: Find $\displaystyle\int_0^1 \frac{dx}{1+3x^2}$.

Solution: Let $u = \sqrt{3}\,x$ so $du = \sqrt{3}\,dx$. Then

$$\int_0^1 \frac{dx}{1+3x^2} = \int_0^{\sqrt{3}} \frac{\frac{1}{\sqrt{3}}\,du}{1+u^2} = \Big[\tfrac{1}{\sqrt{3}}\tan^{-1}u\Big]_0^{\sqrt{3}} = \tfrac{1}{\sqrt{3}}\,\tfrac{\pi}{3} = \tfrac{\pi}{3\sqrt{3}}\,.$$

Integration by Parts

2.13 Theorem: *(Integration by Parts) Let $f(x)$ and $g(x)$ be differentiable in an interval. Then*

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

so

$$\int_{x=a}^{b} f(x)g'(x)\,dx = \left[f(x)g(x) - \int g(x)f'(x)\,dx\right]_{x=a}^{b}.$$

Proof: By the Product Rule, we have $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$ and so

$$\int f'(x)g(x) + f(x)g'(x)\;dx = f(x)g(x) + c\,,$$

which can be rewritten as

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx\,.$$

(We do not need to include the arbitrary constant c since there is now an integral on both sides of the equation).

2.14 Notation: If we write $u = f(x)$, $du = f'(x)\,dx$, $v = g(x)$ and $dv = g'(x)\,dx$, then the top formula in the above theorem becomes

$$\int u\,dv = uv - \int v\,du\,.$$

2.15 Note: To find the integral of a polynomial multiplied by an exponential function or a trigonometric function, try Integrating by parts with u equal to the polynomial (you may need to integrate by parts repeatedly if the polynomial is of high degree).

To integrate a polynomial (or an algebraic) function times a logarithmic or inverse trigonometric function, try integrating by parts letting u be the logarithmic or inverse trigonometric function.

To integrate an exponential function times a sine or cosine function, try integrating by parts twice, letting u be the exponential function both times.

2.16 Example: Find $\int x\sin x\;dx$.

Solution: Integrate by parts using $u = x$, $du = dx$, $v = -\cos x$ and $dv = \sin x\;dx$ to get

$$\int x\sin x\;dx = \int u\,dv = uv - \int v\,du = -x\cos x + \int \cos x\;dx = -x\cos x + \sin x + c\,.$$

2.17 Example: Find $\displaystyle\int (x^2+1)e^{2x}\,dx$.

Solution: Integrate by parts using $u=x^2+1$, $du=2x\,dx$, $v=\frac{1}{2}e^{2x}$ and $dv=e^{2x}\,dx$ to get

$$\int (x^2+1)e^{2x}\,dx=\int u\,dv=uv-\int v\,du=\tfrac{1}{2}(x^2+1)e^{2x}-\int x\,e^{2x}\,dx\,.$$

To find $\displaystyle\int x\,e^{2x}\,dx$ we integrate by parts again, this time using $u_2=x$, $du_2=dx$, $v_2=\frac{1}{2}e^{2x}$ and $dv_2=e^{2x}\,dx$ to get

$$\begin{aligned}\int (x^2+1)e^{2x}\,dx&=\tfrac{1}{2}(x^2+1)e^{2x}-\int x\,e^{2x}\,dx\\&=\tfrac{1}{2}(x^2+1)e^{2x}-\left(\tfrac{1}{2}\,xe^{2x}-\int \tfrac{1}{2}e^{2x}\,dx\right)\\&=\tfrac{1}{2}(x^2+1)e^{2x}-\left(\tfrac{1}{2}\,xe^{2x}-\tfrac{1}{4}e^{2x}\right)+c\\&=\tfrac{1}{4}\left(2x^2-2x+3\right)e^{2x}+c\end{aligned}$$

2.18 Example: Find $\displaystyle\int \ln x\ dx$.

Solution: Integrate by parts using $u=\ln x$, $du=\dfrac{1}{x}\,dx$, $v=x$ and $dv=dx$ to get

$$\int \ln x\ dx=x\ln x-\int 1\,dx=x\ln x-x+c\,.$$

2.19 Example: Find $\displaystyle\int_1^4 \sqrt{x}\,\ln x\ dx$.

Solution: Integrate by parts using $u=\ln x$, $du=\dfrac{1}{x}\,dx$, $v=\frac{2}{3}x^{3/2}$ and $dv=x^{1/2}\,dx$ to get

$$\begin{aligned}\int_1^4\sqrt{x}\,\ln x\ dx&=\left[\tfrac{2}{3}x^{3/2}\ln x-\int\tfrac{2}{3}x^{1/2}\,dx\right]_1^4=\left[\tfrac{2}{3}x^{3/2}\ln x-\tfrac{4}{9}x^{3/2}\right]_1^4\\&=\left(\tfrac{16}{3}\ln 4-\tfrac{32}{9}\right)-\left(\tfrac{2}{3}\ln 1-\tfrac{4}{9}\right)=\tfrac{16}{3}\ln 4-\tfrac{28}{9}\,.\end{aligned}$$

2.20 Example: Find $\displaystyle\int e^x\sin x\ dx$

Solution: Write $I=\displaystyle\int e^x\sin x\,dx$. Integrate by parts twice, first using $u_1=e^x$, $du=e^x\,dx$, $v=-\cos x$ and $dv=\sin x\ dx$, and next using $u_2=e^x$, $du_2=e^x\,dx$, $v_2=\sin x$ and $dv_2=\cos x\ dx$ to get

$$\begin{aligned}I&=-e^x\cos x+\int e^x\cos x\ dx\\&=-e^x\cos x+\left(e^x\sin x-\int e^x\sin x\ dx\right)\,.\\&=-e^x\cos x+e^x\sin x-I\end{aligned}$$

Thus $2I=-e^x\cos x+e^x\sin x+c$ and so $I=\frac{1}{2}(\sin x-\cos x)e^x+d$.

2.21 Example: Let $n \geq 2$ be an integer. Find a formula for $\int \sin^n x\ dx$ in terms of $\int \sin^{n-2} x\ dx$, and hence find $\int \sin^2 x\ dx$ and $\int \sin^4 x\ dx$.

Solution: Let $I = \int \sin^n x\ dx = \int \sin^{n-1} x \sin x\, dx$. Integrate by parts using $u = \sin^{n-1} x$, $du = (n-1)\sin^{n-2} x \cos x\, dx$, $v = -\cos x$ and $dv = \sin x\, dx$ to get

$$\begin{aligned} I &= -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x \cos^2 x\ dx \\ &= -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x (1 - \sin^2 x)\, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)I\,. \end{aligned}$$

Add $(n-1)I$ to both sides to get $nI = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\ dx$, that is

$$\int \sin^n x\ dx = -\tfrac{1}{n}\sin^{n-1} x \cos x + \tfrac{n-1}{n}\int \sin^{n-2} x\ dx\,.$$

In particular, when $n = 2$ we get

$$\int \sin^2 x\ dx = -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}\int 1\, dx = -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x + c$$

and when $n = 4$ we get

$$\int \sin^4 x\ dx = -\tfrac{1}{4}\sin^3 x \cos x + \tfrac{3}{4}\int \sin^2 x\ dx = -\tfrac{1}{4}\sin^3 x \cos x - \tfrac{3}{8}\sin x \cos x + \tfrac{3}{8}x + c\,.$$

2.22 Example: Let $n \geq 2$ be an integer. Find a formula for $\int \sec^n x\ dx$ in terms of $\int \sec^{n-2} x\ dx$, and hence find $\int \sec^3 x\ dx$.

Solution: Let $I = \int \sec^n x\ dx = \int \sec^{n-2} x \sec^2 x\ dx$. Using Integrate by Parts with $u = \sec^{n-2} x$, $du = (n-2)\sec^{n-2} x \tan x\, dx$, $v = \tan x$ and $dv = \tan x\, dx$, we obtain

$$\begin{aligned} I &= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x \tan^2 x\ dx \\ &= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x(\sec^2 x - 1)\ dx \\ &= \sec^{n-2} x \tan x - (n-2)I + (n-2)\int \sec^{n-2} x\ dx \end{aligned}$$

Add $(n-2)I$ to both sides to get $(n-1)I = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\ dx$, that is

$$\int \sec^n x\ dx = \tfrac{1}{n-1}\sec^{n-2} x \tan x + \tfrac{n-2}{n-1}\int \sec^{n-2} x\ dx\,.$$

In particular, when $n = 3$ we get

$$\int \sec^3 x\ dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\int \sec x\ dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln\left|\sec x + \tan x\right| + c$$

Trigonometric Integrals

2.23 Note: To find $\int f(\sin x)\cos^{2n+1} x\ dx$, write $\cos^{2n+1} x = (1-\sin^2 x)^n \cos x$ then try the substitution $u = \sin x$, $du = \cos x\, dx$.

To find $\int f(\cos x)\sin^{2n+1} x\ dx$, write $\sin^{2n+1} x = (1-\cos^2 x)^n \sin x$ then try the substitution $u = \cos x$, $du = -\sin x\ dx$.

To find $\int \sin^{2m} x \cos^{2n} x\, dx$, try using the trigonometric identities $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ and $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$. Alternatively, write $\cos^{2n} x = (1-\sin^2 x)^n$ and use the formula from Example 2.20.

To find $\int f(\tan x)\sec^{2n+2} x\ dx$, write $\sec^{2n+2} x = (1+\tan^2 x)^n \sec^2 x\, dx$ and try the substitution $u = \tan x$, $du = \sec^2 x\, dx$.

To find $\int f(\sec x)\tan^{2n+1} x\ dx$, write $\tan^{2n+1} x = \dfrac{(\sec^2 x - 1)^n}{\sec x} \sec x \tan x\ dx$ and try the substitution $u = \sec x$, $du = \sec x \tan x\ dx$.

To find $\int \sec^{2n+1} x \tan^{2n} x\ dx$, write $\tan^{2n} x = (\sec^2 x - 1)^n$ and use the formula from Example 2.21.

2.24 Example: Find $\displaystyle\int_0^{\pi/3} \frac{\sin^3 x}{\cos^2 x}\, dx$.

Solution: Make the substitution $u = \cos x$ so $du = -\sin x\, dx$. Then

$$\begin{aligned}\int_0^{\pi/3} \frac{\sin^3 x}{\cos^2 x}\, dx &= \int_0^{\pi/3} \frac{(1-\cos^2 x)\sin x\, dx}{\cos^2 x} = \int_1^{1/2} -\frac{(1-u^2)\, du}{u^2} = \int_1^{1/2} -\frac{1}{u^2} + 1\ du \\ &= \Big[\tfrac{1}{u} + u\Big]_1^{1/2} = \left(2 + \tfrac{1}{2}\right) - (1+1) = \tfrac{1}{2}\,.\end{aligned}$$

2.25 Example: Find $\int \sin^6 x\ dx$.

Solution: We could use the method of example 2.20, but we choose instead to use the half-angle formulas. We have

$$\begin{aligned}\int_0^{\pi/4} \sin^6 x\ dx &= \int_0^{\pi/4} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right)^3 dx = \int_0^{\pi/4} \tfrac{1}{8} - \tfrac{3}{8}\cos 2x + \tfrac{3}{8}\cos^2 2x - \tfrac{1}{8}\cos^3 2x\ dx \\ &= \int_0^{\pi/4} \tfrac{1}{8} - \tfrac{3}{8}\cos 2x + \tfrac{3}{8}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right) - \tfrac{1}{8}\left(1 - \sin^2 2x\right)\cos 2x\ dx \\ &= \int_0^{\pi/4} \tfrac{5}{16} - \tfrac{1}{2}\cos 2x + \tfrac{3}{16}\cos 4x + \tfrac{1}{8}\sin^2 2x \cos 2x\ dx \\ &= \Big[\tfrac{5}{15}x - \tfrac{1}{4}\sin 2x + \tfrac{3}{64}\sin 4x + \tfrac{1}{48}\sin^3 2x\Big]_0^{\pi/4} \\ &= \tfrac{5\pi}{64} - \tfrac{1}{4} + \tfrac{1}{48} = \tfrac{5\pi}{64} - \tfrac{11}{48}\,.\end{aligned}$$

2.26 Example: Find $\displaystyle\int_0^{\pi/4} \tan^4 x\ dx$.

Solution: Note first that

$$\tan^4 x = \tan^2 x(\sec^2 x - 1) = \tan^2 x \sec^2 x - \tan^2 x = \tan^2 x \sec^2 x - \sec^2 x + 1\,.$$

To find $\displaystyle\int \tan^2 x \sec^2 x\ dx$, make the substitution $u = \tan\theta$, $du = \sec^2\theta\, d\theta$ to get

$$\int \tan^2 x \sec^2 x\ dx = \int u^2\, du = \tfrac{1}{3}u^3 + c = \tfrac{1}{3}\tan^3 x + c\,.$$

Thus we have

$$\begin{aligned}\int_0^{\pi/4} \tan^4 x\, dx &= \int_0^{\pi/4} \tan^2 x \sec^2 x - \sec^2 x + 1 \\ &= \Big[\tfrac{1}{3}\tan^3 x - \tan x + x\Big]_0^{\pi/4} = \tfrac{1}{3} - 1 + \tfrac{\pi}{4} = \tfrac{\pi}{4} - \tfrac{2}{3}\,.\end{aligned}$$

2.27 Example: Find $\displaystyle\int_0^{\pi/4} \frac{\sec^4 x}{\sqrt{\tan x + 1}}\, dx$.

Solution: Make the substitution $u = \tan x$ so $du = \sec^2 x\, dx$. Then

$$\int_0^{\pi/4} \frac{\sec^4 x}{\sqrt{\tan x + 1}}\, dx = \int_0^{\pi/4} \frac{(\tan^2 x + 1)\sec^2 x\, dx}{\sqrt{\tan x + 1}} = \int_0^1 \frac{(u^2+1)\, du}{\sqrt{u+1}}$$

Now make the substitution $v = u + 1$ so $u = v - 1$ and $du = dv$. Then

$$\begin{aligned}\int_0^1 \frac{u^2+1}{\sqrt{u+1}}\, du &= \int_1^2 \frac{(v-1)^2+1}{\sqrt{v}}\, dv = \int_1^2 v^{3/2} - 2\,v^{1/2} + 2\,v^{-1/2}\ dv \\ &= \Big[\tfrac{2}{5}\,v^{5/2} - \tfrac{4}{3}\,v^{3/2} + 4\,v^{1/2}\Big]_1^2 = \big(\tfrac{2\cdot 4\sqrt{2}}{5} - \tfrac{4\cdot 2\sqrt{2}}{3} + 4\sqrt{2}\big) - \big(\tfrac{2}{5} - \tfrac{4}{3} + 4\big) \\ &= \tfrac{(24-40+60)\sqrt{2}}{15)} - \tfrac{6-20+60}{15} = \tfrac{44\sqrt{2}-46}{15}\,.\end{aligned}$$

2.28 Note: To find $\int \sin(ax)\sin(bx)\,dx$, $\int \cos(ax)\cos(bx)\,dx$, or $\int \sin(ax)\cos(bx)\,dx$, use one of the identities

$$\begin{aligned}\cos(A-B)-\cos(A+B)&=2\sin A\sin B\\ \cos(A-B)+\cos(A+B)&=2\cos A\cos B\\ \sin(A-B)+\sin(A+B)&=2\sin A\cos B\,.\end{aligned}$$

2.29 Example: Find $\displaystyle\int_0^{\pi/6}\cos 3x\cos 2x\,dx$.

Solution: Since $2\cos 3x\cos 2x=\cos(3x-2x)+\cos(3x+2x)=\cos x+\cos 5x$, we have

$$\int_0^{\pi/6}\cos 2x\cos 3x\,dx=\int_0^{\pi/6}\tfrac{1}{2}(\cos x+\cos 5x)\,dx=\Big[\tfrac{1}{2}\sin x+\tfrac{1}{10}\sin 5x\Big]_0^{\pi/6}=\tfrac{1}{4}+\tfrac{1}{20}=\tfrac{3}{10}\,.$$

2.30 Note: The **Weirstrass substitution** $u=\tan\frac{x}{2}$, $x=2\tan^{-1}u$, $dx=\dfrac{2du}{1+u^2}$ converts $\sin x$ and $\cos x$ into rational functions of u: indeed we have $\sin\frac{x}{2}=\frac{u}{\sqrt{1-u^2}}$ and $\cos\frac{x}{2}=\frac{1}{\sqrt{1-u^2}}$ so that $\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}=\frac{2u}{1+u^2}$ and $\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}=\frac{1-u^2}{1+u^2}$.

2.31 Example: Find $\displaystyle\int\frac{dx}{1-\cos x}$.

Solution: We use the Weirstrass substitution $u=\tan\frac{x}{2}$, $dx=\frac{2}{1+u^2}\,du$, and $\cos x=\frac{1-u^2}{1+u^2}$ to get

$$\int\frac{dx}{1-\cos x}=\int\frac{\frac{2}{1+u^2}\,du}{1-\frac{1-u^2}{1+u^2}}=\int\frac{2\,du}{(1+u^2)-(1-u^2)}=\int\frac{du}{u^2}=-\frac{1}{u}+c=-\cot\frac{x}{2}+c\,.$$

Inverse Trigonometric Substitution

2.32 Note: To solve an integral involving $\sqrt{a^2+b^2(x+c)^2}$ or $1/(a^2+b^2(x+c)^2)$, try the substitution $\theta=\tan^{-1}\frac{b(x+c)}{a}$ so that $a\tan\theta=b(x+c)$, $a\sec\theta=\sqrt{a^2+b^2(x+c)^2}$ and $a\sec^2\theta\,d\theta=b\,dx$.

For an integral involving $\sqrt{a^2-b^2(x+c)^2}$, try the substitution $\theta=\sin^{-1}\frac{b(x+c)}{a}$ so that $a\sin\theta=b(x+c)$, $a\cos\theta=\sqrt{a^2-b^2(x+c)^2}$ and $a\cos\theta\,d\theta=b\,dx$.

For an integral involving $\sqrt{b^2(x+c)^2-a^2}$, try the substitution $\theta=\sec^{-1}\frac{b(x+c)}{a}$ so that $a\sec\theta=b(x+c)$, $a\tan\theta=\sqrt{b^2(x+c)^2-a^2}$ and $a\sec\theta\tan\theta\,d\theta=b\,dx$.

2.33 Example: Find $\displaystyle\int_0^1\frac{dx}{(4-3x^2)^{3/2}}$.

Solution: Let $2\sin\theta=\sqrt{3}\,x$ so $2\cos\theta=\sqrt{4-3x^2}$ and $2\cos\theta\,d\theta=\sqrt{3}\,dx$. Then

$$\int_0^1\frac{dx}{(4-3x^2)^{3/2}}=\int_0^{\pi/3}\frac{\frac{2}{\sqrt{3}}\cos\theta\,d\theta}{(2\cos\theta)^3}=\int_0^{\pi/3}\tfrac{1}{4\sqrt{3}}\sec^2\theta\,d\theta=\Big[\tfrac{1}{4\sqrt{3}}\tan\theta\Big]_0^{\pi/3}=\tfrac{1}{4}\,.$$

2.34 Example: Find $\displaystyle\int_1^{\sqrt{3}}\frac{dx}{x^2\sqrt{x^2+3}}$.

Solution: Let $\sqrt{3}\tan\theta=x$ so $\sqrt{3}\sec\theta=\sqrt{x^2+3}$ and $\sqrt{3}\sec^2\theta\,d\theta=dx$, and also let $u=\sin\theta$ so $du=\cos\theta\,d\theta$. Then

$$\begin{aligned}\int_1^{\sqrt{3}}\frac{dx}{x^2\sqrt{x^2+3}}&=\int_{\pi/6}^{\pi/4}\frac{\sqrt{3}\sec^2\theta\,d\theta}{3\tan^2\theta\sqrt{3}\sec\theta}=\int_{\pi/6}^{\pi/4}\frac{1}{3}\,\frac{\sec\theta}{\tan^2\theta}\,d\theta=\int_{\pi/6}^{\pi/4}\frac{1}{3}\,\frac{\cos\theta\,d\theta}{\sin^2\theta}\\&=\int_{1/2}^{1/\sqrt{2}}\frac{1}{3\,u^2}\,du=\Big[-\frac{1}{3u}\Big]_{1/2}^{1/\sqrt{2}}=-\tfrac{\sqrt{2}}{3}+\tfrac{2}{3}=\tfrac{2-\sqrt{2}}{3}\,.\end{aligned}$$

2.35 Example: Find $\displaystyle\int_2^4\frac{\sqrt{x^2-4}}{x^2}\,dx$.

Solution: Let $2\sec\theta=x$ so $2\tan\theta=\sqrt{x^2-4}$ and $2\sec\theta\tan\theta\,d\theta=dx$. Then

$$\begin{aligned}\int_2^4\frac{\sqrt{x^2-4}}{x^2}\,dx&=\int_0^{\pi/3}\frac{\tan^2\theta\sec\theta\,d\theta}{\sec^2\theta}=\int_0^{\pi/3}\frac{\tan^2\theta}{\sec\theta}\,d\theta=\int_0^{\pi/3}\frac{\sec^2\theta-1}{\sec\theta}\,d\theta\\&=\int_0^{\pi/3}\sec\theta-\cos\theta\;d\theta=\Big[\ln|\sec\theta+\tan\theta|-\sin\theta\Big]_0^{\pi/3}=\ln(2+\sqrt{3})-\tfrac{\sqrt{3}}{2}\,.\end{aligned}$$

2.36 Example: Find $\displaystyle\int_2^3(4x-x^2)^{3/2}\,dx$.

Solution: Let $2\sin\theta=x-2$ so $2\cos\theta=\sqrt{4x-x^2}$ and $2\cos\theta\,d\theta=dx$. Then

$$\begin{aligned}\int_2^3(4x-x^2)^{3/2}\,dx&=\int_0^{\pi/6}16\cos^4\theta\,d\theta=\int_0^{\pi/6}4\,(1+\cos2\theta)^2\,d\theta\\&=\int 4+8\cos2\theta+4\cos^2 2\theta\;d\theta=\int 4+8\cos2\theta+2+2\cos4\theta\;d\theta\\&=\Big[6\theta+4\sin2\theta+\tfrac{1}{2}\sin4\theta\Big]_0^{\pi/6}=\pi+2\sqrt{3}+\tfrac{\sqrt{3}}{4}=\pi+\tfrac{9\sqrt{3}}{4}\,.\end{aligned}$$

Partial Fractions

2.37 Note: We can find the integral of a rational function $\dfrac{f(x)}{g(x)}$ as follows:

Step 1: use long division to find polynomials $q(x)$ and $r(x)$ with $\deg r(x) < \deg g(x)$ such that $f(x) = g(x)q(x) + r(x)$ for all x, and note that $\dfrac{f(x)}{g(x)} = q(x) + \dfrac{r(x)}{g(x)}$ so

$$\int \frac{f(x)}{g(x)}\,dx = \int q(x) + \frac{r(x)}{g(x)}\,dx\,.$$

(If $\deg f(x) < \deg g(x)$ then $q(x) = 0$ and $r(x) = f(x)$).

Step 2: factor $g(x)$ into linear and irreducible quadratic factors.

Step 3: write $\dfrac{r(x)}{g(x)}$ as a sum of terms so that for each linear factor $(ax+b)^k$ we have the k terms

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

and for each irreducible quadratic factor $(ax^2+bx+c)^k$ we have the k terms

$$\frac{B_1x+C_1}{(ax^2+bx+c)} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_kx+C_k}{(ax^2+bx+c)^k}\,.$$

Writing $\dfrac{r(x)}{g(x)}$ in this form is called splitting $\dfrac{r(x)}{g(x)}$ into its **partial fractions** decomposition.

Step 4: solve the integral.

2.38 Example: If $g(x) = x(x-1)^3(x^2+2x+3)^2$ then in step 3 we would write

$$\frac{r(x)}{g(x)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} + \frac{Ex+F}{x^2+2x+3} + \frac{Gx+H}{(x^2+2x+3)^2}\,.$$

and then solve for the various constants.

2.39 Example: Find $\displaystyle\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx$.

Solution: In order to get $\dfrac{x-7}{(x-1)^2(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2}$ we need

$$A(x-1)(x+2) + B(x+2) + C(x-1)^2 = x-7\,.$$

Equating coefficients gives $A + C = 0$, $A + B - 2C = 1$ and $-2A + 2B + C = -7$. Solving these three equations gives $A = 1$, $B = -2$ and $C = -1$, and so we have

$$\begin{aligned}
\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx &= \int_2^3 \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} \\
&= \int_2^3 \frac{1}{x-1} - \frac{2}{(x-1)^2} - \frac{1}{x+2}\,dx = \Big[\ln(x-1) + \tfrac{2}{x-1} - \ln(x+2)\Big]_2^3 \\
&= (\ln 2 + 1 - \ln 5) - (2 - \ln 4) = \ln\frac{8}{5} - 1\,.
\end{aligned}$$

2.40 Example: Find $\displaystyle\int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x}\,dx$.

Solution: Use long division of polynomials to show that $\dfrac{x^4-x^3+1}{x^3+x} = x-1+\dfrac{-x^2+x+1}{x^3+x}$. Next, note that to get $\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1} = \dfrac{-x^2+x+1}{x^3+x}$ we need $A(x^2+1)+(Bx+C)(x) = -x^2+x+1$. Equating coefficients gives $A+B=-1$, $C=1$ and $A=1$. Solving these three equations gives $A=1$, $B=-2$ and $C=1$. Thus

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x}\,dx &= \int_1^{\sqrt{3}} x-1+\frac{1}{x}-\frac{2x}{x^2+1}+\frac{1}{x^2+1}\;dx \\
&= \Big[\tfrac{1}{2}\,x^2 - x + \ln x - \ln(x^2+1) + \tan^{-1} x\Big]_1^{\sqrt{3}} \\
&= \left(\tfrac{3}{2} - \sqrt{3} + \ln\sqrt{3} - \ln 4 + \tfrac{\pi}{3}\right) - \left(\tfrac{1}{2} - 1 - \ln 2 + \tfrac{\pi}{4}\right) \\
&= 2 - \sqrt{3} + \ln\tfrac{\sqrt{3}}{2} + \tfrac{\pi}{12}\,.
\end{aligned}$$

2.41 Example: Find $I = \displaystyle\int_1^2 \frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2}\,dx$.

Solution: To get

$$\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2-2x+5}+\frac{Ex+F}{(x^2-2x+5)^2} = \frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2}$$

we need $Ax(x^2-2x+5)^2 + B(x^2-2x+5)^2 + (Cx+D)(x^2)(x^2-2x+5) + (Ex+F)(x^2) = x^5+x^4-2x^3-2x^2-5x-25$. Expanding the left hand side then equating coefficients gives the 5 equations

$$\begin{aligned}
&A+C=1\,,\ -4A+B-2C+D=1\,,\ 14A-4B+5C-2D+E=-2 \\
&-20A+14B+5D+F=-2\,,\ 25A-20B=-5\,,\ 25B=-25
\end{aligned}$$

Solving these equations gives $A=-1$, $B=-1$, $C=2$, $D=2$, $E=2$ and $F=-18$, so

$$\begin{aligned}
I &= \int_1^2 -\frac{1}{x}-\frac{1}{x^2}+\frac{2x+2}{x^2-2x+5}+\frac{2x-18}{(x^2-2x+5)^2}\;dx \\
&= \int_1^2 -\frac{1}{x}-\frac{1}{x^2}+\frac{2x-2+4}{x^2-2x+5}+\frac{2x-2-16}{(x^2-2x+5)^2}\;dx \\
&= \int_1^2 -\frac{1}{x}-\frac{1}{x^2}+\frac{2x-2}{x^2-2x+5}+\frac{4}{x^2-2x+5}+\frac{2x-2}{(x^2-2x+5)^2}-\frac{16}{(x^2-2x+5)^2}\;dx
\end{aligned}$$

We have $\displaystyle\int \frac{1}{x}\,dx = \ln x + c$ and $\displaystyle\int \frac{1}{x^2}\,dx = -\frac{1}{x}+c$. Make the substitution $u = x^2-2x+5$, $du = (2x-2)\,dx$ to get

$$\int \frac{(2x-2)\,dx}{x^2-2x+5} = \int \frac{du}{u} = \ln u + c = \ln(x^2-2x+5)+c$$

and

$$\int \frac{(2x-2)\,dx}{(x^2-2x+5)^2} = \int \frac{du}{u^2} = \frac{-1}{u}+c = \frac{-1}{x^2-2x+5}+c.$$

Make the substitution $2\tan\theta = x-1$, $2\sec\theta = \sqrt{x^2-2x+5}$, $2\sec^2\theta\,d\theta = dx$ to get

$$\int \frac{4\,dx}{x^2-2x+5} = \int \frac{4\cdot 2\sec^2\theta\,d\theta}{(2\sec\theta)^2} = \int 2\,d\theta = 2\theta + c = 2\tan^{-1}\left(\tfrac{x-1}{2}\right)+c$$

and

$$\int \frac{16\,dx}{(x^2-2x+5)^2} = \int \frac{16\cdot 2\sec^2\theta\,d\theta}{(2\sec\theta)^4}\,d\theta = \int \frac{2\,d\theta}{\sec^2\theta} = \int 2\cos^2\theta\,d\theta = \int 1+\cos 2\theta\;d\theta$$
$$= \theta + \tfrac{1}{2}\,\sin 2\theta + c = \theta + \sin\theta\cos\theta + c = \tan^{-1}\left(\tfrac{x-1}{2}\right) + \tfrac{2(x-1)}{x^2-2x+5} + c\,.$$

Thus we have

$$\begin{aligned} I &= \Bigg[-\ln x + \frac{1}{x} + \ln(x^2-2x+5) + 2\tan^{-1}\frac{x-1}{2} \\ &\qquad - \frac{1}{x^2-2x+5} - \tan^{-1}\frac{x-1}{2} - \frac{2(x-1)}{x^2-2x+5}\Bigg]_1^2 \\ &= \left[\ln\frac{x^2-2x+5}{x} + \frac{1}{x} - \frac{2x-1}{x^2-2x+5} + \tan^{-1}\frac{x-1}{2}\right]_1^2 \\ &= \left(\ln\tfrac{5}{2} + \tfrac{1}{2} - \tfrac{3}{5} + \tan^{-1}\tfrac{1}{2}\right) - \left(\ln 4 + 1 - \tfrac{1}{4}\right) \\ &= \ln\tfrac{5}{8} - \tfrac{17}{20} + \tan^{-1}\tfrac{1}{2}\,. \end{aligned}$$

2.42 Example: Find $\displaystyle\int \frac{\sec^3 x\,dx}{\sec x - 1}$.

Solution: Multiply the numerator and denominator by $\sec x + 1$ to get

$$\int \frac{\sec^3 x\,dx}{\sec x - 1} = \int \frac{\sec^3 x(\sec x + 1)}{(\sec^2 x - 1)}\,dx = \int \frac{\sec^4 x + \sec^3 x}{\tan^2 x}\,dx = \int \frac{\sec^4 x}{\tan^2 x}\,dx + \int \frac{\sec^3 x}{\tan^2 x}\,dx\,.$$

Make the substitution $u = \tan x$, $du = \sec^2 x\,dx$ to get

$$\int \frac{\sec^4 x}{\tan^2 x}\,dx = \int \frac{(\tan^2 x + 1)\sec^2 x\,dx}{\tan^2 x} = \int \frac{u^2+1}{u^2}\,du$$
$$= \int 1 + \frac{1}{u^2}\,du = u - \frac{1}{u} + c = \tan x - \cot x + c\,.$$

Make the substitution $v = \sin x$, $dv = \cos x\,dx$ and integrate by parts to get

$$\begin{aligned} \int \frac{\sec^3 x}{\tan^2 x}\,dx &= \int \frac{dx}{\cos x\sin^2 x} = \int \frac{\cos x\,dx}{(1-\sin^2 x)\sin^2 x} = \int \frac{dv}{(1-v^2)\,v^2} \\ &= \int \frac{1}{1-v^2} + \frac{1}{v^2}\;dv = \int \frac{\frac{1}{2}}{1-v} + \frac{\frac{1}{2}}{1+v} + \frac{1}{v^2}\;dv \\ &= -\tfrac{1}{2}\ln|1-v| + \tfrac{1}{2}\ln|1+v| - \tfrac{1}{v} + c = \tfrac{1}{2}\ln\left|\tfrac{1+v}{1-v}\right| - \tfrac{1}{v} + c \\ &= \tfrac{1}{2}\ln\tfrac{1+\sin x}{1-\sin x} - \csc x + c = \tfrac{1}{2}\ln\tfrac{(1+\sin x)^2}{(\cos x)^2} - \csc x + c = \ln\left|\tfrac{1+\sin x}{\cos x}\right| - \csc x + c\,. \end{aligned}$$

Thus $\displaystyle\int \frac{\sec^3 x}{\sec x - 1}\,dx = \tan x - \cot x + \ln|\sec x + \tan x| - \csc x + c.$

Approximate Integration

2.43 Definition: Let f be integrable on $[a,b]$. We can approximate the integral of f on $[a,b]$ by any Riemann sum

$$I = \int_a^b f(x)\,dx \cong \sum_{k=1}^n f(c_k)\Delta_k x$$

where $a = x_0 < x_1 < \cdots < x_n = b$, $\Delta_k x = x_k - x_{k-1}$ and $c_k \in [x_{k-1}, x_k]$. The n^{th} **Left Endpoint Approximation** L_n, the n^{th} **Right Endpoint Approximation** R_n, and the n^{th} **Midpoint Approximation** M_n, for the integral $I = \displaystyle\int_a^b f(x)\,dx$ are the Riemann sums for f obtained by using the partition of $[a,b]$ into n equal sized subintervals and by choosing c_k to be the left endpoint, the right endpoint, or the midpoint of the k^{th} subinterval $[x_{k-1}, x_k]$. We have

$$\begin{aligned}
L_n &= \sum_{k=1}^n f(x_{k-1})\Delta x = \frac{b-a}{n}\Big(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\Big)\\
R_n &= \sum_{k=1}^n f(x_k)\Delta x = \frac{b-a}{n}\Big(f(x_1) + f(x_2) + \cdots + f(x_n)\Big)\\
M_n &= \sum_{k=1}^n f\Big(\frac{x_{k-1}+x_k}{2}\Big)\Delta x = \frac{b-a}{n}\Big(f\left(\tfrac{x_0+x_1}{2}\right) + f\left(\tfrac{x_1+x_2}{2}\right) + \cdots + f\big(\tfrac{x_{n-1}+x_n}{2}\big)\Big)
\end{aligned}$$

where $x_k = a + \frac{b-a}{n}\,k$ and $\Delta x = \frac{b-a}{n}$.

2.44 Definition: Let f be integrable on $[a,b]$. The **Trapezoidal Approximation** T_n for the integral $I = \displaystyle\int_a^b f(x)\,dx$ is defined as follows. We use the partition of $[a,b]$ into n equal-sized subintervals, so we let $x_k = a + \frac{b-a}{n}\,k$ and $\Delta x = \frac{b-a}{n}$. Let g_k be the linear polynomial with $g_k(x_{k-1}) = f(x_{k-1})$ and $g_k(x_k) = f(x_k)$. Let g be the piecewise-linear function defined by $g(x) = g_k(x)$ for $x \in [x_{k-1}, x_k]$. We define

$$T_n = \int_a^b g(x)\,dx\,.$$

Note that

$$\int_{x_{k-1}}^{x_k} g(x)\,dx = \int_{x_{k-1}}^{x_k} g_k(x)\,dx = \frac{f(x_{k-1}) + f(x_k)}{2}\,\Delta x$$

(indeed, the integral measures the area of a trapezoid) so we have

$$\begin{aligned}
T_n &= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} g(x)\,dx = \sum_{k=1}^n \frac{f(x_{k-1}) + f(x_k)}{2}\,\Delta x = \frac{L_n + R_n}{2}\\
&= \frac{b-a}{2n}\Big(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big)\,.
\end{aligned}$$

2.45 Definition: Let f be integrable on $[a,b]$. For an even positive integer n, we define the **Simpson Approximation** S_n for the integral $I = \int_a^b f(x)\,dx$ as follows. We partition $[a,b]$ into n equal-sized subintervals. Let $x_k = a + \frac{b-a}{n}\,k$ and $\Delta x = \frac{b-a}{n}$. For $k = 1, 2, \cdots, \frac{n}{2}$, let g_k be the quadratic polynomial with $g(x_{2k-2}) = f(x_{2k-2})$, $g(x_{2k-1}) = f(x_{2k-1})$ and $g(x_{2k}) = f(x_{2k})$. Let g be the piecewise-quadratic function given by $g(x) = g_k(x)$ for $x \in [x_{2k-2}, x_{2k}]$. We define

$$S_n = \int_a^b g(x)\,dx\,.$$

Note that if $h(x) = Ax^2 + Bx + C$ is the quadratic polynomial with $h(-1) = u$, $h(0) = v$ and $h(1) = w$, then we must have $u = h(-1) = A - B + C$, $v = h(0) = C$ and $w = h(1) = A + B + C$. Solving these three equations gives $A = \frac{u-2v+w}{2}$, $B = \frac{w-u}{2}$ and Cv so we have

$$\begin{aligned}\int_{-1}^1 h(x)\,dx &= \int_{-1}^1 \tfrac{u-2v+w}{2}\,x^2 + \tfrac{w-u}{2}\,x + v\;dx\\ &= \Big[\tfrac{u-2v+w}{6}\,x^3 + \tfrac{w-u}{4}\,x^2 + v\,x\Big]_{-1}^1\\ &= \tfrac{u-2v+w}{3} + 2v = \tfrac{u+4v+w}{3}\,.\end{aligned}$$

It follows, by shifting and scaling, that

$$\int_{x_{2k-2}}^{x_{2k}} g_k(x)\,dx = \frac{f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})}{3}\,\Delta x\,.$$

Thus

$$\begin{aligned}S_n &= \sum_{k=1}^{n/2} \int_{x_{2k-2}}^{x_{2k}} g(x)\,dx = \sum_{k=1}^{n/2} \frac{f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})}{3}\,\Delta x\\ &= \frac{b-a}{3n}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big)\,.\end{aligned}$$

2.46 Theorem: *(Error Bounds for Approximate Integration) Suppose that the higher order derivatives of f exist and are continuous on $[a,b]$. Let $I = \int_a^b f(x)\,dx$. and let L_n, R_n, T_n, M_n and S_n be the left endpoint, right endpoint, midpoint, trapezoidal and Simpson approximation of I. Then*

$$\begin{aligned}\big|L_n - I\big| &\leq \frac{(b-a)^2}{2\,n}\,\max_{a\leq x\leq b}\big|f'(x)\big|\\ \big|R_n - I\big| &\leq \frac{(b-a)^2}{2\,n}\,\max_{a\leq x\leq b}\big|f'(x)\big|\\ \big|T_n - I\big| &\leq \frac{(b-a)^3}{12\,n^2}\,\max_{a\leq x\leq b}\big|f''(x)\big|\\ \big|M_n - I\big| &\leq \frac{(b-a)^3}{24\,n^2}\,\max_{a\leq x\leq b}\big|f''(x)\big|\\ \big|S_n - I\big| &\leq \frac{(b-a)^5}{180\,n^4}\,\max_{a\leq x\leq b}\big|f''''(x)\big|\end{aligned}$$

Proof: We may assign some proofs as exercises and we may provide some proofs later.

2.47 Example: Let $f(x)=\sin^2 x$. Find the exact value $I=\displaystyle\int_0^{4\pi/3} f(x)\,dx$, find the approximations L_8, R_8, M_8, T_8 and S_8, and find a bound on the error for each of these approximations.

Solution: The exact value of the integral is

$$I=\int_0^{4\pi/3}\sin^2 x\;dx=\int_0^{4\pi/3}\tfrac{1}{2}-\tfrac{1}{2}\cos 2x\;dx=\Big[\tfrac{1}{2}\,x-\tfrac{1}{4}\sin 2x\Big]_0^{4\pi/3}=\tfrac{4\pi}{3}-\tfrac{\sqrt{3}}{8}\,.$$

When we divide the interval $\big[0,4\pi/3\big]$ into 8 equal subintervals, the size each of the subintervals is $\Delta x=\frac{\pi}{6}$ and the endpoints of the subintervals are $0,\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2},\frac{2\pi}{3},\frac{5\pi}{6},\pi,\frac{7\pi}{6},\frac{4\pi}{3}$. Thus the approximations are

$$\begin{aligned}
L_8&=\tfrac{\pi}{6}\Big(f(0)+f\big(\tfrac{\pi}{6}\big)+f\big(\tfrac{\pi}{3}\big)+f\big(\tfrac{\pi}{2}\big)+f\big(\tfrac{2\pi}{3}\big)+f\big(\tfrac{5\pi}{6}\big)+f(\pi)+f\big(\tfrac{7\pi}{6}\big)\Big)\\
&=\tfrac{\pi}{6}\big(0+\tfrac14+\tfrac34+1+\tfrac34+\tfrac14+0+\tfrac14\big)=\tfrac{13\pi}{24}\,,\\
R_8&=\tfrac{\pi}{6}\Big(f\big(\tfrac{\pi}{6}\big)+f\big(\tfrac{\pi}{3}\big)+f\big(\tfrac{\pi}{2}\big)+f\big(\tfrac{2\pi}{3}\big)+f\big(\tfrac{5\pi}{6}\big)+f(\pi)+f\big(\tfrac{7\pi}{6}\big)+f\big(\tfrac{4\pi}{3}\big)\Big)\\
&=\tfrac{\pi}{6}\big(\tfrac14+\tfrac34+1+\tfrac34+\tfrac14+0+\tfrac14+\tfrac34\big)=\tfrac{2\pi}{3}\,,\\
T_8&=\tfrac12\big(L_8+R_8\big)=\tfrac{29\pi}{48}\,,\\
M_8&=\tfrac{\pi}{6}\Big(f\big(\tfrac{\pi}{12}\big)+f\big(\tfrac{\pi}{4}\big)+f\big(\tfrac{5\pi}{12}\big)+f\big(\tfrac{7\pi}{12}\big)+f\big(\tfrac{3\pi}{4}\big)+f\big(\tfrac{11\pi}{12}\big)+f\big(\tfrac{13\pi}{12}\big)+f\big(\tfrac{15\pi}{12}\big)\Big)\\
&=\tfrac{\pi}{6}\Big(\tfrac{2-\sqrt3}{4}+\tfrac12+\tfrac{2+\sqrt3}{4}+\tfrac{2+\sqrt3}{4}+\tfrac12+\tfrac{2-\sqrt3}{4}+\tfrac{2-\sqrt3}{4}+\tfrac12\Big)=\tfrac{\pi}{6}\Big(4-\tfrac{\sqrt3}{4}\Big)\,,\\
S_8&=\tfrac{\pi}{18}\Big(f(0)+4f\big(\tfrac{\pi}{6}\big)+2f\big(\tfrac{\pi}{3}\big)+4f\big(\tfrac{\pi}{2}\big)+2f\big(\tfrac{2\pi}{3}\big)+4f\big(\tfrac{5\pi}{6}\big)+2f(\pi)+4f\big(\tfrac{7\pi}{6}\big)+f\big(\tfrac{4\pi}{3}\big)\Big)\\
&=\tfrac{\pi}{18}\big(0+1+\tfrac32+4+\tfrac32+1+0+1+\tfrac34\big)=\tfrac{43\pi}{72}\,.
\end{aligned}$$

Note that to find the values of f needed for the midpoint approximation M_8, we used the identity $f(x)=\sin^2 x=\frac12-\frac12\cos 2x$. From this same identity, we obtain $f'(x)=\sin 2x$ and then $f''(x)=2\cos 2x$, $f'''(x)=-4\sin 2x$ and $f''''(x)=-8\cos 2x$. Thus we find that

$$\max_{0\le x\le 4\pi/3}\big|f'(x)\big|=1\;,\quad \max_{0\le x\le 4\pi/3}\big|f''(x)\big|=2\;\text{ and }\;\max_{0\le x\le 4\pi/3}\big|f''''(x)\big|=8\,.$$

The above theorem gives the following error bounds.

$$\begin{aligned}
\big|L_8-I\big|&\le\frac{(4\pi/3)^2}{16}\cdot 1=\tfrac{\pi^2}{9}\\
\big|R_r-I\big|&\le\frac{(4\pi/3)^2}{16}\cdot 1=\tfrac{\pi^2}{9}\\
\big|T_n-I\big|&\le\frac{(4\pi/3)^3}{12\cdot 6^2}\cdot 2=\tfrac{8\pi^3}{3^6}\\
\big|M_n-I\big|&\le\frac{(4\pi/3)^3}{24\cdot 6^2}\cdot 2=\tfrac{4\pi^3}{3^6}\\
\big|S_n-I\big|&\le\frac{(4\pi/3)^5}{180\cdot 6^4}\cdot 8=\tfrac{2^7\pi^5}{5\cdot 3^{11}}
\end{aligned}$$

Improper Integration

2.48 Definition: Suppose that $f : [a, b) \to \mathbb{R}$ is integrable on every closed interval contained in $[a, b)$. Then we define the **improper integral** of f on $[a, b)$ to be

$$\int_a^b f = \lim_{t \to b^-} \int_a^t f$$

provided the limit exists and, when the improper integral exists and is finite, we say that f is **improperly integrable** on $[a, b)$, (or that the improper integral of f on $[a, b)$ **converges**). In this definition we also allow the case that $b = \infty$, and then we have

$$\int_a^\infty f = \lim_{t \to \infty} \int_a^t f\,.$$

Similarly, if $f : (a, b] \to \mathbb{R}$ is integrable on every closed interval in $(a, b]$ then we define the **improper integral** of f on $(a, b]$ to be

$$\int_a^b f = \lim_{t \to a^+} \int_t^b f$$

provided the limit exists, and we say that f is **improperly integrable** on $(a, b]$ when the improper integral is finite. In this definition we also allow the case that $a = -\infty$. For a function $f : (a, b) \to \mathbb{R}$, which is integrable on every closed interval in (a, b), we choose a point $c \in (a, b)$, then we define the **improper integral** of f on (a, b) to be

$$\int_a^b f = \int_a^c f + \int_c^b f$$

provided that both of the improper integrals on the right exist and can be added, and we say that f is **improperly integrable** on (a, b) when both of the improper integrals on the right are finite. As an exercise, you should verify that the value of this integral does not depend on the choice of c.

2.49 Notation: For a function $F : (a, b) \to \mathbb{R}$ write

$$\Big[F(x)\Big]_{a^+}^{b^-} = \lim_{x \to b^-} F(x) - \lim_{x \to a^+} F(x)\,.$$

We use similar notation when $F : [a, b) \to \mathbb{R}$ and when $F : (a, b] \to \mathbb{R}$.

2.50 Note: Suppose that $f : (a, b) \to \mathbb{R}$ is integrable on every closed interval contained in (a, b) and that F is differentiable with $F' = f$ on (a, b). Then

$$\int_a^b f = \Big[F(x)\Big]_{a^+}^{b^-}.$$

A similar result holds for functions defined on half-open intervals $[a, b)$ and $(a, b]$.

Proof: Choose $c \in (a, b)$. By the Fundamental Theorem of Calculus we have

$$\begin{aligned}
\int_a^b f = \int_a^c f + \int_c^b f &= \lim_{s \to a^+} \int_s^c f + \lim_{t \to b^-} \int_c^t f \\
&= \lim_{s \to a^+} \big(F(c) - F(s)\big) + \lim_{t \to b^-} \big(F(t) - F(c)\big) \\
&= \lim_{t \to b^-} F(t) - \lim_{s \to a^+} F(s) = \Big[F(x)\Big]_{a^+}^{b^-}.
\end{aligned}$$

2.51 Example: Find $\displaystyle\int_0^1 \frac{dx}{x}$ and find $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$.

Solution: We have

$$\int_0^1 \frac{dx}{x} = \Big[\ln x\Big]_{0^+}^1 = 0 - (-\infty) = \infty$$

and

$$\int_0^1 \frac{dx}{\sqrt{x}} = \Big[2\sqrt{x}\Big]_{0^+}^1 = 2 - 0 = 2\,.$$

2.52 Example: Show that $\displaystyle\int_0^1 \frac{dx}{x^p}$ converges if and only if $p < 1$.

Solution: The case that $p = 1$ was dealt with in the previous example. If $p > 1$ so that $p - 1 > 0$ then we have

$$\int_0^1 \frac{dx}{x^p} = \left[\frac{-1}{(p-1)x^{p-1}}\right]_{0^+}^1 = \left(-\tfrac{1}{p-1}\right) - \big(-\infty\big) = \infty$$

and if $p < 1$ so that $1 - p > 0$ then we have

$$\int_0^1 \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_{0^+}^1 = \left(\tfrac{1}{1-p}\right) - (0) = \tfrac{1}{1-p}\,.$$

2.53 Example: Show that $\displaystyle\int_1^\infty \frac{dx}{x^p}$ converges if and only if $p > 1$.

Solution: When $p = 1$ we have

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{1}{x} = \Big[\ln x\Big]_1^\infty = \infty - 0 = \infty\,.$$

When $p > 1$ so that $p - 1 > 0$ we have

$$\int_1^\infty \frac{dx}{x^p} = \left[\frac{-1}{(p-1)x^{p-1}}\right]_1^\infty = (0) - \left(-\tfrac{1}{p-1}\right) = \tfrac{1}{p-1}$$

and if $p < 1$ so that $1 - p > 0$ then we have

$$\int_1^\infty \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^\infty = (\infty) - \left(\tfrac{1}{1-p}\right) = \infty\,.$$

2.54 Example: Find $\displaystyle\int_0^\infty e^{-x}\,dx$.

Solution: We have

$$\int_0^\infty e^{-x}\,dx = \Big[-e^{-x}\Big]_0^\infty = 0-(-1) = 1\,.$$

2.55 Example: Find $\displaystyle\int_0^1 \ln x\,dx$.

Solution: We have

$$\int_0^1 \ln x\;dx = \Big[x\ln x - x\Big]_{0^+}^1 = (-1)-(0) = -1\,,$$

since l'Hôpital's Rule gives $\displaystyle\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$.

2.56 Theorem: *(Comparison) Let f and g be integrable on closed subintervals of (a,b), and suppose that $0\le f(x)\le g(x)$ for all $x\in(a,b)$. If g is improperly integrable on (a,b) then so is f and then we have*

$$\int_a^b f \le \int_a^b g\,.$$

On the other hand, if $\displaystyle\int_a^b f$ diverges then $\displaystyle\int_a^b g$ diverges, too. A similar result holds for functions f and g defined on half-open intervals.

Proof: The proof is left as an exercise.

2.57 Example: Determine whether $\displaystyle\int_0^{\pi/2}\sqrt{\sec x}\,dx$ converges.

Solution: For $0\le x<\frac{\pi}{2}$ we have $\cos x\ge 1-\frac{2}{\pi}\,x$ so $\sec x\le\frac{1}{1-\frac{2}{\pi}\,x}$ hence $\sqrt{\sec x}\le\frac{1}{\sqrt{1-\frac{2}{\pi}\,x}}$. Let $u=1-\frac{2}{\pi}\,x$ so that $du=-\frac{2}{\pi}\,dx$. Then

$$\int_{x=0}^{\pi/2}\frac{1}{\sqrt{1-\frac{2}{\pi}\,x}}\,dx = \int_{u=1}^0 -\tfrac{\pi}{2}\,u^{-1/2} = \Big[-\pi\,u^{1/2}\Big]_1^0 = \pi$$

which is finite. It follows that $\displaystyle\int_0^{\pi/2}\sqrt{\sec x}\,dx$ converges, by comparison.

2.58 Example: Determine whether $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges.

Solution: For $0\le u$ we have $e^u\ge 1+u$, so for $0\le x$ we have $e^{x^2}\ge 1+x^2$, so $e^{-x^2}\le\dfrac{1}{1+x^2}$. Since

$$\int_0^\infty\frac{dx}{1+x^2} = \Big[\tan^{-1}x\Big]_0^\infty = \tfrac{\pi}{2}\,,$$

which is finite, we see that $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges, by comparison.

2.59 Theorem: *(Estimation) Let f be integrable on closed subintervals of (a,b). If $|f|$ is improperly integrable on (a,b) then so is f, and then we have*

$$\left|\int_a^b f\right| \le \int_a^b |f|\,.$$

A similar result holds for functions defined on half-open intervals.

Proof: The proof is left as an exercise.

2.60 Example: Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ converges.

Solution: We shall show that both of the integrals $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ and $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$ converge.

Since $\displaystyle\lim_{x\to 0^+}\frac{\sin x}{x} = 1$, the function f defined by $f(0)=1$ and $f(x)=\dfrac{\sin x}{x}$ for $x>0$ is continuous (hence integrable) on $[0,1]$. By part 1 of the Fundamental Theorem of Calculus, the function $\displaystyle\int_r^1 f(x)\,dx$ is a continuous function of r for $r\in[0,1]$ and so we have

$$\int_0^1 \frac{\sin x}{x}\,dx = \lim_{r\to 0^+}\int_r^1 \frac{\sin x}{x}\,dx = \lim_{r\to 0^+}\int_r^1 f(x)\,dx = \int_0^1 f(x)\,dx\,,$$

which is finite, so $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ converges.

Integrate by parts using $u=\frac{1}{x}$, $du=-\frac{1}{x^2}\,dx$, $v=-\sin x$ and $dv=\cos x\,dx$ to get

$$\int_1^\infty \frac{\sin x}{x}\,dx = \left[-\frac{\cos x}{x}\right]_1^\infty - \int_1^\infty \frac{\cos x}{x^2}\,dx = \cos(1) - \int_1^\infty \frac{\cos x}{x^2}\,dx\,.$$

Since $\left|\dfrac{\cos x}{x^2}\right| \le \dfrac{1}{x^2}$ and $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges, we see that $\displaystyle\int_1^\infty \left|\frac{\cos x}{x^2}\right| dx$ converges too, by comparison. Thus $\displaystyle\int_1^\infty \frac{\cos x}{x^2}\,dx$ also converges by the Estimation Theorem.

Chapter 3. Applications of the Definite Integral

Area Between Curves

3.1 Note: Suppose that f and g are integrable on $[a,b]$ with $f(x) \leq g(x)$ for all $x \in [a,b]$. We can approximate the area of the region R given by

$$a \leq x \leq b \ , \ f(x) \leq y \leq g(x)$$

as follows. Choose a partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

of $[a,b]$ and choose sample points $c_k \in [x_{k-1}, x_k]$. We divide the region R into strips with the k^{th} strip given by

$$x_{k-1} \leq x \leq x_k \ , \ f(x) \leq y \leq g(x) \, .$$

The area $\Delta_k A$ of the k^{th} strip is approximately equal to the area of the rectangle with base $\Delta_k x = x_k - x_{k-1}$ and height $g(c_k) - f(c_k)$, that is

$$\Delta_k A \cong \big(g(c_k) - f(c_k)\big)\Delta_k x \, .$$

The area of the entire region R is

$$A = \sum_{k=1}^{n} \Delta_k A \cong \sum_{k=1}^{n} \big(g(c_k) - f(c_k)\big)\Delta_k x \, .$$

We notice that the sum on the right is a Riemann sum for the function $g(x) - f(x)$ on the interval $[a,b]$, so we define the exact area of the region R to be the limit of these Riemann sums.

3.2 Definition: Suppose that f and g are integrable on $[a,b]$ with $f(x) \leq g(x)$ for all $x \in [a,b]$. We define the **area** of the region R given by

$$a \leq x \leq b \ , \ f(x) \leq y \leq g(x)$$

to be

$$A = \int_a^b g(x) - f(x) \, dx \, .$$

3.3 Example: Find the area of the region R which lies between the x-axis and the parabola $y = 1 - x^2$.

Solution: The region R is given by $-1 \leq x \leq 1$, $0 \leq y \leq 1 - x^2$ and so the area is

$$A = \int_{-1}^{1} 1 - x^2 \, dx = \Big[x - \tfrac{1}{3}\,x^3\Big]_{-1}^{1} = \big(1 - \tfrac{1}{3}\big) - \big(-1 + \tfrac{1}{3}\big) = \tfrac{4}{3} \, .$$

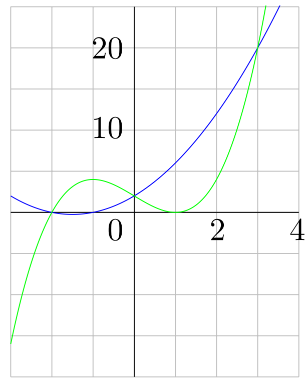
3.4 Example: Find the area of the region R which lies between the curves $y = x^2 + 3x + 2$ and $y = x^3 - 3x + 2$.

Solution: Let $f(x) = x^2 + 3x + 2$ and $g(x) = x^3 - 3x + 2$. First, let us find the points of intersection of the two curves and determine where $f(x) \geq g(x)$. We have

$$f(x) - g(x) = (x^2 + 3x + 2) - (x^3 - 3x + 2) = -(x^3 - x^2 - 6x) = -x(x-3)(x+2)$$

and so $f(x) = g(x)$ when $x \in \{-2, 0, 3\}$ with $f(x) \geq g(x)$ for $x \in (-\infty, -2] \cup [0, 3]$ and $f(x) \leq g(x)$ for $x \in [-2, 0] \cup [3, \infty)$. Next we make a table of values and sketch the curves. The curve $y = f(x)$ is shown in blue and the curve $y = g(x)$ is shown in green.

x	$f(x)$	$g(x)$
-3	2	-16
-2	0	0
-1	0	4
0	2	2
1	6	0
2	12	4
3	20	20



The region R consists of two parts with the first part given by $-2 \leq x \leq 0$, $f(x) \leq y \leq g(x)$ and the second part given by $0 \leq x \leq 3$, $g(x) \leq y \leq f(x)$ and so the total area is

$$\begin{aligned} A &= \int_{-2}^{0} g(x) - f(x)\, dx + \int_0^3 f(x) - g(x)\, dx \\ &= \int_{-2}^{0} x^3 - x^2 - 6x\, dx + \int_0^3 -x^3 + x^2 + 6x\, dx \\ &= \Big[\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 - 3x^2\Big]_{-2}^{0} + \Big[-\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 + 3x^2\Big]_0^3 \\ &= -\big(4 + \tfrac{8}{3} - 12\big) + \big(-\tfrac{81}{4} + 9 + 27\big) = \tfrac{16}{3} + \tfrac{63}{4} = \tfrac{253}{12}\,. \end{aligned}$$

3.5 Example: Find the area of a circle of radius r.

Solution: The area of a circle of radius r is equal to 4 times the area of the region given by $0 \leq x \leq r$, $0 \leq y \leq \sqrt{r^2 - x^2}$, and so the area of the circle is

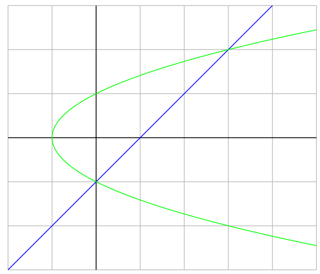
$$A = 4\int_0^r \sqrt{r^2 - x^2}\, dx\,.$$

To solve the integral, we make the substitution $r\sin\theta = x$ so that $r\cos\theta = \sqrt{r^2 - x^2}$ and $r\cos\theta\, d\theta = dx$ to get

$$\begin{aligned} A &= \int_{x=0}^{r} 4\sqrt{r^2 - x^2}\, dx = \int_{\theta=0}^{\pi/2} 4r\cos\theta \cdot r\cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} 4r^2\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) d\theta \\ &= r^2 \int_0^{\pi/2} 2 - 2\cos 2\theta\, d\theta = r^2\Big[2\theta - \sin 2\theta\Big]_0^{\pi/2} = r^2(\pi - 0) = \pi r^2\,. \end{aligned}$$

3.6 Example: Find the area of the region R which lies between the curves $y = x - 1$ and $y^2 = x + 1$.

Solution: The line $y = x - 1$ is shown in blue and the parabola $y^2 = x + 1$ is shown in green.



We now find the area in two ways. For the first solution, we divide the given region along the y-axis into two regions with the first given by $-1 \le x \le 0$, $-\sqrt{x+1} \le y \le \sqrt{x+1}$, and the second given by $0 \le x \le 3$, $x - 1 \le y \le \sqrt{x+1}$. The area of the first region is

$$A_1 = \int_{-1}^{0} \sqrt{x+1} - \left(-\sqrt{x+1}\right)\,dx = \int_{-1}^{0} 2(x+1)^{1/2}\,dx = \left[\tfrac{4}{3}(x+1)^{3/2}\right]_{-1}^{0} = \tfrac{4}{3}\,,$$

and the area of the second region is

$$A_2 = \int_{0}^{3} \sqrt{x+1} - (x-1)\,dx = \left[\tfrac{2}{3}(x+1)^{3/2} - \tfrac{1}{2}\,x^2 + x\right]_{0}^{3} = \left(\tfrac{16}{3} - \tfrac{9}{2} + 3\right) - \left(\tfrac{2}{3}\right) = \tfrac{19}{6}\,,$$

and so the total area is

$$A = A_1 + A_2 = \tfrac{4}{3} + \tfrac{19}{6} = \tfrac{27}{6} = \tfrac{9}{2}\,.$$

For the second solution we shall interchange the roles of x and y, thinking of x as a function of y. The line is given by $x = y + 1$ and the parabola is given by $x = y^2 - 1$, so the region R is given by $-1 \le y \le 2$, $y^2 - 1 \le x \le y + 1$, and hence the area is

$$\begin{aligned} A &= \int_{y=-1}^{2} (y+1) - (y^2 - 1)\,dy = \int_{-1}^{2} -y^2 + y + 2\,dy = \left[-\tfrac{1}{3}\,y^3 + \tfrac{1}{2}\,y^2 + 2y\right]_{-1}^{2} \\ &= \left(-\tfrac{8}{3} + 2 + 4\right) - \left(\tfrac{1}{3} + \tfrac{1}{2} - 2\right) = \tfrac{9}{2}\,. \end{aligned}$$

Volume by Cross-Section

3.7 Note: Suppose that a solid S lies in space between $x = a$ and $x = b$, and its cross-sectional area at x (that is the area of the intersection of the solid with the plane perpendicular to the x-axis at the position x) is equal to $A(x)$, where A is integrable on $[a, b]$. We can approximate the volume of S as follows. Choose a partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

of $[a, b]$ and choose sample points $c_k \in [x_{k-1}, x_k]$. Divide the solid into strips where the k^{th} strip lies between $x = x_{k-1}$ and $x = x_k$ and has thickness $\Delta_k x = x_k - x_{k-1}$. The volume of the k^{th} strip is

$$\Delta_k V \cong A(c_k)\Delta_k x$$

and the total volume of S is

$$V = \sum_{k=1}^{n} \Delta_k V \cong \sum_{k=1}^{n} A(c_k)\Delta_k x\,.$$

We notice that the sum on the right is a Riemann sum for the function $A(x)$ on $[a, b]$, so we define the exact volume of S to be the limit of these Riemann sums.

3.8 Definition: Suppose that a solid S lies in space between $x = a$ and $x = b$, and that its cross-sectional area at x is equal to $A(x)$, where A is integrable on $[a, b]$. We define the **volume** of S to be

$$V = \int_a^b A(x)\,dx\,.$$

3.9 Example: Let f and g be integrable on $[a, b]$ with $0 \le f(x) \le g(x)$ for all $x \in [a, b]$. Let R be the region in the xy-plane given by

$$a \le x \le b\ ,\ \ f(x) \le y \le g(x)$$

and let S be the solid obtained by revolving R about the y-axis. Then the cross-section of S at position x is an annulus (that is the region between two concentric circles) with inner radius $f(x)$ and outer radius $g(x)$, so the cross-sectional area is

$$A(x) = \pi g(x)^2 - \pi f(x)^2\,.$$

Thus the volume of S is

$$V = \int_a^b \pi\big(g(x)^2 - f(x)^2\big)\ dx\,.$$

3.10 Example: Find the volume of a cone of base radius r and height h.

Solution: Such a cone (lying on its side) can be obtained by revolving the triangular region R given by $0 \le x \le h$, $0 \le y \le \frac{r}{h}\,x$ about the x-axis, so the volume is

$$V = \int_0^h \pi\left(\tfrac{r}{h}\,x\right)^2\ dx = \tfrac{\pi\, r^2}{h^2}\left[\tfrac{1}{3}\,x^3\right]_0^h = \tfrac{\pi\, r^2}{h^2}\left(\tfrac{1}{3}\,h^3\right) = \tfrac{1}{3}\,\pi\, r^2 h\,.$$

3.11 Example: Find the volume of a sphere of radius r.

Solution: One half of such a sphere can be obtained by revolving the region R given by $0 \le x \le r$, $0 \le y \le \sqrt{r^2 - x^2}$ about the x-axis, and so the volume is

$$V = 2\int_0^r \pi(r^2 - x^2)\,dx = 2\pi\Big[r^2 x - \tfrac{1}{3}\,x^3\Big]_0^r = 2\pi\left(r^3 - \tfrac{1}{3}\,r^3\right) = \tfrac{4}{3}\,\pi\, r^3\,.$$

3.12 Example: Find the volume of the football-shaped solid S which is obtained by revolving the region R which lies under one arch of the sine curve about the x-axis.

Solution: The region R is given by $0 \leq x \leq \pi$, $0 \leq y \leq \sin x$ and so the volume is

$$V = \int_0^\pi \pi \sin^2 x \, dx = \int_0^\pi \pi \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) dx = \pi \Big[\tfrac{1}{2}\,x - \tfrac{1}{4}\sin 2x\Big]_0^\pi = \tfrac{\pi^2}{2}\,.$$

3.13 Example: Let R be the (infinitely long) region given by $1 \leq x$, $0 \leq y \leq \frac{1}{x}$, and let S be the solid obtained by revolving R about the x-axis.

Solution: The area of the region R is

$$A = \int_1^\infty \frac{1}{x}\, dx = \Big[\ln x\Big]_1^\infty = \infty$$

because $\lim\limits_{x\to\infty} \ln x = \infty$. The volume of the solid, on the other hand, is

$$V = \int_1^\infty \pi \cdot \frac{1}{x^2}\, dx = \pi \left[-\frac{1}{x}\right]_1^\infty = \pi$$

because $\lim\limits_{x\to\infty} \frac{1}{x} = 0$.

3.14 Remark: The above example gives rise to the following amusing paradox. It would require an infinite amount of paint to cover the region R but only a finite amount of paint to fill the solid S. But surely if we fill S with paint we have also covered R with paint! The resolution to this paradox lies in the fact that our calculation holds for mathematical paint which can flow down into an arbitrarily small tube.

3.15 Example: Find the volume of the solid S given by $x^2 + y^2 \leq r^2$, $x^2 + z^2 \leq r^2$ (this is the intersection of two cylinders).

Solution: To find the cross-section at x (where $-r \leq x \leq r$) we treat x as a fixed constant, and then the cross-section is given by $y^2 \leq r^2 - x^2$ and $z^2 \leq r^2 - x^2$, or equivalently by $|y| \leq \sqrt{r^2 - x^2}$ and $|z| \leq \sqrt{r^2 - x^2}$. Thus we see (somewhat surprisingly) that the cross-section at x is the square given by $|y| \leq \sqrt{r^2 - x^2}$ and $|z| \leq \sqrt{r^2 - x^2}$. This square has sides of length $2\sqrt{r^2 - x^2}$ so the cross-sectional area is

$$A(x) = 4(r^2 - x^2)\,.$$

Thus the volume of S is

$$V = \int_{-r}^r A(x)\, dx = \int_{-r}^r 4(r^2 - x^2)\, dx = 4\Big[r^2\, x - \tfrac{1}{3}\, x^3\Big]_{-r}^r = 4\left(\tfrac{2}{3}\, r^3 - \left(-\tfrac{2}{3}\, r^3\right)\right) = \tfrac{16}{3}\, r^3\,.$$

Volume by Cylindrical Shells

3.16 Note: Suppose that f and g are integrable on $[a,b]$ with $f(x) \le g(x)$ for all $x \in [a,b]$, let R be the region in the xy-plane given by

$$a \le x \le b\ ,\ \ f(x) \le y \le g(x)\,,$$

and let S be the solid obtained by revolving R about the y-axis. We can approximate the volume of S as follows. We choose a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$, and we choose intermediate points $c_k \in [x_{k-1}, x_k]$. We divide the region into strips where the k^{th} strip R_k is given by $x_{k-1} \le x \le x_k$, $f(x) \le y \le g(x)$. We divide the solid into "cylindrical shells" where the k^{th} shell S_k is obtained by revolving R_k about the y-axis. The volume of the k^{th} shell is

$$\Delta_k V \cong 2\pi\, c_k \big(g(c_k) - f(c_k)\big) \Delta_k x$$

where $\Delta_k x = x_k - x_{k-1}$. The total volume of S is

$$S = \sum_{k=1}^{n} \Delta_k V \cong \sum_{k=1}^{n} 2\pi\, c_k \big(g(c_k) - f(c_k)\big) \Delta_k x\,.$$

The sum on the right is a Riemann sum for the function $2\pi\, x\big(g(x) - f(x)\big)$ on $[a,b]$.

3.17 Definition: Suppose that f and g are integrable on $[a,b]$ with $f(x) \le g(x)$ for all $x \in [a,b]$, let R be the region in the xy-plane given by

$$a \le x \le b\ ,\ \ f(x) \le y \le g(x)\,,$$

and let S be the solid obtained by revolving R about the y-axis. We define the **volume** of S to be

$$V = \int_a^b 2\pi\, x\big(f(x) - g(x)\big)\, dx\,.$$

3.18 Example: Find the volume of a sphere of radius r.

Solution: One half of such a sphere can be obtained by revolving the region R given by $0 \le x \le r$, $0 \le y \le \sqrt{r^2 - x^2}$ about the y-axis, so the volume is

$$V = 2 \int_0^r 2\pi x \sqrt{r^2 - x^2}\ dx\,.$$

To solve the integral, we let $u = r^2 - x^2$ so that $du = -2x\,dx$, and we have

$$V = \int_0^r 4\pi x \sqrt{r^2 - x^2}\, dx = \int_{r^2}^0 -2\pi\, \sqrt{u}\ du = \Big[\tfrac{4}{3}\,\pi\, u^{3/2}\Big]_0^r = \tfrac{4}{3}\,\pi\, r^3\,.$$

3.19 Example: Find the volume of the discus-shaped solid S obtained by revolving the region R given by $0 \le x \le \frac{\pi}{2}$, $-\cos x \le y \le \cos x$ about the y-axis.

Solution: The volume is

$$V = \int_0^{\pi/2} 2\pi\, x\big(\cos x - (-\cos x)\big)\, dx = \int_0^{\pi/2} 4\pi\, x \cos x\ dx\,.$$

To solve the integral, we integrate by parts using $u = x$ and $dv = \cos x\, dx$ to get

$$V = 4\pi\left[x \sin x - \int \sin x\ dx\right]_0^{\pi/2} = 4\pi\Big[x \sin x + \cos x\Big]_0^{\pi/2} = 4\pi\big(\tfrac{\pi}{2} - 1\big) = 2\pi^2 - 4\pi\,.$$

3.20 Example: A bowl is in the shape of the surface obtained by revolving the part of the parabola $y = x^2$ with $0 \le x \le 2$ about the y-axis. Find the capacity of the bowl.

Solution: The capacity of the bowl is the volume of the liquid in the bowl when it is full. The liquid is in the shape of the solid S obtained by revolving the region R given by $0 \le x \le 2$, $x^2 \le y \le 4$ about the y-axis. We find the volume in two ways. Using the method of cylindrical shells, we have

$$V = \int_{x=0}^{2} 2\pi\, x(4-x^2)\, dx = \pi \int_0^2 8x - 2x^3 \; dx = \pi \Big[4x^2 - \tfrac{1}{2}\, x^4 \Big]_0^2 = 8\pi\, .$$

For the second solution, we interchange the roles of x and y. Note that the region R is also given by $0 \le y \le 4, 0 \le x \le \sqrt{y}$. Using the method of cross-sections we obtain

$$V = \int_{y=0}^{4} \pi \big(\sqrt{y}\big)^2\, dy = \pi \int_0^4 y\, dy = \pi \Big[\tfrac{1}{2}\, y^2 \Big]_0^4 = 8\pi\, .$$

3.21 Example: Find the volume of the solid torus (that is the doughnut-shaped solid) S with inner radius $R - r$ and outer radius $R + r$, where $0 < r < R$.

Solution: Note that such a torus can be obtained by revolving the disc D given by

$$(x - R)^2 + y^2 \le r^2$$

about the y-axis. We find the volume in two ways. First we use the method of cylindrical shells. The disc D is given by

$$R - r \le x \le R + r \quad \text{and} \quad -\sqrt{r^2 - (x-R)^2} \le y \le \sqrt{r^2 - (x-R)^2}$$

so the volume of the taurus is

$$V = \int_{x=R-r}^{R+r} 2\pi\, x \cdot 2\sqrt{r^2 - (x-R)^2}\; dx\, .$$

To solve this integral we let $r \sin\theta = (x - R)$ so that $r\cos\theta = \sqrt{r^2 - (x-R)^2}$ and $r\cos\theta\, d\theta = dx$ to get

$$\begin{aligned} V &= \int_{x=R-r}^{R+r} 4\pi\, x\sqrt{r^2-(x-R)^2}\; dx = \int_{\theta=-\pi/2}^{\pi/2} 4\pi \cdot (R + r\sin\theta)\cdot r\cos\theta \cdot r\cos\theta\, d\theta \\ &= 4\pi\, r^2 \int_{-\pi/2}^{\pi/2} R\cos^2\theta + r\sin\theta\cos^2\theta \; d\theta = 4\pi\, r^2 \int_{-\pi/2}^{\pi/2} R\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right) + r\sin\theta\cos^2\theta \; d\theta \\ &= 4\pi\, r^2 \Big[R\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right) + r\cdot\tfrac{1}{3}\cos^3\theta \Big]_{-\pi/2}^{\pi/2} = 4\pi\, r^2 \cdot R\left(\tfrac{\pi}{4} + \tfrac{\pi}{4}\right) = 2\pi^2\, r^2 R\, . \end{aligned}$$

For the second solution, we interchange the roles of x and y and use the cross-section method. The disc D is given by

$$-r \le y \le r \quad \text{and} \quad R - \sqrt{r^2 - y^2} \le x \le R + \sqrt{r^2 - y^2}$$

and so the volume is

$$\begin{aligned} V &= \int_{y=-r}^{r} \pi\Big((R + \sqrt{r^2-y^2})^2 - (R - \sqrt{r^2-y^2})^2 \Big)\, dy = \int_{-r}^{r} 4\pi R\sqrt{r^2-y^2}\; dy \\ &= 4\pi R \cdot \tfrac{1}{2}\pi r^2 = 2\pi^2 r^2 R\, . \end{aligned}$$

We used the fact that $\displaystyle\int_{-r}^{r} \sqrt{r^2-y^2}\; dy$ measures the area of a semicircle.

Arclength

3.22 Note: Let f be differentiable on $[a,b]$ (or let f be differentiable in (a,b) and continuous on $[a,b]$). Let C be the curve $y=f(x)$ with $a\le x\le b$. We approximate the length of C as follows. Choose a partition $a=x_0<x_1<\cdots<x_n=b$ of $[a,b]$. Write $\Delta_k x=x_k-x_{k-1}$ and $\Delta_k y=f(x_k)-f(x_{k-1})$. By the Mean Value Theorem, we can choose sample points $c_k\in[x_{k-1},x_k]$ so that

$$f'(c_k)=\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}=\frac{\Delta_k y}{\Delta_k x}\,.$$

Let C_k be the part of the curve $y=f(x)$ with $x_{k-1}\le x\le x_k$, and let D_k be the line segment from $\big(x_{k-1},f(x_{k-1})\big)$ to $\big(x_k,f(x_k)\big)$. The length $\Delta_k L$ of C_k is approximately equal to the length of D_k, that is

$$\Delta_k L\cong\sqrt{(\Delta_k x)^2+(\Delta_k y)^2}=\sqrt{1+\left(\frac{\Delta_k y}{\Delta_k x}\right)^2}\cdot\Delta_k x=\sqrt{1+f'(c_k)^2}\cdot\Delta_k x$$

and so the total length of C is

$$L=\sum_{k=1}^{n}L_k\cong\sum_{k=1}^{n}\sqrt{1+f'(c_k)^2}\cdot\Delta_k x\,.$$

The sum on the right is a Riemann sum for the function $\sqrt{1+f'(x)^2}$ on $[a,b]$.

3.23 Definition: Let f be differentiable on $[a,b]$ (or let f be differentiable in (a,b) and continuous on $[a,b]$). We define the **length** (or the **arclength**) of the curve $y=f(x)$ from $x=a$ to $x=b$ to be

$$L=\int_a^b\sqrt{1+f'(x)^2}\;dx\,.$$

We say that f is **rectifiable** when the length L is finite.

3.24 Example: Find the length of the curve $y=x^2$ with $0\le x\le 2$.

Solution: Let $f(x)=x^2$ so $f'(x)=2x$. The length of the curve is

$$L=\int_0^2\sqrt{1+f'(x)^2}\;dx=\int_0^2\sqrt{1+4x^2}\;dx\,.$$

To solve the integral, let $\tan\theta=2x$ so $\sec\theta=\sqrt{1+4x^2}$ and $\sec^2\theta\,d\theta=2\,dx$ to get

$$\begin{aligned}\int\sqrt{1+4x^2}\;dx&=\int\tfrac12\sec^3\theta\;d\theta=\tfrac14\sec\theta\tan\theta+\tfrac14\ln\big|\sec\theta+\tan\theta\big|+c\\&=\tfrac12\,x\sqrt{1+4x^2}+\tfrac14\ln\big|2x+\sqrt{1+4x^2}\big|+c\end{aligned}$$

so that

$$L=\int_0^2\sqrt{1+4x^2}\;dx=\Big[\tfrac12\,x\sqrt{1+4x^2}+\tfrac14\ln\big(2x+\sqrt{1+4x^2}\big)\Big]_0^2=\sqrt{17}+\tfrac14\ln\big(4+\sqrt{17}\big)\,.$$

Surface Area

3.25 Note: The area of (the lateral surface of) a cone of base radius r and slant height l is given by $A = \pi r l$. More generally, the area of a slice of a cone with base radius r, top radius s, and slant height l, is given by

$$A = \pi(r+s)\,l\,.$$

3.26 Note: Let f be differentiable on $[a,b]$ (or let f be differentiable in (a,b) and continuous on $[a,b]$). Let C be the curve in the xy-plane given by $y=f(x)$ with $a \le x \le b$. Let S be the surface obtained by revolving C about the x-axis. We can approximate the area of the surface S as follows. Choose a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$. Write $\Delta_k x = x_k - x_{k-1}$ and $\Delta_k y = f(x_k) - f(x_{k-1})$. Use the Mean Value Theorem to select $c_k \in [x_{k-1}, x_k]$ so that $f'(c_k) = \dfrac{\Delta_k y}{\Delta_k x}$. Let C_k be the part of the curve C with $x_{k-1} \le x \le x_k$, and let S_k denote the slice of the surface S which is obtained by revolving C_k about the x-axis. Let D_k be the line segment from $\big(x_{k-1}, f(x_{k-1})\big)$ to $\big(x_k, f(x_k)\big)$ and let T_k be the slice of a cone obtained by revolving D_k about the x-axis. The area $\Delta_k A$ of the slice S_k is approximately equal to the area of T_k, that is

$$\begin{aligned}\Delta_k A &\cong \pi\big(f(x_{k-1}) + f(x_k)\big)\Delta_k L \\ &= \pi\big(f(x_{k-1}) + f(x_k)\big)\sqrt{1+f'(c_k)^2}\;\Delta_k x \\ &\cong 2\pi f(c_k)\sqrt{1+f'(c_k)^2}\;\Delta_k x\,.\end{aligned}$$

The final sum is a Riemann sum for $2\pi f(x)\sqrt{1+f'(x)^2}$ on $[a,b]$.

3.27 Definition: Let f be differentiable on $[a,b]$ (or let f be differentiable in (a,b) and continuous on $[a,b]$). Let C be the curve given by $y=f(x)$ with $a \le x \le b$. Let S be the surface obtained by revolving C about the x-axis. We define the **area** of the surface S to be

$$A = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\;dx\,.$$

3.28 Note: A similar argument to the one given above shows that we can approximate the area of the surface S obtained by revolving the curve C given by $y=f(x)$ with $a \le x \le b$ about the y-axis by a Riemann sum for the function $2\pi\, x\sqrt{1+f'(x)^2}$ on $[a,b]$.

3.29 Definition: Let f be differentiable on $[a,b]$ (or let f be differentiable in (a,b) and continuous on $[a,b]$). Let C be the curve given by $y=f(x)$ with $a \le x \le b$. Let S be the surface obtained by revolving C about the y-axis. We define the **area** of the surface S to be

$$A = \int_a^b 2\pi\, x\sqrt{1+f'(x)^2}\;dx\,.$$

3.30 Example: Find the area of a sphere of radius r.

Solution: Such a sphere can be obtained by revolving the portion of the curve $y=\sqrt{r^2-x^2}$ with $-r\leq x\leq r$ about the x-axis. Let $f(x)=\sqrt{r^2-x^2}$ so that $f'(x)=-\dfrac{x}{\sqrt{r^2-x^2}}$ and

$$\sqrt{1+f'(x)^2}=\sqrt{1+\frac{x^2}{r^2-x^2}}=\sqrt{\frac{r^2}{r^2-x^2}}=\frac{r}{\sqrt{r^2-x^2}}\,.$$

Using the first of the above two definitions, the surface area is

$$A=\int_{-r}^{r}2\pi\,f(x)\sqrt{1+f'(x)^2}\;dx=\int_{-r}^{r}2\pi\sqrt{r^2-x^2}\cdot\frac{r}{\sqrt{r^2-x^2}}\;dx=\int_{-r}^{r}2\pi\,r\;dx=4\pi\,r^2\,.$$

Alternatively, we can obtain half of such a sphere by revolving the curve $y=\sqrt{r^2-x^2}$ with $0\leq x\leq r$ about the y-axis. Using the second of the above two definitions, the area of the sphere is

$$A=2\int_{x=0}^{r}2\pi\,x\sqrt{1+f'(x)^2}\;dx=\int_{0}^{r}4\pi\,x\cdot\frac{r}{\sqrt{r^2-x^2}}\;dx\,.$$

To solve the integral, we let $u=r^2-x^2$ so $du=-2x\,dx$ to get

$$A=\int_{x=0}^{r}\frac{4\pi r\,x}{\sqrt{r^2-x^2}}\,dx=\int_{u=r^2}^{0}-2\pi\,r\,u^{-1/2}\;du=\Big[-4\pi\,r\,u^{1/2}\Big]_{r^2}^{0}=4\pi\,r^2\,.$$

3.31 Example: Find the area of a torus of inner radius $R-r$ and outer radius $R+r$.

Solution: Half of such a torus can be obtained by revolving the curve $y=\sqrt{r^2-(R-x)^2}$ with $R-r\leq x\leq R+r$ about the y-axis. Let $f(x)=\sqrt{r^2-(x-R)^2}$. Then we have

$$f'(x)=\frac{-(x-R)}{\sqrt{r^2-(x-R)^2}}$$

so

$$\sqrt{1+f'(x)^2}=\sqrt{1+\frac{(x-R)^2}{r^2-(x-R)^2}}=\sqrt{\frac{r^2}{r^2-(x-R)^2}}=\frac{r}{\sqrt{r^2-(x-R)^2}}$$

and so, using the second of the above two definitions, the surface area is

$$A=2\int_{R-r}^{R+r}2\pi\,x\cdot\frac{r}{\sqrt{r^2-(x-R)^2}}\;dx=4\pi\,r\int_{R-r}^{R-r}\frac{x\,dx}{\sqrt{r^2-(x-R)^2}}\,.$$

To solve the integral, make the substitution $r\sin\theta=x-R$ so $r\cos\theta=\sqrt{r^2-(x-R)^2}$ and $r\cos\theta\,d\theta=dx$. Then we obtain

$$\begin{aligned}A&=4\pi\,r\int_{x=R-r}^{R+r}\frac{x\,dx}{\sqrt{r^2-(x-R)^2}}=4\pi\,r\int_{\theta=-\pi/2}^{\pi/2}\frac{(R+r\sin\theta)\cdot r\cos\theta\,d\theta}{r\cos\theta}\\&=4\pi\,r\int_{-\pi/2}^{\pi/2}R+r\sin\theta\;d\theta=4\pi\,r\Big[R\theta-r\cos\theta\Big]_{-\pi/2}^{\pi/2}=4\,\pi^2 r\,R\,.\end{aligned}$$

Mass and Density

3.32 Example: Suppose a rod lies along the x-axis from $x = a$ to $x = b$, and the linear density (that is mass per unit length) of the rod is equal to $\rho(x)$, where $\rho(x)$ is integrable on $[a.b]$. We can approximate the mass of the rod as follows. Choose a partition $a = x_0 < x_1 < \cdots < x_n = b$ and choose sample points $c_k \in [x_{k-1}, x_k]$. The mass of the part of the rod between $x = x_{k-1}$ and $x = x_k$ is

$$\Delta_k M \cong \rho(c_k)\Delta_k x$$

and so the total mass of the rod is

$$M = \sum_{k=1}^{n} \Delta_k M \cong \sum_{k=1}^{n} \rho(c_k)\Delta_k x \,.$$

The sum on the right is a Riemann sum for the function $\rho(x)$. The exact mass of the rod is the limit of these Riemann sums, that is

$$M = \int_a^b \rho(x)\, dx \,.$$

3.33 Example: Suppose that a ball of radius R has varying density, and the density at each point which lies at a distance of r units from the origin is equal to $\rho(r)$, where we suppose that ρ is integrable on $[0, R]$. We can approximate the mass of the ball as follows. Choose a partition $0 = r_0 < r_1 < \cdots < r_n = R$ of the interval $[0, R]$, and choose sample points $c_k \in [r_{k-1}, r_k]$. Divide the sphere into spherical shells using concentric spheres of radius r_k. The volume of the k^{th} spherical shell is $\Delta_k V \cong 4\pi {c_k}^2 \Delta_k r$ so its mass is

$$\Delta_k M \cong \rho(c_k)\Delta_k V \cong 4\pi {c_k}^2 \rho(c_k)\Delta_k r \,.$$

The total mass of the ball is

$$M = \sum_{k=1}^{n} \Delta_k M \cong \sum_{k=1}^{n} 4\pi {c_k}^2 \rho(c_k)\Delta_k r \,.$$

This is a Riemann sum, and the exact mass of the ball is the limit of these Riemann sums, that is

$$M = \int_a^b 4\pi r^2 \rho(r)\; dr \,.$$

Force

3.34 Example: A tank is in the shape of the parabolic sheet given by $y = x^2$, $-2 \le x \le 2$, $-5 \le z \le 5$ together with the two ends given by $-2 \le x \le 2$, $x^2 \le y \le 4$ with $z = \pm 5$ (where the y-axis is pointing upwards). The tank is filled with a liquid of density ρ. The pressure $P(h)$ (force per unit area) exerted by the liquid on each wall at all points which lie at a depth h is given by

$$P = \rho g h$$

where g is the gravitational constant. Find the total force exerted by the liquid on each of the ends of the tank.

Solution: We provide a less formal solution than we gave in previous examples. Although we make no mention of a Riemann sum, it should be apparent that we are in fact approximating the total force by a Riemann sum and then calculating the exact force as a limit of Riemann sums. Along one of the ends of the tank, consider a thin horizontal slice at position y of thickness Δy. The slice is at a depth of $h = 4 - y$ so the pressure at all points is $P = \rho g h = \rho g(4-y)$. The width of the slice is equal to $2\sqrt{y}$, so the area of the slice is $\Delta A = 2\sqrt{y}\,\Delta y$, and so the force exerted by the water on the slice is

$$\Delta F = P\,\Delta A = \rho g(4-y)\cdot 2\sqrt{y}\,\Delta y\,.$$

The total force exerted on the end of the tank is

$$\begin{aligned} F &= \int_{y=0}^{4} \rho g(4-y)\cdot 2\sqrt{y}\;dy \;=\rho g\int_0^4 8\,y^{1/2} - 2\,y^{3/2}\,dy \\ &= \rho g\Big[\tfrac{16}{3}\,y^{3/2} - \tfrac{4}{5}\,y^{5/2}\Big]_0^4 = \rho g\left(\tfrac{128}{3} - \tfrac{128}{5}\right) = \tfrac{256}{15}\,\rho g\,. \end{aligned}$$

3.35 Example: A charged rod, of charge Q (with its charge evenly distributed along its length) lies along the x-axis from $x=0$ to $x=2$. A small object of charge q lies at position $(x,y) = (2,1)$. Find the force exerted by the rod on the object. Use the fact that the force exerted by one small object of charge q_1 at position p_1 on another of charge q_2 at position p_2 is equal to

$$F = \frac{k\,q_1q_2}{|u|^2}\cdot\frac{u}{|u|}$$

where k is a constant and u is the direction vector from p_1 to p_2, that is $u = p_2 - p_1$.

Solution: Again, we provide a less formal solution, making no mention of Riemann sums. Consider a small slice of rod at position x of thickness Δx. Since the rod has length 2, the charge per unit length is $\frac{Q}{2}$ and so the charge on the slice of rod is $\Delta Q = \frac{Q}{2}\,\Delta x$. The distance from the slice, which is at position $(x,0)$ to the small object, which is at position $(2,1)$, is equal to $r = |u| = \sqrt{(2-x)^2+1}$ and so the magnitude of the force exerted by the slice on the object is

$$\Delta F = \frac{kq\cdot\frac{Q}{2}\,\Delta x}{(2-x)^2+1}\,.$$

By similar triangles, the x and y-components of the force, exerted by the slice of rod on the object, are given by

$$\begin{aligned} \Delta F_x &= \frac{2-x}{\sqrt{(2-x)^2+1}}\;\Delta F = \frac{kQ\,(2-x)\,\Delta x}{2\left((2-x)^2+1\right)^{3/2}} \\ \Delta F_y &= \frac{1}{\sqrt{(2-x)^2+1}}\;\Delta F = \frac{kqQ\,\Delta x}{2\left((2-x)^2+1\right)^{3/2}}\,. \end{aligned}$$

The x-component of the total force is

$$F_x = \int_{x=0}^{2} \frac{kqQ\,(2-x)\,dx}{2\left((2-x)^2+1\right)^{3/2}}\,.$$

To solve the integral, we let $u = (2-x)^2+1$ so that $du = -2(2-x)\,dx$ to get

$$\begin{aligned} F_x &= \int_{x=0}^{2} \frac{kqQ\,(2-x)\,dx}{2\left((2-x)^2+1\right)^{3/2}} = \int_{u=5}^{1} \frac{-kqQ\cdot\frac{1}{2}\,du}{2\,u^{3/2}} = kqQ\int_{5}^{1} -\tfrac{1}{4}\,u^{-3/2}\ du \\ &= kqQ\Big[\tfrac{1}{2}\,u^{-1/2}\Big]_5^1 = \tfrac{1}{2}\,kqQ\big(1-\tfrac{1}{\sqrt{5}}\big)\,. \end{aligned}$$

The y-component of the total force is

$$F_y = \int_{x=0}^{2} \frac{kqQ\ dx}{2\left((2-x)^2+1\right)^{3/2}}\,.$$

To solve this integral, we let $\tan\theta = 2-x$ so $\sec\theta = \sqrt{(2-x)^2+1}$ and $\sec^2\theta\,d\theta = -dx$. Then

$$\begin{aligned} \int \frac{kqQ\ dx}{2\left((2-x)^2+1\right)^{3/2}} &= \int \frac{-kqQ\sec^2\theta\,d\theta}{2\sec^3\theta} = \int -\tfrac{1}{2}\,kqQ\cos\theta\ d\theta \\ &= -\tfrac{1}{2}\,kqQ\sin\theta + c = -\tfrac{1}{2}\,kqQ\cdot\tfrac{2-x}{\sqrt{(2-x)^2+1}} + c \end{aligned}$$

and so

$$F_y = \int_{x=0}^{2} \frac{kqQ\ dx}{2\left((2-x)^2+1\right)^{3/2}} = \Big[-\tfrac{1}{2}\,kqQ\cdot\tfrac{2-x}{\sqrt{(2-x)^2+1}}\Big]_0^2 = \tfrac{1}{2}\,kqQ\cdot\tfrac{2}{\sqrt{5}}\,.$$

Thus the total force exerted by the rod on the object, expressed as a vector, is

$$F = \big(F_x, F_y\big) = \tfrac{1}{2}\,kqQ\Big(1-\tfrac{1}{\sqrt{5}}\,,\ \tfrac{2}{\sqrt{5}}\Big)\,.$$

Work

3.36 Example: A tank is in the shape of the parabolic sheet given by $y = x^2$, $-2 \le x \le 2$, $-5 \le z \le 5$ together with the two ends given by $-2 \le x \le 2$, $x^2 \le y \le 4$ with $z = \pm 5$ (where the y-axis is pointing vertically). The tank is filled with a liquid of density ρ. Find the work required to pump all the liquid out of the tank, bringing it all to the level of the top of the tank. Use the fact that the work required to raise a small object of mass m from height h_1 to height h_2 is equal to

$$W = mgh$$

where $h = h_2 - h_1$.

Solution: We provide an informal solution. Consider a thin slice of liquid at position y of thickness Δy. The slice is in the shape of a thin rectangle of length $l = 10$, width $w = 2\sqrt{y}$ and thickness Δy, so its volume is $\Delta V = 20\sqrt{y}\,\Delta y$, and so its mass is given by $\Delta M = \rho \Delta V = 20\rho\sqrt{y}\,\Delta y$. All the water in this slice must be raised from height $h_1 = 4 - y$ to height $h_2 = 4$, and so the work done in pumping the water in this slice is

$$\Delta W = gh\,\Delta M = 20\rho g\,(4-y)\sqrt{y}\,\Delta y\,.$$

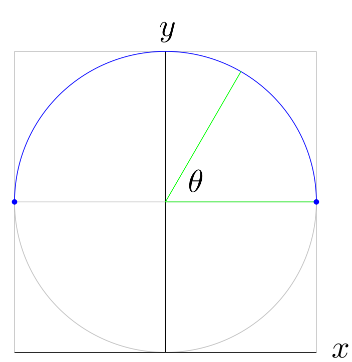
The total work required to pump all the water in the tank is

$$\begin{aligned} W &= \int_{y=0}^{4} 20\rho g\,(4-y)\sqrt{y}\;dy = 20\rho g \int_0^4 4\,y^{1/2} - y^{3/2}\;dy \\ &= 20\rho g\Big[\tfrac{8}{3}\,y^{3/2} - \tfrac{2}{5}\,y^{5/2}\Big]_0^4 = 20\rho g\big(\tfrac{64}{3} - \tfrac{64}{5}\big) = \tfrac{2560}{3}\,\rho g\,. \end{aligned}$$

3.37 Example: A chain, of length π and mass M, lies along the x-axis. Find the work required to lift the chain and lie it along the top half of the circle $x^2 + (y-1)^2 = 1$ (where the y-axis points upwards).

Solution: Let θ be as shown below. For a thin slice of the chain (when it is lying on the top half of the circle) at position θ of thickness $\Delta\theta$, the mass of the slice is $\Delta M = \frac{M}{\pi}\Delta\theta$, and the height of the slice above the x-axis is $y = 1 + \sin\theta$, so the work done in lifting the slice from the x-axis is $\Delta W = gy\,\Delta M = \frac{gM}{\pi}(1+\sin\theta)\Delta\theta$. The total work is

$$W = \int_{\theta=0}^{\theta=\pi} \tfrac{gM}{\pi}(1+\sin\theta)\,d\theta = \tfrac{gM}{\pi}\Big[\theta - \cos\theta\Big]_{\theta=0}^{\pi} = \tfrac{gM}{\pi}(\pi+2)\,.$$



Chapter 4. Parametric and Polar Curves

Parametric Curves

4.1 Definition: For a function $f : I \subseteq \mathbf{R} \to \mathbf{R}^2$, where I is an interval in $\mathbf{R}$, we define the **graph** of f to be the set

$$\text{Graph}(f) = \left\{ (x, f(x)) \middle| x \in I \right\}.$$

When f is continuous, this set is a **curve** in $\mathbf{R}^2$, and we call it the curve defined **explicitly** by the equation $y = f(x)$, or simply the curve $y = f(x)$.

For a function $f : U \subseteq \mathbf{R}^2 \to \mathbf{R}$, where U is a connected set in $\mathbf{R}^2$, we define the **null set** of f to be the set

$$\text{Null(f)} = \left\{ (x, y) \in U \middle| f(x, y) = 0 \right\}.$$

When f is continuous, this set is typically a curve in $\mathbf{R}^2$, and we call it the curve defined **implicitly** by the equation $f(x, y) = 0$, or simply the curve $f(x, y) = 0$.

For a function $f : I \subseteq \mathbf{R} \to \mathbf{R}^2$ given by $f(t) = \big(x(t), y(t)\big)$, where I is an interval in $\mathbf{R}$, we define the **range** (or **image**) of f to be the set

$$\text{Range}(f) = \left\{ f(t) \middle| t \in I \right\} = \left\{ \big(x(t), y(t)\big) \middle| t \in I \right\}.$$

When $x(t)$ and $y(t)$ are continuous, this set is typically a curve, and we call it the curve defined **parametrically** by the equation $p = (x, y) = f(t)$ $\big($or by the equations $x = x(t)$ and $y = y(t)\big)$, or simply the curve $p = f(t)$. The variable t is called the **parameter**.

4.2 Example: The circle of radius 1 centred at $(0, 0)$ is the curve defined implicitly by the equation

$$x^2 + y^2 = 1.$$

The top half of the circle is given explicitly by the equation

$$y = \sqrt{1 - x^2}\ ,\ -r \leq x \leq r.$$

The entire circle can also be described parametrically by the equation

$$(x, y) = \big(\cos t, \sin t\big)\ ,\ 0 \leq t \leq 2\pi.$$

4.3 Remark: These concepts can be generalized to higher dimensions. For example, the top half of the sphere of radius 1 centred at $(0, 0, 0)$ can be given explicitly by the equation $z = \sqrt{1 - x^2 - y^2}$, and the entire sphere can be given implicitly by $x^2 + y^2 + z^2 = 1$, and the entire sphere can be given parametrically by expressing the coordinates x, y and z of a point on the sphere in terms of the point's latitude ϕ and longitude θ; in mathematics it is common practice to take $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$ with $\phi = 0$ at the north pole and $\phi = \pi$ at the south pole, and then x, y and z are given by

$$(x, y, z) = \big(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi\big).$$

4.4 Example: Given two points $a, b \in \mathbf{R}^2$, the line segment from a to b is given parametrically by

$$p = f(t) = a + t(b - a)\ ,\ 0 \leq t \leq 1.$$

4.5 Example: An arc of the circle of radius r centred at (a, b) can be given parametrically by

$$(x, y) = \big(a + r\cos t, b + r\sin t\big) \text{ with } \alpha \leq t \leq \beta.$$

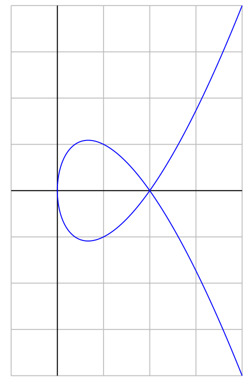
Taking $\alpha = 0$ and $\beta = 2\pi$ yields the entire circle.

4.6 Note: We can always sketch a parametric curve simply by making a table of values and plotting points.

4.7 Example: Sketch the the parametric curve $(x, y) = \big(t^2, t^3 - 2t\big)$.

Solution: We make a table of values and plot points. The points should be connected in order according to the value of t.

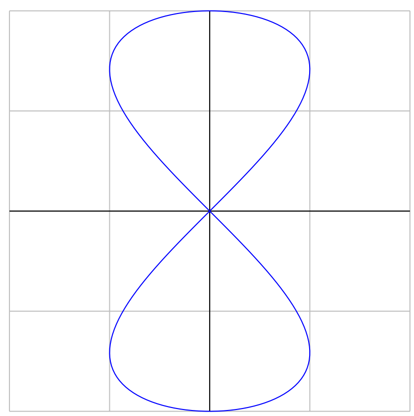
t	x	y
-2	4	-4
$-\sqrt{2}$	2	0
-1	1	1
0	0	0
1	1	-1
$\sqrt{2}$	2	0
2	4	4



4.8 Example: Sketch the parametric curve $(x, y) = \big(\sin 2t, 2\sin t\big)$.

Solution: We make stable of values and plot points.

t	x	y
0	0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	1
$\frac{\pi}{4}$	1	$\sqrt{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	0	2
$\frac{2\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{3\pi}{4}$	-1	$\sqrt{2}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	1
π	0	0



4.9 Definition: The curve $(x, y) = \big(t^2, t^3 - t\big)$ is called an **alpha curve** because it is shaped like the greek letter α. The curve $(x, y) = \big(\sin 2t, 2\sin t\big)$ is called a **figure eight curve**.

4.10 Note: Sometimes (but not always), given a parametric equation for a curve, we can eliminate the parameter (using some algebraic manipulations) to obtain an implicit or an explicit equation for the curve.

4.11 Example: Eliminate the parameter to find an implicit equation for each of the curves $(x, y) = (t^2 + 1, t^3)$ and $(x, y) = (\sin t, \sec t)$.

Solution: For the curve $(x, y) = (t^2 + 1, t^3)$, we have $(x - 1)^3 = (t^2)^3 = t^6 = y^2$, so the curve is given implicitly by $(x - 1)^3 = y^2$. For the curve $(x, y) = (\sin t, \sec t)$ we have $y^2 = \sec^2 t = \frac{1}{\cos^2 t} = \frac{1}{1-\sin^2 t} = \frac{1}{1-x^2}$ and so the curve is given implicitly by $y^2 = \frac{1}{1-x^2}$.

4.12 Example: Eliminate the parameter to find an implicit equation for the alpha curve $(x,y)=\big(t^2,t^3-2t\big)$ and for the figure eight curve $(x,y)=(\sin 2t,2\sin t)$.

Solution: For the alpha curve $(x,t)=\big(t^2,t^3-2t\big)$ we have

$$y^2=(t^3-2t)^2=(t^6-4t^4+4t^2=x^3-4x^2+4x=x(x-2)^2$$

and so this alpha curve is given implicitly by $y^2=x(x-2)^2$. For the figure eight curve $(x,y)=\big(\sin 2t,2\sin t\big)$ we have

$$\begin{aligned}x^2&=(\sin 2t)^2=(2\sin t\cos t)^2=4\sin^2 t\cos^2 t=4\sin^2 t(1-\sin^2 t)\\&=y^2\big(1-(y/2)^2\big)=\tfrac14\,y^2(4-y^2)=\tfrac14\,y^2(2-y)(2+y)\,,\end{aligned}$$

and so this figure eight curve is given implicitly by $x^2=\frac14\,y^2(2-y)(2+y)$.

4.13 Note: For a parametric curve $(x,y)=f(t)=\big(x(t),y(t)\big)$, when the variable t represents time, the point $f(t)=\big(x(t),y(t)\big)$ represents the position of a moving point.

4.14 Example: A **cycloid** is the curve followed by a point on a circle in the xy-plane which rolls, without slipping, along the x-axis. Find a parametric equation for a cycloid

Solution: Say the circle has radius r, it begins with its centre at position $(0,r)$, and it rolls in the direction of the positive x-axis at speed s, and say we are interested with the point on the circle initially at position $(0,0)$. At time t, the centre will be at position (st,r). Let $\theta=\theta(t)$ be the angle through which the circle has revolved about its centre at time t. Since the circle revolves at a constant rate, we have $\theta(t)=ct$ for some constant c. Since the circle rolls without slipping, it makes one full revolution about its centre when $x(t)=2\pi r$ (the circumference of the circle), so we have $\theta(t)=2\pi$ when $st=2\pi r$, that is $ct=2\pi$ when $t=2\pi\,r/s$, and so $c=s/r$. Since, at time t, the centre of the circle is at $(st,0)=(r\theta(t),r)$ and the circle has rotated (clockwise) by the angle $\theta(t)=\frac{s}{r}\,t$, it follows that the point on the circle which was originally at $(0,0)$ will have moved to the position

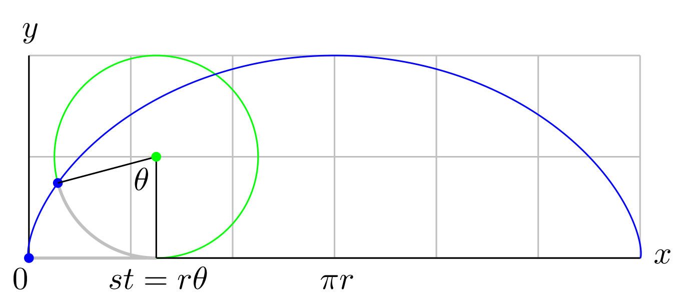
$$(x,y)=(r\,\theta(t),r)-\big(r\sin\theta(t),r\cos\theta(t)\big)\,.$$

We can use the angle θ as our parameter and write this as

$$(x,y)=\big(x(\theta),y(\theta)\big)=(r\theta,r)-(r\sin\theta,r\cos\theta)=r\big(\theta-\sin\theta,1-\cos\theta\big)$$

or we can use time t as our parameter and write

$$(x,y)=\big(x(t),y(t)\big)=(st,r)-\Big(r\sin\left(\tfrac{s}{r}\,t\right),r\cos\left(\tfrac{s}{r}\,t\right)\Big)$$



4.15 Definition: The **tangent vector** to the parametric curve $(x,y)=f(t)=\big(x(t),y(t)\big)$ at the point where $t=t_0$ is the vector

$$f'(t_0)=\big(x'(t_0),y'(t_0)\big)\,.$$

The **linearization** of f at $t=t_0$ is the function $L(t)$ given by

$$L(t)=f(t_0)+f'(t_0)(t-t_0)$$

and when $f'(t_0)\neq 0$, the **tangent line** to the curve $p=f(t)$ at the point $f(t_0)$ is the line given parametrically by

$$(x,y)=L(t)=f(t_0)+f'(t_0)(t-t_0)\,.$$

When t represents time and $f(t)$ represents the position of a moving point, the tangent vector $f'(t)=\big(x'(t),y'(t)\big)$ is also called the **velocity** of the moving point at time t. The **speed** of the moving point is the length of the velocity vector. We also define the **acceleration** of the moving point at time t to be the vector $f''(t)=\big(x''(t),y''(t)\big)$.

4.16 Example: Consider the alpha curve $(x,y)=\big(t^2,t^3-2t\big)$. Find an explicit equation for the tangent line at the point where $t=1$.

Solution: We have $\big(x(t),y(t)\big)=(t^2,t^3-2t)$ and so $\big(x'(t),y'(t)\big)=(2t,3t^2-2)$. When $t=1$ we have $(x,y)=(1,-1)$ and $(x',y')=(2,1)$, and so the tangent line is the line through $(1,-1)$ in the direction of the vector $(2,1)$. This line has slope $\frac{1}{2}$, and its equation is $y+1=\frac{1}{2}(x-1)$, that is $y=\frac{1}{2}\,x-\frac{3}{2}$.

4.17 Example: A small stone is stuck in the tread of the tire of a car. The tire has radius $r=0.25$ (in meters) and the car moves at speed $s=10$ (in meters per second). The stone moves along a cycloid with its position (in meters) at time t (in seconds) given by

$$(x,y)=\big(x(t),y(t)\big)=(st,r)-\left(r\sin\left(\tfrac{s}{r}\,t\right),r\cos\left(\tfrac{s}{r}\,t\right)\right).$$

Find the position, the velocity, and the speed of the stone at time $t=\pi/120$.

Solution: Put $r=\frac{1}{4}$ and $s=10$ into the parametric equations to get

$$\begin{aligned}(x,y)&=\big(10\,t,\tfrac{1}{4}\big)-\big(\tfrac{1}{4}\sin 40t,\tfrac{1}{4}\cos 40t\big)\\(x',y')&=\big(10,0\big)-\big(10\cos 40t,-10\sin 40t\big)\,.\end{aligned}$$

When $t=\frac{\pi}{120}$ the position, velocity and speed are

$$\begin{aligned}p=(x,y)&=\big(\tfrac{\pi}{12},\tfrac{1}{4}\big)-\big(\tfrac{\sqrt{3}}{8},\tfrac{1}{8}\big)=\big(\tfrac{\pi}{12}-\tfrac{\sqrt{3}}{8},\tfrac{1}{8}\big)\\v=(x',y')&=\big(10,0\big)-\big(5,-5\sqrt{3}\big)=(5,5\sqrt{3})\\|v|&=\sqrt{(x')^2+(y')^2}=\sqrt{25+75}=10\,.\end{aligned}$$

Is it surprising that the stone is moving at the same speed as the car?

4.18 Note: Consider the parametric curve $(x,y) = f(t) = \big(x(t), y(t)\big)$ with $r \le t \le s$. Suppose that we are able to eliminate the parameter to express the curve explicitly by $y = g(x)$. Then for all $t \in [r,s]$ we have

$$y(t) = g\big(x(t)\big)\,.$$

Taking the derivative (with respect to t) on both sides, we obtain $y'(t) = g'\big(x(t)\big)x'(t)$ and so

$$g'\big(x(t)\big) = \frac{y'(t)}{x'(t)}$$

whenever $x'(t) \ne 0$. This formula should come as no surprise because both sides measure the slope of the tangent line to the given curve at the point $\big(x(t), y(t)\big)$. Taking the derivative again, we obtain $g''\big(x(t)\big)x'(t) = \frac{d}{dt}\big(y'(t)/x'(t)\big)$, that is

$$g''\big(x(t)\big) = \frac{\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)}{x'(t)}$$

whenever $x'(t) \ne 0$. We could also obtain a formula for $g'''\big(x(t)\big)$ in terms of $x(t)$ and $y(t)$.

4.19 Example: Consider the figure-eight curve $(x,y) = \big(\sin 2t, 2\sin t\big)$. Suppose that the portion of the curve with $-\frac{\pi}{4} \le t \le \frac{\pi}{4}$ is given explicitly by $y = g(x)$ with $-1 \le x \le 1$. Find $g'\big(\frac{\sqrt{3}}{2}\big)$ and $g''\left(\frac{\sqrt{3}}{2}\right)$.

Solution: Note that for $-\frac{\pi}{4} \le t \le \frac{\pi}{4}$ we have

$$x(t) = \tfrac{\sqrt{3}}{2} \iff \sin 2t = \tfrac{\sqrt{3}}{2} \iff 2t = \tfrac{\pi}{3} \iff t = \tfrac{\pi}{6}\,.$$

Also, we have

$$\begin{aligned}
\big(x(t), y(t)\big) &= \big(\sin 2t, 2\sin t\big)\\
\big(x'(t), y'(t)\big) &= \big(2\cos 2t, 2\cos t\big)\\
g'\big(x(t)\big) &= \frac{y'(t)}{x'(t)} = \frac{\cos t}{\cos 2t}\\
g''\big(x(t)\big) &= \frac{\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)}{x'(t)} = \frac{\frac{d}{dt}\left(\frac{\cos t}{\cos 2t}\right)}{2\cos 2t} = \frac{-\sin t\cos 2t + 2\cos t\sin 2t}{2\cos^3 2t}\,.
\end{aligned}$$

Put in $t = \frac{\pi}{6}$ to get

$$\begin{aligned}
g'\left(\tfrac{\sqrt{3}}{2}\right) &= \frac{\cos\frac{\pi}{6}}{\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}\\
g''\left(\tfrac{\sqrt{3}}{2}\right) &= \frac{-\sin\frac{\pi}{6}\cos\frac{\pi}{3} + 2\cos\frac{\pi}{6}\sin\frac{\pi}{3}}{2\cos^3\frac{\pi}{3}} = \frac{-\frac{1}{2}\cdot\frac{1}{2} + 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}}{\frac{1}{4}} = 5\,.
\end{aligned}$$

4.20 Note: Consider the curve given parametrically by $(x,y)=f(t)=\big(x(t),y(t)\big)$ with $r\leq t\leq s$. Suppose that $y(t)\geq 0$ and $x'(t)\geq 0$ for all $t\in[r,s]$ and let $a=x(r)$ and $b=x(s)$. Note that $a\geq b$ since $x'(t)\geq 0$ for all t. Suppose that we can eliminate the parameter to express the curve explicitly by $y=g(x)$ with $a\leq x\leq b$. Note that for all $t\in[r,s]$ we have $y(t)=g\big(x(t)\big)$. Using the Substitution Rule, we obtain the following formulas.

The area of the region R given by $a\leq x\leq b$, $0\leq y\leq g(x)$ is

$$A=\int_{x=a}^{b} g(x)\,dx=\int_{t=r}^{s} g\big(x(t)\big)\,x'(t)\,dt=\int_{t=r}^{s} y(t)\,x'(t)\,dt\,,$$

the volume of the solid obtained by revolving R about the x-axis is

$$V=\int_{x=a}^{b}\pi\,g(x)^2\,dx=\int_{t=r}^{s}\pi\,y(t)^2x'(t)\,dt\,,$$

in the case that $a\geq 0$, the volume of the solid obtained by revolving R about the y-axis is

$$V=\int_{x=a}^{b}2\pi\,xg(x)\,dx=\int_{t=r}^{s}2\pi\,x(t)\,y(t)\,x'(t)\,dt\,,$$

the length of the curve C given by $y=g(x)$ with $a\leq x\leq b$ is

$$L=\int_{x=a}^{b}\sqrt{1+g'(x)^2}\;dx=\int_{t=r}^{s}\sqrt{1+\tfrac{y'(t)^2}{x'(t)^2}}\;x'(t)\,dt=\int_{t=r}^{s}\sqrt{x'(t)^2+y'(t)^2}\;dt\,,$$

the area of the surface obtained by revolving C about the x-axis is

$$A=\int_{x=a}^{b}2\pi\,g(x)\sqrt{1+g'(x)^2}\;dx=\int_{t=r}^{s}2\pi\,y(x)\sqrt{x'(t)^2+y'(t)^2}\;dt\,,$$

and when $a\geq 0$, the area of the surface obtained by revolving C about the y-axis is

$$A=\int_{x=a}^{b}2\pi\,x\sqrt{1+g'(x)^2}\;dx=\int_{t=r}^{s}2\pi\,x(t)\sqrt{x'(t)^2+y'(t)^2}\;dt\,.$$

4.21 Example: Consider the curve C given by $(x,y)=\big(t^2,t^3\big)$ with $0\leq t\leq 2$. Find the area of the region R which lies below C and above the x-axis between $x=0$ and $x=4$. Find the volume of the solid obtained by revolving R about the x-axis. Find the length of the curve C.

Solution: The required area, volume and length are

$$A=\int_0^2 y(t)\,x'(t)\,dt=\int_0^2 t^3\cdot 2t\;dt=\int_0^2 2t^4\;dt=\Big[\tfrac{2}{5}\,t^5\Big]_0^2=\tfrac{64}{5}$$

$$V=\int_{t=0}^{2}\pi\,y(t)^2\,x'(t)\,dt=\int_0^2\pi\,t^6\cdot 2t\;dt=\int_0^2 2\pi\,t^7\;dt=\Big[\tfrac{\pi}{4}\,t^8\Big]_0^2=64\pi$$

$$L=\int_{t=0}^{2}\sqrt{x'(t)^2+y'(t)^2}\;dt=\int_0^2\sqrt{(2t)^2+(3t^2)^2}\;dt=\int_0^2 t\sqrt{4+9t^2}\;dt\,.$$

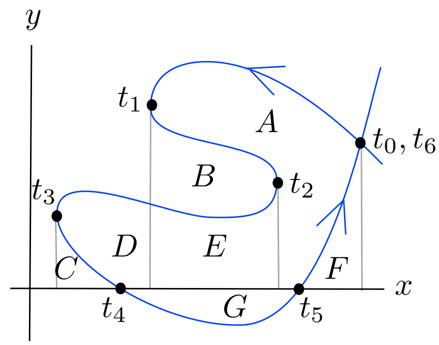
To solve the last integral, let $u=4+9t^2$ so that $du=18t\,dt$. Then the length is

$$L=\int_{t=0}^{2}t\sqrt{4+9t^2}\;dt=\int_{u=4}^{40}\tfrac{1}{18}\,u^{1/2}\,du=\Big[\tfrac{1}{27}\,u^{3/2}\Big]_4^{40}=\tfrac{1}{27}\big(40^{3/2}-8\big)\,.$$

4.22 Note: We can obtain similar formulas to the ones above in the case that $y(t) \leq 0$ or in the case that $x'(t) \leq 0$. For example, when $y(t) \geq 0$ and $x'(t) \leq 0$ for all $t \in [r,s]$, if we let $a = x(s)$ and $b = x(r)$ so that $a \leq b$, then the area of the region R given by $a \leq x \leq b$, $0 \leq y \leq g(x)$ is equal to

$$A = \int_{x=a}^{b} g(x)\,dx = \int_{t=s}^{r} y(t)\,x'(t)\,dt = -\int_{t=r}^{s} y(t)\,x'(t)\,dt\,.$$

4.23 Example: Consider the parametric curve $(x,y) = \big(x(t), y(t)\big)$ with $r \leq t \leq s$ whose image is shown below. Let $r = t_0 < t_1 < \cdots < t_6 = s$ be the values of t corresponding to the indicated points, and let A, B, C, D, E, F and G be the areas of the indicated regions.



For $t_0 \leq t \leq t_1$ we have $y(t) \geq 0$ and $x'(t) \leq 0$ so the integral $\int_{t_0}^{t_1} y(y)\,x'(t)\,dt$ measures the *negative* of the area under the curve, that is $\int_{t_0}^{t_1} y(y)\,x'(t)\,dt = -(A+B+E+F)\,.$ For $t_1 \leq t \leq t_2$ we have $y(t) \geq 0$ and $x'(t) \geq 0$ so the integral $\int_{t_1}^{t_2} y(y)\,x'(t)\,dt$ measures the positive area under the curve, that is $\int_{t_1}^{t_2} y(y)\,x'(t)\,dt = (B+E)\,.$ Similarly, we find that $\int_{t_2}^{t_3} y(t)\,x'(t)\,dt = -(C+D+E)$, $\int_{t_3}^{t_4} y(t)\,x'(t)\,dt = C$, $\int_{t_4}^{t_5} y(t)\,x'(t)\,dt = -G$, and $\int_{t_5}^{t_6} y(t)\,x'(t)\,dt = F$. Thus we have

$$\begin{aligned}\int_r^s y(t)\,x'(t)\,dt &= -(A+B+E+F)+(B+E)-(C+D+E)+(C)-(G)+(F)\\ &= -(A+D+E+G)\end{aligned}$$

so the integral measures the negative of the area inside the loop. If the loop had been traversed in the opposite direction (clockwise instead of anticlockwise) the integral would have given the positive area inside the loop.

4.24 Example: Consider the alpha curve given by $(x,y) = \big(t^2, t^3-2t\big)$. Find the area of the region R inside the loop, and find the volume of the solid obtained by revolving R about the x-axis.

Solution: With the help of the table of values and plot from example 4.7, we see that the loop is the portion of the curve with $-\sqrt{2} \le t \le \sqrt{2}$. By symmetry, the bottom half of the region R lies between the axis and the portion of the curve with $0 \le t \le \sqrt{2}$. For that part of the curve we have $y(t) = t(t-\sqrt{2})(t+\sqrt{2}) \le 0$ and $x'(t) = 2t \ge 0$ and so the area of R is

$$\begin{aligned} A &= -2\int_{t=0}^{\sqrt{2}} y(t)\,x'(t)\,dt = -2\int_0^{\sqrt{2}} (t^3-2t)\cdot 2t\,dt = 4\int_0^{\sqrt{2}} 2t^2 - t^4\ dt \\ &= 4\Big[\tfrac{2}{3}\,t^3 - \tfrac{1}{5}\,t^5\Big]_0^{\sqrt{2}} = 4\big(\tfrac{4\sqrt{2}}{3} - \tfrac{4\sqrt{2}}{5}\big) = \tfrac{32\sqrt{2}}{15} \end{aligned}$$

and (since $x'(t) > 0$) the volume of the solid obtained by revolving R about the x-axis is

$$\begin{aligned} V &= \int_{t=0}^{\sqrt{2}} \pi\,y(t)^2 x'(t)\,dt = \int_0^{\sqrt{2}} \pi(t^3-2t)^2\cdot 2t\,dt = 2\pi\int_0^{\sqrt{2}} t^7 - 4t^5 + 4t^3\ dt \\ &= 2\pi\Big[\tfrac{1}{8}\,t^8 - \tfrac{2}{3}\,t^6 + t^4\Big]_0^{\sqrt{2}} = 2\pi\big(2 - \tfrac{16}{3} + 4\big) = \tfrac{4\pi}{3}\,. \end{aligned}$$

4.25 Example: Consider the cycloid $(x,y) = (t-\sin t, 1-\cos t)$. Find the length of one arch of the cycloid, and find the area of the surface obtained by revolving this arch about the x-axis.

Solution: Note that one arch of the cycloid is the part of the cycloid given by $0 \le t \le 2\pi$. We have $x'(t) = 1-\cos t$ and $y'(t) = \sin t$ and so for $0 \le t \le 2\pi$ we have

$$\begin{aligned} \sqrt{x'(t)^2 + y'(t)^2} &= \sqrt{(1-\cos t)^2 + (\sin t)^2} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \\ &= \sqrt{2-2\cos t} = \sqrt{4\sin^2\left(\tfrac{1}{2}\,t\right)} = 2\sin\left(\tfrac{1}{2}\,t\right) \end{aligned}$$

(since $\sin(t/2) \ge 0$ for $0 \le t \le 2\pi$) and so the length of one arch is

$$L = \int_{t=0}^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\ dt = \int_0^{2\pi} 2\sin\left(\tfrac{1}{2}\,t\right)\ dt = \Big[-4\cos\left(\tfrac{1}{2}\,t\right)\Big]_0^{2\pi} = 4+4 = 8$$

and the area of the surface obtained by revolving this arch about the x-axis is

$$\begin{aligned} A &= \int_{t=0}^{2\pi} 2\pi\,y(t)\sqrt{x'(t)^2 + y'(t)^2}\ dt = \int_0^{2\pi} 2\pi(1-\cos t)\cdot 2\sin\left(\tfrac{1}{2}\,t\right)\ dt \\ &= 4\pi\int_0^{2\pi} \Big(1 - \big(1 - 2\sin^2\left(\tfrac{1}{2}\,t\right)\big)\Big)\sin\left(\tfrac{1}{2}\,t\right)\ dt = 8\pi\int_0^{2\pi} \sin^3\left(\tfrac{1}{2}\,t\right)\ dt\,. \end{aligned}$$

To solve the integral, we let $u = \cos\left(\tfrac{1}{2}\,t\right)$ so that $du = -\tfrac{1}{2}\sin\left(\tfrac{1}{2}\,t\right)\ dt$.Then

$$\begin{aligned} A &= 8\pi\int_{t=0}^{2\pi} \sin^3\left(\tfrac{1}{2}\,t\right)\ dt = 8\pi\int_{t=0}^{2\pi} \Big(1 - \cos^2\left(\tfrac{1}{2}\,t\right)\Big)\sin\left(\tfrac{1}{2}\,t\right)\ dt \\ &= 8\pi\int_{u=1}^{-1} -2(1-u^2)\,du = -16\pi\Big[u - \tfrac{1}{3}\,u^3\Big]_1^{-1} = -16\pi\big(-\tfrac{2}{3} - \tfrac{2}{3}\big) = \tfrac{64\pi}{3}\,. \end{aligned}$$

Polar Coordinates

4.26 Definition: A point in the plane is most commonly represented by an ordered pair (x,y) with $x\in\mathbf{R}$ and $y\in\mathbf{R}$, where x represents the horizontal position of the point and y represents the vertical position. The numbers x and y are called the **cartesian coordinates** of the point. To plot the position of a point represented in cartesian coordinates, it is convenient to use a **cartesian grid** which usually shows the x-axis pointing to the right and the y-axis pointing upwards along with some horizontal lines $y=$constant and some vertical lines $x=$constant.

The second most common way to represent a point in the plane is by an ordered pair (r,θ) with $0\le r\in\mathbf{R}$ and $\theta\in\mathbf{R}$ where r represents the distance from the point to the origin and (when $r\neq 0$) θ represents the angle from the positive x-axis to the point in the counterclockwise direction. The numbers r and θ are called the **polar coordinates** of the point. To plot the position of a point represented i polar coordinates, it is convenient to use a **polar grid** which usually shows the x and y-axes along with some circles $r=$constant and some rays $\theta=$constant.

4.27 Note: When a point is represented in cartesian coordinates by (x,y) and in polar coordinates by (r,θ), the coordinates x, y, r and θ satisfy the following relationships.

$$\begin{array}{ll} x=r\cos\theta & r^2=x^2+y^2 \\ y=r\sin\theta & \tan\theta=\frac{y}{x} \end{array}$$

4.28 Note: Given a point in the plane, the values of its cartesian coordinates x and y uniquely determine and are uniquely determined by the position of the point. On the other hand, although the values of the polar coordinates r and θ uniquely determine the position of the point and the position of the point uniquely determines the value of r, the position of the point does not uniquely determine the value of θ. At the origin $(x,y)=(0,0)$, we have $r=0$ while $\theta\in\mathbf{R}$ is arbitrary, and at points $(x,y)\neq(0,0)$, we have $r=\sqrt{x^2+y^2}$ and θ is only determined uniquely up to a multiple of 2π. To be explicit, we have

$$\theta=\begin{cases} \tan^{-1}\frac{y}{x} & +2\pi\,k \quad \text{for some } k\in\mathbf{Z} & \text{, if } x>0 \\ \cos^{-1}\frac{x}{\sqrt{x^2+y^2}} & +2\pi\,k \quad \text{for some } k\in\mathbf{Z} & \text{, if } y>0 \\ \pi+\tan^{-1}\frac{y}{x} & +2\pi\,k \quad \text{for some } k\in\mathbf{Z} & \text{, if } x>0 \\ -\cos^{-1}\frac{x}{\sqrt{x^2+y^2}} & +2\pi\,k \quad \text{for some } k\in\mathbf{Z} & \text{, if } y>0 \end{cases}$$

4.29 Note: Sometimes, when expressing points in polar coordinates, we allow r to take negative values. For all values of $r\in\mathbf{R}$, the point given in polar coordinates by (r,θ) corresponds to the point given in cartesian coordinates by $(x,y)=\big(r\cos\theta,r\sin\theta\big)$.

4.30 Example: Express each of the polar points $(r,\theta)=\big(2,-\frac{\pi}{6}\big)$, $\big(-6,\frac{3\pi}{4}\big)$ and $\big(10,\tan^{-1}3\big)$ in cartesian form.

Solution: The point given in polar coordinates by $(r,\theta)=\big(2,-\frac{\pi}{6}\big)$ is given in cartesian coordinates by $(x,y)=(\sqrt{3},-1)$. The polar point given by $(r,\theta)=\big(-6,\frac{3\pi}{4}\big)$ is given in cartesian coordinates by $(x,y)=(3\sqrt{2},-3\sqrt{2})$. Note that when $\theta=\tan^{-1}3$ we have $\sin\theta=\frac{3}{\sqrt{10}}$ and $\cos\theta=\frac{1}{\sqrt{10}}$, and so the polar point $(r,\theta)=(10,\tan^{-1}3)$ is given in cartesian coordinates by $(x,y)=\big(\sqrt{10},3\sqrt{10}\big)$.

4.31 Example: Express each of the cartesian points $(x,y)=(1,\sqrt{3})$, $(-2,2)$ and $(-3,-4)$ in polar form.

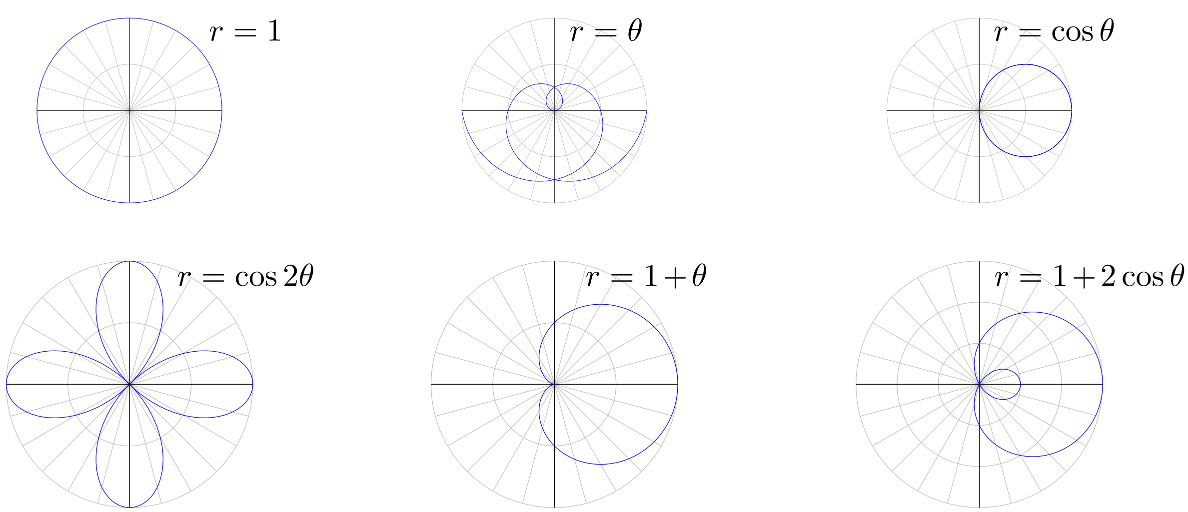
Solution: The point given in cartesian coordinates by $(x,y)=(1,\sqrt{3})$ can be given in polar coordinates by $(r,\theta)=\left(2,\frac{\pi}{3}\right)$. The cartesian point $(x,y)=(-2,2)$ can be given in polar coordinates by $(r,\theta)=\left(2\sqrt{2},\frac{3\pi}{4}\right)$. The cartesian point $(x,y)=(-3,-4)$ can be given in polar coordinates by $(r,\theta)=\left(5,\pi+\tan^{-1}\frac{4}{3}\right)$. The third point $(x,y)=(-3,-4)$ could also be given, for example, by $(r,\theta)=\left(-5,\tan^{-1}\frac{4}{3}\right)=\left(-5,\sin^{-1}\frac{4}{5}\right)$.

4.32 Note: A curve in the plane can be described in polar coordinates either explicitly, implicitly or parametrically. The explicit curve given by $r=f(\theta)$ for $\theta\in I$ is the set of polar points $\left\{(r,\theta)\middle|r=f(\theta)\text{ for some }\theta\in I\right\}$, the implicit curve $f(r,\theta)=0$ is the set of polar points $\left\{(r,\theta)\middle|f(r,\theta)=0\right\}$, and the parametric curve given by $(r,\theta)=f(t)=\big(r(t),\theta(t)\big)$ for $t\in I$ is the set of polar points $\left\{\big(r(t),\theta(t)\big)\middle|t\in I\right\}$. Such curves are most easily sketched using a polar grid.

4.33 Example: Sketch each of the explicit polar curves $r=1$, $r=\theta$, $r=\cos\theta$, $r=\cos 2\theta$, $r=1+\cos\theta$ and $r=1+2\cos\theta$.

Solution: For each curve, we make a table of values and plot points in a polar grid.

θ	$r=1$	$r=\theta$	$r=\cos\theta$	$r=\cos 2\theta$	$r=1+\cos\theta$	$r=1+2\cos\theta$
0	1	0	1	1	2	3
$\pi/6$	1	$\pi/6$	$\sqrt{3}/2$	$1/2$	$1+\sqrt{3}/2$	$1+\sqrt{3}$
$\pi/4$	1	$\pi/4$	$\sqrt{2}/2$	0	$1+\sqrt{2}/2$	$1+\sqrt{2}$
$\pi/3$	1	$\pi/3$	$1/2$	$-1/2$	$3/2$	2
$\pi/2$	1	$\pi/2$	0	-1	1	1
$2\pi/3$	1	$2\pi/3$	$-1/2$	$-1/2$	$1/2$	0
$3\pi/4$	1	$3\pi/4$	$-\sqrt{2}/2$	0	$1-\sqrt{2}/2$	$1-\sqrt{2}$
$5\pi/6$	1	$5\pi/6$	$-\sqrt{3}/2$	$1/2$	$1-\sqrt{3}/2$	$1-\sqrt{3}$
π	1	π	-1	1	0	-2



4.34 Definition: A **limaçon** is a polar curve of the form $r=a+b\cos\theta$ (or $r=a+b\sin\theta$), where $a,b\in\mathbf{R}^+$. A **cardioid** is a limaçon of the form $r=a+a\cos\theta$ (or $r=a+a\sin\theta$). A polar **rose** is a polar curve of the form $r=a\cos(n\theta)$ $\big($or $r=a\sin(n\theta)\big)$ where $a\in\mathbf{R}^+$ and $n\in\mathbf{Z}^+$. When n is odd, the polar rose $r=a\cos(n\theta)$ has n petals, but when n is even, the polar rose has $2n$ petals,

4.35 Note: If a curve is described explicitly in cartesian coordinates by $y = f(x)$, then the same curve is described implicitly in polar coordinates (using the formulas $x = r\cos\theta$ and $y = r\sin\theta$) by $r\sin\theta = f(r\cos\theta)$.

If a curve is described implicitly in cartesian coordinates by $f(x,y) = 0$, then the same curve is given implicitly in polar coordinates by $f\big(r\cos\theta, r\sin\theta\big) = 0$.

If a curve is described parametrically in cartesian coordinates by $(x,y) = \big(x(t),y(t)\big)$ then sometimes (but not always) we can use algebraic manipulation (making use of the formulas $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2+y^2$ and $\tan\theta = \frac{y}{x}$) to express the curve in polar coordinates.

4.36 Note: If a curve is given parametrically in polar coordinates by $(r,\theta) = \big(r(t),\theta(t)\big)$, then it is given parametrically in cartesian coordinates by $(x,y) = \big(r(t)\cos\theta(t), r(t)\sin\theta(t)\big)$.

If a curve is described explicitly in polar coordinates by $r = r(\theta)$ then the same curve is given parametrically, still in polar coordinates, by $(r,\theta) = \big(r(t),t\big)$, and so it is given parametrically in cartesian coordinates by $(x,y) = \big(r(t)\cos t, r(t)\sin t\big)$.

If a curve is given implicitly in polar coordinates by $f(r,\theta) = 0$, then sometimes (but not always) we can use algebraic manipulation to express the curve in cartesian coordinates.

4.37 Example: Express each of the cartesian curves $y = x+1$, $y = x^2$ and $y^2 - x^2 = 1$ explicitly in polar coordinates.

Solution: The line $y = x+1$ is given implicitly in polar coordinates by $r\sin\theta = r\cos\theta + 1$, or equivalently by $r(\sin\theta - \cos\theta) = 1$, and so it is given explicitly by

$$r = r(\theta) = \frac{1}{\sin\theta - \cos\theta}\,.$$

The parabola $y = x^2$ is given implicitly in polar coordinates by $r\sin\theta = (r\cos\theta)^2$, that is $r\big(\sin\theta = r\cos^2\theta\big)$. or equivalently $r = 0$ or $r = \sin\theta/\cos^2\theta = \sec\theta\tan\theta$. Since the origin $r = 0$ already lies on the polar curve $r = \sec\theta\tan\theta$, the curve is given explicitly simply by

$$r = r(\theta) = \sec\theta\tan\theta\,.$$

Finally the hyperbola, given in cartesian coordinates by $y^2 - x^2 = 1$, is given implicitly in polar coordinates by $r^2\sin^2\theta - r^2\cos^2\theta = 1$, that is $r^2(\sin^2\theta - \cos^2\theta) = 1$, or equivalently $r^2(-\cos 2\theta) = 1$, that $r^2 = -\sec 2\theta$. When $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, and then again when $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$, we have $\sec 2\theta < 0$ and we can take $r = \sqrt{-\sec 2\theta}$, so the hyperbola is given explicitly by

$$r = r(\theta) = \sqrt{-\sec 2\theta}\,.$$

4.38 Example: Express each of the polar curves $r = 1$, $r = \theta$, $r = \cos\theta$, $r = \cos 2\theta$ and $r = 1 + \cos\theta$ in cartesian coordinates.

Solution: The polar curve $r = 1$ is the unit circle, which can be given in cartesian coordinates, either implicitly by $x^2 + y^2 = 1$, or parametrically by $(x.y) = (\cos t, \sin t)$. The polar curve $r = \theta$ can be given parametrically in polar coordinates by $(r,\theta) = (t,t)$, and it can be given parametrically in cartesian coordinates by

$$(x,y) = (r\cos\theta, r\sin\theta) = (t\cos t, t\sin t)\,.$$

The polar curve $r = \cos\theta$ can also be given in polar coordinates by $r^2 = r\cos\theta$, and in cartesian coordinates this becomes $x^2 + y^2 = x$. Completing the square, the equation can also be written as

$$\left(x - \tfrac{1}{2}\right)^2 + y^2 = \tfrac{1}{4}$$

and so we see that the polar curve $r = \cos\theta$ is in fact the circle of radius $\frac{1}{2}$ centered at $(x, y) = \left(\frac{1}{2}, 0\right)$ (you might have guessed this by looking at its graph, which we plotted in Example 4.33). Finally, the four-petaled rose given in polar coordinates by $r = \cos 2\theta = \cos^2\theta - \sin^2\theta$ can also be given in polar coordinates by $r^3 = (r\cos\theta)^2 + (r\sin\theta)^2$, which we can write in cartesian coordinates as $(x^2 + y^2)^{3/2} = x^2 - y^2$, or equivalently as

$$(x^2 + y^2)^3 = x^2 - y^2 .$$

4.39 Note: Given a polar curve which is described either explicitly or parametrically, we can describe the curve parametrically in cartesian coordinates, and then we can perform various calculations related to the curve, for example we can find the slope of the curve at a point or we can find the tangent line to the curve at a point, or we can find the area of the region R which lies between the curve and the x-axis with $a \le x \le b$, or we can find the volume of the solid obtained by revolving R about either the x or the y-axis.

4.40 Example: Find a formula for the slope of the polar curve $r = r(\theta)$ at the point where $\theta = t$ (that is, the slope of the tangent line to the curve at the point where $\theta = t$).

Solution: The curve can be given parametrically in polar coordinates by $(r, \theta) = \big(r(t), t\big)$ and so it can be given parametrically in cartesian coordinates by

$$(x, y) = \big(r(t)\cos t, r(t)\sin t\big) .$$

Using the Product Rule, the slope at the point where $\theta = t$ is equal to

$$\frac{y'(t)}{x'(t)} = \frac{r'(t)\sin t + r(t)\cos t}{r'(t)\cos t - r(t)\sin t} .$$

4.41 Example: Find the cartesian coordinates of all of the horizontal and vertical points on the cardioid $r = 1 + \cos\theta$.

Solution: The cardioid can be given parametrically in cartesian coordinates by

$$(x, y) = \big((1 + \cos t)\cos t, (1 + \cos t)\sin t\big) .$$

Since $x(t) = \cos t + \cos^2 t$, we have

$$x'(t) = -\sin t - 2\sin t\cos t = -(\sin t)(1 + 2\cos t)$$

and so $x'(t) = 0$ when $\sin t = 0$ and when $\cos t = -\frac{1}{2}$. that is when $t = 0, \pi, \pm\frac{2\pi}{3}$ plus integer multiples of 2π. Since $y(t) = \sin t + \sin t\cos t$, we have

$$y'(t) = \cos t + \cos^2 t - \sin^2 t = 2\cos^2 t + \cos t - 1 = (2\cos t - 1)(\cos t + 1)$$

and so $y'(t) = 0$ when $\cos t = \frac{1}{2}$ and when $\cos t = -1$, that is when $t = \pm\frac{\pi}{3}, \pi$ plus multiples of 2π. When $t = 0, \pm\frac{2\pi}{3}$, that is at the points $(x, y) = (2, 0), \left(-\frac{1}{4}, \pm\frac{\sqrt{3}}{4}\right)$, we have $x'(t) = 0$ and $y'(t) = 0$, so the curve is vertical at these points. When $t = \pm\frac{\pi}{3}$, that is at the points $(x, y) = \left(\frac{3}{4}, \frac{3\sqrt{3}}{4}\right)$, we have $y'(t) = 0$ and $x'(t) \ne 0$, so the curve is horizontal at these points. When $t = \pi$, that is at the point $(x, y) = (0, 0)$, we have both $x'(t)$ and $y'(t) = 0$ so some care is needed. Using L'Hôpital's Rule, we have

$$\begin{aligned}\lim_{t\to\pi}\frac{y'(t)}{x'(t)} &= \lim_{t\to\pi}\frac{(2\cos t - 1)(\cos t + 1)}{-(\sin t)(1 + 2\cos t)} = \lim_{t\to\pi}\frac{2\cos t - 1}{1 + 2\cos t}\cdot\lim_{t\to\pi}\frac{\cos t + 1}{-\sin t} \\ &= \tfrac{-2-1}{1-2}\cdot\lim_{t\to\pi}\frac{\cos t + 1}{-\sin t} = 3\cdot\lim_{t\to\pi}\frac{-\sin t}{\cos t} = 3\cdot 0 = 0\end{aligned}$$

and so we can consider the point $(x, y) = (0, 0)$ to be a horizontal point.

4.42 Example: Find the area of the region R which lies inside the limaçon $r = 1+2\cos\theta$ with $1 \le x \le 2$.

Solution: The limçon can be given parametrically in cartesian coordinates by

$$(x,y) = \big((1+2\cos t)\cos t, (1+2\cos t)\sin t\big)\,.$$

Note that since $x(t) = \cos t + 2\cos^2 t$ we have

$$x'(t) = -\sin t - 4\sin t\cos t = -\sin t\,(1+4\cos t)\,.$$

The top half of the region R is the region which lies below the portion of the limaçon with $0 \le t \le \frac{\pi}{3}$ and above the x-axis with $1 \le x \le 3$. Note that $y(t) \ge 0$ and $x'(t) \le 0$ for $0 \le t \le \frac{\pi}{3}$, and so the area of R is

$$A = -2\int_{t=0}^{\pi/3} y(t)\,x'(t)\,dt = 2\int_0^{\pi/3} (1+2\cos t)\cos t\cdot\sin t\,(1+4\cos t)\,dt\,.$$

We let $u = \cos t$ so $du = -\sin t\,dt$ to get

$$\begin{aligned} A &= \int_{u=1}^{1/2} -2u(1+2u)(1+4u)\,du = \int_{1/2}^{1} 2u + 12u^2 + 16u^3\ du \\ &= \Big[u^2 + 4u^3 + 4u^4\Big]_{1/2}^{1} = \big(1+4+4\big) - \big(\tfrac14 + \tfrac12 + \tfrac14\big) = 8\,. \end{aligned}$$

4.43 Example: Find a formula for the length of the polar curve $r = r(\theta)$ with $\alpha \le \theta \le \beta$.

Solution: The curve can be given parametrically, in cartesian coordinates, by

$$(x,y) = \big(r(t)\cos t, r(t)\sin t\big)$$

so we have

$$\begin{aligned} x'(t) &= r'(t)\cos t - r(t)\sin t \\ y'(t) &= r'(t)\sin t + r(t)\sin t \end{aligned}$$

and hence

$$\begin{aligned} x'(t)^2 + y'(t)^2 &= \big(r'(t)^2\cos^2 t - 2\,r(t)r'(t)\sin t\cos t + r(t)^2\sin^2 t\big) \\ &\quad + \big(r'(t)^2\sin^2 t - 2\,r(t)r'(t)\sin t\cos t + r(t)^2\cos^2 t\big) \\ &= r'(t)^2 + r(t)^2\,. \end{aligned}$$

Thus the length of the curve is

$$L = \int_{t=\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2}\ dt = \int_{t=\alpha}^{\beta} \sqrt{r'(t)^2 + r(t)^2}\ dt\,.$$

4.44 Example: Find the length of the cardioid $r = 1 + \cos\theta$.

Solution: We have $r(t) = 1+\cos t$ and $r'(t) = -\sin t$. The top half of the cardioid is given by $0 \le t \le \pi$, and note that when $0 \le t \le \pi$ we have $\cos(t/2) \ge 0$. Using the above formula, the length of the cardioid is

$$\begin{aligned} L &= 2\int_{t=0}^{\pi} \sqrt{r(t)^2 + r'(t)^2}\ dt = 2\int_0^{\pi} \sqrt{1 + 2\cos t + \cos^2 t + \sin^2 t}\ dt \\ &= \int_0^{\pi} 2\sqrt{2+2\cos t}\ dt = \int_0^{\pi} 2\sqrt{4\cos^2(t/2)}\ dt = \int_0^{\pi} 4\cos(t/2)\ dt \\ &= \Big[8\sin(t/2)\Big]_0^{\pi} = 8\,. \end{aligned}$$

4.45 Note: There is an alternative (and often preferable) way to calculate the area of a region which is described using polar coordinates. Consider the region R given in polar coordinates by $\alpha \le \theta \le \beta$, $f(\theta) \le r \le g(\theta)$. We can approximate the area of R as follows. Choose a partition $\alpha = \theta_1 < \theta_2 < \cdots < \theta_n = \beta$ of the interval $[\alpha, \beta]$. Choose sample points $c_k \in [\theta_{k-1}, \theta_k]$. Slice the region R into thin wedges with the k^{th} wedge given by $\theta_{k-1} \le \theta \le \theta_k$, $f(\theta) \le r \le g(\theta)$. The area of the k^{th} wedge is approximately

$$\Delta_k A \cong \tfrac{1}{2}\big(g(c_k)^2 - f(c_k)^2\big)\Delta_k\theta$$

where $\Delta_k\theta = \theta_k - \theta_{k-1}$. The total area is approximately

$$A \cong \sum_{k=1}^{n} \tfrac{1}{2}\left(g(c_k)^2 - f(c_k)^2\right)\Delta_k\theta\,.$$

The sum is a Riemann sum for the function $\frac{1}{2}\big(g(\theta)^2 - f(\theta)^2\big)$ on the interval $[\alpha, \beta]$, and so the exact area of R is the limit of these Riemann sums, that is

$$A = \int_{\theta=\alpha}^{\beta} \tfrac{1}{2}\big(g(\theta)^2 - f(\theta)^2\big)\,d\theta\,.$$

4.46 Example: Find the area of the region R which lies inside the cardioid $r = 1 + \cos\theta$.

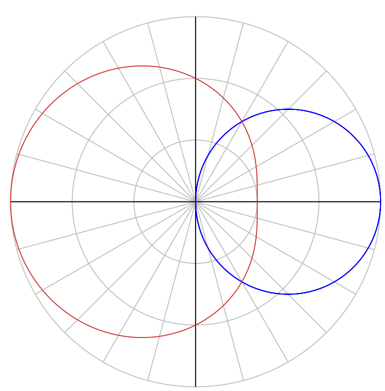
Solution: Using the above formula, the area is

$$\begin{aligned} A &= \int_{\theta=0}^{2\pi} \tfrac{1}{2}(1+\cos\theta)^2\,d\theta = \int_0^{2\pi} \tfrac{1}{2} + \cos\theta + \tfrac{1}{2}\cos^2\theta\;d\theta \\ &= \int_0^{2\pi} \tfrac{1}{2} + \cos\theta + \tfrac{1}{4} + \tfrac{1}{4}\cos 2\theta\;d\theta = \int_0^{2\pi} \tfrac{3}{4} + \cos\theta + \tfrac{1}{4}\cos 2\theta\;d\theta \\ &= \Big[\tfrac{3}{4}\,\theta + \sin\theta + \tfrac{1}{8}\sin 2\theta\Big]_0^{2\pi} = \tfrac{3\pi}{2}\,. \end{aligned}$$

4.47 Exercise: Find the area of the region which lies inside both the circle $r = 1$ and the rose $r = 2\cos 2\theta$.

4.48 Example: Find the area of the region R which lies inside both the circle $r = 3\cos\theta$ and the limaçon $r = 2 - \cos\theta$.

Solution: First we make a sketch (by plotting points). The curve $r = 3\cos\theta$ is shown in blue (it is a circle) and the curve $r = 2 - \cos\theta$ is shown in red.



The sketch helps to set up the integral. The total area is twice the area of the portion above the x-axis, which we divide into the portion with $0 \le \theta \le \frac{\pi}{3}$ and the portion with

$\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$. The total area is

$$\begin{aligned}
A &= 2\left(\int_0^{\pi/3} \tfrac{1}{2}(2-\cos\theta)^2\,d\theta + \int_{\pi/3}^{\pi/2} \tfrac{1}{2}(3\cos\theta)^2\,d\theta\right)\\
&= \int_0^{\pi/3} \left(4 - 4\cos\theta + \cos^2\theta\right)d\theta + \int_{\pi/3}^{\pi/2} 9\cos^2\theta\ d\theta\\
&= \int_0^{\pi/3} \left(4 - 4\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right)d\theta + \int_{\pi/3}^{\pi/2} \tfrac{9}{2}(1+\cos 2\theta)\,d\theta\\
&= \left[\tfrac{9}{2}\theta - 4\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/3} + \left[\tfrac{9}{2}\theta + \tfrac{9}{4}\sin 2\theta\right]_{\pi/3}^{\pi/2}\\
&= \left(\tfrac{3\pi}{2} - 2\sqrt{3} + \tfrac{\sqrt{3}}{8}\right) + \left(\tfrac{9\pi}{4}\right) - \left(\tfrac{3\pi}{2} + \tfrac{9\sqrt{3}}{8}\right)\\
&= \tfrac{9\pi}{4} - 3\sqrt{3}
\end{aligned}$$

Chapter 5. Differential Equations

Differential Equations

5.1 Definition: An (ordinary) **differential equation**, or **DE**, is an equation which involves a function, say $y = y(x)$, of a single variable x, along with some of its derivatives $y'(x)$, $y''(x)$, etc. The **order** of a DE is the highest of the orders of the derivatives which occur in the equation. For example, the equation $y''(x) + 2y'(x)y(x)^3 = \sin x$ is a second order DE.

A **solution** to a DE is a function $y = y(x)$ which makes the equation true for all x in some interval. A DE can have many solutions. To solve a DE you must find the **general solution**, which means to find all possible solutions. Often, the general solution will involve arbitrary constants and the number of arbitrary constants will be equal to the order of the DE.

Sometimes we require that a solution to a DE satisfies one or more additional conditions, called **initial conditions**. A DE together with an initial condition (or a set of initial conditions) is called an **initial value problem**, or an **IVP**. Often, in an IVP, the number of initial conditions is equal to the order of the DE, and there is exactly one solution.

5.2 Example: Find a solution of the form $y = ax^2 + bx + c$ to the DE $y''y' + x^2 = y$.

Solution: Let $y = ax^2 + bx + c$. Then $y' = 2ax + b$ and $y'' = 2a$ and so we have $y''y' + x^2 = y \iff 2a(2ax+b) + x^2 = ax^2 + bx + c \iff x^2 + 4a^2\,x + 2ab = ax^2 + bx + c$. Equating coefficients gives $1 = a$, $4a^2 = b$ and $2ab = c$, and so we must have $a = 1$, $b = 4$ and $c = 8$. Thus the only such solution is $y = x^2 + 4x + 8$.

5.3 Example: Find two distinct constants r_1 and r_2 such that $y = e^{r_1 x}$ and $e^{r_2 x}$ are both solutions to the DE $y'' + 3y' + 2y = 0$, show that $y = a\,e^{r_1 x} + b\,e^{r_2 x}$ is a solution for any constants a and b, and then find a solution to the DE with $y(0) = 1$ and $y'(0) = 0$.

Solution: Let $y = e^{rx}$. Then $y' = r\,e^{rx}$ and $y'' = r^2\,e^{rx}$ and so $y'' + 3y' + 2y = 0 \iff r^2\,e^{rx} + 3r\,e^{rx} + 2\,e^{rx} = 0 \iff (r^2+3r+2)e^{rx} = 0 \iff (r+1)(r+2)\,e^{rx} = 0 \iff r = -1$ or $r = -2$. Thus we can take $r_1 = -1$ and $r_2 = -2$.

Now, let $y = a\,e^{r_1 x} + b\,e^{r_2 x} = a\,e^{-x} + b\,e^{-2x}$. Then $y' = -a\,e^{-x} - 2b\,e^{-2x}$ and $y'' = a\,e^{-x} + 4b\,e^{-2x}$ and so we have

$$y'' + 3y' + 2y = a\,e^{-x} + 4b\,e^{-2x} - 3a\,e^{-x} - 6b\,e^{-2x} + 2a\,e^{-x} + 2b\,e^{-2x} = 0\,.$$

This shows that $y = a\,e^{-x} + b\,e^{-2x}$ is a solution to the DE. Also, note that $y(0) = a + b$ and $y'(0) = -a - 2b$, and so to get $y(0) = 1$ and $y'(0) = 0$ we need $a + b = 1$ and $-a - 2b = 0$. Solve these two equations to get $a = 2$ and $b = -1$. Thus the required solution is $y = 2\,e^{-x} - e^{-2x}$.

5.4 Example: A rock is thrown downwards at 5 m/s from the top of a 100 m cliff and it falls to the ground. Assuming that the rock accelerates downwards at 10 m/s^2, find the speed of the rock when it lands.

Solution: Let $x(t)$ be the height of the rock, in meters, after t seconds. We must solve the IVP which consists of the 2$^{\text{nd}}$ order DE $x''(t) = -10$ and the two initial conditions $x'(t) = -5$ and $x(0) = 100$. We have

$$\begin{aligned} x''(t) &= -10 \\ \int x''(t)\,dt &= \int -10\,dt \\ x'(t) &= -10t + c_1 \end{aligned}$$

where c_1 is a constant. Since $x'(0) = -5$ we find that $c_1 = -5$, so we have

$$\begin{aligned} x'(t) &= -10t - 5 \\ \int x'(t)\,dt &= \int -10t - 5\,dt \\ x(t) &= -5t^2 - 5t + c_2 \end{aligned}$$

where c_2 is another constant. Since $x(0) = 100$ we have $c_2 = 100$ and so the solution to the IVP is $x(t) = -5t^2 - 5t + 100$. To find out when the rock lands, we solve $x(t) = 0$:

$$\begin{aligned} 0 &= -5t^2 - 5t + 100 \\ 0 &= t^2 + t - 20 \\ &= (t+5)(t-4) \end{aligned}$$

so it lands when $t = 4$. Since $x'(4) = -45$, the rock lands at a speed of 45 m/s.

Direction Fields

5.5 Definition: The graph of a solution $y = y(x)$ to a DE is called a **solution curve**.

5.6 Note: It is easy to sketch the solution curves to any DE of the form

$$y'(x) = F(x, y(x))$$

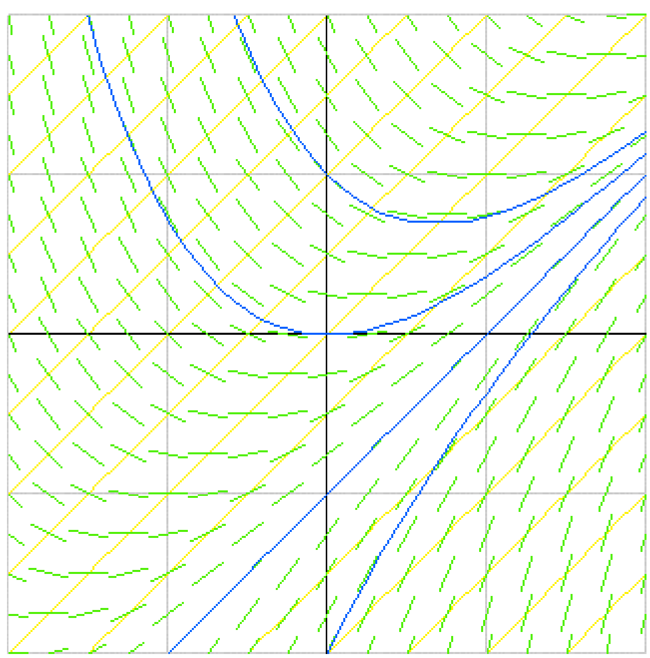
in the following way. First choose many points (x, y), and for each point (x, y) find the value of $F(x, y)$. If $y = y(x)$ is any solution to the DE, so that $y'(x) = F(x, y)$, then $F(x, y)$ is the slope of the solution curve at the point (x, y). At each point (x, y), draw a short line segment with slope $F(x, y)$. The resulting picture is called the **slope field** or the **direction field** of the DE. If we choose enough points (x, y) it should be possible to visualize the solution curves; they follow the direction of the short line segments.

To draw the direction field of the DE $y'(x) = F(x, y)$ by hand, it helps to first lightly draw several **isoclines**; these are the curves $F(x, y) = m$, where m is a constant. Along the isocline $F(x, y) = m$ we then draw many short line segments of slope m.

To draw the graph of the solution to the IVP $y'(x) = F(x, y)$, with $y(x_0) = y_0$, sketch the direction field for the DE $y'(x) = F(x, y)$ and then draw the solution curve which passes through the point (x_0, y_0).

5.7 Example: Sketch the direction field for the DE $y' = x - y$, then sketch the solution curves through each of the points $(x_0, y_0) = (0, -2)$, $(0, -1)$, $(0, 0)$ and $(0, 1)$.

Solution: The isoclines are the lines $x - y = m$. To sketch the direction field, we first lightly draw the lines $x - y = m$ for several values of m. These are shown below in yellow for $m = -\frac{7}{2}, -\frac{6}{2}, -\frac{5}{2}, \cdots, \frac{5}{2}, \frac{6}{2}, \frac{7}{2}$. Then, along each isocline, we draw many short line segments of the appropriate slope; on the isocline $x - y = m$ we draw line segments of slope m. These are shown in green. The solution curves through each of the points $(x_0, y_0) = (0, -2)$, $(0, -1)$, $(0, 0)$ and $(0, 1)$ are shown below in blue.



Euler's Method

5.8 Note: We can approximate the solution to the IVP $y'(x) = F\big(x, y(x)\big)$ with $y(a) = b$ using the following method, which is known as **Euler's Method**. Pick a small value Δx, which we call the **step size**. Let $x_0 = a$ and $y_0 = b$. Having found x_n and y_n, we let

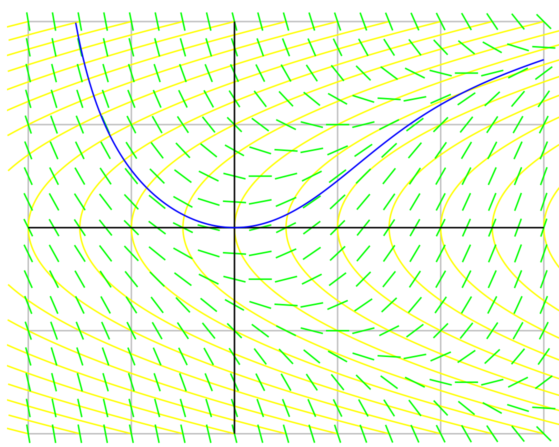
$$\begin{aligned} x_{n+1} &= x_n + \Delta x \\ y_{n+1} &= y_n + F(x_n, y_n)\,\Delta x\,. \end{aligned}$$

The solution curve $y = f(x)$ is then approximated for values $x \geq a$ by the piecewise linear curve whose graph has vertices at the points (x_n, y_n). Note that the slope of the line segment from (x_n, y_n) to (x_{n+1}, y_{n+1}) is equal to the slope of the direction field at the point (x_n, y_n). If we also wish to approximate the solution for values $x \leq a$, we can construct points (x_n, y_n) with $n < 0$ by letting

$$\begin{aligned} x_{n-1} &= x_n - \Delta x \\ y_{n-1} &= y_n - F(x_n, y_n)\,\Delta x\,. \end{aligned}$$

5.9 Example: Consider the IVP $y' = x - y^2$ with $y(0) = 0$. Sketch the direction field for the given DE along with the graph of the solution curve $y = f(x)$. With the help of a calculator, apply Euler's method with step size $\Delta x = \frac{1}{2}$ to approximate the value of $f(3)$.

Solution: The isocline (curve of constant slope) $y' = m$ is the sideways parabola $m = x - y^2$, or $x = y^2 + m$. The isoclines are shown in yellow, the slope field is shown in green, and the solution curve with $y(0) = 0$ is shown in blue.



We let $x_0 = 0$ and $y_0 = 0$. For $k \geq 0$ we set $x_{k+1} = x_k + \Delta x$ and $y_{k+1} = y_k + F(x_k, y_k)\Delta x$, where $F(x, y) = x - y^2$. We make a table listing the values of x_k, y_k and $F(x_k, y_k)$.

k	x_k	y_k	$F(x_k, y_k) = x_k - {y_k}^2$
0	0	0	0
1	0.5	0	0.5
2	1.0	0.25	0.9375
3	1.5	0.71875	0.9833984375
4	2.0	1.210449219	0.534812688
5	2.5	1.477855563	0.315942935
6	3.0	1.635827030	

Thus we have $f(3) \cong y_6 \cong 1.6$.

Separable First Order Equations

5.10 Definition: A **separable** first order DE is a DE which can be written in the form

$$f\big(y(x)\big)\,y'(x) = g(x)\,.$$

for some continuous functions $f(y)$ and $g(x)$.

5.11 Note: $y = y(x)$ is a solution to the separable DE $f(y)\,y' = g(x)$ when

$$\int f(y(x))\,y'(x)\,dx = \int g(x)\,dx\,,$$

and by the change of variables formula, we have $\int f(y(x))\,y'(x)\,dx = \int f(y)\,dy$. So to solve the DE, we rewrite it as $f(y)\,dy = g(x)\,dx$ and then integrate both sides.

5.12 Example: Solve the DE $y' = x^2 y$.

Solution: We write the DE as $\dfrac{dy}{y} = x^2\,dx$, assuming $y \neq 0$, and integrate both sides to get

$$\begin{aligned} \ln|y| &= \tfrac{1}{3}x^3 + c \\ |y| &= e^{\frac{1}{3}x + c} \\ y &= \pm e^c e^{x^3/3} = Ae^{x^3/3}\,, \end{aligned}$$

where c is an arbitrary constant, and we set $A = \pm e^c$, so A is an arbitrary non-zero constant. Notice that $y = 0$ is also a solution to the DE, so the general solution is $y = Ae^{x^3/3}$, where A is an arbitrary constant.

5.13 Example: Solve the IVP $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$ with $y(1) = \pi$.

Solution: We rewrite the DE as $(2y + \cos y)dy = 6x^2\,dx$ then integrate both sides to get

$$\begin{aligned} \int (2y + \cos y)dy &= \int 6x^2\,dx \\ y^2 + \sin y &= 2x^3 + c \end{aligned}$$

where c is any constant. In this example we cannot solve for y explicitly as a function of x. To find the solution which satisfies the initial condition $y(1) = \pi$, we substitute $x = 1$ and $y = \pi$ into the above implicit solution to get $\pi^2 + \sin\pi = 2 + c$ so we find that $c = \pi^2 - 2$. Thus the solution to the IVP $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$, $y(1) = \pi$ is given implicitly by

$$y^2 + \sin y = 2x^3 + \pi^2 - 2\,.$$

Linear First Order Equations

5.14 Definition: A **linear** first order DE is a DE which can be written in the form

$$y'(x) + p(x)\,y(x) = q(x)$$

for some continuous functions $p(x)$ and $q(x)$.

5.15 Note: There is a trick which can be used to solve the linear DE $y' + py = q$. The trick is to find a function $\lambda = \lambda(x)$, called an **integrating factor**, such that $\lambda' = \lambda\, p$ so that $(\lambda\, y)' = \lambda\, y' + \lambda' y = \lambda\, y' + \lambda\, p\, y$. If we can find λ then we can solve the DE as follows:

$$\begin{aligned} y' + p\,y &= q \\ \lambda\, y' + \lambda\, p\, y &= \lambda\, q \\ (\lambda\, y)' &= \lambda\, q \\ \lambda\, y &= \int \lambda\, q\, dx \\ y &= \frac{1}{\lambda}\int \lambda\, q\, dx \end{aligned}$$

To find an integrating factor λ we must find a solution to the DE $\lambda'(x) = \lambda(x)p(x)$. This is a separable DE, so we rewrite it as $(1/\lambda)d\lambda = p(x)\,dx$ and integrate both sides

$$\begin{aligned} \int \frac{d\lambda}{\lambda} &= \int p(x)\,dx \\ \ln|\lambda| &= \int p(x)\,dx \\ |\lambda| &= e^{\int p(x)\,dx} \\ \lambda &= \pm e^{\int p(x)\,dx} \end{aligned}$$

Since any integrating factor will do , we can take $\lambda = e^{\int p(x)\,dx}$, and when we solve the integral $\int p(x)\,dx$ it is not necessary to keep track of the constant of integration. We summarize this in the following theorem.

5.16 Theorem: *The general solution to the linear DE* $y'(x) + p(x)y(x) + q(x) = 0$ *is*

$$y(x) = \frac{1}{\lambda(x)}\int \lambda(x)q(x)\,dx \text{ , where } \lambda(x) = e^{\int p(x)\,dx}\,.$$

5.17 Example: Find the general solution to the DE $y' - x^2 y = 0$.

Solution: This DE is both separable and linear. We already found the solution to this DE in the previous section by treating it at a separable DE. Now we will solve it again, using our method for solving linear DEs. An integrating factor is $\lambda = e^{\int -x^2\,dx} = e^{-x^3/3}$, and the solution is $y = \dfrac{1}{\lambda}\displaystyle\int 0\,dx = e^{x^3/3}\,c$, where c is a constant.

5.18 Example: Find the solution to the IVP $y' + 2y = e^{-5x}$, $y(0) = 1$.

Solution: An integrating factor is $\lambda = e^{\int 2\,dx} = e^{2x}$, and the solution to the DE is

$$y = \frac{1}{\lambda}\int \lambda\, e^{-5x}\, dx = e^{-2x}\int e^{-3x}\, dx = e^{-2x}\left(-\tfrac{1}{3}e^{-3x} + c\right) = -\tfrac{1}{3}e^{-5x} + c\,e^{-2x}\,,$$

where c is an arbitrary constant. Since $y(0) = 1$, we have $1 = -\frac{1}{3} + c$ and so $c = \frac{4}{3}$. Thus the solution to the IVP is $y = \frac{4}{3}e^{-2x} - \frac{1}{3}e^{-5x}$.

5.19 Example: Find the solution to the IVP $y' - 2xy = x$, $y(0) = 0$.

Solution: An integrating factor is $\lambda = e^{\int -2x\,dx} = e^{-x^2}$, and the solution to the DE is

$$y = \frac{1}{\lambda}\int x\,\lambda\, dx = e^{x^2}\int x\,e^{-x^2}\, dx = e^{x^2}\left(-\tfrac{1}{2}e^{-x^2} + c\right) = c\,e^{x^2} - \tfrac{1}{2}\,.$$

Since $y(0) = 0$ we have $c = \frac{1}{2}$ so the solution to the IVP is $y = \frac{1}{2}(e^{x^2} - 1)$.

5.20 Example: Solve the IVP $xy'' + y' = 4x$ with $y(1) = y(2) = 1$.

Solution: Write $u = y'$ so that $u' = y''$. Then the DE can be written as $xu' + u = 4x$. This is linear since we can write it as $u' + \frac{1}{x}\,u = 4$. An integrating factor is $\lambda = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$, and the solution is

$$u = \frac{1}{x}\int 4x\, dx = \frac{1}{x}(2x^2 + a) = 2x + \frac{a}{x}$$

where a is a constant, that is $y' = 2x + \dfrac{a}{x}$. Thus

$$y = \int 2x + \frac{a}{x}\; dx = x^2 + a\ln x + b$$

where b is a constant. To get $y(1) = 1$ we need $1 + b = 1$ so $b = 0$, and to get $y(2) = 1$ we need $4 + a\ln 2 = 1$ so $a = -\frac{3}{\ln 2}$. Thus the solution is

$$y = x^2 - \frac{3\ln x}{\ln 2} = x^2 - 3\log_2 x\,.$$

Applications

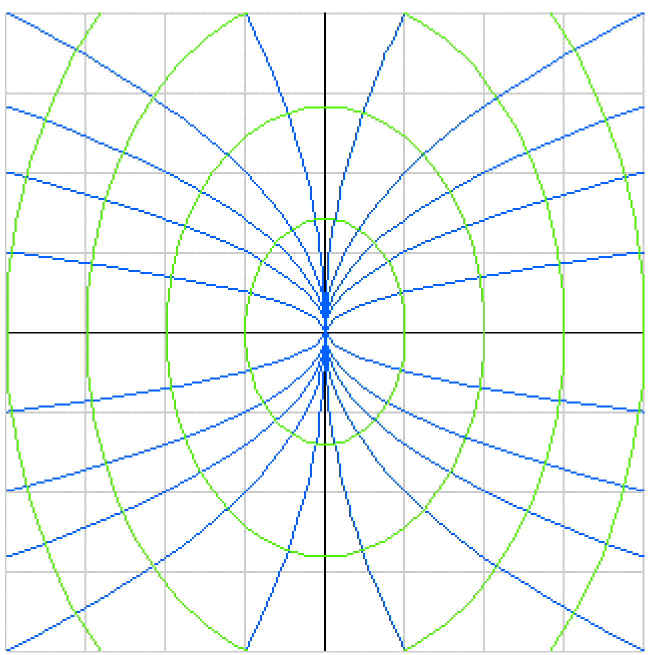
5.21 Definition: An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family orthogonally. For example, each straight line $y = mx$ through the origin is an orthogonal trajectory of the family of circles $x^2 + y^2 = r^2$, where r can be any positive constant.

5.22 Example: Find the orthogonal trajectories of the family of parabolas $x = k\,y^2$, where k is an arbitrary constant.

Solution: Differentiating $x = k\,y^2$ we obtain $1 = 2ky\,y'$ so the parabola $x = k\,y^2$ has slope $y' = \dfrac{1}{2ky}$ at each point. Since $k = \dfrac{x}{y^2}$, the parabola has slope $y' = \dfrac{1}{2ky} = \dfrac{y}{2x}$. Since the orthogonal trajectories are perpendicular to the parabolas, their slope is $y' = -\dfrac{2x}{y}$. So to find the orthogonal trajectories, we solve the DE $y' = -\dfrac{2x}{y}$. This is a separable DE, so we rewrite it as $y\,dy = -2x\,dx$ and integrate:

$$\begin{aligned} \int y\,dy &= \int -2x\,dx \\ \frac{1}{2}y^2 &= -x^2 + c \\ x^2 + \frac{y^2}{2} &= c \end{aligned}$$

Thus the orthogonal trajectories are the ellipses $x^2 + \dfrac{y^2}{2} = c$, where c is an arbitrary positive constant. Some of the parabolas and ellipses in these families are shown below.



5.23 Definition: A quantity $y = y(t)$ is said to **grow or decay exponentially** if it satisfies the DE $y'(t) = k\,y(t)$ for some constant k. This DE is both separable and linear. Let us solve it as a linear DE (as an exercise, try solving it as a separable DE). An integrating factor is $\lambda = e^{\int -k\,dt} = e^{-kt}$, and the general solution is

$$y = \frac{1}{\lambda}\int \lambda \cdot 0\,dt = e^{kt}\int 0\,dt = c\,e^{kt}$$

where c is an arbitrary constant. Notice that c is equal to $y(0)$, so the solution is

$$y(t) = y(0)\,e^{kt}\,.$$

When $f(0)$ and k are positive, we say that y **grows exponentially**. When $f(0) > 0$ and $k < 0$ we say that y **decays exponentially**.

5.24 Example: Suppose that a bacteria culture grows exponentially. After 2 hours there are 600 bacteria and after 8 hours the count is 75,000. Find a formula for the number of bacteria after t hours, and determine when the count was 3,000.

Solution: Let $y(t)$ be the count after t hours. Since the count grows exponentially, we have $y(t) = c\,e^{kt}$ for some positive constants c and k. Since $y(2) = 600$ we have $c\,e^{2k} = 600$, and since $y(8) = 75,000$ we have $c\,e^{8k} = 75,000$. We divide these two equations to get

$$\begin{aligned}\frac{c\,e^{8k}}{c\,e^{2k}} &= \frac{75,000}{6,000}\\ e^{6k} &= 125\\ e^{2k} &= 5\end{aligned}$$

so $k = \frac{1}{2}\ln 5$. Since $c\,e^{2k} = 600$ we have $5c = 600$ so $c = 120$. Thus $y(t) = 120e^{(\frac{1}{2}\ln 5)t} = 120(5^{t/2})$. The count was 3,000 when $120 \cdot 5^{t/2} = 3000$. Solve this to find that $t = 4$.

5.25 Example: A certain proportion of the carbon in all living plant material is the radioactive isotope C^{14}. It is believed that this proportion has not changed in the last several hundred thousand years. After a plant dies, the amount of C^{14} decays exponentially. The **half-life** of C^{14} is about 5730 years, which means that after 5730 years, one half of the initial C^{14} will be left.

Suppose that a parchment is found which contains 70% as much C^{14} as it did initially. Determine the age of the parchment.

Solution: Let $y(t)$ be the amount of C^{14} remaining in the parchment after t years. Since it decays exponentially, we have $y(t) = y(0)\,e^{kt}$. Since $y(5730) = \frac{1}{2}y(0)$, we have

$$\begin{aligned}y(0)\,e^{5730\,k} &= \frac{y(0)}{2}\\ e^{5730\,k} &= \frac{1}{2}\\ 5730\,k &= -\ln 2\\ k &= -\frac{\ln 2}{5730}\,.\end{aligned}$$

We want to find the value of t such that $y(t) = .70\, y(0)$, so we solve for t:

$$\begin{aligned} y(0)e^{kt} &= .7\, y(0) \\ e^{kt} &= .7 \\ kt &= \ln(.7) \\ t = \frac{\ln(.7)}{k} &= -\frac{5730\, \ln(.7)}{\ln 2} \cong 2950\,. \end{aligned}$$

Thus the parchment is about 2950 years old.

5.26 Note: Newton's Law of Cooling (or Warming) states that the rate of cooling (or warming) of an object is proportional to the temperature difference between the object and its surroundings. That is, if $T(t)$ is the temperature of the object at time t, and if K is the constant temperature of the surroundings, then

$$T'(t) = k(K - T)$$

for some constant k.

5.27 Example: A glass of water is taken from the refrigerator, where the temperature is 4°, and placed on a table, where the temperature is 20°. After 6 minutes, the water is found to be 11°. Find the temperature of the water after another 3 minutes.

Solution: Let $T(t)$ be the temperature of the water after t minutes. Then $T'(t) = k(20-T)$ for some constant k. This is a linear DE since we can write it as $T' + kT = 20k$ (its also separable). An integrating factor is $\lambda(t) = e^{\int k\, dt} = e^{kt}$, and the general solution is

$$T = \frac{1}{\lambda(t)} \int 20k\, \lambda(t)\, dt = e^{-kt} \int 20k\, e^{kt}\, dt = e^{-kt}(20e^{kt} + c) = 20 + c\, e^{kt}\,,$$

where c is a constant. Since $T(0) = 4$ we have $4 = 20 + c$ so $c = -16$. Thus $T(t) = 20 - 16e^{kt}$. Since $T(6) = 11$ we have $11 = 20 - 16e^{6k}$, so $e^{6k} = \frac{9}{16}$, and $k = \frac{1}{6}\ln(\frac{9}{16})$. Thus

$$T(t) = 20 - 16e^{(\frac{1}{6}\ln\frac{9}{16})t} = 20 - 16\big(\tfrac{9}{16}\big)^{t/6} = 20 - 16\big(\tfrac{3}{4}\big)^{t/3}\,,$$

and so $T(9) = 20 - 16\big(\frac{3}{4}\big)^3 = 20 - \frac{27}{4} = \frac{53}{4}$. The temperature after 9 minutes is 13.25°.

5.28 Example: In a simple electric circuit with a battery, producing a voltage of E volts, a resistor, of resistence R ohms, and an inductor, with an inductance of L henries, the current $I(t)$ at time t satisfies the DE

$$L\, I'(t) + R\, I(t) = E\,.$$

Given that $E = 12$, $R = 4$, $L = 2$ and $I(0) = 0$, find $I(t)$.

Solution: This DE is quite similar to the one that appears in Newton's Law of Cooling (or Warming). Put in the given values for E, R and L into the given DE to get $2\, I' + 4\, I = 12$. This is linear since we can write it as $I' + 2\, I = 6$. An integrating factor is $\lambda = e^{\int 2\, dt} = e^{2t}$ and the general solution is

$$I = e^{-2t} \int 6\, e^{2t}\, dt = e^{-2t}\big(3e^{2t} + c\big) = 3 + c\, e^{-2t}\,.$$

Put in $I(0) = 0$ to get $3 + c = 0$ so that $c = -3$, and so we have $I(t) = 3 - 3\, e^{-2t}$.

5.29 Note: In a typical **mixing problem**, a solution containing a given concentration c_1 of some substance (maybe salt in water) enters a tank at a fixed rate r_1. The mixture is kept stirred, and it is drained at another rate r_2. The problem is to find the amount $y(t)$ of substance in the tank at time t. We solve it by solving the DE $y'(t) = r_1c_1 - r_2c_2$, where c_2 is the concentration of the substance in the tank; that is $c_2 = \dfrac{y}{V}$, where V is the volume of the solution in the tank. If r_1 and r_2 are constant, then the volume is $V = V(0) + (r_1 - r_2)t$. In general, V satisfies the DE $V'(t) = r_1(t) - r_2(t)$.

5.30 Example: A tank contains 20 kg of salt dissolved in 5000 L of water. Brine, at a concentration of .03 kg/L, enters that tank at a rate of 25 L/min. The solution is kept well mixed and drains from the tank at the same rate. Find the amount of salt in the tank after 5 hours.

Solution: Let $y(t)$ be the amount of salt in the tank, in kilograms, after t minutes. We must solve the IVP $y' = r_1c_1 - r_2c_2$, with $y(0) = 20$, where $r_1 = r_2 = 25$, $c_1 = .03$, and $c_2 = \dfrac{y}{V} = \dfrac{y}{5000}$. The DE becomes $y' = (25)(.03) - (25)\dfrac{y}{5000} = .75 - \dfrac{y}{200}$, or equivalently $y' + \frac{1}{200}\,y = \frac{3}{4}$. This DE is linear (its also separable). An integrating factor is $\lambda = e^{\int \frac{1}{200}\,dt} = e^{t/200}$, and the general solution to the DE is

$$y(t) = \frac{1}{\lambda(t)}\int \tfrac{3}{4}\lambda(t)\,dt = e^{-t/200}\int \tfrac{3}{4}e^{t/200}\,dt = e^{-t/200}\big(150\,e^{t/200} + c\big) = 150 + c\,e^{-t/200}\,.$$

Since $y(0) = 20$ we have $150 + c = 20$ and so $c = -130$. Thus the solution is

$$y(t) = 150 - 130\,e^{-t/200}\,.$$

After 5 hours (that's 300 minutes), we have $y(300) = 150 - 130\,e^{-3/2} \cong 121$, so there will be about 121 kg of salt in the tank.

5.31 Note: According to **Toricelli's Law**, when a liquid drains through a hole in a tank of liquid, it flows through the hole at a speed which is proportional to the square root of the depth of the water above the hole. If the liquid is non-viscous, the speed is

$$v \cong \sqrt{2gy}$$

where g is the gravitational constant and y is the depth.

5.32 Example: A tank, in the shape of an inverted cone of radius 1 m and height 4 m, is filled with water. Water then drains from a hole of area 25 cm^2 at the bottom tip of the tank. If the water drains at a velocity of $v = 4\sqrt{y}$ m/s, where y m is the depth of the water in the tank, then find the time at which the tank will be empty.

Solution: Since the water drains at speed $v = 4\sqrt{y}$ from a hole of area $A = \frac{25}{10000} = \frac{1}{400}$ (in m^2), we have $V' = -Av = -\frac{1}{100}\sqrt{y}$, where $V = V(t)$ is the volume of water in the tank at time t. On the other hand, when the water is y m deep, the water in the tank forms a cone of height y and radius $\frac{1}{4}\,y$, so the volume is $V = \frac{1}{3}\,\pi\left(\frac{1}{4}\,y\right)^2 y = \frac{1}{48}\,\pi y^3$, and so we have $V' = \frac{1}{16}\,\pi y^2\,y'$. Equating these two expressions for V' we find that $\frac{1}{16}\,\pi y^2\,y' = -\frac{1}{100}\,y^{1/2}$, so y satisfies the DE $\frac{\pi}{16}\,y^2\,y' = -\frac{1}{100}\sqrt{y}$ which we can write as $y^{\frac{3}{2}}\,dy = -\frac{4}{25\pi}\,dt$. Integrate both sides to get $\frac{2}{5}\,y^{5/2} = -\frac{4}{25\pi}\,t + c$. Put in $y(0) = 4$ to get $c = \frac{2}{5}\cdot 32$, so we have $\frac{2}{5}\,y^{5/2} = \frac{2}{5}\cdot 32 - \frac{4}{25\pi}\,t$, that is $y = \left(32 - \frac{2}{5\pi}\,t\right)^{2/5}$. The tank will be empty when $y = 0$, and this happens when $\frac{2}{5\pi}\,t = 32$, that is when $t = 80\pi$ so it takes 80π seconds (that is about 4 minutes and 11 seconds) for the tank to empty.

Chapter 6. Sequences and Series

Sequences (Review)

6.1 Definition: A **sequence** (of real numbers) is a function $a : \{k, k+1, k+2, \cdots\} \to \mathbf{R}$ for some integer k. For a sequence $a : \{k, k+1, \cdots\} \to \mathbf{R}$, we write $a_n = a(n)$ for $n \geq k$, we refer to the function a as the sequence $\{a_n\}$ or the sequence $\{a_n\}_{n\geq k}$, and we write

$$\{a_n\}_{n\geq k} = a_k, a_{k+1}, a_{k+2}, \cdots$$

We say that $\{a_n\}_{n\geq k}$ lies in the set $I \subset \mathbf{R}$ when $a_n \in I$ for every $n \geq k$.

6.2 Definition: We say the sequences $\{a_n\}_{n\geq k}$ **converges** to the real number l, or that the **limit** of the sequence $\{a_n\}_{n\geq k}$ is equal to l, and we write $\lim\limits_{n\to\infty} a_n = l$ or we write $a_n \to l$ (as $n \to \infty$), when for every $\epsilon > 0$ there exists $N \geq k$ such that for every integer n we have

$$n > N \Longrightarrow |a_n - l| < \epsilon\,.$$

We say the sequence $\{a_n\}$ **converges** if it converges to some real number l.

We say the limit of $\{a_n\}$ is equal to infinity, and write $\lim\limits_{n\to\infty} a_n = \infty$ or $a_n \to \infty$ when for every $R \in \mathbf{R}$ there exists $N \geq k$ such that for every integer n we have

$$n > N \Longrightarrow a_n > R\,.$$

We say the limit of $\{a_n\}$ is equal to negative infinity and write $\lim\limits_{n\to\infty} a_n = -\infty$ or $a_n \to -\infty$ when for every $R \in \mathbf{R}$ there exists $N \geq k$ such that for every integer n we have

$$n > N \Longrightarrow a_n < R\,.$$

6.3 Theorem: *(First Finitely Many Terms do Not Affect Convergence) Let l be a positive integer. Then $\lim\limits_{n\to\infty} a_n$ exists if and only if $\lim\limits_{n\to\infty} a_{n+l}$ exists, and in this case the limits are equal.*

6.4 Theorem: *(Linearity, Products and Quotients) If $\{a_n\}$ and $\{b_n\}$ are convergent sequences then*
(1) for any real number c, $\{c\,a_n\}$ converges with $\lim\limits_{n\to\infty} c\,a_n = c \lim\limits_{n\to\infty} a_n$,
(2) the sequence $\{a_n + b_n\}$ converges with $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$,
(3) the sequence $\{a_n b_n\}$ converges with $\lim\limits_{n\to\infty} (a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right)$, and
(4) if $\lim\limits_{n\to\infty} b_n \neq 0$ then the sequence $\left\{\dfrac{a_n}{b_n}\right\}$ converges with $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$.

6.5 Note: By defining algebraic operations on the extended real line $\mathbf{R}\cup\{\pm\infty\}$, the above theorem can be extended to include many cases in which $\lim\limits_{n\to\infty} a_n = \pm\infty$ or $\lim\limits_{n\to\infty} b_n = \pm\infty$, but some care is needed for the so called **indeterminate forms** $\infty - \infty$, $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$.

6.6 Theorem: *(Comparison and Squeeze) Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences.*
(1) If $a_n \leq b_n$ for all n and $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ both exist, then $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$
(2) If $a_n \leq b_n \leq c_n$ for all $n \geq k$ and if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n$ then $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} a_n$.

6.7 Theorem: *(Sequences and Absolute Values) Let* $\{a_n\}$ *be a sequence.*
(1) If $\lim_{n\to\infty} a_n$ *exists then* $\lim_{n\to\infty} |a_n| = \left|\lim_{n\to\infty} a_n\right|$.
(2) If $\lim_{n\to\infty} |a_n| = 0$ *then* $\lim_{n\to\infty} a_n = 0$.
(3) If $|a_n| \le b_n$ *for all* $n \ge k$ *and* $\lim_{n\to\infty} b_n = 0$ *then* $\lim_{n\to\infty} a_n = 0$.

6.8 Definition: The sequence $\{a_n\}_{n\ge k}$ is called **increasing** when $a_n \le a_{n+1}$ for all $n \ge k$, or equivalently when $n \le m \Longrightarrow a_n \le a_m$ for all integers $n, m \ge k$. The sequence $\{a_n\}_{n\ge k}$ is called **strictly increasing** when $a_n < a_{n+1}$ for all $n \ge k$. The sequence $\{a_n\}_{n\ge k}$ is **bounded above** by the real number b when $a_n \le b$ for all $n \ge k$, and in this case b is called an **upper bound** for the sequence. We say that $\{a_n\}$ is **bounded above** when it is bounded above by some real number b. We have similar definitions for the terms **decreasing**, **strictly decreasing**, **bounded below** and **lower bound**.

6.9 Theorem: *(Monotone Convergence)*
(1) If $\{a_n\}$ *is increasing and bounded above by* b*, then* $\{a_n\}$ *converges and* $\lim_{n\to\infty} a_n \le b$.
(2) If $\{a_n\}$ *is increasing and is not bounded above, then* $\lim_{n\to\infty} a_n = \infty$.
(3) If $\{a_n\}$ *is decreasing and bounded below by* c*, then* $\{a_n\}$ *converges and* $\lim_{n\to\infty} a_n \ge c$.
(4) If $\{a_n\}$ *is decreasing and is not bounded below, then* $\lim_{n\to\infty} a_n = -\infty$.

6.10 Definition: A sequence $\{a_n\}_{n\ge k}$ is said to be **Cauchy** when for every $\epsilon > 0$ there exists N such that for all integers n, m we have

$$n, m > N \Longrightarrow |a_n - a_m| < \epsilon\,.$$

6.11 Theorem: *(Cauchy Criterion) A sequence* $\{a_n\}$ *converges if and only if it is Cauchy.*

6.12 Theorem: *(Sequences and Functions) Let* I *be an interval which either contains* a *or has* a *as an endpoint (in which case* a *can be finite or infinite), and let* $f : I \to \mathbf{R}$. *Then* $\lim_{x\to a} f(x) = l$ *(where* l *can be finite or infinite) if and only if* $\lim_{n\to\infty} f(a_n) = l$ *for every sequence* $\{a_n\}$ *in* $I \setminus \{a\}$ *with* $\lim_{n\to\infty} a_n = a$.

Series

6.13 Definition: Let $\{a_n\}_{n\geq k}$ be a sequence. The **series** $\sum_{n\geq k} a_n$ is defined to be the sequence $\{S_l\}_{l\geq k}$ where

$$S_l = \sum_{n=k}^{l} a_n = a_k + a_{k+1} + \cdots + a_l\,.$$

The term S_l is called the l^{th} **partial sum** of the series $\sum_{n\geq k} a_n$. The **sum** of the series, denoted by

$$S = \sum_{n=k}^{\infty} a_n = a_k + a_{k+1} + a_{k+2} + \cdots\,,$$

is the limit of the sequence of partial sums, if it exists, and we say the series **converges** when the sum exists and is finite.

6.14 Example: (Geometric Series) Show that for $a \neq 0$, the series $\sum_{n\geq k} a_n$ converges if and only if $|r| < 1$, and that in this case

$$\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}\,.$$

Solution: The l^{th} partial sum is

$$S_l = \sum_{n=k}^{\infty} ar^n = ar^k + ar^{k+1} + ar^{k+2} + \cdots + ar^l\,.$$

When $r = 1$ we have $S_l = a(l-k+1)$ and so $\lim\limits_{l\to\infty} S_l = \pm\infty$ ($+\infty$ when $a > 0$ and $-\infty$ when $a < 0$). When $r \neq 1$ we have $rS_l = ar^{k+1} + ar^{k+2} + \cdots + ar^l + ar^{l+1}$, so $S_l - rS_l = ar^k - ar^{l+1} = ar^k\big(1 - r^{l-k+1}\big)$ and so

$$S_l = \frac{ar^k(1 - r^{l-k+1})}{1-r}\,.$$

When $r > 1$, $\lim\limits_{l\to\infty} r^{l-k+1} = \infty$ and so $\lim\limits_{l\to\infty} S_l = \pm\infty$ ($+\infty$ when $a > 0$ and $-\infty$ when $a < 0$). When $r \leq -1$, $\lim\limits_{l\to\infty} r^{l-k+1}$ does not exist, and so neither does $\lim\limits_{l\to\infty} S_l$. When $|r| < 1$, we have $\lim\limits_{l\to\infty} r^{l-k+1} = 0$ and so $\lim\limits_{l\to\infty} S_l = \dfrac{ar^k}{1-r}$, as required.

6.15 Example: Find $\displaystyle\sum_{n=-1}^{\infty} \frac{3^{n+1}}{2^{2n-1}}$.

Solution: This is a geometric series. By the formula in the previous example, we have

$$\sum_{n=-1}^{\infty} \frac{3^{n+1}}{2^{2n-1}} = \sum_{n=-1}^{\infty} \tfrac{9}{2}\left(\tfrac{3}{4}\right)^n = \frac{\frac{9}{2}\left(\frac{3}{4}\right)^{-1}}{1-\frac{3}{4}} = \tfrac{9}{2}\cdot\tfrac{4}{3}\cdot\tfrac{4}{1} = 24\,.$$

6.16 Example: (Telescoping Series) Find $\displaystyle\sum_{i=1}^{\infty} \frac{1}{n^2+2n}$.

Solution: We use a partial fractions decomposition. The l^{th} partial sum is

$$\begin{aligned} S_l &= \sum_{n=1}^{l} \frac{1}{n(n+2)} = \sum_{n=1}^{l} \left(\frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2} \right) = \tfrac{1}{2} \sum_{n=1}^{l} \left(\tfrac{1}{n} - \tfrac{1}{n+2} \right) \\ &= \tfrac{1}{2} \Big(\big(1 - \tfrac{1}{3}\big) + \big(\tfrac{1}{2} - \tfrac{1}{4}\big) + \big(\tfrac{1}{3} - \tfrac{1}{5}\big) + \cdots + \left(\tfrac{1}{n-2} - \tfrac{1}{n}\right) + \left(\tfrac{1}{n-1} - \tfrac{1}{n+1}\right) + \left(\tfrac{1}{n} - \tfrac{1}{n+2}\right) \Big) \\ &= \tfrac{1}{2} \left(1 + \tfrac{1}{2} - \tfrac{1}{n+1} - \tfrac{1}{n+2} \right) , \end{aligned}$$

since all the other terms cancel. Thus the sum of the series is

$$S = \lim_{l \to \infty} S_l = \tfrac{1}{2}\left(1 + \tfrac{1}{2}\right) = \tfrac{3}{4} \,.$$

6.17 Theorem: *(First Finitely Many Terms do Not Affect Convergence) Let $\{a_n\}_{n \geq k}$ be a sequence. Then for any integer $m \geq k$, the series $\displaystyle\sum_{n \geq k} a_n$ converges if and only if the series $\displaystyle\sum_{n \geq m} a_n$ converges, and in this case*

$$\sum_{n=k}^{\infty} a_n = \big(a_k + a_{k+1} + \cdots + a_{m-1}\big) + \sum_{n=m}^{\infty} a_n \,.$$

Proof: Let $S_l = \displaystyle\sum_{n=k}^{l} a_n$ and let $T_l = \displaystyle\sum_{n=m}^{l} a_n$. Then for all $l \geq m$ we have

$$S_l = \big(a_k + a_{k+1} + \cdots + a_{m-1}\big) + T_l \,,$$

and so $\{S_l\}$ converges if and only if $\{T_l\}$ converges, and in this case

$$\lim_{l \to \infty} S_l = \big(a_k + a_{k+1} + \cdots + a_{m-1}\big) + \lim_{l \to \infty} T_l \,.$$

6.18 Note: Since the first finitely many terms do not affect the convergence of a series, we often omit the subscript $n \geq k$ in the expression $\displaystyle\sum_{n \geq k} a_n$ when we are interested in whether or not the series converges. On the other hand, we cannot omit the subscript $n = k$ when we are interested in the value of the sum $\displaystyle\sum_{n=k}^{\infty} a_n$.

6.19 Definition: When we approximate a value x by the value y, the **error** in our approximation is $|x - y|$.

6.20 Note: If $\displaystyle\sum_{n \geq k} a_n$ converges and $l \geq k$ then, by the above theorem, so does $\displaystyle\sum_{n \geq l+1}^{\infty} a_n$. If we approximate the sum $S = \displaystyle\sum_{n=k}^{\infty} a_n$ by the l^{th}partial sum $S_l = \displaystyle\sum_{n=k}^{l} a_n$, then the **error** in our approximation is

$$\big|S - S_l\big| = \left| \sum_{n=l+1}^{\infty} a_n \right| \,.$$

6.21 Theorem: *(Linearity) If* $\sum a_n$ *and* $\sum b_n$ *are convergent series then*

(1) for any real number c, $\sum c a_n$ *converges and* $\displaystyle\sum_{n=k}^{\infty} c a_n = c \sum_{n=k}^{\infty} a_n$, *and*

(2) the series $\sum (a_n + b_n)$ *converges and* $\displaystyle\sum_{n=k}^{\infty}(a_n + b_n) = \sum_{n=k}^{\infty} a_n + \sum_{n=k}^{\infty} b_n$.

Proof: This follows immediately from the Linearity Theorem for sequences.

6.22 Theorem: *(Series of Positive Terms) Let* $\sum a_n$ *be a series.*

(1) If $a_n \geq 0$ *for all* $n \geq k$ *then either* $\sum a_n$ *converges or* $\displaystyle\sum_{n=k}^{\infty} a_n = \infty$.

(2) If $a_n \leq 0$ *for all* $n \geq k$ *then either* $\sum a_n$ *converges or* $\displaystyle\sum_{n=k}^{\infty} a_n = -\infty$.

Proof: This follows from the Monotone Convergence Theorem for sequences. Indeed if $a_n \geq 0$ for all $n \geq k$, then $\{S_l\}$ is increasing (since $S_{l+1} = S_l + a_{l+1} \geq S_l$ for all l). Either $\{S_l\}$ is bounded above, in which case $\{S_l\}$ converges hence $\sum a_n$ converges, or $\{S_l\}$ is unbounded, in which case $\displaystyle\lim_{n\to\infty} S_l = \infty$ hence $\displaystyle\sum_{n=k}^{\infty} a_n = \infty$.

6.23 Theorem: *(Cauchy Criterion) Let* $\sum a_n$ *be a series. Then* $\sum a_n$ *converges if and only if for all* $\epsilon > 0$ *there exists* N *such that for all* $l, m \in \mathbf{Z}$, $m > l > N \Longrightarrow \left| \displaystyle\sum_{n=l+1}^{m} a_n \right| < \epsilon$.

Proof: This follows from the Cauchy Criterion for the convergence of the sequence of partial sums. Indeed $\{S_l\}$ converges if and only if for all $\epsilon > 0$ there exists N such that $m > l > N \Longrightarrow |S_m - S_l| < \epsilon$, and we have $|S_m - S_l| = \left| \displaystyle\sum_{n=k}^{m} a_n - \sum_{n=k}^{l} a_n \right| = \left| \displaystyle\sum_{n=l+1}^{m} a_n \right|$.

Convergence Tests

6.24 Theorem: *(Divergence Test) If* $\sum a_n$ *converges then* $\lim\limits_{n\to\infty} a_n = 0$*. Equivalently, if* $\lim\limits_{n\to\infty} a_n$ *either does not exist, or exists but is not equal to 0, then* $\sum a_n$ *diverges.*

Proof: Suppose that $\sum a_n$ converges, and say $\sum\limits_{n=k}^{\infty} a_n = S$. Let S_l be the l^{th}partial sum. Then $\lim\limits_{l\to\infty} S_l = S = \lim\limits_{l\to\infty} S_{l-1}$, and we have $a_l = S_l - S_{l-1}$, and so

$$\lim_{l\to\infty} a_l = \lim_{l\to\infty} S_l - \lim_{l\to\infty} S_{l-1} = S - S = 0\,.$$

6.25 Example: Determine whether $\sum e^{1/n}$ converges.

Solution: Since $\lim\limits_{n\to\infty} e^{1/n} = e^0 = 1$, $\sum e^{1/n}$ diverges by the Divergence Test.

6.26 Note: The converse of the Divergence Test is false. For example, as we shall see below, $\sum \frac{1}{n}$ diverges even though $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

6.27 Theorem: *(Integral Test) Let* $f(x)$ *be positive and decreasing for* $x \ge k$*, and let* $a_n = f(n)$ *for all integers* $n \ge k$*. Then* $\sum a_n$ *converges if and only if* $\int_k^\infty f(x)\,dx$ *converges, and in this case, for any* $l \ge k$ *we have*

$$\int_{l+1}^{\infty} f(x)\,dx \le \sum_{n=l+1}^{\infty} a_n \le \int_l^\infty f(x)\,dx\,.$$

Proof: Let T_m be the m^{th} partial sum for $\sum\limits_{n\ge l+1} a_n$, so $T_m = \sum\limits_{n=l+1}^{m} a_n$. Note that since $f(x)$ is decreasing, it is integrable on any closed interval. Also, for each $n \ge l$ we have $a_n = f(n) \le f(x)$ for all $x \in [n-1, n]$, so $\int_{n-1}^{n} f(x)\,dx \ge \int_{n-1}^{n} a_n\,dx = a_n$ and so

$$T_m = \sum_{n=l+1}^{m} a_n \le \sum_{n=l+1}^{m} \int_{n-1}^{n} f(x)\,dx = \int_l^m f(x)\,dx \le \int_l^\infty f(x)\,dx\,.$$

Since $f(n) = a_n$ is positive, the sequence $\{T_m\}$ is increasing. If $\int_k^\infty f$ converges, then $\{T_n\}$ is bounded above by $\int_l^\infty f(x)\,dx$, and so it converges with $\lim\limits_{m\to\infty} T_m \le \int_l^\infty f(x)\,dx$.

Similarly, for each $n \ge l$ we have $a_n = f(n) \ge f(x)$ for all $x \in [n, n+1]$ so that $\int_n^{n+1} f(x)\,dx \le \int_n^{n+1} a_n dx = a_n$ and so

$$T_m = \sum_{n=l+1}^{m} a_n \ge \sum_{n=l+1}^{m} \int_n^{n+1} f(x)\,dx = \int_{l+1}^{m+1} f(x)\,dx\,.$$

If $\int_k^\infty f$ converges, then $\lim\limits_{m\to\infty} T_m \ge \lim\limits_{m\to\infty} \int_{l+1}^{m+1} f(x)\,dx = \int_{l+1}^{\infty} f(x)\,dx$. If $\int_k^\infty f = \infty$ then $\lim\limits_{m\to\infty} \int_{l+1}^{m+1} f(x)\,dx = \infty$, and so $\lim\limits_{m\to\infty} T_m = \infty$ too, by Comparison.

6.28 Example: (p-Series) Show that the series $\displaystyle\sum_{n\geq 1}\frac{1}{n^p}$ converges if and only if $p>1$. In particular, the **harmonic series** $\sum \frac{1}{n}$ diverges.

Solution: If $p<0$ then $\displaystyle\lim_{n\to\infty}\frac{1}{n^p}=\infty$ and if $p=0$ then $\displaystyle\lim_{n\to\infty}\frac{1}{n^p}=1$, so in either case $\sum\frac{1}{n^p}$ diverges by the Divergence Test. Suppose that $p>0$. Let $a_n=\frac{1}{n^p}$ for integers $n\geq 1$, and let $f(x)=\frac{1}{x^p}$ for real numbers $x\geq 1$. Note that $f(x)$ is positive and decreasing for $x\geq 1$ and $a_n=f(n)$ for all $n\geq 1$. Since we know that $\displaystyle\int_1^\infty f(x)\,dx$ converges if and only if $p>1$, it follows from the Integral Test that $\sum a_n$ converges if and only if $p>1$.

6.29 Example: Approximate $S=\displaystyle\sum_{n=1}^\infty\frac{1}{2n^2}$ so that the error is at most $\frac{1}{100}$.

Solution: We let $a_n=\frac{1}{2n^2}$ and $f(x)=\frac{1}{2x^2}$ so that we can apply the Integral Test. If we choose to approximate the sum S by the l^{th}partial sum S_l, then the error is

$$E=S-S_l=\sum_{n=l+1}^\infty a_n\leq\int_l^\infty\frac{1}{2x^2}\,dx=\left[-\frac{1}{2x}\right]_l^\infty=\frac{1}{2l}\,,$$

and so to insure that $E\leq\frac{1}{100}$ we can choose l so that $\frac{1}{2l}\leq\frac{1}{100}$, that is $l\geq 50$. Since it would be tedious to add up the first 50 terms of the series, we take an alternate approach. The Integral Test gives us upper and lower bounds: we have

$$\begin{aligned}\int_{l+1}^\infty f(x)\,dx\leq S-S_l&\leq\int_l^\infty f(x)\,dx\\ \frac{1}{2(l+1)}\leq S-S_l&\leq\frac{1}{2l}\\ S_l+\frac{1}{2(l+1)}\ \leq S\leq&\ S_l+\frac{1}{2l}\,.\end{aligned}$$

If approximate S using the midpoint of the upper and lower bounds, that is if we make the approximation $S\cong S_l+\frac{1}{2}\left(\frac{1}{2l}+\frac{1}{2(l+1)}\right)$, then the error E will be at most half of the difference of the bounds:

$$E\leq\tfrac{1}{2}\left(\tfrac{1}{2l}-\tfrac{1}{2(l+1)}\right)=\tfrac{1}{4l(l+1)}\,.$$

To get $E\leq\frac{1}{100}$ we want $\frac{1}{4\,l\,(l+1)}\leq\frac{1}{100}$, that is $l\,(l+1)\geq 25$, and so we can take $l=5$. Thus we estimate

$$S\cong S_5+\tfrac{1}{2}\left(\tfrac{1}{10}+\tfrac{1}{12}\right)=\tfrac{1}{2}+\tfrac{1}{8}+\tfrac{1}{18}+\tfrac{1}{32}+\tfrac{1}{50}+\tfrac{1}{20}+\tfrac{1}{24}=\tfrac{5929}{7200}\,.$$

$\big($Incidentally, the exact value of this sum is $\frac{\pi^2}{12}\big)$.

6.30 Theorem: *(Comparison Test) Let $0 \leq a_n \leq b_n$ for all $n \geq k$. Then if $\sum b_n$ converges then so does $\sum a_n$ and in this case,*

$$\sum_{n=k}^{\infty} a_n \leq \sum_{n=k}^{\infty} b_n \,.$$

Proof: Let $S_l = \displaystyle\sum_{n=k}^{l} a_n$ and let $T_l = \displaystyle\sum_{n=k}^{l} b_n$. Since $0 \leq a_n, b_n$ for all n, the sequences $\{S_l\}$ and $\{T_l\}$ are increasing. Since $a_n \leq b_n$ for all n we have $S_l \leq T_l$ for all l. Suppose that $\sum b_n$ converges with say $\displaystyle\sum_{n=k}^{\infty} b_n = T$ so that $\displaystyle\lim_{l\to\infty}\{T_l\} = T$. Then $S_l \leq T_l \leq T$ for all l, so $\{S_l\}$ is increasing and bounded above, hence convergent, and $\displaystyle\lim_{l\to\infty} S_l \leq \lim_{l\to\infty} T_l$.

6.31 Example: Determine whether $\displaystyle\sum_{n\geq 0} \frac{1}{\sqrt{n^3+1}}$ converges.

Solution: Note that $0 \leq \frac{1}{\sqrt{n^3+1}} \leq \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$ for all $n \geq 1$, and $\sum \frac{1}{n^{3/2}}$ converges since it is a p-series with $p = \frac{3}{2}$, and so $\sum \frac{1}{\sqrt{n^3+1}}$ also converges, by the Comparison Test.

6.32 Example: Determine whether $\displaystyle\sum_{n\geq 1} \tan \tfrac{1}{n}$ converges.

Solution: For $0 < x < \frac{\pi}{2}$ we have $x < \tan x$, so for $n \geq 1$ we have $0 < \frac{1}{n} < \tan \frac{1}{n}$. Since the harmonic series $\sum \frac{1}{n}$ diverges, the series $\sum \tan \frac{1}{n}$ also diverges by the Comparison Test.

6.33 Example: Approximate $S = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ so that the error is at most $\frac{1}{100}$.

Solution: If we make the approximation $S \cong S_l = \displaystyle\sum_{n=0}^{l} \frac{1}{n!}$ then the error is

$$\begin{aligned}
E = S - S_l &= \sum_{n=l+1}^{\infty} \frac{1}{n!} \\
&= \tfrac{1}{(l+1)!} + \tfrac{1}{(l+2)!} + \tfrac{1}{(l+3)!} + \tfrac{1}{(l+4)!} + \cdots \\
&= \tfrac{1}{(l+1)!}\left(1 + \tfrac{1}{l+2} + \tfrac{1}{(l+2)(l+3)} + \tfrac{1}{(l+2)(l+3)(l+4)} + \cdots\right) \\
&\leq \tfrac{1}{(l+1)!}\left(1 + \tfrac{1}{l+2} + \tfrac{1}{(l+2)^2} + \tfrac{1}{(l+2)^3} + \cdots\right) \\
&= \tfrac{1}{(l+1)!}\frac{1}{1 - \frac{1}{l+2}} \\
&= \tfrac{l+2}{(l+1)(l+1)!}
\end{aligned}$$

where we used the Comparison Test and the formula for the sum of a geometric series. To get $E \leq \frac{1}{100}$ we can choose l so that $\frac{l+2}{(l+1)(l+1)!} \leq \frac{1}{100}$. By trial and error, we find that we can take $l = 4$, so we make the approximation

$$S \cong S_4 = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} = \tfrac{65}{24}\,.$$

(Incidentally, we shall see later that the exact value of this sum is e).

6.34 Theorem: *(Limit Comparison Test) Let $a_n \geq 0$ and let $b_n > 0$ for all $n \geq k$. Suppose that $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = r$. Then*
(1) if $r = \infty$ and $\sum a_n$ converges then so does $\sum b_n$,
(2) if $r = 0$ and $\sum b_n$ converges then so does $\sum a_n$, and
(3) if $0 < r < \infty$ then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Proof: If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$, then for large n we have $\frac{a_n}{b_n} > 1$ so that $a_n > b_n$, and so if $\sum a_n$ converges, then so does $\sum b_n$ by the Comparison Test. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ then for large n we have $\frac{a_n}{b_n} < 1$ so $a_n < b_n$, and so if $\sum b_n$ converges then so does $\sum a_n$ by the Comparison Test. Suppose that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = r$ with $0 < r < \infty$. Choose N so that when $n > N$ we have $\left|\frac{a_n}{b_n} - r\right| < \frac{r}{2}$ so that $\frac{r}{2} < \frac{a_n}{b_n} < \frac{3r}{2}$ and hence

$$0 < \tfrac{r}{2} b_n \leq a_n \leq \tfrac{3r}{2} b_n\,.$$

If $\sum a_n$ converges, then $\sum \frac{r}{2} b_n$ converges by the Comparison Test, and hence $\sum b_n$ converges by linearity. If $\sum b_n$ converges, then $\sum \frac{3r}{2} b_n$ converges by linearity, and hence so does $\sum a_n$ by the Comparison Test.

6.35 Example: Determine whether $\sum \frac{1}{\sqrt{n^3-1}}$ converges.

Solution: Note that we cannot use the same argument that we used earlier to show that $\sum \frac{1}{\sqrt{n^3+1}}$ converges, because $\frac{1}{\sqrt{n^3+1}} < \frac{1}{n^{3/2}}$ but $\frac{1}{\sqrt{n^3-1}} > \frac{1}{n^{3/2}}$. We use a different approach. Let $a_n = \frac{1}{\sqrt{n^3-1}}$ and let $b_n = \frac{1}{n^{3/2}}$. Then $\lim \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^{3/2}}{\sqrt{n^3-1}} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1-\frac{1}{n^3}}} = 1$, and $\sum b_n = \sum \frac{1}{n^{3/2}}$ converges (its a p-series with $p = \frac{3}{2}$), and so $\sum a_n$ converges too, by the Limit Comparison Test.

6.36 Theorem: *(Ratio Test) Let* $a_n > 0$ *for all* $n \geq k$*. Suppose* $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = r$*. Then*
(1) if $r < 1$ *then* $\sum a_n$ *converges, and*
(2) if $r > 1$ *then* $\lim\limits_{n\to\infty} a_n = \infty$ *so* $\sum a_n = \infty$*.*

Proof: Suppose that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$. Choose s with $r < s < 1$, and then choose N so that when $n > N$ we have $\frac{a_{n+1}}{a_n} < s$ and hence $a_{n+1} < s\,a_n$. Fix $k > N$. Then $a_{k+1} < sa_k$, $a_{k+2} < sa_{k+1} < s^2 a_k$, $a_{k+3} < sa_{k+2} < s^3 a_k$, and so on, so we have $a_n < b_n = s^{n-k} a_k$ for all $n \geq k$. Since $\sum b_n$ is geometric with ratio $s < 1$, it converges, and hence so does $\sum a_n$ by the Comparison Test.

Now suppose that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = r > 1$. Choose s with $1 < s < r$, then choose N so that when $n > N$ we have $\frac{a_{n+1}}{a_n} > s$ and hence $a_{n+1} > sa_n$. Fix $k > N$. Then as above $a_n > b_n = s^{n-k} a_k$ for all $n \geq k$, and $\lim\limits_{n\to\infty} b_n = \infty$, so $\lim\limits_{n\to\infty} a_n = \infty$ too.

6.37 Example: Determine whether $\sum \frac{5^n}{n!}$ converges.

Solution: Let $a_n = \frac{5^n}{n!}$. Then $\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \frac{5}{n+1} \to 0$ as $n \to \infty$, and so $\sum a_n$ converges by the Ratio Test.

6.38 Note: If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, then $\sum a_n$ could converge or diverge. For example, if $a_n = \frac{1}{n}$ then $\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \to 1$ as $n \to \infty$ and $\sum a_n$ diverges, but if $b_n = \frac{1}{n^2}$ then $\frac{b_{n+1}}{b_n} = \frac{n^2}{(n+1)^2} \to 1$ as $n \to \infty$ and $\sum b_n$ converges.

6.39 Theorem: *(Root Test) Let* $a_n \geq 0$ *for all* $n \geq k$*. Let* $r = \limsup\limits_{n\to\infty} \sqrt[n]{a_n}$*. Then*
(1) if $r < 1$ *then* $\sum a_n$ *converges, and*
(2) if $r > 1$ *then* $\lim\limits_{n\to\infty} a_n = \infty$ *so* $\sum a_n = \infty$*.*

Proof: The proof is left as an exercise. It is similar to the proof of the Ratio Test.

6.40 Example: Determine whether $\sum \left(\frac{n}{n+1}\right)^{n^2}$ converges.

Solution: Let $a_n = \left(\frac{n}{n+1}\right)^{n^2}$. Then $\sqrt[n]{a_n} = \left(\frac{n}{n+1}\right)^n = e^{n \ln\left(\frac{n}{n+1}\right)}$, and by l'Hôpital's Rule we have $\lim\limits_{n\to\infty} n \ln\left(\frac{n}{n+1}\right) = \lim\limits_{x\to\infty} \dfrac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{(x+1)^2}}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \dfrac{-x^2}{(x+1)^2} = -1$, and so $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = e^{-1} < 1$. Thus $\sum a_n$ converges by the Root Test.

6.41 Definition: A sequence $\{a_n\}_{n\geq k}$ is said to be **alternating** when either we have $a_n = (-1)^n |a_n|$ for all $n \geq k$ or we have $a_n = (-1)^{n+1}|a_n|$ for all $n \geq k$.

6.42 Theorem: *(Aternating Series Test) Let $\{a_n\}_{n\geq k}$ be an alternating series. If $\{|a_n|\}$ is decreasing with $\lim\limits_{n\to\infty} |a_n| = 0$ then $\sum\limits_{n\geq k} a_n$ converges, and in this case*

$$\left|\sum_{n=k}^{\infty} a_n\right| \leq |a_k| \,.$$

Proof: To simplify notation, we give the proof in the case that $k = 0$ and $a_n = (-1)^n|a_n|$. Suppose that $\big\{|a_n|\big\}$ is decreasing with $|a_n| \to 0$. Let $S_l = \sum\limits_{n=0}^{l} a_n$. We consider the sequences $\{S_{2l}\}$ and $\{S_{2l-1}\}$ of even and odd partial sums. Note that since $\big\{|a_n|\big\}$ is decreasing, we have

$$S_{2l} - S_{2l-1} = |a_{2l}| - |a_{2l-1}| \leq 0$$

so $\big\{S_{2l}\big\}$ is decreasing, and we have

$$\begin{aligned} S_{2l} &= |a_0| - |a_1| + |a_2| - |a_3| + \cdots + |a_{2l-2}| - |a_{2l-1}| + |a_{2l}| \\ &= \big(|a_0| - |a_1|\big) + \big(|a_2| - |a_3|\big) + \cdots + \big(|a_{2l-2}| - |a_{2l-1}|\big) + |a_{2l}| \\ &\geq |a_0| - |a_1| \end{aligned}$$

and so $\{S_{2l}\}$ is bounded below by $|a_0| - |a_1|$. Thus $\big\{S_{2l}\big\}$ converges by the Monotone Convergence Theorem. Similarly, $\big\{S_{2l-1}\big\}$ is increasing and bounded above by $|a_0|$, so it also converges, and we have $\lim\limits_{l\to\infty} S_{2l-1} \leq |a_0|$.

Finally we note that since $|a_n| \to 0$, taking the limit on both sides of the equality $|a_{2l}| = S_{2l} - S_{2l-1}$ gives $0 = \lim\limits_{l\to\infty} S_{2l} - \lim\limits_{l\to\infty} S_{2l-1}$. and so we have $\lim\limits_{l\to\infty} S_{2l} = \lim\limits_{l\to\infty} S_{2l-1}$. It follows that $\big\{S_l\big\}$ converges, and we have $\lim\limits_{l\to\infty} S_l = \lim\limits_{l\to\infty} S_{2l} = \lim\limits_{l\to\infty} S_{2l-1} \leq |a_0|$.

6.43 Example: Determine whether $\displaystyle\sum_{n\geq 2} \frac{(-1)^n \ln n}{\sqrt{n}}$ converges.

Solution: Let $a_n = \dfrac{(-1)^n \ln n}{\sqrt{n}}$. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$ so that $|a_n| = f(n)$. Note that

$$f'(x) = \frac{\frac{1}{x}\cdot\sqrt{x} - \ln x \cdot \frac{1}{2\sqrt{x}}}{x} = \frac{2 - \ln x}{2x^{3/2}}\,,$$

so we have $f'(x) < 0$ for $x > e^2$. Thus $f(x)$ is decreasing for $x > e^2$, and so $\big\{|a_n|\big\}$ is decreasing for $n \geq 8$. Also, by l'Hôpital's Rule, we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \tfrac{2}{\sqrt{x}} = 0$$

and so $|a_n| \to 0$ as $n \to \infty$. Thus $\sum a_n$ converges by the Alternating Series Test.

6.44 Example: Approximate the sum $S = \sum_{n=0}^{\infty} \dfrac{(-2)^n}{(2n)!}$ so that the error is at most $\frac{1}{2000}$.

Solution: Let $a_n = \dfrac{(-2)^n}{(2n)!}$. Note that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{2^n} = \frac{2}{(2n+2)(2n+1)} = \frac{1}{(n+1)(2n+1)} \,.$$

Since $\frac{|a_{n+1}|}{|a_n|} \leq 1$ for all $n \geq 0$, we know that $\{|a_n|\}$ is decreasing. Since $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$, we know that $\sum |a_n|$ converges by the Ratio Test, and so $|a_n| \to 0$ by the Divergence Test. This shows that we can apply the Alternating Series Test.

If we approximate S by the l^{th}partial sum $S_l = \sum_{n=0}^{l} a_n$, then by the Alternating Series Test, the error is

$$E = \big|S - S_l\big| = \left|\sum_{n=l+1}^{\infty} a_n\right| \leq \big|a_{l+1}\big| = \frac{2^{l+1}}{(2l+2)!} \,.$$

To get $E \leq \frac{1}{2000}$ we can choose l so that $\frac{2^{l+1}}{(l+1)!} \leq \frac{1}{2000}$. By trial and error we find that we can take $l = 3$. Thus we make the approximation

$$S \cong S_3 = 1 - \tfrac{2}{2!} + \tfrac{2^2}{4!} - \tfrac{2^3}{6!} = 1 - 1 + \tfrac{1}{6} + \tfrac{1}{90} = \tfrac{7}{45} \,.$$

(We shall see later that the exact value of this sum is $\cos\sqrt{2}$).

6.45 Definition: A series $\sum_{n\geq k} a_n$ is said to **converge absolutely** when $\sum_{n\geq k} |a_n|$ converges. The series is said to **converge conditionally** if $\sum_{n\geq k} a_n$ converges but $\sum_{n\geq k} |a_n|$ diverges.

6.46 Example: For $0 < p \leq 1$, the p-series $\sum \frac{1}{n^p}$ diverges, but since $\left\{\frac{1}{n^p}\right\}$ is decreasing towards 0, $\sum \frac{(-1)^n}{n^p}$ converges by the Alternating Series Test. Thus for $0 < p \leq 1$, the alternating p-series $\sum \dfrac{(-1)^n}{n^p}$ converges conditionally.

6.47 Theorem: *(Absolute Convergence Implies Convergence) If $\sum |a_n|$ converges then so does $\sum a_n$.*

Proof: Suppose that $\sum |a_n|$ converges. Note that $-|a_n| \leq a_n \leq |a_n|$ so that

$$0 \leq a_n + |a_n| \leq 2|a_n| \text{ for all } n\,.$$

Since $\sum |a_n|$ converges, $\sum 2|a_n|$ converges by linearity, and so $\sum \big(a_n + |a_n|\big)$ converges by the Comparison Test. Since $\sum |a_n|$ and $\sum \big(a_n + |a_n|\big)$ both converge, $\sum a_n$ converges by linearity.

6.48 Example: Determine whether $\sum \dfrac{\sin n}{n^2}$ converges.

Solution: Let $a_n = \dfrac{\sin n}{n^2}$. Then $|a_n| = \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges (its a p-series with $p = 2$), $\sum |a_n|$ converges by the Comparison Test, and hence $\sum a_n$ converges too, since absolute convergence implies convergence.

6.49 Theorem: *(Multiplication of Series) Suppose that* $\sum_{n\geq 0}|a_n|$ *converges and* $\sum_{n\geq 0} b_n$ *converges and define* $c_n=\sum_{k=0}^{n}a_kb_{n-k}$*. Then* $\sum_{n\geq 0} c_n$ *converges and*

$$\sum_{n=0}^{\infty}c_n=\left(\sum_{n=0}^{\infty}a_n\right)\left(\sum_{n=0}^{\infty}b_n\right).$$

Proof: Let $A_l=\sum_{n=0}^{l}a_n$, $B_l=\sum_{n=0}^{l}b_n$, $C_l=\sum_{n=0}^{l}c_n$, $A=\sum_{n=0}^{\infty}a_n$, $B=\sum_{n=0}^{\infty}b_n$, $K=\sum_{n=0}^{\infty}|a_n|$ and $E_l=B-B_l$. Then we have

$$\begin{aligned}C_l&=a_0b_0+(a_0b_1+a_1b_0)+(a_0b_2+a_1b_1+a_2b_0)+\cdots+(a_0b_l+\cdots+a_lb_0)\\&=a_0B_l+a_1B_{l-1}+a_2B_{l-2}+\cdots+a_lB_0\\&=a_0(B-E_l)+a_1(B-E_{l-1})+\cdots+a_l(B-E_0)\\&=A_lB-\big(a_0E_l+a_1E_{l-1}+\cdots+a_lE_0\big)\end{aligned}$$

and so

$$\big|AB-C_l\big|\leq\big|(A-A_l)B\big|+\big|a_0E_l+a_1E_{l-1}+\cdots+a_lE_0\big|.$$

Let $\epsilon>0$. Choose m so that $j>m\Longrightarrow E_j<\frac{\epsilon}{3K}$. Let $E=\max\big\{|E_0|,\cdots,|E_m|\big\}$. Choose $L>m$ so that when $l>L$ we have $\sum_{n=l-m}^{l}|a_n|<\frac{\epsilon}{3E}$ and we have $|A_l-A||B|<\frac{\epsilon}{3}$. Then for $l>L$,

$$\begin{aligned}\big|C_l-AB\big|&<\big|(A_l-A)B\big|+\big|a_0E_l+\cdots+a_{l-m-1}E_{m+1}\big|+\big|a_{l-m}E_m+\cdots+a_lE_0\big|\\&\leq\tfrac{\epsilon}{3}+\left(\sum_{n=0}^{l-m-1}|a_n|\right)\tfrac{\epsilon}{3K}+\left(\sum_{n=l-m+1}^{l}|a_n|\right)E\\&<\tfrac{\epsilon}{3}+K\tfrac{\epsilon}{3K}+\tfrac{\epsilon}{3E}E=\epsilon\,.\end{aligned}$$

6.50 Example: Find an example of sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ such that $\sum_{n\geq 0}a_n$ and $\sum_{n\geq 0}b_n$ both converge, but $\sum_{n\geq 0}c_n$ diverges where $c_n=\sum_{k=0}^{n}a_kb_{n-k}$.

Solution: Let $a_n=b_n=\dfrac{(-1)^n}{\sqrt{n+1}}$ for $n\geq 0$, and let

$$c_n=\sum_{k=0}^{n}a_kb_{n-k}=(-1)^n\sum_{k=0}^{n}\frac{1}{\sqrt{(k+1)(n-k+1)}}\,.$$

Recall that for $p,q\geq 0$ we have $\sqrt{pq}\leq\frac{1}{2}(p+q)$ (indeed $(p+q)^2-4pq=p^2-2pq+q^2=(p-q)^2\geq 0$, so $(p+q)^2\geq 4pq$). In particular $\sqrt{(k+1)(n-k+1)}\leq\frac{1}{2}(n+2)$ and so $|c_n|\geq\sum_{k=0}^{n}\frac{2}{n+2}=\frac{2(n+1)}{n+2}$. Thus $\lim_{n\to\infty}|c_n|\neq 0$ so $\sum c_n$ diverges by the Divergence Test.

6.51 Theorem: *(Fubini's Theorem for Series) Let $a_{n,m} \in \mathbf{R}$ for all $n, m \geq 0$. Suppose that $\sum\limits_{m\geq 0} |a_{n,m}|$ converges for each $n \geq 0$ and that $\sum\limits_{n\geq 0} \left(\sum\limits_{m=0}^{\infty} |a_{n,m}| \right)$ converges. Then $\sum\limits_{m\geq 0} a_{n,m}$ converges for all $n \geq 0$, $\sum\limits_{n\geq 0} \left(\sum\limits_{m=0}^{\infty} a_{n,m} \right)$ converges, $\sum\limits_{n\geq 0} a_{n,m}$ converges for all $m \geq 0$, $\sum\limits_{m\geq 0} \left(\sum\limits_{n=0}^{\infty} a_{n,m} \right)$ converges, and*

$$\sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} a_{n,m} \right) = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} a_{n,m} \right) .$$

Proof: First we claim that $\sum\limits_{n\geq 0} |a_{n,m}|$ converges for all $m \geq 0$, $\sum\limits_{m\geq 0} \left(\sum\limits_{n=0}^{\infty} |a_{n,m}| \right)$ converges, and $\sum\limits_{n=0}^{\infty} \left(\sum\limits_{m=0}^{\infty} |a_{n,m}| \right) = \sum\limits_{m=0}^{\infty} \left(\sum\limits_{n=0}^{\infty} |a_{n,m}| \right)$. For all n, m we have $|a_{n,m}| \leq \sum\limits_{k=0}^{\infty} |a_{n,k}|$, and $\sum\limits_{n\geq 0} \left(\sum\limits_{k=0}^{\infty} |a_{n,k}| \right)$ converges, so we know that $\sum\limits_{n\geq 0} |a_{n,m}|$ converges for all $m \geq 0$, by the Comparison Test. Let $k \geq 0$ and let $\epsilon > 0$. Since each sum $\sum\limits_{n\geq 0} |a_{n,m}|$ converges, we can choose L so that when $l > L$ we have $\sum\limits_{n=l+1}^{\infty} |a_{n,m}| < \frac{\epsilon}{k+1}$ for all $m = 0, 1, \cdots, k$. Then for $l > L$ we have

$$\begin{aligned}
\sum_{m=0}^{k} \left(\sum_{n=0}^{\infty} |a_{n,m}| \right) &= \sum_{m=0}^{k} \left(\sum_{n=0}^{l} |a_{n,m}| + \sum_{n=l+1}^{\infty} |a_{n,m}| \right) < \sum_{m=0}^{k} \left(\sum_{n=0}^{l} |a_{n,m}| + \tfrac{\epsilon}{k+1} \right) \\
&= \sum_{m=0}^{k} \left(\sum_{n=0}^{l} |a_{n,m}| \right) + \epsilon = \sum_{n=0}^{l} \left(\sum_{m=0}^{k} |a_{m,n}| \right) + \epsilon \\
&\leq \sum_{n=0}^{l} \left(\sum_{m=0}^{\infty} |a_{m,n}| \right) + \epsilon \leq \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} |a_{m,n}| \right) + \epsilon
\end{aligned}$$

Since ϵ was arbitrary, we have $\sum\limits_{m=0}^{k} \left(\sum\limits_{n=0}^{\infty} |a_{n,m}| \right) \leq \sum\limits_{n=0}^{\infty} \left(\sum\limits_{m=0}^{\infty} |a_{m,n}| \right)$. Since the sequence of partial sums $\sum\limits_{m=0}^{k} \left(\sum\limits_{n=0}^{\infty} |a_{n,m}| \right)$ is increasing and bounded above by $\sum\limits_{n=0}^{\infty} \left(\sum\limits_{m=0}^{\infty} |a_{m,n}| \right)$, it converges and we have $\sum\limits_{m=0}^{\infty} \left(\sum\limits_{n=0}^{\infty} |a_{n,m}| \right) \leq \sum\limits_{m=0}^{\infty} \left(\sum\limits_{n=0}^{\infty} |a_{n,m}| \right)$. By symmetry, we obtain the opposite inequality, and the claim is proved

For all $n \geq 0$, since $\sum\limits_{m\geq 0} |a_{n,m}|$ converges we know that $\sum\limits_{m\geq 0} a_{n,m}$ converges and that $\left| \sum\limits_{m=0}^{\infty} a_{n,m} \right| \leq \sum\limits_{m=0}^{\infty} |a_{n,m}|$ by the Absolute Convergence Theorem. Since $\sum\limits_{n\geq 0} \left(\sum\limits_{m=0}^{\infty} |a_{n,m}| \right)$ converges, $\sum\limits_{n\geq 0} \left| \sum\limits_{m=0}^{\infty} a_{n,m} \right|$ converges by the Comparison Test, and so $\sum\limits_{n\geq 0} \left(\sum\limits_{m=0}^{\infty} a_{n,m} \right)$ also converges by the Absolute Convergence Theorem. Similarly, $\sum\limits_{n\geq 0} a_{n,m}$ converges for all $m \geq 0$ and $\sum\limits_{m\geq 0}\left(\sum\limits_{n=0}^{\infty} a_{n,m}\right)$ converges.

Let $\epsilon > 0$. Since $\sum\limits_{n\geq 0}\left(\sum\limits_{m=0}^{\infty} |a_{n,m}|\right)$ and $\sum\limits_{m\geq 0}\left(\sum\limits_{n=0}^{\infty} |a_{n,m}|\right)$ both converge, we can choose k and l so that $\sum\limits_{n=l+1}^{\infty}\left(\sum\limits_{m=0}^{\infty} |a_{n,m}|\right) < \frac{\epsilon}{4}$ and $\sum\limits_{m=k+1}^{\infty}\left(\sum\limits_{n=0}^{\infty} |a_{n,m}|\right) < \frac{\epsilon}{4}$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{n,m}\right) &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{k} a_{n,m} + \sum_{m=k+1}^{\infty} a_{n,m}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{k} a_{n,m}\right) + \sum_{n=0}^{\infty}\left(\sum_{m=k+1}^{\infty} a_{n,m}\right)\\
&= \sum_{n=0}^{l}\left(\sum_{m=0}^{k} a_{n,m}\right) + \sum_{n=l+1}^{\infty}\left(\sum_{m=0}^{k} a_{n,m}\right) + \sum_{n=0}^{\infty}\left(\sum_{m=k+1}^{\infty} a_{n,m}\right)
\end{aligned}$$

and so we have

$$\begin{aligned}
\left|\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{n,m}\right) - \sum_{n=0}^{l}\left(\sum_{m=0}^{k} a_{n,m}\right)\right| &\leq \left|\sum_{n=l+1}^{\infty}\left(\sum_{m=0}^{k} a_{n,m}\right)\right| + \left|\sum_{n=0}^{\infty}\left(\sum_{m=k+1}^{\infty} a_{n,m}\right)\right|\\
&\leq \sum_{n=l+1}^{\infty}\left|\sum_{m=0}^{k} a_{n,m}\right| + \sum_{n=0}^{\infty}\left|\sum_{m=k+1}^{\infty} a_{n,m}\right|\\
&\leq \sum_{n=l+1}^{\infty}\left(\sum_{m=0}^{k} |a_{n,m}|\right) + \sum_{n=0}^{\infty}\left(\sum_{m=k+1}^{\infty} |a_{n,m}|\right)\\
&= \sum_{n=l+1}^{\infty}\left(\sum_{m=0}^{k} |a_{n,m}|\right) + \sum_{m=k+1}^{\infty}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right)\\
&\leq \sum_{n=l+1}^{\infty}\left(\sum_{m=0}^{\infty} |a_{n,m}|\right) + \sum_{m=k+1}^{\infty}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right)\\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}\,.
\end{aligned}$$

Similarly we have $\left|\sum\limits_{m=0}^{\infty}\left(\sum\limits_{n=0}^{\infty} a_{n,m}\right) - \sum\limits_{m=0}^{l}\left(\sum\limits_{n=0}^{k} a_{n,m}\right)\right| < \frac{\epsilon}{2}$, and so

$$\left|\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{n,m}\right) - \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{n,m}\right)\right| < \epsilon\,.$$

Since ϵ was arbitrary, we have $\sum\limits_{n=0}^{\infty}\left(\sum\limits_{m=0}^{\infty} a_{n,m}\right) = \sum\limits_{m=0}^{\infty}\left(\sum\limits_{n=0}^{\infty} a_{n,m}\right)$ as required.

Chapter 7. Sequences and Series of Functions

Pointwise Convergence

7.1 Definition: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and for each integer $n \geq p$ let $f_n : A \to \mathbb{R}$. We say that the sequence of functions $(f_n)_{n\geq p}$ **converges pointwise** to f on A, and we write $f_n \to f$ pointwise on A, when $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in A$, that is when for all $x \in A$ and for all $\epsilon > 0$ there exists $m \geq p$ such that for all integers n we have

$$n \geq m \Longrightarrow |f_n(x) - f(x)| < \epsilon \, .$$

7.2 Note: By the Cauchy Criterion for convergence, the sequence $(f_n)_{n\geq p}$ converges pointwise to some function $f(x)$ on A if and only if for all $x \in A$ and for all $\epsilon > 0$ there exists $m \geq p$ such that for all integers k, ℓ we have

$$k, \ell \geq m \Longrightarrow \big|f_k(x) - f_\ell(x)\big| < \epsilon \, .$$

7.3 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is continuous but f is not.

Solution: Let $f_n(x) = x^n$. Then $\lim\limits_{n\to\infty} f_n(x) = \begin{cases} 0 \text{ if } x \neq 1 \\ 1 \text{ if } x = 1 \, . \end{cases}$

7.4 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is differentiable and f is differentiable, but $\lim\limits_{n\to\infty} {f_n}' \neq f'$.

Solution: Let $f_n(x) = \frac{1}{n}\tan^{-1}(nx)$. Then $\lim\limits_{n\to\infty} f_n(x) = 0$, and ${f_n}'(x) = \dfrac{1}{1+(nx)^2}$ so $\lim\limits_{n\to\infty} {f_n}'(x) = \begin{cases} 0 \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \, . \end{cases}$

7.5 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is integrable but f is not.

Solution: We have $\mathbb{Q} \cap [0,1] = \{a_1, a_2, a_3, \cdots\}$ where

$$(a_n)_{n\geq 1} = \left(\tfrac{0}{1}, \tfrac{1}{1}, \tfrac{0}{2}, \tfrac{1}{2}, \tfrac{2}{2}, \tfrac{0}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, \tfrac{0}{4}, \cdots, \tfrac{4}{4}, \cdots\right).$$

(as an exercise, you can check that $a_n = \frac{k}{\ell}$ where $\ell = \left\lceil \frac{-3+\sqrt{9-8n}}{2} \right\rceil$ and $k = n - \frac{\ell^2+\ell}{2}$). For $x \in [0,1]$, let $f_n(x) = \begin{cases} 0 \text{ if } x \notin \{a_1, a_2, \cdots, a_n\} \\ 1 \text{ if } x \in \{a_1, a_2, \cdots, a_n\} \, . \end{cases}$ Then $\lim\limits_{n\to\infty} f_n(x) = \begin{cases} 0 \text{ if } x \notin \mathbb{Q} \\ 1 \text{ if } x \in \mathbb{Q} \, . \end{cases}$

7.6 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is integrable and f is integrable but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx \neq \int_0^1 f(x)\, dx \, .$$

Solution: Let $f_1(x) = \begin{cases} 48\left(x - \frac{1}{2}\right)(1-x) \text{ if } \frac{1}{2} \leq x \leq 1, \\ 0 \text{ otherwise} \, . \end{cases}$ For $n \geq 1$ let $f_n(x) = n f_1(nx)$.

Then each f_n is continuous with $\displaystyle\int_0^1 f_n(x)\, dx = 1$, and $\lim\limits_{n\to\infty} f_n(x) = 0$ for all x.

Uniform Convergence

7.7 Definition: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and for each integer $n \geq p$ let $f_n : A \to \mathbb{R}$. We say that the sequence of functions $(f_n)_{n\geq p}$ **converges uniformly** to f on A, and we write $f_n \to f$ uniformly on A, when for all $\epsilon > 0$ there exists $m \in \mathbb{Z}_{\geq p}$ such that for all $x \in A$ and for all integers $n \in \mathbb{Z}_{\geq p}$ we have

$$n \geq m \Longrightarrow \big|f_n(x) - f(x)\big| < \epsilon\,.$$

7.8 Theorem: *(Cauchy Criterion for Uniform Convergence of Sequences of Functions) Let $(f_n)_{n\geq p}$ be a sequence of functions on $A \subseteq \mathbb{R}$. Then (f_n) converges uniformly (to some function f) on A if and only if for all $\epsilon > 0$ there exists $m \in \mathbb{Z}_{\geq p}$ such that for all $x \in A$ and for all integers $k, \ell \in \mathbb{Z}_{\geq p}$ we have*

$$k, \ell \geq m \Longrightarrow \big|f_k(x) - f_\ell(x)\big| < \epsilon\,.$$

Proof: Suppose that (f_n) converges uniformly to f on A. Let $\epsilon > 0$. Choose m so that for all $x \in A$ we have $n \geq m \Longrightarrow \big|f_n(x) - f(x)\big| < \frac{\epsilon}{2}$. Then for $k, \ell \geq m$ we have $\big|f_k(x) - f(x)\big| < \frac{\epsilon}{2}$ and $\big|f_\ell(x) - f(x)\big| < \frac{\epsilon}{2}$ and so

$$\big|f_k(x) - f_\ell(x)\big| \leq \big|f_k(x) - f(x)\big| + \big|f_\ell(x) - f(x)\big| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,.$$

Conversely, suppose that (f_n) satisfies the Cauchy Criterion for uniform convergence, that is for all $\epsilon > 0$ there exists m such that for all $x \in A$ and all integers n, ℓ we have

$$n, \ell \geq m \Longrightarrow \big|f_n(x) - f_\ell(x)\big| < \epsilon\,.$$

For each fixed $x \in A$, $(f_n(x))$ is a Cauchy sequence, so $(f_n(x))$ converges, and we can define $f(x)$ by

$$f(x) = \lim_{n\to\infty} f_n(x)\,.$$

We know that $f_n \to f$ pointwise on A, but we must show that $f_n \to f$ uniformly on A. Let $\epsilon > 0$. Choose m so that for all $x \in A$ and for all integers n, ℓ we have

$$n, \ell \geq m \Longrightarrow \big|f_n(x) - f_\ell(x)\big| < \tfrac{\epsilon}{2}\,.$$

Let $x \in A$. Since $\lim\limits_{\ell\to\infty} f_\ell(x) = f(x)$, we can choose $\ell \geq m$ so that $\big|f_\ell(x) - f(x)\big| < \frac{\epsilon}{2}$. Then for $n \geq m$ we have

$$\big|f_n(x) - f(x)\big| \leq \big|f_n(x) - f_\ell(x)\big| + \big|f_\ell(x) - f(x)\big| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,.$$

7.9 Theorem: *(Uniform Convergence, Limits and Continuity) Suppose that $f_n \to f$ uniformly on A. Let x be a limit point of A. If $\lim\limits_{y\to x} f_n(y)$ exists for each n, then*

$$\lim_{y\to x}\lim_{n\to\infty} f_n(y) = \lim_{n\to\infty}\lim_{y\to x} f_n(y)\,.$$

In particular, if each f_n is continuous in A, then so is f.

Proof: Suppose that $\lim\limits_{y\to x} f_n(y)$ exists for all n. Let $b_n = \lim\limits_{y\to x} f_n(y)$. We must show that $\lim\limits_{y\to x} f(y) = \lim\limits_{n\to\infty} b_n$. We claim first that (b_n) converges. Let $\epsilon > 0$. Choose m so that $k, \ell \geq m \Longrightarrow \big|f_k(y) - f_\ell(y)\big| < \frac{\epsilon}{3}$ for all $y \in A$. Let $k, \ell \geq m$. Choose $y \in A$ so that $\big|f_k(y) - b_k\big| < \frac{\epsilon}{3}$ and $\big|f_\ell(y) - b_\ell\big| < \frac{\epsilon}{3}$. Then we have

$$\big|b_k - b_\ell\big| \leq \big|b_k - f_k(y)\big| + \big|f_k(y) - f_\ell(y)\big| + \big|f_\ell(y) - b_\ell\big| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon\,.$$

By the Cauchy Criterion for sequences, (b_n) converges, as claimed.

Now, let $b = \lim_{n\to\infty} b_n$. We must show that $\lim_{y\to x} f(x) = b$. Let $\epsilon > 0$. Choose m so that when $n \geq m$ we have $\left|f_n(y) - f(y)\right| < \frac{\epsilon}{3}$ for all $y \in A$ and we have $|b_n - b| < \frac{\epsilon}{3}$. Let $n \geq m$. Since $\lim_{y\to x} f_n(y) = b_n$ we can choose $\delta > 0$ so that $0 < |y - x| < \delta \implies \left|f_n(y) - b_n\right| < \frac{\epsilon}{3}$. Then when $0 < |y - x| < \delta$ we have

$$\left|f(y) - b\right| \leq \left|f(y) - f_n(y)\right| + \left|f_n(y) - b_n\right| + \left|b_n - b\right| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon\,.$$

Thus $\lim_{y\to x} f(x) = b$, as required.

In particular, if $x \in A$ and each f_n is continuous at x then we have

$$\lim_{y\to x} f(y) = \lim_{y\to x}\lim_{n\to\infty} f_n(y) = \lim_{n\to\infty}\lim_{y\to x} f_n(y) = \lim_{n\to\infty} f_n(x) = f(x)$$

so f is continuous at x.

7.10 Theorem: *(Uniform Convergence and Integration) Suppose that $f_n \to f$ uniformly on $[a, b]$. If each f_n is integrable on $[a, b]$ then so is f. In this case, if $g_n(x) = \int_a^x f_n(t)\,dt$ and $g(x) = \int_a^x f(t)\,dt$, then $g_n \to g$ uniformly on $[a, b]$. In particular, we have*

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx\,.$$

Proof: Suppose that each f_n is integrable on $[a, b]$. We claim that f is integrable on $[a, b]$. Let $\epsilon > 0$. Choose N so that $n \geq N \implies \left|f_n(x) - f(x)\right| < \frac{\epsilon}{4(b-a)}$ for all $x \in [a, b]$. Fix $n \geq N$. Choose a partition X of $[a, b]$ so that $U(f_n, X) - L(f_n, X) < \frac{\epsilon}{2}$. Note that since $\left|f_n(x) - f(x)\right| < \frac{\epsilon}{4(b-a)}$ we have $M_i(f) < M_i(f_n) + \frac{\epsilon}{4(b-a)}$ and $m_i(f) > m_i(f_n) - \frac{\epsilon}{4(b-a)}$, and so

$$\begin{aligned}
U(f, X) - L(f, X) &= \sum_{i=1}^{n}\big(M_i(f) - m_i(f)\big)\Delta_i x < \sum_{i=1}^{n}\Big(M_i(f_n) - m_i(f_n) + \tfrac{\epsilon}{2(b-a)}\Big)\,\Delta_i x \\
&= U(f_n, X) - L(f_n, X) + \tfrac{\epsilon}{2} < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,.
\end{aligned}$$

Thus f is integrable on $[a, b]$.

Now define $g_n(x) = \int_a^x f_n(t)\,dt$ and $g(x) = \int_a^x f(t)\,dt$. We claim that $g_n \to g$ uniformly on $[a, b]$. Let $\epsilon > 0$. Choose N so that $n \geq N \implies \left|f_n(t) - f(t)\right| < \frac{\epsilon}{2(b-a)}$ for all $t \in I$. Let $n \geq N$. Let $x \in [a, b]$. Then we have

$$\begin{aligned}
\left|g_n(x) - g(x)\right| &= \left|\int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt\right| = \left|\int_a^x f_n(t) - f(t)\;dt\right| \\
&\leq \int_a^x \left|f_n(t) - f(t)\right|dt \leq \int_a^x \tfrac{\epsilon}{2(b-a)}\;dt = \tfrac{\epsilon}{2(b-a)}(x - a) \leq \tfrac{\epsilon}{2} < \epsilon\,.
\end{aligned}$$

Thus $g_n \to g$ uniformly on $[a, b]$, as required.

In particular, we have $\lim_{n\to\infty} g_n(b) = g(b)$, that is

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx\,.$$

7.11 Theorem: *(Uniform Convergence and Differentiation) Let (f_n) be a sequence of functions on $[a,b]$. Suppose that each f_n is differentiable on $[a,b]$, $\big({f_n}'\big)$ converges uniformly on $[a,b]$, and $\big(f_n(c)\big)$ converges for some $c\in[a,b]$. Then (f_n) converges uniformly on $[a,b]$, $\lim\limits_{n\to\infty} f_n(x)$ is differentiable, and*

$$\frac{d}{dx}\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{d}{dx}f_n(x)\,.$$

Proof: We claim that (f_n) converges uniformly on $[a,b]$. Let $\epsilon>0$. Choose N so that when $n,m\geq N$ we have $\big|{f_n}'(t)-{f_m}'(t)\big|<\frac{\epsilon}{2(b-a)}$ for all $t\in[a,b]$ and we have $\big|f_n(c)-f_m(c)\big|<\frac{\epsilon}{2}$. Let $n,m\geq N$. Let $x\in[a,b]$. By the Mean Value Theorem applied to the function $f_n(x)-f_m(x)$, we can choose t between c and x so that

$$\big(f_n(x)-f_m(x)-f_n(c)+f_m(c)\big)=\big({f_n}'(t)-{f_m}'(t)\big)(x-c)\,.$$

Then we have

$$\begin{aligned}\big|f_n(x)-f_m(x)\big|&\leq\big|f_n(x)-f_m(x)-f_n(c)+f_m(c)\big|+\big|f_n(c)-f_m(c)\big|\\&=\big|{f_n}'(t)-{f_m}'(t)\big||x-c|+\big|f_n(c)-f_m(c)\big|\\&<\tfrac{\epsilon}{2(b-a)}(b-a)+\tfrac{\epsilon}{2}=\epsilon\,.\end{aligned}$$

Thus (f_n) converges uniformly on $[a,b]$.

Let $f(x)=\lim\limits_{n\to\infty} f_n(x)$. We claim that f is differentiable with $f'(x)=\lim\limits_{n\to\infty}{f_n}'(x)$ for all $x\in[a,b]$. Fix $x\in[a,b]$. Note that

$$\begin{aligned}f'(x)=\lim_{n\to\infty}{f_n}'(x)&\iff\lim_{y\to x}\frac{f(y)-f(x)}{y-x}=\lim_{n\to\infty}\lim_{y\to x}\frac{f_n(y)-f_n(x)}{y-x}\\&\iff\lim_{y\to x}\lim_{n\to\infty}\frac{f_n(y)-f_n(x)}{y-x}=\lim_{n\to\infty}\lim_{y\to x}\frac{f_n(y)-f_n(x)}{y-x}\end{aligned}$$

so it suffices to show that $\big(g_n\big)$ converges uniformly on $[a,b]\setminus\{x\}$, where

$$g_n(y)=\frac{f_n(y)-f_n(x)}{y-x}\,.$$

Let $\epsilon>0$. Choose N so that $n,m\geq N\implies\big|{f_n}'(t)-{f_m}'(t)\big|<\epsilon$ for all $t\in[a,b]$. Let $n,m\geq N$. Let $y\in[a,b]\setminus\{x\}$. By the Mean Value Theorem, we can choose t between x and y so that

$$\big(f_n(y)-f_m(y)-f_n(x)+f_m(x)\big)=\big({f_n}'(t)-{f_m}'(t)\big)(y-x)\,.$$

Then

$$\big|g_n(y)-g_m(y)\big|=\left|\frac{f_n(y)-f_m(y)-f_n(x)+f_m(x)}{y-x}\right|=\big|{f_n}'(t)-{f_m}'(t)\big|<\epsilon\,.$$

Thus $\big(g_n\big)$ converges uniformly on $[a,b]\setminus\{x\}$, as required.

Series of Functions

7.12 Definition: Let $(f_n)_{n\geq p}$ be a sequence of functions on $A\subseteq\mathbb{R}$. The **series of functions** $\sum_{n\geq p} f_n(x)$ is defined to be the sequence $\big(S_l(x)\big)$ where $S_l(x)=\sum_{n=p}^{l} f_n(x)$. The function $S_l(x)$ is called the l^{th} **partial sum** of the series. We say the series $\sum_{n\geq p} f_n(x)$ converges pointwise (or uniformly) on A when the sequence $\{S_l\}$ converges, pointwise (or uniformly) on A. In this case, the **sum** of the series of functions is defined to be the function

$$f(x)=\sum_{n=p}^{\infty} f_n(x)=\lim_{l\to\infty} S_l(x)\,.$$

7.13 Theorem: *(Cauchy Criterion for the Uniform Convergence of a Series of Functions) The series* $\sum_{n\geq p} f_n(x)$ *converges uniformly (to some function* f*) on* A *if and only if for every* $\epsilon>0$ *there exists* $N\geq p$ *such that for all* $x\in A$ *and for all* $k,\ell\geq p$ *we have*

$$\ell>k\geq N\Longrightarrow\left|\sum_{n=k+1}^{\ell} f_n(x)\right|<\epsilon\,.$$

Proof: This follows immediately from the analogous theorem for sequences of functions.

7.14 Theorem: *(Uniform Convergence, Limits and Continuity) Suppose that* $\sum_{n\geq p} f_n(x)$ *converges uniformly on* A*. Let* x *be a limit point of* A*. If* $\lim_{y\to x} f_n(y)$ *exists for all* $n\geq p$*, then*

$$\lim_{y\to x}\sum_{n=p}^{\infty} f_n(y)=\sum_{n=p}^{\infty}\lim_{y\to x} f_n(y)\,.$$

In particular, if each $f_n(x)$ *is continuous on* A *then so is* $\sum_{n=p}^{\infty} f_n(x)$.

Proof: This follows immediately from the analogous theorem for sequences of functions.

7.15 Theorem: *(Uniform Convergence and Integration) Suppose that* $\sum_{n\geq p} f_n(x)$ *converges uniformly on* $[a,b]$*. If each* $f_n(x)$ *is integrable on* $[a,b]$*, then so is* $\sum_{n=p}^{\infty} f_n(x)$*. In this case, if we define* $g_n(x)=\int_a^x f_n(t)\,dt$ *and* $g(x)=\int_a^x\sum_{n=p}^{\infty} f_n(t)\,dt$*, then* $\sum_{n\geq p} g_n(x)$ *converges uniformly to* $g(x)$ *on* A*. In particular, we have*

$$\int_a^b\sum_{n=p}^{\infty} f_n(x)\,dx=\sum_{n=p}^{\infty}\int_a^b f_n(x)\,dx\,.$$

Proof: This follows immediately from the analogous theorem for sequences of functions.

7.16 Theorem: *(Uniform Convergence and Differentiation) Suppose that each $f_n(x)$ is differentiable on $[a,b]$, $\sum_{n\geq p} {f_n}'(x)$ converges uniformly on $[a,b]$, and $\sum_{n\geq p} f_n(c)$ converges for some $c\in[a,b]$. Then $\sum_{n\geq p} f_n(x)$ converges uniformly on $[a,b]$ and*

$$\frac{d}{dx}\sum_{n=p}^{\infty} f_n(x) = \sum_{n=p}^{\infty}\frac{d}{dx}f_n(x)\,.$$

Proof: This follows immediately from the analogous theorem for sequences of functions.

7.17 Theorem: *(The Weierstrass M-Test) Suppose that f_n is bounded with $|f_n(x)|\leq M_n$ for all $n\geq p$ and $x\in A$, and $\sum_{n\geq p} M_n$ converges. Then $\sum_{n\geq p} f_n(x)$ converges uniformly on A.*

Proof: Let $\epsilon>0$. Choose N so that $\ell>k\geq N \implies \sum_{n=k+1}^{\ell} M_n<\epsilon$. Let $\ell>k\geq N$. Let $x\in A$. Then

$$\left|\sum_{n=k+1}^{\ell} f_n(x)\right| \leq \sum_{n=k+1}^{\ell}\big|f_n(x)\big| \leq \sum_{n=k+1}^{\ell} M_n < \epsilon\,.$$

7.18 Example: Find a sequence of functions $\big(f_n(x)\big)_{n\geq 0}$, each of which is differentiable on $\mathbb{R}$, such that $\sum_{n\geq 0} f_n(x)$ converges uniformly on $\mathbb{R}$, but the sum $f(x)=\sum_{n=0}^{\infty} f_n(x)$ is nowhere differentiable.

Solution: Let $f_n(x)=\frac{1}{2^n}\sin^2(8^n x)$. Since $|f_n(x)|\leq\frac{1}{2^n}$ and $\sum\frac{1}{2^n}$ converges, $\sum_{n\geq 0} f_n(x)$ converges uniformly on $\mathbb{R}$. Let $f(x)=\sum_{n=0}^{\infty} f_n(x)$. We claim that $f(x)$ is nowhere differentiable. Let $x\in\mathbb{R}$. For each n, let m, a_n and b_n be such that $a_n=\frac{m\pi}{2\cdot 8^n}$, $b_n=\frac{(m+1)\pi}{2\cdot 8^n}$ and $x\in[a_n,b_n)$. Note that one of $f_n(a_n)$ and $f_n(b_n)$ is equal to $\frac{1}{2^n}$ and the other is equal to 0 so we have $\big|f_n(b_n)-f_n(a_n)\big|=\frac{1}{2^n}$. Note also that for $k>n$ we have $f_k(a_n)=f_k(b_n)=0$. Also, for all k we have $f_k(x)=\frac{1}{2^k}\sin^2(8^k x)$, ${f_k}'(x)=4^k\sin(2\cdot 8^k x)$, and $\big|{f_k}'(x)\big|\leq 4^k$, so by the Mean Value Theorem,

$$\big|f_k(b_n)-f_k(a_n)\big|\leq 4^k|b_n-a_n|\,.$$

Finally, note that if $f'(x)$ did exist, then we would have $f'(x)=\lim_{n\to\infty}\frac{f(b_n)-f(a_n)}{b_n-a_n}$, but

$$\begin{aligned}\left|\frac{f(b_n)-f(a_n)}{b_n-a_n}\right| &= \left|\sum_{k=0}^{\infty}\frac{f_k(b_n)-f_k(a_n)}{b_n-a_n}\right| = \left|\sum_{k=0}^{n}\frac{f_k(b_n)-f_k(a_n)}{b_n-a_n}\right| \\ &\geq \left|\frac{f_n(b_n)-f_n(a_n)}{b_n-a_n}\right| - \sum_{k=0}^{n-1}\left|\frac{f_k(b_n)-f_k(a_n)}{b_n-a_n}\right| \\ &\geq \frac{\frac{1}{2^n}}{\frac{\pi}{2\cdot 8^n}} - \sum_{k=0}^{n-1}4^k = \tfrac{2\cdot 4^n}{\pi}-\tfrac{4^n-1}{3} = \left(\tfrac{2}{\pi}-\tfrac{1}{3}\right)4^n+\tfrac{1}{3}\to\infty \text{ as } n\to\infty\end{aligned}$$

Power Series

7.19 Definition: A **power series centred at** a is a series of the form $\sum_{n\geq 0} a_n(x-a)^n$ for some real numbers a_n, where we use the convention that $(x-a)^0 = 1$.

7.20 Example: The geometric series $\sum_{n\geq 0} x^n$ is a power series centred at 0. It converges when $|x| < 1$ and for all such x the sum of the series is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,.$$

7.21 Lemma: *(Abel's Formula) Let $\{a_n\}$ and $\{b_n\}$ be sequences. Then we have*

$$\sum_{n=m}^{l} a_n b_n + \sum_{p=m}^{l-1}\left(\sum_{n=m}^{p} a_n\right)(b_{p+1}-b_p) = \left(\sum_{n=m}^{l} a_n\right) b_l\,.$$

Proof: We have

$$\begin{aligned}\sum_{p=m}^{l-1}\left(\sum_{n=m}^{p} a_n\right)(b_{p+1}-b_p) =\ & a_m(b_{m+1}-b_m) + (a_m + a_{m+1})(b_{m+2}-b_{m+1}) \\ & + (a_m + a_{m+1} + a_{m+2})(b_{m+3}-b_{m+2}) \\ & + \cdots + (a_m + a_{m+1} + a_{m+2} + \cdots + a_{l-1})(b_l - b_{l-1}) \\ =\ & -a_m b_m - a_{m+1}b_{m+1} - \cdots - a_{l-1}b_{l-1} \\ & + (a_m + a_{m+1} + \cdots a_{l-1})b_l - a_l b_l + a_l b_l \\ =\ & \left(\sum_{n=m}^{l} a_n\right) b_l - \sum_{n=m}^{l} a_n b_n\,.\end{aligned}$$

7.22 Definition: Let (a_n) be a sequence in $\mathbb{R}$. We define $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} s_n$ where $s_n = \sup\{a_k \,|\, k \geq n\}$ (with $\limsup_{n\to\infty} a_n = \infty$ when (a_n) is not bounded above).

7.23 Theorem: *(The Interval and Radius of Convergence) Let $\sum_{n\geq 0} a_n(x-a)^n$ be a power series and let $R = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \in [0,\infty]$. Then the set of $x \in \mathbb{R}$ for which the power series converges is an interval I centred at a of radius R. Indeed*

(1) if $|x-a| > R$ then $\lim_{n\to\infty} a_n(x-a) \neq 0$ so $\sum_{n\geq 0} n_n(x-a)^n$ diverges,

(2) if $|x-a| < R$ then $\sum_{n\geq 0} a_n(x-a)^n$ converges absolutely,

(3) if $0 < r < R$ then $\sum_{n\geq 0} a_n(x-a)^n$ converges uniformly in $[a-r, a+r]$, and

(4) (Abel's Theorem) if $\sum_{n\geq 0} a_n(x-a)^n$ converges when $x = a+R$ then the convergence is uniform on $[a, a+R]$, and similarly if $\sum_{n\geq 0} a_n(x-a)^n$ converges when $x = a-R$ then the convergence is uniform on $[a-R, a]$.

Proof: To prove part (1), suppose that $|x-a|>R$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n(x-a)^n|} = |x-a| \limsup_{n\to\infty} \sqrt[n]{|a_n|} > R\cdot \tfrac{1}{R} = 1\,,$$

and so $\lim\limits_{n\to\infty} a_n(x-a)^n \neq 0$ and $\sum a_n(x-a)^n$ diverges, by the Root Test.

To prove part (2), suppose that $|x-a|<R$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n(x-a)^n|} = |x-a| \limsup_{n\to\infty} \sqrt[n]{|a_n|} < R\cdot \tfrac{1}{R} = 1\,,$$

and so $\sum \left|a_n(x-a)^n\right|$ converges, by the Root Test.

To prove part (3), fix $0<r<R$. By part (2), $\sum \left|a_n(x-a)^n\right|$ converges when $x=a+r$, that is $\sum |a_nr^n|$ converges. Let $x\in[a-r,a+r]$. Then $|a_n(x-a)^n|\leq|a_nr^n|$ and $\sum|a_nr^n|$ converges, and so $\sum|a_n(x-a)^n|$ converges uniformly by the Weierstrass M-Test.

Now let us prove part (4). Suppose that $\sum a_n(x-a)^n$ converges when $x=a+R$, that is $\sum a_nR^n$ converges. Let $\epsilon>0$. Choose N so that $l>m>N \Longrightarrow \left|\sum\limits_{n=m}^{l} a_nR^n\right| < \epsilon$.

Then by Abel's Formula and using telescoping we have

$$\begin{aligned}
\left|\sum_{n=m}^{l} a_n(x-a)^n\right| &= \left|\sum_{n=m}^{l} a_nR^n \left(\tfrac{x-a}{R}\right)^n\right| \\
&= \left|\left(\sum_{n=m}^{l} a_nR^n\right)\left(\tfrac{x-a}{R}\right)^l - \sum_{p=m}^{l-1}\left(\sum_{n=m}^{p} a_nR^n\right)\left(\left(\tfrac{x-a}{R}\right)^{p+1} - \left(\tfrac{x-a}{R}\right)^p\right)\right| \\
&\leq \left|\sum_{n=m}^{l} a_nR^n\right|\left(\tfrac{x-a}{R}\right)^l + \sum_{p=m}^{l-1}\left|\sum_{n=m}^{p} a_nR^n\right|\left(\left(\tfrac{x-a}{R}\right)^p - \left(\tfrac{x-a}{R}\right)^{p+1}\right) \\
&< \epsilon\left(\tfrac{x-a}{R}\right)^l + \epsilon\left(\left(\tfrac{x-a}{R}\right)^m - \left(\tfrac{x-a}{R}\right)^l\right) = \epsilon\left(\tfrac{x-a}{R}\right)^m < \epsilon\,.
\end{aligned}$$

7.24 Definition: The number R in the above theorem is called the **radius of convergence** of the power series, and the interval I is called the **interval of convergence** of the power series.

7.25 Example: Find the interval of convergence of the power series $\sum\limits_{n\geq1}\dfrac{(3-2x)^n}{\sqrt{n}}$.

Solution: First note that this is in fact a power series, since $\dfrac{(3-2x)^n}{\sqrt{n}} = \frac{(-2)^n}{\sqrt{n}}\left(x-\frac{3}{2}\right)^n$, and so $\sum\limits_{n\geq1}\dfrac{(3-2x)^n}{\sqrt{n}} = \sum\limits_{n\geq0} c_n(x-a)^n$, where $c_0=0$, $c_n=\frac{(-2)^n}{\sqrt{n}}$ for $n\geq1$ and $a=\frac{3}{2}$.

Now, let $a_n=\dfrac{(3-2x)^n}{\sqrt{n}}$. Then $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{(3-2x)^{n+1}}{\sqrt{n+1}}\,\dfrac{\sqrt{n}}{(3-2x)^n}\right| = \sqrt{\frac{n}{n+1}}\,|3-2x|$, so $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = |3-2x|$. By the Ratio Test, $\sum a_n$ converges when $|3-2x|<1$ and diverges when $|3-2x|>1$. Equivalently, it converges when $x\in(1,2)$ and diverges when $x\notin[1,2]$. When $x=1$ so $(3-2x)=1$, we have $\sum a_n=\sum\frac{1}{\sqrt{n}}$, which diverges (its a p-series), and when $x=2$ so $(3-2x)=-1$, we have $\sum a_n=\sum\frac{(-1)^n}{\sqrt{n}}$ which converges by the Alternating Series Test. Thus the interval of convergence is $I=(1,2\,]$.

Operations on Power Series

7.26 Theorem: *(Continuity of Power Series) Suppose that the power series* $\sum a_n(x-a)^n$ *converges in an interval* I*. Then the sum* $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ *is continuous in* I*.*

Proof: This follows from uniform convergence of $\sum a_n(x-a)^n$ in closed subintervals of I.

7.27 Theorem: *(Addition and Subtraction of Power Series) Suppose that the power series* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in the interval* I*. Then* $\sum(a_n+b_n)(x-a)^n$ *and* $\sum(a_n-b_n)(x-a)^n$ *both converge in* I*, and for all* $x \in I$ *we have*

$$\left(\sum_{n=0}^{\infty} a_n(x-a)^n\right) \pm \left(\sum_{n=0}^{\infty} b_n(x-a)^n\right) = \sum_{n=0}^{\infty}(a_n \pm b_n)(x-a)^n .$$

Proof: This follows from Linearity.

7.28 Theorem: *(Multiplication of Power Series) Suppose the power series* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in an open interval* I *with* $a \in I$*. Let* $c_n = \sum_{k=0}^{n} a_k b_{n-k}$*. Then* $\sum c_n(x-a)^n$ *converges in* I *and for all* $x \in I$ *we have*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \left(\sum_{n=0}^{\infty} a_n(x-a)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-a)^n\right) .$$

Proof: This follows from the Multiplication of Series Theorem, since the power series converge absolutely in I.

7.29 Theorem: *(Division of Power Series) Suppose that* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in an open interval* I *with* $a \in I$*, and that* $b_0 \neq 0$*. Define* c_n *by*

$$c_0 = \tfrac{a_0}{b_0} \text{ , and for } n > 0,\ c_n = \tfrac{a_n}{b_0} - \tfrac{b_n c_0}{b_0} - \tfrac{b_{n-1}c_1}{b_0} - \cdots - \tfrac{b_1 c_{n-1}}{b_0} .$$

Then there is an open interval J *with* $a \in J$ *such that* $\sum c_n(x-a)^n$ *converges in* J *and for all* $x \in J$*,*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \frac{\displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n}{\displaystyle\sum_{n=0}^{\infty} b_n(x-a)^n} .$$

Proof: Choose $r > 0$ so that $a + r \in I$. Note that $\sum |a_n r^n|$ and $\sum |b_n r^n|$ both converges. Since $|a_n r^n| \to 0$ and $|b_n r^n| \to 0$ and $b_0 \neq 0$, we can choose M so that $M \geq \left|\frac{a_n r^n}{b_0}\right|$ and $M \geq \left|\frac{b_n r^n}{b_0}\right|$ for all n. Note that $|c_0| = \left|\frac{a_0}{b_0}\right| \leq M$ and since $c_1 = \frac{a_1}{b_0} + \frac{b_1 c_0}{b_0}$ we have

$$|c_1 r| \leq \left|\tfrac{a_1 r}{b_0}\right| + \left|\tfrac{b_1 r}{b_0}\right| |c_0| \leq M + M^2 = M(1+M) .$$

Suppose, inductively, that $|c_k r^k| \leq M(1+M)^k$ for all $k < n$. Then since

$$a_n = b_n c_0 + b_{n-1}c_1 + \cdots + b_1 c_{n-1} + b_0 c_n ,$$

we have

$$\begin{aligned}|c_n r^n| &\le \left|\tfrac{a_n r^n}{b_0}\right| + \left|\tfrac{b_n r^n}{b_0}\right| |c_0| + \left|\tfrac{b_{n-1} r^{n-1}}{b_0}\right| |c_1 r| + \cdots + \left|\tfrac{b_1 r}{b_0}\right| |c_{n-1} r^{n-1}| \\ &\le M + M^2 + M^2(1+M) + M^2(1+M)^2 + M^2(1+M)^3 + \cdots + M^2(1+M)^{n-1} \\ &= M + M^2\left(\frac{(1+M)^n - 1}{M}\right) = M(1+M)^n\,.\end{aligned}$$

Bu induction, we have $|c_n r^n| \le M(1+M)^n$ for all $n \ge 0$. Let $J_1 = \left(a - \frac{r}{1+M}, a + \frac{r}{1+M}\right)$. Let $x \in J_1$ so $|x-a| < \frac{r}{1+M}$. Then for all n we have

$$|c_n(x-a)^n| = |c_n r^n| \cdot \frac{1}{(1+M)^n} \cdot \left|\frac{x-a}{r/(1+M)}\right|^n \le M\left|\frac{x-a}{r/(1+M)}\right|^n$$

and so $\sum |c_n(x-a)^n|$ converges by the Comparison Test.

Note that from the definition of c_n we have $a_n = \sum\limits_{k=0}^{n} c_k b_{n-k}$, and so by multiplying power series, we have

$$\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-a)^n\right) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for all $x \in I \cap J_1$. Finally note that $f(x) = \sum\limits_{n=0}^{\infty} b_n(x-a)^n$ is continuous in I and we have $f(0) = b_0 \ne 0$, and so there is an interval $J \subset I \cap J_1$ with $a \in J$ such that $f(x) \ne 0$ in J.

7.30 Theorem: *(Composition of Power Series) Let $f(x) = \sum\limits_{n=0}^{\infty} a_n(x-a)^n$ in an open interval I with $a \in I$, and let $g(y) = \sum\limits_{m=0}^{\infty} b_m(y-b)^m$ in an open interval J with $b \in J$ and with $a_0 \in J$. Let K be an open interval with $a \in K$ such that $f(K) \subset J$. For each $m \ge 0$, let $c_{n,m}$ be the coefficients, found by multiplying power series, such that $\sum\limits_{n=0}^{\infty} c_{n,m}(x-a)^n = b_n\left(\sum\limits_{n=0}^{\infty} a_n(x-a)^n - b\right)^m$. Then $\sum\limits_{m\ge 0} c_{n,m}$ converges for all $m \ge 0$, and for all $x \in K$, $\sum\limits_{n\ge 0}\left(\sum\limits_{m=0}^{\infty} c_{n,m}\right)(x-a)^n$ converges and*

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} c_{n,m}\right)(x-a)^n = g\big(f(x)\big)\,.$$

Proof: This follows from Fubini's Theorem for Series since

$$g\big(f(x)\big) = \sum_{m=0}^{\infty} b_m\big(f(x)-b\big)^m = \sum_{m=0}^{\infty} b_m\left(\sum_{n=0}^{\infty} a_n(x-a)^n - b\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} c_{n,m}(x-a)^n\right).$$

7.31 Theorem: *(Integration of Power Series) Suppose that $\sum a_n(x-a)^n$ converges in the interval I. Then for all $x\in I$, the sum $f(x)=\sum_{n=0}^\infty a_n(x-a)^n$ is integrable on $[a,x]$ (or $[x,a]$) and*

$$\int_a^x \sum_{n=0}^\infty a_n(t-a)^n\,dt = \sum_{n=0}^\infty \int_a^x a_n(t-a)^n\,dt = \sum_{n=0}^\infty \frac{a_n}{n+1}(x-a)^{n+1}\,.$$

Proof: This follows from uniform convergence.

7.32 Theorem: *(Differentiation of Power Series) Suppose that $\sum a_n(x-a)^n$ converges in the open interval I. Then the sum $f(x)=\sum_{n=0}^\infty a_n(x-a)^n$ is differentiable in I and*

$$f'(x)=\sum_{n=1}^\infty n\,a_n(x-a)^{n-1}\,.$$

Proof: We claim that the radius of convergence of $\sum a_n(x-a)^n$ is equal to the radius of convergence of $\sum na_n(x-a)^{n-1}$. Let R be the radius of convergence of $\sum a_n(x-a)^n$ and let S be the radius of convergence of $\sum na_n(x-a)^{n-1}$. Fix $x\in(a-R,a+R)$ so $|x-a|<R$ and $\sum\left|a_n(x-a)^n\right|$ converges. Choose r,s with $|x-a|<r<s<R$. Since $\lim\limits_{n\to\infty}\dfrac{(r/s)^n}{(1/n)}=0$, we can choose N so that $n\geq N \implies \left(\frac{r}{s}\right)^n<\frac{1}{n}$. Then for $n\geq N$ we have

$$\left|na_n(x-a)^n\right| = \left|n\left(\tfrac{r}{s}\right)^n\left(\tfrac{x-a}{r}\right)^n a_n s^n\right| \leq 1\cdot 1\cdot |a_n s^n|\,.$$

Since $\sum|a_ns^n|$ converges, $\sum\left|na_n(x-a)^n\right|$ converges by the Comparison Test, and so $\sum\left|na_n(x-a)^{n-1}\right|$ converges by Linearity. Thus $R\leq S$.

Now fix $x\in(a-S,a+s)$ so that $|x-a|<S$ and $\sum\left|na_n(x-a)^{n-1}\right|$ converges. Then $\sum\left|na_n(x-a)^n\right|$ converges by Linearity, and $\left|a_n(x-a)^n\right|\leq\left|na_n(x-a)^n\right|$ so $\sum\left|a_n(x-a)^n\right|$ converges by Comparison. Thus $S\leq R$ and so $R=S$ as claimed.

The theorem now follows from the uniform convergence of $\sum na_n(x-a)^{n-1}$.

7.33 Example: We have $\frac{1}{1+x}=\sum\limits_{n=0}^\infty(-1)^nx^n$ for $|x|<1$. By Integration of Power Series, $\ln x=\sum\limits_{n=0}^\infty\frac{(-1)^n}{n+1}x^{n+1}=\sum\limits_{n=1}^\infty\frac{(-1)^n}{n}x^n$ for $|x|<1$. In particular, we can take $x=\frac12$ to get $\ln\frac32=\sum\limits_{n=1}^\infty\frac{(-1)^n}{n\cdot 2^n}$ and we can take $x=-\frac12$ to get $\ln\frac12=\sum\limits_{n=1}^\infty\frac{-1}{n\cdot2^n}$, that is $\ln 2=\sum\limits_{n=1}^\infty\frac{1}{n\cdot2^n}$.

Let us also argue that we can also take $x=1$. Note that the series $\sum\limits_{n=1}^\infty\frac{(-1)^{n+1}}{n}x^n$ diverges when $x=-1$ (by the Integral Test) and converges when $x=1$ (by the Alternating Series Test), so the interval of convergence is $(-1,1]$. Thus the sum $f(x)=\sum\limits_{n=1}^\infty\frac{(-1)^{n+1}}{n}x^n$ is defined for $-1<x\leq1$. We know already that $f(x)=\ln(1+x)$ for $-1<x<1$. By Abel's Theorem, the series converges uniformly on $[0,1]$, so by the Continuity of Power Series Theorem, the sum $f(x)=\sum\limits_{n=0}^\infty\frac{(-1)^{n+1}}{n}x^n$ is continuous on $[0,1]$ and in particular $f(x)$ is continuous at $x=1$. Since $f(x)=\ln(1+x)$ for $|x|<1$ and and since both $f(x)$ and $\ln(1+x)$ are continuous at 1 it follows that $f(1)=\ln 2$. Thus we have $\ln 2=\sum\limits_{n=1}^\infty\frac{(-1)^{n+1}}{n}$.

7.34 Example: Find a power series centered at 0 whose sum is $f(x) = \dfrac{1}{x^2+3x+2}$, and find its interval of convergence.

Solution: We have

$$\begin{aligned} f(x) &= \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{1+x} - \frac{\frac{1}{2}}{1+\frac{x}{2}} \\ &= \sum_{n=0}^{\infty}(-x)^n - \sum_{n=0}^{\infty} \tfrac{1}{2}\left(-\tfrac{x}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n x^n - \sum_{n=0}^{\infty} \tfrac{(-1)^n}{2^{n+1}} x^n \\ &= \sum_{n=0}^{\infty}(-1)^n \left(1 - \tfrac{1}{2^{n+1}}\right) x^n \, . \end{aligned}$$

Since $\displaystyle\sum_{n=0}^{\infty}(-x)^n$ converges if and only if $|x|<1$ and $\displaystyle\sum_{n=0}^{\infty} \tfrac{1}{2}\left(-\tfrac{x}{2}\right)^n$ converges when $|x|<2$, it follows from Linearity the the sum of these two series converges if and only if $|x|<1$.

7.35 Example: Find a power series centered at -4 whose sum is $f(x) = \dfrac{1}{x^2+3x+2}$, and find its interval of convergence.

Solution: We have

$$\begin{aligned} f(x) &= \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{(x+4)-3} - \frac{1}{(x+4)-2} \\ &= \frac{-\frac{1}{3}}{1-\frac{x+4}{3}} + \frac{\frac{1}{2}}{1-\frac{x+4}{2}} = \sum_{n=0}^{\infty} -\tfrac{1}{3}\left(\tfrac{x+4}{3}\right)^n + \sum_{n=0}^{\infty} \tfrac{1}{2}\left(\tfrac{x+4}{2}\right)^n \\ &= \sum_{n=0}^{\infty} \left(\tfrac{1}{2^{n+1}} - \tfrac{1}{3^{n+1}}\right)(x+4)^n \, . \end{aligned}$$

Since $\displaystyle\sum_{n=0}^{\infty} -\tfrac{1}{3}\left(\tfrac{x+4}{3}\right)^n$ converges when $|x+4|<3$ and $\displaystyle\sum_{n=0}^{\infty} \tfrac{1}{2}\left(\tfrac{x+4}{2}\right)^n$ converges if and only if $|x+4|<2$, it follows that their sum converges if and only if $|x+4|<2$.

7.36 Example: Find a power series centered at 0 whose sum is $\ln(1+x)$.

Solution: For $|x|<1$ we have

$$\begin{aligned} \frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots \\ \ln(1+x) &= \int 1 - x + x^2 - x^3 + + \cdots \; dx \\ &= c + x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots \end{aligned}$$

Put in $x=0$ to get $0=c$, and so

$$\ln(1+x) = \sum_{n=1}^{\infty} \tfrac{(-1)^{n+1}}{n} x^n = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots \, .$$

7.37 Example: Find a power series centered at 0 whose sum is $f(x) = \dfrac{1}{(1-x)^2}$.

Solution: We provide three solutions. For the first solution, we multiply two power series. For $|x| < 1$ we have

$$\begin{aligned} f(x) &= \frac{1}{1-x} \cdot \frac{1}{1-x} \\ &= \big(1 + x + x^2 + x^3 + \cdots\big)\big(1 + x + x^2 + x^3 + \cdots\big) \\ &= 1 + (1+1)x + (1+1+1)x^2 + (1+1+1+1)x^3 + \cdots \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots \\ &= \sum_{n=0}^{\infty} (n+1)x^n\,. \end{aligned}$$

For the second solution, we note that $f(x) = \dfrac{1}{1 - 2x + x^2}$ and we use long division.

$$\begin{array}{r|l} & 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots \\ \hline 1 - 2x + x^2 & 1 + 0x + 0x^2 + 0x^3 + 0x^4 - \cdots \\ & \underline{1 - 2x + x^2} \\ & \qquad 2x - x^2 \\ & \qquad \underline{2x - 4x^2 + 2x^3} \\ & \qquad\qquad 3x^2 - 2x^3 \\ & \qquad\qquad \underline{3x^2 - 6x^3 + 3x^4} \\ & \qquad\qquad\qquad 4x^3 - 8x^4 + \cdots \\ & \qquad\qquad\qquad \underline{4x^3 - 8x^4 + \cdots} \\ & \qquad\qquad\qquad\qquad 5x^4 + \cdots \end{array}$$

For the third solution, we note that $\displaystyle\int \frac{1}{(1-x)^2} = \frac{1}{1-x}$ and we use differentiation.

$$\begin{aligned} \frac{1}{1-x} &= 1 + x^2 + x^3 + x^4 + x^5 + \cdots \\ \frac{d}{dx}\left(\frac{1}{1-x}\right) &= \frac{d}{dx}\big(1 + x + x^2 + x^3 + x^4 + x^5 + \cdots\big) \\ \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots\,. \end{aligned}$$

Taylor Series

7.38 Theorem: *Suppose that* $f(x)=\displaystyle\sum_{n=0}^{\infty}a_n(x-a)^n$ *in an open interval* I *centred at* a. *Then* f *is infinitely differentiable at* a *and for all* $n\geq 0$ *we have*

$$a_n=\frac{f^{(n)}(a)}{n!}\ ,$$

where $f^{(n)}(a)$ *denotes the* n^{th} *derivative of* f *at* a.

Proof: By repeated application of the Differentiation of Power Series Theorem, for all $x\in I$, we have $f'(x)=\sum_{n=1}^{\infty}na_n(x-a)^{n-1}$, $f''(x)=\sum_{n=2}^{\infty}n(n-1)a_n(x-a)^{n-2}$ and $f'''(x)=\sum_{n=3}^{\infty}n(n-1)(n-2)a_n(x-a)^{n-3}$, and in general

$$f^{(k)}(x)=\sum_{n=k}^{\infty}n(n-1)\cdots(n-k+1)a_n(x-a)^{n-k}$$

and so $f(a)=a_0$, $f'(a)=a_1$, $f''(a)=2\cdot 1\,a_2$ and $f'''(a)=3\cdot 2\cdot 1\,a_3$, and in general

$$f^{(n)}(a)=n!\ a_n$$

7.39 Definition: Given a function $f(x)$ whose derivatives of all order exist at $x=a$, we define the **Taylor series** of $f(x)$ centered at a to be the power series

$$T(x)=\sum_{n\geq 0}a_n(x-a)^n\quad\text{ where }a_n=\frac{f^{(n)}(a)}{n!}$$

and we define the l^{th} **Taylor Polynomial** of $f(x)$ centered at a to be the l^{th} partial sum

$$T_l(x)=\sum_{n=0}^{l}a_n(x-a)^n\quad\text{ where }a_n=\frac{f^{(n)}(a)}{n!}$$

7.40 Example: Find the Taylor series centered at 0 for $f(x)=e^x$.

Solution: We have $f^{(n)}(x)=e^x$ for all n, so $f^{(n)}(0)=1$ and $a_n=\frac{1}{n!}$ for all $n\geq 0$. Thus the Taylor series is

$$T(x)=\sum_{n=0}^{\infty}\tfrac{1}{n!}\,x^n=1+x+\tfrac{1}{2!}x^2=\tfrac{1}{3!}x^3+\tfrac{1}{4!}x^4+\cdots\,.$$

7.41 Example: Find the Taylor series centered at 0 for $f(x)=\sin x$.

Solution: We have $f'(x)=\cos x$, $f''(x)=-\sin x$, $f'''(x)=-\cos x$, $f''''(x)=\sin x$ and so on, so that in general $f^{(2n)}(x)=(-1)^n\sin x$ and $f^{(2n+1)}(x)=(-1)^n\cos x$. It follows that $f^{(2n)}(0)=0$ and $f^{(2n+1)}(0)=(-1)^n$, so we have $a_{2n}=0$ and $a_{2n+1}=\frac{(-1)^n}{(2n+1)!}$. Thus

$$T(x)=\sum_{n=0}^{\infty}\tfrac{(-1)^n}{(2n+1)!}x^{2n+1}=x-\tfrac{1}{3!}x^3+\tfrac{1}{5!}x^5-\tfrac{1}{7!}x^7+\cdots\,.$$

7.42 Example: Find the Taylor series centered at 0 for $f(x) = (1+x)^p$ where $p \in \mathbb{R}$.

Solution: $f'(x) = p(1+x)^{p-1}$, $f''(x) = p(p-1)(1+x)^{p-2}$, $f'''(x) = p(p-1)(p-2)(1+x)^{p-3}$, and in general

$$f^{(n)}(x) = p(p-1)(p-2)\cdots(p-n+1)(1+x)^{p-n}\,,$$

so $f(0) = 1$, $f'(0) = p$, $f''(0) = p(p-1)$, and in general $f^{(n)}(0) = p(p-1)(p-2)\cdots(p-n+1)$, and so we have $a_n = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$. Thus the Taylor series is

$$T(x) = \sum_{n=0}^{\infty} \binom{p}{n} x^n = 1 + px + \tfrac{p(p-1)}{2!}x^2 + \tfrac{p(p-1)(p-2)}{3!}x^3 + \tfrac{p(p-1)(p-2)(p-3)}{4!}x^4 + \cdots$$

where we use the notation

$$\binom{p}{0} = 1\ ,\ \text{and for } n \geq 1,\ \binom{p}{n} = \tfrac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$$

7.43 Theorem: *(Taylor) Let $f(x)$ be infinitely differentiable in an open interval I with $a \in I$. Let $T_l(x)$ be the l^{th} Taylor polynomial for $f(x)$ centered at a. Then for all $x \in I$ there exists a number c between a and x such that*

$$f(x) - T_l(x) = \frac{f^{(l+1)}(c)}{(l+1)!}(x-a)^{l+1}\,.$$

Proof: When $x = a$ both sides of the above equation are 0. Suppose that $x > a$ (the case that $x < a$ is similar). Since $f^{(l+1)}$ is differentiable and hence continuous, by the Extreme Value Theorem it attains its maximum and minimum values, say M and m. Since $m \leq f^{(l+1)}(t) \leq M$ for all $t \in I$, we have

$$\int_a^{t_1} m\,dt \leq \int_a^{t_1} f^{(l+1)}(t)\,dt \leq \int_a^{t_1} M\,dt$$

that is

$$m(t_1 - a) \leq f^{(l)}(t_1) - f^{(l)}(a) \leq M(t_1 - a)$$

for all $t_1 > a$ in I. Integrating each term with respect to t_1 from a to t_2, we get

$$\tfrac{1}{2}m(t_2 - a)^2 \leq f^{(l-1)}(t_2) - f^{(l)}(a)(t_2 - a) \leq \tfrac{1}{2}M(t_t - a)^2$$

for all $t_2 > a$ in I. Integrating with respect to t_2 from a to t_3 gives

$$\tfrac{1}{3!}m(t_3 - a)^3 \leq f^{(l-2)}(t_3) - f^{(l-2)}(a) - \tfrac{1}{2}f^{(l)}(a)(t_3 - a)^3 \leq \tfrac{1}{3!}M(t_3 - a)^3$$

for all $t_3 > a$ in I. Repeating this procedure eventually gives

$$\tfrac{1}{(l+1)!}m(t_{l+1} - a)^{l+1} \leq f(t_{l+1}) - T_l(t_{l+1}) \leq \tfrac{1}{(l+1)!}M(t_{l+1} - a)^{l+1}$$

for all $t_{l+1} > a$ in I. In particular $\frac{1}{(l+1)!}m(x-a)^{l+1} \leq f(x) - T_l(x) \leq \frac{1}{(l+1)!}M(x-a)^{l+1}$, so

$$m \leq \big(f(x) - T_l(x)\big)\tfrac{(l+1)!}{(x-a)^{l+1}} \leq M\,.$$

By the Intermediate Value Theorem, there is a number $c \in [a, x]$ such that

$$f^{(l+1)}(c) = \big(f(x) - T_l(x)\big)\frac{(l+1)!}{(x-a)^{l+1}}$$

.

7.44 Theorem: *The functions e^x, $\sin x$ and $(1+x)^p$ are all exactly equal to the sum of their Taylor series centered at 0 in the interval of convergence.*

Proof: First let $f(x)=e^x$ and let $x\in\mathbb{R}$. By Taylor's Theorem, $f(x)-T_l(x)=\dfrac{e^c x^{l+1}}{(l+1)!}$ for some c between 0 and x, and so

$$\big|f(x)-T_l(x)\big|\leq\frac{e^{|x|}|x|^{l+1}}{(l+1)!}\,.$$

Since $\sum\dfrac{e^{|x|}|x|^{l+1}}{(l+1)!}$ converges by the Ratio Test, we have $\lim\limits_{l\to\infty}\dfrac{e^{|x|}|x|^{l+1}}{(l+1)!}=0$ by the Divergence Test, so $\lim\limits_{l\to\infty}\big(f(x)-T_l(x)\big)=0$, and so $f(x)=\lim\limits_{l\to\infty}T_l(x)=T(x)$.

Now let $f(x)=\sin x$ and let $x\in\mathbb{R}$. By Taylor's Theorem, $f(x)-T(x)=\dfrac{f^{(l+1)}(c)\,x^{l+1}}{(l+1)!}$ for some c between 0 and x. Since $f^{(l+1)}(x)$ is one of the functions $\pm\sin x$ or $\pm\cos x$, we have $\big|f^{(l+1)}(c)\big|\leq 1$ for all c and so

$$\big|f(x)-T(x)\big|\leq\frac{|x|^{l+1}}{(l+1)!}\,.$$

Since $\sum\dfrac{|x|^{l+1}}{(l+1)!}$ converges by the Ratio Test, $\lim\limits_{l\to\infty}\dfrac{|x|^{l+1}}{(l+1)!}=0$ by the Divergence Test, and so we have and $f(x)=T(x)$ as above.

Finally, let $f(x)=(1+x)^p$. The Taylor series centered at 0 is

$$T(x)=1+px+\tfrac{p(p-1)}{2!}x^2+\tfrac{p(p-1)(p-2)}{3!}x^3+\tfrac{p(p-1)(p-2)(p-3)}{4!}x^4+\cdots$$

and it converges for $|x|<1$. Differentiating the power series gives

$$T'(x)=p+\tfrac{p(p-1)}{1!}x+\tfrac{p(p-1)(p-2)}{2!}x^2+\tfrac{p(p-1)(p-2)(p-3)}{3!}x^3+\cdots$$

and so

$$\begin{aligned}(1+x)T'(x)&=p+\left(p+\tfrac{p(p-1)}{1!}\right)x+\left(\tfrac{p(p-1)}{1!}+\tfrac{p(p-1)(p-2)}{2!}\right)x^2\\&\qquad+\left(\tfrac{p(p-1)(p-2)}{2!}-\tfrac{p(p-1)(p-2)(p-3)}{3!}\right)x^3+\cdots\\&=p+\tfrac{p\cdot p}{1!}x+\tfrac{p\cdot p(p-1)}{2!}x^2+\tfrac{p\cdot p(p-1)(p-2)}{3!}x^3+\cdots\\&=p\,T(x)\,.\end{aligned}$$

Thus we have $(1+x)T'(x)=pT(x)$ with $T(0)=1$. This DE is linear since we can write it as $T'(x)-\frac{p}{1+x}T(x)=0$. An integrating factor is $\lambda=e^{\int-\frac{p}{1+x}\,dx}=e^{-p\ln(1+x)}=(1+x)^{-p}$ and the solution is $T(x)=(1+x)^{-p}\displaystyle\int 0\,dx=b(1+x)^p$ for some constant b. Since $T(0)=1$ we have $b=1$ and so $T(x)=(1+x)^p=f(x)$.

Applications

7.45 Example: Let $f(x)=\sin\left(\frac{1}{2}x^2\right)$. Find the 10^{th} derivative $f^{(10)}(0)$.

7.46 Example: Find $\displaystyle\lim_{x\to 0}\frac{e^{-2x^2}-\cos 2x}{\left(\tan^{-1}x+\ln(1+x)\right)^2}$

7.47 Example: Approximate the value of $\frac{1}{\sqrt{e}}$ so the error is at most $\frac{1}{100}$.

7.48 Example: Approximate the value of $\sqrt{e}$ so the error is at most $\frac{1}{100}$.

7.49 Example: Approximate the value of $\ln 2$ so the error is at most $\frac{1}{50}$

7.50 Example: Approximate the value of $10^{2/3}$ so the error is at most $\frac{1}{100}$.

7.51 Example: Approximate the value of π so the error is at most $\frac{1}{100}$.

7.52 Example: Approximate the value of $\sin\left(10^\circ\right)$ so the error is at most $\frac{1}{1000}$.

7.53 Example: Approximate the value of $\displaystyle\int_0^1 e^{-x^2}\,dx$ so the error is at most $\frac{1}{100}$.

7.54 Example: Approximate the value of $\displaystyle\int_0^{\sqrt{2}}\frac{\sin x}{x}\,dx$ so the error is at most $\frac{1}{50}$.

7.55 Example: Find the exact value of the sum $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n}{(2n)!}$.

7.56 Example: Find the exact value of the sum $\displaystyle\sum_{n=1}^{\infty}\frac{n-2}{(-3)^n}$.

7.57 Example: Find the exact value of the sum $\displaystyle\sum_{n=0}^{\infty}\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)}{5^n\,n!}$.

7.58 Example: Find the solution to the IVP $y''=xy$ with $y(0)=2$ and $y'(0)=3$.

7.59 Example: Find the power series solution to the IVP $y'=x+y^2$ with $y(0)=0$, and show that the power series has a positive radius of convergence.