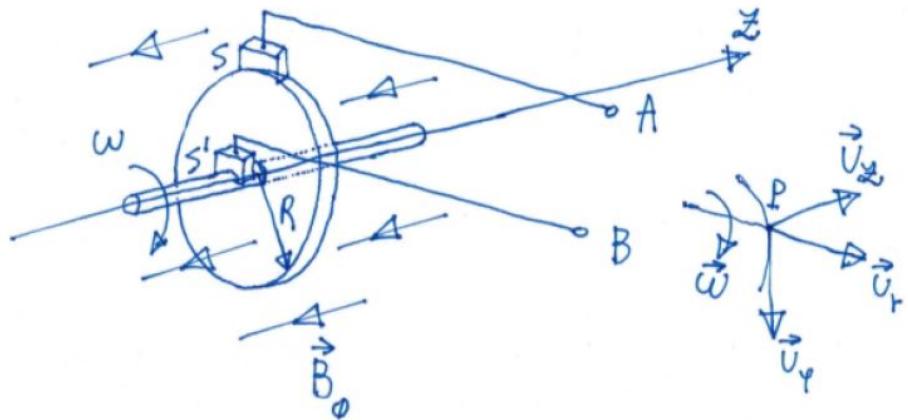


LECTURES

PHYS 342



## Electromagnetism II

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## Contents

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<b>1</b>		<b>7</b>
1.1	Review of Electrostatics . . . . .	7
1.2	Perfect Conductors in Vacuum . . . . .	8
<b>2</b>		<b>11</b>
2.1	Review of Electric Current . . . . .	11
2.2	Review of Magnetostatic Field in Vacuum . . . . .	13
<b>3</b>		<b>17</b>
3.1	On the Rotational Character of Ampère's Law . . . . .	17
3.2	Laplace's Elementary Law (or Biot-Savart Law) . . . . .	20
<b>4</b>		<b>23</b>
4.1	Lorentz Force . . . . .	23
4.1.1	The Relativity of Electric and Magnetic Fields . . . . .	23
<b>5</b>		<b>29</b>
5.1	On the Vector Potential of $\vec{B}$ . . . . .	29
5.1.1	Vector Potential for a Uniform Field $\vec{B}$ . . . . .	29
5.1.2	Magnetostatic Field of an Infinite, Straight Wire Revisited . . . . .	33
<b>6</b>		<b>37</b>
6.1	Weak Formulation of the Electrostatic Problem . . . . .	37
<b>7</b>		<b>43</b>
7.1	Electrostatic Energy of a System of Charged Conductors . . . . .	43
7.2	The Principle of Least Action for the Capacitance . . . . .	44
<b>8</b>		<b>49</b>
8.1	Neumann Integrals . . . . .	49
<b>9</b>		<b>53</b>
9.1	Self Inductance of a Quasi-Filiform Circular Ring (Maxwell's Inductance) . . . . .	53

---

<b>10</b>	<b>59</b>
10.1 Forces on Electrically Charged Conductors . . . . .	59
10.2 Force for Parallel-Plate Condensers . . . . .	62
<b>11</b>	<b>67</b>
11.1 Forces on Electric Current-Carrying Circuits in an External Magnetic Field . . . . .	67
11.2 Force Between Two Straight, Indefinite (Infinitely Long) Current-Carrying Conductors, Parallel to Each Other . . . . .	69
11.3 Work to Move a Current-Carrying Circuit . . . . .	71
<b>12</b>	<b>75</b>
12.1 Multipole Series Expansion of the Electrostatic Potential. . . . .	75
12.2 The Electrostatic Dipole. . . . .	79
<b>13</b>	<b>83</b>
13.1 Electrostatic Induction (Revisited) . . . . .	83
13.2 The Method of Images. . . . .	86
<b>14</b>	<b>93</b>
14.1 Multipole Series Expansion of the Electrostatic Potential for a Continuous Volume Charge Distribution . . . . .	93
14.2 Ideal Electrostatic Dipole . . . . .	97
14.3 Interaction Between an Ideal Electrostatic Dipole and an Electrostatic Field . . . . .	97
14.4 Field Dipole Interaction Energy . . . . .	100
<b>15</b>	<b>101</b>
15.1 Distribution of Dipoles . . . . .	101
15.2 Charge Distributions Equivalent to Dipole Distributions . . . . .	102
<b>16</b>	<b>107</b>
16.1 Multipole Series Expansion of the Magnetostatic Vector Potential for a Continuous and Limited Volume Electric Current Distribution . . . . .	107
<b>17</b>	<b>113</b>
17.1 The Ideal Magnetic Dipole . . . . .	113
17.2 Rectangular Circuit with Steady Current in an External Uniform Magnetic Field . . . . .	117
<b>18</b>	<b>121</b>
18.1 Polarization Phenomena: The Macroscopic Viewpoint . . . . .	121
18.1.1 Introduction . . . . .	121
18.1.2 Polarization Vector and Polarization Charge . . . . .	123
18.2 Electric Field due to Known Polarization Distributions . . . . .	130
18.3 Examples on Depolarization Field . . . . .	131

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## CONTENTS

---

18.4	Gauss' Theorem for Dielectrics . . . . .	136
18.4.1	Boundary Conditions for Vector $\vec{D}$ and $\vec{E}$ . . . . .	139
18.5	Linear Dielectric Materials: Electric Susceptibility and Dielectric Constant	141
18.6	Electric Field in Presence of Linear and Homogeneous Dielectrics . . . . .	144
18.6.1	Relationships between Charge Densities . . . . .	145
18.6.2	General Equations for the Electric Field and Potential . . . . .	146
18.6.3	Presence of a Single Dielectric . . . . .	147
18.6.4	Presence of Many Dielectrics . . . . .	149
18.7	Indefinite Dielectric Slab in Parallel-Plate Condensator . . . . .	150
18.8	Dielectric Sphere of Radius $a$ in an External Electric Field (Assumed to be Uniform in Absence of the Sphere and with Intensity $\vec{E}_0$ ) . . . . .	157
18.9	Dielectric of Arbitrary geometrical Shape in a Parallel-Plate Condensator	158
<b>19</b>		<b>161</b>
19.1	The Magnetostatic Field in Presence of Magnetic Materials . . . . .	161
19.1.1	Introduction . . . . .	161
19.1.2	The Magnetization Field . . . . .	162
<b>20</b>		<b>167</b>
20.1	Laws of Magnetostatics for Magnetic Materials . . . . .	167
20.2	Classification of Magnetic Materials . . . . .	169
<b>21</b>		<b>173</b>
21.1	Boundary Conditions for Vector $\vec{B}$ and $\vec{H}$ . . . . .	173
21.2	Magnetization of Isotropic, Linear, and Homogeneous Materials . . . . .	175
21.2.1	Finite Cylindrical Magnet . . . . .	175
21.2.2	Infinite Cylindrical Magnet . . . . .	178
21.2.3	Magnetic Slab . . . . .	180
<b>22</b>		<b>181</b>
22.1	Magnetization of Isotropic, Linear, and Homogeneous Materials . . . . .	181
22.1.1	Spherical Magnet . . . . .	181
22.2	Ampère's Equivalence Theorem . . . . .	188
<b>23</b>		<b>189</b>
23.1	The Fundamental Laws of the Electromagnetic Field in Vacuum - Integral Form . . . . .	189
23.1.1	Time-Derivative Swap . . . . .	190
23.1.2	Solenoidal Properties . . . . .	191
<b>24</b>		<b>195</b>
24.1	Electromagnetic Induction . . . . .	195
24.1.1	Faraday-Neumann's Law for Two Circuits . . . . .	196
24.1.2	Lenz's Law . . . . .	196
24.1.3	Electromagnetic Induction in an Open Circuit . . . . .	198

---

<b>25</b>		<b>199</b>
25.1	Barlow's Wheel . . . . .	199
25.2	Displacement Current . . . . .	202
	25.2.1 Charging Capacitor . . . . .	202
<b>26</b>		<b>205</b>
26.1	Magnetic Energy Associated with a Current . . . . .	205
	26.1.1 Single Circuit with Current . . . . .	205
26.2	Two Circuits with Current . . . . .	211
	26.2.1 $N$ Circuits with Currents . . . . .	215
<b>27</b>		<b>217</b>
27.1	The Electromagnetic Field Laws in Presence of Materials . . . . .	217
27.2	Maxwell's Laws in Local Form . . . . .	220
	27.2.1 Case 1 . . . . .	220
	27.2.2 Case 2 . . . . .	221
<b>28</b>		<b>225</b>
28.1	Electromagnetic Potentials . . . . .	225
<b>29</b>		<b>231</b>
29.1	Magnetic Field Energy . . . . .	231
<b>30</b>		<b>235</b>
30.1	Electromagnetic Field Energy . . . . .	235
<b>31</b>		<b>239</b>
31.1	An Oscillating Circuit . . . . .	239

# Chapter 1

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## 1.1 Review of Electrostatics

Consider a volume charge distribution with density  $\rho \in C^0(\Omega)$  in a domain  $\Omega$  of the 3D Euclidean space.

According to Gauss' theorem

$$\iint_{\Sigma} \vec{E} \cdot \vec{n} dA = \frac{1}{\epsilon_0} \iiint_{\Omega} \rho dV \quad (1.1)$$

Where  $\Sigma$  is a surface enclosing  $\Omega$ . This equation can be written in differential form as (if  $\rho$  is continuous and limited)

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho \quad (1.2)$$

which is also valid when  $\rho = 0$ .

The irrotational property of  $\vec{E}$  can be written in integral form as

$$\oint \vec{E} \cdot \vec{t} d\ell = 0 \quad (1.3)$$

and, in differential form, as

$$\vec{\nabla} \times \vec{E} = \vec{0} \quad (1.4)$$

In the case of a surface charge distribution with density  $\rho$  in  $\Omega$ , the local form of Gauss' theorem and the irrotational property is

$$(\vec{E}_1 - \vec{E}_2) \cdot \vec{n} = \frac{1}{\epsilon_0} \sigma \quad (1.5)$$

and

$$(\vec{E}_1 - \vec{E}_2) \cdot \vec{t} = 0 \quad (1.6)$$

respectively (see conventions in the next figure).

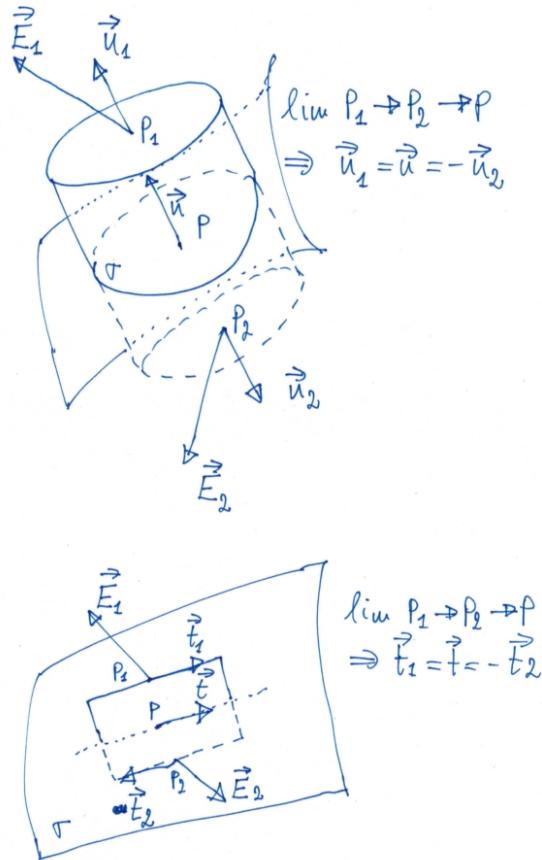


Figure 1.1

In a simply connected (or star) domain, the electrostatic field  $\vec{E}$  is not only irrotational, but also conservative. In this case, a scalar potential  $\phi$  can be defined (often, the symbol  $V$  instead of  $\phi$  is used), such that

$$\vec{E} = -\vec{\nabla}\phi \quad (1.7)$$

From this definition and the differential form of Maxwell's equations for  $\vec{E}$ , it can be shown that

$$\nabla^2\phi = 0 \quad (1.8)$$

when  $\rho = 0$  (Laplace equation) and

$$\nabla^2\phi = -\frac{1}{\epsilon_0}\rho \quad (1.9)$$

when  $\rho \neq 0$  (Poisson equation).

## 1.2 Perfect Conductors in Vacuum

Consider a perfect and homogeneous conductor in vacuum.

The conductor is said to be in electrostatic equilibrium when no macroscopic motion of charges exists in the conductor. Under these conditions

$$\vec{E} = \vec{0} \quad (1.10)$$

at each point inside the conductor. Following the notation in the figure below,

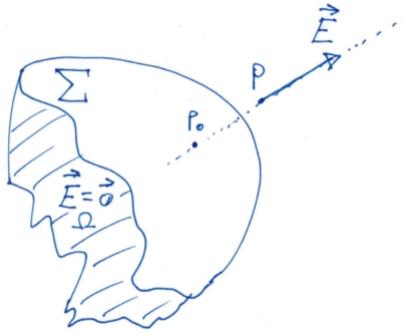


Figure 1.2

when the conductor is in an external field  $\vec{E}_{ex}$ , a reaction field  $\vec{E}_r$  is generated until

$$\vec{E} = \vec{E}_{ex} + \vec{E}_r = \vec{0} \quad (1.11)$$

at each point inside the conductor. Under these conditions, a surface charge density  $\sigma$  is generated on the conductor surface.  $\sigma$  depends on the point  $P_0$  on the surface. According to Coulomb's theorem

$$\lim_{P \rightarrow P_0^+} E_n(P) = \frac{\sigma(P_0)}{\epsilon_0} \quad (1.12)$$

where  $P_0^+$  indicates that the point is located on the outer skin of the conductor.

From the definition of  $\phi$ , it follows

$$\phi = \text{const} \Big|_{\Omega \cup \Sigma} \quad (1.13)$$

and

$$E_n = - \frac{\partial}{\partial n} \phi \Big|_{\Sigma^+} \quad (1.14)$$

where  $\Sigma^+$  indicates the outer skin of the conductor.

If the conductor is charged with  $Q$ ,

$$Q = \epsilon_0 \iint_{\Sigma' \supset \Sigma} E_n dA = \iint_{\Sigma} \sigma dA \quad (1.15)$$

In general,

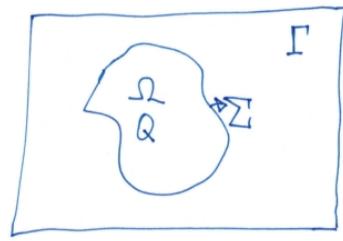


Figure 1.3

- i)  $\phi \in C^\circ(\Gamma \cup \Sigma)$
- ii)  $\vec{\nabla}^2 \phi = 0 \quad \forall P \in \Gamma$
- iii)  $\phi = \text{const} \quad \forall P \in \Omega \cup \Sigma$
- iv)  $-\iint_{\Sigma^+} \frac{\partial}{\partial n} \phi dA = \frac{1}{\epsilon_0} Q$
- v)  $\lim_{P \rightarrow \infty} \phi(P) = 0$

- Definition of capacitance

Given two conductors,  $A$  and  $B$ , with charges  $+|Q|$  and  $-|Q|$ , respectively, the capacitance of the system is defined as

$$C \equiv \frac{|Q|}{\Delta\phi} \quad (1.16)$$

where  $\Delta\phi = \phi_A - \phi_B$ .

# Chapter 2

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## 2.1 Review of Electric Current

The electric current intensity is defined as

$$i(t) = \frac{d}{dt} q(t) \quad (2.1)$$

A metallic conductor can be thought as a lattice of positive fixed charges and a cloud of electrons that are free to move. The electrons can collide with themselves and the positive charges. Under electrostatic equilibrium conditions, the electrons undergo a chaotic thermal motion. Thus, given a cross-section  $A$  of a conductor, the same number of electrons crosses  $A$  in one direction as in the opposite direction. This means that on a macroscopic time scale,  $q(t) = 0$  at all times. This dynamics is called diffusion.

When a force is applied to the electrons, the center of mass of the cloud of electrons can move at a certain velocity. This dynamics, called drift, gives rise to conduction currents.

Consider a region  $\Omega$  of a conductor characterized by an electric current.  $\rho_f$  is the volume density of the moving charges (charge free carriers) and  $\vec{v}$  their velocity at point  $P$  and time  $t$ . Given a surface element with area  $dA$  and a normal unit vector  $\vec{n}$  to  $dA$ , the total charge that crosses  $dA$  in a time  $dt$  is

$$dq = \rho_f \vec{v} \cdot \vec{n} dt dA \quad (2.2)$$

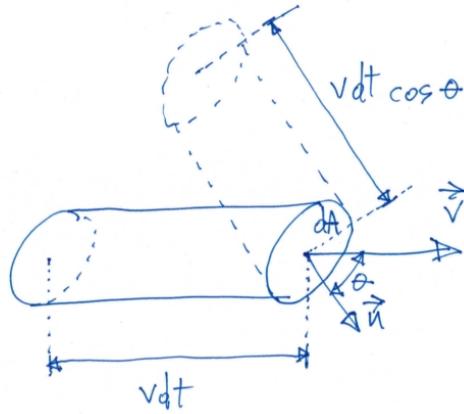


Figure 2.1

The current volume density is defined as

$$\vec{J} = \rho_f \vec{v} \quad (2.3)$$

Given a generic open surface with area  $A$ , the current intensity through  $A$  with respect to an arbitrary direction  $\vec{n}$  is

$$i = \iint_A \rho \vec{v} \cdot \vec{n} dA = \iint_A \vec{J} \cdot \vec{n} dA \quad (2.4)$$

Charge transport fulfils the charge conservation principle, i.e., given a closed surface  $\Sigma$ , the charge that crosses  $\Sigma$  in a generic time window corresponds to the change of charge within the volume  $\Omega$  enclosed by  $\Sigma$ . Mathematically,

$$\iint_{\Sigma} \vec{J} \cdot \vec{n} dA = \iiint_{\Omega} \vec{\nabla} \cdot \vec{J} dV = - \iiint_{\Omega} \frac{\partial}{\partial t} \rho dV \quad (2.5)$$

where  $\rho$  is the volume charge density in  $\Omega$ .

Given the arbitrariness of  $\Sigma$  and  $\Omega$ , this means that

$$\vec{\nabla} \cdot \vec{J} = - \frac{\partial}{\partial t} \rho \quad (2.6)$$

This is called continuity equation.

An electric current is said to be stationary when both charge and current densities are time independent at each point in a conductor. In this case,

$$\vec{\nabla} \cdot \vec{J} = 0 \quad (2.7)$$

or

$$\iint_{\Sigma} \vec{J} \cdot \vec{n} dA = 0 \quad (2.8)$$

The field  $\vec{J}$  is said to be solenoidal.

Under stationary conditions, the laws for the electric field inside and outside a conductor with current are

$$\iint_{\Sigma} \vec{E} \cdot \vec{n} dA = \frac{1}{\epsilon_0} Q \quad (2.9)$$

$$\oint_{\gamma} \vec{E} \cdot \vec{t} d\ell = 0 \quad (2.10)$$

In general,

$$\vec{J} = f(\vec{E}) \quad (2.11)$$

where the function  $f$  depends on the conductor. This is called constitutive relation.

For certain conductors at constant temperature, the constitutive relation is linear and is called Ohm's law (in local form):

$$\vec{J} = g\vec{E} \quad (2.12)$$

where  $g$  is called electric conductivity. In integral form, Ohm's law reads

$$\Delta\phi = RI \quad (2.13)$$

where  $\Delta\phi$  is the potential difference (drop) across a conductor of resistance  $R$  that carries a current  $I$ .

## 2.2 Review of Magnetostatic Field in Vacuum

The magnetostatic field generated by a generic distribution of stationary current is solenoidal

$$\iint_{\Sigma} \vec{B} \cdot \vec{n} dA = 0 \quad (2.14)$$

This means that magnetic charge (i.e., magnetic monopoles) were never found. The field  $\vec{B}$  is also rotational (Ampère's law):

$$\oint_{\gamma} \vec{B} \cdot \vec{t} d\ell = \mu_0 I \quad (2.15)$$

where  $I$  is a stationary current. In particular,  $I$  is the algebraic sum of all currents linked with  $\gamma$ .

In local form, the solenoidal property can be written as

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (2.16)$$

in the case of a stationary current with finite (or zero) volume density  $\vec{J}$ . For a current with surface density  $\vec{J}_S$

$$\vec{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0 \quad (2.17)$$

where  $\vec{B}_1$  and  $\vec{B}_2$  are the fields slightly above and below the surface where  $\vec{J}_S$  is defined.

Similarly, Ampère's law can be written as

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad (2.18)$$

and

$$\vec{u} \times (\vec{B}_1 - \vec{B}_2) = \mu_0 \vec{J}_S \quad (2.19)$$

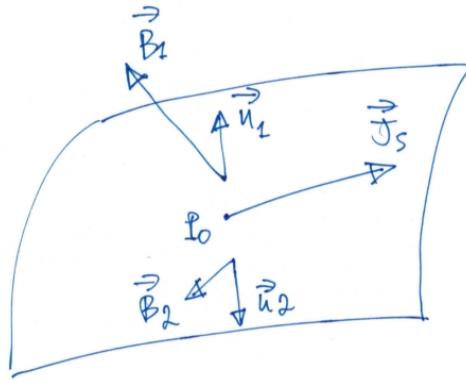


Figure 2.2

We remind that a solenoidal field is characterized by a vector potential. The vector potential  $\vec{A}$  of  $\vec{B}$  is such that

$$\vec{\nabla} \times \vec{A} = \vec{B} \quad (2.20)$$

It is always possible to find a field  $\vec{A}$  such that

$$\vec{\nabla} \cdot \vec{A} = 0 \quad (2.21)$$

Thus, from (2.18) and (2.20)

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} \\ &= -\vec{\nabla}^2 \vec{A} = \mu_0 \vec{J} \end{aligned} \quad (2.22)$$

or

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J} \quad (2.23)$$

where we used (2.21) and a well-known vector calculus identity.

In a Cartesian coordinate system

$$\begin{cases} \vec{\nabla}^2 A_x = -\mu_0 J_x \\ \vec{\nabla}^2 A_y = -\mu_0 J_y \\ \vec{\nabla}^2 A_z = -\mu_0 J_z \end{cases} \quad (2.24)$$

Assuming the functions  $A_x$ ,  $A_y$ , and  $A_z$  go to zero at infinity (e.g., for a limited  $\vec{J}$ ), the solutions to the above system are of “Coulomb type:”

$$\begin{cases} A_x(P) = \frac{\mu_0}{4\pi} \iiint_{\Omega} \frac{J_x(Q)}{r_{QP}} dV \\ A_y(P) = \frac{\mu_0}{4\pi} \iiint_{\Omega} \frac{J_y(Q)}{r_{QP}} dV \\ A_z(P) = \frac{\mu_0}{4\pi} \iiint_{\Omega} \frac{J_z(Q)}{r_{QP}} dV \end{cases} \quad (2.25)$$

At last, we remind the definition of inductance. Given a field  $B$  generated by a current  $I$  on a line  $\gamma$ , a new current

$$I' = kI \quad (2.26)$$

where  $k$  is a proportionality constant, also on  $\gamma$  will generate a field

$$B' = kB \quad (2.27)$$

If  $\Phi_\gamma$  is the flux through a surface bordered by  $\gamma$  due to  $B$ ,

$$\Phi'_\gamma = k\Phi_\gamma = \frac{I'}{I} \Phi_\gamma \quad (2.28)$$

If  $I = 1A$ ,

$$\Phi'_\gamma = I' \frac{\Phi_\gamma}{1A} = I'L \quad (2.29)$$

where  $L$  is the inductance associated with  $\gamma$ .



# Chapter 3

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## 3.1 On the Rotational Character of Ampère's Law

The magnetostatic field  $\vec{B}$  is rotational, i.e.,

$$\oint_{\gamma} \vec{B} \cdot \vec{t} d\ell = \mu_0 I \quad (3.1)$$

The current  $I$  is the algebraic sum of all currents linked with  $\gamma$ . Two closed lines are linked to each other if it is impossible to disconnect them without cutting one of them.

In Fig. 3.1 below two currents  $I_1$  and  $I_2$  are linked with an oriented curve  $\gamma$ .

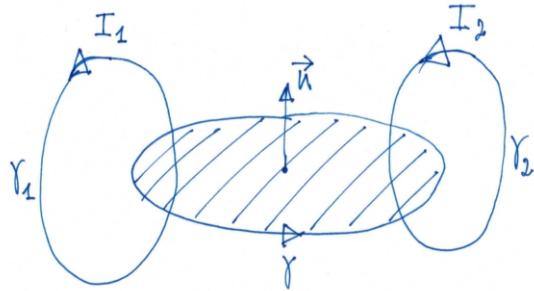


Figure 3.1

In this case, the direction of  $I_1$  is the same as  $\vec{n}$  when  $I_1$  crosses the surface associated with  $\vec{n}$ . Thus,  $I_1$  is assumed to be positive in Ampère's law. On the contrary,  $I_2$  is negative. In summary, for this example

$$\oint_{\gamma} \vec{B} \cdot \vec{t} d\ell = \mu_0 (I_1 - I_2) \quad (3.2)$$

When  $\gamma$  coincides with a vector line of  $\vec{B}$ , the product  $\vec{B} \cdot \vec{t}$  is either always positive or negative, but non-zero. Hence, each vector line of  $\vec{B}$  must be linked with at least one current. We remind that, because of the solenoidal property of  $\vec{B}$ , the vector lines of  $\vec{B}$  are closed.

When  $\gamma$  does not link any current,  $\vec{B}$  appears to be a rotational field, i.e.,

$$\oint_{\gamma} \vec{B} \cdot \vec{t} d\ell = 0 \quad (3.3)$$

exactly as for the electrostatic field  $\vec{E}$ . Consider two equally oriented lines  $\gamma_1$  and  $\gamma_2$  and assume the lines are open with equal limits  $A$  and  $B$ . Assuming a total current  $I$  is linked with the closed line  $A\gamma_1 B(-\gamma_2)A$  (see Fig. 3.2),

$$\int_{A\gamma_1 B} \vec{B} \cdot \vec{t} d\ell - \int_{A\gamma_2 B} \vec{B} \cdot \vec{t} d\ell = \mu_0 I \quad (3.4)$$

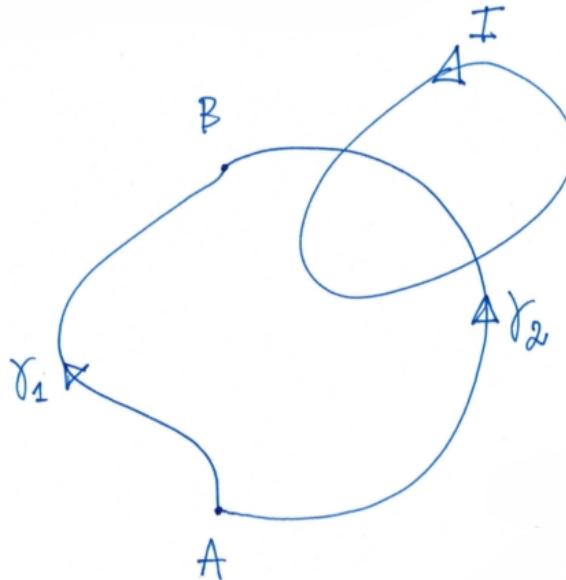


Figure 3.2

When a curve as  $A\gamma_1 B(-\gamma_2)A$  does not link any current or a total current  $I = 0$ , then

$$\int_{A\gamma_1 B} \vec{B} \cdot \vec{t} d\ell = \int_{A\gamma_2 B} \vec{B} \cdot \vec{t} d\ell \quad (3.5)$$

which is equivalent to

$$\oint_{A\gamma_1 B(-\gamma_2)A} \vec{B} \cdot \vec{t} d\ell = 0 \quad (3.6)$$

As a consequence, within internally connected regions of space with no currents, it is possible to find a potential function for  $\vec{B}$ . Note that an internally connected region of space is such that given a closed loop in the region is always possible to contract it to a point without ever exiting the region (also called simply-connected region).

In Fig. 3.3, point  $O$  is the trace of a current  $I$  perpendicular to the page. The current  $I$  is a uniform current on a straight filiform infinite conductor. This current clearly generates a field  $\vec{B}$  in space. In the plane of the page and in all planes parallel to it imagine to cut a hole with closed border  $\tilde{\gamma}$  surrounding  $O$ . When now considering  $\vec{B}$  in the region of space with the hole around  $O$ , the field is considered in a region that does not contain the current which generates it!

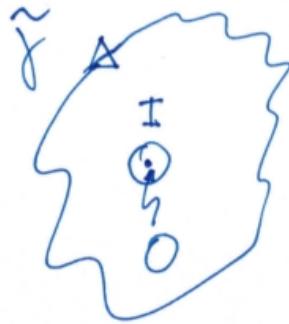


Figure 3.3

This leads to an apparent difficulty. In fact, each closed line entirely contained in the region with the hole does not link any current and, thus, the circulation of  $\vec{B}$  must be zero. However, the field  $\vec{B}$  generated by  $I$  is characterized by circular field lines with center in  $O$  (see PHYS 242) and considering  $\vec{B}$  outside  $\tilde{\gamma}$  cannot change this fact. Therefore, the circulation of  $\vec{B}$  around a closed, e.g., circular, line with  $O$  as center (i.e., a vector line of  $\vec{B}$ ) outside  $\tilde{\gamma}$  must be different than zero! This difficulty is resolved by considering only internally connected regions of space, where cutting hole is not allowed.

Consider again an infinite straight line with current  $I$  crossing a plane at point  $O$ . This time, however, we cut from the plane a region  $\Omega_1$  delimited by a curve  $\gamma_1$  that starts at infinite, goes around  $O$ , and goes back to infinite.

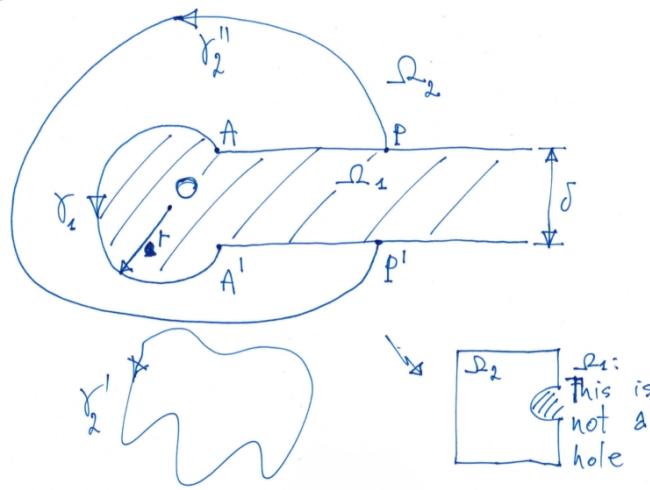


Figure 3.4

Since  $\Omega_1$  is not a hole,  $\Omega_2$  is an internally connected region. Additionally, any closed line in  $\Omega_2$  (e.g.,  $\gamma'_2$  or  $\gamma''_2 = PP'A'AP$ ) does not link  $I$ . Thus, a scalar potential  $\varphi$  can be associated with  $\vec{B}$  in  $\Omega_2$ .

Referring to Fig. 3.4, the radius  $r$  and distance  $\delta$  can be made arbitrary small (so long  $r, \delta \neq 0$ ). In this way, it would seem that  $\varphi$  can be defined almost everywhere in space. In the limit  $r, \delta \rightarrow 0^+$ ,  $\varphi$  would be defined everywhere! This is not entirely correct. Consider  $P$  and  $P'$  in Fig. 3.4. By definition,

$$\varphi(P) - \varphi(P') = \int_{PAA'P \text{ on } \gamma_1} \vec{B} \cdot \vec{t} d\ell \quad (3.7)$$

When  $\delta \rightarrow 0^+$ ,  $PAA'P$  on  $\gamma_1$  tends to close around  $O$  and

$$\varphi(P) - \varphi(P') = \mu_0 I \quad (3.8)$$

The scalar potential  $\varphi$  of  $\vec{B}$  is characterized by a discontinuity of the first kind equal to  $\mu_0 I$  between two infinitesimally closed points  $P$  and  $P'$ . This happens every time we attempt to extend the region where  $\varphi$  is defined to include the entire space (including the current generating  $\vec{B}$ ). This means the scalar potential of  $\vec{B}$  is a multivalued function. Thus, a scalar potential is not a good representation of  $\vec{B}$ .

## 3.2 Laplace's Elementary Law (or Biot-Savart Law)

In PHYS 242 and in the summary in Chapter 2, we introduced a vector potential  $\vec{A}$  for  $\vec{B}$ . We also saw that the vector potential  $\vec{A}$  at a generic point  $P$  due to a current density  $\vec{J}$  at points  $Q$  in  $\tau$  is given by

$$\vec{A}(P) = \frac{\mu_0}{4\pi} \iiint_{\tau} \frac{\vec{J}(Q)}{r_{QP}} dV \quad (3.9)$$

By the definition  $\vec{B} = \vec{\nabla} \times \vec{A}$ , if follows

$$\vec{B}(P) = \frac{\mu_0}{4\pi} \vec{\nabla}_P \times \iiint_{\tau} \frac{\vec{J}(Q)}{r_{QP}} dV \quad (3.10)$$

Due to the linearity of “ $\vec{\nabla}_P \times$ ”, we can write

$$\vec{B}(P) = \frac{\mu_0}{4\pi} \iiint_{\tau} \vec{\nabla}_P \times \frac{\vec{J}(Q)}{r_{QP}} dV \quad (3.11)$$

From the vector relation

$$\vec{\nabla}_P \times \frac{\vec{J}(Q)}{r_{QP}} = \frac{1}{r_{QP}} \vec{\nabla}_P \times \vec{J} - \vec{J} \times \vec{\nabla}_P \frac{1}{r_{QP}} \quad (3.12)$$

and being  $\vec{\nabla}_P \times \vec{J}(Q) = \vec{0}$  we readily obtain

$$\vec{B}(P) = \frac{-\mu_0}{4\pi} \iiint_{\tau} \vec{J}(Q) \times \vec{\nabla}_P \frac{1}{r_{QP}} dV \quad (3.13)$$

From the definition of gradient,

$$\vec{\nabla}_P \frac{1}{r_{QP}} = -\frac{1}{r_{QP}^2} \vec{u}_{QP} = -\frac{\vec{r}_{QP}}{r_{QP}^3} \quad (3.14)$$

Finally,

$$\vec{B}(P) = \frac{\mu_0}{4\pi} \iiint_{\tau} \frac{\vec{J}(Q) \times \vec{r}_{QP}}{r_{QP}^3} dV \quad (3.15)$$

Consider a filiform conductor with transversal cross-section  $dA$  and with a stationary current  $I$  (see Fig. 3.5). The volume of an infinitesimal element  $d\ell$  of the conductor is

$$dV = dAd\ell \quad (3.16)$$

From Eq. (3.15), we find

$$\begin{aligned} \vec{B}(P) &= \frac{\mu_0}{4\pi} \iiint_{\tau} \frac{\vec{J}(Q)dA \times \vec{r}_{QP}}{r_{QP}^3} d\ell \\ &= \frac{\mu_0}{4\pi} I \oint_{\gamma} \frac{\vec{t} \times \vec{r}_{QP}}{r_{QP}^3} d\ell \end{aligned} \quad (3.17)$$

Where  $I = \iint_{dA} J(Q)dA$ . Note that  $\vec{J}(Q) = J(Q)\vec{t}$  and

$$\begin{aligned} I &= \iint_{dA} \vec{J}(Q) \cdot \vec{n} dA = \iint_{dA} J(Q)\vec{t} \cdot \vec{t} dA \\ &= \iint_{dA} J(Q)dA \end{aligned}$$

### 3.2. LAPLACE'S ELEMENTARY LAW (OR BIOT-SAVART LAW)

The infinitesimal contribution to  $\vec{B}$  due to  $d\ell$  is thus

$$d\vec{B}(P) = \frac{\mu_0}{4\pi} I d\ell \frac{\vec{t} \times \vec{t}_{QP}}{r_{QP}^3} \quad (3.18)$$

This is known as Laplace's elementary law. This equation does not have independent value from Eq. (3.17) because it has no physical meaning to consider an “infinitesimal current element”  $Id\ell$  regardless form the closed circuit it is part of.

The resemblance between Eq. (3.18) and

$$d\vec{E}(P) = \frac{1}{4\pi\epsilon_0} dq \frac{\vec{r}_{QP}}{r_{QP}^3}$$

valid in electrostatic is remarkable.

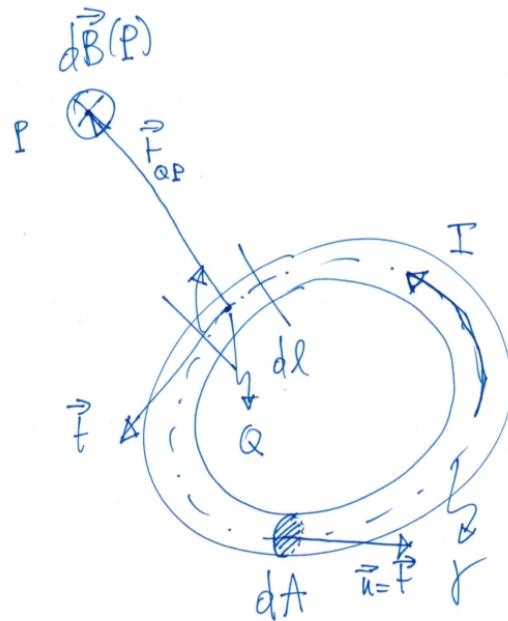


Figure 3.5

# Chapter 4

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## 4.1 Lorentz Force

As we saw in PHYS 242, the force on an electric charge depends on where the charge is in space and on how fast is moving.

The first part of the force is the electric force, which is independent of the motion of the charge  $q$  and is described by the electric field  $\vec{E}$ . The second part is the magnetic force, which depends on the velocity  $\vec{v}$  of the charge and on the magnetic field  $\vec{B}$ . In summary, the total electromagnetic force can be written as

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (4.1)$$

This force is called Lorentz force and is as fundamental as Newton's law of mechanics.

### 4.1.1 The Relativity of Electric and Magnetic Fields

In Eq. (4.1) we did not specify the reference frame with respect to which the velocity  $\vec{v}$  is defined.

To begin with, we assume the relativity principle is applicable to electromagnetism. We mean that, in the framework of special relativity, Maxwell's equations have the same form in all inertial frames of reference.

Consider a negative charge  $-|q|$  moving with velocity  $\vec{v}_0$  parallel to a wire carrying a steady, uniform current  $I$ , as shown in Fig. 4.1. The wire has cylindrical shape with cross-section  $A$  and is assumed to be straight and indefinitely long.

We will study this problem in two different frames of reference,  $\mathcal{S}$  and  $\mathcal{S}'$ . Frame  $\mathcal{S}$  is fixed with respect to the wire (see Fig. 4.1a) and frame  $\mathcal{S}'$  is fixed with respect to the charge (see Fig. 4.1b).

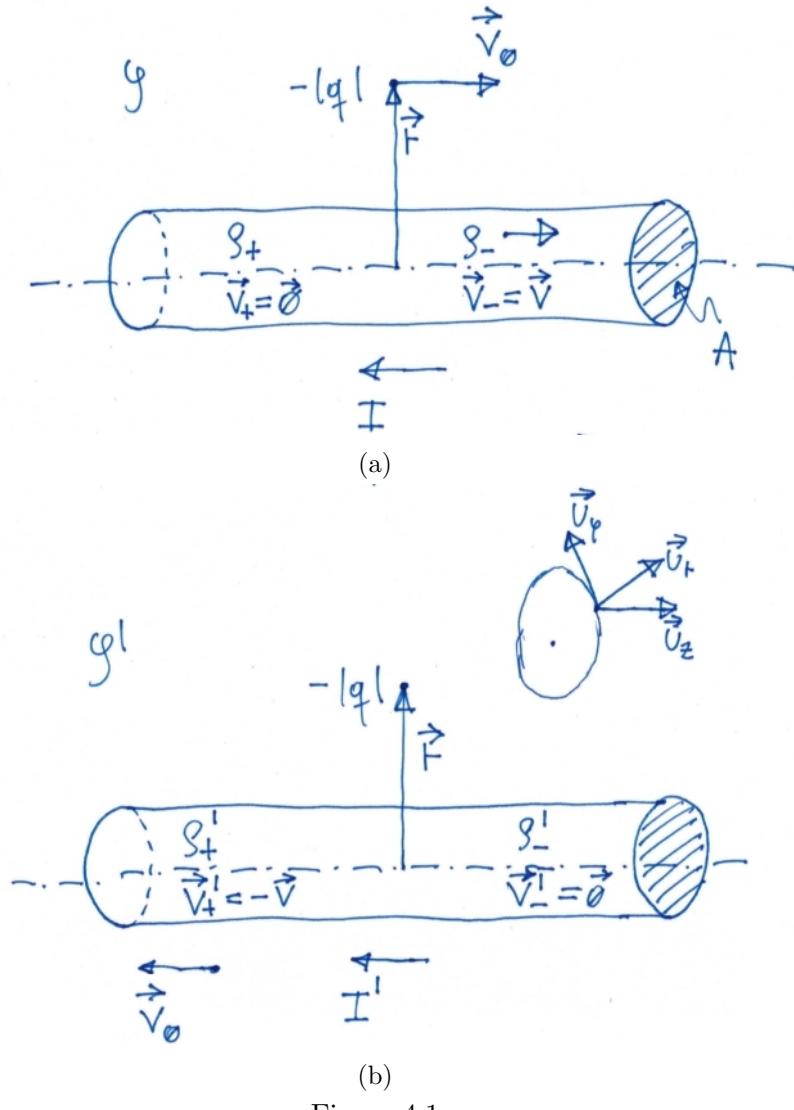


Figure 4.1

It is clear that in the  $\mathcal{S}$ -frame, the charged particle is subjected to a magnetic force due to the magnetic field generated by  $I$  [we derived such a magnetic field in PHYS 242 and we will re-derive it again from a different perspective in the next chapter, Chapter 5, see Eq. (5.25)] acting on the moving particle. If the particles were moving freely, it would curve towards the wire (the charge is negative). In the  $\mathcal{S}'$ -frame, however, there can be no magnetic force on the particle because the particle is at rest in that frame. Does the particle stay where it is in  $\mathcal{S}'$ ? According to the principle of relativity we should see the particle move also in  $\mathcal{S}'$ .

In a normal conductor (e.g., copper) the electric currents come from the motion of some of the negative electrons (called the conduction electrons), while the positive nuclear charges and the rest of the electrons (together forming the core ions) stay fixed in the body of the material.

We assume the volume density of the conduction electrons to be  $\rho_-$  and their velocity

in  $\mathcal{S}$  to be  $\vec{v}_- = \vec{v}$ . The density of the charges at rest in  $\mathcal{S}$  is  $\rho_+(\vec{v}_+ = \vec{0})$ . Since we are considering an uncharged wire, it must be  $\rho_+ + \rho_- = 0$ . As a consequence, there is no electric field outside the wire. The force on the moving particle  $-|q|$  is thus

$$\vec{F} = -|q| \vec{v}_0 \times \vec{B} \quad (4.2)$$

Using the result (5.25) for  $\vec{B}$  and noting that  $I$  points in the opposite direction of  $\vec{v}$ , we obtain

$$\vec{F} = -|q| v_0 \vec{u}_z \times \vec{u}_\varphi \frac{\mu_0(-|I|)}{2\pi} \frac{1}{r}, \quad r > 0 \quad (4.3)$$

where we represented the various vectors in a cylindrical coordinate system, as indicated in Fig. 4.1. Hence,

$$\vec{F} = -\frac{|q| v_0 \mu_0 |I|}{2\pi} \frac{1}{r} \vec{u}_r, \quad r > 0 \quad (4.4)$$

By means of Eq. (2.3) and Eq. (2.4) in the case of a uniform, steady current, we have

$$I = \rho_- v A \quad \text{or} \quad |I| = -\rho_- v A = \rho_+ v A \quad (4.5)$$

Then,

$$\vec{F} = -\frac{|q| v_0 \mu_0 \rho_+ v A}{2\pi} \frac{1}{r} \vec{u}_r, \quad r > 0 \quad (4.6)$$

Without loosing generality, we further assume  $v_0 = v$ . Hence,

$$\vec{F} = -\frac{|q|}{2\pi \epsilon_0} \frac{\rho_+ A}{r} \frac{v^2}{c^2} \vec{u}_r, \quad r > 0 \quad (4.7)$$

In  $\mathcal{S}'$ , the particle is at rest and the wire moves towards the left with velocity  $v$  (see Fig. 4.1b). The positive charges moving with the wire will generate a magnetic field  $\vec{B}'$ . However, the particle is now at rest and, thus,  $\vec{B}'$  generates no magnetic force on it. If there is any force on the particle, it must be an electric force. Due to the relativity principle for Maxwell's equations, there must be such a force in  $\mathcal{S}'$ , otherwise the particle would experience different forces in  $\mathcal{S}$  and  $\mathcal{S}'$ : This cannot be. *It must be that an uncharged (neutral) wire with a current appears to be charged when set in motion.*

We must try to compute the charge density in the wire in  $\mathcal{S}'$  from our knowledge about it in  $\mathcal{S}$ . At first glance, one may think the charge densities are the same in  $\mathcal{S}$  and  $\mathcal{S}'$ . However, from special relativity we know that lengths are changed between  $\mathcal{S}$  and  $\mathcal{S}'$ . Since charge densities depend on the volume occupied by charges, the densities must also change.

Going from  $\mathcal{S}$  to  $\mathcal{S}'$ , the apparent mass of a particle changes by  $(1 - v^2/c^2)^{-1/2}$ . An important question is whether charge undergoes a similar change between  $\mathcal{S}$  and  $\mathcal{S}'$ . As it turns out, because of charge conservation, charges are always the same, moving or not. This statement is fully confirmed by empirical evidence. The charge  $q$  of a particle is an invariant scalar quantity, independent from the frame of reference. In a given frame, the charge density of a distribution of charges is proportional to the number of charges per unit volume. Thus, we only need to worry about the fact that such a volume can change because of the relativistic contraction of distances.

Consider now our moving wire. For the sake of generality, consider a piece of wire of length  $L$ , in which there is a generic charge density  $\rho$  of stationary charges. The total charge in this piece of wire is

$$q = \rho L A \quad (4.8)$$

If the same charges are observed in a different frame that moves at velocity  $v$ , they are found in a piece of the material with the shorter length

$$L' = L \sqrt{1 - \frac{v^2}{c^2}} \quad (4.9)$$

but with the same area  $A$  since dimensions transverse to the motion are unchanged (see Fig. 4.2).

If we call  $\rho'$  the density of charges in the frame in which they are moving, the total charge will be

$$q' = \rho' L' A \quad (4.10)$$

It must be

$$q' = q \quad (4.11)$$

because of the total charge invariance. Thus,

$$\rho' L' = \rho L \quad (4.12)$$

and finally, from (4.9),

$$\rho' = \frac{\rho}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (4.13)$$

The charge density of a moving distribution of charges varies in the same way as the relativistic mass of a particle. Note this result is totally general; in our specific example, we will need to use it carefully for the positive and negative charge densities  $\rho_+$  and  $\rho_-$ , respectively. In particular, we will need to use it for each of them *separately* and keeping in mind whether we are in frame  $\mathcal{S}$  or  $\mathcal{S}'$ . The frame makes a big difference as in  $\mathcal{S}$  the positive charges in the wire are fixed and, thus, the role of  $\rho$  and  $\rho'$  is exactly the same as in Eq. (4.13). However, when considering the negative charges in the wire, these are moving in  $\mathcal{S}$  and fixed in  $\mathcal{S}'$ ; in this case, thus, the role of  $\rho$  and  $\rho'$  is the *opposite* of those in Eq. (4.13): This is a typical mistake done when studying this problem! Another important caveat is that the total positive  $q_+ = \rho_+ L A$  and negative  $q_- = \rho_- L A$  charges must *individually and separately* be conserved when moving from  $\mathcal{S}$  to  $\mathcal{S}'$ . In other words, it is not the total charge  $q = (q_+ + q_-)$  that is conserved between the two frames, but each of them individually. In fact, if the total charge was the conserved charge, we would have  $q = (q_+ + q_-) = 0$  in  $\mathcal{S}$  and  $q' = (q'_+ + q'_-) = 0$  in  $\mathcal{S}'$ , which would mean there was no electric force in  $\mathcal{S}'$  and Maxwell's equations would give a different result in the two frames (clearly against the relativity principle for Maxwell's equations).

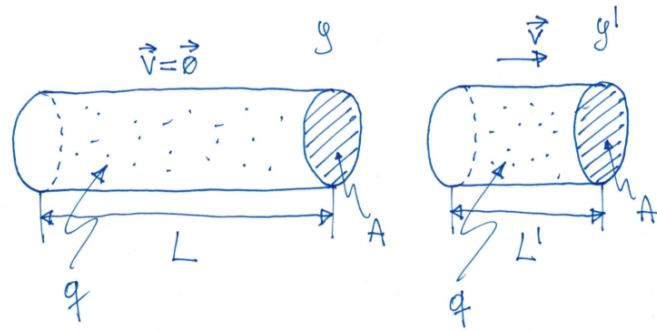


Figure 4.2

We can now use the general result of Eq. (4.13) for the positive charge density  $\rho_+$  of our wire. These charges are at rest in  $\mathcal{S}(\vec{v}_+ = \vec{0})$ . However, in  $\mathcal{S}'$  we find

$$\rho'_+ = \frac{\rho_+}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (4.14)$$

because  $\vec{v}'_+ = -\vec{v}$ .

The positive charges are at rest in  $\mathcal{S}$  and move at speed  $-\vec{v}$  in  $\mathcal{S}'$ . The negative charges, on the contrary, are at rest in  $\mathcal{S}'$  and move at speed  $\vec{v}$  in  $\mathcal{S}$ . The two type of charges are characterized by a dual behaviour. The general result of Eq. (4.13) must reflect this fact and, thus, the role of  $\rho_-$  and  $\rho'_-$  must be swapped compared to that of  $\rho_+$  and  $\rho'_+$ :

$$\rho_- = \frac{\rho'_-}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (4.15)$$

or

$$\rho'_- = \rho_- \sqrt{1 - \frac{v^2}{c^2}} \quad (4.15')$$

Note that the actual sign of  $\vec{v}$  does not matter here since the square of the speed only enters in Eq. (4.13).

In  $\mathcal{S}'$ , the net charge density  $\rho'$  is given by

$$\rho' = \rho'_+ + \rho'_- \quad (4.16)$$

Using Eqs. (4.14) and (4.15'), we get

$$\rho' = \frac{\rho_+}{\sqrt{1 - \frac{v^2}{c^2}}} + \rho_- \sqrt{1 - \frac{v^2}{c^2}} \quad (4.17)$$

Because of the neutrality of the wire in  $\mathcal{S}$  (stationary wire),  $\rho_- = -\rho_+$ , we have

$$\rho' = \rho_+ \frac{\frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (4.18)$$

The wire in  $\mathcal{S}'$  (moving wire) is positively charged with density  $\rho'$  given by (4.18) and, thus, generates an electric field  $\vec{E}'$  at the external stationary charged particle. We found the solution to this simple problem in PHYS 242:

$$\begin{aligned}\vec{E}' &= \frac{\lambda}{2\pi\epsilon_0 r} \vec{u}_r = \frac{\rho' A}{2\pi\epsilon_0 r} \vec{u}_r \\ &= \frac{\rho_+ A \frac{v^2}{c^2}}{2\pi\epsilon_0 \sqrt{1 - \frac{v^2}{c^2}}} \frac{1}{r} \vec{u}_r\end{aligned}\quad (4.19)$$

Thus, the force on  $-|q|$  in  $\mathcal{S}'$  is

$$\vec{F}' = -\frac{|q|}{2\pi\epsilon_0} \frac{\rho_+ A}{r} \frac{\frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \vec{u}_r, \quad r > 0 \quad (4.20)$$

This force is directed towards the wire, precisely as the magnetic force (4.7) in  $\mathcal{S}$ . By comparing the magnitude of the electric force  $\vec{F}'$  in  $\mathcal{S}'$  given by Eq. (4.7), we find that

$$F' = \frac{F}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (4.21)$$

For small velocities, i.e.,  $v \ll c$ , the two forces are equal. In this case, we can now state that electricity and magnetism are two different representations of the same phenomenon, the electromagnetic interaction of particles. The separation of this interaction into electric and magnetic parts depends on the reference frame chosen for the description. Note that Lorentz force Eq. (4.1) is not to be altered if the source of  $\vec{E}$  and  $\vec{B}$  is moving. The values of  $\vec{E}$  and  $\vec{B}$ , however, will be altered by the motion.

# Chapter 5

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## 5.1 On the Vector Potential of $\vec{B}$

In PHYS 242 we saw that a complete definition of a vector potential  $\vec{A}$  for a magnetostatic field  $\vec{B}$  comprises a set of two equations:

$$\vec{\nabla} \times \vec{A} = \vec{B} \quad (5.1a)$$

$$\vec{\nabla} \cdot \vec{A} = 0 \quad (5.1b)$$

To solve these equations, we must also specify how  $\vec{A}$  behaves on some boundary, or at large distances. For example, given a current density  $\vec{J}$  limited in space, we can assume a field that goes to zero at infinite.

### 5.1.1 Vector Potential for a Uniform Field $\vec{B}$

Consider a uniform magnetostatic field  $\vec{B}_0$  along the z-axis of a Cartesian coordinate system. From Eq. (5.1a),

$$\left\{ \begin{array}{l} B_x = \frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y = 0 \\ B_y = \frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z = 0 \\ B_z = \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x = B_0 \end{array} \right. \quad (5.2)$$

By inspection, we find one possible solution of these equations:

$$A_x = 0, \quad A_y = B_0 x, \quad A_z = 0 \quad (5.3a)$$

Another possible solution is:

$$A_x = -B_0 y, \quad A_y = 0, \quad A_z = 0 \quad (5.3b)$$

A linear combination of (5.3a) and (5.3b) is also a valid solution:

$$A_x = \beta(-B_0 y), \quad A_y = \alpha B_0 x, \quad A_z = 0 \quad (5.3c)$$

with  $\alpha, \beta \in \mathbb{R}$ . For a given  $\vec{B}$ ,  $\vec{A}$  is not unique.

When  $\alpha = \beta = 1/2$ , from (5.3c) it follows that the x-component of  $\vec{A}$  is proportional to  $-y$  and the y-component to  $x$ . The parametric equations of a unit circle with center on the z-axis and parallel to the xy-plane are

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}, t \in [0, 2\pi) \quad (5.4)$$

that is, a point  $P$  on the circle can be represented by the vector

$$(\cos t)\vec{u}_x + (\sin t)\vec{u}_y \quad (5.5)$$

By deriving Eqs. (5.4) with respect to  $t$ , we obtain

$$\begin{cases} \dot{x} = -\sin t \\ \dot{y} = \cos t \end{cases} \quad (5.6)$$

From which it follows the tangent vector to the circle at  $P$  is

$$(-\sin t)\vec{u}_x + (\cos t)\vec{u}_y \quad (5.7)$$

For a circle of radius  $r$ , the coordinates  $x$  and  $y$  in Eqs. (5.3c) are

$$\begin{cases} x = r \cos t \\ y = r \sin t \end{cases} \quad (5.8)$$

from which it follows that the components of  $\vec{A}$  in (5.3c) are those of a tangent vector to the points of the circle of radius  $r$ .

As a consequence,  $\vec{A}$  must be at right angles to vector  $\vec{r}$  (see Fig. 5.1).

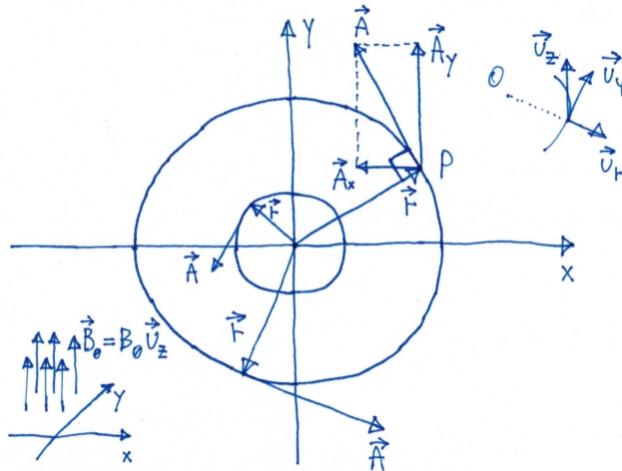


Figure 5.1

For  $\alpha = \beta = 1/2$ , the magnitude of  $\vec{A}$  is

$$||\vec{A}|| = \frac{1}{2} B_0 \sqrt{x^2 + y^2} = \frac{1}{2} B_0 r \quad (5.9)$$

Hence, for a uniform field  $\vec{B}_0$

$$\vec{A} = \frac{1}{2} \vec{B}_0 \times \vec{r} = \frac{1}{2} B_0 r \vec{u}_\varphi \quad (5.10)$$

where  $\vec{u}_\varphi$  is the tangent unit vector of a cylindrical coordinate system  $Or\varphi z$ . The vector potential  $\vec{A}$  has the magnitude  $B_0 r/2$  and rotates about the z-axis as shown in Fig. 5.1. For example, if  $\vec{B}$  is the axial field inside a long, straight solenoid,  $\vec{A}$  circulates in the same sense as do the currents of the solenoid.

In cylindrical coordinates

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial}{\partial \varphi} A_\varphi + \frac{\partial}{\partial z} A_z \quad (5.11)$$

The only component of  $\vec{A}$  in (5.10) is  $A_\varphi$ , which does not depend on  $\varphi$ . Thus, the vector potential for a uniform field  $\vec{B}_0$  fulfils condition (5.1b).

The vector potential for a uniform field can be obtained in another way. Given a surface  $\Sigma_\gamma$  with border  $\gamma$ , from Stokes' theorem

$$\begin{aligned} \Phi_\gamma = \iint_{\Sigma_\gamma} \vec{B} \cdot \vec{n} dA &= \iint_{\Sigma_\gamma} (\vec{\nabla} \times \vec{A}) \cdot \vec{n} dA \\ &= \oint_{\gamma} \vec{A} \cdot \vec{t} d\ell \end{aligned} \quad (5.12)$$

In the case of  $\vec{B}_0$ , we can attempt to use Eq. (5.12) to find  $\vec{A}$ . In general, we must assume  $\vec{A} = A_r(r, \varphi, z)\vec{u}_r + A_\varphi(r, \varphi, z)\vec{u}_\varphi + A_z(r, y, z)\vec{u}_z$ , where we represented  $\vec{A}$  in  $Or\varphi z$ .

The configuration of  $\vec{B}_0$  gives rise to three symmetries for  $A$ : Rotational, translational, and anti-rotational symmetry (see Fig. 5.2).

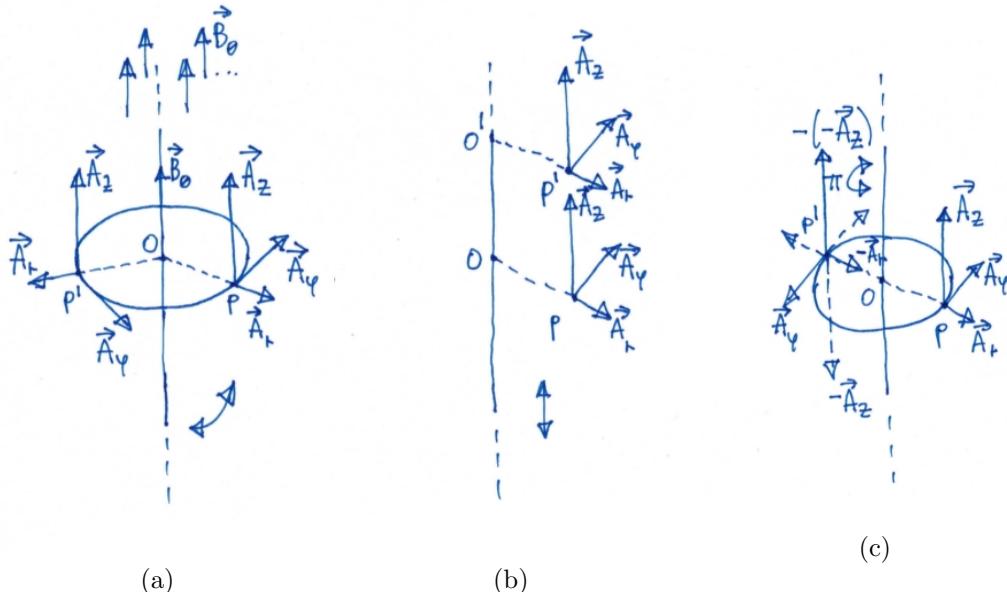


Figure 5.2

Because of the anti-rotational symmetry shown in Fig. 5.2c,  $\vec{A}_r(P)$  gets transformed into  $-\vec{A}_r(P')$ . This is because upon rotating the z-axis by  $\mp\pi$  about  $O$ ,  $\vec{B}_0$  changes sign. Thus, in order to keep the same field configuration, the sign of all vectors of interest must be flipped. Applying the rotational symmetry from  $P'$  to  $P$  in Fig. 5.2c, we find that  $\vec{A}_r = \vec{0}$ . Note that if the pivot for the rotation of the z-axis was  $0' \neq 0$ , we should have applied both the rotational and translational symmetry to go back to  $P$  in Fig. 5.2c. The symmetry argument also shows that the components  $\vec{A}_\varphi$  and  $\vec{A}_z$  can both exist and, if they do, they can only depend on  $r$ .

From Eq. (5.12), assuming  $\gamma$  to be a circle of radius  $r$  and center on the z-axis, we obtain

$$\begin{aligned}
 \Phi_\gamma &= \iint_{\Sigma_\gamma} B_0 \vec{u}_z \cdot \vec{u}_z dA = B_0 \pi r^2 \\
 &= \oint_{\gamma} (A_\varphi(r) \vec{u}_\varphi + A_z(r) \vec{u}_z) \cdot \vec{u}_\varphi d\ell \\
 &= A_\varphi(r) 2\pi r
 \end{aligned} \tag{5.13}$$

Thus,

$$\vec{A}_\varphi = \frac{1}{2} B_0 r \vec{u}_\varphi \tag{5.14}$$

confirming the result of Eq. (5.10). However, nothing can be said on  $A_z(r) \vec{u}_z$ .

We can try to impose conditions (5.1a) and (5.1b) and learn more about  $\vec{A}_z$ . From

the definition of curl in cylindrical coordinates,

$$\begin{aligned}
 \vec{\nabla} \times \vec{A} &= \left( \frac{1}{r} \frac{\partial}{\partial \varphi} A_z - \frac{\partial}{\partial z} A_\varphi \right) \vec{u}_r \\
 &+ \left( \frac{\partial}{\partial z} A_r - \frac{\partial}{\partial r} A_z \right) \vec{u}_\varphi \\
 &+ \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_\varphi) - \frac{\partial}{\partial \varphi} A_r \right] \vec{u}_z
 \end{aligned} \tag{5.15}$$

from

$$\vec{A} = \frac{1}{2} B_0 r \vec{u}_\varphi + A_z(r) \vec{u}_z \tag{5.16}$$

and from

$$\vec{\nabla} \times \vec{A} = B_0 \vec{u}_z \tag{5.17}$$

We find

$$\begin{cases} -\frac{\partial}{\partial r} A_z(r) \vec{u}_\varphi = \vec{0} \\ B_0 \vec{u}_z = B_0 \vec{u}_z \end{cases} \tag{5.18}$$

From which it follows that if a component  $A_z$  exists it must not depend on  $r$ . Hence, if  $A_z$  exists has to be a constant. Nothing new is learnt from Eq. (5.1b), which is fulfilled even if  $A_z = \text{const.}$  As always, when using symmetry arguments one has to be particularly careful when making assumptions. In the example of a uniform magnetostatic field, the symmetry arguments do not allow us to assume that  $A_z = 0$ . Thus, the approach to find  $\vec{A}$  based on symmetries and on Eq. (5.12) is less general than that based on solving Eq. (5.1a).

In the example we have just given, we have calculated  $\vec{A}$  from  $\vec{B}$ . This is the opposite of what one normally does.

In PHYS 242 and in Chapter 2, we have seen that, given a current density  $\vec{J}$  in  $\tau$ ,

$$\vec{A}(P) = \frac{\mu_0}{4\pi} \iiint_{\tau} \frac{\vec{J}(Q)}{r_{QP}} dV \tag{5.19}$$

We remind that here  $\vec{J}$  is assumed to be a stationary (or steady) current, i.e.,  $\vec{\nabla} \cdot \vec{J} = 0$ . Equation (5.19) means that we can find each component of  $\vec{A}$  by solving three imaginary electrostatic problems for the charge distributions  $\rho_1(Q) = J_x(Q)/c^2$ ,  $\rho_2(Q) = J_y(Q)/c^2$ , and  $\rho_3(Q) = J_z(Q)/c^2$ , where  $c \equiv 1/\sqrt{\mu_0 \epsilon_0}$ .

### 5.1.2 Magnetostatic Field of an Infinite, Straight Wire Revisited

We solved this problem in PHYS 242, where we used Ampère's law and symmetry arguments. As before, we take an infinite, straight wire carrying a steady and uniform current  $I$  along the bottom-top direction of the wire, assumed to be an ideal filiform wire.

In order to solve this problem by means of the vector potential, at first we must assume the wire to have a finite cross-section of radius  $R$  and a cylindrical shape.

This assumption allows us to define a current density, which would be ill-defined for an ideal filiform conductor (where  $R \rightarrow 0^+$ ). We will solve the problem for each point outside the cylinder and, at the end, impose  $R \rightarrow 0^+$ .

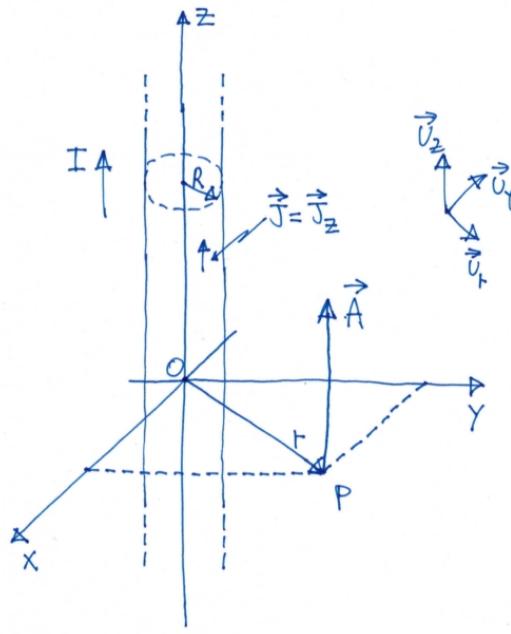


Figure 5.3

Following the notation in Fig. 5.3, and due to the uniformity assumption on  $I$ , it is easy to convince oneself that

$$\begin{cases} J_x = J_y = 0; J_z = \frac{I}{\pi R^2} & , \quad r \in [0, R] \\ J_x = J_y = J_z = 0 & , \quad r \in (R, +\infty) \end{cases} \quad (5.20)$$

with  $r = \sqrt{x^2 + y^2}$ . The current source points  $Q$  in Eq. (5.19) are for  $r \in [0, R]$ , where  $\vec{J} = (I/\pi R^2) \vec{u}_z$ . From Eqs. (5.20), it is straightforward to find the three components of the vector integral of Eq. (5.19) for each point  $P$  outside the cylindrical conductor, i.e., for  $r \in (R, +\infty)$ :

$$\begin{cases} A_x = 0 & (5.21a) \\ A_y = 0 & (5.21b) \\ A_z = \frac{\mu_0}{4\pi} \iiint_{cylinder} \frac{J_z}{r_{QP}} dV & (5.21c) \end{cases}$$

$$\begin{cases} A_x = 0 & (5.21a) \\ A_y = 0 & (5.21b) \\ A_z = \frac{\mu_0}{4\pi} \iiint_{cylinder} \frac{J_z}{r_{QP}} dV & (5.21c) \end{cases}$$

$$\begin{cases} A_x = 0 & (5.21a) \\ A_y = 0 & (5.21b) \\ A_z = \frac{\mu_0}{4\pi} \iiint_{cylinder} \frac{J_z}{r_{QP}} dV & (5.21c) \end{cases}$$

We encounter an integral similar to that of Eq. (5.21c) when calculating the electrostatic field  $\vec{E}$  and potential  $\phi$  of a wire as the one in Fig. 5.3, however, with a uniform charge distribution with volume density  $\rho$ . In that case, we found the solution by means of Gauss' theorem and symmetry arguments. In the special case  $R \rightarrow 0^+$  (i.e.,  $\rho \rightarrow \lambda$ , a linear charge density), we also found the solution by direct integration. We found (see Appendix 5.1.2)

$$\phi = -\frac{\lambda}{2\pi\epsilon_0} \ln r, \quad r \in (R, +\infty) \quad (5.22)$$

where, in general,  $\lambda = \pi R^2 \rho$ .

The component  $A_z$  can then be obtained by assuming a fictitious volume charge density  $\rho = J_z/c^2$  in (5.22). We finally find

$$A_z = -\frac{\pi R^2 J_z}{2\pi\epsilon_0 c^2} \ln r = -\frac{\mu_0 I}{2\pi} \ln r, \quad r \in (R, +\infty) \quad (5.23)$$

Note that, assuming  $R \rightarrow 0^+$  (i.e., ideal filiform condition) does not affect the last expression of Eq. (5.23), which, thus, can be used for the ideal filiform wire originally considered in this problem:

$$A_z = -\frac{\mu_0 I}{2\pi} \ln r, \quad r \in (0, +\infty) \quad (5.23')$$

From Eq. (5.1a), the definition of curl in Cartesian coordinates, and (5.23')

$$\left\{ \begin{array}{l} B_x = \frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y = -\frac{\mu_0 I}{2\pi} \frac{\partial}{\partial y} \ln \sqrt{x^2 + y^2} \\ \quad = -\frac{\mu_0 I}{2\pi} \frac{y}{r^2} \\ B_y = \frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z = \frac{\mu_0 I}{2\pi} \frac{x}{r^2} \\ B_z = 0 \end{array} \right. \quad (5.24)$$

As for the case of Eqs. (5.3c) for  $\alpha = \beta = 1/2$ , the components  $B_x$  and  $B_y$  in Eqs. (5.24) are those of a vector tangent to each point of a circle of radius  $r$ , center on the z-axis, and parallel to the xy-plane, that is

$$\vec{B} = \frac{\mu_0 I}{2\pi} \frac{1}{r} \vec{u}_\varphi, \quad r \in (0, +\infty) \quad (5.25)$$

This is exactly the same result found from Ampère's law and symmetry arguments.

### Appendix 5.1.1

The electrostatic potential at a generic point  $P$  due to a charge distribution with volume density  $\rho$  defined at each point  $Q$  in a region  $\tau$  is

$$\phi(P) = \frac{1}{4\pi\epsilon_0} \iiint_{\tau} \frac{\rho(Q)}{r_{QP}} dV \quad (5.1A)$$

The three Cartesian components of the vector integral of Eq. (5.19) can be cast into Eq. (5.1A) by noting that

$$\frac{1}{4\pi\epsilon_0} \rho(Q) = \frac{\mu_0}{4\pi} J_i(Q) \quad (5.2A)$$

with  $i = x, y, z$ . That is

$$\rho(Q) = \mu_0\epsilon_0 J_i(Q) = \frac{J_i(Q)}{c^2} \quad (5.3A)$$

Same equations have same solutions: Magnetostatic problems can be thus solved from known electrostatic ones.

### Appendix 5.1.2

Note that the electrostatic potential of Eq. (5.22) is not well-defined because the argument of the logarithm is not dimensionless. In general, we should have defined

$$\phi = -\frac{\lambda}{2\pi\epsilon_0} \ln \frac{r}{r_0} \quad (5.4A)$$

with  $r_0$  a constant with units  $m$ . This is equivalent to

$$\phi = -\frac{\lambda}{2\pi\epsilon_0} (\ln r - \ln r_0) \quad (5.4A')$$

The constant part of this expression vanishes every time we calculate a derivative to find  $\vec{B}$  from  $\vec{A}$ . Thus, we decided to neglect it in (5.22).

# Chapter 6

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In Chapter 4, we have seen that electric and magnetic forces are different representations of the same phenomenon: The electromagnetic interaction.

In the spirit of that finding, we will next consider a few electric and magnetic problems back to back and in much more depth than in PHYS 242.

## 6.1 Weak Formulation of the Electrostatic Problem

In PHYS 242, we demonstrated that the total electrostatic energy associated with a charge distribution with volume density  $\rho$  in a region  $\Omega$  of space is

$$U_e = \frac{1}{2} \iiint_{\Omega} \rho \phi dV \quad (6.1)$$

where  $\phi$  is the electrostatic potential. Assuming  $\rho$  to be limited in space, we also demonstrated that

$$U_e = \frac{1}{2} \iiint_{\Omega_{\infty}} \epsilon_0 E^2 dV \quad (6.2)$$

where  $\vec{E}$  is the field generated by  $\rho$ . In Eq. (6.2),  $\Omega_{\infty}$  represents the entire Euclidean space. The region  $\Omega$  where  $\rho$  is defined is thus a subspace of  $\Omega_{\infty}$ ,  $\Omega \subset \Omega_{\infty}$ . Hence, (6.1) can also be written as

$$U_e = \frac{1}{2} \iiint_{\Omega_{\infty}} \rho \phi dV \quad (6.1')$$

A different way to write the total electrostatic energy is then

$$U_e = \iiint_{\Omega_{\infty}} \rho \phi dV - \frac{1}{2} \epsilon_0 \iiint_{\Omega_{\infty}} E^2 dV \quad (6.3)$$

From the definition of potential,  $\vec{E} = -\vec{\nabla}\phi$ . Thus,

$$U_e = \iiint_{\Omega_{\infty}} \rho \phi dV - \frac{1}{2} \epsilon_0 \iiint_{\Omega_{\infty}} ||\vec{\nabla}\phi||^2 dV \quad (6.4)$$

When  $\phi$  is an unknown arbitrary function, expression (6.4) becomes a functional of  $\phi$ ,  $U_e(\phi)$ . As we will show in Appendix 6.1, among all possible solutions, the special function  $\bar{\phi}$  that minimizes  $[-U_e(\phi)]$  is the actual electrostatic potential associated with  $\rho$ . The electrostatic problem can thus be stated as finding the function  $\bar{\phi}$  that minimizes

$$\frac{1}{2} \epsilon_0 \iiint_{\Omega_\infty} ||\vec{\nabla} \phi||^2 dV - \iiint_{\Omega_\infty} \rho \phi dV \quad (6.4')$$

This is called the weak formulation of the electrostatic problem. Incidentally, the weak formulation is a special case of the least action principle!

More in general, given a so-called strong Poisson problem

$$\begin{cases} \vec{\nabla}^2 u = -f & \text{in } \Omega \\ u = 0 & \text{on } \Sigma \end{cases} \quad (6.5)$$

where  $\Sigma$  is the surface enclosing  $\Omega$ , the corresponding weak problem is written as

$$\begin{cases} \text{find } u \in H : J(u) = \min J(v), & v \in H \\ J(v) = \frac{1}{2} \int_{\Omega} ||\vec{\nabla} v||^2 dV - \int_{\Omega} f v dV \end{cases} \quad (6.6)$$

In these equations, the function  $u$  can be the vertical displacement of an elastic membrane acted upon by the external force  $f$  or, as in our case, the electrostatic potential  $\phi$  associated with a charge density  $\rho = \epsilon_0 f$ .

Note that the space  $H$  in (6.6) is a special Sobolev space.  $H$  is the space of functions  $v$  which are square-integrable and with square-integrable derivative. We remind that, given a domain  $\Omega \subset \mathbb{R}^n$ , a square-integrable function is defined in a space

$$L^2(\Omega) = \left\{ v : \Omega \rightarrow \mathbb{R} : \int_{\Omega} v^2 dV < +\infty \right\}$$

Therefore

$$H = H^1(\Omega) = \left\{ v \in L^2(\Omega); v' \in L^2(\Omega) \right\}$$

Consider now Poisson's equation (6.5) for  $u = 0$  and  $f = \rho/\epsilon_0$ . Multiplying both sides of (6.5) by a trial function  $\varphi$ , by neglecting the boundary condition  $\phi \neq 0$  on  $\Sigma$  and, instead, extending  $\Omega$  to  $\Omega_\infty$  (more general case), and by integrating over  $\Omega_\infty$ , we obtain

$$\iiint_{\Omega_\infty} \varphi (\vec{\nabla}^2 \phi) dV + \frac{1}{\epsilon_0} \iiint_{\Omega_\infty} \varphi \rho dV = 0 \quad (6.7)$$

From Green's first identity

$$\iiint_{\Omega_\infty} \varphi (\vec{\nabla}^2 \phi) dV + \iiint_{\Omega_\infty} (\vec{\nabla} \varphi) \cdot (\vec{\nabla} \phi) dV = \iint_{\Sigma_\infty} \varphi (\vec{\nabla} \phi) \cdot \vec{n} dA \quad (6.8)$$

where  $\Sigma_\infty$  can be assumed to be the surface of a sphere of infinite radius, enclosing entirely  $\Omega_\infty$ .

The trial function  $\varphi$ , while, in general, is not the potential associated with  $\rho$ , still maintains its nature of electrostatic potential. Thus, for a limited  $\rho$  we expect  $\varphi \sim 1/r$  for large  $r$ . From definition,  $\vec{\nabla}\phi = -\vec{E}$ , which we expect to go as  $\sim 1/r^2$  for larger. The area of  $\Sigma_\infty$  goes as  $\sim r^2$ , thus

$$\iint_{\Sigma_\infty} \varphi(\vec{\nabla}\phi) \cdot \vec{n} dA \sim \frac{1}{r^3} r^2 = \frac{1}{r} \quad (6.9)$$

for large  $r$ . For  $r \rightarrow +\infty$ , this integral tends to zero. Therefore,

$$\iiint_{\Omega_\infty} \varphi(\vec{\nabla}^2\phi) dV = - \iiint_{\Omega_\infty} (\vec{\nabla}\phi) \cdot (\vec{\nabla}\phi) dV \quad (6.10)$$

Finally, we can re-write Eq. (6.5) as

$$\epsilon_0 \iiint_{\Omega_\infty} (\vec{\nabla}\phi) \cdot (\vec{\nabla}\varphi) dV - \iiint_{\Omega_\infty} \rho\varphi dV = 0 \quad (6.11)$$

The electrostatic problem, thus, can be equivalently stated as to finding the function  $\phi$  for all possible trial functions  $\varphi$  that satisfies the condition of Eq. (6.11).

In one dimension and with respect to a Cartesian coordinate system, Eq. (6.11) can be written as

$$\int_{\lambda} \phi' \varphi' dx - \int_{\lambda} \rho \varphi dx = 0 \quad (6.11')$$

where we assumed  $\epsilon_0 = 1$  and the “primers” indicate a spatial derivatives with respect to  $x$ . Similarly, Eq. (6.4) in one dimension reads

$$U_e(\phi) = \int_{\lambda} \rho \phi dx - \frac{1}{2} \epsilon_0 \int_{\lambda} (\phi')^2 dx \quad (6.12)$$

For pedagogical reasons, we will use this equation in the proof of Appendix 6.1. By the end of the proof, we will convince ourselves that the minimization of  $[-U_e(\phi)]$  gives the actual solution to the electrostatic problem.

The fact that we need to minimize the negative of the total energy should not be a concern or surprise us. In fact, the electric energy is defined as the work due to the *external* forces. Since we want to solve the problem for the internal forces (which makes sense, as these are the electric forces we ultimately intent to find), we need to remember to change the sign of the electric energy!

It is worth mentioning that weak formulations are the starting point to implement the finite element method, allegedly, one of the most powerful numerical methods for solving partial differential equations.

## Appendix 6.1

Consider Eq. (6.12) and assume that  $\phi$  is a solution that minimizes this equation. Thus, a function  $\tilde{\phi} = \phi + \delta\varphi$ , with  $\delta \in \mathbb{R}$ , must be such that

$$U_e(\phi) \leq U_e(\phi + \delta\varphi), \quad \forall \varphi \quad (6.1A)$$

The function  $\psi(\delta) = U_e(\phi + \delta\varphi)$  is a quadratic function of  $\delta$  with minimum at  $\delta = 0$ . Thus,

$$\psi'(\delta)|_{\delta=0} = \frac{\partial}{\partial \delta} U_e(\phi + \delta\varphi)|_{\delta=0} = 0 \quad (6.2A)$$

By definition

$$\frac{\partial}{\partial \delta} U_e(\phi + \delta\varphi) = \lim_{\delta \rightarrow 0} \frac{U_e(\phi + \delta\varphi) - U_e(\phi)}{\delta}, \quad \forall \varphi \quad (6.3A)$$

Consider the term  $U_e(\phi + \delta\varphi)$ :

$$\begin{aligned} U_e(\phi + \delta\varphi) &= \int_{\lambda} \rho(\phi + \delta\varphi) dx - \frac{1}{2} \epsilon_0 \int_{\lambda} [(\phi + \delta\varphi)'] dx \\ &= \int_{\lambda} \rho\phi dx + \int_{\lambda} \rho\delta\varphi dx - \frac{1}{2} \epsilon_0 \int_{\lambda} [(\phi')^2 + 2\delta\phi'\varphi' + (\delta\varphi')^2] dx \\ &= U_e(\phi) + \int_{\lambda} \rho\delta\varphi dx - \frac{1}{2} \epsilon_0 \int_{\lambda} [\delta^2(\varphi')^2 + 2\delta\phi'\varphi'] dx \end{aligned} \quad (6.4A)$$

Thus,

$$\frac{U_e(\phi + \delta\varphi) - U_e(\phi)}{\delta} = \int_{\lambda} \rho\varphi dx - \frac{1}{2} \epsilon_0 \int_{\lambda} [\delta(\varphi')^2 + 2\phi'\varphi'] dx \quad (6.5A)$$

In the limit  $\delta \rightarrow 0$  and imposing that such limit for Eq. (6.5A) is zero, we obtain (assume  $\epsilon_0 = 1$ )

$$\int_{\lambda} \phi'\varphi' dx - \int_{\lambda} \rho\varphi dx = 0, \quad \forall \varphi \quad (6.6A)$$

Vice-versa, if  $\phi$  is a solution of (6.11'), assuming  $\tilde{\varphi} = \delta\varphi$  we have

$$\int_{\lambda} \phi'\tilde{\varphi}' dx - \int_{\lambda} \rho\tilde{\varphi} dx = 0 \quad (6.7A)$$

or

$$\int_{\lambda} \phi'\delta\varphi' dx - \int_{\lambda} \rho\delta\varphi dx = 0 \quad (6.8A)$$

As a consequence,

$$\int_{\lambda} \delta(\phi'\varphi' - \rho\varphi) dx = 0 \quad (6.8A')$$

which is true iff  $\rho\varphi = \rho'\varphi'$ . Thus,

$$\begin{aligned}
 U_e(\phi + \delta\varphi) &= \int_{\lambda} \rho(\phi + \delta\varphi) dx - \frac{1}{2} \int_{\lambda} [(\phi + \delta\varphi)']^2 dx \\
 &= \int_{\lambda} \rho\phi dx - \frac{1}{2} \int_{\lambda} (\phi')^2 dx + \int_{\lambda} \delta\rho\varphi dx - \int_{\lambda} \delta\phi'\varphi' dx - \frac{1}{2} \int_{\lambda} \delta^2(\varphi')^2 dx \\
 &= U_e(\phi) - \frac{1}{2} \int_{\lambda} \delta^2(\varphi')^2 dx
 \end{aligned}$$

It is evident that

$$\frac{1}{2} \int_{\lambda} \delta^2(\varphi')^2 dx \geq 0, \quad \forall \varphi \text{ and } \forall \delta \in \mathbb{R} \quad (6.9A)$$

This would mean that

$$U_e(\phi) \geq U_e(\phi + \delta\varphi) \quad (6.10A)$$

This result is the opposite of what stated in (6.1A). The apparent problem can be resolved by inverting the signs in Eq. (6.12) and, thus, (6.4).



# Chapter 7

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In Chapter 6, we formulated the electrostatic problem as the minimization of a functional representing the total energy of a system of charges. The functional was defined as

$$U_e^* = \frac{1}{2} \epsilon_0 \iiint_{\Omega} (\vec{\nabla} \phi)^2 dV - \iiint_{\Omega} \rho \phi dV \quad (7.1)$$

In PHYS 242, given two conductors charged with opposite charges  $+q$  and  $-q$  and with potentials  $V_1$  and  $V_2$ , respectively, we defined the capacitance of this system of conductors as

$$C \equiv \frac{q}{\Delta\phi} \quad (7.2)$$

where  $\Delta\phi = V_1 - V_2$ .

We will now attempt to use the weak formulation (7.1) to find an approximate solution for the capacitance of a system of conductors.

## 7.1 Electrostatic Energy of a System of Charged Conductors

Given a charge distribution with surface density  $\sigma$ , the electrostatic energy associated with the distribution is

$$U_e = \frac{1}{2} \iint_{\Sigma} \sigma \phi dA \quad (7.3)$$

where  $\sigma$  is defined on the surface  $\Sigma$  and  $\phi$  is the potential of the distribution.

We remind that given a charged perfect conductor in electrostatic equilibrium, the charge at each point inside the conductor is zero, i.e.,

$$\rho = 0 \quad \forall p \in \Omega \quad (7.4)$$

where  $\Omega$  is the region delimited by  $\Sigma$ . The charge must be distributed on the conductor surface only.

Consider now a system of  $N$  conductors  $\Omega_i$ , with  $i = 1, \dots, N$ . Each of the conductors is characterized by a surface  $\Sigma_i$  with charge surface density  $\sigma_i$ . By means of Eq. (7.3), we find the electrostatic energy of the system of conductors:

$$U_e^N = \frac{1}{2} \sum_{i=1}^N \iint_{\Sigma_i} \sigma_i \phi_i dA \quad (7.5)$$

Note that  $\phi_i$  is constant on the surface of each conductor and, thus, it can be taken outside the sign of integral. Moreover, the charge on the  $i$ -th conductor can be written as

$$q_i = \iint_{\Sigma_i} \sigma_i dA \quad (7.6)$$

Hence,

$$U_e^N = \frac{1}{2} \sum_{i=1}^N q_i \phi_i \quad (7.7)$$

In the special case of a capacitor,  $N = 2$ ,  $q_1 = +q$  and  $q_2 = -q$ , and  $\phi_1 = V_1$  and  $\phi_2 = V_2$ .

Thus,

$$\begin{aligned} U_e^C &= \frac{1}{2} qV_1 - \frac{1}{2} qV_2 = \frac{1}{2} q\Delta\phi \\ &= \frac{1}{2} C(\Delta\phi)^2 = \frac{1}{2} \frac{q^2}{C} \end{aligned} \quad (7.8)$$

where we have used the definition (7.2).

## 7.2 The Principle of Least Action for the Capacitance

In the case of perfect conductors, Eq. (7.1) reduces to

$$U_e^*(\phi) = \frac{1}{2} \epsilon_0 \iiint_{\Omega} (\vec{\nabla} \phi)^2 dV \quad (7.9)$$

because of condition (7.4). The electrostatic problem for perfect conductors can be thus stated as to finding the special potential  $\phi$  that minimizes the electrostatic energy of Eq. (7.9). Note that we could now remove the “\*” from  $U_e^*(\phi)$  since, in this case, the functional to be minimized coincides with the electrostatic energy and not with the opposite of the total energy.

Consider an indefinitely long cylindrical capacitor, the cross-section of which is shown in Fig. 7.1.

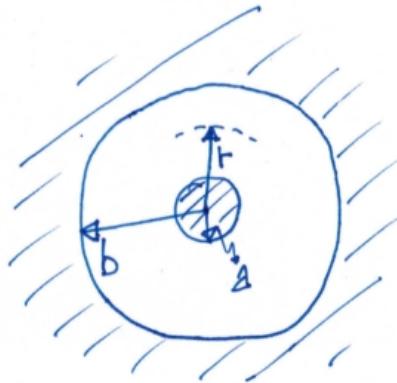


Figure 7.1

The inner conductor has radius  $a$  and potential  $V$  and the outer conductor radius  $b$  and potential zero.

By using the solution  $\bar{\phi}$  that minimizes the functional (7.9), we would be able to calculate the capacitance  $C$  of the capacitor from

$$\frac{1}{2} CV^2 = \frac{1}{2} \epsilon_0 \iiint_{\Omega} (\vec{\nabla} \bar{\phi})^2 dV \quad (7.10)$$

Note that here  $dV$  is an infinitesimal volume in  $\Omega$  and not an infinitesimal potential.

If instead of  $\bar{\phi}$  we were to use a wrong solution, we would find a capacitance that is larger than the correct value. In fact, in Eq. (7.10)  $V$  is fixed and must correspond to the potential that minimizes the electrostatic energy.

Note, however, that even if the potential  $\phi$  used in Eq. (7.10) is a rough approximation of  $\bar{\phi}$ , the corresponding  $C$  will be a good approximation because the error in  $C$  is second order in the error in  $\phi$ . Thus is one of the strengths of the weak formulation.

In PHYS 242, we calculated the capacitance of a cylindrical capacitor,

$$C = \frac{2\pi\epsilon_0}{\ln \frac{b}{a}} \quad (7.11)$$

Assume we did not know this result and assume we did not even know the correct distribution of  $\phi$  and  $\vec{E}$  inside the capacitor. We can try to use (7.10) to guess for  $C$ .

Suppose the potential goes linearly with distance  $r$  inside the capacitor. This means the field is constant. This assumption is consistent with the symmetries associated with charged indefinite cylindrical structures.

In order to fulfil the boundary conditions at the inner and outer conductor, it must be

$$\phi = V \left( 1 - \frac{r-a}{b-a} \right), \quad r \in [a, b] \quad (7.12)$$

This function is  $V$  at  $r = a$ , zero at  $r = b$ , and in between has a constant slope equal to  $-V/(b-a)$ . To use Eq.(7.10) to calculate  $C$ , we must square this gradient and integrate over all volume. Assuming a length  $l = 1$  m of the cylinders, a volume element at radius  $r$  is  $2\pi r dr$ . We obtain,

$$\frac{1}{2} C_1 V^2 \lesssim \frac{1}{2} \epsilon_0 \int_a^b \frac{v^2}{(b-a)^2} 2\pi r dr \quad (7.13)$$

which gives

$$C_1 \lesssim \pi \epsilon_0 \frac{b+a}{b-a} \quad (7.14)$$

Following Feynman 19 – 12, Table 7.1 shows a comparison between the correct capacitance  $C$  calculated from (7.11) and the approximate capacitance  $C_1$  calculated from (7.14) for various values of the ratio  $b/a$ .

$\frac{b}{a}$	$\frac{C}{2\pi\epsilon_0}$	$\frac{C_1}{2\pi\epsilon_0}$
2	1.4423	1.500
4	0.721	0.833
10	0.434	0.612
100	0.267	0.51
1.5	2.4662	2.50
1.1	10.492070	10.500000

Table 7.1

For a relatively big  $b/a = 2$ , which gives a pretty big variation in the field compared with a linearly varying field, we get a pretty good approximation. As expected, the value of  $C_1$  is slightly larger than  $C$ . This is obviously consistent with the least action principle. In the case of a tiny wire inside a big cylinder, as for  $b/a = 100$  for example, the approximation is much tougher and we are off by a factor of nearly 2. This is because the actual field has huge variations inside the two walls and a constant is a very coarse approximation. On the contrary, for conductors close to each other, e.g., for  $b/a = 1.1$ , we obtain a  $C_1$  which is within 0.1% from  $C$ .

We can now try to refine our approximation by guessing a more complicated potential. For example, assume the potential is quadratic in  $r$  (i.e., the field is linear in  $r$ ). The most general quadratic form that fits  $\phi = 0$  at  $r = b$  and  $\phi = V$  at  $r = a$  is

$$\phi = V \left[ 1 + \alpha \left( \frac{r-a}{b-a} \right) - (1+\alpha) \left( \frac{r-a}{b-a} \right)^2 \right] \quad (7.15)$$

where  $r \in [a, b]$  and  $\alpha$  is an arbitrary constant number.

In this case, the magnitude of the field is simply,

$$E = -\frac{d}{dr} \phi = -\frac{\alpha V}{b-a} + 2(1+\alpha) \frac{(r-a)V}{(b-a)^2} \quad (7.16)$$

We can now calculate the capacitance for an arbitrary  $\alpha$  by using this  $E$  in Eq. (7.10). We obtain (see Feynman 19 – 13)

$$C_2 = 2\pi\epsilon_0 \frac{a}{b-a} \left[ \frac{b}{a} \left( \frac{\alpha^2}{6} + \frac{2\alpha}{3} + 1 \right) + \frac{1}{6} \alpha^2 + \frac{1}{3} \right] \quad (7.17)$$

From the least action principle we know that among all possible capacitances for a given configuration (i.e., for given parameters  $a$  and  $b$ ), the best approximation is the smallest capacitance. In general, we can compute the minimum of Eq. (7.17) by deriving with respect to  $\alpha$  and equating to zero (the function (7.17) is definite positive; the minimum from the derivative is a global minimum). We get

$$\frac{b}{b-a} \frac{2\alpha}{6} + \frac{b}{b-a} \frac{2}{3} + \frac{a}{b-a} \frac{2}{6} \alpha = 0 \quad (7.18)$$

from which  $\alpha = -2b/(b+a)$ . By substituting this value into Eq. (7.17), we obtain the minimum capacitance

$$C_2 \lesssim 2\pi\epsilon_0 \frac{b^2 + 4ab + a^2}{3(b^2 - a^2)} \quad (7.19)$$

Table 7.2 shows a comparison between  $C$  from (7.11) and  $C_2$  from (7.19) for a few values of  $b/a$ .

$\frac{b}{a}$	$\frac{C}{2\pi\epsilon_0}$	$\frac{C_2}{2\pi\epsilon_0}$
2	1.4423	1.444
4	0.721	0.733
10	0.434	0.475
100	0.267	0.346
1.5	2.4662	2.4667
1.1	10.4921	10.4921

Table 7.2

The new approximation  $C_2$  is much better than  $C_1$  for all values of  $b/a$ .

In this example, we knew the correct result for the capacitance. We thus used the least action principle just to confirm it is right. However, for capacitors of very complicated shape it could be extremely hard to find a correct analytical solution from the strong Poisson problem. The least action principle, instead, gives a relatively easy way to guess a reasonable approximation for the capacitance.



# Chapter 8

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## 8.1 Neumann Integrals

Consider a quasi-filiform (closed) circuit  $\tau$  with cross-section  $dA$  and carrying a steady current  $I$ , as shown in Fig. 8.1. The current is generated by an emf, which is not shown in the figure.

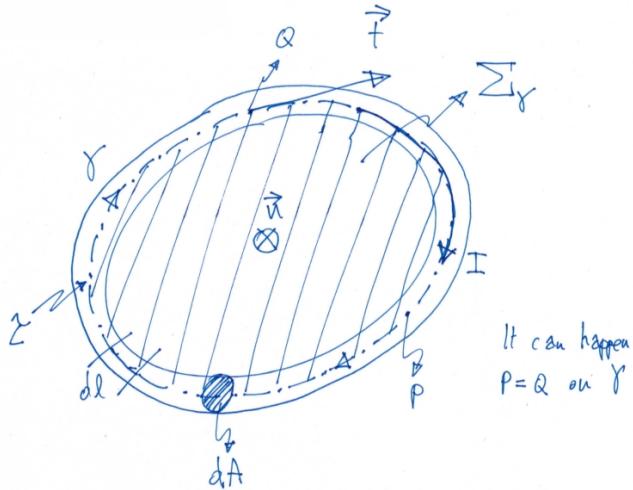


Figure 8.1

The surface  $\Sigma_\gamma$  is any open surface having  $\gamma$  as a border. The normal unit vector  $\vec{n}$  to  $\Sigma_\gamma$  and the orientation of  $\gamma$ , which is used to evaluate the sign of  $I$ , are chosen consistently with the right-hand rule. The line  $\gamma$  is one of the infinite possible longitudinal axes associated with  $\tau$ . The quasi-filiform nature of the latter, in fact, makes it possible to define longitudinal axes that differ by an infinitesimal distance from each other. The axis  $\gamma$  could be the central longitudinal axis of  $\tau$ , i.e., the axis passing through the center of  $dA$  at each cross-section of  $\tau$ .

In PHYS 242, we defined a constant called the inductance coefficient of  $\tau$  with respect to  $\gamma$  as

$$L \equiv \frac{\Phi_\gamma}{I} \quad (8.1)$$

For a single circuit, as  $\tau$ , this is also called the self-inductance coefficient of the circuit.

In the definition (8.1), the flux  $\Phi_\gamma$  is calculated with respect to the special line  $\gamma$  chosen as longitudinal axis of  $\tau$ . Thus, a different inductance  $L'$  will result from a different choice of axis, say  $\gamma'$ . The definition (8.1) leads to an infinite set of inductances, one for each possible axis.

Under quasi-filiform conditions, however, the length of  $\tau$  is much bigger than its diameter. Hence, the fluxes  $\Phi_\gamma$  and  $\Phi_{\gamma'}$  linked with  $\gamma$  and  $\gamma'$  (and those for any other longitudinal axis) and generated by the same current  $I$  are almost identical. As a consequence, the infinite inductance associated with all possible axes of  $\tau$  are approximately the same.

It is worth noting that if we were to assume a rigorously filiform conductor, i.e., a conductor with zero diameter, it would lead to a meaningless definition of  $L$ . In this case, in fact, even in presence of a finite current, the field  $\vec{B}$  would diverge to infinite when approaching the conductor. This can be formally seen from Eq. (3.18), where  $r_{QP} \rightarrow 0$  when considering field points  $P$  closer and closer to source points  $Q$  on  $\tau$ . Since  $\tau$  is considered to be filiform,  $Q$  and  $P$  eventually tend to coincide. Therefore, the flux of  $\vec{B}$  linked with the circuit would also diverge so would  $L$ .

Consider now two quasi-filiform circuits  $\tau_1$  and  $\tau_2$ , as shown in Fig. 8.2.

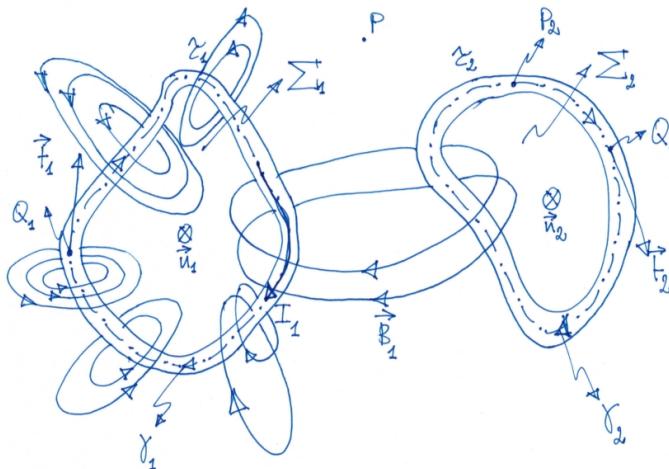


Figure 8.2

The two circuits, which are close to each other, but do not touch, form a circuit system. Assume a steady current  $I_1$  (the sign of which is evaluated with respect to an arbitrary reference direction) flows in  $\tau_1$  and no current flows in  $\tau_2$ . We choose arbitrarily a positive direction on  $\gamma_2$ , which is a longitudinal axis of  $\tau_2$ . The generic surface  $\Sigma_2$  that has  $\gamma_2$  as a border is oriented such that its normal unit vector  $\vec{n}_2$  follows the right-hand rule. The flux due to the magnetic field  $\vec{B}_1$  generated by  $I_1$  and linked with  $\gamma_2$  can be written as

$$\Phi_{21} = M_{21} I_1 \quad (8.2)$$

where  $M_{21}$  is called the mutual-inductance coefficient of  $\tau_1$  on  $\tau_2$ .

Similarly, when a current  $I_2$  flows in  $\tau_2$  and no current in  $\tau_1$ , we find

$$\Phi_{12} = M_{12} I_2 \quad (8.2')$$

where  $\Phi_{12}$  is the flux due to  $I_2$  and linked with  $\tau_1$  and  $M_{12}$  is the mutual inductance of  $\tau_2$  on  $\tau_1$ .

For a formal calculation of  $M_{21}$  and  $M_{12}$  we can proceed as follows. The vector potential due to  $I_1$ ,  $\vec{A}_1$ , can be calculated at any point in space  $P$  from the integral

$$\vec{A}_1(P) = \frac{\mu_0}{4\pi} \iiint_{\tau_1} \frac{\vec{J}_1}{r} dV \quad (8.3)$$

where  $\vec{J}_1$  is the electric current volume density associated with  $I_1$  and evaluated at each source point  $Q_1$  on  $\gamma_1$  (we remind that  $\tau_1$  is quasi filiform; we can thus assume a point  $Q_1$  inside  $\tau_1$  to be a point on  $\gamma_1$  up to an infinitesimal distance). In (8.3),  $r = \|\vec{r}_{Q_1 P}\|$  is the Euclidean norm of the distance between the generic field point  $P$  and  $Q_1$ . Under stationary conditions, the quasi-filiform conductor is a flux tube for  $\vec{J}_1$ .

Therefore,  $I_1 = J_1 dA$  and (8.3) can be rewritten as

$$\vec{A}_1(P) = \frac{\mu_0 I_1}{4\pi} \oint_{\gamma_1} \frac{\vec{t}_1}{r} d\ell_1 \quad (8.4)$$

where  $\vec{t}_1$  is the tangent unit vector at each point  $Q_1$  on  $\gamma_1$  (see Fig. 8.2) and also the direction of  $J_1$ ,  $\vec{J}_1 = J_1 \vec{t}_1$ , and  $d\ell_1$  is an infinitesimal element on  $\gamma_1$ .

From (5.12), we know that

$$\Phi_{21} = \oint_{\gamma_2} \vec{A}_1 \cdot \vec{t}_2 d\ell_2 \quad (8.5)$$

where  $\vec{t}_2$  is indicated in Fig. 8.2 and  $d\ell_2$  is an infinitesimal element on  $\gamma_2$ . By combining (8.4) and (8.5), we find

$$\begin{aligned} M_{21} &= \frac{\Phi_{21}}{I_1} = \frac{\mu_0}{4\pi} \oint_{\gamma_2} \left( \oint_{\gamma_1} \frac{\vec{t}_1}{r} d\ell_1 \right) \cdot \vec{t}_2 d\ell_2 \\ &= \frac{\mu_0}{4\pi} \oint_{\gamma_2} \oint_{\gamma_1} \frac{\vec{t}_1 \cdot \vec{t}_2}{r} d\ell_1 d\ell_2 \end{aligned} \quad (8.6)$$

The quantity  $r$  in the integrand of (8.6) is always different from zero. In fact, as clear from Fig. 8.2, any source point  $Q_1$  on  $\gamma_1$  is always different from any field point  $P_2$  on  $\gamma_2$ . Since  $\gamma_1$  and  $\gamma_2$  do not share any point, the integrand in (8.6) is well defined. Thus, the integral, which is  $M_{21}$ , is also well defined.

Following a similar path, it is possible to calculate  $M_{12}$ :

$$M_{12} = \frac{\mu_0}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\vec{t}_2 \cdot \vec{t}_1}{r} d\ell_2 d\ell_1 \quad (8.6')$$

From the linearity of integration and for the commutative property of the scalar product, it follows that  $M_{21} = M_{12}$ .

The expressions (8.6) and (8.6') are called Neumann integrals. These integrals are only valid under the assumption that the flux is proportional to the current that generates it. In the case of ferromagnetic materials, for example, the Neumann-type integral cannot be used.

Can we use a Neumann-type integral for the calculation of the self inductance of a single, closed circuit with steady current  $I$ ?

Consider again the circuit  $\tau$  of Fig. 8.1. In order to calculate  $L$  by means of Eq. (8.6), we could assume  $\gamma_1 = \gamma_2 = \gamma$ ,  $\vec{t}_1 = \vec{t}_2 = \vec{t}$ , and  $d\ell_1 = d\ell_2 = d\ell$ . We would thus find

$$\begin{aligned} L &= \frac{\mu_0}{4\pi} \oint_{\gamma} \oint_{\gamma} \frac{\vec{t} \cdot \vec{t}}{r} d\ell d\ell \\ &= \frac{\mu_0}{4\pi} \oint_{\gamma} \oint_{\gamma} \frac{1}{r} d\ell d\ell \end{aligned} \quad (8.7)$$

For each source point  $Q$  on  $\gamma$ , the integrand of (8.7) will diverge every time a field point  $P$ , also  $\gamma$ , coincides with  $Q$ . The inductance given by (8.7) is not well defined.

We must find a trick to circumvent this problem. We will so do by means of an example. We will try to calculate the inductance of a circular ring.

# Chapter 9

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## 9.1 Self Inductance of a Quasi-Filiform Circular Ring (Maxwell's Inductance)

Circular rings are simple structures commonly encountered in applications and, yet, the calculation of their self inductance is not an easy problem.

Consider a quasi-filiform circular conductor (i.e., a ring) with radius  $a$  and circular cross-section  $\pi r^2$ , where  $r \ll a$  (quasi filiform condition). The ring carries a steady current  $I$ .

We want to use a Neumann-type integral to calculate the self-inductance coefficient  $L$  of the ring. The simple integral (8.7), however, would diverge. Instead, we will try to emulate a single ring as a pair of coaxial rings with radii  $a$  and  $A$ , respectively (see Fig. 9.1), and calculate the mutual inductance  $M$  between them. In the limit  $a = A$ ,  $M$  should become the self inductance  $L$  of a single ring.

As it turns out, it is easier to solve this problem by assuming a vertical displacement  $\delta$  between the centers of the two coaxial rings (also shown in Fig. 9.1). It is clear that the two planes containing the rings are assumed to be parallel to each other. The quantity  $\delta$  is eventually used to approximate the fact that the wire has a finite cross section with area  $\pi r^2$ . In first approximation, the distance between any pairs of points inside this cross section is on average  $r$ ; a better approximation is the harmonic mean  $e^{-1/4}r$ ; we will use this approximation (introduced in this problem by Maxwell) at the end of this chapter.

For the calculation of  $M$ , we will assume the two rings to be rigorously filiform conductors. The fact that the original single ring has cross-section with radius  $r$  will reappear at the very end of the calculation, where we will find a mathematical connection between  $r$  and  $\delta$ .

In summary, we will calculate the self inductance as

$$L = \lim_{A \rightarrow a} M(A, a; \text{small } \delta) \quad (9.1)$$

and then express  $\delta$  as a function of  $r$  by simple geometric arguments. The limit (9.1) is key to avoid a diverging Neumann integral.

## 9.1. SELF INDUCTANCE OF A QUASI-FILIFORM CIRCULAR RING (MAXWELL'S INDUCTANCE)

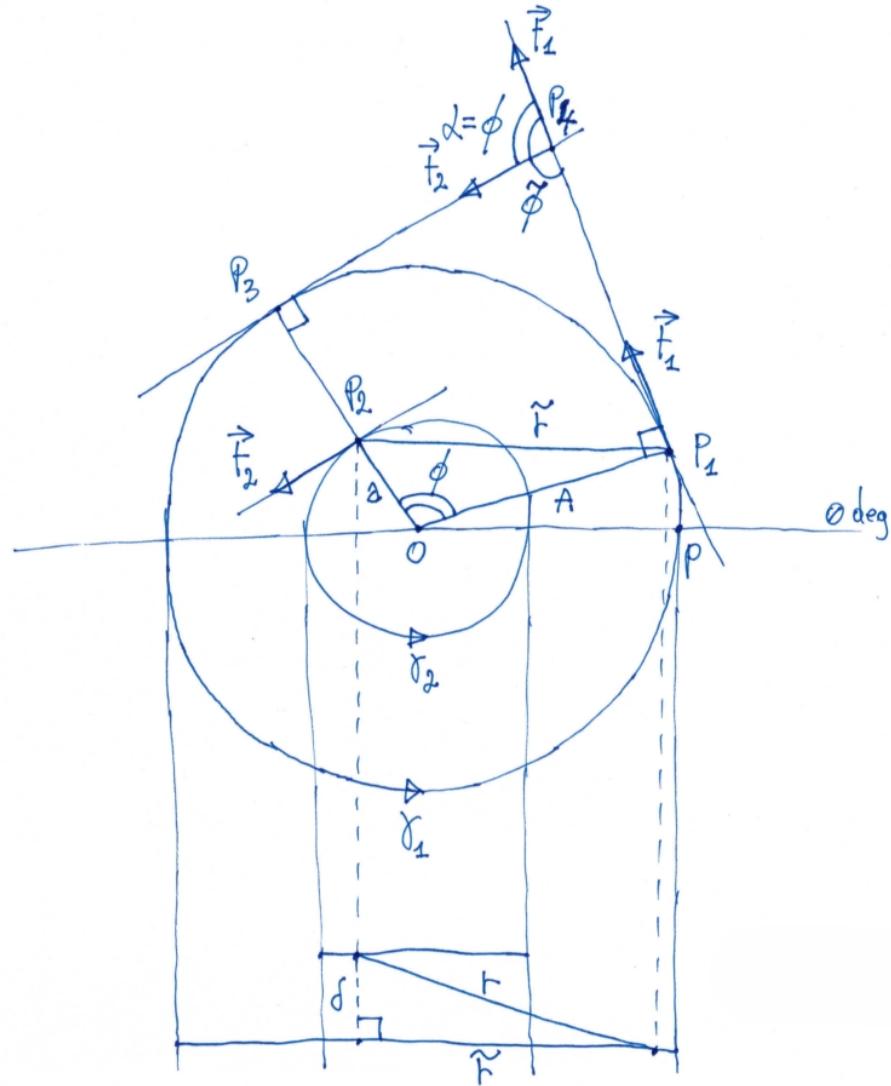


Figure 9.1

Referring to the notation in Fig. 9.1, we can readily obtain each element in the Neumann integral (8.6) for the calculation of the mutual inductance  $M_{21}$  between the filiform rings  $\gamma_1$  and  $\gamma_2$ . We find:

$$r = \sqrt{\tilde{r}^2 + \delta^2} \quad (9.2a)$$

$$\tilde{r} = \overline{P_1 P_2} = \sqrt{A^2 + a^2 - 2Aa \cos \phi} \quad (9.2b)$$

$$\phi = \widehat{POP}_2 - \widehat{POP}_1 \equiv \varphi_2 - \varphi_1 \quad (9.2c)$$

for which

$$r = \sqrt{A^2 + a^2 + \delta^2 - 2Aa \cos(\varphi_2 - \varphi_1)} \quad (9.2d)$$

$$d\ell_1 = Ad\varphi_1, \quad \varphi_1 \in [0, 2\pi] \quad (9.2e)$$

$$d\ell_2 = Ad\varphi_2, \quad \varphi_2 \in [0, 2\pi] \quad (9.2f)$$

From Fig. 9.1, we also find that

$$\vec{t}_1 \cdot \vec{t}_2 = \cos \alpha \quad (9.2g)$$

where  $\alpha$  can easily be found considering the quadrilateral  $OP_1P_4P_3$ . In fact, the interior angles of a simple and planar quadrilateral add up to  $2\pi$ . Thus,

$$\phi + \widehat{OP_1P_4} + \tilde{\phi} + \widehat{P_4P_3O} = \phi + \frac{\pi}{2} + \tilde{\phi} + \frac{\pi}{2} = 2\pi \quad (9.3)$$

for which

$$\tilde{\phi} = \pi - \phi \quad (9.4)$$

From Fig. 9.1, it also must be

$$\tilde{\phi} + \alpha = \pi \quad (9.5)$$

and, thus, using (9.4)

$$\alpha = \pi - \tilde{\phi} = \phi \quad (9.6)$$

We can finally write (9.2g) as

$$\vec{t}_1 \cdot \vec{t}_2 = \cos \phi = \cos(\varphi_2 - \varphi_1) \quad (9.2h)$$

By using (9.2d), (9.2e), (9.2f), and (9.2h), we can write the integral (8.6) as

$$M_{21} = \frac{\mu_0}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{Aa \cos(\varphi_2 - \varphi_1) d\varphi_1 d\varphi_2}{\sqrt{A^2 + a^2 + \delta^2 - 2Aa \cos(\varphi_2 - \varphi_1)}} \quad (9.7)$$

This integral can be exactly solved in the form

$$M_{21} = -\mu_0 \sqrt{Aa} \left[ \left( k - \frac{2}{k} \right) F(k) + \frac{2}{k} E(k) \right] \quad (9.8)$$

where

$$k = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2 + \delta^2}} \quad (9.9)$$

is the modulus of the complete elliptic integrals of the first and second kind

$$\left\{ \begin{array}{l} F(k) = \int_0^{\pi/2} \frac{1}{\Delta} d\varphi \\ E(k) = \int_0^{\pi/2} \Delta d\varphi \end{array} \right. \quad (9.10a)$$

$$\left\{ \begin{array}{l} F(k) = \int_0^{\pi/2} \frac{1}{\Delta} d\varphi \\ E(k) = \int_0^{\pi/2} \Delta d\varphi \end{array} \right. \quad (9.10b)$$

respectively, with  $\Delta = \sqrt{1 - k^2 \sin^2 \varphi}$ .

It can be shown that, in general,  $F(k)$  and  $E(k)$  can be expanded in series:

$$\left\{ \begin{array}{l} F(k) = \frac{\pi}{2} \left[ 1 + \left( \frac{1}{2} \right)^2 k^2 + \left( \frac{1 * 3}{2 * 4} \right)^2 k^4 + \left( \frac{1 * 3 * 5}{2 * 4 * 6} \right)^2 k^6 + \dots \right] \end{array} \right. \quad (9.11a)$$

$$\left\{ \begin{array}{l} E(k) = \frac{\pi}{2} \left[ 1 - \left( \frac{1}{2} \right)^2 k^2 - \left( \frac{1 * 3}{2 * 4} \right)^2 \frac{k^4}{3} - \left( \frac{1 * 3 * 5}{2 * 4 * 6} \right)^2 \frac{k^6}{5} \dots \right] \end{array} \right. \quad (9.11b)$$

Hence, when  $k$  is small  $F(k)$  and  $E(k)$  can easily be evaluated from (9.11a) and (9.11b).

When  $A \rightarrow a$  and  $\delta$  is small, which are the conditions used in the limit (9.1) for the calculation of  $L$ ,  $k \rightarrow 1^-$  as it can readily be shown from (9.9). Indeed, the maximum value of  $k$  is  $k = 1$ . When  $k \sim 1$ , the series (9.11b) converges quite rapidly due to the minus sign in front of all terms with  $k$ . On the contrary, the series (9.11a) converges very slowly because all its terms are positive and add all up. As it turns out, this series converges \*very\* slowly and a huge number of terms must be considered to obtain a reasonable approximation for  $F(k)$ .

When  $k$  is nearly equal to unity, a different type of series expansion must be used for  $F$  and  $E$ :

$$\left\{ \begin{array}{l} F = \ln \frac{4}{k_1} + \left( \frac{1}{2} \right)^2 k_1^2 \left( \ln \frac{4}{k_1} - \frac{2}{1 * 2} \right) \\ \quad + \left( \frac{1 * 3}{2 * 4} \right)^2 k_1^4 \left( \ln \frac{4}{k_1} - \frac{2}{1 * 2} - \frac{2}{3 * 4} \right) + \dots \end{array} \right. \quad (9.12a)$$

$$\left\{ \begin{array}{l} E = 1 + \frac{1}{2} k_1^2 \left( \ln \frac{4}{k_1} - \frac{1}{1 * 2} \right) \\ \quad + \frac{1^2 * 3}{2^2 * 4} k_1^4 \left( \ln \frac{4}{k_1} - \frac{2}{1 * 2} - \frac{1}{3 * 4} \right) + \dots \end{array} \right. \quad (9.12b)$$

where

$$k_1^2 = 1 - k^2 \quad (9.13)$$

which is a small quantity for  $k \sim 1$ . The expansion (9.12a) and (9.12b) are due to Legendre and play a pivotal role in our calculation of  $L$  for the ring.

By approximating

$$\left\{ \begin{array}{l} F \sim \ln \frac{4}{k_1} + \frac{1}{4} k_1^2 \ln \frac{4}{k_1} - \frac{1}{4} k_1^2 \end{array} \right. \quad (9.14a)$$

$$\left\{ \begin{array}{l} E \sim 1 + \frac{1}{2} k_1^2 \ln \frac{4}{k_1} - \frac{1}{4} k_1^2 \end{array} \right. \quad (9.14b)$$

that is, by considering only the first line in (9.12a) and (9.12b) and with simple algebra, we can give an approximate solution for  $M_{21}$ . By inserting (9.14a) and (9.14b) into (9.8),

we obtain

$$\begin{aligned}
 M_{21} &= -\mu_0 \sqrt{Aa} \left( kF - \frac{2}{k} F + \frac{2}{k} E \right) \\
 &\sim -\mu_0 \sqrt{Aa} \left( k \ln \frac{4}{k_1} + k \frac{1}{4} k_1^2 \ln \frac{4}{k_1} - k \frac{1}{4} k_1^2 \right. \\
 &\quad \left. - \frac{2}{k} \ln \frac{4}{k_1} - \frac{2}{k} \frac{1}{4} k_1^2 \ln \frac{4}{k_1} + \frac{2}{k} \frac{1}{4} k_1^2 \right. \\
 &\quad \left. + \frac{2}{k} + \frac{2}{k} \frac{1}{2} k_1^2 \ln \frac{4}{k_1} - \frac{2}{k} \frac{1}{4} k_1^2 \right) \\
 &= -\mu_0 \sqrt{Aa} \left[ -k \frac{1}{4} k_1^2 + \frac{2}{k} \right. \\
 &\quad \left. + \ln \frac{4}{k_1} \left( k + k \frac{1}{4} k_1^2 - \frac{2}{k} + \frac{1}{2k} k_1^2 \right) \right] \\
 &= \mu_0 \sqrt{Aa} \left[ \frac{kk_1^2}{4} - \frac{2}{k} \right. \\
 &\quad \left. - \left( k + \frac{kk_1^2}{4} - \frac{2}{k} + \frac{k_1^2}{2k} \right) \ln \frac{4}{k_1} \right] \tag{9.15}
 \end{aligned}$$

Assume  $A = a$ , from (9.9)

$$k = \frac{2a}{\sqrt{4a^2 + \delta^2}} \tag{9.16}$$

and so

$$k^2 = \frac{4a^2}{4a^2 \left( 1 + \frac{\delta^2}{4a^2} \right)} = \frac{1}{1 + \left( \frac{\delta}{2a} \right)^2} = \frac{1}{1 + \xi} \tag{9.17}$$

where

$$\xi \equiv \left( \frac{\delta}{2a} \right)^2 \tag{9.18}$$

For small values of  $\delta$  (as in the limit (9.1)), a simple Maclaurin series allows us to approximate  $k^2$  as

$$k^2 \sim 1 - \xi \tag{9.19}$$

From (9.13) it then follows that

$$k_1^2 \sim 1 - \xi + \xi = \xi \tag{9.20}$$

Finally,

$$k = \frac{2a}{\sqrt{4a^2 \left( 1 + \frac{\delta^2}{4a^2} \right)}} = \frac{1}{\sqrt{1 + \xi}} \sim 1 - \frac{1}{2} \xi \tag{9.21}$$

By inserting (9.21) and (9.20) into (9.15), we obtain

$$M_{21} \sim L \sim \mu_0 a \left[ \frac{1}{4} \left( 1 - \frac{1}{2} \xi \right) \xi - \frac{2}{1 - \xi/2} \right. \\ \left. - \left( \left( 1 - \frac{1}{2} \xi \right) + \frac{1}{4} \left( 1 - \frac{1}{2} \xi \right) \xi \right. \right. \\ \left. \left. - \frac{2}{1 - \xi/2} + \frac{\xi}{2(1 - \xi/2)} \right) \ln \frac{4}{\sqrt{\xi}} \right] \quad (9.22)$$

In this equation,  $\xi \sim \delta^2$  and  $\sqrt{\xi} \sim \delta$ . As a consequence, in the limit  $\delta \rightarrow 0^+$ ,  $\xi$  is an infinitesimal of higher order compared to  $\sqrt{\xi}$ . In other words, in this limit  $\xi$  goes to zero faster than  $\sqrt{\xi}$ . Thus, the result of Eq. (9.22) for small  $\delta$ ,

$$L \sim \mu_0 a \left[ -2 - (1 - 2) \ln \frac{4}{\sqrt{\xi}} \right] \\ = \mu_0 a \left( \ln \frac{4}{\delta/2a} - 2 \right) \\ = \mu_0 a \left( \ln \frac{8a}{\delta} - 2 \right) \quad (9.23)$$

The result of Eq. (9.23) is an approximate solution for the self inductance  $L$  of the ring. Note that this result must be used with the grain of salt. In fact, for example, for  $\delta = 0$ , Eq. (9.23) goes to  $+\infty$ . However, the approximation is very good so long  $\delta \neq 0$ , even for very small values of  $\delta$  ( $\ln x$  increases very slowly for increasing values of  $x$ ).

The last bit we need to work out is the connection between  $\delta$  and  $r$ . This connection is due to Maxwell. Maxwell demonstrated that the most accurate approximation for  $L$  is obtained when assuming  $\delta$  to be the geometrical mean distance of every pair of points in the section of the conductor. For a quasi-filiform conductor with radius  $r$ , the geometrical mean distance is (we will not prove it):

$$\delta = e^{-1/4} r = \mathcal{G}r \quad (9.24)$$

By inserting this value into (9.23), we find

$$L \sim \mu_0 a \left( \ln \frac{8a}{\mathcal{G}r} - 2 \right) = \mu_0 a \left( \ln \frac{8a}{r} - \ln \mathcal{G} - 2 \right) \\ \sim \mu_0 a \left( \ln \frac{8a}{r} - 1.75 \right) \quad (9.25)$$

This result is known as the Maxwell's inductance for a circular quasi-filiform conductor with radius  $a$  and cross-section  $\pi r^2$ .

# Chapter 10

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In the next couple of chapters, we will focus on the study of the forces acting on electrically charged conductors and on circuits carrying electrical current.

## 10.1 Forces on Electrically Charged Conductors

The knowledge of the electrostatic energy of a system of conductors given by Eq. (7.7) makes it possible to calculate the forces acting on the conductors. This task can easily be achieved by using the principle of conservation of energy.

The first case we consider is when the system of conductors is an isolated system. This means the total charge of the system is constant in time. Under the further assumption that each conductor in the system is spatially isolated, i.e., all conductors do not touch each other, we conclude that the charge on each conductor is also a constant in time.

Among all conductors in the system, suppose that one of them is free to translate by a distance  $d\vec{r}$  under the action of the electric forces due to the other conductors, which are considered to be fixed at their positions in space. To visualize the process, we can imagine to loosen the constraints on one conductor, thereby enabling small spatial displacements. In this case, the mechanical work produced by the field forces is

$$dW = \vec{F} \cdot d\vec{r} \tag{10.1}$$

where  $\vec{F}$  is the net force of all electric forces acting on the conductor. Figure 10.1 shows a schematic of the system under consideration.

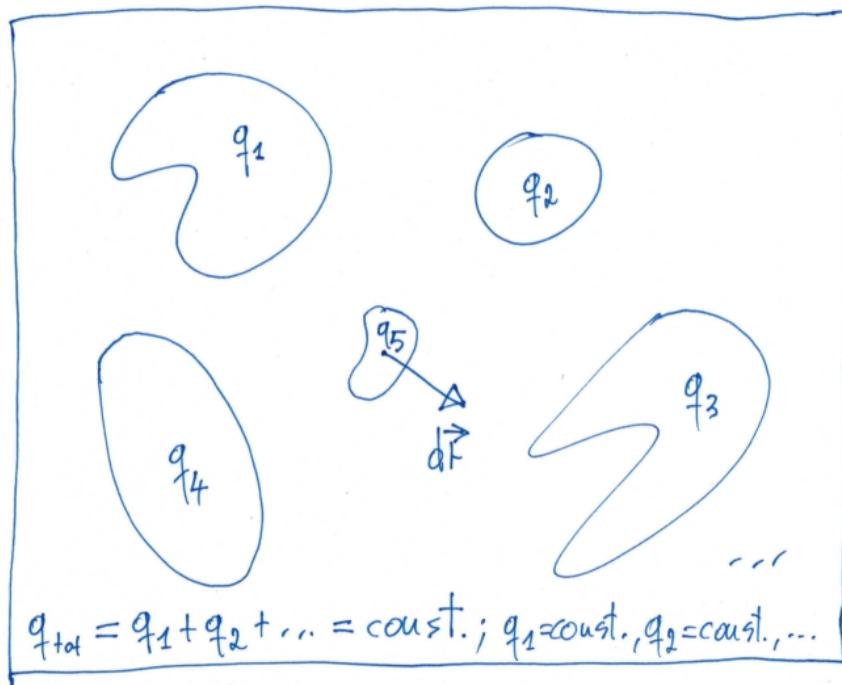


Figure 10.1

Since the system is isolated, the work (10.1) must be against the electrostatic energy  $U_e$  of the system

$$dW = -dU_e|_q \quad (10.2)$$

where the differential is calculated assuming constant charges.

By combining (10.1) and (10.2), we obtain

$$\vec{F} \cdot d\vec{r} = -dU_e|_q \quad (10.3)$$

Then, by using the definition of unit vector  $\vec{u}_r$  along the generic direction  $r$  and that of total differential, we find

$$\begin{aligned} \vec{F} \cdot \left( \frac{\vec{r}}{r} \right) dr &= \vec{F} \cdot \vec{u}_r dr = F_r dr \\ &= - \frac{\partial}{\partial r} U_e|_q dr \end{aligned} \quad (10.4)$$

where  $F_r$  is the component of  $\vec{F}$  along  $r$ . The result (10.4) can finally be simplified as

$$F_r = - \frac{\partial}{\partial r} U_e|_q \quad (10.5)$$

which is the result we were looking for.

The second case is when the conductors form an open system, i.e., a non-isolated system. In such an open system, each conductor is maintained at a constant potential by means of external sources of electric energy, i.e., electric batteries (see PHYS 242).

Figure 10.2 shows the same system as in Fig. 10.1, but in open configuration: Electric charge can move from the sources onto the conductors, thus keeping them at a constant potential.

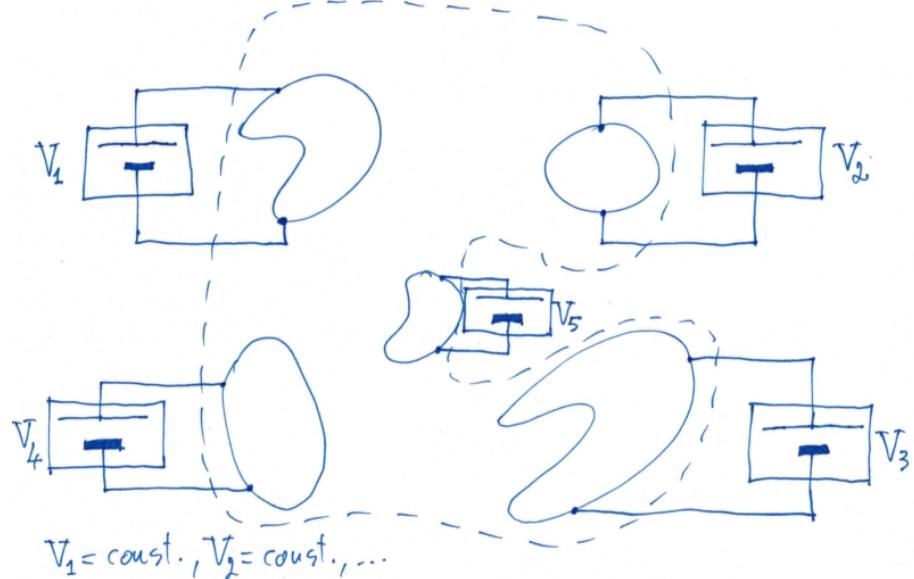


Figure 10.2

In this case, Eq. (10.1) is still valid. This time, however, the mechanical work is against both the electric energy of the system of conductors and that of the external sources. Thus, the equation for the conservation of energy, which is given by Eq. (10.2) in the first case, now becomes

$$dW = -dU_e \Big|_{\phi} + dU_{ext} \quad (10.6)$$

where  $dU_e|_{\phi}$  is the differential of the electrostatic energy of the system of conductors calculated assuming constant potentials and  $dU_{ext}$  indicates the differential of the electric energy due to the external sources.

In order to write the net force  $\vec{F}$  as a function of the electrostatic energy  $U_e$ , it is necessary to find a mathematical connection between  $dU_e|_{\phi}$  and  $dU_{ext}$ . The energy generated by the external sources to increase by  $dq_i$  the charge of the  $i$ -th conductor is  $\phi_i dq_i$ , where  $\phi_i$  is the electrostatic potential for that conductor. Hence,

$$dU_{ext} = \sum_{i=1}^N \phi_i dq_i \quad (10.7)$$

where  $N$  is the total number of conductors. Assuming constant potentials, by differentiating Eq. (7.7), we find

$$dU_e \Big|_{\phi} = \frac{1}{2} \sum_{i=1}^N \phi_i dq_i \quad (10.8)$$

By comparing Eqs. (10.7) and (10.8), we then obtain

$$dU_{ext} = 2dU_e \Big|_{\phi} \quad (10.9)$$

Using (10.9) in (10.6), we have

$$dW = -dU_e \Big|_{\phi} + 2dU_e \Big|_{\phi} = dU_e \Big|_{\phi} \quad (10.10)$$

We note that, because of (10.6) and (10.10) the electric work  $dU_{ext}$  is equally distributed between mechanical work and electrostatic energy. From the definition of work and Eq. (10.10) it follows that

$$\vec{F} \cdot d\vec{r} = dU_e \Big|_{\phi} \quad (10.11)$$

and, finally,

$$F_r = \frac{\partial}{\partial r} U_e \Big|_{\phi} \quad (10.12)$$

where we followed a derivation similar to that in Eqs. (10.4) and (10.5).

In summary, given a system of electric conductors having a specific geometrical configuration and electric state (i.e., given electric charges and potentials for the conductors), in static conditions the forces acting between conductors depend only on the geometrical configuration and electric state of the system. As a consequence, since Eqs. (10.5) and (10.12) are different expressions of the same forces, they must give the same result.

## 10.2 Force for Parallel-Plate Condensers

Consider a simple parallel-plate capacitor, the capacitance of which was calculated in PHYS 242. We want to calculate the force between the two plates of the capacitor.

Assuming the area  $A$  of each plate to be much larger than the distance  $d$  between the plates,  $\sqrt{A} \gg d$ , from simple symmetry arguments (and our previous knowledge of the field  $\vec{E}$  for this type of capacitor) it follows that the force must be directed normally to the plates. In addition, from the very definition of capacitor it also follows that such a force must be attractive. As shown in Fig. 10.3, we choose a Cartesian coordinate system  $Oxyz$ , with the  $yz$ -plane coinciding with one of the two plates and the  $x$ -axis directed towards the other plate.

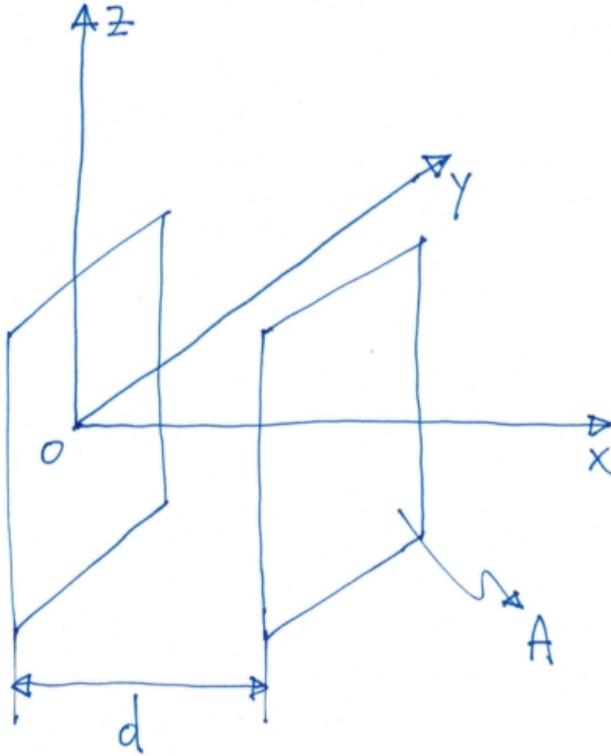


Figure 10.3

We slightly move the two plates with respect to each other along the  $x$  direction, which coincides with the direction of the force  $\vec{F}$  between the plates.

First, we assume the capacitor to be isolated. For example, we charge the capacitor by means of a battery. We then disconnect it from the charge source and imagine to perform the displacement under these conditions.

If  $C$  is the capacitance of the capacitor, from Eqs. (7.8) and (10.5), we have

$$F_x = -\frac{d}{dx} \left( \frac{1}{2} \frac{q^2}{C} \right) \quad (10.13)$$

where  $q$ , which is the positive charge on one of the two plates, must be constant for an isolated system. Thus,

$$\begin{aligned} F_x dx &= -d \left( \frac{1}{2} \frac{q^2}{C} \right) = -\frac{1}{2} q^2 d \left( \frac{1}{C} \right) \\ &= \frac{1}{2} \frac{q^2}{C^2} dC \end{aligned} \quad (10.14)$$

For an arbitrary distance  $x$ , the capacitance of a parallel-plate capacitor is

$$C = \epsilon_0 \frac{A}{x} \quad (10.15)$$

For the capacitor in Fig. 10.3, a small displacement  $dx$  along the  $x$  direction results in (Taylor)

$$\tilde{C} \simeq C + dC = \epsilon_0 \frac{A}{d} - \epsilon_0 \frac{A}{d^2} dx \quad (10.16)$$

From which

$$dC = -\epsilon_0 \frac{A}{d^2} dx \quad (10.17)$$

Using this result in (10.14), we obtain

$$F_x dx = -\frac{1}{2} \frac{q^2}{C^2} \epsilon_0 \frac{A}{d^2} dx \quad (10.18)$$

or

$$\begin{aligned} F_x &= -\frac{1}{2} \frac{q^2}{\epsilon_0^2 A^2} \epsilon_0 \frac{A}{d^2} = -\frac{1}{2} \frac{q^2}{\epsilon_0 A} \\ &= -\frac{1}{2} \frac{\sigma^2}{\epsilon_0} A \end{aligned} \quad (10.19)$$

where we used the fact that  $C = \epsilon_0 A/d$  and  $\sigma = q/A$ . The latter is the surface charge density on the plates. The force with respect to surface is thus

$$\frac{F_x}{A} = -\frac{1}{2} \frac{\sigma^2}{\epsilon_0} \quad (10.20)$$

The minus sign indicates that the force, when applied to the plate we assumed to displace by a quantity  $dx$ , is directed towards the negative side of the  $x$ -axis.

Second, we assume the capacitor to be non isolated. For example, we assume it is always connected to a battery.

From Eqs. (7.8) and (10.12), we have

$$F_x dx = d \left[ \frac{1}{2} C (\Delta\phi)^2 \right] \quad (10.21)$$

where  $\Delta\phi$ , which is the potential difference between the two plates calculated from the positive to the negative plate, must be constant for a non-isolated system. Thus,

$$F_x dx = \frac{1}{2} (\Delta\phi)^2 dC \quad (10.22)$$

Using again (10.17) this time in (10.22), we obtain

$$F_x = -\frac{1}{2} (\Delta\phi)^2 \epsilon_0 \frac{A}{d^2} \quad (10.23)$$

From  $C = \epsilon_0 A/d$  and  $\sigma = q/A$ , we finally find

$$\begin{aligned} F_x &= -\frac{1}{2} (\Delta\phi)^2 \frac{1}{\epsilon_0 A} \left( \epsilon_0 \frac{A}{d} \right)^2 \\ &= -\frac{1}{2} (\Delta\phi)^2 \frac{1}{\epsilon_0 A} c^2 \\ &= -\frac{1}{2} \frac{q^2}{\epsilon_0 A} \\ &= -\frac{1}{2} \frac{\sigma^2}{\epsilon_0} / A \end{aligned} \tag{10.24}$$

where we used the definition of capacitance  $q = C\Delta\phi$ .

As expected, this result is exactly the same as Eq. (10.19).



# Chapter 11

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In Chapter 10, we studied the forces acting on electrically charged conductors. We will now consider a similar problem, but for current-carrying circuits in magnetic fields.

## 11.1 Forces on Electric Current-Carrying Circuits in an External Magnetic Field

Consider a quasi-filiform circuit carrying a current  $I$  in a region of space characterized by an external magnetic field  $\vec{B}_0$ . Consider a circuit element of length  $\Delta l$ , much smaller than the total length of the circuit (here, “circuit element” refers to a small piece of conductor instead of a capacitor, resistor, or inductor). The circuit element defines two cross sections with area  $A_1$  and  $A_2$ , respectively, as shown in Fig. 11.1.

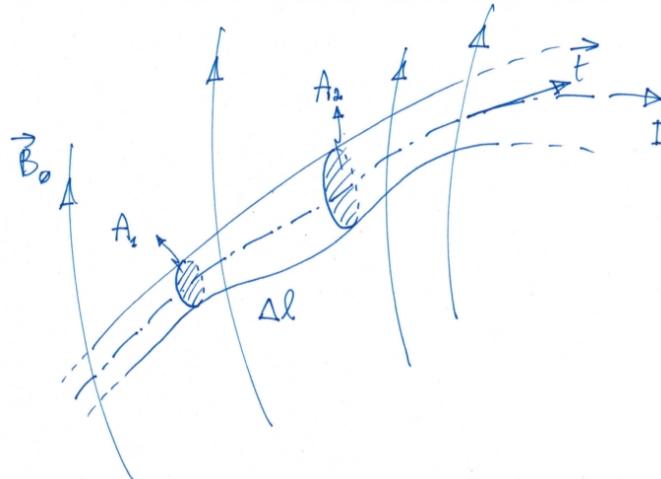


Figure 11.1

Imagine to be able to observe the motion of the charge carriers in the conductor. We indicate with  $q$  the value of electric charge of each carrier and with  $\vec{v}$  its velocity. Note that such a velocity corresponds to the ordered motion which determines the current. If  $\vec{B}$  is the net magnetostatic field in the considered circuit element ( $\vec{B}$  is due to  $\vec{B}_0$  as well as  $I$ ), the force acting on each moving charge carrier is (from Lorentz's force)

$$\vec{F}_q = q\vec{v} \times \vec{B} \quad (11.1)$$

As we saw in PHYS 242, this force gets transferred to the lattice of fixed charges in the conductor (i.e.,  $\vec{F}_q$  is a ponderomotive force). Thus,  $\vec{F}_q$  acts globally on the entire conductor.

We now want to calculate the force  $\Delta\vec{F}$  acting on the conductor element of length  $\Delta\ell$ . To this end, we imagine to take a snapshot at a certain instant in time of the charge carriers existing in  $\Delta\ell$ . The force  $\Delta\vec{F}$  is the net of the forces acting on the carriers at the given time. Assuming that all charge carriers have the same velocity  $\vec{v}$  we have

$$\Delta\vec{F} = q\vec{v} \times \vec{B}\Delta n \quad (11.2)$$

where  $\Delta n$  is the number of carriers in  $\Delta\ell$  at the considered time instant. If  $\Delta t$  is the time interval necessary for each carrier to travel through  $\Delta\ell$ , we have

$$\vec{v} = \frac{\Delta\ell}{\Delta t} \vec{t} \quad (11.3)$$

where  $\vec{t}$  is the unit vector along the central longitudinal axis of the circuit (see Fig. 11.1). As a consequence, we can write (11.2) as

$$\Delta\vec{F} = q \frac{\Delta\ell}{\Delta t} \vec{t} \times \vec{B}\Delta n \quad (11.4)$$

It must be that the carriers going through  $A_2$  in the time window  $\Delta t$  are equal to all and only those contained within the piece of length  $\Delta\ell$ . Therefore, the total charge crossing  $A_2$  during  $\Delta t$  is

$$\Delta q = q\Delta n \quad (11.5)$$

The current is thus

$$I = \frac{\Delta q}{\Delta t} = q \frac{\Delta n}{\Delta t} \quad (11.6)$$

and (11.4) becomes

$$\Delta\vec{F} = I\Delta\ell \vec{t} \times \vec{B} \quad (11.7)$$

In the limit for  $\Delta\ell \rightarrow 0^+$  we obtain the expression for the infinitesimal force acting on an infinitesimal element  $d\ell$  of a conductor carrying a current  $I$  in a magnetostatic field  $\vec{B}$ :

$$d\vec{F} = Id\ell \vec{t} \times \vec{B} \quad (11.8)$$

This force is normal to both the field  $\vec{B}$  and the conductor itself; the force is directed according to a corkscrew rotation from  $Id\ell \vec{t}$  to  $\vec{B}$  by the smallest angle.

In order to obtain the net force acting on the entire current-carrying circuit, it is sufficient to sum up all contributions relative to each infinitesimal element, obtaining

$$\vec{F} = \oint_{\gamma} I\vec{t} \times \vec{B} d\ell \quad (11.9)$$

where  $\gamma$  is a closed oriented line corresponding to the central (main) longitudinal axis of the circuit (indeed, any other longitudinal axis would work as well for a quasi-filiform conductor).

Equation (11.9) can be used, for example, to determine the dynamics of a rigid circuit under the action of the magnetostatic field. In this case, the net force can be calculated from (11.9) considering only the external field  $\vec{B}_0$ . In fact, the field generated by  $I$  itself can only result in a zero force on the circuit. This is similar to a single charge, which cannot produce a nonzero force on itself.

The scenario changes drastically when considering a non-rigid circuit, when studying the deforming effects on the circuit due to forces of type (11.9). In this case, we must determine the distribution of forces acting on each infinitesimal circuit element, taking into account the effective field  $\vec{B}$  acting on that element. Such a field is the sum of the field generated by external electric currents and that due to the current on the circuit itself.

We now want to consider the general case of a bulk current-carrying conductor. To this end, we can consider a generic piece of the bulk conductor, assuming to know the current density distribution  $\vec{J}$  and the magnetostatic field  $\vec{B}$  within the piece. Given a generic flux tube for vector  $\vec{J}$ , consider an infinitesimal element of right-angle cross-section  $dA$  and length  $d\ell$ . Assuming that such a flux tube is a filiform conductor carrying the infinitesimal current

$$dI = \vec{J} \cdot \vec{n} dA \quad (11.10)$$

where  $\vec{n}$  is the normal unit vector to  $dA$  and is parallel to  $\vec{J}$ , by means of Eq. (11.8) we obtain

$$d\vec{F} = (\vec{J} \cdot \vec{n} dA) d\ell \vec{t} \times \vec{B} \quad (11.11)$$

Since  $\vec{J}$  and  $\vec{n}$  have the same direction and  $\vec{t} = \vec{n}$ , we have

$$d\vec{F} = dA d\ell \vec{J} \times \vec{B} \quad (11.12)$$

By defining  $dV = dA d\ell$  the volume of the infinitesimal flux tube, we have

$$\vec{f} = \frac{d\vec{F}}{dV} = \vec{J} \times \vec{B} \quad (11.13)$$

This result gives the force per unit volume acting at a generic point in the conductor. The net force acting on a finite piece of conductor is sufficient to sum up the contributions due to each infinitesimal volume element, obtaining

$$\vec{F} = \iiint_{\tau} (\vec{J} \times \vec{B}) dV \quad (11.14)$$

where  $\tau$  is the region of conductor under consideration.

## 11.2 Force Between Two Straight, Indefinite (Infinitely Long) Current-Carrying Conductors, Parallel to Each Other

Consider two straight and indefinite filiform conductors. Assume the conductors are parallel to each other and placed at a distance  $d$  from each other, in vacuum. Furthermore, assume they carry steady currents  $I_1$  and  $I_2$ , respectively. Figure 11.2 shows a sketch of the problem under consideration.

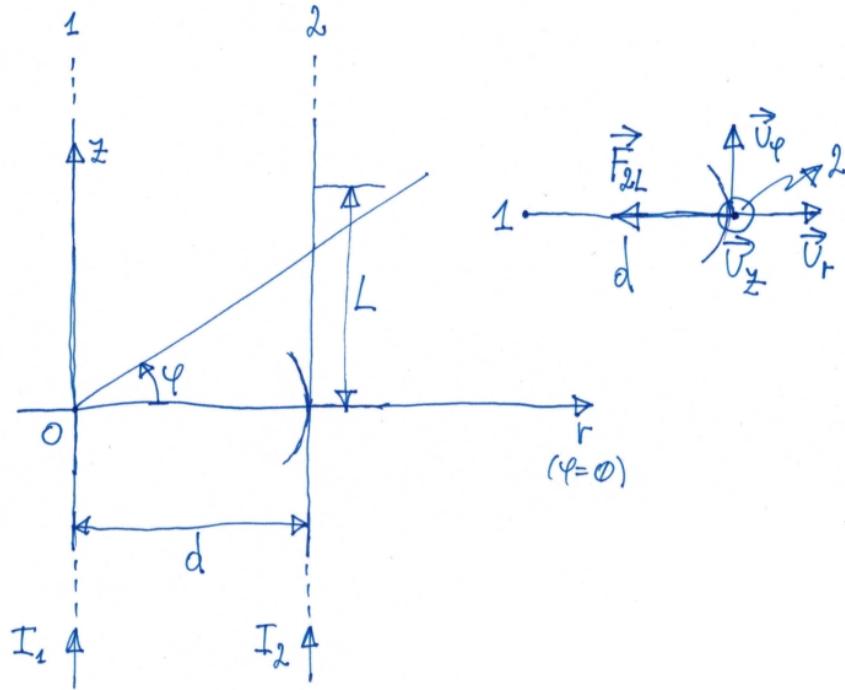


Figure 11.2

We intend to calculate the force acting on each of the two conductors due to the current carried by the other conductor.

We will focus, for simplicity, on the force acting on the second conductor (similar arguments would apply to the force on the first conductor).

First, we choose a cylindrical coordinate system  $Or\varphi z$ , as shown in Fig. 11.2.

Second, we calculate the magnetic field due to  $I_1$  on conductor 1 at a distance  $d$  from it. This is

$$\vec{B}_1 = \frac{\mu_0}{2\pi} \frac{I_1}{d} \vec{u}_\varphi \quad (11.15)$$

Then, we consider a finite segment of length  $L$  of conductor 2 and calculate the force acting on it from (11.9)

$$\begin{aligned} \vec{F}_{2L} &= I_2 \int_0^L \vec{u}_z \times \vec{u}_\varphi \frac{\mu_0}{2\pi} \frac{I_1}{d} d\ell \\ &= -\frac{\mu_0}{2\pi} \frac{I_1 I_2}{d} L \vec{u}_r \end{aligned} \quad (11.16)$$

where  $\vec{t} = \vec{u}_z$ . This force is normal to both conductors and attractive when the two currents have the same sign or repulsive when they have opposite sign.

Equation (11.16) is used to define the *SI* current unit, ampère ( $A$ ): This is the current intensity carried by two straight, indefinite (i.e.,  $L \gg d$ ), and parallel conductors at  $d = 1\text{m}$  generating a force  $\vec{F}_{2L} = \vec{F}_{1L} = 2 \times 10^{-7}\text{Nm}^{-1}$ .

### 11.3 Work to Move a Current-Carrying Circuit

Consider a filiform circuit  $\gamma$  carrying a steady current  $I$  and placed in a region with a magnetic field  $\vec{B}_0$  due to other currents, as shown in Fig. 11.3. As shown in the figure, an infinitesimal element  $d\ell$  of  $\gamma$  is free to move along the conducting tracks  $p$  and  $q$ . Suppose to apply a force on  $d\ell$  that moves it from the position  $PQ$  to the position  $P'Q'$  in a quasi-static fashion. This means the force applied to move the element is counter-balanced at each time by the force  $d\vec{F}$  due to the magnetic field.

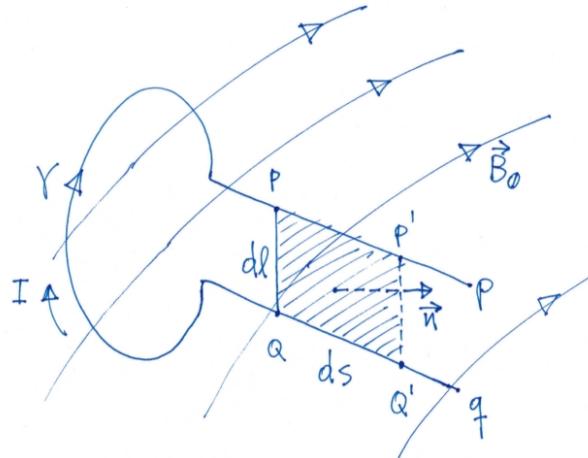


Figure 11.3

Under these conditions, we want to calculate the work of the force  $d\vec{F}$  during the movement. We will assume that during a circuit movement the current on it remains the same, i.e., all circuit displacements take place under steady current conditions.

Following the notation in Fig. 11.3, if the displacement length is  $ds$ , we have

$$dW = d\vec{F} \cdot \vec{t}_{PP'} ds \quad (11.17)$$

where  $\vec{t}_{PP'}$  is the tangent unit vector along  $p$ . If  $\vec{B}$  is the net magnetic field on  $d\ell$  due to both the external currents ( $\vec{B}_0$ ) and  $I$ , from (11.8) we have

$$dW = (Id\ell \vec{t}_{PQ} \times \vec{B}) \cdot \vec{t}_{PP'} ds \quad (11.18)$$

where  $\vec{t}_{PQ}$  is the tangent unit vector to  $\gamma$  at  $d\ell$ . Since

$$\vec{t}_{PP'} ds \times \vec{t}_{PQ} d\ell = (\vec{t}_{PP'} \times \vec{t}_{PQ}) ds d\ell = \vec{n} dA \quad (11.19)$$

where  $dA = ds d\ell$  is the area of the rectangle  $PQQ'P'$  and  $\vec{n}$  the normal unit vector for this rectangle, we find

$$dW = I(\vec{t}_{PP'} \times \vec{t}_{PQ} \cdot \vec{B}) ds d\ell = I(\vec{n} \cdot \vec{B}) dA \quad (11.20)$$

where we used the cyclic properties of the triple product. The quantity  $\vec{n} \cdot \vec{B} dA$  is the infinitesimal flux  $d\Phi_t$  of  $\vec{B}$  through  $dA$ , thus

$$dW = Id\Phi_t \quad (11.21)$$

Consider now a rigid (filiform) circuit that moves with respect to the currents generating  $\vec{B}_0$ . The discussion that led to Eq. (11.21) can be extended to each infinitesimal element in which the rigid circuit can be decomposed. Figure 11.4 shows a rigid circuit that moved from an initial position  $A$  to a final position  $B$ .

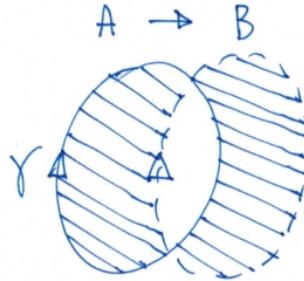


Figure 11.4

In this case, the total work of the field forces is

$$W = I\Delta\Phi_t \quad (11.22)$$

where  $I$  is the steady current carried by  $\gamma$  and  $\Delta\Phi_t$  the flux through shaded area in Fig. 11.4,

$$\Delta\Phi_t = \Phi_B - \Phi_A \quad (11.23)$$

In this expression,  $\Phi_A$  and  $\Phi_B$  are the fluxes linked with  $\gamma$  for position  $A$  and  $B$ , respectively. Note that, when calculating  $\Phi_A$  and  $\Phi_B$  it is not required to take into account the self-induction flux generated by  $I$  because this flux remains constant when moving from  $A$  to  $B$ . The only fluxes to be considered are those due to  $\vec{B}_0$ , i.e., the mutual-induction fluxes between  $\gamma$  and the external currents. In fact, a rigid circuit cannot generate work on itself.

We can finally extend all our results to the general case of a bulk circuit, the lateral dimensions of which are non-negligible compared to its length. Suppose to divide the entire bulk conductor carrying a total current  $I$  (steady) into  $N$  arbitrary flux tubes of the current field, each tube with current intensity  $I_1, I_2, \dots, I_N$ . Assuming that each of these tubes is a quasi-filiform conductor, we can write

$$\begin{aligned} W_1 &= I_1(\Phi'_1 - \Phi_1), & W_2 &= I_2(\Phi'_2 - \Phi_2), \dots, \\ W_N &= I_N(\Phi'_N - \Phi_N) \end{aligned} \quad (11.24)$$

where  $W_i$  is the work of the field forces on the  $i$ -th flux tube during the displacement and  $\Phi_i$  and  $\Phi'_i$  are, respectively, the linked fluxes for the  $i$ -th tube at the initial and final position.

The total work  $W$  on the entire circuit is

$$\begin{aligned}
 W &= W_1 + W_2 + \cdots + W_N \\
 &= I_1(\Phi'_1 - \Phi_1) + I_2(\Phi'_2 - \Phi_2) + \cdots + I_N(\Phi'_N - \Phi_N) \\
 &= I_1\Phi'_1 + I_2\Phi'_2 + \cdots + I_N\Phi'_N \\
 &\quad - (I_1\Phi_1 + I_2\Phi_2 + \cdots + I_N\Phi_N)
 \end{aligned} \tag{11.25}$$

We can thus define a flux  $\Phi$  linked with the entire bulk circuit such that

$$I\Phi = I_1\Phi_1 + I_2\Phi_2 + \cdots + I_N\Phi_N \tag{11.26}$$

By definition,

$$\Phi = \frac{I_1}{I} \Phi_1 + \frac{I_2}{I} \Phi_2 + \cdots + \frac{I_N}{I} \Phi_N \tag{11.27}$$

from which

$$W = I(\Phi' - \Phi) \tag{11.28}$$

where  $\Phi$  and  $\Phi'$  are the fluxes linked with the entire bulk circuit at the initial and final position.

Equation (11.27) extends the concept of linked flux to the case of a non-filiform circuit and, thus, makes it possible to define the self- and mutual-inductance coefficients to the case of bulk circuits.

In addition, the results in this section allows us to calculate readily the net magnetic force acting on the parts of a generic current-carrying circuit as well as the net torque acting on it. In fact, if the (rigid) circuit undergoes a translation  $d\vec{s}$ , the corresponding total work of the magnetic forces must be equal to the work of the net force  $\vec{R}$ . Hence,

$$dW = \vec{R} \cdot d\vec{s} = Id\Phi \tag{11.29}$$

When  $d\vec{s} = dx\vec{u}_x$  in a Cartesian coordinate system  $Oxyz$ , we have

$$R_x dx = Id\Phi \tag{11.30}$$

where  $d\Phi$  is the variation of the flux linked with the circuit carrying a steady current  $I$ , during the displacement. Therefore,

$$R_x = I \frac{\partial}{\partial x} \Phi \tag{11.31a}$$

and, similarly,

$$R_y = I \frac{\partial}{\partial y} \Phi \tag{11.31b}$$

$$R_z = I \frac{\partial}{\partial z} \Phi \tag{11.31c}$$

Similarly, the net torque acting on the circuit is

$$\Gamma_x = I \frac{\partial}{\partial \alpha} \Phi \tag{11.32a}$$

### 11.3. WORK TO MOVE A CURRENT-CARRYING CIRCUIT

$$\Gamma_y = I \frac{\partial}{\partial \beta} \Phi \quad (11.32b)$$

$$\Gamma_z = I \frac{\partial}{\partial \gamma} \Phi \quad (11.32c)$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are the rotation angles about the  $x$ -,  $y$ -, and  $z$ -axis, respectively (see Fig. 11.5).

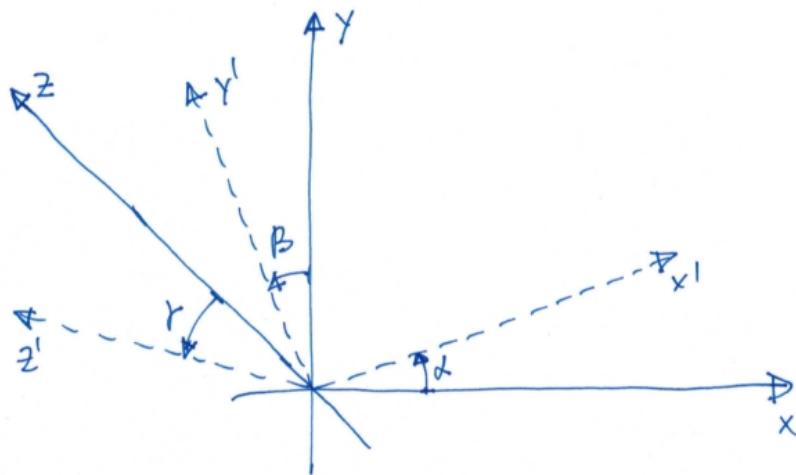


Figure 11.5

# Chapter 12

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## 12.1 Multipole Series Expansion of the Electrostatic Potential.

We now intend to show a powerful method that makes it possible to calculate the approximate potential generated by a given charge distribution. This method, known as multipole series expansion, makes it possible to introduce in a transparent fashion specific type of charge distributions, such as dipoles and multipoles, which are of fundamental interest in many sectors of atomic physics, quantum optics, and in the study of insulating materials. Even circuit quantum electrodynamics, which studies the interaction between superconducting quantum bits (qubits) with a microwave electric field, can be reduced to electric dipoles in a field.

Consider a generic charge distribution in a limited region of space, so that it can be entirely contained within a sphere of radius  $a$  (see Fig. 12.1). We intend to study the effects of such a charge distribution at a large distance from the charge (i.e., at points  $P$  the distance of which from the origin  $O$  in Fig. 12.1 is much larger than  $a$ ).

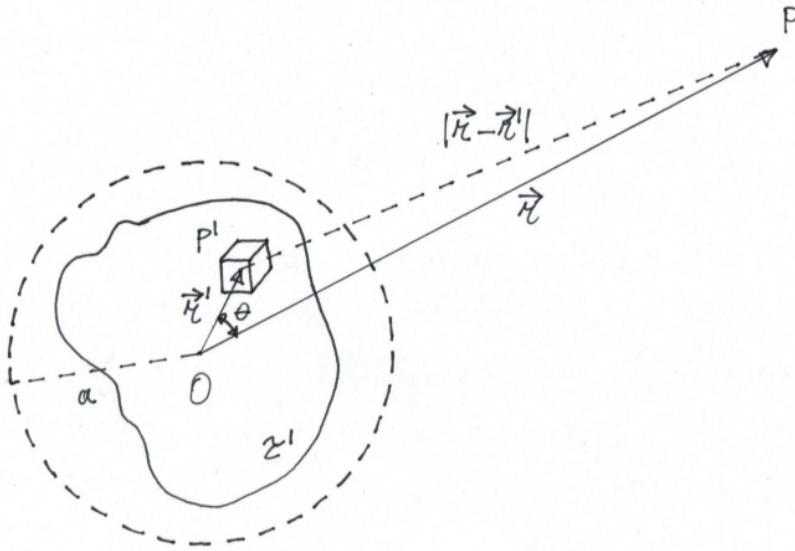


Figure 12.1

we indicate with  $\vec{r}'$  the vector from the origin  $O$  to the point  $P'$  within the region  $\tau'$  occupied by the charges, with  $\rho(\vec{r}')$  the volume charge density at  $P'$ , and with  $\vec{r}$  the vector from  $O$  to the generic observation point  $P$ .

The potential at  $P$  is given by

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{\tau'} \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau' , \quad (12.1)$$

where  $d\tau'$  is an infinitesimal volume element in  $\tau'$  and  $|\vec{r} - \vec{r}'|$  indicates the distance between  $P$  and  $P'$ , which, by means of Carnot's theorem, is given by

$$\begin{aligned} r_{PP'} &= |\vec{r} - \vec{r}'| \\ &= (r^2 + r'^2 - 2rr' \cos \theta)^{1/2} , \end{aligned} \quad (12.2)$$

where  $r$  and  $r'$  represent the absolute values of vector  $\vec{r}$  and  $\vec{r}'$ , respectively, and  $\theta$  the angle between them. Since we are interested in the values of the potential at points very far from the origin, we can assume  $r'/r \ll 1$  and, thus, we can expand the expression  $|\vec{r} - \vec{r}'|$  in power series of  $r'/r$  limited to the first terms. Noting that

$$\begin{aligned} |\vec{r} - \vec{r}'|^{-1} &= (r^2 + r'^2 - 2rr' \cos \theta)^{-1/2} \\ &= (r^2)^{-1/2} \left( \frac{r^2 + r'^2 - 2rr' \cos \theta}{r^2} \right)^{-1/2} \\ &= \frac{1}{r} \left[ 1 + \left( \frac{r'^2}{r^2} - 2 \frac{r'}{r} \cos \theta \right) \right]^{-1/2} \end{aligned}$$

and defining

$$\xi = \left( \frac{r'}{r} \right)^2 - 2 \left( \frac{r'}{r} \right) \cos \theta \quad ,$$

we can write

$$|\vec{r} - \vec{r}'|^{-1} = \frac{1}{r} (1 + \xi)^{-1/2} \quad .$$

By expanding in Maclaurin series, we then obtain

$$(1 + \xi)^{-1/2} = 1 - \frac{1}{2} \xi + \frac{3}{8} \xi^2 + \dots$$

and, thus,

$$\begin{aligned} |\vec{r} - \vec{r}'|^{-1} &= \frac{1}{r} \left[ 1 + \left( \frac{r'}{r} \right) \cos \theta - \frac{1}{2} \left( \frac{r'}{r} \right)^2 + \frac{3}{8} 4 \left( \frac{r'}{r} \right)^2 \cos^2 \theta \right. \\ &\quad \left. - \frac{3}{8} 2 \cdot 2 \left( \frac{r'}{r} \right)^2 \left( \frac{r'}{r} \right) \cos \theta + \frac{3}{8} \left( \frac{r'}{r} \right)^4 + \dots \right] \\ &= \frac{1}{r} \left[ 1 + \left( \frac{r'}{r} \right) \cos \theta + \left( \frac{r'}{r} \right)^2 \frac{3 \cos^2 \theta - 1}{2} + \dots \right] \\ &= \frac{1}{r} + \frac{r'}{r^2} \cos \theta + \frac{r'^2}{r^3} \frac{3 \cos^2 \theta - 1}{2} + \dots \\ &= \frac{1}{r} + \frac{\vec{r}' \cdot \vec{r}}{r^3} + \frac{3(\vec{r}' \cdot \vec{r})^2}{2r^5} - \frac{r'^2}{2r^3} + \dots \end{aligned}$$

By substituting this expansion into (12.1) and, for example, stopping at the first two terms, we obtain

$$V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \iiint_{\tau'} \rho(\vec{r}') \left( \frac{1}{r} + \frac{\vec{r}' \cdot \vec{r}}{r^3} \right) \cdot d\tau' \quad . \quad (12.3)$$

Since the vector  $\vec{r}$  is independent from the variable of integration  $\vec{r}'$ , we obtain

$$V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \left[ \frac{1}{r} \iiint_{\tau'} \rho(\vec{r}') \cdot d\tau' + \frac{\vec{r}}{r^3} \cdot \iiint_{\tau'} \rho(\vec{r}') \vec{r}' \cdot d\tau' \right] \quad . \quad (12.4)$$

The first term in this expression corresponds to the potential that would be generated at  $P$  if the entire charge in region  $\tau'$ ,

$$Q = \iiint_{\tau'} \rho(\vec{r}') \cdot d\tau' \quad ,$$

was concentrated at  $O$ . It is evident that this should be the first approximation term of the potential “seen” by a distant observer from the charge.

The second term in (12.4) gives a contribution that varies with the inverse distance squared. In this term appears a vector quantity

$$\vec{p} = \iiint_{\tau'} \rho(\vec{r}') \vec{r}' \cdot d\tau' \quad , \quad (12.5)$$

which is called dipole moment of the charge distribution. The reason to call this term a “moment” is because the function  $\rho$  is multiplied times vector  $\vec{r}'$ . The dipole moment accounts for the specific distribution of the charge in  $\tau'$ . This goes beyond what the only charge  $Q$  can account for. We can thus rewrite (12.4) as

$$V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \left( \frac{Q}{r} + \frac{\vec{r} \cdot \vec{p}}{r^3} \right) ; \quad (12.6)$$

the two terms in (12.6) are called the term of monopole and dipole, respectively.

Note that the monopole term can be zero. This happens when the charge distribution is made up by the same amount of positive and negative charge. However, even if  $Q = 0$ , the simple approximation (12.6) shows that the electrostatic effects are non-zero. This is due to the presence of the dipole term, which can be different than zero.

We now focus on the dipole term  $\vec{p}$  defined by (12.5). When considering systems with zero total charge  $Q$ , the dipole moment can be calculated assuming the entire charge concentrated in the system centre of charge. Similar to the centre of mass, the centre of charge is defined by

$$\vec{r}'_c = \frac{\iiint_{\tau'} \rho(\vec{r}') \vec{r}' \cdot d\tau'}{\iiint_{\tau'} \rho(\vec{r}') \cdot d\tau'} ,$$

where  $\vec{r}'_c$  is the vector distance of the centre of charge from the origin. As a consequence,

$$\vec{r}'_c Q = \iiint_{\tau'} \rho(\vec{r}') \vec{r}' \cdot d\tau' .$$

Hence, the dipole moment can be calculated by concentrating the charge  $Q$  in the centre of charge and multiplying it by  $\vec{r}'_c$ .

In general, the dipole moment depends on the point with respect to which is calculated (i.e., the origin of the vectors  $\vec{r}'$ ). In particular, it is zero the dipole moment with respect to the centre of charge  $\vec{r}'_c$ . In the next lecture, we will see that, for systems of zero total charge, the dipole moment is independent from the point with respect to which is calculated: In those cases, the dipole moment is an intrinsic property of the system itself.

We note again that (12.4) is an approximation of (12.1) limited to the first two terms only of the series expansion of (12.1). Retaining higher order terms, called multipole terms, we would have

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left( \frac{k_0}{r} + \frac{k_1}{r^2} + \frac{k_2}{r^3} + \dots \right) , \quad (12.7)$$

where

$$\begin{aligned} k_0 &= \iiint_{\tau'} \rho(\vec{r}') d\tau' , \\ k_1 &= \left[ \iiint_{\tau'} \rho(\vec{r}') \vec{r}' \cdot d\tau' \right] \cdot \frac{\vec{r}'}{r} , \\ k_3 &= \iiint_{\tau'} \rho(\vec{r}') \left[ \frac{3}{2} \left( \frac{\vec{r}' \cdot \vec{r}}{r} \right)^2 - \frac{r'^2}{2} \right] \cdot d\tau' , \quad \dots \end{aligned}$$

The third term contains the quadrupole moment of the distribution, the fourth term the octupole moment, and so forth. In the following, we will only use the first two moments  $Q$  and  $\vec{p}$ . Note that higher moments play an important role, e.g., in nuclear physics. However, it is important to remember that when the first two terms are zero, the potential at large distance is mostly given by the quadrupole moment. When the quadrupole moment is also zero, by the octupole moment, and so on. The series (12.7) shows that, for a limited charge distribution in space, the potential at infinity goes to zero at least as the inverse distance.

## 12.2 The Electrostatic Dipole.

The dipole moment of a given charge distribution,

$$\vec{p} = \iiint_{\tau'} \rho(\vec{r}') \vec{r}' \cdot d\tau' ,$$

depends, in general, from the origin chosen to measure distances. It does not depend on the origin when the total charge of the distribution is zero. In this case  $\vec{p}$  is an intrinsic characteristic of the system. This can be demonstrated by displacing the origin  $O$  by a vector  $\vec{R}$ . The new dipole moment is given by

$$\begin{aligned} \vec{p}' &= \iiint_{\tau'} (\vec{r}' - \vec{R}) \rho(\vec{r}') \cdot d\tau' \\ &= \iiint_{\tau'} \vec{r}' \rho(\vec{r}') \cdot d\tau' - Q \vec{R} = \vec{p} - Q \vec{R} , \end{aligned} \quad (12.8)$$

where, as always,

$$Q = \iiint_{\tau'} \rho(\vec{r}') \cdot d\tau' .$$

It is obvious that when  $Q = 0$ , from (12.8) follows that

$$\vec{p}' = \vec{p} . \quad (12.9)$$

A particularly simple charge distribution that satisfies the condition  $Q = 0$  is that of two equal point-like charges of opposite sign, and located at a distance  $d$  from each other.

The dipole moment for the case of the two charges can be calculated choosing any origin. For example, we can choose the origin to coincide with the point occupied by the negative charge. The dipole moment can be thus calculated straightforwardly and is given by

$$\begin{aligned}\vec{p} &= \iiint_{\tau'} \left[ -q\delta(\vec{r}')\vec{r}' + q\delta(\vec{r}' - \vec{d})\vec{r}' \right] \cdot d\tau' \\ &= -q\left[\vec{r}'\right]_0 + q\left[\vec{r}'\right]_{\vec{d}} = q\vec{d} \quad ,\end{aligned}\quad (12.10)$$

where  $\vec{d}$  is the vector distance between the negative and positive charge. In other words, the dipole moment for the two charges is a vector having: a) the direction of the line connecting the two charges; b) orientation from the negative to the positive charge; c) absolute value given by the product of the positive charge times the distance between the charges. In first approximation, the potential generated at a generic point in space, at a distance much larger than the separation between the two charges,  $d$ , is given by (12.6) for  $Q = 0$ :

$$V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \frac{\vec{r} \cdot \vec{p}}{r^3} \quad . \quad (12.11)$$

The electric field can be obtained from (12.11) knowing that

$$\vec{E} = - \operatorname{grad} V \quad .$$

Choosing as the origin of the coordinates the position occupied by the negative charge and indicating with  $\vec{r}$  the vector from the origin to a point  $P$  where the field is evaluated, we obtain

$$\vec{E}(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \left[ \frac{3(\vec{r} \cdot \vec{p})\vec{r}}{r^5} - \frac{\vec{p}}{r^3} \right] \quad . \quad (12.12)$$

The first term in (12.12) represents the radial component of the field. The second term the component along the direction of the dipole moment. The field lines are shown.

The specific charge distribution here considered becomes particularly interesting in the following limiting case. Assume to make extremely small, in the limit zero, the separation  $d$  between the two charges (the positive and negative charge we are considering). At the same time, assume to increase the charge  $q$ , in the limit making it infinite, so that the absolute value  $qd$  of the dipole moment remains finite (it is clear that if we were to make the two charges infinitesimally close without increasing their value, the potential and field would both go to zero everywhere in space). Under these assumption, it is easy to show that the only surviving term in the series expansion of the potential, Eq. (12.7), is that containing the dipole moment. In fact, from (12.7) for  $r' = d$ , the quadrupole moment is given by

$$\frac{1}{8\pi\epsilon_0} q \left[ \frac{3(\vec{r} \cdot \vec{d})^2}{r^5} - \frac{d^2}{r^3} \right]$$

(note that this can easily be shown by integration:

$$\begin{aligned}
 k_2 &= \iiint_{\tau'} \left\{ -q\delta(\vec{r}') \left[ \frac{3}{2} \left( \frac{\vec{r}' \cdot \vec{r}}{r} \right)^2 - \frac{r'^2}{2} \right] \right. \\
 &\quad \left. + q\delta(\vec{r}' - \vec{d}) \left[ \frac{3}{2} \left( \frac{\vec{r}' \cdot \vec{r}}{r} \right)^2 - \frac{r'^2}{2} \right] \right\} \cdot d\tau' \\
 &= -q \left[ \frac{3}{2} \left( \frac{0 \cdot \vec{r}}{r} \right)^2 - \frac{0^2}{2} \right] + q \left[ \frac{3}{2} \left( \frac{\vec{d} \cdot \vec{r}}{r} \right)^2 - \frac{d^2}{2} \right] \\
 &= q \frac{3}{2} \left( \frac{\vec{r} \cdot \vec{d}}{r} \right)^2 - \frac{d^2}{2}
 \end{aligned}$$

and, thus,

$$\frac{1}{4\pi\epsilon_0} \frac{k_2}{r^3} = \frac{1}{8\pi\epsilon_0} \left[ \frac{3(\vec{r} \cdot \vec{d})^2}{r^5} - \frac{d^2}{r^3} \right] \quad .$$

This term contains the product  $qd^2$ , which, for  $d \rightarrow 0$  and  $q \rightarrow \infty$  (so to keep  $qd$  constant), goes to zero ( $d^2$  wins over  $q$ ). Similarly, higher order multipole terms contain the products  $qd^3$ ,  $qd^4$ , and so forth, which also go to zero.

The so obtained system is called ideal electrostatic dipole. The potential and field of an ideal dipole are given by (12.11) and (12.12), respectively, exactly:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{r} \cdot \vec{p}}{r^3}$$

and

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{3(\vec{r} \cdot \vec{p})\vec{r}}{r^5} - \frac{\vec{p}}{r^3} \right] \quad .$$

The relevance of the ideal dipole resides in the fact that often we are interested in systems of charges with zero global charge and for which only the effects at large distances compared to their dimensions are needed. Examples of such systems are atoms and molecules. These are globally neutral, however, they often have non-zero dipole moment or, under specific conditions, they acquire a dipole moment due to the action of external forces.



# Chapter 13

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## 13.1 Electrostatic Induction (Revisited)

We now intend to give a more quantitative description of the concept of electrostatic induction introduced in PHYS 242. We consider to bring an initially neutral conducting body in a region with an electrostatic field generated by a set of fixed charges outside the conductor. Under the action of the field, the charge carriers in the conductor move towards the conductor surface  $\Sigma$  in a very short time and distributes on  $\Sigma$  in a way that counter-balances point by point inside the conductor the external field. As a consequence, the field inside the conductor becomes zero, while outside it gets deformed compare to the configuration without conductor. Figure 13.1 shows this phenomenon in the case of a uniform external field.

At this point, we must calculate the induced charge distribution on the conducting body. We proceed assuming a full knowledge on the value and position of the inducing (external) charges  $q_{\text{ext}}$ . We also assume that their distribution does not change upon introducing the conductive body. This assumption corresponds to consider all charges  $q_{\text{ext}}$  (external to the conductor) to be fixed in the lab reference frame. Consider now the (known) potential  $V_{\text{ext}}(P)$  generated at a generic point  $P$  by  $q_{\text{ext}}$ . In general, this potential changes point by point on  $\Sigma$ . In fact, it is the total potential  $V(P)$ , given by the sum of  $V_{\text{ext}}$  and the potential generated by the surface charge distribution induced on  $\Sigma$ ,  $V_{\text{ind}}$ , to be constant in the conductor including its surface  $\Sigma$ . The function  $V_{\text{ind}}(P)$  must satisfy Laplace equation at each point in the region  $\Omega^+$  outside  $\Sigma$ ,

$$\vec{\nabla}^2 V_{\text{ind}}(P) = 0 \quad (13.1)$$

(because the charges generating  $V_{\text{ind}}(P)$  are distributed only on the surface  $\Sigma$ ). In addition, at each point on  $\Sigma$  must be

$$\begin{aligned} V(P) \Big|_{\Sigma} &= V_{\text{ext}}(P) \Big|_{\Sigma} + V_{\text{ind}}(P) \Big|_{\Sigma} \\ &= \text{const} \quad . \end{aligned} \quad (13.2)$$

If we assume to know the value of  $V_0$  (note that  $V_0$  cannot arbitrarily chosen because of the overall neutrality of the conductor; we will not show how to determine  $V_0$  here), (13.2) reads

$$V_{\text{ind}}(P) \Big|_{\Sigma} = V_0 - V_{\text{ext}}(P) \Big|_{\Sigma} \quad . \quad (13.3)$$

### 13.1. ELECTROSTATIC INDUCTION (REVISITED)

We must thus find a function  $V_{\text{ind}}(P) \in C^0(\Omega^+ \cup \Sigma)$ , that verifies Laplace equation at each point in  $\Omega^+$ , assumes on  $\Sigma$  the values given by (13.3), and has a regular behaviour at infinity. This is an external Dirichlet problem for the boundary condition (13.3). Suppose we manage to find such a solution  $V_{\text{ind}}(P)$ . By summing this solution with  $V_{\text{ext}}(P)$ , we find the potential  $V(P)$  outside the conductor. We can then calculate the derivative  $(\partial V/\partial n)$  at each point on the exterior of  $\Sigma$ ,  $\Sigma^+$ . Since the derivative at each point on the interior of  $\Sigma$ ,  $\Sigma^-$ , must be zero (because  $V$  is constant inside the conductor), we can conclude that

$$\frac{\partial}{\partial n} V(P) \Big|_{\Sigma^+} = -\frac{\sigma_{\text{ind}}}{\epsilon_0} . \quad (13.4)$$

This result allows us to find the induced surface charge distribution density  $\sigma_{\text{ind}}$  at each point on  $\Sigma$ .

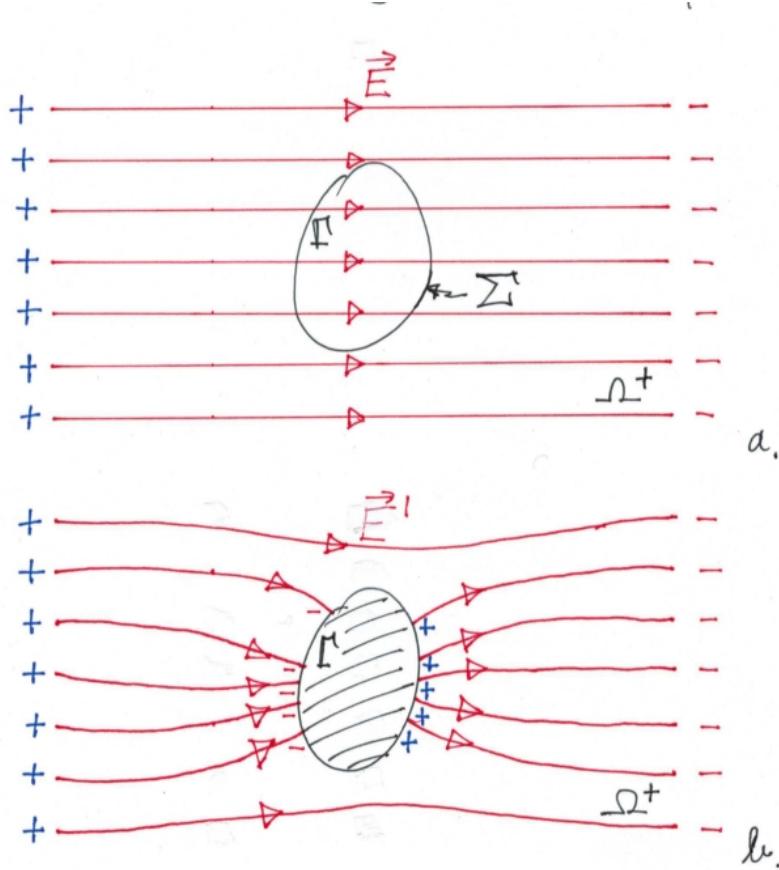


Figure 13.1

The concept of electrostatic induction also allows us to revisit the operative definition of electrostatic field.

If a set of charged conductors is fixed in space by maintaining each of the conductors fixed at its position by suitable forces, the charge distribution on each of them is determined by the position and charge on all the other conductors. It is sufficient to

move one of the conductors to modify the charge distribution on the other conductors because of electrostatic induction. Thus, the electrical charges are not fixed and the electrostatic field cannot be measured by means of any arbitrary test charge. A small enough test charge that does not modify the pre-existing charge distribution appreciable must be used. The presence of such a test charge should only produce a negligible perturbation.

For example, consider an infinite plane that separates the space into two regions. On the left side of the plane, the space is filled with a conducting material. On the right side there is vacuum.

The conductor and, thus, the plane that represents its surface are assumed to be neutral. Hence, the field is zero at each point in space. Suppose we want to “measure” such a field by means of a test charge  $q$  at a generic point  $P$ . After locating  $q$  at  $P$ , because of electrostatic induction, the surface of the conductor gets charged with sign opposite with respect to that of  $q$ . If we choose  $q$  to be positive, the plane gets negatively charged and a force due to the induced negative charge acts on  $q$ . By measuring the force  $\vec{F}$  and by dividing it by  $q$ , we obtain a field

$$\vec{E} = \frac{\vec{F}}{q}$$

different from zero! Obviously, this cannot be the null field we originally intended to measure in absence of the test charge  $q$ . In order to minimize the error, we should consider  $\vec{F}/q$  ratios for values of  $q$  as small as possible.

We now intend to calculate the force acting on  $q$ . To this end we must find a potential function  $V$  such that

a) satisfies Poisson equation

$$\vec{\nabla}^2 V = -\frac{1}{\epsilon_0} q \delta(P) \quad (13.5)$$

in the region on the right side of the conductor. Note that  $\delta(P)$  is the delta-Dirac centered at  $P$ ;

- b) is continuous until the conducting plane;
- c) takes a constant value on such a plane;
- d) has a regular behavior at infinity.

As it turns out, finding the solution to this problem is rather complicated.

## 13.2 The Method of Images.

An easy way to circumvent the Poisson problem outlined above is to resort to a trick known as the method of images.

In the introduction to this course, we studied the problem of the balanced electrostatic dipole consisting of two point-like charges, a positive charge  $+q$  and a negative charge  $-q$ , at a distance  $d$  from each other. Figure 13.2 shows the field lines associated with the dipole together with a few equipotential surfaces. We remind the latter must be normal to the field lines.

Consider the equipotential surface marked  $\Sigma_A$  in the figure. Assume to shape a thin conducting film so that it fits perfectly with  $\Sigma_A$ . The potential on the conducting film must be constant. If we now adjust such a potential to be exactly the constant potential on  $\Sigma_A$ ,  $V_A$ , and we insert the film right where  $\Sigma_A$  is defined, there would no manner for an observer to measure any effect due to the presence of the film, or its absence (other than by “touching” the film!). Following Feynman, from the simple problem of an electrostatic dipole we have now solved a new problem: The problem of a complicated curved conductor with a given potential near a point-like charge  $q$ . If the bending radius of the conducting film is infinite (or, at least very large), we have effectively constructed the case of a closed conducting shell with a point-like charge  $q$  nearby. The shell behaves as an electrostatic shield and, thud, the regions inside and outside the shell behave independently. In particular, we could fill the entire shell with a conducting material and obtain the same result. In other words, we have solved the problem of a full conducting sphere close to a charge  $q$  (Fig. 13.3). In the region outside the conductor the field is equal to that of two point-like charges, while inside the conductor is zero. In addition, due to Coulomb’s theorem the field lines must be normal to the conductor surface. This is obviously the case here because of the definition of equipotential surface.

In summary, we can compute the field in Fig. 13.3 by simply calculating the field due to a charge  $+q$  and to a charge  $-q$  positioned at a suitable point. The negative charge  $-q$  we position behind the conducting surface is called an *image charge*.

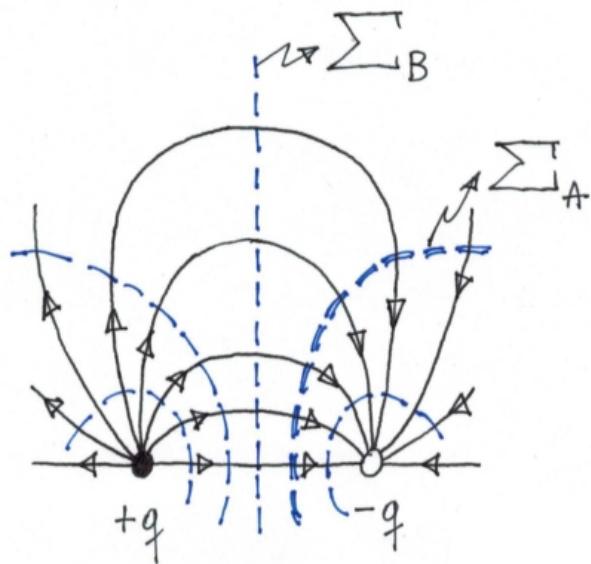


Figure 13.2

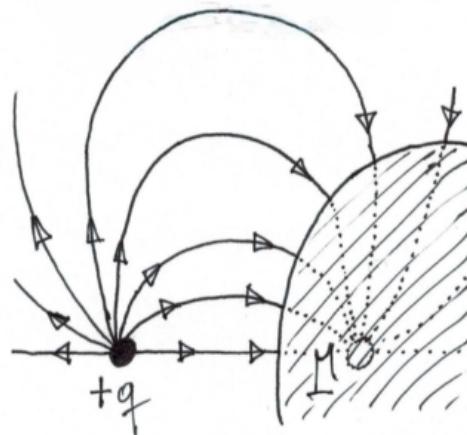


Figure 13.3

The simplest example on the use of the method of images is for the equipotential surface  $\Sigma_B$  in Fig. 13.2. This makes it possible to solve the problem of a thin infinite conducting plane facing a point-like positive charge  $+q$ . The field is obtained from Fig. 13.2 by crossing out the entire region on the right side of  $\Sigma_B$  (see Fig. 13.4). Note

that the potential due to the charges  $+q$  and  $-q$  can readily be obtained from the superposition principle,

$$V(P) = \frac{1}{4\pi\epsilon_0} \left( \frac{q}{r_1} - \frac{q}{r_2} \right) , \quad (13.6)$$

where  $r_1$  and  $r_2$  are the absolute values of the distances between  $+q$  and  $-q$  and the generic point  $P$  where we want to evaluate the potential. When  $P \in \Sigma_B$ ,  $r_1 = r_2$  and, thus,

$$V\Big|_{\Sigma_B} = 0 .$$

In other words, we have solved the problem of a positive point-like charge nearby a “grounded” conductor (we studied the concept of grounding in PHYS 242).

As we know from the theory of electrostatic induction, the positive point-like charge  $+q$  induces a negative charge on the surface of the conductor. Given a point  $P_0$  on the outer surface (i.e., the surface pointing towards  $+q$ ) of the conductor, the surface density  $\sigma$  associated with the induced negative charges can be calculated from Coulomb’s theorem as

$$\lim_{P \rightarrow P_0^-} E_n(P) = \frac{\sigma(P_0)}{\epsilon_0} , \quad (13.7)$$

where  $E_n$  is the normal and only possible component of the field at each point on the outer surface of the conductor. In order to calculate  $\sigma(P_0)$  we must then calculate the field  $\vec{E}$  at  $P_0$  due to both the positive (real) charge  $+q$  and the negative (image) charge  $-q$ . The summation of the two fields,  $\vec{E}_{e+}(P_0)$  and  $\vec{E}_{e-}(P_0)$ , respectively, coincides with the total field at  $P_0$  due to the conducting plane next to  $+q$ . Referring to Fig. 13.4, assume a cylindrical coordinate system with origin  $O$  at the intersection between the  $z$  axis, which is the normal to the conductor surface passing through  $+q$  and directed from  $+q$  to  $-q$ , and the conductor itself. The generic distance between  $O$  and  $P_0$  is  $r$  and  $\varphi$  is the angle measured counter-clockwise from a reference vertical axis on the conductor surface and passing through  $O$  (see the figure). Note that the normal unit vector  $\vec{n}$  to the conductor surface is directed from the negative to the positive charge and, thus,

$$\vec{n} = -\vec{u}_z \quad (13.8)$$

with respect to the chosen coordinate system  $Or\varphi z$ .

As shown in Fig. 13.4, due to Coulomb’s *law* and the specularity of  $+q$  and  $-q$ , the tangent components  $\vec{E}_{te-}(P_0)$  and  $\vec{E}_{te+}(P_0)$  cancel each other,  $\vec{E}_{te-}(P_0) = -\vec{E}_{te+}(P_0)$ . As expected from Coulomb’s *theorem*, only a normal component  $\vec{E}_n(P_0)$  survives: The total field at  $P_0$  is normal to the conductor surface and directed into it (i.e., towards

$-q$ ). At  $P_0$ , the normal component of the field due to  $+q$  is given by

$$\begin{aligned}\vec{E}_{ze+}(P_0) &= \frac{1}{4\pi\epsilon_0} \frac{q}{\left[\left(\frac{d}{2}\right)^2 + r^2\right]^{1/2}} \cos\alpha \cdot \vec{u}_z \\ &= \frac{1}{4\pi\epsilon_0} \frac{q\left(\frac{d}{2}\right)}{\left[\left(\frac{d}{2}\right)^2 + r^2\right]^{3/2}} \cdot \vec{u}_z \quad ,\end{aligned}\quad (13.9)$$

where  $d$  is the distance between  $+q$  and  $-q$  and  $\alpha$  is the angle indicated in Fig. 13.4. By projecting this field onto  $\vec{n}$ , we obtain the magnitude

$$\begin{aligned}E_{ne+}(P_0) &= E_{ze+}(P_0) \cdot \vec{u}_z \cdot \vec{n} \\ &= E_{ze+}(P_0) \cdot \vec{u}_z \cdot (-\vec{u}_z) \\ &= -\frac{1}{4\pi\epsilon_0} \frac{q\left(\frac{d}{2}\right)}{\left[\left(\frac{d}{2}\right)^2 + r^2\right]^{3/2}} \quad .\end{aligned}\quad (13.10)$$

Similarly, the normal component at  $P_0$  due to  $-q$  is given by

$$\vec{E}_{ze-}(P_0) = \vec{E}_{ze+}(P_0)$$

and, thus,

$$E_{ne-}(P_0) = E_{ne+}(P_0) \quad . \quad (13.11)$$

Hence, the magnitude of the total normal component of the field at  $P_0$  is given by

$$E_n(P_0) = \frac{1}{4\pi\epsilon_0} \frac{qd}{\left[\left(\frac{d}{2}\right)^2 + r^2\right]^{3/2}} \quad (13.12)$$

(the total component is double each component).

Finally, from (13.7) we obtain

$$\begin{aligned}\sigma(P_0) &= \epsilon_0 E_n(P_0) \\ &= -\frac{qd}{4\pi \left[\left(\frac{d}{2}\right)^2 + r^2\right]^{3/2}} \quad .\end{aligned}\quad (13.13)$$

A good check to find if our calculation is correct is to integrate  $\sigma$  over the entire outer surface of the conductor. Proceeding in a similar fashion as for the case of the potential

of an infinite charged plane derived from that of a disk, we obtain

$$\begin{aligned}
 q|_{\Sigma_B} &= \iint_{\Sigma_B} \sigma \cdot dA \\
 &= -\frac{qd}{4\pi} \int_0^{2\pi} d\varphi \int_0^{+\infty} \frac{r \cdot dr}{\left[\left(\frac{d}{2}\right)^2 + r^2\right]^{3/2}} \\
 &= \frac{qd}{2} \left[ \frac{1}{\sqrt{\left(\frac{d}{2}\right)^2 + r^2}} \right]_0^{+\infty} \\
 &= -q \frac{d}{2} \frac{2}{d} = -q \quad ,
 \end{aligned} \tag{13.14}$$

which confirms the presence of an induced negative charge on the conductor.

The last open question is the force  $\vec{F}$  acting on  $+q$  due to the presence of the charged conducting plane. This can be calculated either by integration from the surface charge density (13.13) or, more simply as the force due to the image charge  $-q$ . We find

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{d^2} \cdot \vec{u}_z \quad . \tag{13.15}$$

As a last remark, we notice that the potential function (13.6) fulfills all conditions, a)-d). It can be shown that this is the unique solution to our Dirichlet problem.

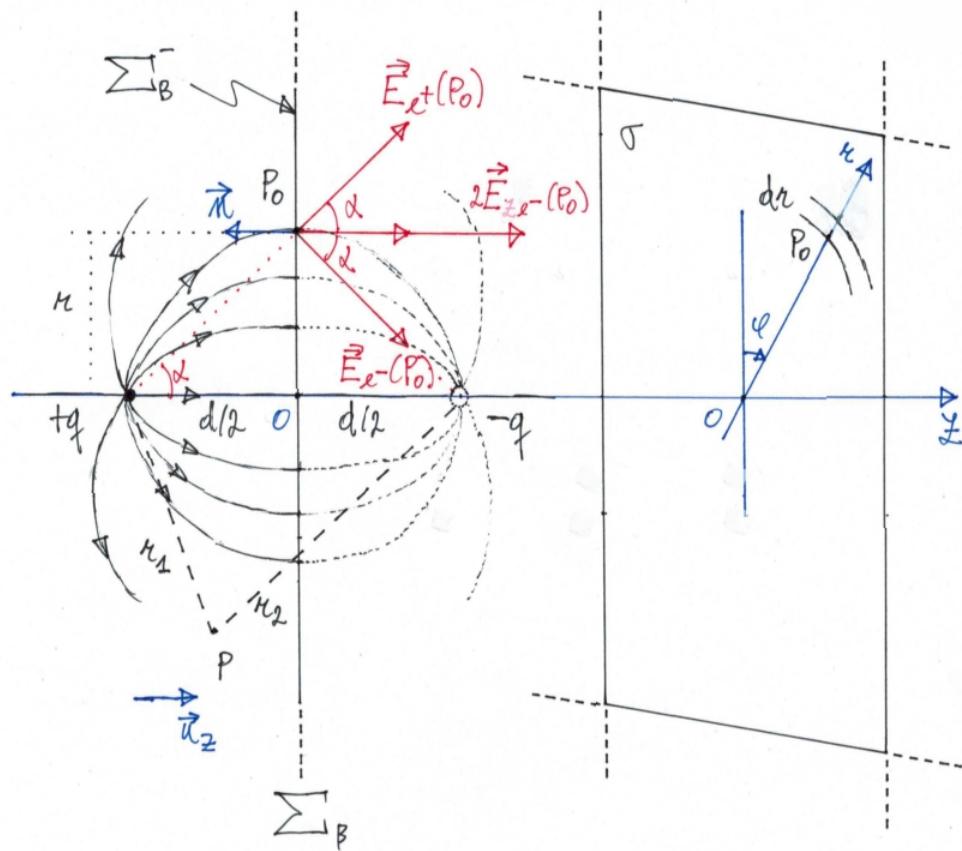


Figure 13.4



# Chapter 14

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We now present an alternative approach, based on Legendre polynomials, to derive the multipole series expansion of the electrostatic potential. In two chapters, we will consider the same problem in magnetostatics. **IMPORTANT:** The method based on Legendre polynomials gives exactly the same result as that based on Taylor series. Thus, you can go directly to Sec. 14.2, without loosing any information. We believe presenting the method based on Legendre polynomials will be useful later in your career and, thus, we present it here.

## 14.1 Multipole Series Expansion of the Electrostatic Potential for a Continuous Volume Charge Distribution

Consider a charge distribution with continuous and limited density  $\rho$  in a region  $\Omega$  of the three-dimensional Euclidean space (see Fig. 14.1).

The electrostatic potential at a point  $P$  outside  $\Omega$  is given by

$$\phi(P) = \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \frac{\rho(Q)}{r_{QP}} dV \quad (14.1)$$

where  $r_{QP}$  is the absolute value of the distance between  $P$  and the arbitrary infinitesimal volume element  $dV$  centered at  $Q$ .

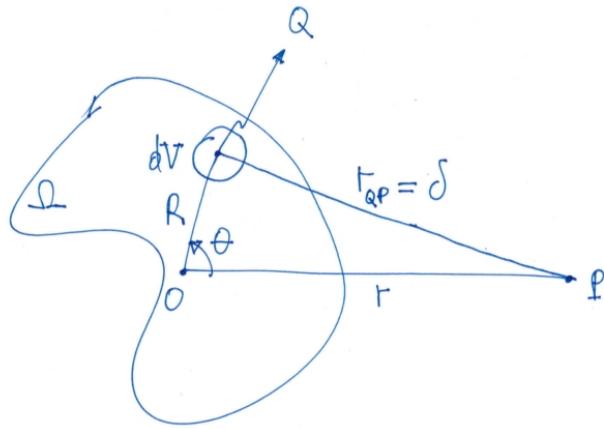


Figure 14.1

For simplicity, we can rename  $r_{QP} = \delta$  and define  $r$  to be the magnitude of the distance between  $O$  and  $P$ , where  $O$  is a point inside  $\Omega$  with  $O \neq Q$ , and  $R$  the magnitude of the distance between  $O$  and  $Q$ , where  $Q$  is also inside  $\Omega$ . By calling  $\theta = \hat{P} \hat{O} \hat{Q}$ , from the theorem of cosines

$$\delta^2 = r^2 + R^2 - 2rR \cos \theta \quad (14.2)$$

Since  $r \neq 0$  because  $P$  is outside  $\Omega$ , (14.2) can be rewritten as

$$\delta = r \sqrt{1 + \left(\frac{R}{r}\right)^2 - 2\left(\frac{R}{r}\right) \cos \theta} \quad (14.3)$$

By substituting

$$\begin{cases} \frac{R}{r} = u \\ \cos \theta = v \end{cases} \quad (14.4a)$$

$$(14.4b)$$

we have

$$\delta = r \sqrt{1 + u^2 - 2uv} \quad (14.5)$$

Using (14.5) in (14.1), we obtain

$$\phi(P) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \iiint_{\Omega} \frac{\rho}{\sqrt{1 + u^2 - 2uv}} dV \quad (14.6)$$

The integrand of (14.6) can be expanded by means of a power series:

$$\frac{1}{\sqrt{1 + u^2 - 2uv}} = \sum_{n=0}^{\infty} P_n(v) u^n \quad (14.7)$$

where  $P_n(v)$  is the Legendre polynomial of order  $n$ .

Note that Eq. (14.7) is one way to define the Legendre's polynomials. These can also be defined as the solutions of the ordinary differential equations with variable coefficients

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \quad (14.8)$$

The Legendre's polynomial can be calculated by means of the Rodriguez's identity

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (14.9)$$

Using this identity, the first few polynomials are

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2} (3x^2 - 1)$$

$$P_3(x) = \frac{1}{2} (5x^3 - 3x)$$

The series (14.7) converges uniformly in the range  $v \in (-1, +1)$ ; also,  $u < 1$  at each point if point  $P$  is very far from the region  $\Omega$  (in which case  $r \gg R$ ). Using (14.7) in (14.6), we have

$$\phi(P) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \iiint_{\Omega} \sum_{n=0}^{\infty} \rho P_n(v) u^n dV \quad (14.10)$$

The series in the integrand is uniformly convergent and the terms of the series are continuous functions. Thus, the sign of integral and that of series can be safely exchanged,

$$\phi(P) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \sum_{n=0}^{\infty} \iiint_{\Omega} \rho P_n(v) u^n dV \quad (14.11)$$

The first few terms of this series of integrals are

$$\phi_0 = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \iiint_{\Omega} \rho dV = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \quad (14.12a)$$

$$\phi_1 = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \iiint_{\Omega} \frac{\rho R \cos \theta}{r} dV \quad (14.12b)$$

$$\phi_2 = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \iiint_{\Omega} \frac{\rho R^2 (3 \cos^2 \theta - 1)}{2r^2} dV \quad (14.12c)$$

where we used the Legendre's polynomials  $P_0$ ,  $P_1$ , and  $P_2$ . In summary, we first found  $\phi(P)$ , we then showed how  $\phi$  depends on  $r$ , and, at last, we expanded in power series the part of the potential that does not depend on  $r$  (note that this is a legitimate procedure even if  $r$  appears as a scaling factor in the definition of  $u$ ).

#### 14.1. MULTIPOLE SERIES EXPANSION OF THE ELECTROSTATIC POTENTIAL FOR A CONTINUOUS VOLUME CHARGE DISTRIBUTION

The term  $\phi_0$  is the electrostatic potential at  $P$  as if the total charge  $q$  of  $\Omega$  was concentrated at  $O$ .

The term

$$\phi_1 = \frac{1}{4\pi\epsilon_0} \frac{P}{r^2} \quad (14.13)$$

is the term of electrostatic dipole, where

$$P = \iiint_{\Omega} \rho R \cos \theta dV \quad (14.14)$$

is the electrostatic dipole moment along the  $OP$  direction.

The next term

$$\phi_2 = \frac{1}{4\pi\epsilon_0} \frac{m}{r^3} \quad (14.15)$$

is the term of electrostatic quadrupole, where

$$m = \iiint_{\Omega} \frac{1}{2} R^2 (3 \cos^2 \theta - 1) \rho dV \quad (14.16)$$

is the electrostatic quadrupole moment.

In conclusion,

$$\phi(P) = \frac{1}{4\pi\epsilon_0} \left( \frac{q}{r} + \frac{p}{r^2} + \frac{m}{r^3} + \dots \right) \quad (14.17)$$

By defining the vectors  $\vec{r}$  and  $\vec{R}$ , another way to write the potential of (14.17) considering only the first two terms is

$$\phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \left[ \frac{1}{r} \iiint_{\Omega} \rho(\vec{R}) dV + \frac{\vec{r}}{r^3} \cdot \iiint_{\Omega} \rho(\vec{R}) \vec{R} dV \right] \quad (14.18)$$

where, again, we can define

$$\left\{ \begin{array}{l} q = \iiint_{\Omega} \rho(\vec{R}) dV \end{array} \right. \quad (14.19a)$$

$$\left\{ \begin{array}{l} \vec{p} = \iiint_{\Omega} \rho(\vec{R}) \vec{R} dV \end{array} \right. \quad (14.19b)$$

This form unveils the vector nature of  $\vec{p}$ , the dipole moment. Thus,

$$\phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \left( \frac{q}{r} + \frac{\vec{r} \cdot \vec{p}}{r^3} \right) \quad (14.20)$$

## 14.2 Ideal Electrostatic Dipole

In the case of two point-like charges  $+q$  and  $-q$  at a distance  $\vec{d}$  from each other, from (14.19a) and (14.19b)

$$\begin{cases} q &= \iiint_{\Omega} \left[ -q\delta(\vec{R}) + q\delta(\vec{R} - \vec{d}) \right] dV = 0 \end{cases} \quad (14.21a)$$

$$\begin{cases} \vec{p} &= \iiint_{\Omega} \left[ -q\delta(\vec{R})\vec{R} + q\delta(\vec{R} - \vec{d})\vec{R} \right] dV \\ &= -q\left[\vec{R}\right]_0 + q\left[\vec{R}\right]_{\vec{d}} = q\vec{d} \end{cases} \quad (14.21b)$$

From (14.20)

$$\phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \frac{\vec{r} \cdot \vec{p}}{r^3} \quad (14.22)$$

and

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \left[ \frac{3(\vec{r} \cdot \vec{p})\vec{r}}{r^5} - \frac{\vec{p}}{r^3} \right] \quad (14.23)$$

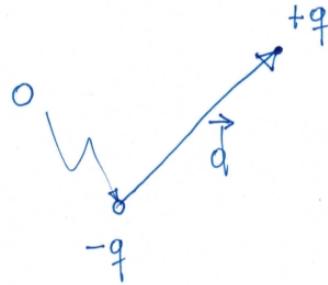


Figure 14.2

When  $d \rightarrow 0$  and  $qd = \text{const.}$ , the electrostatic dipole is said to be ideal. In this case, the sign of approximate in Eqs. (14.22) and (14.23) can be substituted by the sign of equal (see previous Chapter).

## 14.3 Interaction Between an Ideal Electrostatic Dipole and an Electrostatic Field

When an ideal electrostatic dipole is placed in an electrostatic field generated by a set of external charge distributions, the dipole is acted upon by mechanical actions. If we assume the two point-like charges in the dipole to be rigidly attached to each other, the system can be mechanically regarded as a rigid body. With respect to a given pivot, those mechanical actions can be completely described by a net force and torque.

### Net Force

Consider the ideal electrostatic dipole of Fig. 14.3.

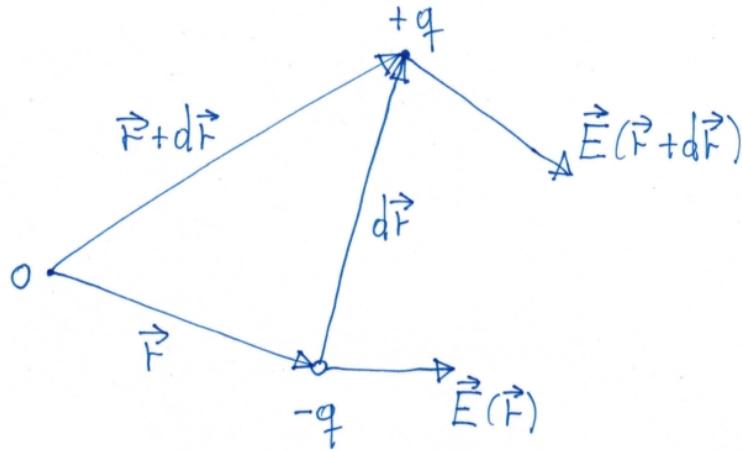


Figure 14.3

The distance between  $O$  and the point occupied by the negative charge  $(-q)$  is  $\vec{r}$ . The electrostatic field at that point is  $\vec{E}(\vec{r})$ . The field at  $(+q)$ , which is  $d\vec{r}$  apart from  $(-q)$  (ideal dipole condition), is

$$\vec{E}(\vec{r} + d\vec{r}) = \vec{E}(\vec{r}) + d\vec{E} \quad (14.24)$$

where  $d\vec{E}$  is the field variation corresponding to the infinitesimal displacement  $d\vec{r}$ .

With respect to a Cartesian coordinate system  $Oxyz$ ,

$$d\vec{r} = dx \vec{u}_x + dy \vec{u}_y + dz \vec{u}_z \quad (14.25)$$

and

$$d\vec{E} = dE_x \vec{u}_x + dE_y \vec{u}_y + dE_z \vec{u}_z \quad (14.26)$$

Thus, from simple vector calculus

$$dE_x = \frac{\partial}{\partial x} E_x dx + \frac{\partial}{\partial y} E_y dy + \frac{\partial}{\partial z} E_z dz \quad (14.27a)$$

$$dE_y = \frac{\partial}{\partial x} E_y dx + \frac{\partial}{\partial y} E_y dy + \frac{\partial}{\partial z} E_z dz \quad (14.27b)$$

$$dE_z = \frac{\partial}{\partial x} E_z dx + \frac{\partial}{\partial y} E_z dy + \frac{\partial}{\partial z} E_z dz \quad (14.27c)$$

which, in matrix form, can be written as

$$\begin{bmatrix} dE_x \\ dE_y \\ dE_z \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} E_x & \frac{\partial}{\partial y} E_x & \frac{\partial}{\partial z} E_x \\ \frac{\partial}{\partial x} E_y & \frac{\partial}{\partial y} E_y & \frac{\partial}{\partial z} E_y \\ \frac{\partial}{\partial x} E_z & \frac{\partial}{\partial y} E_z & \frac{\partial}{\partial z} E_z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \quad (14.28)$$

It is worth mentioning that the field studied here is that in absence of the dipole. In fact, since we assumed the dipole to be a rigid body, the reciprocal actions of the two charges in the dipole are ineffective from a mechanical point of view.

By defining (14.28) as the gradient of the vector field  $\vec{E}$  (this gradient is a second order tensor), we can write (14.24) as

$$d\vec{E} = (\vec{\nabla} \vec{E}) \cdot d\vec{r} \quad (14.29)$$

Calling the forces acting on the negative and positive charges  $\vec{F}_-$  and  $\vec{F}_+$ , respectively we have the net force

$$\begin{aligned} \vec{F} &= \vec{F}_- + \vec{F}_+ = -q\vec{E} + q(\vec{E} + d\vec{E}) \\ &= qd\vec{E} = qd\vec{r} \cdot \vec{\nabla} \vec{E} = \vec{p} \cdot \vec{\nabla} \vec{E} \end{aligned} \quad (14.30)$$

where  $\vec{p} = qd\vec{r}$  is the dipole moment.

### Net Torque

We choose  $O$  in Fig. 14.3 as pivot.

Thus, the torque with respect to  $O$  is

$$\begin{aligned} \vec{\tau}_0 &= \vec{r} \times \vec{F}_- + (\vec{r} + d\vec{r}) \times \vec{F}_+ \\ &= -\vec{r} \times \vec{E}_q + (\vec{r} + d\vec{r}) \times (\vec{E} + d\vec{E})q \\ &= qd\vec{r} \times \vec{E} + q\vec{r} \times d\vec{E} + qd\vec{r} \times d\vec{E} \\ &= \vec{p} \times \vec{E} + \vec{r} \times \vec{F} + \vec{p} \times d\vec{E} \end{aligned} \quad (14.31)$$

where  $\vec{F}$  is given by (14.30). The last term in (14.31) is  $\sim d\vec{r} \times d\vec{E}$ , i.e., is of higher order compared to the other terms. This term can thus be neglected. Hence,

$$\vec{\tau}_0 = \vec{r} \times \vec{F} + \vec{p} \times \vec{E} \quad (14.32)$$

When  $O$  is chosen to be at  $(-q)$ ,

$$\vec{\tau}'_0 = \vec{p} \times \vec{E} \quad (14.32')$$

In summary, the net force acting on an ideal electrostatic dipole is different from zero only for a non uniform electrostatic field in the neighborhood occupied by the dipole. In fact, for a uniform field  $d\vec{E} = \vec{0}$  and, from (14.30),  $\vec{F} = \vec{0}$ . For a uniform field, the system is fully described by (14.32'). In this case, if the dipole is free from any other mechanical action, it rotates until it aligns with the field, at which point  $\vec{\tau}'_0 = \vec{0}$ . There are two possible equilibrium conditions for which  $\vec{\tau}'_0 = \vec{0}$ . One when  $\vec{p}$  and  $\vec{E}$  are parallel with same sign and one when they are parallel with opposite sign (antiparallel). The first position is stable, i.e., for any small rotations the dipole goes back to its original position. The second position is unstable.

## 14.4 Field Dipole Interaction Energy

We want to calculate the interaction energy between an ideal electrostatic dipole and an external electrostatic field. The dipole is characterized by a moment  $\vec{p}$  and the field by a vector  $\vec{E}$ .

The interaction energy of interest is not the energy of a system of two point-like charges that form the dipole. That energy would be the work required to build the dipole starting from two charges at a very large distance. Instead, we want to calculate the energy for the interaction between an already built dipole and all the other charges that generate the field acting on it. The energy we are seeking corresponds to the work necessary to move an already built dipole from a configuration where the dipole does not interact with the field (e.g., because the charges generating the field are very far from the dipole) to a configuration where the dipole interacts with the field. We remind that the dipole is supposed to be a rigid body and, thus, the action due to the two charges are mechanically ineffective.

Following the nomenclature in 14.3, assume  $\phi$  is the electrostatic potential at the point occupied by  $(-q)$ . The potential at the point occupied by  $(+q)$  is, then,

$$\tilde{\phi} = \phi(\vec{r} + \vec{dr}) = \phi(\vec{r}) + d\phi \quad (14.33)$$

where  $d\phi$  is the potential variation corresponding to the infinitesimal displacement  $\vec{dr}$ .

From (14.33) and the definition of directional derivative, it follows that

$$\phi(\vec{r}) + d\phi = \phi(\vec{r}) + \text{grad } \phi(\vec{r}) \cdot \vec{dr} \quad (14.34)$$

Thus, the electrostatic interaction energy of dipole in the field is given by

$$\begin{aligned} U_I &= U_- + U_+ \\ &= -q\phi + q(\phi + d\phi) \\ &= q\vec{dr} \cdot \text{grad } \phi \\ &= -\vec{p} \cdot \vec{E} \end{aligned} \quad (14.35)$$

Note that  $U_-$  is not the energy of a system of two charges. In fact,  $U_-$  and  $U_+$  are the electrostatic potential energies of a single (negative or positive, respectively) charge in the potential generated by some other charge distribution (this is why there is no factor  $1/2$  in (14.35)).

# Chapter 15

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We will now consider the important case of  $N$  dipoles in space.

## 15.1 Distribution of Dipoles

Thanks to the superposition principle, the electrostatic potential and field for a system of  $N$  dipoles is given by the sum of the potential and field of each dipole.

As for electric charges, it is also possible to consider continuous distributions of dipoles. To this end, consider a region of space with volume  $\Delta V$  (assumed to be very small compared to the typical scale of the experiments, but still finite). Suppose the region contains a very large number  $N$  of quasi-ideal dipoles. In this case, quasi-ideal means the characteristic distance  $d$  of the dipoles is much smaller than  $\sqrt[3]{\Delta V}$ . The total dipole moment  $\Delta \vec{\Pi}$  in  $\Delta V$  is given by the vector sum of the dipole moments  $\vec{p}_i$  of each quasi-ideal dipole,

$$\Delta \vec{\Pi} = \sum_{i=1}^N \vec{p}_i \quad (15.1)$$

Suppose all quasi-ideal dipoles are identical and oriented in the same way, i.e.,  $\vec{p}_i = \vec{p}$ . Thus,

$$\Delta \vec{\Pi} = N \vec{p} \quad (15.2)$$

Introducing the dipole moment per unit volume,

$$\vec{P} = \frac{\Delta \vec{\Pi}}{\Delta V} \quad (15.3)$$

we readily obtain

$$\vec{P} = n \vec{p} \quad (15.4)$$

where  $n$  is the number of dipoles per unit volume.

The definition of  $\vec{P}$  can be easily extended to the case where the dipoles are not identical and have different orientations,

$$\vec{P} = \frac{1}{\Delta V} \sum_{i=1}^N \vec{p}_i \quad (15.5)$$

For electric charges, a charge distribution is characterized by a scalar function  $\rho(Q)$ , which represents the charge per unit volume at a generic point  $Q$ . Similarly, a dipole distribution is characterized by a vector function  $\vec{P}(Q)$  that gives the dipole moment per unit volume at  $Q$ .

We now want to calculate the electrostatic potential and field due to a dipole distribution. The latter can be regarded as a dipole continuum with a vector density  $\vec{P}(Q)$ . Consider a region of space  $\Omega$  with a continuous dipole distribution with vector density  $\vec{P}(Q)$ . We then consider a small volume  $\Delta V$  centered at a generic point  $Q'$  in  $\Omega$  and substitute the set of dipoles in  $\Delta V$  with a single effective dipole with moment

$$\Delta \vec{\Pi} = \vec{P}(Q') \Delta V \quad (15.6)$$

From

$$\phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \frac{\vec{r} \cdot \vec{p}}{r^3} \quad (15.7)$$

we can find the potential at point  $Q$  due to the effective dipole,

$$\Delta\phi \approx \frac{1}{4\pi\epsilon_0} \frac{\Delta \vec{\Pi} \cdot (\vec{r} - \vec{r}')}{\|\vec{r} - \vec{r}'\|^3} \quad (15.8)$$

By repeating the same argument for each volume  $\Delta V$  in which the region  $\Omega$  can be divided and superimposing the effects, we obtain the potential  $\phi$  at  $Q$  for the entire dipole distribution,

$$\phi(\vec{r}) = \phi(Q) \approx \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{\|\vec{r} - \vec{r}'\|^3} dV' \quad (15.9)$$

Note that the approximate sign is because the dipoles are supposed to be quasi ideal.

The field can be calculated as  $\vec{E}(\vec{r}) = -\text{grad } \phi(\vec{r})$ .

## 15.2 Charge Distributions Equivalent to Dipole Distributions

We want to show that a continuous dipole distribution in a given region of space is equivalent to a suitable charge distribution in the same region, as far as potential and field are concerned.

From simple vector calculus,

$$\text{grad } \frac{1}{\|\vec{r} - \vec{r}'\|} = \text{grad } \frac{1}{\|\vec{r}\|} = \text{grad } \frac{1}{\rho} \quad (15.10)$$

where  $\|\rho\| = \rho > 0$ . In spherical coordinates,

$$\vec{\rho} = \vec{r} - \vec{r}' = \rho \vec{u}_r \quad (15.11)$$

and, thus,

$$\begin{aligned}\operatorname{grad} \frac{1}{\rho} &= -\frac{1}{\rho^2} \vec{u}_r = -\frac{1}{\|\vec{\rho}\|^2} \frac{\vec{\rho}}{\|\vec{\rho}\|} \\ &= \frac{\vec{r} - \vec{r}'}{\|\vec{r} - \vec{r}'\|^3}\end{aligned}\quad (15.12)$$

it follows that

$$\frac{\vec{r} - \vec{r}'}{\|\vec{r} - \vec{r}'\|^3} = -\operatorname{grad} \frac{1}{\|\vec{r} - \vec{r}'\|} \quad (15.13)$$

The operator  $\operatorname{grad}$  acts on the variable  $\vec{r}$ . Indicating with  $\operatorname{grad}'$  the operator acting on the variable  $\vec{r}'$ , we have

$$\frac{\vec{r} - \vec{r}'}{\|\vec{r} - \vec{r}'\|^3} = +\operatorname{grad}' \frac{1}{\|\vec{r} - \vec{r}'\|} \quad (15.14)$$

By substituting (15.14) into (15.9), we find

$$\phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \vec{P}(\vec{r}') \cdot \operatorname{grad}' \frac{1}{\|\vec{r} - \vec{r}'\|} dV' \quad (15.15)$$

From vector calculus,

$$\vec{P}(\vec{r}') \cdot \operatorname{grad}' \frac{1}{\|\vec{r} - \vec{r}'\|} = \operatorname{div}' \cdot \left[ \frac{\vec{P}(\vec{r}')}{\|\vec{r} - \vec{r}'\|} \right] - \frac{1}{\|\vec{r} - \vec{r}'\|} \operatorname{div}' \vec{P}(\vec{r}') \quad (15.16)$$

Finally,

$$\begin{aligned}\phi(\vec{r}) &\approx \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \operatorname{div}' \cdot \left[ \frac{\vec{P}(\vec{r}')}{\|\vec{r} - \vec{r}'\|} \right] dV' \\ &\quad - \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \frac{1}{\|\vec{r} - \vec{r}'\|} \operatorname{div}' \vec{P}(\vec{r}') dV'\end{aligned}\quad (15.17)$$

Using the divergence theorem for the first integral, we have

$$\begin{aligned}\phi(\vec{r}) &\approx \frac{1}{4\pi\epsilon_0} \iint_{\Sigma} \frac{\vec{P}(\vec{r}') \cdot \vec{n}}{\|\vec{r} - \vec{r}'\|} dA \\ &\quad - \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \frac{\operatorname{div}'[\vec{P}(\vec{r}')]}{\|\vec{r} - \vec{r}'\|} dV',\end{aligned}\quad (15.18)$$

where  $\Sigma$  is the closed surface associated with  $\Omega$  and  $\vec{n}$  its normal unit vector.

The two integrals in (15.18) correspond to the potentials due to a surface charge distribution on  $\Sigma$  with density

$$\sigma_{eq} = \vec{P} \cdot \vec{n} = P_n \quad (15.19)$$

and a volume charge distribution in  $\Omega$  with density

$$\sigma_{eq} = -\operatorname{div}' \vec{P} \quad (15.20)$$

Given a closed surface  $\Sigma$  that contains the entire distribution of dipoles, the flux of  $\vec{E}$  through  $\Sigma$  is zero. This can be directly deduced from Gauss' theorem because the sum of the opposite charges in each dipole is zero and, thus, the sum of all charges for the dipoles in the entire distribution is zero. This conclusion is consistent with (15.19) and (15.20). In fact, the total charge for the distribution of dipoles in  $\Omega$  is given by

$$\begin{aligned} q_{tot} &= \iiint_{\Omega} \rho_{eq} dV + \iint_{\Sigma} \sigma_{eq} dA \\ &= \iiint_{\Omega} (-\operatorname{div}' \cdot \vec{P}) dV + \iint_{\Sigma} \vec{P} \cdot \vec{n} dA \end{aligned} \quad (15.21)$$

Using the divergence theorem for the volume integral, we have

$$q_{tot} = - \iint_{\Sigma} \vec{P} \cdot \vec{n} dA + \iint_{\Sigma} \vec{P} \cdot \vec{n} dA = 0 \quad (15.22)$$

In this case,  $\Sigma$  contains  $\Omega$  completely. If, instead, we were to consider a surface  $\Sigma'$  that cuts some of the dipoles, as shown in Fig. 15.1, the sum of all charges within  $\Sigma'$  would be different from zero and so would be the flux of  $\vec{E}$  through it. This case explains how, given a set of dipoles (each of which has zero total charge), it is possible to find an equivalent charge distribution with a nonzero local density.

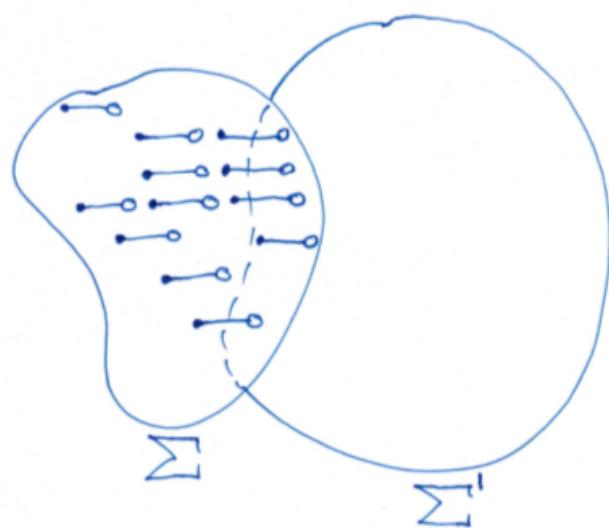


Figure 13.1.



Figure 15.1



# Chapter 16

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We will now turn to the case of the multipole expansion for the magnetostatic field.

## 16.1 Multipole Series Expansion of the Magnetostatic Vector Potential for a Continuous and Limited Volume Electric Current Distribution

Using the same nomenclature as in Fig. 14.1, consider a limited region of space  $\Omega$  where is defined a continuous volume electric current distribution with density  $\vec{J}$ . We want to study the effects due to the current at a very large distance from it.

The magnetostatic vector potential  $\vec{A}$  at point  $P$  due to  $\vec{J}$  is

$$\vec{A}(P) = \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint_{\Omega} \frac{\vec{J}(\vec{R})}{\|\vec{r} - \vec{R}\|} dV \quad (16.1)$$

Considering the points  $P$  such that

$$\frac{\vec{R}}{\vec{r}} \ll 1 \quad (16.2)$$

we can use the same approximation that led to Eq. 14.18. By substituting  $\phi(\vec{r})$  with  $A(\vec{r})$ ,  $1/4\pi\epsilon_0$  with  $\mu_0/4\pi$ , and  $\rho(\vec{R})$  with  $\vec{J}(\vec{R})$ , we find

$$\vec{A}(\vec{r}) \approx \frac{\mu_0}{4\pi} \iiint_{\Omega} \vec{J}(\vec{R}) \left( \frac{1}{r} + \frac{\vec{R} \cdot \vec{r}}{r^3} \right) dV \quad (16.3)$$

By defining

$$\vec{A}_0(\vec{r}) = \frac{\mu_0}{4\pi} \iiint_{\Omega} \vec{J}(\vec{R}) \frac{1}{r} dV \quad (16.4a)$$

$$\vec{A}_1(\vec{r}) = \frac{\mu_0}{4\pi} \iiint_{\Omega} \vec{J}(\vec{R}) \frac{\vec{R} \cdot \vec{r}}{r^3} dV \quad (16.4b)$$

we can write

$$\vec{A}(\vec{r}) \approx \vec{A}_0(\vec{r}) + \vec{A}_1(\vec{r}) \quad (16.5)$$

### 16.1. MULTIPOLE SERIES EXPANSION OF THE MAGNETOSTATIC VECTOR POTENTIAL FOR A CONTINUOUS AND LIMITED VOLUME ELECTRIC CURRENT DISTRIBUTION

The first term in (16.5) is called the term of magnetostatic monopole and the second term of magnetostatic dipole.

In the magnetostatic case, the term of monopole is always zero. We can demonstrate this as follows. Since the integral (16.4a) is calculated with respect to  $\vec{R}$ , we can take  $r$  outside the sign of integral,

$$\vec{A}_0(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r} \iiint_{\Omega} \vec{J}(\vec{R}) dV \quad (16.6)$$

In order to calculate this integral, we divide the entire region  $\Omega$  into many infinitesimal (closed) circuits, each of which is realized as a flux tube of  $\vec{J}$ . Thus, the integral becomes the vector sum of the contributions from each flux tube. Consider the generic infinitesimal contribution due to a flux tube. Suppose the closed and oriented line  $\gamma$  is the main axis of the flux tube under consideration and  $dA$  the area, generally variable along  $\gamma$ , of the normal cross section of the tube. We have

$$\begin{aligned} d\vec{A}_0 &= \frac{\mu_0}{4\pi} \frac{1}{r} \oint_{\gamma} \vec{J}(\vec{R}) dA \, dl \\ &= \frac{\mu_0}{4\pi} \frac{1}{r} \oint_{\gamma} J(\vec{R}) dA \, \vec{t} \, dl \end{aligned} \quad (16.7)$$

where  $\vec{t}$  is the tangent unit vector at each point on  $\gamma$ . We assume  $\vec{t}$  to have the same direction and sign of  $\vec{J}$  at each point on  $\gamma$ . Note that all vectors  $\vec{t} dl$  are consecutive along  $\gamma$ . Altogether, they form a closed polygon and, hence,

$$\oint_{\gamma} \vec{t} dl = 0 \quad (16.8)$$

It is easy to be convinced that (16.8) is correct: This is nothing but the vector sum of a continuous series of very short adjacent vectors, from one point to the very same point. Then, because of the solenoidal property of  $\vec{J}$ , the infinitesimal current  $J(\vec{R}) dA$  must be constant along  $\gamma$ . Thus,

$$d\vec{A}_0 = \frac{\mu_0}{4\pi} \frac{1}{r} J(\vec{R}) dA \oint_{\gamma} \vec{t} dl = 0 \quad (16.9)$$

where we used the property (16.8). Since the contribution from a single infinitesimal flux tube is zero, it must be

$$\iiint_{\Omega} \vec{J}(\vec{R}) dV = 0 \quad (16.10)$$

from which it follows that

$$\vec{A}_0(\vec{r}) = 0 \quad (16.11)$$

This important result is one incarnation of a fundamental principle of magnetism, that is the absence of so called magnetic monopoles. This statement must be intended

as follows: So far, there has been no empirical evidence of the existance of magnetic monopoles (which, theoretically, could exist). Consider a magnet with two poles, North and South (we will come back to this concept when discussing magnetic materials). After breaking the magnet into two pieces, each sub-magnet will still have both a North and a South pole. This means that it is impossible to find a nonzero “magnetic charge,” i.e., a piece of matter with only a North or a South pole (those to be assumed as net quantities in an algebraic sense). The amount of North and South poles always balances to zero. The term of magnetic monopole is equivalent to the term of electric monopole, which represents the total electric charge of the system [see Eq. (14.19a)]. The term of magnetic monopole is, thus, the total magnetic charge of the system, which must be zero.

Consider now the term of magnetic dipole in (16.5). From simple vector calculus,

$$\begin{aligned}\frac{\vec{R} \cdot \vec{r}}{r^3} &= \vec{R} \cdot \frac{\vec{r}}{r} \frac{1}{r^2} = \vec{R} \cdot \vec{u}_r \frac{1}{r^2} \\ &= -\vec{R} \cdot \text{grad} \frac{1}{r}\end{aligned}\quad (16.12)$$

Thus,

$$\begin{aligned}\vec{A}_1(\vec{r}) &= \frac{\mu_0}{4\pi} \iiint_{\Omega} \vec{J}(\vec{R}) \frac{\vec{R} \cdot \vec{r}}{r^3} dV \\ &= -\frac{\mu_0}{4\pi} \iiint_{\Omega} \vec{J}(\vec{R}) \left( \vec{R} \cdot \text{grad} \frac{1}{r} \right) dV\end{aligned}\quad (16.13)$$

By writing

$$\begin{aligned}\vec{J}(\vec{R}) \left( \vec{R} \cdot \text{grad} \frac{1}{r} \right) &= \frac{1}{2} \vec{J}(\vec{R}) \left( \vec{R} \cdot \text{grad} \frac{1}{r} \right) + \frac{1}{2} \vec{J}(\vec{R}) \left( \vec{R} \cdot \text{grad} \frac{1}{r} \right) \\ &\quad - \frac{1}{2} \vec{R} \left[ \vec{J}(\vec{R}) \cdot \text{grad} \frac{1}{r} \right] + \frac{1}{2} \vec{R} \left[ \vec{J}(\vec{R}) \cdot \text{grad} \frac{1}{r} \right]\end{aligned}\quad (16.14)$$

we have

$$\begin{aligned}\vec{A}_1(\vec{r}) &= -\frac{\mu_0}{4\pi} \iiint_{\Omega} \frac{1}{2} \left\{ \vec{J}(\vec{R}) \left( \vec{R} \cdot \text{grad} \frac{1}{r} \right) - \vec{R} \left[ \vec{J}(\vec{R}) \cdot \text{grad} \frac{1}{r} \right] \right\} dV \\ &\quad - \frac{\mu_0}{4\pi} \iiint_{\Omega} \frac{1}{2} \left\{ \vec{J}(\vec{R}) \left( \vec{R} \cdot \text{grad} \frac{1}{r} \right) + \vec{R} \left[ \vec{J}(\vec{R}) \cdot \text{grad} \frac{1}{r} \right] \right\} dV\end{aligned}\quad (16.15)$$

The second integral on the right-hand side of (16.15) is zero. In fact, by dividing once again the entire current distribution into infinitesimal flux tubes (i.e., the region  $\Omega$  is divided into an infinite set of flux tubes), we have

$$\vec{J}(\vec{R}) dV = J(\vec{R}) dA \vec{t} dl = dI \vec{t} dl \quad (16.16)$$

### 16.1. MULTIPOLE SERIES EXPANSION OF THE MAGNETOSTATIC VECTOR POTENTIAL FOR A CONTINUOUS AND LIMITED VOLUME ELECTRIC CURRENT DISTRIBUTION

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where  $dI$  is the current carried by a generic infinitesimal flux tube (as always, under steady conditions). The contribution from such a flux tube of axis  $\gamma$  to the second integral in (16.15) is

$$d\vec{A}_1^* = -\frac{\mu_0}{4\pi} dI \oint_{\gamma} \frac{1}{2} \left[ \vec{t} \left( \vec{R} \cdot \text{grad} \frac{1}{r} \right) + \vec{R} \left( \vec{t} \cdot \text{grad} \frac{1}{r} \right) \right] dl \quad (16.17)$$

Since  $\vec{t}dl = d\vec{R}$ , we then have

$$d\vec{A}_1^* = -\frac{\mu_0}{4\pi} dI \oint_{\gamma} \frac{1}{2} \left[ d\vec{R} \left( \vec{R} \cdot \text{grad} \frac{1}{r} \right) + \vec{R} \left( d\vec{R} \cdot \text{grad} \frac{1}{r} \right) \right] \quad (16.18)$$

Note that  $d\vec{R}$  is an infinitesimal at any point inside  $\Omega$ , which has been divided into infinite  $\vec{t}dl$  elements. Moreover,

$$d\vec{R} \left( \vec{R} \cdot \text{grad} \frac{1}{r} \right) + \vec{R} \left( d\vec{R} \cdot \text{grad} \frac{1}{r} \right) = d' \left[ \vec{R} \left( \vec{R} \cdot \text{grad} \frac{1}{r} \right) \right] \quad (16.19)$$

where  $d'$  indicates the operation of differentiation with respect to the variable  $\vec{R}$ . This demonstrates that vector  $d\vec{A}_1^*$  is the line integral along  $\gamma$  of an exact differential. Since  $\gamma$  is a closed line  $d\vec{A}_1^*$  must be zero. This result can be extended to each flux tube in  $\Omega$ . Therefore, the second integral in (16.15) must be zero and we have

$$\vec{A}_1(\vec{r}) = -\frac{\mu_0}{4\pi} \iiint_{\Omega} \frac{1}{2} \left\{ \vec{J}(\vec{R}) \left( \vec{R} \cdot \text{grad} \frac{1}{r} \right) - \vec{R} \left[ \vec{J}(\vec{R}) \cdot \text{grad} \frac{1}{r} \right] \right\} dV \quad (16.20)$$

From the vector identity

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c} \quad (16.21)$$

which is valid for each term of vectors  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$ , we have

$$\begin{aligned} \vec{J}(\vec{R}) \left( \vec{R} \cdot \text{grad} \frac{1}{r} \right) - \vec{R} \left[ \vec{J}(\vec{R}) \cdot \text{grad} \frac{1}{r} \right] &= \vec{R} \times \left[ \vec{J}(\vec{R}) \times \text{grad} \frac{1}{r} \right] \\ &= \left[ \vec{R} \times \vec{J}(\vec{R}) \right] \times \text{grad} \frac{1}{r} \end{aligned} \quad (16.22)$$

and, thus,

$$\vec{A}_1(\vec{r}) = -\frac{\mu_0}{4\pi} \frac{1}{2} \iiint_{\Omega} \left\{ [\vec{R} \times \vec{J}(\vec{R})] \times \text{grad} \frac{1}{r} \right\} dV \quad (16.23)$$

Since the vector  $\text{grad}(1/r)$  is independent from the variable of integration, we find

$$\vec{A}_1(\vec{r}) = -\frac{\mu_0}{4\pi} \left\{ \frac{1}{2} \iiint_{\Omega} [\vec{R} \times \vec{J}(\vec{R})] dV \right\} \times \text{grad} \frac{1}{r} \quad (16.24)$$

The vector

$$\vec{m} = \frac{1}{2} \iiint_{\Omega} [\vec{R} \times \vec{J}(\vec{R})] dV \quad (16.25)$$

in the integral (16.24) is called magnetostatic (or simply magnetic) dipole moment for  $\vec{J}$  with respect to  $O$  (in Fig. 14.1). Using (16.25) in (16.24) we have

$$\vec{A}_1(\vec{r}) = -\frac{\mu_0}{4\pi} \vec{m} \times \text{grad} \frac{1}{r} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} \quad (16.26)$$

The vector  $\vec{m}$  defined by (16.25) is independent from the point  $O$  from which the vectors  $\vec{R}$  originate. In fact, given a new origin  $O'$  and calling  $\rho'$  the distances from it, it follows that

$$\vec{\rho}' = \vec{R} + \vec{r}_{O'O} \quad (16.27)$$

where  $\vec{r}_{O'O}$  is the vector between points  $O'$  and  $O$  (directed from  $O'$  to  $O$ ). By substituting (16.27) in (16.25), we obtain

$$\begin{aligned} \vec{m}' &= \frac{1}{2} \iiint_{\Omega} \vec{\rho}' \times \vec{J}(\vec{\rho}') dV = \frac{1}{2} \iiint_{\Omega} (\vec{R} + \vec{r}_{O'O}) \times \vec{J}(\vec{\rho}') dV \\ &= \frac{1}{2} \iiint_{\Omega} \vec{R} \times \vec{J}(\vec{\rho}') dV + \frac{1}{2} \iiint_{\Omega} \vec{r}_{O'O} \times \vec{J}(\vec{\rho}') dV \end{aligned} \quad (16.28)$$

The vector  $\vec{r}_{O'O}$  is independent from the variable of integration. Hence,

$$\vec{m}' = \frac{1}{2} \iiint_{\Omega} \vec{R} \times \vec{J}(\vec{\rho}') dV + \frac{1}{2} \vec{r}_{O'O} \times \iiint_{\Omega} \vec{J}(\vec{\rho}') dV \quad (16.29)$$

From (16.10), we then find

$$\vec{m}' = \frac{1}{2} \iiint_{\Omega} \vec{R} \times \vec{J}(\vec{\rho}') dV \quad (16.30)$$

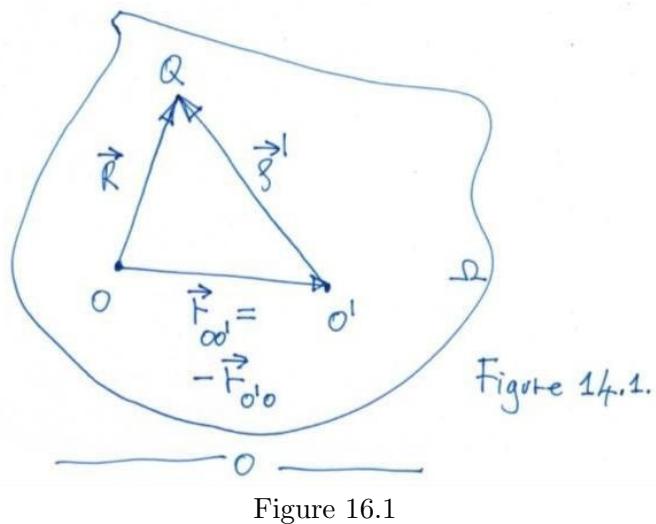
If  $\vec{\rho}'$  indicates point  $Q$  in  $\Omega$  with respect to  $O'$  and  $\vec{R}$  indicates the same point  $Q$  with respect to  $O$  (see Fig. 16.1), from (16.30) it follows that

$$\vec{m}' = \vec{m} \quad (16.31)$$

In summary, in the electrostatic case the dipole moment for a given charge distribution is independent from the chosen origin only when the total charge is zero. In the magnetostatic case this is always true because of (16.10).

16.1. MULTIPOLE SERIES EXPANSION OF THE MAGNETOSTATIC VECTOR POTENTIAL FOR A CONTINUOUS AND LIMITED VOLUME ELECTRIC CURRENT DISTRIBUTION

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# Chapter 17

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## 17.1 The Ideal Magnetic Dipole

An important case of what studied in the previous chapter is found when the electric current distribution becomes a steady current on a conductive filiform wire forming a loop  $\gamma$ .

Suppose the loop resides entirely on a plane, as shown in Fig. 17.1. The magnetic dipole moment, which is indicated as  $\vec{\mu}$  in this case, becomes particularly simple. In fact,

$$\vec{J}dV = \vec{J}A_C d\ell = JA_C \vec{t} d\ell = I \vec{t} d\ell \quad (17.1)$$

where  $A_C$  is the area of the normal cross section of the conductor and  $I$  is the current intensity carried by it. Using the notation in Fig. 17.1,  $\vec{t}$  is the tangent unit vector to  $\gamma$  at the mid point of an infinitesimal element  $d\ell$ ;  $\vec{R}$  is the vector connecting the center  $O$  of the loop to the same mid point and  $\vec{n}$  is the normal unit vector of the surface having  $\gamma$  as border, on the plane of  $\gamma$ , oriented according to the clockwise direction chosen on  $\gamma$ .

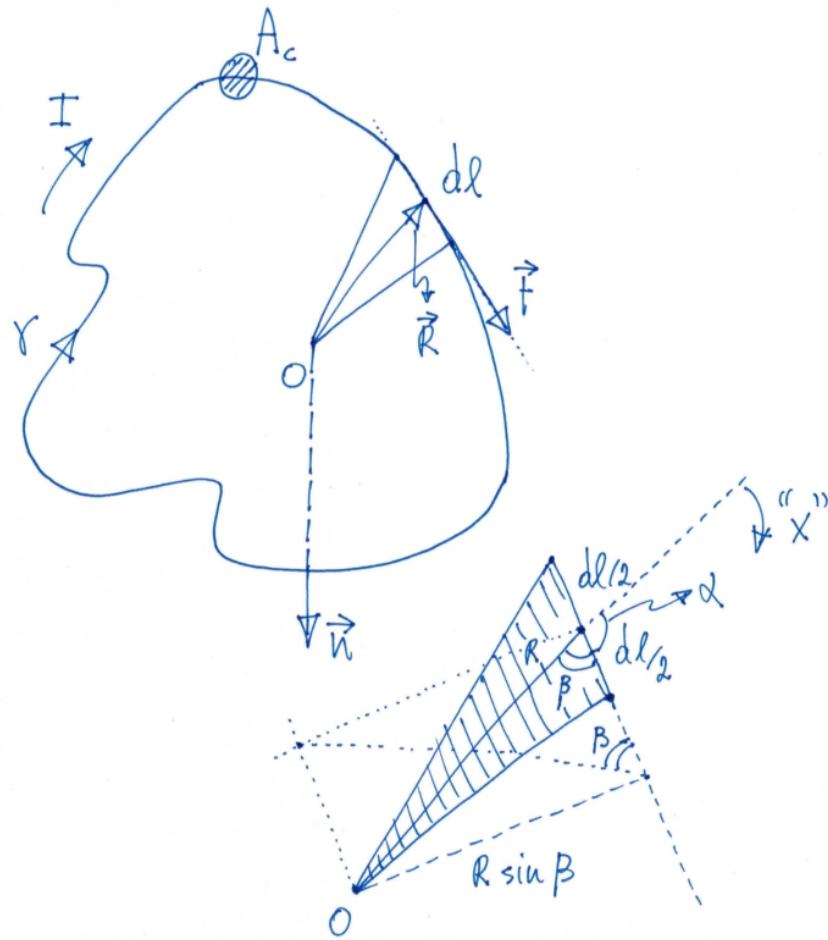


Figure 17.1

By substituting (17.1) into (16.25), we immediately find

$$\vec{\mu} = I \oint_{\gamma} \frac{1}{2} \vec{R} \times \vec{t} d\ell \quad (17.2)$$

From Fig. 17.1, it follows that

$$\vec{R} \times \vec{t} d\ell = R d\ell \sin \alpha \vec{n} = (R \sin \alpha) d\ell \vec{n} \quad (17.3)$$

From the same figure, the area of the hashed triangle is given by

$$\begin{aligned} dA_t &= \frac{1}{2} d\ell R \sin \beta = \frac{1}{2} d\ell R \sin(\pi - \alpha) \\ &= \frac{1}{2} (R \sin \alpha) d\ell \end{aligned} \quad (17.4)$$

Thus,

$$\vec{\mu} = I \oint_{\gamma} dA_t \vec{n} = IA_{\ell} \vec{n} \quad (17.5)$$

where  $A_\ell$  is the area of the planar region bordered by  $\gamma$ .

As in the electrostatic case, also in magnetostatics we can define an ideal magnetic dipole. This is defined in the limit  $A_\ell \rightarrow 0^+$  while, at the same time,  $I \rightarrow +\infty$  so that  $IA_\ell = \text{const} = \mu$ .

The magnetic field due to an ideal magnetic dipole can readily be calculated from (16.26) and the definition of vector potential in magnetostatics. In fact, for an ideal magnetic dipole, the vector potential is

$$A_\mu(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{\mu} \times \vec{r}}{r^3} \quad (17.6)$$

This expression can be considered to be exact because, in the ideal case, the current distribution of the dipole is concentrated in one point and  $r$  is infinitely larger than  $R$ , which is actually zero.

From (17.6),

$$\vec{B}_\mu = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left( \frac{\mu_0}{4\pi} \frac{\vec{\mu} \times \vec{r}}{r^3} \right) \quad (17.7)$$

Since  $\vec{r}/r^3 = -\vec{\nabla}(1/r)$ , we have

$$\vec{B}_\mu = -\frac{\mu_0}{4\pi} \vec{\nabla} \times (\vec{\mu} \times \vec{\nabla} \frac{1}{r}) \quad (17.8)$$

From vector calculus

$$\vec{\nabla} \times (U \vec{A}) = U \vec{\nabla} \times \vec{A} + (\vec{\nabla} U) \times \vec{A} \quad (17.9)$$

and, thus,

$$\vec{\mu} \times \vec{\nabla} \frac{1}{r} = -\vec{\nabla} \times \left( \frac{\vec{\mu}}{r} \right) + \frac{1}{r} \vec{\nabla} \times \vec{\mu} \quad (17.10)$$

from which, being  $\mu = \text{const}$ , it follows that

$$\vec{\mu} \times \vec{\nabla} \frac{1}{r} = -\vec{\nabla} \times \frac{\vec{\mu}}{r} \quad (17.11)$$

Equation (17.8) thus becomes

$$\vec{B}_\mu = \frac{\mu_0}{4\pi} \vec{\nabla} \times \left( \vec{\nabla} \times \frac{\vec{\mu}}{r} \right) \quad (17.12)$$

From vector calculus, for any vector field  $\vec{A}$ ,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} \quad (17.13)$$

we find

$$\vec{B}_\mu = \frac{\mu_0}{4\pi} \left[ \vec{\nabla} \left( \vec{\nabla} \cdot \frac{\vec{\mu}}{r} \right) - \vec{\nabla}^2 \frac{\vec{\mu}}{r} \right] \quad (17.14)$$

By writing term  $\vec{\nabla}^2(\vec{\mu}/r)$  in Cartesian coordinates, we have

$$\left( \vec{\nabla}^2 \frac{\vec{\mu}}{r} \right)_x = \vec{\nabla}^2 \frac{\mu_x}{r}, \quad \left( \vec{\nabla}^2 \frac{\vec{\mu}}{r} \right)_y = \vec{\nabla}^2 \frac{\mu_y}{r}, \quad \left( \vec{\nabla}^2 \frac{\vec{\mu}}{r} \right)_z = \vec{\nabla}^2 \frac{\mu_z}{r} \quad (17.15)$$

Since  $\mu = \text{const}$

$$\left(\vec{\nabla}^2 \frac{\vec{\mu}}{r}\right)_x = \mu_x \vec{\nabla}^2 \frac{1}{r}, \quad \left(\vec{\nabla}^2 \frac{\vec{\mu}}{r}\right)_y = \mu_y \vec{\nabla}^2 \frac{1}{r}, \quad \left(\vec{\nabla}^2 \frac{\vec{\mu}}{r}\right)_z = \mu_z \vec{\nabla}^2 \frac{1}{r}, \quad (17.16)$$

Since  $r \neq 0$ , in spherical coordinates,

$$\begin{aligned} \vec{\nabla}^2 \frac{1}{r} &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \frac{1}{r} \right) \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( -\frac{1}{r^2} \right) = 0 \end{aligned} \quad (17.17)$$

Thus,

$$\vec{B}_\mu(\vec{r}) = \frac{\mu_0}{4\pi} \vec{\nabla} \left( \vec{\nabla} \cdot \frac{\vec{\mu}}{r} \right) \quad (17.18)$$

Equation (17.18) gives  $\vec{B}_\mu$  at all points in space, except for the point where the dipole is located, where the magnetic field is not defined. This result also shows that the magnetic field due to an ideal magnetic dipole can be written as the gradient of a suitable scalar function  $\psi$ , given by

$$\psi = \psi(\vec{r}) = \frac{\mu_0}{4\pi} \vec{\nabla} \cdot \frac{\vec{\mu}}{r} \quad (17.19)$$

we have thus found a current distribution for which the magnetic field is characterized by a single valued scalar potential at all points in space, except for the point where the current is defined. In fact, any closed line  $\gamma$  that does not go through the dipole itself does not link the dipole current. Thus, from the second law of magnetostatics

$$\oint_{\gamma} \vec{B} \cdot \vec{t} d\ell = 0 \quad (17.20)$$

which guarantees that  $\vec{B}$  derives from a scalar potential. However, when considering a closed line linked with the dipole, the circulation of  $\vec{B}$  is different from zero.

Consider again (17.19). From a well-known vector calculus identity, we have

$$\vec{\nabla} \cdot \frac{\vec{\mu}}{r} = \frac{1}{r} \vec{\nabla} \cdot \vec{\mu} + \vec{\mu} \cdot \vec{\nabla} \frac{1}{r} \quad (17.21)$$

Since  $\vec{\nabla} \cdot \vec{\mu} = 0$ , we obtain

$$\vec{\nabla} \cdot \frac{\vec{\mu}}{r} = \vec{\mu} \cdot \vec{\nabla} \frac{1}{r} \quad (17.22)$$

We thus have

$$\psi(\vec{r}) = \frac{\mu_0}{4\pi} \vec{\mu} \cdot \vec{\nabla} \frac{1}{r} \quad (17.23)$$

from which

$$\psi(\vec{r}) = -\frac{\mu_0}{4\pi} \frac{\vec{\mu} \cdot \vec{r}}{r^3} \quad (17.24)$$

which is similar to the expression

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \vec{r}}{r^3} \quad (17.25)$$

for the electrostatic potential of an ideal electric dipole with moment  $\vec{p}$ .

At very large distances from an electric and magnetic ideal dipoles, the field lines for  $\vec{E}$  and  $\vec{B}$  are also similar. Of course, these lines are quite different in proximity of the dipoles.

Finally, the magnetic field due to a generic limited current distribution can always be represented as the linear superposition of the fields generated by a suitable distribution of ideal magnetic dipoles. As a consequence, the field of any limited current distribution goes to zero at least as  $1/r^3$ , as the field of each dipole.

## 17.2 Rectangular Circuit with Steady Current in an External Uniform Magnetic Field

Consider a rectangular rigid circuit  $\gamma = ABCD$  as shown in Fig. 17.2. The circuit is free to move about its vertical axis  $z$  and carries a steady current  $I$ . The circuit is placed in an external uniform magnetic field  $\vec{B}_0$ , which is perpendicular to  $z$ . The normal unit vector  $\vec{n}$  to the circuit surface makes an angle  $\alpha$  with  $\vec{B}_0$ . We consider only the effects due to  $\vec{B}_0$  and not those due to  $I$  since the circuit is rigid.

From

$$\vec{F} = \oint_{\gamma} I\vec{t} \times \vec{B} d\ell \quad (17.26)$$

it is possible to calculate the forces acting on the four segments of  $\gamma$ .

The force along  $AB$  is given by

$$\begin{aligned} \vec{F}_{AB} &= I\vec{u}_z \int_{AB} B_0 \sin\left(\frac{\pi}{2} - \alpha\right) d\ell \\ &= I\vec{u}_z \int_{AB} B_0 \cos \alpha d\ell \end{aligned} \quad (17.27)$$

Similarly, on  $CD$

$$\begin{aligned} \vec{F}_{CD} &= -I\vec{u}_z \int_{CD} B_0 \sin\left(\frac{\pi}{2} + \alpha\right) d\ell \\ &= -I\vec{u}_z \int_{CD} B_0 \cos \alpha d\ell \end{aligned} \quad (17.28)$$

The integrals (17.27) and (17.28) have equal absolute value ( $\overline{AB} = \overline{CD}$ ), but opposite sign. Thus, their contribution to the force on  $\gamma$  is zero.

Along  $BC$ , we have

$$\vec{F}_{BC} = IB_0 \vec{u}_B \int_{BC} d\ell = IB_0 L \vec{u}_B \quad (17.29)$$

where  $\overline{BC} = \overline{DA} = L$  and  $\vec{u}_B$  is a unit vector in the plane of  $\vec{B}_0$  and normal to  $\vec{B}_0$  (see Fig. 17.2). Similarly, along  $DA$

$$\vec{F}_{DA} = -IB_0L\vec{u}_B \quad (17.30)$$

The forces  $\vec{F}_{BC}$  and  $\vec{F}_{DA}$  are equal at each point on  $BC$  and  $DA$ . The torque generated by these two forces with respect to the  $z$ -axis is given by

$$\begin{aligned} \vec{\tau}_z &= \vec{r}_{OB} \times \vec{F}_{BC} + \vec{r}_{OA} \times \vec{F}_{DA} \\ &= \frac{\ell}{2} \vec{u}_{OB} \times \vec{u}_B IB_0 L - \frac{\ell}{2} \vec{u}_{OA} \times \vec{u}_B IB_0 L \\ &= \frac{\ell}{2} IB_0 L \sin \beta \vec{u}_z + \frac{\ell}{2} IB_0 L \sin(\pi - \beta) \vec{u}_z \\ &= IB_0 L \ell \sin \alpha \vec{u}_z = IL \ell B_0 \sin \alpha \vec{u}_z \\ &= \vec{\mu} \times \vec{B}_0 \end{aligned} \quad (17.31)$$

where  $\overline{OB} = \overline{OA} = \ell/2$  and  $\vec{u}_{OB}$ ,  $\vec{u}_{OA}$ , and the angles  $\alpha$  and  $\beta$  are shown in Fig. 17.2. The torque  $\vec{\tau}_z$  makes the circuit rotate till  $\vec{n}$  and  $\vec{B}_0$  have the same direction.

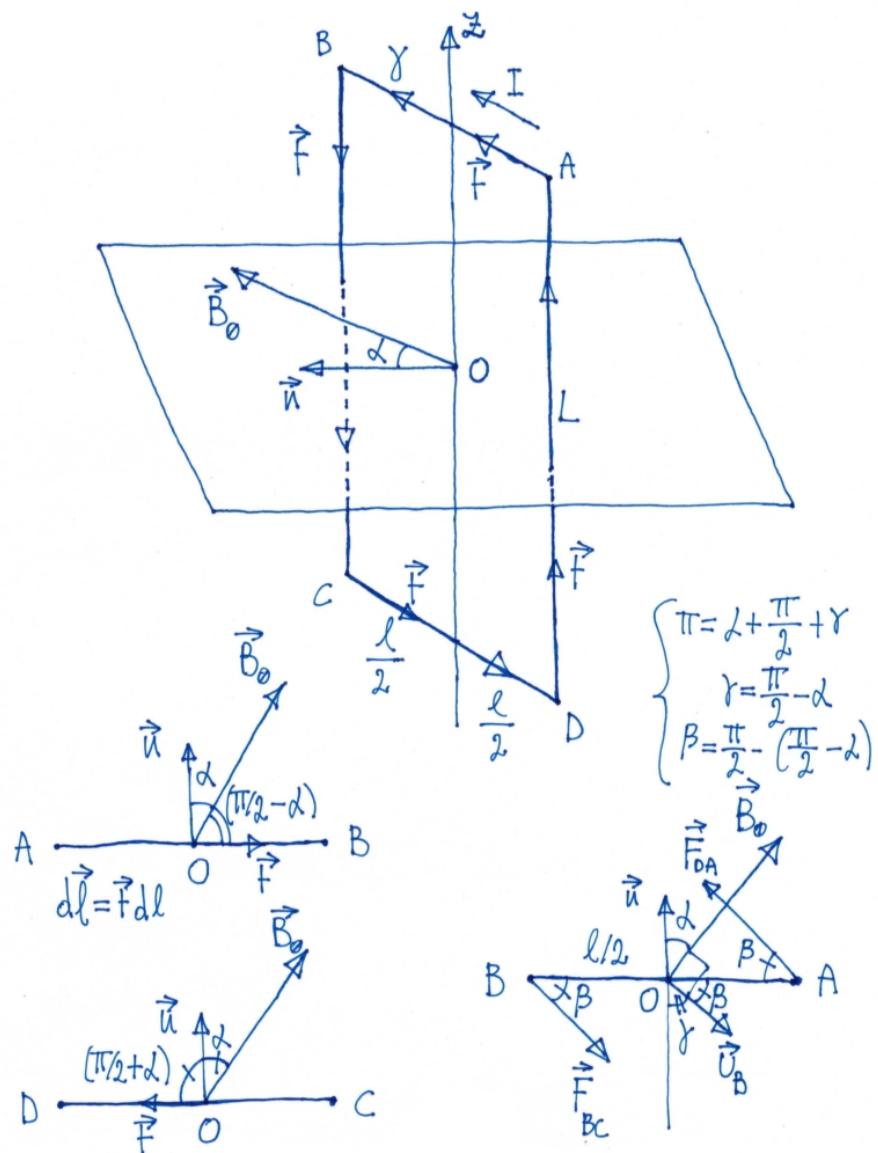


Figure 17.2

17.2. RECTANGULAR CIRCUIT WITH STEADY CURRENT IN AN EXTERNAL  
UNIFORM MAGNETIC FIELD

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# Chapter 18

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This chapter will be devoted to the study of the electrostatic field in presence of insulating materials, also called dielectric materials.

## 18.1 Polarization Phenomena: The Macroscopic Viewpoint

### 18.1.1 Introduction

So far, we studied the electrostatic fields generated by fixed charges located in known positions or distributed on conducting bodies in vacuum. When insulating bodies - dielectrics - are also considered, experimental evidence shows that the electrostatic field in the vicinity of such dielectrics is strongly modified by them.

For example, consider an uncharged dielectric body of spherical shape and suppose to place the body in proximity of a positively charged body, so that an electrostatic field acts on the dielectric, as shown in Fig. 18.1. The electrostatic field in proximity of the charged body gets modified by the presence of the dielectric sphere, the surface of which gets negatively charged in the region close to the positively charged body and positively charged far from the charged body. This phenomenon would seem at first similar to the induction in the case of a conducting sphere. We will show that the two phenomena are in reality quite different. For example, the dielectric sphere is not an equipotential surface as in the case of conductors.

Suppose the sphere of Fig. 18.1 is conductive and imagine to split into two parts after placing it close to a positive charge  $+q$ . One of the two parts will be globally positive and one negative. When using instead a dielectric sphere, we would find that each of the two parts is globally uncharged. In fact, on the region of the cut appear charges that neutralize the rest of the charges on the piece. The global neutrality persists even when two parts are further split and for non-spherical dielectric objects.

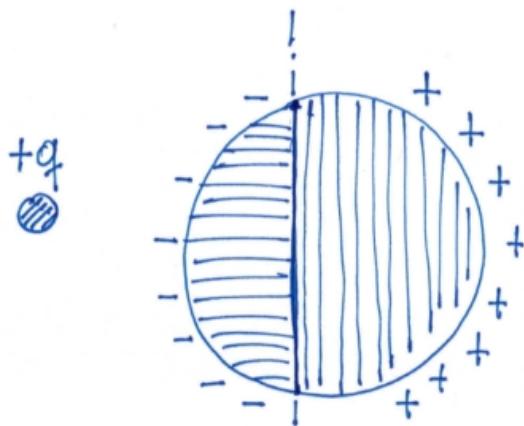


Figure 18.1

This experience can easily be explained remembering that the molecules of a dielectric, which are globally neutral, are made of charge carriers of opposite sign bound together to equilibrium positions by very strong restoring forces that allow only small displacements. In equilibrium conditions (i.e., when the microscopic charges are not acted upon by any net force), the electrostatic field generated by a dielectric is zero because the charge distributions of opposite sign balance each other. When the dielectric is in an external electrostatic field, the positive carriers are displaced in the direction of the field and the negative carriers in the opposite direction, thus creating a set of effective electric dipoles. While the displacements are small (on the order of the molecular dimensions), the global charge distribution is affected by them. Under these conditions the dielectric is said to be polarized. In conductors, the free charges can make macroscopic movements and, thus, can be macroscopically separated. In dielectrics, the charge carriers are bound to their equilibrium positions (besides small displacements) and, thus, a macroscopic separation is impossible.

A physics infinitesimal volume element of a dielectric contains a very large number of molecules. For example, for solids and liquids a cubic volume of side  $10^{-3}\text{mm}$  contains on the order of ten billion molecules. Such a volume, remains globally neutral even when polarized. Without any polarization, the charge distribution is characterized by a zero dipole moment. That is, the electrostatic field due to the distribution is negligible, even at a small distance from the dielectric element. When polarized, the element is characterized by a nonzero dipole moment due to the displacement of the charges of opposite sign.

The contribution of a piece of a polarized dielectric object (volume element) to the electrostatic field at large distance from the object is determined from the dipole moment of the piece. The dielectric is then regarded as a medium the properties of which vary continuously from one of its points to another. It is worth noting that this dielectric model cannot be used on the atomic scale. The model, for example, cannot be used to calculate the electrostatic field acting on a given atom of the dielectric. In fact, the atomic charge carriers are acted on by forces due to charges that are so close that cannot be treated as a continuous distribution. As a consequence, the actual field inside the material is locally characterized by very large intensities and very large variations

between points at distances on the order of the atomic dimensions.

A macroscopic description of a dielectric material treats the dielectric as a continuum. In this case, the field at a point inside the dielectric has to be regarded as the average of the values of the microscopic field in the neighborhood of the point. In other words, if  $\vec{E}_{\text{micro}}$  is the effective microscopic electrostatic field inside the material, the average field on a volume element  $\Delta V$  is

$$\vec{E} = \langle \vec{E}_{\text{micro}} \rangle = \frac{1}{\Delta V} \iiint_{\Delta V} \vec{E}_{\text{micro}} dV \quad (18.1)$$

According to the operative definition,  $\vec{E}$  is given by the ratio between the force acting on a test charge and the charge itself. This charge must be a physics infinitesimal: The charge must be small enough not to perturb macroscopically the distribution of the source charges and large enough on a microscopic scale so that the force acting on it can be approximated by the average force in the neighborhood of the point being considered.

In summary, a polarized dielectric contributes to the electric field, both inside and outside the volume occupied by the dielectric. The polarization depends on the total electric field inside the dielectric; the total field inside the dielectric, in turn, depends in part on the polarization. Moreover, the fact that the polarization also contributes to the field outside the dielectric can affect other charge distributions outside the dielectric, e.g., distributions on conducting bodies; the modified charge distributions outside the dielectric can then change the electric field acting on the dielectric itself.

### 18.1.2 Polarization Vector and Polarization Charge

Macroscopically, an unpolarized dielectric can be represented as a continuous medium with zero charge density at each point. However, we know that the dielectric contains charges of both positive and negative sign and that the polarization phenomena are due to small displacements of those charges. We can thus model the dielectric as the linear superposition of two continuous electric charge distributions, one positive  $\rho_+$  and one negative  $\rho_-$ . The former is due to the contribution from the positive charges of the atomic nuclei of the material and the latter is due to the electrons. Because of the very large number of charges per unit volume,  $\rho_+$  and  $\rho_-$  have very large values.

For an unpolarized or polarized dielectric, the charge densities are equal and of opposite sign at each point in the material,

$$\rho_+(x, y, z) + \rho_-(x, y, z) = 0 \quad (18.2)$$

where the coordinates  $(x, y, z)$  of a generic point  $P$  in the dielectric refer to a Cartesian coordinate system. Thus, given a generic volume element  $\Delta V$  in an unpolarized or polarized dielectric, the charges within  $\Delta V$  are equal and opposite,  $q_+ = \rho_+ \Delta V = -\rho_- \Delta V = q_-$ .

Suppose to polarize a dielectric. From a macroscopic point of view this means to apply a small displacement  $\vec{\ell}_+(x, y, z)$  to the positive charges in the neighborhood of a generic point  $(x, y, z)$  and a small displacement  $\vec{\ell}_-(x, y, z)$  to the negative charges.

## 18.1. POLARIZATION PHENOMENA: THE MACROSCOPIC VIEWPOINT

Hence, we have a small relative displacement

$$\vec{\ell} = \vec{\ell}_+ - \vec{\ell}_- \quad (18.3)$$

between the centers of charge of the two charges. Note that in this description, the relative displacement  $\vec{\ell}$  must be thought as an infinitesimal of higher order with respect to the linear dimensions of  $\Delta V$  (i.e.,  $|\vec{\ell}| \ll \sqrt[3]{\Delta V}$ ); the density  $\rho_+$  is supposed to be very large, in the limit infinite, so that the product  $\rho_+ \vec{\ell}$  can be assumed to be finite. Thus,

$$\vec{D} = (\vec{\ell}_+ - \vec{\ell}_-) \rho_+ \Delta V = \rho_+ \vec{\ell} \Delta V \quad , \quad (18.4)$$

which is the dipole moment of a quasi-ideal electric dipole.

Figure 18.2 shows a section of a cubic element of volume  $\Delta V$  of a dielectric. The positive charge is indicated by a set of parallel lines inclined to the right and the negative charge by lines inclined to the left. Panel (a) shows an unpolarized element; panel (b) shows the element under polarization with displacement  $\vec{\ell}_+$  and  $\vec{\ell}_-$  parallel to the  $AB$  direction. Panels (c) and (d) show the qualitative behavior of the molecules on the edges of the element.

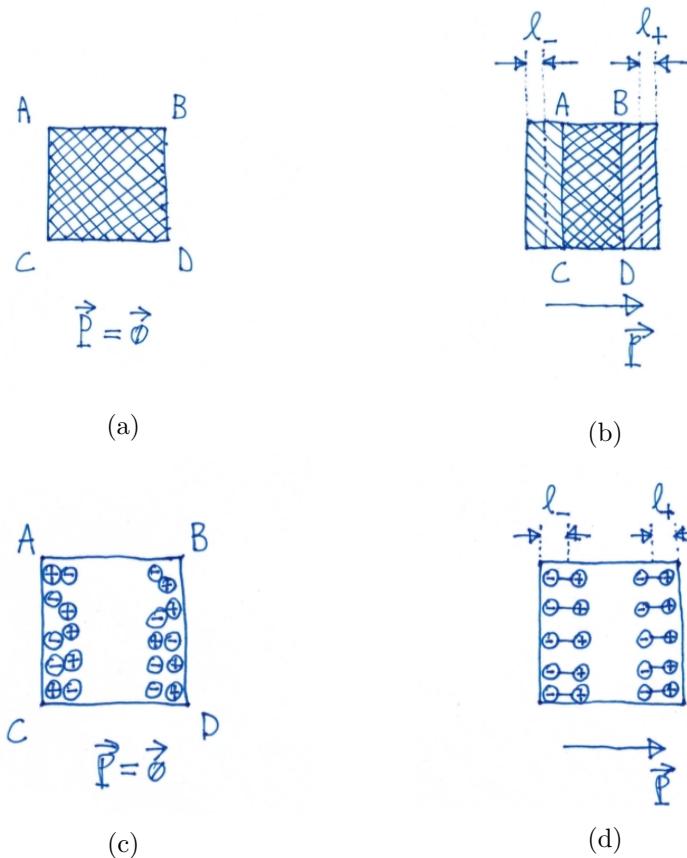


Figure 18.2

As mentioned in the introduction, the macroscopic field can be calculated from the dipole moment of each volume element. We can thus define a polarization vector  $\vec{P}$  as

the electric dipole moment per unit volume,

$$\vec{P} = \lim_{\Delta V \rightarrow 0^+} \frac{\Delta \vec{p}}{\Delta V} \quad (18.5)$$

where the limit must be thought as a “physics” limit because  $\vec{P}$  is a macroscopic quantity. Using (18.4) in (18.5), we have

$$\vec{P} = \rho_+ \vec{\ell} \quad (18.6)$$

The contribution due to the polarized material to the macroscopic electrostatic field can be calculated from a continuous dipole moment distribution. Each volume element generates a dipole field. The total field is the sum of all these infinitesimal contributions. From chapter 4 we know that we can substitute a distribution of dipoles with an equivalent distribution of volume and surface charges. In the case of a dielectric, these charges are called polarization charges and can be localized both on the external surface with density  $\sigma_p$  and in the volume with density  $\rho_p$ .

Note that a description of polarization by means of a charge distribution with nonzero volume density may seem incompatible with the description based on a set of dipoles having zero total charge.

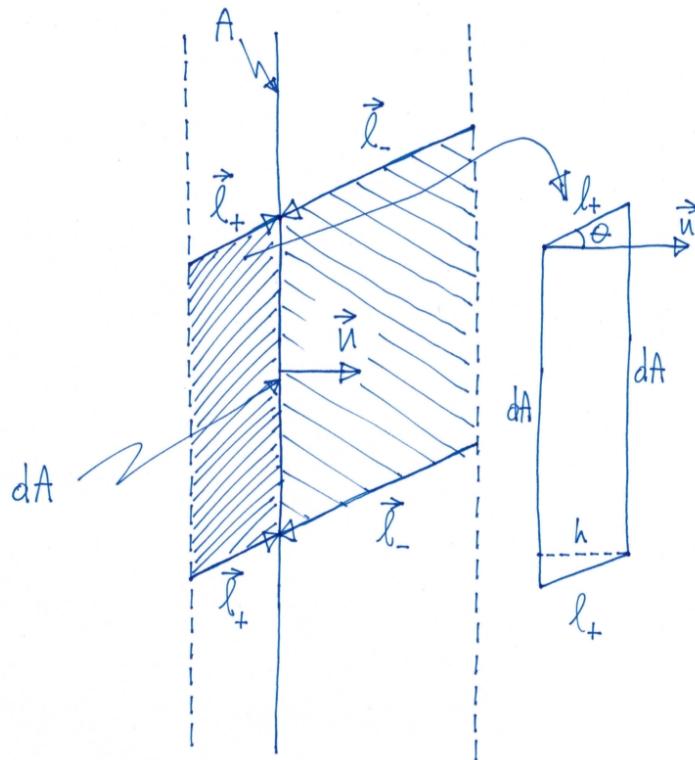


Figure 18.3

The existence of a charge on the surface of the dielectric is easy to explain. For example, referring to Fig. 18.2, suppose the face  $AC$  of the element is part of the

external surface of the dielectric. It is clear that when the dielectric is polarized, the face  $AC$  is characterized by a negative distribution.

The existence of a charge in the volume of the dielectric is less transparent. Consider a generic surface of area  $A$  and normal  $\vec{n}$  inside the dielectric, as shown in Fig. 18.3. During the polarization of the dielectric, macroscopically the surface  $A$  is crossed in both directions by charges of opposite sign. We want to calculate the net charge crossing a surface element  $dA$  in the dielectric (see Fig. 18.3), consistently with  $\vec{n}$ .

Due to the displacement  $\vec{\ell}_+$ ,  $dA$  is crossed by the positive charge that was adjacent to it before the polarization. This charge is contained in a cylinder with  $dA$  and a surface parallel to it (also of area  $dA$ ) as bases; the lateral surface of the cylinder is generated by lines that are parallel to  $\vec{\ell}_+$  and of length  $\ell_+ = \|\vec{\ell}_+\|$ . It is evident that all the charges within such a cylinder cross  $dA$ ; in fact, the charges adjacent to  $dA$  definitely cross it after being displaced; similarly, the charges further from  $dA$  on the left also cross it as they get displaced by a vector long enough to bring them just on the other side of  $dA$ . From simple geometry, the volume of such a cylinder is  $dAh$ , where  $h = \ell_+ \cos \theta = \vec{\ell}_+ \cdot \vec{n}$  (see Fig. 18.3). Thus, the positive charge crossing  $dA$  is given by

$$dq_+ = \rho_+ \vec{\ell}_+ \cdot \vec{n} dA \quad (18.7)$$

For the negative charge we have

$$dq_- = \rho_- \vec{\ell}_- \cdot \vec{n} dA = -\rho_+ \vec{\ell}_- \cdot \vec{n} dA \quad (18.8)$$

The net charge crossing  $dA$  is thus

$$\begin{aligned} dq_p &= dq_+ + dq_- = \rho_+ (\vec{\ell}_+ - \vec{\ell}_-) \cdot \vec{n} dA \\ &= \rho_+ \vec{\ell} \cdot \vec{n} dA \end{aligned} \quad (18.9)$$

From (18.6), we can then write

$$dq_p = \rho_+ \vec{\ell} \cdot \vec{n} dA = \vec{P} \cdot \vec{n} dA \quad (18.10)$$

This means that the net charge crossing  $dA$  at the time of the polarization is given by the flux of  $\vec{P}$  through  $dA$ .

Consider now a generic closed surface  $\Sigma$  inside the dielectric;  $\Omega$  is the volume enclosed by  $\Sigma$ , as shown in Fig. 18.4.

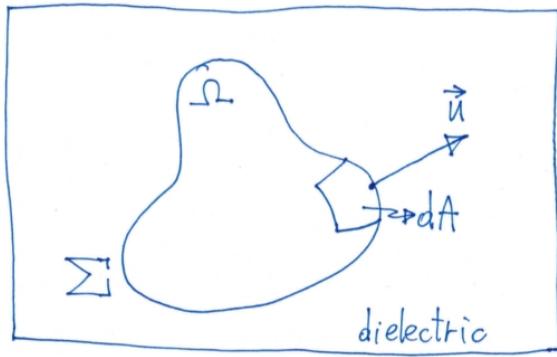


Figure 18.4

When the dielectric is unpolarized,  $\Sigma$  contains a zero global charge  $q_p$ .

When the dielectric is polarized, by integrating (18.10) it is evident that  $\Sigma$  contains a charge  $q_p \neq 0$ . We define this positive or negative extra charge within  $\Sigma$  as a polarization charge. Due to the charge conservation principle, the extra charge  $q_p$  is given by the total charge that entered  $\Omega$  through  $\Sigma$  during the polarization process. From (18.10), given  $\vec{n}$  as shown in Fig. 18.4, the total charge entered  $\Omega$  is

$$q_p = - \iint_{\Sigma} \vec{P} \cdot \vec{n} dA \quad (18.11)$$

where the negative sign is due to the charge flow that is opposite compared to  $\vec{n}$ . The charge  $q_p$  is distributed in  $\Omega$  with density  $\rho_p$ , thus

$$q_p = \iiint_{\Omega} \rho_p dV \quad (18.12)$$

By comparing (18.11) and (18.12) and observing that these two expressions must give the same result for  $q_p$  for any given surface  $\Sigma$  (and relative volume  $\Omega$ ), we can readily obtain the relationship between  $\rho_p$  and  $\vec{P}$ . To this end, by means of the divergence theorem for (18.11)

$$q_p = - \iiint_{\Omega} \vec{\nabla} \cdot \vec{P} dV \quad (18.13)$$

Then, from (18.12)

$$- \iiint_{\Omega} \vec{\nabla} \cdot \vec{P} dV = \iiint_{\Omega} \rho_p dV \quad (18.14)$$

and, finally,

$$\rho_p = - \vec{\nabla} \cdot \vec{P} \quad (18.15)$$

## 18.1. POLARIZATION PHENOMENA: THE MACROSCOPIC VIEWPOINT

due to the arbitrariness of  $\Omega$ .

If surface  $A$  in Fig. 18.3 is not inside the dielectric, but is part of its outer surface, we can easily calculate the polarization charge on the dielectric surface, i.e., the surface density  $\sigma_p$ . From (18.10)

$$dq_{ps} = \vec{P} \cdot \vec{n} dA \quad (18.16)$$

and the fact that

$$dq_{ps} = \sigma_p dA \quad (18.17)$$

we find

$$\sigma_p = \vec{P} \cdot \vec{n} = P_n \quad (18.18)$$

This result can be generalized to the case where  $A$  is a surface discontinuity between two different dielectrics 1 and 2. Consider a coin-type surface such that half of the surface is in dielectric 1 and the other half in dielectric 2; the coin surface has bases of area  $dA$ , both parallel to  $A$  (see Fig. 18.5).

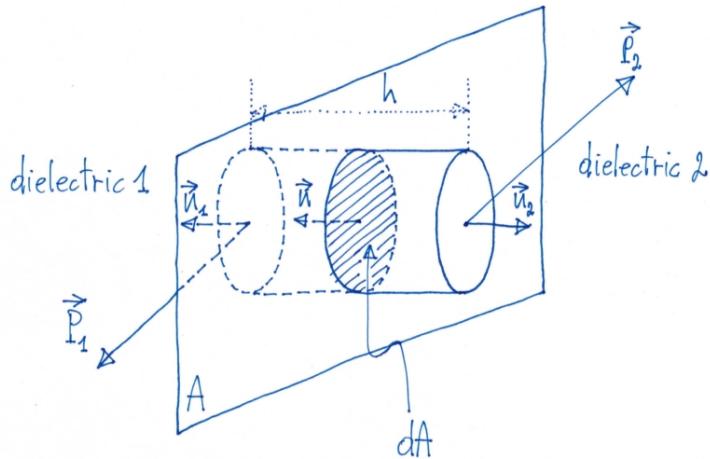


Figure 18.5

From (18.11), the polarization charge  $dq_p$  within the volume of the coin is equal to the flux of  $\vec{P}$  entering the same volume. Note that (for small  $h$ ) the coin volume is  $dV = h dA$ . The polarization charge can be distributed with density  $\sigma_p$  on the surface of discontinuity  $A$  and with densities  $\rho_{p1}$  and  $\rho_{p2}$  in the volume of each dielectric. Thus, in the coin volume we have a surface charge  $dq_{ps} = \sigma_p dA$  on  $A$  and a volume charge  $dq_p = (\rho_{p1} + \rho_{p2})dV = (\rho_{p1} + \rho_{p2})h dA$  in the volume. When  $h \rightarrow 0^+$ ,  $dq_p \rightarrow 0^+$ . In the same limit, also the flux of  $\vec{P}$  through the lateral surface of the coin is negligible (because the surface goes to zero). The only contributions to the flux are from the two bases. Following the notation in Fig. 18.5, when  $h \rightarrow 0^+$ ,  $\vec{n}_1 = +\vec{n} = -\vec{n}_2$  and the total flux is

$$\vec{P}_1 \cdot \vec{n}_1 dA + \vec{P}_2 \cdot \vec{n}_2 dA = (\vec{P}_1 - \vec{P}_2) \cdot \vec{n} dA \quad (18.19)$$

From the integral from (18.11), which we must use when considering a coin-type structure (we cannot use (18.16)), we have

$$dq_p = \sigma_p dA = -(\vec{P}_1 - \vec{P}_2) \cdot \vec{n} dA \quad (18.20)$$

and, finally,

$$\sigma_p = (\vec{P}_2 - \vec{P}_1) \cdot \vec{n} = P_{n2} - P_{n1} \quad (18.21)$$

In particular, if dielectric 1 is the vacuum,  $\vec{P}_1 = \vec{0}$  and we find again (18.18); if dielectric 2 is the vacuum,  $\vec{P}_2 = \vec{0}$  and we find again (18.18) noting that now  $\vec{n}$  is opposite of the convention used in (18.18).

We want to verify that the dielectric description based on dipoles with densities  $\rho_+$  and  $\rho_-$  and that based on the polarization charges  $\rho_p$  and  $\sigma_p$  are equivalent.

Each piece of dielectric is characterized by surface charges with density  $\sigma_p$  given by (18.18) and volume charges with density  $\rho_p$  given by (18.15). The total charge of the piece of dielectric is given by

$$q = \iint_{\Sigma} \sigma_p dA + \iiint_{\Omega} \rho_p dV \quad (18.22)$$

Using (18.15), (18.18), and the divergence theorem, we have

$$\begin{aligned} q &= \iint_{\Sigma} \vec{P} \cdot \vec{n} dA - \iiint_{\Omega} \vec{\nabla} \cdot \vec{P} dV \\ &= \iiint_{\Omega} \vec{\nabla} \cdot \vec{P} dV - \iiint_{\Omega} \vec{\nabla} \cdot \vec{P} dV = 0 \end{aligned} \quad (18.23)$$

This must be true for any volume of dielectric, for example  $\Delta V$  of Fig. 18.2. A finite piece of dielectric is made by many adjacent volume elements  $\Delta V$ . The surface charges of the internal elements neutralize, so that only the volume charge of these elements count. Hence, the only contributions to the field are due to the volume charges and the external surface charges. Note that the reason why the internal surface charges neutralize is because, under polarization, adjacent elements have surface with opposite sign (i.e., the positive charges on  $BD$  neutralize the negative charges on  $A'C'$  of the nearby element).

In many cases,  $\vec{\nabla} \cdot \vec{P} = 0$  and, thus, only the charges on the external surfaces must be considered. This happens in all uniformly polarized dielectrics and in isotropic, linear, and homogeneous dielectrics.

The charge densities  $\sigma_p$  and  $\rho_p$  that form in the dielectric under polarization are not due to external charges; they are due to the displacement of microscopic charges from their equilibrium conditions in the dielectric. For this reason, they are also called bound charge densities to distinguish them from the free charge densities in conductors. The total charge in space, in general, associated with the sum of a free and a bound density are

$$\rho = \rho_f + \rho_p \quad (18.24a)$$

$$\sigma = \sigma_f + \sigma_p \quad (18.24b)$$

where  $\rho_f$  and  $\sigma_f$  are the free charge densities in a volume and surface, respectively. Typically,  $\sigma_f$  and  $\rho_f$  are distributed only on conductors. For example,  $\sigma_f$  could be

## 18.2. ELECTRIC FIELD DUE TO KNOWN POLARIZATION DISTRIBUTIONS

distributed on a thin metallic sheet inserted at the interface between two different dielectrics, as shown in Fig. 18.6.

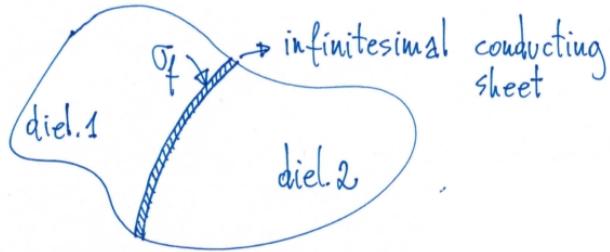


Figure 18.6

## 18.2 Electric Field due to Known Polarization Distributions

We now want to calculate the contribution to the macroscopic electric field due to known polarization distributions.

Consider a dielectric in a region of space  $\Omega$  with surface  $\Sigma$ , as shown in Fig. 18.7. Point  $O$  is a generic origin in space,  $\vec{r}$  is the vector between  $O$  and a field point  $P$ , and  $\vec{r}'$  is the vector between  $O$  and a point  $Q$  in  $\Omega$ .

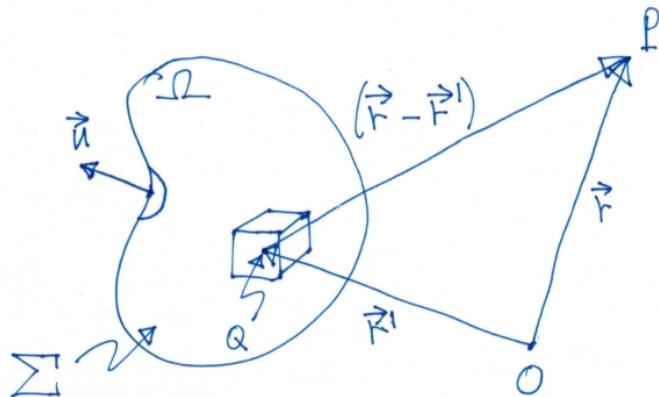


Figure 18.7

Assume the polarization is characterized by the vector  $\vec{P}(\vec{r}')$ . We will first calculate the electric potential  $\phi(\vec{r})$  and, then, the field  $\vec{E}(\vec{r})$ . The volume element  $\Delta V$  centered in  $Q$  has an electric dipole moment  $\Delta \vec{p} = \vec{P} \Delta V$ . From the expression of the potential for an electric dipole (ideal case)

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{r} \cdot \vec{p}}{r^3} \quad (18.25)$$

we find the potential associated with  $\Delta V$ ,

$$\begin{aligned}\Delta\phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \frac{\Delta\vec{p} \cdot (\vec{r} - \vec{r}')}{| \vec{r} - \vec{r}' |^3} \\ &= \frac{1}{4\pi\epsilon_0} \frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{| \vec{r} - \vec{r}' |^3} \Delta V\end{aligned}\quad (18.26)$$

By summing all contributions in the dielectric, the total potential is

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \vec{P}(\vec{r}') \cdot \frac{\vec{r} - \vec{r}'}{| \vec{r} - \vec{r}' |^3} dV \quad (18.27)$$

The polarization can also be described as a distribution of charges on  $\Sigma$  with density  $\sigma_p$  and in  $\Omega$  with density  $\rho_p$ . Hence, the potential can also be written as the superposition of the infinitesimal contributions  $\sigma_p dA$  and  $\rho_p dV$ :

$$\begin{aligned}\phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \iint_{\Sigma} \sigma_p \frac{1}{| \vec{r} - \vec{r}' |} dA \\ &\quad + \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \rho_p \frac{1}{| \vec{r} - \vec{r}' |} dV\end{aligned}\quad (18.28)$$

From (18.16) and (18.17), we then have

$$\begin{aligned}\phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \iint_{\Sigma} \frac{\vec{P}(\vec{r}') \cdot \vec{n}}{| \vec{r} - \vec{r}' |} dA \\ &\quad - \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \frac{\vec{\nabla}' \cdot \vec{P}(\vec{r}')}{| \vec{r} - \vec{r}' |} dV\end{aligned}\quad (18.29)$$

where  $\vec{n}$  is shown in Fig. 18.7 and the operator “ $\vec{\nabla}' \cdot$ ” refers to the coordinates  $\vec{r}'$ .

The field  $\vec{E}(\vec{r})$  can be obtained from  $(-\vec{\nabla}\phi(\vec{r}))$  or, directly, from superposition:

$$\begin{aligned}\vec{E}(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \iint_{\Sigma} \frac{\vec{r} - \vec{r}'}{| \vec{r} - \vec{r}' |^3} \sigma_p dA \\ &\quad + \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} \frac{\vec{r} - \vec{r}'}{| \vec{r} - \vec{r}' |^3} \rho_p dV\end{aligned}\quad (18.30)$$

### 18.3 Examples on Depolarization Field

We will now consider a few examples.

**Example 1.** Circular cylinder of length  $a$ , cross-section  $A$ , and uniformly polarized along the axis direction.

Figure 18.8 shows the dielectric along with the uniform distribution  $\vec{P}$  and the polarization charge.

Due to the uniformity of  $\vec{P}$ ,

$$\rho_p = -\vec{\nabla} \cdot \vec{P} = 0 \quad (18.31)$$

Referring to the normal unit vectors  $\vec{n}_1$  and  $\vec{n}_2$  in the figure, we have

$$\sigma_{p1} = \vec{P} \cdot \vec{n}_1 = |\vec{P}| \vec{n}_1 \cdot \vec{n}_1 = |\vec{P}| \quad (18.32a)$$

and

$$\sigma_{p2} = \vec{P} \cdot \vec{n}_2 = -|\vec{P}| \vec{n}_1 \cdot \vec{n}_1 = -|\vec{P}| \quad (18.32b)$$

on the surfaces  $A_1$  and  $A_2$ , respectively.

Because of symmetry reasons, the electric field inside the dielectric due to the polarization,  $\vec{E}_p$ , must be the same on each plane passing through the cylinder's axis. The field  $\vec{E}_p$  is due to two simple charged layers of opposite sign on the surfaces  $A_1$  and  $A_2$ . Note that  $\vec{E}_p$  acts also outside the dielectric. At large distance,  $\vec{E}_p$  is similar to the field generated by an electric dipole moment equal to the total electric dipole moment of the dielectric,  $\vec{p} = \vec{P}V$ , with  $V = aA$ . Thus,  $\vec{E}_p$  due to a uniform  $\vec{P}$  is not uniform in the dielectric and is opposite of  $\vec{P}$ .

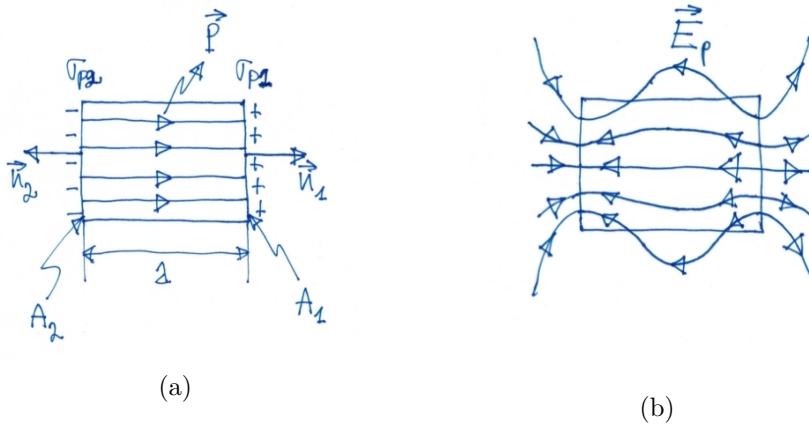


Figure 18.8

**Example 2.** Circular cylinder of infinite length, cross-section  $A$ , and uniformly polarized along the axis direction.

This can be considered as the limiting case of the cylinder of Fig. 18.8a for  $a \rightarrow +\infty$ . In this case, the two lateral surfaces  $A_1$  and  $A_2$  are infinitely far from each other.

Since the areas  $A_1$  and  $A_2$  are supposed to be constant and so are the densities  $\sigma_{p1}$  and  $\sigma_{p2}$ , the electric field  $\vec{E}_p$  in the central region of the cylinder becomes smaller and smaller for larger values of  $a$ . Finally, in the limit  $a \rightarrow +\infty$ ,  $\vec{E}_p = \vec{0}$  at each point in space.

**Example 3.** Indefinite dielectric slab, with thickness  $a$ , and uniformly polarized in the direction normal to its faces.

This can be considered as the limiting case of the cylinder of Fig. 18.8a for  $A_1 = A_2 = A \rightarrow +\infty$ . In this case, the electric field is generated by an indefinite planar double layer with uniform charge distribution of density  $\sigma_p = |\vec{P}|$ ; each layer has charges of opposite sign,  $\sigma_{p1} = +|\vec{P}|$  and  $\sigma_{p2} = -|\vec{P}|$ , as shown in Fig. 18.9. The two layers are the faces of the dielectric slab under consideration.

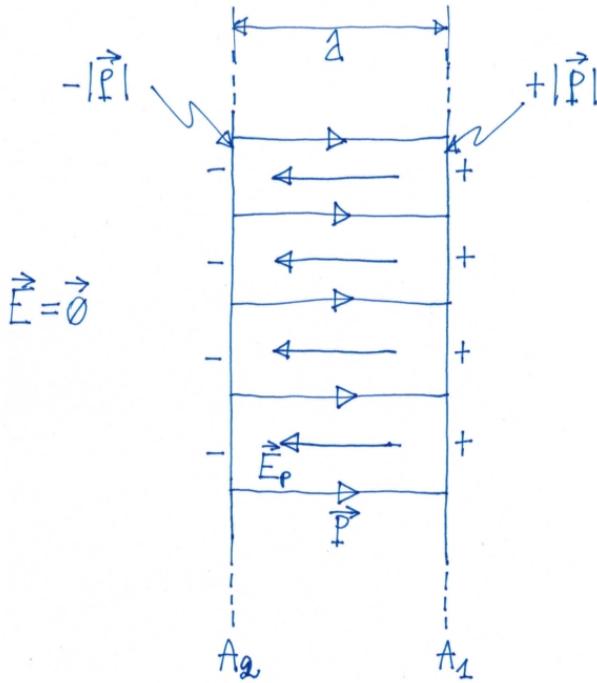


Figure 18.9

The field  $\vec{E}_p = \vec{0}$  at each point outside the dielectric. Inside the dielectric  $\vec{E}_p$  is uniform and directed from the positive to the negative charges (i.e., opposite of  $\vec{P}$ ) and with magnitude  $|\vec{E}_p| = |\vec{P}|/\epsilon_0$ . In summary,

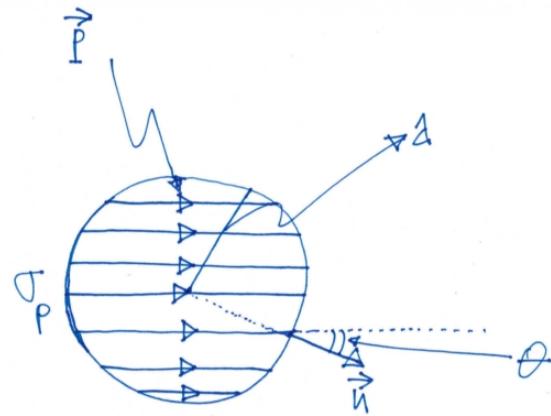
$$\begin{cases} \vec{E}_p = \vec{0} & \forall P \text{ outside} \end{cases} \quad (18.33a)$$

$$\begin{cases} \vec{E}_p = -\frac{\vec{P}}{\epsilon_0} & \forall P \text{ inside} \end{cases} \quad (18.33b)$$

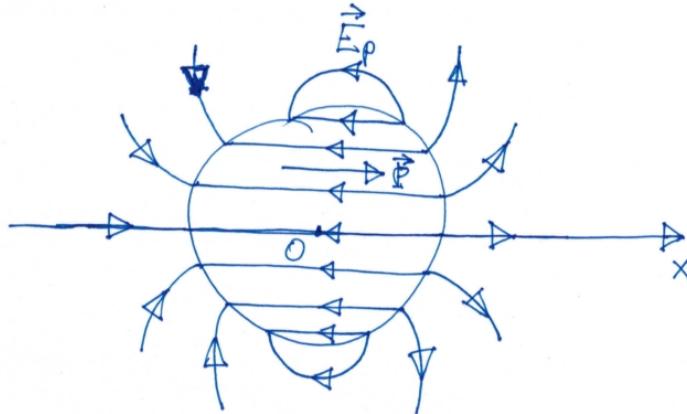
**Example 4.** Dielectric sphere, with radius  $a$ , and uniformly polarized. A uniformly polarized sphere with polarization  $\vec{P}$  is shown in Fig. 18.10.

The polarization generates a surface charge density with cylindrical symmetry with respect to the direction of  $\vec{P}$ . As shown in the figure, the charge density  $\sigma_p$ , at each point on the surface of the sphere, depends only on the angle  $\theta$  between the normal to the spherical surface and the direction of  $\vec{P}$ ,

$$\sigma_p = \vec{P} \cdot \vec{n} = |\vec{P}| \cos \theta \quad (18.34)$$



(a)



(b)

Figure 18.10

As always for a uniform polarization, in the sphere volume

$$\rho_p = -\vec{\nabla} \cdot \vec{P} = 0 \quad (18.35)$$

We could find  $\vec{E}_p$  at any point in space (inside and outside the sphere) by substituting (18.34) into (18.30). This approach, however, presents some analytical difficulties. It is easier to use the macroscopic model of polarization. Polarizing the sphere is equivalent to displace the uniform charge densities  $\rho_+$  and  $\rho_-$  by  $\vec{\ell} = \vec{\ell}_+ - \vec{\ell}_-$  so that  $\vec{P} = \rho_+ \vec{\ell}$ . Thus, the field  $\vec{E}$  is the net of the fields due to these two equal and of opposite sign spherical charge distributions at distance  $\vec{\ell}$  from each other. We call  $\tau = 4\pi a^3/3$  the volume of the sphere,  $q = \tau \rho_+$  the total positive charge in the sphere and  $(-q)$  the total negative charge.

The field outside a spherical charge distribution is equivalent to the field generated by a point-like charge located at the geometrical center of the sphere with charge equal to the total charge of the sphere. Thus, the field outside the polarized sphere is equal

to the field due to two point-like charges  $q$  and  $(-q)$  located in proximity of the center of the sphere and at a small distance  $\vec{\ell}$  from each other. As a consequence, the field is that of an electric dipole with moment  $\vec{p} = q\vec{\ell}$  right at the center of the sphere. In summary,

$$\vec{p} = q\vec{\ell} = \tau\rho_+\vec{\ell} = \tau\vec{P} \quad (18.36)$$

Note that (because of the particular symmetry), the field outside the sphere is that of a dipole not only at large distance (as in the example of the finite cylinder), but also in proximity of the sphere.

From the knowledge of the field and potential at each point outside the sphere and at each point on the sphere itself, we can easily calculate the field inside the sphere. Given  $\vec{r}$  the vector distance between  $O$  and a generic point at potential  $\phi(\vec{r})$ , we have

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \vec{r}}{|\vec{r}|^3} = \frac{1}{4\pi\epsilon_0} \tau \frac{\vec{P} \cdot \vec{r}}{|\vec{r}|^3} \quad (18.37)$$

For the points on the sphere surface we have

$$\begin{cases} |\vec{r}| = a \end{cases} \quad (18.38a)$$

$$\begin{cases} \vec{P} \cdot \vec{r} = |\vec{P}| a \cos \theta \end{cases} \quad (18.38b)$$

With respect to the coordinate  $x$  in Fig. 18.10b,

$$x = a \cos \theta \quad (18.39)$$

Thus, on the surface

$$\phi(\vec{a}) = \frac{1}{4\pi\epsilon_0} \tau \frac{|\vec{P}|}{a^3} x = \frac{P}{3\epsilon_0} x \quad (18.40)$$

Because of (18.35), for each point inside the sphere  $\phi$  is the special solution to Laplace's equation

$$\vec{\nabla}^2 \phi = \frac{d^2}{dx^2} \phi(x) = 0 \quad (18.41)$$

that satisfies the boundary condition (18.40). It is easy to show that

$$\phi(x) = Kx = \frac{P}{3\epsilon_0} x \quad (18.42)$$

is the wanted solution. Equation (18.42) shows that  $\phi$  increases linearly with  $x$  (i.e., in the direction of  $\vec{P}$ ) and is independent from the other coordinates. Thus, the field  $\vec{E}_p$  inside the sphere is uniform, parallel to  $\vec{P}$ , but of opposite sign:

$$\vec{E}_p = -\frac{\vec{P}}{3\epsilon_0} \quad \forall P \text{ inside} \quad (18.43)$$

Note that in all these examples the field  $\vec{E}_p$  due to  $\vec{P}$  is opposed to  $\vec{P}$  and is called for this reason, depolarizing electric field. In the last example of this chapter, we will calculate also the total field  $\vec{E}$  inside and outside the sphere.

## 18.4 Gauss' Theorem for Dielectrics

We demonstrated Gauss' theorem for a distribution of fixed charges in vacuum. From our discussion on polarization phenomena, it is evident that Gauss' theorem is valid also in presence of dielectric, so long among all charges in space we consider also the polarization charges. In this case,

$$\iint_{\Sigma} \vec{E} \cdot \vec{n} dA = \frac{1}{\epsilon_0} (q_f + q_p) \quad (18.44)$$

where  $q_f$  and  $q_p$  are the total free and polarization charge within the region  $\Omega$  enclosed by the closed surface  $\Sigma$ . From

$$q_p = - \iint_{\Sigma} \vec{P} \cdot \vec{n} dA \quad (18.45)$$

we have

$$\iint_{\Sigma} \vec{E} \cdot \vec{n} dA = \frac{1}{\epsilon_0} q_f - \frac{1}{\epsilon_0} \iint_{\Sigma} \vec{P} \cdot \vec{n} dA \quad (18.46)$$

and, thus,

$$\iint_{\Sigma} (\epsilon_0 \vec{E} + \vec{P}) \cdot \vec{n} dA = q_f \quad (18.47)$$

This means that the outgoing flux of vector  $(\epsilon_0 \vec{E} + \vec{P})$  through  $\Sigma$  is equal to the total free charge in  $\Omega$ . By defining the electric displacement vector

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad (18.48)$$

Eq. (18.47) becomes

$$\iint_{\Sigma} \vec{D} \cdot \vec{n} dA = q_f \quad (18.49)$$

which is Gauss' theorem in integral form for  $\vec{D}$ . Note that the *SI* units of  $\vec{D}$  are the same of  $\vec{P}$ , i.e.,  $\text{C/m}^2$ .

Equations (18.44) and (18.49) can be written in differential form under the usual conditions that  $\rho_f$  and  $\rho_p$ , which are the charge densities associated with  $q_f$  and  $q_p$ , respectively, are both  $C^0(\Omega)$  and therein limited. In this case, we have

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} (\rho_f + \rho_p) \quad (18.50)$$

and

$$\vec{\nabla} \cdot \vec{D} = \rho_f \quad (18.51)$$

where we used the divergence theorem for (18.44) and (18.49) and the definitions

$$q_f = \iiint_{\Omega_f} \rho_f dV \quad (18.52)$$

and

$$q_p = \iiint_{\Omega_p} \rho_p dV \quad (18.53)$$

where  $\Omega_f$  and  $\Omega_p$  are the regions where  $\rho_f$  and  $\rho_p$  are defined, respectively.

From Eq. (18.51), we could think that  $\vec{D}$  behaves exactly as a field  $\vec{E}$  generated in vacuum by a distribution of free charges  $\rho_f$ , in total absence of any dielectric. This is not true in general. There is a fundamental difference between  $\vec{E}$  and  $\vec{D}$ : While  $\vec{E}$  is always irrotational,  $\vec{D}$  is not. In fact, from the definition (18.48), we have

$$\oint_{\gamma} \vec{D} \cdot d\vec{\ell} = \epsilon_0 \oint_{\gamma} \vec{E} \cdot d\vec{\ell} + \oint_{\gamma} \vec{P} \cdot d\vec{\ell} \quad (18.54)$$

Then, from the irrotational property of  $\vec{E}$ :

$$\oint_{\gamma} \vec{D} \cdot d\vec{\ell} = \oint_{\gamma} \vec{P} \cdot d\vec{\ell} \quad (18.55)$$

where  $\gamma$  is any closed line in space. As always, for continuous and limited charge distributions (18.55) can be written in differential form by means of Stokes' theorem

$$\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P} \quad (18.56)$$

In general, it is not always true that  $\vec{P}$  is irrotational. In fact, it is easy to find examples where integrals of  $\vec{P}$  on closed lines are non zero. Even if  $\vec{P}$  were irrotational in the entire volume of a dielectric, the discontinuity of  $\vec{P}$  at the dielectric external surface must be taken into account.

Consider the example of a circular cylinder of length  $a$ , cross-section  $A$ , and uniformly polarized along the axis direction, as shown in Fig. 18.11. The circulation of  $\vec{P}$  along the closed oriented line  $ABCD$  indicated in Fig. 18.11 is clearly non zero. In fact, there is a positive contribution from the integral along the segment  $AB$  ( $\vec{P}$  is directed as  $AB$ ) and zero contributions from the other three segments ( $\vec{P}$  is normal to both  $BC$  and  $DA$  and  $\vec{P} = \vec{0}$  for each point on  $CD$ ).

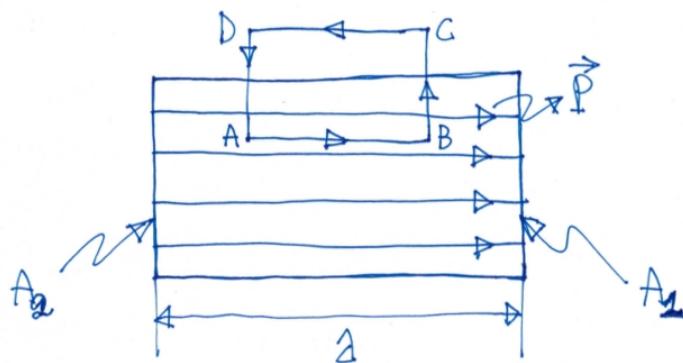


Figure 18.11

Consider now the example of an indefinite dielectric slab, with thickness  $a$ , and uniformly polarized in the direction normal to its faces, as shown in Fig. 18.12. In this case, it can be shown that the circulation of  $\vec{P}$  along any closed line is zero. This is clear for lines completely outside the slab, where  $\vec{P} = \vec{0}$ , and for lines completely inside the slab, where  $\vec{P}$  is uniform (note that for each point inside the slab  $\vec{P}$  is continuously derivable and  $\vec{\nabla} \times \vec{P} = \vec{0}$ ). Consider a closed oriented line  $A\gamma_1B\gamma_2A$ , as shown in Fig. 18.12a. This line is partly inside and partly outside the dielectric. The contribution from  $A\gamma_1B$  to the circulation of  $\vec{P}$  is clearly zero because  $\vec{P} = \vec{0}$  outside the dielectric. The only nonzero contribution could be from  $B\gamma_2A$ , which is an open line inside the dielectric. In order to show that also the contribution from  $B\gamma_2A$  is zero, we can consider the closed line  $B\gamma_2AB$ , obtained by choosing the points  $A$  and  $B$  to be at the same infinitesimal distance from the lateral surface  $A_1$  of the slab. In this case,  $AB$  is a straight line parallel to  $A_1$  and infinitesimally close to it. The closed oriented line  $B\gamma_2AB$  is entirely contained within the volume of the dielectric. Hence, because of the uniformity of  $\vec{P}$ , the circulation of  $\vec{P}$  along  $B\gamma_2AB$  must be equal to zero,

$$\oint_{B\gamma_2AB} \vec{P} \cdot d\vec{\ell} = 0 \quad (18.57)$$

The contribution to this integral due to the line segment  $AB$  is zero because  $\vec{P}$  is normal to  $AB$  at each point on  $AB$ ,

$$\int_{AB} \vec{P} \cdot d\vec{\ell} = 0 \quad (18.58)$$

Since,

$$\begin{aligned} \oint_{B\gamma_2AB} \vec{P} \cdot d\vec{\ell} &= \int_{B\gamma_2A} \vec{P} \cdot d\vec{\ell} + \int_{AB} \vec{P} \cdot d\vec{\ell} \\ &= \int_{B\gamma_2A} \vec{P} \cdot d\vec{\ell} = 0 \end{aligned} \quad (18.59)$$

the circulation of  $\vec{P}$  on any closed line both inside and outside the slab must be zero. For the case of a uniformly polarization slab,

$$\oint_{\gamma} \vec{P} \cdot d\vec{\ell} = 0 \quad (18.60)$$

for any closed line  $\gamma$ .

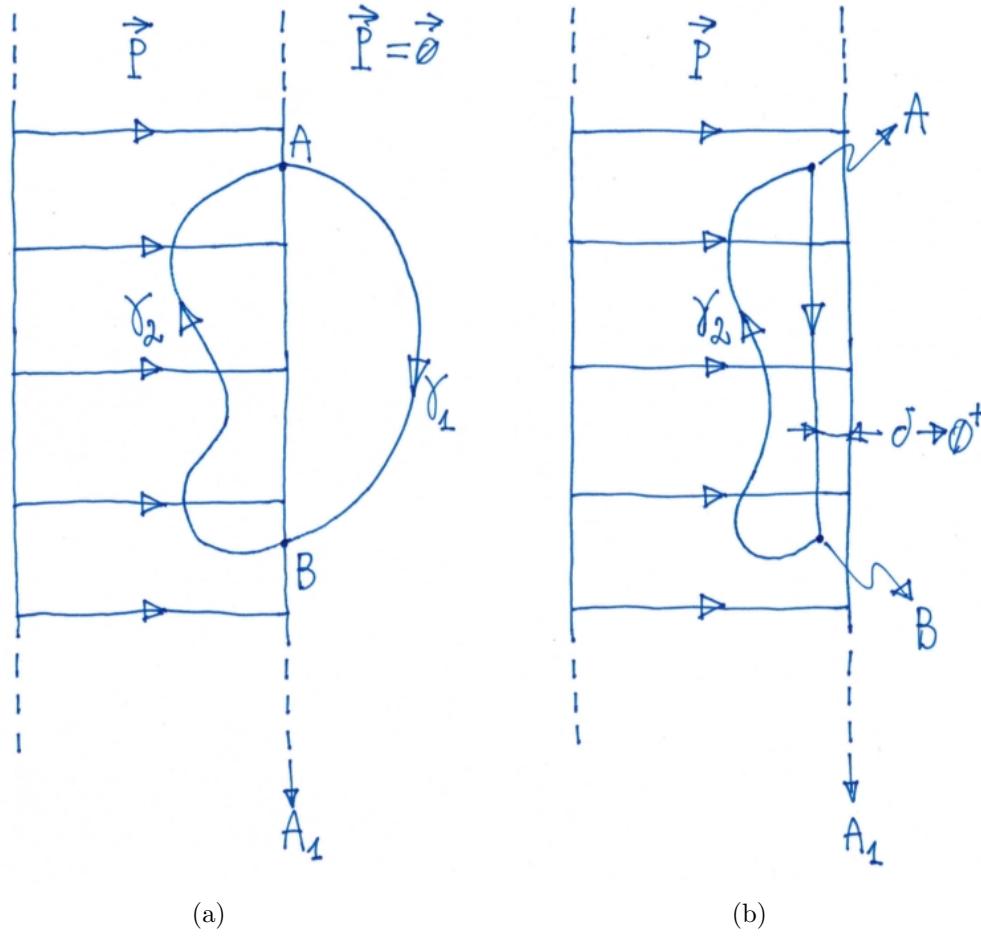


Figure 18.12

From these examples is clear that the only knowledge of  $\rho_f$  is insufficient to calculate  $\vec{D}$  from (18.51) because, in general,  $\vec{\nabla} \times \vec{D} \neq \vec{0}$ . In fact, we also must know the type of dielectric and its geometry (or, equivalently, the distribution of  $\vec{P}$ ). Even if  $\vec{P}$  is conservative at each point inside the material, the fact that is conservative everywhere depends on the dielectric geometry. As we will soon demonstrate, this is true when the discontinuity surfaces of the dielectric are normal to  $\vec{P}$  at each of their points.

In conclusion, in electrostatics  $\vec{D}$  does not have a fundamental role as  $\vec{E}$ . Even from a practical point of view, experimentally it is usually not possible to control the charges on conductors and dielectrics, but it is possible to control the potential differences and, thus,  $\vec{E}$  and not  $\vec{D}$ .

#### 18.4.1 Boundary Conditions for Vector $\vec{D}$ and $\vec{E}$

We now want to study the behavior of  $\vec{D}$  and  $\vec{E}$  in proximity of the separation surface  $A$  between two dielectrics.

Consider two different dielectrics 1 and 2 separated by  $A$ . As always, consider a coin-type surface  $\Sigma_c$  between 1 and 2, as shown in Fig. 18.13a. Each base of  $\Sigma_c$  has area  $dA$ ; the height of  $\Sigma_c$  is  $h$ . Suppose a free charge with surface density  $\sigma_f$  is distributed on  $A$  and with volume densities  $\rho_{f1}$  and  $\rho_{f2}$  in the volume of 1 and 2 contained in  $\Sigma_c$ .

Assuming the directions of  $\vec{D}_1$ ,  $\vec{n}_1$ ,  $\vec{D}_2$ ,  $\vec{n}_2$ , and  $\vec{n}$  are those in Fig. 18.13a, in the limit  $h \rightarrow 0^+$  the volume charge in 1 and 2 within  $\Sigma_c$  is negligible compared to the surface charge on  $A$  and the flux of  $\vec{D}$  through the lateral surface is negligible compared to that through the bases (see, e.g., the derivation leading to (18.21)). Hence, from Eq. (18.49) we find

$$\begin{aligned} \vec{D}_1 \cdot \vec{n}_1 dA + \vec{D}_2 \cdot \vec{n}_2 dA &= (\vec{D}_1 - \vec{D}_2) \cdot \vec{n} dA \\ &= \sigma_f dA \end{aligned} \quad (18.61)$$

which finally gives

$$D_{n1} - D_{n2} = \sigma_f \quad (18.62)$$

If there is a free charge on  $A$ , the normal component of  $\vec{D}$  has a discontinuity of the first kind equal to  $\sigma_f$ . If  $\sigma_f = 0$ ,  $D_n$  is continuous,

$$D_{n1} = D_{n2} \quad (18.63)$$

The situation is different for  $\vec{E}$ , the normal component of which is discontinuous, in general, even if  $\sigma_f = 0$ . This can be understood from (18.48) because of the continuity of  $D_n$  and the discontinuity of  $\vec{P}$ . In fact, a polarization charge is present on  $A$  such that

$$P_{n2} - P_{n1} = \sigma_p \quad (18.64)$$

From (18.48),

$$\begin{aligned} D_{n1} - D_{n2} &= 0 \\ &= \epsilon_0 (E_{n1} - E_{n2}) + (P_{n1} - P_{n2}) \end{aligned} \quad (18.65)$$

and, thus,

$$\begin{aligned} \epsilon_0 (E_{n1} - E_{n2}) &= P_{n2} - P_{n1} \\ &= \sigma_p \end{aligned} \quad (18.66)$$

Consider now a cut-type line  $\gamma_c$  between 1 and 2, as shown in Fig. 18.13b. The line comprises two segments  $AB$  and  $CD$  of infinitesimal length  $d\ell$ , parallel to  $A$ , one in 1 and the other in 2. The other two segments,  $BC$  and  $DA$ , have length  $h$ . Because of the irrotational property of  $\vec{E}$ , the integral of  $\vec{E}$  along any closed line must be zero. In the particular case of line  $\gamma_c$ , in the limit  $h \rightarrow 0^+$  the contribution to the circulation of  $\vec{E}$  along  $BC$  and  $DA$  is negligible and, thus,

$$\begin{aligned} \vec{E}_1 \cdot \vec{t}_1 d\ell + \vec{E}_2 \cdot \vec{t}_2 d\ell &= (\vec{E}_1 - \vec{E}_2) \cdot \vec{t} d\ell \\ &= 0 \end{aligned} \quad (18.67)$$

from which

$$E_{t1} - E_{t2} = 0 \quad (18.68)$$

or

$$E_{t1} = E_{t2} \quad (18.69)$$

That is, the tangent component of  $\vec{E}$  is continuous.

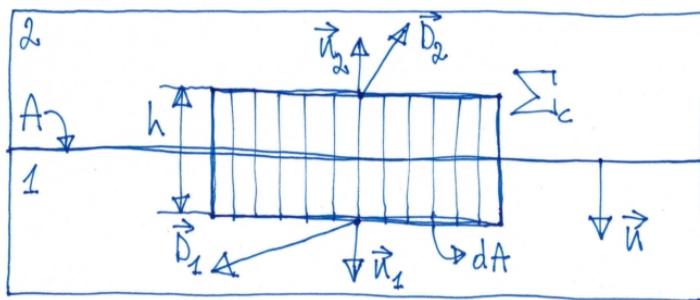
From (18.69) and (18.48),

$$D_{t1} - D_{t2} = \epsilon_0 (E_{t1} - E_{t2}) + (P_{t1} - P_{t2}) \quad (18.70)$$

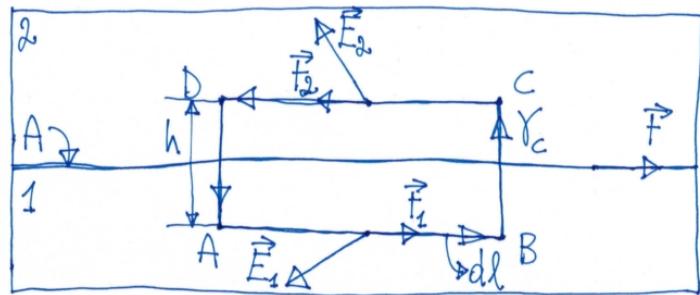
and, thus,

$$D_{t1} - D_{t2} = P_{t1} - P_{t2} \quad (18.71)$$

That is, the tangent component of  $\vec{D}$  is characterized by the same discontinuity of the tangent component of  $\vec{P}$  at each point on  $A$ .



(a)



(b)

Figure 18.13

The boundary conditions (18.62), (18.66), (18.69), and (18.71) can also be regarded as the local form of Maxwell's equations at the boundary between two different dielectrics.

## 18.5 Linear Dielectric Materials: Electric Susceptibility and Dielectric Constant

From a macroscopic point of view, the behavior of a dielectric material can be fully characterized by the relationship  $\vec{P} = \vec{P}(\vec{E})$  between the macroscopic quantities  $\vec{P}$  and  $\vec{E}$  that are measured at the same point in the dielectric. This relationship can be found experimentally by means of macroscopic measurement.

In most materials,  $\vec{P} = \vec{0}$  when  $\vec{E} = \vec{0}$ ; in some cases,  $\vec{P}$  and  $\vec{E}$  have the same direction and the relationship is equivalent to a simple scalar equation; in some other (more general cases),  $\vec{P}$  and  $\vec{E}$  do not have the same direction and the relationship  $\vec{P} = \vec{P}(\vec{E})$  has tensorial character. This happens in anisotropic materials (for example, most of the monocrystalline solids). In the following, we will focus on materials that can be assumed to be isotropic from a macroscopic point of view; examples include polycrystalline solids, amorphous solids, and fluids. In these materials,  $\vec{P}$  has the same direction of the field  $\vec{E}$  that generates it:

$$\vec{P} = \chi(E) \vec{E} \quad (18.72)$$

where the scalar quantity  $\chi$ , in general function of the intensity of  $\vec{E}$ , is called the electric susceptibility.

The dimensions of  $\chi$  are equal to those of the vacuum dielectric constant  $\epsilon_0$  and, thus, the SI units of  $\chi$  are  $\text{Cs}^2/(\text{Kg m}^3)$ , i.e.,  $\text{F/m}$ . In the literature, an adimensional susceptibility  $\chi_r$  is defined, where

$$\chi_r = \frac{\chi}{\epsilon_0} \quad (18.73)$$

Using (18.72) in the definition of  $\vec{D}$  given by (18.76), we find

$$\begin{aligned} \vec{D} &= \epsilon_0 \vec{E} + \chi(E) \vec{E} \\ &= \epsilon(E) \vec{E} \end{aligned} \quad (18.74)$$

where the scalar  $\epsilon(E)$  is defined as

$$\epsilon(E) = \epsilon_0 + \chi(E) \quad (18.75)$$

and is called dielectric constant or permittivity of the material. In general, the quantities  $\chi$  and  $\epsilon$  are functions of the mass density  $\rho_m$ , of the temperature  $T$ , and of the electric field intensity  $E$  for the considered material.

However, it is possible to experimentally verify that for most materials the dependence from  $E$  is very weak; these are called linear materials. In these cases, given  $\rho_m$  and  $T$ ,  $\chi$  and  $\epsilon$  are independent from  $E$  in a wide range of parameters and, thus, the electric behavoir of the material is completely determined by  $\chi$  and  $\epsilon$  themselves (with units). In these cases, it is custom to introduce an adimensional constant  $\epsilon_r$

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \frac{\chi}{\epsilon_0} \quad (18.76)$$

called the dielectric constant relative to vacuum. In this course, we will almost always consider linear materials. For these materials it is easy to specify the conditions for which  $\vec{\nabla} \times \vec{D} = \vec{0}$ . In fact, we have

$$\begin{aligned} \vec{\nabla} \times \vec{D} &= \vec{\nabla} \times (\epsilon \vec{E}) = \epsilon \vec{\nabla} \times \vec{E} + \vec{\nabla} \epsilon \times \vec{E} \\ &= \vec{\nabla} \epsilon \times \vec{E} \end{aligned} \quad (18.77)$$

## CHAPTER 18.

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When  $\vec{\nabla}\epsilon$  is zero or parallel to  $\vec{E}$  at each point,  $\vec{D}$  is irrotational ( $\vec{\nabla} \times \vec{D} = \vec{0}$ ). This can happen, for example, for a homogeneous material where  $\vec{\nabla}\epsilon = 0$  at each point inside the material and the external surfaces of the material (at which  $\epsilon$  has a discontinuity) are normal to  $\vec{E}$  at each of their point.

A material cannot be considered to be linear for any arbitrary value of the applied electric field. In general, a linear relationship between  $\vec{P}$  and  $\vec{E}$  can only exist for applied electric fields of moderate intensity. For each dielectric material corresponds an electric field intensity at which the dielectric undergoes an electrical breakdown. This intensity, which is typically very high, depends on the material and its physical conditions (e.g., pressure and temperature). The break down value of the electric field  $E_{\max}$  is called dielectric strength. The value of  $\epsilon_r$  and, in some cases, of  $E_{\max}$  for a few dielectrics is reported in Table 18.1.

Material	$\epsilon_r$	$E_{\max}$ (V/m)
air (1 bar)	1.00059	$3 \times 10^6$
air (100 bars)	1.0548	
distilled water (20°C)	80.0	
ethanol (0°C)	28.4	
nylon	6.0	$(5, 20) \times 10^6$
mica	3.5	$16 \times 10^6$
glasses (various)	(5,10)	$9 \times 10^6$

Table 18.1

Consider now two linear dielectrics 1 and 2 separated by a surface  $A$  with zero surface charge,  $\sigma_f = 0$ , as shown in Fig. 18.14; it is possible to find a simple relationship between the directions of the field vectors in 1 and 2.

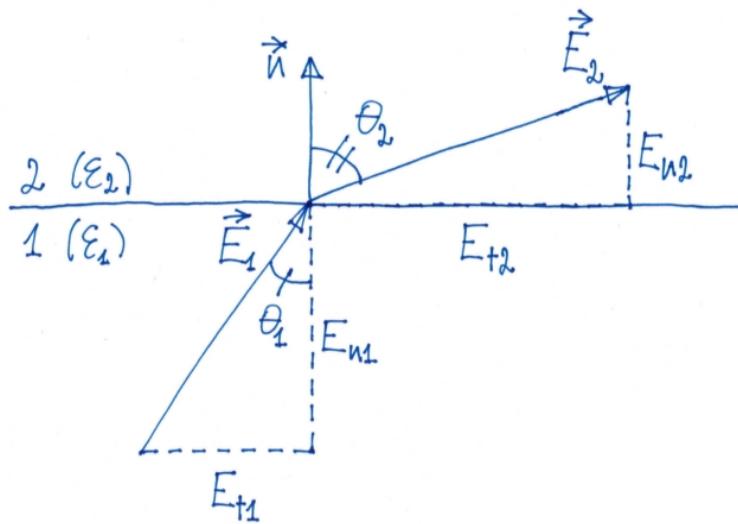


Figure 18.14

The angles between  $\vec{E}_1$  (and, thus,  $\vec{D}_1$  and  $\vec{P}_1$ ) and the normal unit vector  $\vec{n}$  and between  $\vec{E}_2$  (and, thus  $\vec{D}_2$  and  $\vec{P}_2$ ) and  $\vec{n}$  are called  $\theta_1$  and  $\theta_2$ , respectively. The dielectric constants of the two materials are  $\epsilon_1$  and  $\epsilon_2$ , respectively. Under these conditions, (18.63) can be written as

$$\epsilon_1 E_{n1} = \epsilon_2 E_{n2} \quad (18.78)$$

Dividing this equation by (18.69), we find

$$\epsilon_1 \frac{E_{n1}}{E_{t1}} = \epsilon_2 \frac{E_{n2}}{E_{t2}} \quad (18.79)$$

By definition

$$\left\{ \tan \theta_1 = \frac{E_{t1}}{E_{n1}} \right. \quad (18.80a)$$

$$\left. \tan \theta_2 = \frac{E_{t2}}{E_{n2}} \right. \quad (18.80b)$$

we obtain

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\epsilon_1}{\epsilon_2} = \frac{\epsilon_{r1}}{\epsilon_{r2}} \quad (18.81)$$

This result is called the law of refraction of the field lines. In particular, if dielectric 1 has a constant  $\epsilon_1 \ll \epsilon_2$ , for example, 1 is air ( $\epsilon_{r1} \sim 1$ ) and 2 is water ( $\epsilon_{r2} = 80$ ), from (18.81) we have  $\theta_1 \ll \theta_2$ . Thus, the field lines in the dielectric with smaller  $\epsilon$  tend to be directed normally to the separation surface.

## 18.6 Electric Field in Presence of Linear and Homogeneous Dielectrics

We now want to study the problem of how to calculate the electric field when both conductors and dielectrics of given characteristics are considered. The conductors are

assumed to be perfect and in electrostatic equilibrium. The dielectrics are considered to be linear and homogeneous. In addition, the geometrical shape and dimensions of all bodies are supposed to be known; the relationship between polarization and electric field is known at each point in any dielectric; free charge distributions, total charges on conductors, and potential difference between conductors are also known.

In the dielectrics,

$$\vec{P} = \chi \vec{E} \quad (18.82)$$

and

$$\vec{D} = \epsilon \vec{E} \quad (18.83)$$

where  $\chi$  and  $\epsilon$  are independent from  $\vec{E}$  and are constant at any point in the dielectrics. The electric field is thus calculated in regions of space with uniform characteristics separated by surfaces of discontinuity (a very common case in applied physics). It is worth noting that because of the linear relationship between  $\vec{P}$  and  $\vec{E}$ , it is possible to extend to the case of linear and homogeneous dielectrics the results obtained in electrostatics in the case of perfect conductors in vacuum. For the same reason it is possible to apply the superposition principle.

### 18.6.1 Relationships between Charge Densities

At each point inside the dielectrics, which are assumed to be linear and homogeneous, the free charge and polarization densities are proportional. This makes qualitatively sense as a polarization charge is expected to be generated by an external electrostatic field due to a given free charge. In fact, from (18.82) and (18.83) we have

$$\begin{aligned} \rho_p &= -\vec{\nabla} \cdot \vec{P} = -\vec{\nabla} \cdot (\chi \vec{E}) = -\chi \vec{\nabla} \cdot \vec{E} \\ &= -\frac{\epsilon_0 \epsilon_r - \epsilon_0}{\epsilon_0} \rho = -(\epsilon_r - 1) \rho \end{aligned} \quad (18.84)$$

and

$$\begin{aligned} \rho_f &= \vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (\epsilon \vec{E}) = \epsilon \vec{\nabla} \cdot \vec{E} \\ &= \frac{\epsilon_r \epsilon_0}{\epsilon_0} \rho = \epsilon_r \rho \end{aligned} \quad (18.85)$$

By combining (18.84) and (18.85) and eliminating  $\rho$ , we obtain

$$\rho_p = -\frac{\epsilon_r - 1}{\epsilon_r} \rho_f \quad (18.86)$$

Hence, at each point inside a linear and homogeneous dielectric there is polarization charge only when and where there is free charge. Since  $\epsilon_r \geq 1$ ,  $\rho_p$  is always smaller than  $\rho_f$  in absolute value (this means that the polarization partially screens the external field that generates it).

A similar result applies to the surface of separation between dielectric and conductors. Assuming conductors in electrostatic equilibrium (as always, conductors are also assumed to be perfect, homogeneous, and isotherm), it must be  $\vec{E} = \vec{0}$  at each point

inside the conductors. From (18.82), it must also be  $\vec{P} = \vec{0}$  at each point inside the conductors. Following the notation in Figs. 18.5 and 18.13a, suppose material 1 is a dielectric and material 2 a conductor. In this case, the normal unit vector  $\vec{n}$  is oriented from the conductor towards the dielectric. From (18.21), imposing  $P_{n2} = 0$  inside the conductor, we have

$$\begin{aligned}\sigma_p &= -P_{n1} = -P_n = -\chi E_n = -\chi \frac{\sigma}{\epsilon_0} \\ &= -(\epsilon_r - 1)\sigma\end{aligned}\tag{18.87}$$

where we used (18.82) for the normal components of  $\vec{P}$  and  $\vec{E}$  and Coulomb's theorem for the total surface charge density  $\sigma$ . From (18.83) for the normal components of  $D_n$  and  $E_n$ , Coulomb's theorem for  $\sigma$ , and (18.62) imposing  $D_{n2} = 0 (= \epsilon E_{n2})$  inside the conductor, we find

$$\begin{aligned}\sigma_f &= D_{n1} = D_n = \epsilon E_n = \epsilon \frac{\sigma}{\epsilon_0} \\ &= \epsilon_r \sigma\end{aligned}\tag{18.88}$$

Finally, solving the system of equations (18.87) and (18.88), we obtain

$$\sigma_p = -\frac{\epsilon_r - 1}{\epsilon_r} \sigma_f\tag{18.89}$$

At last, for the surface of separation between different dielectrics we must use the boundary conditions given by Eqs. (18.21), (18.62), (18.68), and (18.71).

### 18.6.2 General Equations for the Electric Field and Potential

It is straightforward to determine the equations for the electric fields at each point inside single pieces of dielectric materials. In fact, in the volume of these materials the vectors  $\vec{D}$ ,  $\vec{P}$ , and  $\vec{E}$  are proportional, with proportionality constants that are independent from the point (see Eqs. (18.82) and (18.83)). We have,

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \frac{\vec{D}}{\epsilon} = \frac{1}{\epsilon} \vec{\nabla} \cdot \vec{D}\tag{18.90}$$

and, thus,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_f}{\epsilon}\tag{18.91}$$

This equation is analogous to that we would write in vacuum in presence of only free charges. In this case, the effect of the polarization results in changing  $\epsilon_0$  to  $\epsilon$  for the dielectric.

By defining the electric scalar potential (as usual),

$$\vec{E} = -\vec{\nabla}\phi\tag{18.92}$$

we finally obtain

$$\vec{\nabla}^2\phi = -\frac{\rho_f}{\epsilon_0\epsilon_r}\tag{18.93}$$

When  $\rho_f = 0$  in the volume, which is quite typical in a dielectric, (18.93) becomes

$$\vec{\nabla}^2 \phi = 0 \quad (18.94)$$

i.e., Laplace's equation.

In the volume of a single dielectric,  $\vec{D}$  is the same as in vacuum. In presence of a surface of discontinuity between two different dielectrics, however, this is not true in general. If a free charge with density  $\sigma_f$  is distributed on the surface of separation (i.e., an infinitesimally thin conducting sheet with charge  $\sigma_f$  is placed between the two dielectrics at their separation surface), the normal components of  $\vec{D}$  have a discontinuity that would be the same with or without dielectrics (see Eq. (18.62)). However, the tangent components of  $\vec{D}$  would have a discontinuity that depends on the presence of the two dielectrics (see Eq. (18.71)). The only scenario where  $\vec{D}$  would still behave as in vacuum is when the surface of discontinuity between the two dielectrics is normal to  $\vec{D}$  at each point; in this case, in fact,  $\vec{D}$  has no tangent components and, thus, its distribution is independent from the dielectrics.

### 18.6.3 Presence of a Single Dielectric

The simplest case for the calculation of the electric field  $\vec{E}$  is when a single linear and homogeneous dielectric with constant  $\epsilon_r$  fills the inner region of a hollow conducting cavity (electric shield). In general, as shown in Fig. 18.15, it is also assumed that a charged conductor (not hollow) is embedded in the dielectric region. Note that this problem is a general case of a charge conductor embedded in an infinitely large dielectric (which could be air). In this case, the electric shield is an equipotential sphere with infinite radius.

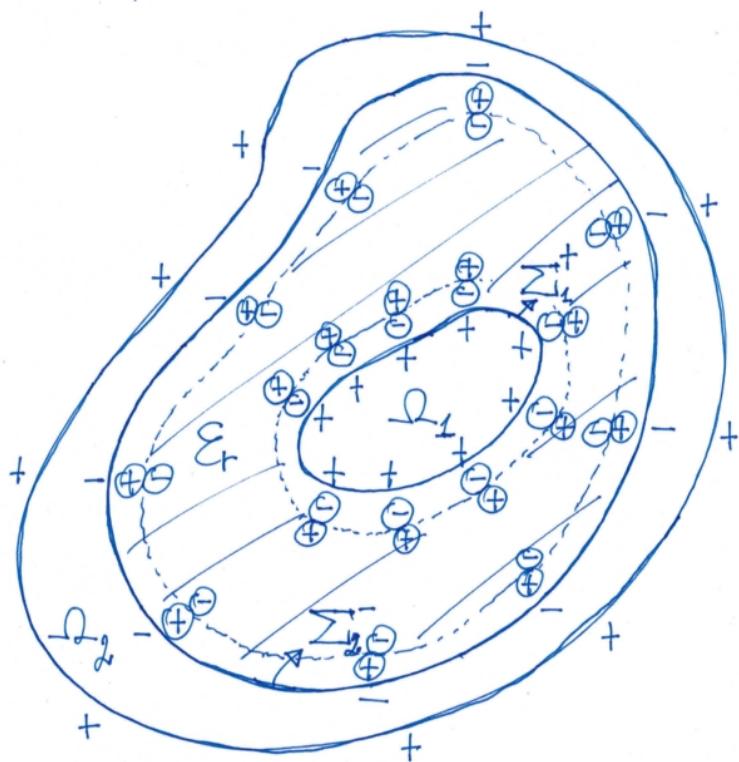


Figure 18.15

For the structure in Fig. 18.15, once, e.g., a positive charge  $+q$  is placed on the internal conductor  $\Omega_1$ , the field distribution in the hollow region between  $\Omega_1$  and  $\Omega_2$  is unequivocally determined, regardless of the charge (even if different from  $-q$ ) on  $\Omega_2$  (definition of electric shield). By means of total electrostatic induction, the charge  $+q$  on  $\Omega_1$  generates a charge  $-q$  on  $\Omega_2$ , where  $+q$  is distributed on the outer surface  $\Sigma_1^+$  of  $\Omega_1$  and  $-q$  on the inner surface  $\Sigma_2^-$  of  $\Omega_2$ .

Assume the total free charge  $+q$  and  $-q$  on the conductors are known. In presence or absence of the dielectric in the cavity, i.e., either with a dielectric or vacuum, each conductor is characterized by a constant potential, the free charges are distributed only on the conductors' surfaces ( $\Sigma_1^+$  and  $\Sigma_2^-$  in the example of Fig. 18.15), and the electric potential satisfies Laplace's equation at each point in the cavity (with dielectric or vacuum). The solution to Laplace equation must be unique. Thus, for the same total free charge on each conductor, also the free charge densities  $\sigma_f$  must be the same (i.e., they must have the same functional dependence on the point in the case of a dielectric or vacuum).

In presence of a dielectric though, the dielectric gets polarized. Hence, the dielectric surfaces adjacent to each charged conductor develop a polarization charge with surface density  $\sigma_p$ , which has opposite sign of the corresponding  $\sigma_f$  on the conductor surface and is given by (18.89). In the example of Fig. 18.15, the dielectric surface nearby  $\Omega_1$  develops a negative polarization charge, while the surface nearby  $\Omega_2$  a positive one. This example clarifies the origin of the negative sign in (18.89): Intuitively, e.g., the positive charge on  $\Omega_1$  "attracts" the negative charges of the quasi-ideal electric dipoles

that make up the dielectric (see Fig. 18.15).

The polarization charge  $\sigma_p$  tends to shield  $\sigma_f$ , i.e., to reduce the field  $\vec{E}$  in the dielectric. In fact, from (18.88) it is evident that the total  $\sigma$  is reduced by a factor  $\epsilon_r$  with respect to  $\sigma_f$ . Thus, the field  $\vec{E}_d$  at each point is equal to the field  $\vec{E}_0$  in vacuum divided by  $\epsilon_r$ :

$$\vec{E}_d = \frac{\vec{E}_0}{\epsilon_r} \quad (18.95)$$

From this result also follows that

$$\vec{D}_d = \epsilon_0 \epsilon_r \vec{E}_d = \epsilon_0 \vec{E}_0 = \vec{D}_0 \quad (18.96)$$

as expected from our previous considerations. All these results are independent from the shape and dimensions of the charged conductors.

From (18.95) it follows that also all potential differences reduce by a factor  $\epsilon_r$ . From the definition of capacitance this means that, for the same free charges on the conductors, the capacitance in presence of a dielectric increases by  $\epsilon_r$ . Thus, inserting a dielectric makes it possible to increase the capacitance of a condensator and to effectively measure the dielectric constant.

#### 18.6.4 Pressence of Many Dielectrics

We now want to discuss the method required to calculate  $\vec{E}$  in a region with multiple dielectrics separated by discontinuity surfaces (the dielectrics are assumed to be different) and conductors (with given potentials, total free charges,  $\sigma$ , etc.).

In general, the potential at each point inside the dielectrics obeys Eq. (18.93), which, in the more common case where the free charges are only in the conductors, becomes Laplace's equation. However, in the case of multiple dielectrics both field and potential change when going through one of the discontinuity surfaces between different dielectrics. For these problems, boundary conditions must be imposed not only on the conductors, but also at the discontinuity surfaces between dielectrics.

In particular, from (18.69) we must impose the continuity of the tangent components of  $\vec{E}$  at the discontinuity surfaces. This condition corresponds to the continuity of the corresponding potentials at the surface,

$$\phi_1 = \phi_2 \quad (18.97)$$

The equivalence between the continuity of the tangent components of  $\vec{E}$  and  $\phi$  is clear when observing that, considered two congruent and parallel lines on each side of a surface of discontinuity, the continuity of the tangent components of  $\vec{E}$  implies the continuity of the line integrals of  $\vec{E}$  and, thus, of the potentials.

Moreover, when  $\sigma_f = 0$  on the discontinuity surfaces, from (18.63) we have

$$\epsilon_1 \frac{\partial}{\partial n} \phi_1 = \epsilon_2 \frac{\partial}{\partial n} \phi_2 \quad (18.98)$$

We will now consider two examples that will help clarify the macroscopic theory of dielectrics developed in the previous chapters.

## 18.7 Indefinite Dielectric Slab in Parallel-Plate Condensator

Consider an indefinite planar slab made from a dielectric material; the thickness of the slab is  $\ell_1$ . We imagine to place the slab in the middle of a parallel-plate condensator; the slab is placed parallel to the condensator plates. Figure 18.16a shows a lateral cross-section of the system. The plates of the condensator, which are also indefinite, are placed at a distance  $\ell$  from each other, with  $\ell \geq \ell_1$ . The vacuum region between each plate and the dielectric, thus, has a length  $\ell_0$ . The left plate of the condensator is positively charged with a surface charge density  $\sigma_f$  on the page  $\Sigma_A^+$  (see Fig. 18.16a); the other plate has charge  $-\sigma_f$  (on  $\Sigma_D^-$ ). The surfaces of discontinuity are named  $A$ ,  $B$ ,  $C$ , and  $D$ , where  $A$  and  $D$  are the conductors' surfaces and  $B$  and  $C$  the left and right interface between vacuum and dielectric.

We consider a Cartesian coordinate system with  $x$ -axis normal to the discontinuity surfaces; the origin is chosen on  $A$ .

In absence of the dielectric, the electric field  $\vec{E}$  is uniform between the two plates; because of symmetry reasons the vector field lines of  $\vec{E}$  are oriented normally with respect to the condensators' plates and the intensity  $|\vec{E}|$  is equal at each point on any plane parallel to the plates (see Fig. 18.16a, inset). The symmetries remain valid even in presence of the dielectric slab, and thus, the arguments for  $\vec{E}$  apply also to  $\vec{D}$ . Because of the geometry of the dielectric discontinuities,  $\vec{D}$  with dielectric is the same as  $\vec{D}$  in vacuum for the same total free charge on the conductors. In fact, in the region  $x \in (0, \ell)$ , we have  $\sigma_f = 0$ ; hence, the normal and only component of  $\vec{D}$  is continuous in that region. As a consequence  $\vec{D}$  is uniform in the space between the two conductors. Calling  $\vec{D}_0$  and  $\vec{D}_1$  the electric displacement fields in vacuum and in the dielectric, respectively, it must be

$$\vec{D}_0 = \vec{D}_1 \quad (18.99)$$

The same cannot be said for  $\vec{E}$ , which is characterized by a discontinuity of the first kind at each dielectric interface. If the dielectric is linear and homogeneous, we can write

$$\vec{D}_0 = \epsilon_0 \vec{E}_0 \quad (18.100)$$

and

$$\vec{D}_1 = \epsilon_1 \vec{E}_1 = \epsilon_0 \epsilon_{r1} \vec{E}_1 \quad (18.101)$$

From (18.99) it follows that  $\epsilon_0 \vec{E}_0 = \epsilon_0 \epsilon_{r1} \vec{E}_1$  and, thus,

$$\vec{E}_0 = \epsilon_{r1} \vec{E}_1 \quad (18.102)$$

This means that the field  $\vec{E}_0$  in vacuum is  $\epsilon_{r1}$  times larger than the field  $\vec{E}_1$  in the dielectric. The results (18.102) also allows us to calculate the polarization charge at each vacuum/dielectric interface. From (18.66) and following the notation in Fig. 18.13a, at

interface  $B$  we find

$$\begin{aligned}\sigma_p|_B &= \epsilon_0(E_0\vec{u}_x \cdot (-\vec{u}_x) - E_1\vec{u}_x \cdot (-\vec{u}_x)) \\ &= \epsilon_0E_1 - \epsilon_0\epsilon_{r1}E_1 \\ &= (1 - \epsilon_{r1})\epsilon_0E_1\end{aligned}\tag{18.103}$$

which, as expected, is a negative quantity [in fact,  $(1 - \epsilon_{r1}) < 0$  when  $\epsilon_{r1} \neq \epsilon_0$  ]. The negative sign can be explained qualitatively: The uniform electric field inside the condensator is directed from  $A$  to  $D$  and, thus, it displaces towards the right the positive charge of each electric dipole in the dielectric; as a consequence, a layer of negative charge appears inside the dielectric in proximity of  $B$  (see Fig. 18.16a). For similar reasons, a layer of positive charge appears at interface  $C$  inside the dielectric. In this case,

$$\begin{aligned}\sigma_p|_C &= \epsilon_0(E_{n0} - E_{n1}) \\ &= \epsilon_0\epsilon_{r1}E_1 - \epsilon_0E_1 \\ &= (\epsilon_{r1} - 1)\epsilon_0E_1\end{aligned}\tag{18.104}$$

which, as expected, is a positive quantity because  $(\epsilon_{r1} - 1) > 0$  when  $\epsilon_{r1} \neq \epsilon_0$ . The surface charge distributions at  $B$  and  $C$  with densities  $\sigma_p|_B$  and  $\sigma_p|_C$  given by (18.103) and (18.104), respectively, generate a depolarization field directed from  $C$  to  $B$  inside the dielectric; in the dielectric, such a field diminishes the intensity of the electric field inside the condensator in absence of dielectric,  $\vec{E}_0$ , as already shown by Eq. (18.102). Note that the results of Eqs. (18.103) and (18.104) were obtained by using Eq. (18.102) and, thus,  $\sigma_p|_B$  and  $\sigma_p|_C$  are both function of the generic field intensity  $E_1$ . A more detailed calculation must be performed in order to obtain an expression for the surface polarization charge densities which only function of the boundary conditions of the problem.

Before proceeding with this calculation, we want to show how to use simple arguments to find important properties of the system under consideration.

From the definitions of electric potential and work, the potential difference between the condensator's plates (electrodes)  $A$  and  $D$  is given by

$$\begin{aligned}\Delta\phi &= \phi_A - \phi_D \\ &= (\phi_A - \phi_B) + (\phi_B - \phi_C) + (\phi_C - \phi_D) \\ &= \int_{AB} E_0\vec{u}_x \cdot \vec{u}_x dx + \int_{BC} E_1\vec{u}_x \cdot \vec{u}_x dx + \int_{CD} E_0\vec{u}_x \cdot \vec{u}_x dx \\ &= E_0\ell_0 + E_1\ell_1 + E_0\ell_0 \\ &= \epsilon_{r1}E_1\ell_0 + E_1\ell_1 + \epsilon_{r1}E_1\ell_0 \\ &= (\ell_1 + 2\epsilon_{r1}\ell_0)E_1\end{aligned}\tag{18.105}$$

where we wrote the vector fields with respect to the Cartesian coordinate system in Fig. 18.16a ( $\vec{n} = \vec{u}_x$ ) we used (18.102), and we defined the potential difference from the positive to the negative electrodes. Also note that we used the linearity of the potential function to split  $\Delta\phi$  into three sub-potential differences.

From Coulomb's theorem, the surface charge density at  $\Sigma_A^+$  is given by

$$\sigma_f|_{\Sigma_A^+} = \epsilon_0 E_0 = \epsilon_0 \epsilon_{r1} E_1 \quad (18.106)$$

Assuming the area of  $\Sigma_A^+$  to be  $A$  and supposing that  $\sqrt{A} \gg \ell$ , the total free charge on  $\Sigma_A^+$  is

$$\begin{aligned} q_f|_{\Sigma_A^+} &= \sigma_f|_{\Sigma_A^+} A \\ &= \epsilon_0 \epsilon_{r1} E_1 A \end{aligned} \quad (18.107)$$

The capacitance of the system under consideration is thus given by

$$C = \frac{q_f|_{\Sigma_A^+}}{\Delta\phi} = \frac{\epsilon_0 \epsilon_{r1} A}{\ell_1 + 2\epsilon_{r1}\ell_0} \quad (18.108)$$

In the special case when  $\ell_1 = 0$ , we find the capacitance of a condensator with plates  $A$  and  $D$  in absence of the dielectric:

$$C_0 = \epsilon_0 \frac{A}{2\ell_0} = \epsilon_0 \frac{A}{\ell} \quad (18.109)$$

When the space between plates  $A$  and  $D$  is entirely occupied by a dielectric, i.e., when  $\ell_0 = 0$ , we find

$$\begin{aligned} C_1 &= \epsilon_{r1} \epsilon_0 \frac{A}{\ell_1} = \epsilon_{r1} \epsilon_0 \frac{A}{\ell} \\ &= \epsilon_{r1} C_0 \end{aligned} \quad (18.110)$$

By measuring  $C_0$  and  $C_1$  it is possible to obtain in practice the value of  $\epsilon_{r1}$ .

These results extend to the case of many dielectric layers with different dielectric constants, all confined within the plates of a condensator. The electric field  $\vec{E}$  has larger intensity in the layers where  $\epsilon_r$  is smaller and the potential difference tends to localize in these layers.

A more rigorous solution to this problem can be obtained by solving Laplace's equation and imposing suitable boundary conditions.

The only free charge in the system of Fig. 18.16a is the charge distribution with surface density  $\sigma_f$  on  $\Sigma_A^+$  and that with surface density  $-\sigma_f$  on  $\Sigma_D^+$ . Hence, in the region  $x \in (0, \ell)$  we have

$$\frac{d^2}{dx^2} \phi(x) = 0 \quad (18.111)$$

where  $\phi(x)$  is the electric potential at each point inside the condensator; the potential is only function of  $x$  for symmetry reasons. We can divide the region  $x \in (0, \ell)$  in

three sub-regions: Region 1 for  $x \in (0, \ell_0)$  in vacuum; region 2 for  $x \in (\ell_0, \ell_0 + \ell_1)$  in the dielectric; region 3 for  $x \in (\ell_0 + \ell_1, \ell)$  again in vacuum. In a Cartesian coordinate system  $Oxyz$  and with respect to only one variable ( $x$ ), the general solution to (18.111) is

$$\phi(x) = Ax + B \quad (18.112)$$

For the system under consideration, we can write one solution of the type (18.112) for each of the three regions 1, 2, and 3, and, then, connect them by means of the boundary conditions. We have,

$$\phi(x) = \begin{cases} A_1x + B_1, & x \in (0, \ell_0) \end{cases} \quad (18.113a)$$

$$\phi(x) = \begin{cases} A_2x + B_2, & x \in (\ell_0, \ell_0 + \ell_1) \end{cases} \quad (18.113b)$$

$$\phi(x) = \begin{cases} A_3x + B_3, & x \in (\ell_0 + \ell_1, \ell) \end{cases} \quad (18.113c)$$

We will now proceed from 0 to  $\ell$  on the  $x$ -axis, impose the boundary conditions, and field  $A_i$  and  $B_i$  with  $i = 1, 2, 3$ .

In order to find  $A_1$  we use Coulomb's theorem at  $\Sigma_A^+$ ,

$$\begin{aligned} - \iint_{\Sigma_A^+} \frac{\partial}{\partial x} \phi_1(x) dA &= - \iint_{\Sigma_A^+} A_1 dA \\ &= \frac{1}{\epsilon_0} q \Big|_{\Sigma_A^+} \\ &= \frac{1}{\epsilon_0} \iint_{\Sigma_A^+} \sigma_f dA \end{aligned} \quad (18.114)$$

from which

$$A_1 = - \frac{\sigma_f}{\epsilon_0} \quad (18.115)$$

Note that the potentials in regions 1, 2, and 3 are defined  $\phi_1(x)$ ,  $\phi_2(x)$ , and  $\phi_3(x)$ , respectively.

In order to find  $B_1$  we note that  $\Sigma_A^+$  is an equipotential surface, the potential of which can be set to an arbitrary value  $\phi_A$ . Thus,

$$\phi_1(x = 0) = B_1 = \phi_A \quad (18.116)$$

In summary, for  $x \in [0, \ell_0]$

$$\phi(x) = - \frac{\sigma_f}{\epsilon_0} x + \phi_A \quad (= \phi_1(x)) \quad (18.117)$$

At the discontinuity surface  $B$  we must impose condition (4.98) with  $\epsilon_1 = \epsilon_0$  and  $\epsilon_2 = \epsilon_0 \epsilon_1$  and with  $n = x$ ,

$$\epsilon_0 \frac{\partial}{\partial x} \phi_1(x) \Big|_{x=\ell_0^-} = \epsilon_0 \epsilon_1 \frac{\partial}{\partial x} \phi_2(x) \Big|_{x=\ell_0^+} \quad (18.118)$$

from which, using (18.117),

$$-\frac{\sigma_f}{\epsilon_0} = \epsilon_{r1} A_2 \quad (18.119)$$

and, thus,

$$A_2 = -\frac{\sigma_f}{\epsilon_0 \epsilon_{r1}} \quad (18.120)$$

Imposing condition (4.97) at  $B$ ,

$$\begin{aligned} \lim_{x \rightarrow \ell_0^-} \phi_1(x) &= -\frac{\sigma_f}{\epsilon_0} \ell_0 + \phi_A \\ &= \lim_{x \rightarrow \ell_0^+} \phi_2(x) = -\frac{\sigma_f}{\epsilon_0 \epsilon_{r1}} \ell_0 + B_2 \end{aligned} \quad (18.121)$$

from which,

$$\begin{aligned} B_2 &= \frac{\sigma_f}{\epsilon_0} \left( \frac{1}{\epsilon_{r1}} - 1 \right) \ell_0 + \phi_A \\ &= \frac{\sigma_f}{\epsilon_0} \frac{1 - \epsilon_{r1}}{\epsilon_{r1}} \ell_0 + \phi_A \end{aligned} \quad (18.122)$$

In summary, for  $x \in [\ell_0, \ell_0 + \ell_1]$

$$\begin{aligned} \phi(x) &= -\frac{\sigma_f}{\epsilon_0 \epsilon_{r1}} x + \frac{\sigma_f}{\epsilon_0} \frac{1 - \epsilon_{r1}}{\epsilon_{r1}} \ell_0 + \phi_A \\ &= \phi_2(x) \end{aligned} \quad (18.123)$$

At  $C$  we have

$$\epsilon_{r1} \frac{\partial}{\partial x} \phi_2(x) \Big|_{x=(\ell_0+\ell_1)^-} = \frac{\partial}{\partial x} \phi_3(x) \Big|_{x=(\ell_0+\ell_1)^+} \quad (18.124)$$

from which,

$$-\epsilon_{r1} \frac{\sigma_f}{\epsilon_0 \epsilon_{r1}} = A_3 \quad (18.125)$$

and, thus,

$$A_3 = -\frac{\sigma_f}{\epsilon_0} \quad (18.126)$$

As expected, in vacuum  $A_1 = A_3$ . The slope  $A_2$  is less steep compared to  $A_1$  and  $A_3$ .

From the continuity if the potential at  $C$ ,

$$\begin{aligned} \lim_{x \rightarrow (\ell_0 + \ell_1)^-} \phi_2(x) &= -\frac{\sigma_f}{\epsilon_0 \epsilon_{r1}} (\ell_0 + \ell_1) + \frac{\sigma_f}{\epsilon_0} \frac{1 - \epsilon_{r1}}{\epsilon_{r1}} \ell_0 + \phi_A \\ &= -\frac{\sigma_f}{\epsilon_0} \left( \frac{1}{\epsilon_{r1}} \ell_1 + \ell_0 \right) + \phi_A \\ &= \lim_{x \rightarrow (\ell_0 + \ell_1)^+} \phi_3(x) \\ &= -\frac{\sigma_f}{\epsilon_0} (\ell_0 + \ell_1) + B_3 \end{aligned} \quad (18.127)$$

from which,

$$B_3 = \frac{\sigma_f}{\epsilon_0} \frac{\epsilon_{r1} - 1}{\epsilon_{r1}} \ell_1 + \phi_A \quad (18.128)$$

In summary, for  $x \in [\ell_0 + \ell_1, \ell]$

$$\begin{aligned} \phi(x) &= -\frac{\sigma_f}{\epsilon_0} x + \frac{\sigma_f}{\epsilon_0} \frac{\epsilon_{r1} - 1}{\epsilon_{r1}} \ell_1 + \phi_A \\ & \quad (= \phi_3(x)) \end{aligned} \quad (18.129)$$

It is easy to verify that

$$\phi_A > \phi(x = \ell_0) > \phi(x = \ell_0 + \ell_1) > \phi(x = 2\ell_0 + \ell_1) \quad (18.130)$$

As a sanity check, we apply Coulomb's theorem at  $\Sigma_D^-$ ,

$$\begin{aligned} - \iint_{\Sigma_D^-} -\frac{\partial}{\partial x} \phi_3(x) dA &= -\frac{\sigma_f}{\epsilon_0} \iint_{\Sigma_D^-} dA \\ &= -\frac{1}{\epsilon_0} q \Big|_{\Sigma_A^+} \end{aligned} \quad (18.131)$$

as expected (the double negative sign is because  $\vec{n}_D = -\vec{u}_x$ ).

In summary

$$\phi(x) = \begin{cases} -\frac{\sigma_f}{\epsilon_0} x + \phi_A , & x \in [0, \ell_0] \end{cases} \quad (18.132a)$$

$$\phi(x) = \begin{cases} -\frac{\sigma_f}{\epsilon_0 \epsilon_{r1}} x + \frac{\sigma_f}{\epsilon_0} \frac{1 - \epsilon_{r1}}{\epsilon_{r1}} \ell_0 + \phi_A , & x \in [\ell_0, \ell_0 + \ell_1] \end{cases} \quad (18.132b)$$

$$\phi(x) = \begin{cases} -\frac{\sigma_f}{\epsilon_0} x + \frac{\sigma_f}{\epsilon_0} \frac{\epsilon_{r1} - 1}{\epsilon_{r1}} \ell_1 + \phi_A , & x \in [\ell_0 + \ell_1, \ell] \end{cases} \quad (18.132c)$$

This potential is plotted in Fig. 18.16b.

The field  $\vec{E}$  can be obtained from  $\vec{E} = -\vec{\nabla}\phi = -d\phi/dx$ ,

$$\vec{E}(x) = \begin{cases} \frac{\sigma_f}{\epsilon_0} \vec{u}_x , & x \in [0, \ell_0] \end{cases} \quad (18.133a)$$

$$\vec{E}(x) = \begin{cases} \frac{\sigma_f}{\epsilon_0 \epsilon_{r1}} \vec{u}_x , & x \in [\ell_0, \ell_0 + \ell_1] \end{cases} \quad (18.133b)$$

$$\vec{E}(x) = \begin{cases} \frac{\sigma_f}{\epsilon_0} \vec{u}_x , & x \in [\ell_0 + \ell_1, \ell] \end{cases} \quad (18.133c)$$

From the knowledge of  $\vec{E}$ , we can finally calculate the polarization surface charge densities at  $B$  and  $C$ :

$$\sigma \Big|_B = \frac{1 - \epsilon_{r1}}{\epsilon_{r1}} \sigma_f \quad (18.134)$$

## 18.7. INDEFINITE DIELECTRIC SLAB IN PARALLEL-PLATE CONDENSATOR

and

$$\sigma|_C = -\sigma|_B \quad (18.135)$$

where we used (18.103) and (18.104).

The field  $\vec{E}$  inside the condensator is plotted in Fig. 18.16c (intensity).

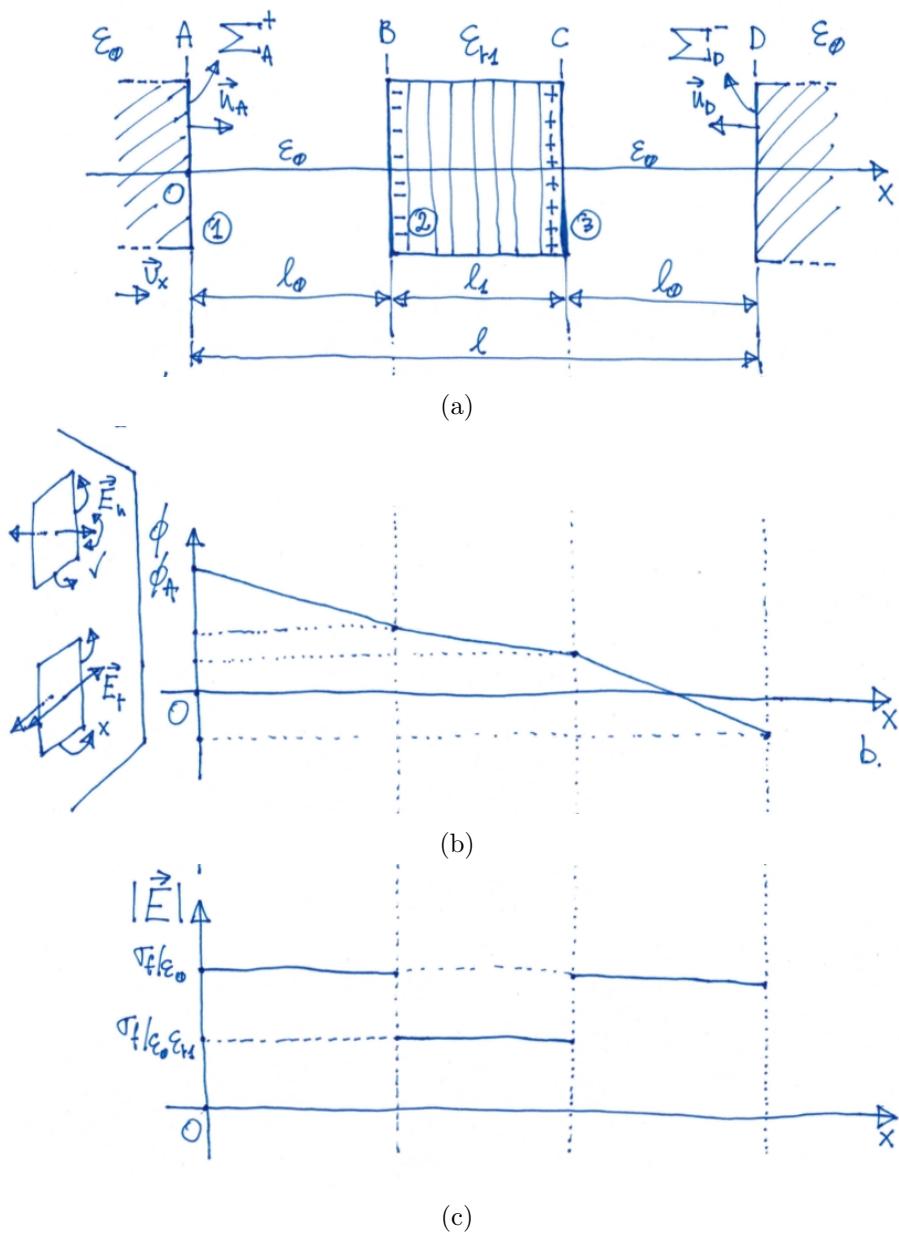


Figure 18.16

## 18.8 Dielectric Sphere of Radius $a$ in an External Electric Field (Assumed to be Uniform in Absence of the Sphere and with Intensity $\vec{E}_0$ )

Consider a dielectric sphere of radius  $a$  and with dielectric constant  $\epsilon_r$ ; the sphere is placed within the plates of a planar condensator, which is assumed to be charged and with dimensions much larger than the sphere.

Under these conditions, the free charges on the plates are sufficiently far from the sphere at the time the sphere is introduced that their distribution practically does not change. This means the electric field at distance sufficiently far from the sphere remains uniform. In proximity of the sphere, however, this is not true; the sphere gets polarized with a distribution  $\vec{P}$ , which, in turn, generates a depolarizing electric field  $\vec{E}_p$ .

The total electric field inside and outside the sphere is given by

$$\vec{E} = \vec{E}_0 + \vec{E}_p \quad (18.136)$$

where  $\vec{E}_0$  us the external field (uniform). The polarization vector  $\vec{P}$  depends on  $\vec{E}$ ,

$$\vec{P} = \chi \vec{E} = (\epsilon_r - 1) \epsilon_0 \vec{E} \quad (18.137)$$

where we assumed the dielectric to be linear and homogeneous.

Formally, we could solve this problem finding the electric potential by means of Laplace's equation, with the boundary conditions (4.97) and (4.98) at the surface of the sphere and the surface of the sphere and the condition that the potential is that of a uniform field with intensity  $E_0$  at large distance from the sphere.

However, from the results for  $\vec{E}_p$  inside a uniformly polarized sphere, we can determine  $\vec{E}$  in a more direct fashion.

We can suppose the sphere polarizes uniformly and verify this is a good assumption a posteriori. We can thus assume the depolarization field  $\vec{E}_p$  inside the sphere is uniform and given by (18.43),

$$\vec{E}_p = -\frac{\vec{P}}{3\epsilon_0} \quad (18.138)$$

By combining (18.138), (18.136), and (18.137) we have

$$\vec{E} = \vec{E}_0 - \frac{\vec{P}}{3\epsilon_0} = \vec{E}_0 - \frac{\epsilon_r - 1}{3} \vec{E} \quad (18.139)$$

from which we find that  $\vec{E}$  inside the sphere is uniform and given by

$$\vec{E} = \frac{3}{2 + \epsilon_r} \vec{E}_0 \quad (18.140)$$

Since  $\epsilon_r > 1$ , the electric field inside the dielectric is smaller than  $\vec{E}_0$ .

The expression for  $\vec{P}$  as a function of  $\vec{E}_0$ s can be obtained from (18.140) and (18.137),

$$\vec{P} = \frac{3(\epsilon_r - 1)}{2 + \epsilon_r} \epsilon_0 \vec{E}_0 \quad (18.141)$$

## 18.9. DIELECTRIC OF ARBITRARY GEOMETRICAL SHAPE IN A PARALLEL-PLATE CONDENSATOR

since  $\vec{E}_0$  is uniform in the region occupied by the sphere, so is  $\vec{P}$ , thus confirming our hypothesis.

The electric field in the points outside the sphere can be found from (18.136) remembering that in these points  $\vec{E}_p$  is the dipole field of a dipole  $|\vec{P}|4\pi a^3/3$  located at the center of the sphere

As shown in Fig. 18.17, the field lines of  $\vec{E}$  tend to go through the material with higher dielectric constant.

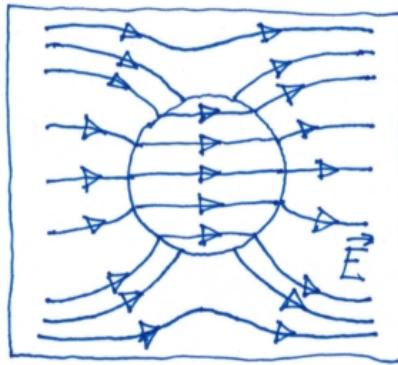


Figure 18.17

## 18.9 Dielectric of Arbitrary geometrical Shape in a Parallel-Plate Condensator

From the previous example, we could be tempted to conclude that any linear and homogeneous dielectric in a uniform electric field gets polarized uniformly. This is only true for dielectric with geometrical shape such that  $\vec{E}_p$  inside them is uniform (proof not shown here). It can be proven that this is only possible for ellipsoid (and, thus, spheres).

For arbitrary dielectrics, e.g., a finite cylinder, inside a charged condensator,  $\vec{P}$  is in general non uniform (even for linear and homogeneous dielectrics).

If the potential difference  $\Delta\phi$  between the condensator's plates and the relative dielectric constant  $\epsilon_r$  are known,  $\phi$  is determined by the following conditions:

- 1) At each point outside the dielectric (i.e., in vacuum),  $\phi = \phi_1$  and  $\vec{\nabla}^2\phi_1 = 0$ ;
- 2) At each point inside the dielectric,  $\phi = \phi_2$  and  $\vec{\nabla}^2\phi_2 = 0$ ;
- 3) At each point on the external surface of the condensator's plates  $A$  and  $B$ ,  $\phi_1$  must have constant values  $\phi_A$  and  $\phi_B$ , respectively (we can set  $\phi_B = 0$  so that  $\Delta\phi = \phi_A - \phi_B = \phi_A$ );
- 4) At each point on  $A$  and  $B$   $\phi_1$  satisfies Coulomb's theorem (if the charge density on  $A$  and  $B$  is given):

5) At each point on the cylinder,

$$\left\{ \begin{array}{l} \phi_1 = \phi_2 \end{array} \right. \quad (18.142a)$$

$$\left\{ \begin{array}{l} \frac{\partial}{\partial n} \phi_1 = \epsilon_r \frac{\partial}{\partial n} \phi_2 \end{array} \right. \quad (18.142b)$$



# Chapter 19

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## 19.1 The Magnetostatic Field in Presence of Magnetic Materials

### 19.1.1 Introduction

In the next few chapters we will study the behavior of magnetic materials. We will follow an approach similar to that followed for dielectrics. However, when studying magnetic materials, we will encounter major differences compared to the dielectrics.

Linear dielectrics behave quantitatively very differently than vacuum; linear magnetic materials, however, behave practically like vacuum and, thus, are usually neglected in practical applications.

Nonlinear dielectrics (ferroelectric materials) are less relevant than linear dielectrics; on the contrary, nonlinear magnetic materials (ferromagnetic materials) are extremely important in applications. For example, common magnets are made from ferromagnetic materials.

The macroscopic model of dielectrics was obtained assuming dielectric materials are made from a continuous distribution of electric dipoles. Similarly, a macroscopic model of magnetic materials can be obtained by representing the materials as a continuous distribution of magnetic dipoles, i.e., as a continuum of infinitesimally small (in a physics sense) closed loops of electric current.

Consider the Rutherford-Bohr model of the hydrogen atom. In this model, the electron that rotates about the proton is equivalent to a current of intensity

$$i = \frac{e}{T} \quad (19.1)$$

where  $e$  is the electron charge and  $T$  is the orbit period. In fact, the electron passes through each ideal cross-section of the orbit  $1/T$  times.

The magnetostatic dipole moment associated with electron is thus

$$\begin{aligned} m_{\text{el}} = m_B &= iA = \frac{e}{T} \pi r^2 \\ &= e \frac{\omega}{2\pi} \pi r^2 \\ &= e \frac{m_e}{m_e} \frac{\omega}{2} r^2 \end{aligned}$$

$$= (m_e \omega r^2) \frac{e}{2m_e} = \hbar \frac{\ell}{2m_e} \quad (19.2)$$

where  $m_e$  is the electron mass,  $\hbar = m_e \omega r^2 = r^2 m$ ,  $\omega = I\omega$  is the unit of orbital angular momentum in atomic physics, and  $m_B$  is called Bohr magneton (unit of the atomic magnetic moments).

### 19.1.2 The Magnetization Field

Consider a point  $P$  in a generic magnetic material; the volume element  $\Delta V$  is centered in  $P$ . If  $\Delta \vec{m}$  is the magnetic moment of the material contained within  $\Delta V$ , the magnetization vector field  $\vec{M}$  or  $\vec{M}$ -field at point  $P$  is defined as

$$\vec{M} = \lim_{\Delta V \rightarrow 0^+} \frac{\Delta \vec{m}}{\Delta V} \quad (19.3)$$

This field is, thus, the magnetic moment per unit volume associated with the material. Note the analogy between the so defined vector  $\vec{M}$  and the polarized vector  $\vec{P}$  for dielectrics. As for  $\vec{P}$ ,  $\vec{M}$  can be written as

$$\vec{M} = N \overline{\vec{m}} \quad (19.4)$$

where  $N$  is the number of atoms per unit volume and  $\overline{\vec{m}}$  is the mean value of the atomic magnetic moment.

We can now define the current density vector  $\vec{J}_m$  in the material. This vector is called magnetization current or bound (volumetric) current and is analogous to the polarization charge or bond (volumetric) charge  $\rho_p$  in dielectrics. Consider a physics infinitesimal surface  $\Delta A$  containing a point  $P$ . At point  $P$ , we can define  $\vec{J}_m$  as

$$(\vec{J}_m)_n = \vec{J}_m \cdot \vec{n} = \lim_{\Delta A \rightarrow 0^+} \frac{I_m}{\Delta A} \quad (19.5)$$

where  $\vec{n}$  is the normal unit vector for  $\Delta A$  at  $P$  and  $I_m$  is the magnetization current intensity through  $\Delta A$ ;  $I_m$  is the algebraic sum of all microscopic currents crossing  $\Delta A$ . As always, the limit in (19.5) must be regarded as a macroscopic limit.

Consider a surface  $A_\gamma$  that has as a border a closed line  $\gamma$ . The surface is entirely contained in a material with  $N_c$  elementary current loops per unit volume. The total current through  $A_\gamma$  is due to all and only the currents of the elementary current loops linked with  $\gamma$ , as shown in Fig. 19.1a. For all the other loops, which are cut by  $A_\gamma$ , the contribution to the total current is zero because sum of two contributions of equal absolute value, but opposite sign (in correspondence to the two points at which the loop is cut by  $A_\gamma$ ). We assume the current loops to be circular with orientation given by their respective magnetic moment.

In order to calculate the total current through  $A_\gamma$  we can assume all current loops to have the same orientation along the mean direction of all loops and with area  $A_e$  equal to the average of the projection of all loops' areas onto the plane normal to the mean direction. This corresponds to consider all the equivalent dipoles to have the same direction and intensity equal to the mean equivalent dipole. If the material is not magnetized, the atomic magnetic moments all have random orientations and, thus,

$\vec{m} = 0$  and  $\vec{M} = \vec{0}$ . Hence, an element  $d\vec{\ell}$  crosses all the current loops the center of which is contained within the cylindrical volume obtained by translating one loop parallel to itself and with center on  $d\vec{\ell}$ , as shown in Fig. 19.1b. This volume is

$$A_e = \frac{\vec{m}}{|\vec{m}|} \cdot d\vec{\ell} \quad (19.6)$$

where vector  $\vec{m}$  is the mean atomic magnetic moment. If  $N_c$  is the number of loops per unit volume and  $i$  the current of one loop, the total current of the loops crossed by  $d\vec{\ell}$  is

$$\begin{aligned} N_c i A_e \frac{\vec{m}}{|\vec{m}|} \cdot d\vec{\ell} &= \frac{i A_e}{|\vec{m}|} N_c \vec{m} \cdot d\vec{\ell} \\ &= N_c \vec{m} \cdot d\vec{\ell} = \vec{M} \cdot d\ell \end{aligned} \quad (19.7)$$

Therefore, the current associated with the entire surface  $A_\gamma$  is

$$\iint_{A_\gamma} \vec{J}_m \cdot \vec{n} dA = \oint_{\gamma} \vec{M} \cdot d\vec{\ell} \quad (19.8)$$

By means of Stokes' theorem, the line integral in (19.8) can be transformed in a surface integral, thus obtaining

$$\iint_{A_\gamma} \vec{J}_m \cdot \vec{n} dA = \iint_{A_\gamma} \vec{\nabla} \times \vec{M} \cdot \vec{n} dA \quad (19.9)$$

Since  $A_\gamma$  can be arbitrarily chosen, we finally obtain

$$\vec{\nabla} \times \vec{M} = \vec{J}_m \quad (19.10)$$

This result is valid at each point of a material, where the material is continuous. It cannot be used at each point of the external surface of a magnetic material because, at these points,  $\vec{M}$  is discontinuous and, thus,  $\vec{\nabla} \times \vec{M}$  is not well-defined (at least in the usual sense).

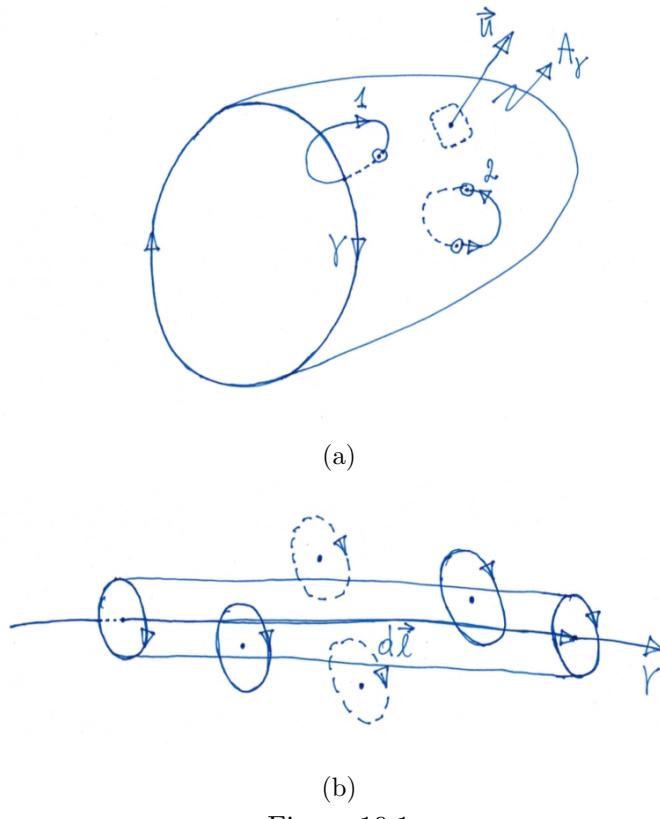


Figure 19.1

The local form of (19.8) at the points on the external surface of a magnetic material can be obtained by means of a cut-type curve  $\gamma_c$ . It is simpler to first obtain the local form for two magnetic materials with different magnetization, as shown in Fig. 19.2, where  $\vec{J}_{ms}$  is the surface magnetization current.

Figure 19.2a shows a cut-type curve  $\vec{J}'_c$  and Fig. 19.2b a cut-type curve  $\gamma''_c$ . Two curves are needed because  $\vec{J}_{ms}$  has two components,  $\vec{J}_{mx}$  and  $\vec{J}_{my}$ .

The curve  $\gamma'_c$  is oriented clockwise with respect to the chosen direction of  $\vec{J}_{my}$ . In particular,  $\vec{J}_{my} = J_{my}(P)\vec{u}_y = J_{my}(P)\vec{t}'$ . We assume  $\overline{AB} = d\ell'_2$ ,  $\overline{CD} = d\ell'_1$ , and  $\overline{BC} = \overline{DA} = h$ .

From (19.8)

$$\begin{aligned}
 \oint_{\gamma'_c} \vec{M} \cdot \vec{t} d\ell &= \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} \\
 &= \int_{\gamma'_s} \vec{J}_{ms} \cdot \vec{t} d\ell = J_{my} d\ell' \tag{19.11}
 \end{aligned}$$

where  $\gamma'_s$  is the cross-section between the rectangular surface enclosed by  $\gamma'_c$  and  $\Sigma$ , and  $d\ell'$  its length.

In the limit  $h \rightarrow 0$ ,  $d\ell'_1 = d\ell'_2 = d\ell'$ ,  $\overline{BC} = \overline{DA} = 0$ , and  $-\vec{t}'_1 = +\vec{t}'_2 = \vec{t}'' = \vec{u}_x$ .

Thus, (19.11) becomes

$$\vec{M}_2 \cdot \vec{t}'' d\ell' - \vec{M}_1 \cdot \vec{t}'' d\ell' = J_{my} d\ell' \quad (19.12)$$

from which

$$M''_{+2} - M''_{+1} = J_{my} \quad (19.13)$$

A similar argument applies to  $\gamma''_c$ , which is  $\gamma'_c$  rotating by an angle  $\pi/2$  about  $\vec{n}$ . For consistency with the signs for  $\vec{t}''_1$ ,  $\vec{t}''_2$ , and  $\vec{t}''$  (i.e.,  $\vec{t}''_1 = -\vec{t}''_2 = \vec{t}''$ ),  $\gamma''_c$  must be oriented counterclockwise with respect to the chosen direction of  $\vec{J}_{mx}$ . Equation (19.8) in the case of  $\gamma''_c$  gives

$$M'_{t2} - M'_{t1} = -J_{mx} \quad (19.14)$$

Note that if there were volume current densities  $\vec{J}_{m1}$  and  $\vec{J}_{m2}$  in the neighborhood of  $P$ , the corresponding current intensities through the surfaces of the rectangles  $A'DCB'$  and  $AA'B'B$ , respectively, would be infinitesimal of higher order compared to the intensity  $I'm$  through  $\gamma'_s$ . This is because the areas of these rectangles go to zero when  $h \rightarrow 0$ . A similar argument applies to the case of  $\gamma''_c$ .

Since  $\vec{t}'' = \vec{u}_x$ ,  $\vec{t}' = \vec{u}_y$ , and  $\vec{n} = \vec{u}_z$ , (19.13) and (19.14) can be combined in a single relationship

$$(\vec{M}_1 - \vec{M}_2) \times \vec{n} = \vec{J}_{ms} \quad (19.15)$$

where  $(\vec{M}_1 - \vec{M}_2) = (M''_{t1} - M''_{t2}, M'_{t1} - M'_{t2}, M_{n1} - M_{n2})$ ,  $\vec{n} = (0, 0, 1)$ , and  $\vec{J}_{ms} = (J_{mx}, J_{my}, 0)$ . In fact

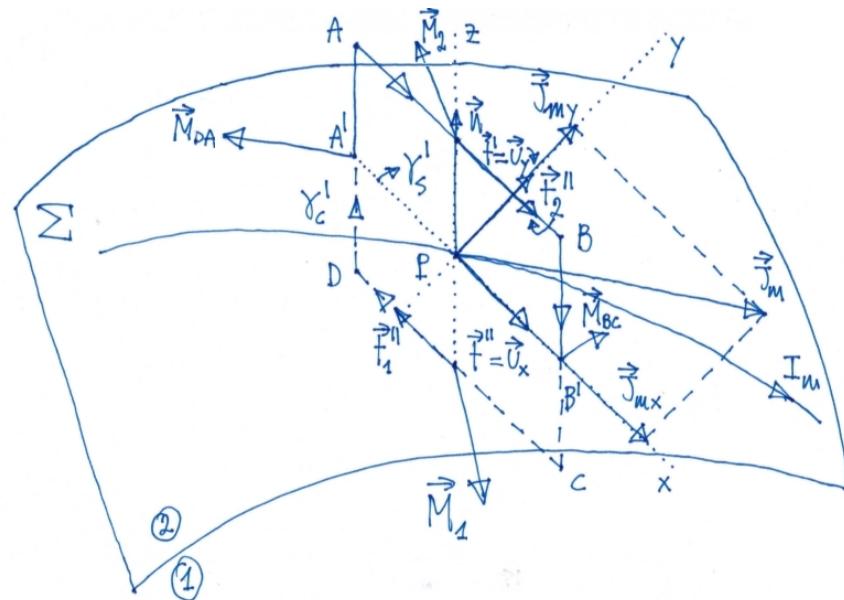
$$\begin{aligned} (\vec{M}_1 - \vec{M}_2) \times \vec{n} &= \begin{vmatrix} \vec{t}'' = \vec{u}_x & \vec{t}' = \vec{u}_y & \vec{u} = \vec{z} \\ (M''_{t1} - M''_{t2}) & (M'_{t1} - M'_{t2}) & (M_{n1} - M_{n2}) \\ 0 & 0 & 1 \end{vmatrix} \\ &= (M'_{t1} - M'_{t2})\vec{u}_x - (M''_{t1} - M''_{t2})\vec{u}_y \\ &= J_{mx}\vec{u}_x + J_{my}\vec{u}_y \end{aligned} \quad (19.16)$$

When the region ② in Fig. 19.2 is the vacuum, from (19.15) with  $\vec{M}_2 = \vec{0}$  we have

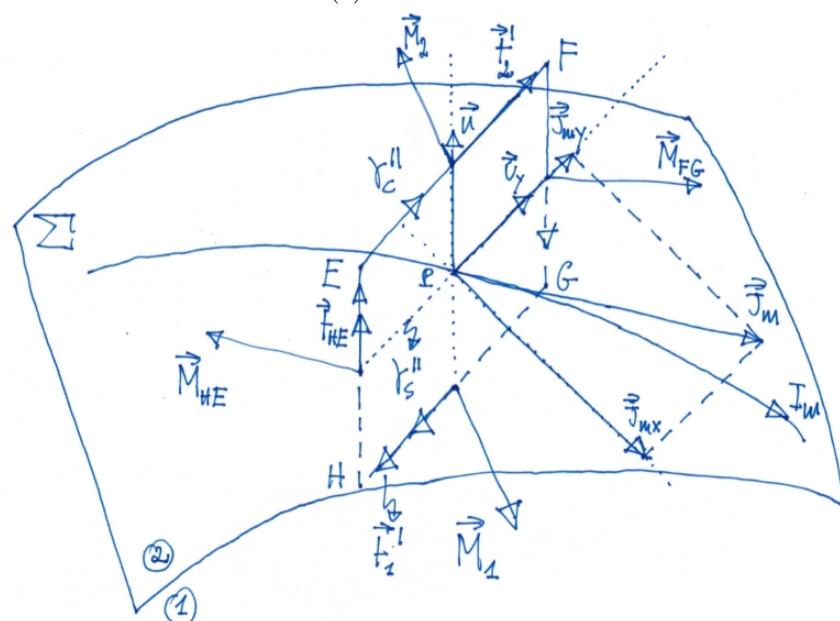
$$\vec{M}_1 \times \vec{n} = \vec{M} \times \vec{n} = \vec{J}_{ms} \quad (19.17)$$

In summary, a magnetic material is equivalent to a system of currents given by:

- a) A volume current distribution with density (19.10);
- b) A surface current distribution with density (19.17).



(a)



(b)

Figure 19.2

# Chapter 20

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## 20.1 Laws of Magnetostatics for Magnetic Materials

We now want to extend the laws of magnetostatics from the case of vacuum to that of a generic magnetic material.

As for the dielectrics, we define the macroscopic vectors  $\vec{B}$  and  $\vec{J}$  as the mean values of the corresponding microscopic fields,  $\vec{B} = \langle \vec{B}_{\text{micro}} \rangle$  and  $\vec{J} = \langle \vec{J}_{\text{micro}} \rangle$ .

In presence of molecular currents, the solenoidal property of  $\vec{B}$  remains valid:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (20.1)$$

The total macroscopic current density  $\vec{J}$  is given by the algebraic sum in a vector sense of the conduction current density  $\vec{J}_c$  and the magnetization current density  $\vec{J}_m$ . The former is due to the oriented motion of the charge free carriers, while the latter is due to the magnetization of the material. We remind that the fundamental properties of  $\vec{J}_c$  are (see PHYS 242 for details)

$$\iint_{\Sigma} \vec{J}_c \cdot \vec{n} dA = 0 \quad (20.2)$$

in integral form,

$$\vec{\nabla} \cdot \vec{J}_c = 0 \quad (20.3)$$

in differential form in the points in a volume, and

$$J_{n1} = J_{n2} \quad (20.4)$$

in local form at the discontinuity points on the surface separating two different materials, 1 and 2. The integral, differential, and local form for the properties of  $\vec{J}_m$  are

$$\iint_{A_{\gamma}} \vec{J}_m \cdot \vec{n} dA = \oint_{\gamma} \vec{M} \cdot d\ell \quad (20.5)$$

$$\vec{\nabla} \times \vec{M} = \vec{J}_m \quad (20.6)$$

and

$$(\vec{M}_1 - \vec{M}_2) \times \vec{n} = \vec{J}_{ms} \quad (20.7)$$

as we found in the previous chapter.

We can thus extend Ampère's law to the case of magnetic materials, obtaining

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_{\text{tot}} = \mu_0 (\vec{J}_c + \vec{J}_m) \quad (20.8)$$

The equivalent integral form of (20.1) and (20.8) are

$$\oint_{\Sigma} \vec{B} \cdot \vec{n} dA = 0 \quad (20.9)$$

where  $\Sigma$  is a generic closed surface, and

$$\oint_{\gamma} \vec{B} \cdot \vec{t} d\ell = \mu_0 I_{\text{tot}} = \mu_0 (I_c + I_m) \quad (20.10)$$

where  $I_{\text{tot}}$  is the total current (algebraic sum of  $I_c$  and  $I_m$ , the conduction and magnetization current intensities, respectively) linked with the closed line  $\gamma$ .

Using (20.6) in (20.8), we have

$$\vec{\nabla} \times \left( \frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_c \quad (20.11)$$

By defining

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \quad (20.12)$$

as the magnetic field strength or  $\vec{H}$ -field, (20.11) can be rewritten as

$$\vec{\nabla} \times \vec{H} = \vec{J}_c$$

or, equivalently, in integral form as

$$\oint_{\gamma} \vec{H} \cdot \vec{t} d\ell = I_c \quad (20.13)$$

Thus, the SI units of  $H$  are  $\text{Am}^{-1}$ . Note the analogy between  $\vec{H}$  and vector  $\vec{D}$  for the dielectrics. The  $\vec{\nabla} \cdot \vec{D}$  is only due to the free charges; similarly, the  $\vec{\nabla} \times \vec{H}$  is only due to the conduction currents.

From (20.1), we find

$$\vec{\nabla} \cdot [\mu_0 (\vec{H} + \vec{M})] = 0 \quad (20.14)$$

and, thus,

$$\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M} \quad (20.15)$$

Therefore, in presence of magnetic materials,  $\vec{B}$  remains solenoidal, whereas  $\vec{M}$  and  $\vec{H}$ , in general, are not (we will return to this point later). In fact, in general it is not possible to calculate  $\vec{H}$  from the sole knowledge of the conduction currents. Vector  $\vec{H}$  also depends on its sources (i.e., the points where the divergence of the field is

different than zero). The sources depend on the geometry of the given system and the nonuniformities of the magnetic materials. Once again, the analogy to  $\vec{D}$  is evident.

In order to describe the magnetic behavior of a material from a macroscopic point of view, it is sufficient to give the relationship between vector  $\vec{B}$  (magnetic induction field) and  $\vec{H}$ . As for the dielectrics, the field responsible for the magnetization of the material is due to the current external to the material as well as the magnetization currents.

Note that we will study the case of surface discontinuity in the next chapter.

## 20.2 Classification of Magnetic Materials

In order to determine experimentally the so-called magnetic characteristic function  $B = B(H)$  for a certain material, it is necessary to measure  $B$  and  $H$  separately at each point in the material and for different magnetization conditions. One possible method is to use a solenoidal toroid with a total number of loops  $N$  and current  $I$ ; the magnetic material to be studied can be made of toroidal shape and used as the core of the solenoid (see Fig. 20.1). If the magnetic material is isotropic and homogeneous (but, in general, nonlinear), because of symmetry reasons (see PHYS 242) the vector lines of  $\vec{H}$  are contained within the solenoid and are circular with same axis as the revolution axis of the solenoid ( $\vec{n}$ ). From (20.13), the intensity of  $\vec{H}$  at a distance  $r$  from  $\vec{n}$  is given by

$$H = \frac{NI}{2\pi r} \quad (20.16)$$

Due to the symmetry of the system,  $H$  is independent from the material and can be arbitrarily changed by tuning  $I$ . Thus, from (20.16) and the measurement of  $B$  it is possible to find  $B(H)$  for any magnetic material.

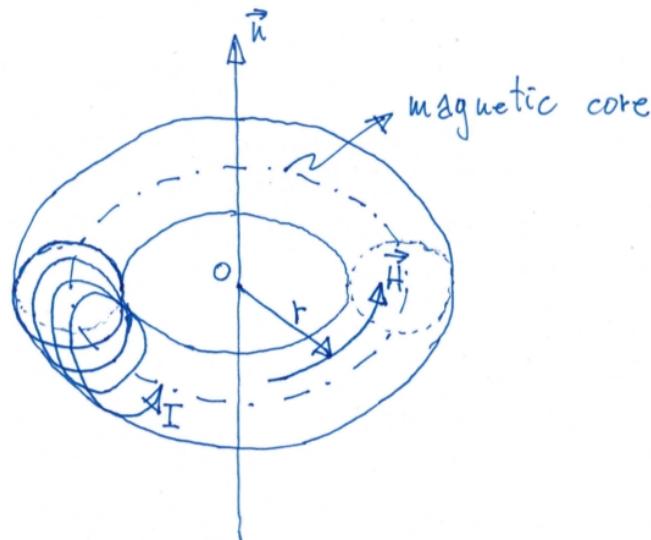


Figure 20.1

From the knowledge of  $B(H)$  it is possible to classify magnetic materials in three major categories: Diamagnetic materials (e.g., copper, silver, bismuth, and graphite);

paramagnetic materials (e.g., air, aluminium, and platinum); ferromagnetic materials (e.g., iron, nickel, and cobalt).

Diamagnetic materials and paramagnetic materials are characterized by a linear  $B(H)$  characteristic, i.e.,

$$\vec{B} = \mu \vec{H} \quad (20.17)$$

where  $\mu$  is a characteristic constant for the material, which is called magnetic permeability. The SI units of  $\mu$  are those of  $B/H$ , Tm/A.

From (20.17) it follows that there is a linear relationship also between  $\vec{M}$  and  $\vec{B}$ . In fact, from (20.12) we have

$$\vec{B} = \mu_0 \left( \vec{H} + \vec{M} \right) = \mu_0 \left( \frac{\vec{B}}{\mu} + \vec{M} \right) \quad (20.18)$$

and, thus,

$$\vec{M} = \left( \frac{1}{\mu_0} - \frac{1}{\mu} \right) \vec{B} \quad (20.19)$$

For diamagnetic materials

$$\mu < \mu_0 \quad (20.20)$$

and for paramagnetic materials

$$\mu > \mu_0 \quad (20.21)$$

The relative permeability is defined as

$$\mu_r = \frac{\mu}{\mu_0} \quad (20.22)$$

Thus, (20.20) and (20.21) can be written as

$$\mu_r < 1 \quad (20.23)$$

and

$$\mu_r > 1 \quad (20.24)$$

respectively.

As it turns out, the relative permeabilities of diamagnetic and paramagnetic materials are very close to unity. For diamagnetic materials  $\epsilon_r \sim 1 - 10^{-5}$  and for paramagnetic materials  $\epsilon_r \sim 1 + 10^{-4}$ . Hence, in most cases both diamagnetic and paramagnetic materials are considered as if they were in vacuum.

Figure 20.2a shows the qualitative behavior of the  $B(H)$  characteristics for diamagnetic and paramagnetic materials with respect to that of vacuum.

Quite different is the behavior for ferromagnetic materials, which, for the same value of  $H$ , can have values of  $B$  even  $10^6$  larger. In addition, for these materials the  $B(H)$  characteristic is highly nonlinear, as qualitatively shown in Fig. 20.2b.

Remarkably, in the case of cyclic magnetization (i.e.,  $H$  alternates between positive and negative values), the  $B(H)$  characteristic becomes multi-valued. This means that for a given value of  $H$ , there can be different (multiple) values of  $B$ , depending on the magnetization “history” of the material. Figure 20.2c shows the co-called hysteresis

cycle of a ferromagnetic material. The cycle starts at point  $P$ , which is characterized by the values  $H_{\max}$  and  $B_{\max}$ ; by progressively reducing the values of  $H$  in the material, the values of  $B$  follow the path  $PRCP'$  according to the orientation shown in the figure; the first part of the cycle ends at point  $P'$ , which is the symmetric of  $P$  with respect to  $O$ ; then, by increasing  $H$  from  $-H_{\max}$  to  $+H_{\max}$ , the values of  $B$  follow the path  $P'R'C'P$  according to the shown direction, until reaching point  $P$  again.

Thus, for a single value of  $H$  corresponds two values of  $B$ . Note that the case of Fig. 20.2b corresponds to a non-cyclic magnetization cycle.

Particularly interesting is point  $R$  (see Fig. 20.2c), where the material is magnetized ( $B \neq 0$ ) even if there is no external current acting on it ( $H = 0$ ). This is a permanent magnetization phenomena, which is quite common for many materials. A typical example is magnetite ( $\text{Fe}_3\text{O}_4$ ), from which common magnets are made.

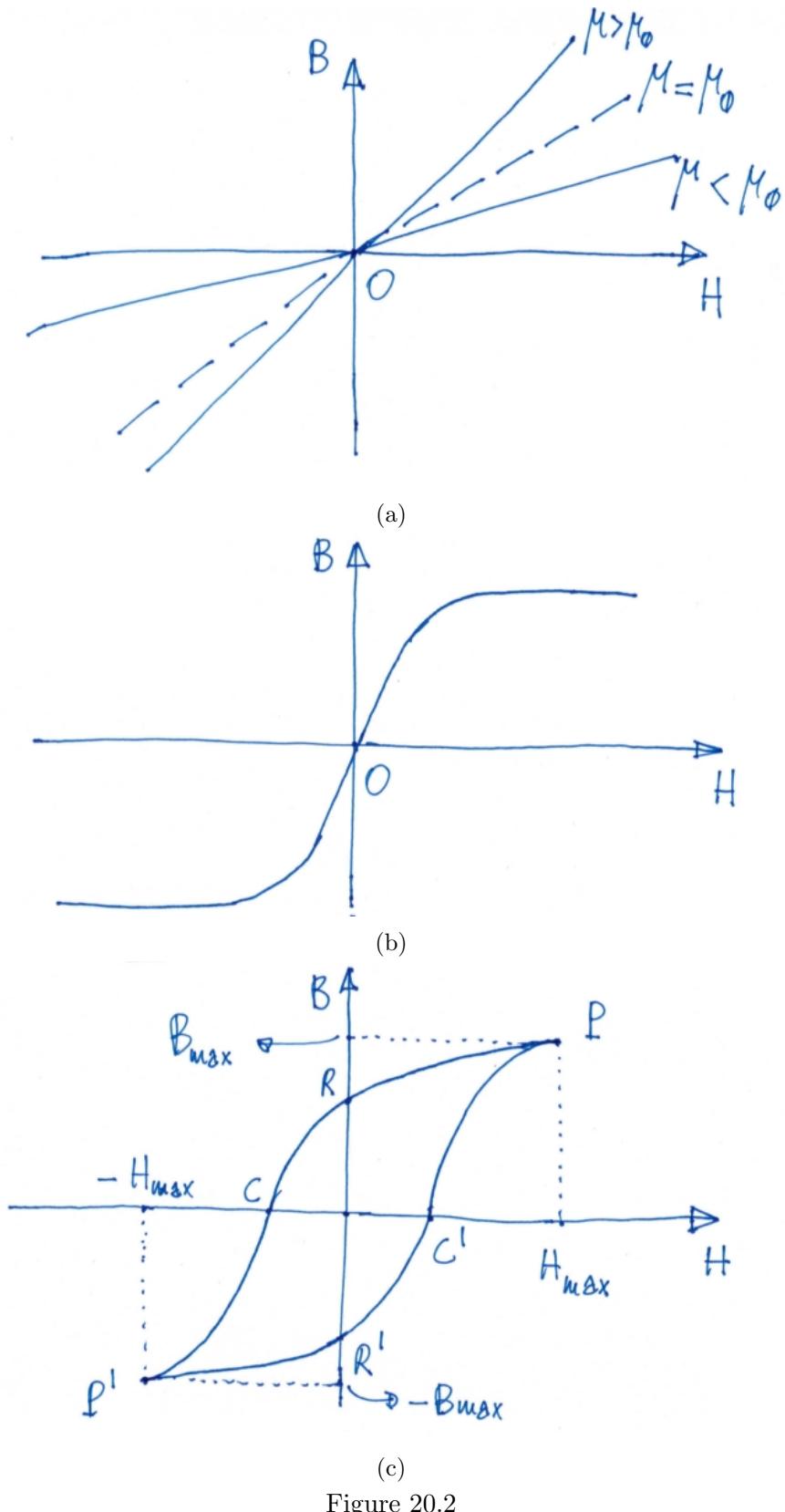


Figure 20.2

# Chapter 21

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## 21.1 Boundary Conditions for Vector $\vec{B}$ and $\vec{H}$

As always, after studying the laws for the vector fields  $\vec{B}$  and  $\vec{H}$  in integral form and in differential form at each point in the volume of a magnetic material, we now want to find the corresponding laws in local form in proximity of the separation surface  $A$  between two different magnetic materials, 1 and 2. The local forms in proximity of  $A$  are also called the boundary conditions for  $\vec{B}$  and  $\vec{H}$ .

In vacuum or in presence of magnetic materials, vector  $\vec{B}$  is always solenoidal. Thus, we can readily find the boundary conditions for the normal components of  $\vec{B}$  at any point on  $A$  by means of a coin-type surface  $\Sigma_c$ , as shown in Fig. 21.1.

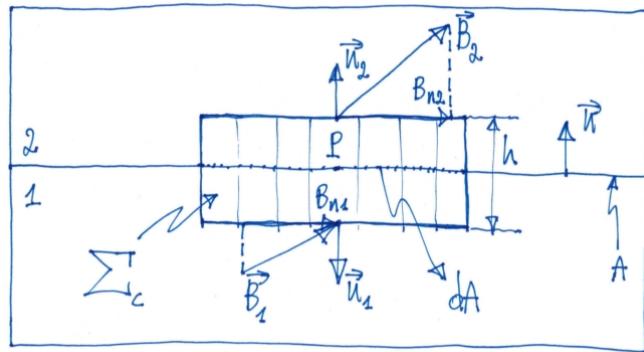


Figure 21.1

The surface  $\Sigma_c$  comprises a lateral surface of height  $h$  and two bases with infinitesimal area  $dA_1$  and  $dA_2$ . When  $h \rightarrow 0^+$ , the lateral surface becomes negligible,  $dA_1 = dA = dA_2$ , and, following the notation in Fig. 21.1,  $\vec{n}_1 = -\vec{n} = \vec{n}_2$ . The solenoidal property of  $\vec{B}$  in integral form can be written as

$$\begin{aligned}
 \iint_{\Sigma_c} \vec{B} \cdot \vec{n} dA &= \vec{B}_1 \cdot \vec{n}_1 dA_1 + \vec{B}_2 \cdot \vec{n}_2 dA_2 \\
 &= \vec{B}_1 \cdot (-\vec{n}) dA + \vec{B}_2 \cdot \vec{n} dA \\
 &= -B_{n1} dA + B_{n2} dA = 0
 \end{aligned} \tag{21.1}$$

or, equivalently,

$$B_{n2} - B_{n1} = 0 \quad (21.2)$$

From Ampère's theorem for vector  $\vec{B}$  along a cut-type line  $\gamma_c$

$$\oint_{\gamma_c} \vec{B} \cdot \vec{t} d\ell = \mu_0 I_t \quad (21.3)$$

where  $I_t$  is the total current linked with  $\gamma_c$ . Assuming this current is associated with a volume current density  $\vec{J}_t$  and a surface current density  $\vec{J}_{ts}$ , the contribution of  $\vec{J}_t$  to the line integral (21.3) is negligible because the flux through the area of  $\gamma_c$  tends to 0 when the cut-type line is squeezed around the separation surface  $A$ . However, the contribution of  $\vec{J}_{ts}$  is non-negligible and, thus,

$$(\vec{B}_1 - \vec{B}_2) \times \vec{n} = \vec{J}_{ts} \quad (21.4)$$

We remind that this boundary condition is actually obtained using two cut-type curves  $\gamma'_c$  and  $\gamma''_c$  and from the definition

$$I_{ts} = \int_{\gamma'_s, \gamma''_s} \vec{J}_{ts} \cdot \vec{t} d\ell \quad (21.5)$$

where  $\gamma'_s$  and  $\gamma''_s$  are the intersection between  $\gamma'_c$  and  $A$  and  $\gamma''_c$  and  $A$ , respectively, as shown in Fig 19.2.

The same two cut-type lines  $\gamma'_c$  and  $\gamma''_c$  can be used to obtain the boundary conditions for the tangent components of  $\vec{H}$ . Assume the volume conduction current densities  $\vec{J}_{c1}$  and  $\vec{J}_{c2}$  and the surface conduction current density  $\vec{J}_{cs}$  exist in the neighborhood of a point  $P$  of the boundary  $A$ . Ampère's law for  $\vec{H}$

$$\oint_{\gamma_c} \vec{H} \cdot \vec{t} d\ell = I_{ct} \quad (21.6)$$

applied to  $\gamma_c = \gamma'_c$  and  $\gamma_c = \gamma''_c$  gives

$$(\vec{H}_1 - \vec{H}_2) \times \vec{n} = \vec{J}_{cs} \quad (21.7)$$

where, as always,  $\vec{H}_1$  and  $\vec{H}_2$  are the values of  $\vec{H}$  in the neighborhood of  $P$  just below and above  $A$ .

When  $\vec{J}_{cs} = 0$ , we have

$$H_{t2} - H_{t1} = 0 \quad (21.8)$$

along any tangent direction at  $P$ . In this case, if both materials 1 and 2 are linear we can write

$$B_{n1} = \mu_{r1} \mu_0 H_{n1} = \mu_{r2} \mu_0 H_{n2} = B_{n2} \quad (21.9)$$

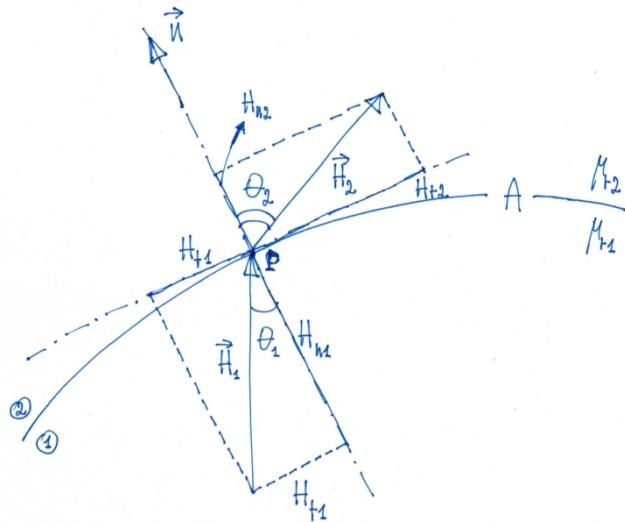


Figure 21.2

where  $\mu_{r1}$  and  $\mu_{r2}$  are the relative magnetic permeabilities of the two materials. Following the notation in Fig. 21.2 and using Eqs. (21.8) and (21.9), we have

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\frac{H_{t1}}{H_{n1}}}{\frac{H_{t2}}{H_{n2}}} = \frac{\mu_{r1}}{\mu_{r2}} \quad (21.10)$$

which represents the refraction law for the vector lines of  $\vec{H}$  at  $A$ . If one of the two materials has magnetic permeability larger than the other (e.g.,  $\mu_1 > \mu_2$ ) the field lines exiting that material tend to be normal to  $A$ , i.e.,  $\tan \theta_1 > \tan \theta_2$ .

## 21.2 Magnetization of Isotropic, Linear, and Homogeneous Materials

In general, the magnetization vector  $\vec{M}$  of an isotropic, linear and homogeneous magnetic material is a function of the induction field  $\vec{B}$ . In turn,  $\vec{B}$  is the sum of a field  $\vec{B}_0$  due to external conduction currents  $\vec{J}_c$  and a field  $\vec{B}_m$  due to the distribution of the magnetization currents  $\vec{J}_m$ ,

$$\vec{B} = \vec{B}_0 + \vec{B}_m \quad (21.11)$$

Hence, if  $\vec{B}_0$  is uniform,  $\vec{M}$  is not necessarily uniform. The magnetization vector  $\vec{M}$  is uniform only if also  $\vec{B}_m$  is uniform.

We shall now consider a few examples where the field  $\vec{B}_m$  is generated by a given  $\vec{M}$  under the assumption that the magnetic materials are rigid, i.e.,  $\vec{M}$  does not depend on  $\vec{B}_m$ . Note that hereafter we will drop the subscript “ $m$ ” from  $\vec{B}_m$  for simplicity.

### 21.2.1 Finite Cylindrical Magnet

Consider a permanent cylindrical magnet of finite length and uniformly magnetized along its longitudinal axes (see Fig. 21.3a).

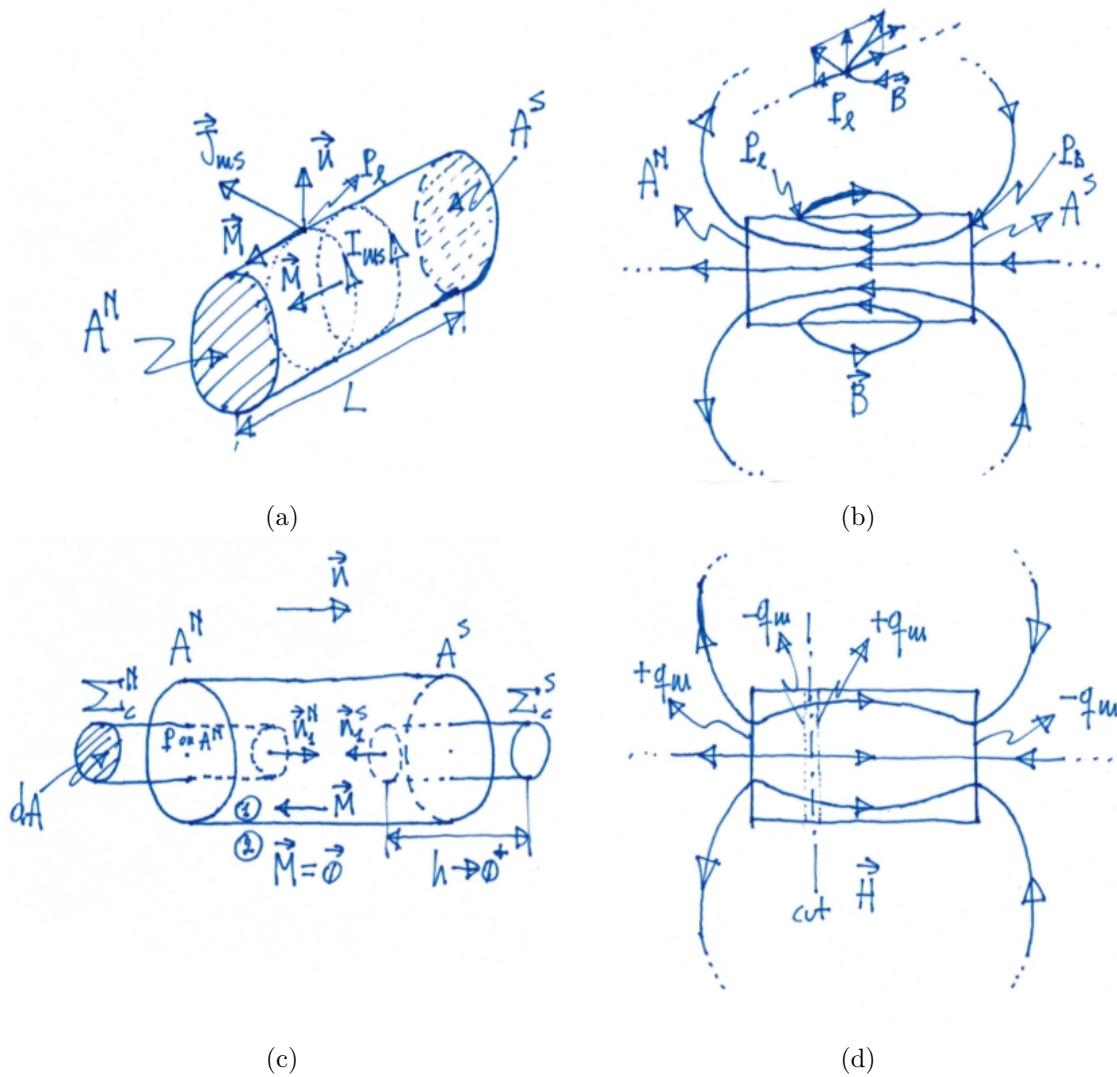


Figure 21.3

In this case the volume magnetization current density  $\vec{J}_m$  is given by

$$\vec{J}_m = \vec{\nabla} \times \vec{M} = \vec{0} \quad (21.12)$$

because  $\vec{M}$  is uniform. The surface magnetization current density  $\vec{J}_{ms}$ , instead, is given by

$$\vec{J}_{ms} = \vec{M} \times \vec{n} \quad (21.13)$$

thus generating a surface magnetization current with intensity  $I_{ms}$  as shown in Fig. 21.3a. As a consequence, the permanent magnet can be considered as a finite solenoid with current  $I_{ms}$ . From the right-hand rule, the direction of  $\vec{B}$  inside the magnet is that shown in Fig. 21.3b. Since  $\vec{\nabla} \cdot \vec{B} = 0$ , the field lines of  $\vec{B}$  must be closed. In addition, on the magnet bases  $\Sigma_c^N$  and  $\Sigma_c^S$  both the tangent and normal components of  $\vec{B}$  are continuous

$$\begin{cases} B_{t1} = B_{t2} \end{cases} \quad (21.14a)$$

$$\begin{cases} B_{n1} = B_{n2} \end{cases} \quad (21.14b)$$

because  $\vec{J}_{ms}|_{A^N} = \vec{0}$  and  $\vec{J}_{ms}|_{A^S} = \vec{0}$ . On the lateral surfaces of the magnet  $\vec{J}_{ms}$  is given by Eq. 21.13 and, thus, the normal components of  $\vec{B}$  are continuous, but its tangent components are discontinuous with discontinuity given by  $|\vec{J}_{ms}|$ . A qualitative behavior of the field lines of  $\vec{B}$  inside and outside the magnet is shown in Fig. 21.3b.

Outside the magnet  $\vec{M} = \vec{0}$  and

$$\vec{H} = \frac{\vec{B}}{\mu_0} \quad (21.15)$$

Therefore, the qualitative behavior of field lines of  $\vec{H}$  is similar to that of the field lines of  $\vec{B}$ .

At each point inside the permanent magnet the field lines of  $\vec{H}$  can be found from the relationship

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \quad (21.16)$$

Since any conduction current inside the magnet is assumed to be absent,

$$\vec{J}_c = \vec{J}_{cs} = \vec{0} \quad (21.17)$$

the field  $\vec{H}$  is irrotational

$$\vec{\nabla} \times \vec{H} = \vec{0} \quad (21.18)$$

that is  $\vec{H}$  can be written as a gradient of a magnetic scalar potential  $\phi_m$ ,

$$\vec{H} = -\vec{\nabla} \phi_m \quad (21.19)$$

In this case, vector  $\vec{H}$  behaves as the electrostatic field  $\vec{E}$ .

At each point in the volume  $\Omega$  of a magnetic material

$$\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M} \quad (21.20)$$

A magnetic charge volume density  $\rho_m$  can be defined as

$$\rho_m = \vec{\nabla} \cdot \vec{M} \quad (21.21)$$

clearly showing that the divergence of  $\vec{M}$  can be regarded as a source for the  $\vec{H}$ -field. In the case of a uniformly magnetized material,  $\rho_m = 0$ . In general Eq. (21.20) can be written in integral form as

$$\begin{aligned} \iiint_{\Omega} \vec{\nabla} \cdot \vec{H} dV &= \iint_{\Sigma} \vec{H} \cdot \vec{n} dA \\ &= - \iiint_{\Omega} \vec{\nabla} \cdot \vec{M} dV \\ &= - \iint_{\Sigma} \vec{M} \cdot \vec{n} dA \end{aligned} \quad (21.22)$$

where  $\Sigma$  is the total lateral surface of the material. The end result of Eq. (21.21) was obtained by using the divergence theorem. We can thus define a magnetic charge  $q_m$  as

$$\iint_{\Sigma} \vec{H} \cdot \vec{n} dA = - \iint_{\Sigma} \vec{M} \cdot \vec{n} dA = q_m \quad (21.23)$$

with the caveat that positive and negative magnetic charges cannot be separated. This result can be regarded as Gauss' theorem for the  $\vec{H}$ -field.

Figure 21.3c shows two coin-type surfaces  $\Sigma_c^N$  and  $\Sigma_c^S$  in correspondence of a point on  $A^N$  and  $A^S$ , respectively. In the limit  $h \rightarrow 0^+$ ,  $n_1^N = \vec{n} = -n_1^S$  and the area of each cross-section for both coin-type surfaces becomes  $dA$ . Under these conditions, Gauss' theorem gives

$$\iint_{\Sigma_c^S} \vec{H} \cdot \vec{n} dA = - \vec{M} \cdot \vec{n}_1^S dA = \vec{M} \cdot \vec{n} dA = -M_n dA = -q_m \quad (21.24)$$

$$\iint_{\Sigma_c^N} \vec{H} \cdot \vec{n} dA = - \vec{M} \cdot \vec{n}_1^N dA = - \vec{M} \cdot \vec{n} dA = +M_n dA = +q_m \quad (21.25)$$

where  $\vec{M} = -M\vec{n}$ .

The scalar potential  $\phi_m$ , thus, is equivalent to that of an electric dipole with charges  $\pm q_m$ , as shown in Fig. 21.3d. Hence, the vector lines of  $\vec{H}$  inside the magnet are directed from the north to the south pole, i.e., opposite to  $\vec{B}$ . Figure 21.3d shows the qualitative behavior of the the field lines of  $\vec{H}$  inside and outside the magnet.

### 21.2.2 Infinite Cylindrical Magnet

Consider a permanent cylindrical magnet of infinite length and uniformly magnetized along its longitudinal axes (see Fig. 21.4a).

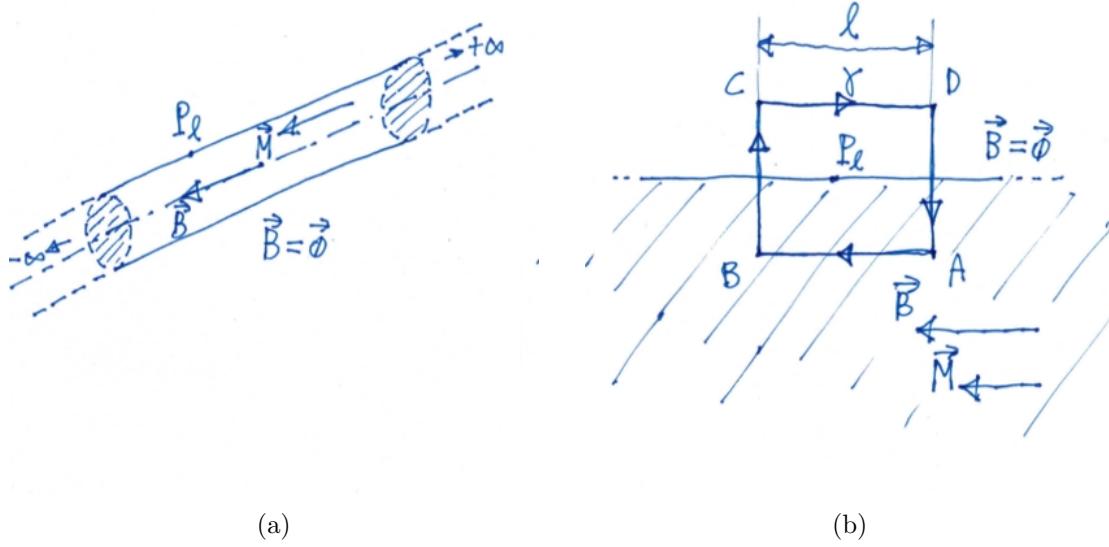


Figure 21.4

Because of the symmetry of the problem this magnet is equivalent to an infinite solenoid. Inside the magnet, the induction field  $\vec{B}$  is uniform and directed as  $\vec{M}$ ; outside the magnet the  $\vec{B}$ -field is zero.

By means of Ampère's theorem for  $\vec{B}$  along the line  $\gamma$  shown in Fig. 21.4b, we obtain

$$\begin{aligned}
 \oint \vec{B} \cdot d\vec{\ell} &= \mu_0 I_{ms} \\
 &= \mu_0 |\vec{J}_{ms}| \ell = \mu_0 |\vec{M} \times \vec{n}| \ell \\
 &= \mu_0 M \ell
 \end{aligned} \tag{21.26}$$

In this case, at each point inside the magnet we obtain

$$\left\{ \begin{array}{l} \vec{B} = \mu_0 \vec{M} \end{array} \right. \tag{21.27a}$$

$$\left\{ \begin{array}{l} \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = \vec{0} \end{array} \right. \tag{21.27b}$$

It is clear that if the magnet was not infinitely long the induction field  $\vec{B}$  generated at its interior would be smaller than (21.27a) because the total surface current would be smaller. The case of the infinite cylindrical magnet represents the configuration with maximum efficiency for the creation of a field  $\vec{B}$  due to a uniform magnetization  $\vec{M}$ . Hence, (21.27a) is an upper bound for  $\vec{B}$ , i.e., in the general case of a finite cylindrical magnet

$$|\vec{B}| \leq \mu_0 |\vec{M}| \tag{21.28}$$

due to the linearity assumption vectors  $\vec{B}$  and  $\vec{H}$  are parallel to each other both inside and outside the magnet, and so are the vectors  $\vec{M}$  and  $\vec{B}$ . As a consequence for a finite

magnet we can write

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \leq \vec{0} \quad (21.29)$$

showing one more time that the  $\vec{H}$  and  $\vec{B}$  have opposite directions inside the magnet.

### 21.2.3 Magnetic Slab

Consider a cylindrical magnet of finite length and bases of infinite radius. Such an infinite disk is equivalent to a slab. In this case, there exists a finite  $I_{ms}$  on the lateral surface of the disk given by  $J_{ms}\ell$ , where  $\ell$  is the width of the disk. Since  $I_{ms}$  is finite, but at the infinite distance from any other point in the disk, at each point inside the disk (note that there are no points outside the disk) it must be

$$\begin{cases} \vec{B} = \vec{0} \\ \vec{H} = -\vec{M} \end{cases} \quad (21.30a)$$

$$\begin{cases} \vec{B} = \vec{0} \\ \vec{H} = -\vec{M} \end{cases} \quad (21.30b)$$

# Chapter 22

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## 22.1 Magnetization of Isotropic, Linear, and Homogeneous Materials

We continue with the study of isotropic, linear, and homogeneous materials.

### 22.1.1 Spherical Magnet

Consider a sphere of radius  $R$  made from a magnetic material. The sphere's volume is called  $\Omega$  and its surface  $\Sigma$ . Assume the sphere is uniformly magnetized with  $\vec{M}$ , as shown in Fig. 22.1a.

For each point  $P \in \Omega$  we have

$$\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M} = \rho_m = 0 \quad (22.1)$$

that is, there is no volume magnetic charge density  $\rho_m$  because  $\vec{M}$  is uniform.

Consider a point  $P \in \Sigma$ . The surface magnetic charge density at  $P$ ,  $\sigma_m$ , can be found employing a coin-type surface  $\Sigma_c$  as shown in Fig. 22.1a. As always, the bases of  $\Sigma_c$  are the infinitesimals  $dA$  and its height is  $h$ . Following the notation in the figure, when  $h \rightarrow 0^+$  we have

$$\begin{aligned} \iint_{\Sigma_c} \vec{H} \cdot \vec{n} dA &= -\vec{M} \cdot \vec{n}_1 dA \\ &= -\vec{M} \cdot (-\vec{n}) dA \\ &= M \cos \theta dA \\ &= \sigma_m dA \end{aligned} \quad (22.2)$$

where  $\theta$  is the angle between  $\vec{M}$  and  $\vec{n}$ .

Under the assumption that all conduction current densities  $\vec{J}_c = \vec{0}$  inside and outside the sphere, field  $\vec{H}$  is irrotational,

$$\vec{\nabla} \times \vec{H} = \vec{J}_c = \vec{0} \quad (22.3)$$

This implies that, within the star domain,  $\vec{H}$  is conservative and can be described by a scalar potential  $\phi_m$ ,

$$\vec{H} = -\vec{\nabla} \phi_m \quad (22.4)$$

By inserting Eq. (22.4) into (22.1), which represents the solenoidal property of  $\vec{B}$ , we find

$$\mu_0 \vec{\nabla} \cdot (-\vec{\nabla} \phi_m + \vec{M}) = 0 \quad (22.5)$$

from which

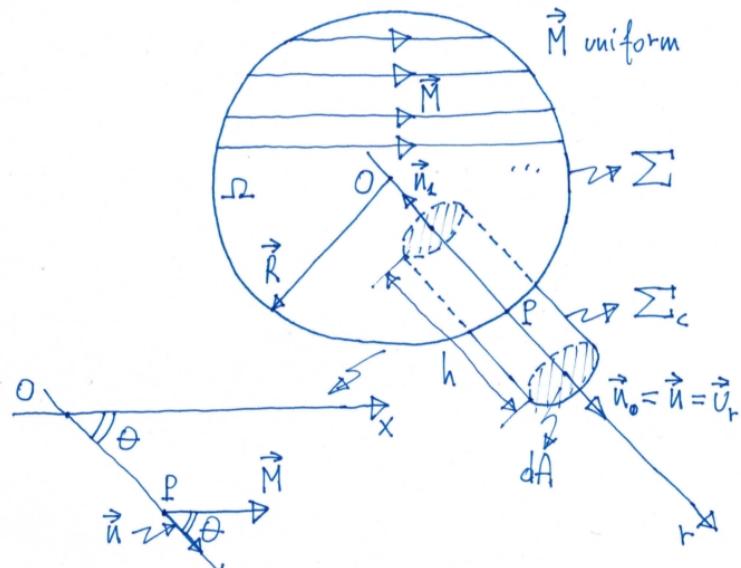
$$\mu_0 \vec{\nabla} \cdot \vec{M} - \mu_0 \vec{\nabla} \cdot (\vec{\nabla} \phi_m) = 0 \quad (22.6)$$

and finally

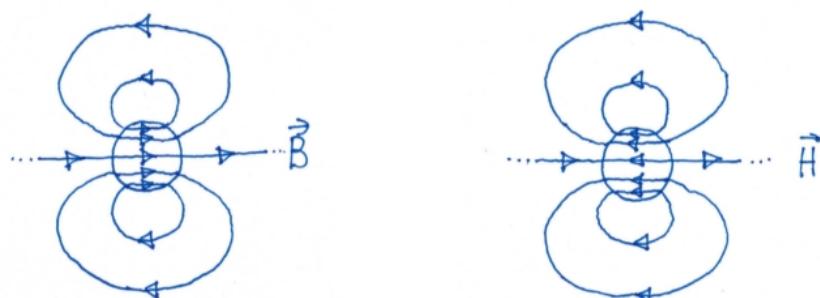
$$\vec{\nabla}^2 \phi_m = -\rho_m \quad (22.7)$$

This is Poisson's equation for  $\vec{H}$ . Note that no factor  $\mu_0$  appears in this equation.

At this point, Eq. (22.7) for  $\rho_m = 0$  (i.e., Laplace's equation) in combination with the boundary condition given by Eq. (22.2) can be solved at each point in space to obtain  $\phi_m$ ,  $\vec{H}$ , and  $\vec{B}$ . This calculation is mathematically involved and requires the knowledge of spherical harmonics.



(a)



(b)

(c)

Figure 22.1

A simple solution to the problem can be obtained in analogy to the case of the uniformly polarized dielectric sphere. Given the volume of the sphere  $V = 4\pi R^3/3$ , the magnetic dipole moment associated with the sphere is given by

$$\vec{m} = V \vec{M} \quad (22.8)$$

The dipole scalar potential  $\phi_m(\vec{r})$  for  $\vec{r} > R$  can be found from the equivalent dielectric problem exchanging  $\vec{P}$  with  $\vec{M}$ ,

$$\phi_m(\vec{r}) = \frac{1}{4\pi} \frac{\vec{M} \cdot \vec{r}}{r^3} V \quad (22.9)$$

At each point on  $\Sigma$ , i.e., for  $|\vec{r}| = R$  we have

$$\vec{M} \cdot \vec{R} = MR \cos \theta = Mx \quad (22.10)$$

where  $x$  is the parametric coordinate of a point on  $\Sigma$  along the  $x$ -axis shown in Fig. 22.1a. As a consequence

$$\phi(\vec{R}) = \frac{1}{4\pi} \frac{Mx}{R^3} \frac{4}{3} \pi R^3 = \frac{1}{3} Mx \quad (22.11)$$

Note that because of the special spherical symmetry, the dipole solution (22.9) is valid not only far away from the sphere, but also in its proximity. This can be formally shown by means of spherical harmonics.

For small  $\vec{r} < R$ , we can write Laplace's equation

$$\vec{\nabla}^2 \phi_m = 0 \quad (22.12)$$

The sphere is uniformly magnetized along the direction of  $\vec{M}$ . This means that with respect to magnetization the sphere appears as the cross-section of the cylinder of radius  $R$ . For this reason  $\phi_m$  is expected to be only a function of  $x$ -coordinate. Hence,

$$\phi_m = Ax + B \quad (22.13)$$

Due to the arbitrariness of  $B$ , we can set  $B = 0$ . We can thus guess

$$K = \frac{1}{3} M \quad (22.14)$$

and so

$$\phi_m(x) = \frac{1}{3} Mx \quad (22.15)$$

at each point in  $\Omega$ . Finally

$$\vec{H} = -\vec{\nabla} \phi_m = -\frac{1}{3} \vec{M} \quad (22.16)$$

Remembering that  $\vec{M} = \vec{0}$  for  $\vec{r} > R$ , for  $\vec{r} > R$

$$\begin{cases} \vec{B} = \mu_0 \vec{H} \\ \vec{H} = -\frac{1}{4\pi} \vec{\nabla} \left( \frac{\vec{m} \cdot \vec{r}}{r^3} \right) \end{cases} \quad (22.17a) \quad (22.17b)$$

For  $\vec{r} \leq R$

$$\begin{cases} \vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} = \frac{2}{3} \mu_0 \vec{M} \end{cases} \quad (22.18a)$$

$$\begin{cases} \vec{H} = -\frac{1}{3} \vec{M} \end{cases} \quad (22.18b)$$

Figure 22.1b shows the qualitative behavior of the field lines of  $\vec{B}$  and  $\vec{H}$ . Again,  $\vec{H}$  and  $\vec{B}$  are opposite of each other inside the magnetized sphere.

It is worth finding the  $\vec{B}$ -field associated with the  $\vec{H}$ -field (22.17b). Since  $\vec{M} = \vec{0}$  outside the sphere,  $\vec{B} = \mu_0 \vec{H}$ , thus, for  $\vec{r} > \vec{R}$ ,

$$\vec{B} = -\frac{\mu_0}{4\pi} \vec{\nabla} \left( \frac{\vec{m} \cdot \vec{r}}{r^3} \right) \quad (22.19)$$

Note that

$$\begin{aligned} \vec{\nabla} \left( \frac{\vec{m} \cdot \vec{r}}{r^3} \right) &= \frac{1}{r^3} \vec{\nabla}(\vec{m} \cdot \vec{r}) + (\vec{m} \cdot \vec{r}) \vec{\nabla} \left( \frac{1}{r^3} \right) \\ &= \frac{1}{r^3} \vec{\nabla}(m_x x + m_y y + m_z z) - \frac{(\vec{m} \cdot \vec{r}) \vec{r}}{r^5} \\ &= \frac{\vec{m}}{r^3} - \frac{(\vec{m} \cdot \vec{r}) \vec{r}}{r^5} \end{aligned} \quad (22.20)$$

where we use the fact that the components of  $\vec{m}$  are all constant and  $\vec{\nabla} x = \vec{u}_x$  and similarly for  $y$  and  $z$ .

Assume an external uniform induction field  $\vec{B}_0$  exists in the region of the magnetized sphere. By means of the superposition principle for  $\vec{r} > \vec{R}$

$$\begin{cases} \vec{B} = \vec{B}_0 + \frac{\mu_0}{4\pi} \left[ -\frac{\vec{m}}{r^3} + \frac{(\vec{m} \cdot \vec{r}) \vec{r}}{r^5} \right] \\ \vec{H} = \frac{\vec{B}}{\mu_0} \end{cases} \quad (22.21a)$$

$$(22.21b)$$

for  $\vec{r} \leq \vec{R}$

$$\begin{cases} \vec{B} = \vec{B}_0 + \frac{2}{3} \mu_0 M \end{cases} \quad (22.22a)$$

$$\begin{cases} \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = \frac{\vec{B}_0}{\mu_0} + \frac{2}{3} \vec{M} - \vec{M} = \vec{H}_0 - \frac{1}{3} \vec{M} \end{cases} \quad (22.22b)$$

where  $\vec{H}_0 = \vec{B}_0 / \mu_0$ .

Assuming a linear material with relative permeability  $\mu_r$ ,

$$\vec{B} = \mu_r \mu_0 \vec{H} \quad (22.23)$$

By substituting (22.23) into (22.22a) and further using (22.22b) for  $\vec{H}$ , we find

$$\vec{M}(\vec{B}) = \frac{\vec{B}_0}{\mu_0} \frac{3(\mu_r - 1)}{(2 + \mu_r)} \quad (22.24)$$

In the limiting case  $\mu_r \sim 1$  (i.e., almost in vacuum), at each point inside the sphere we find

$$\begin{cases} \vec{M} \sim \vec{0} \end{cases} \quad (22.25a)$$

$$\begin{cases} \vec{B} \sim \vec{B}_0 \end{cases} \quad (22.25b)$$

$$\begin{cases} \vec{H} \sim \vec{H}_0 \end{cases} \quad (22.25c)$$

In the limiting case  $\mu_r = 0$  (highly diamagnetic material), at each point inside the sphere we find

$$\begin{cases} \vec{M} = -\frac{3\vec{B}_0}{2\mu_0} \end{cases} \quad (22.26a)$$

$$\begin{cases} \vec{B} = \vec{0} \end{cases} \quad (22.26b)$$

$$\begin{cases} \vec{H} = \frac{3}{2} \vec{H}_0 \end{cases} \quad (22.26c)$$

In the limiting case  $\mu_r \rightarrow \infty$  (highly paramagnetic material), we can write

$$\vec{M}(\vec{B}) = \frac{\vec{B}_0}{\mu_0} \frac{\mu_r \left(3 - \frac{3}{\mu_r}\right)}{\mu_r \left(\frac{2}{\mu_r} + 1\right)} \quad (22.27)$$

and, thus, at each point inside the sphere we find

$$\begin{cases} \vec{M} \sim \frac{3\vec{B}_0}{\mu_0} \end{cases} \quad (22.28a)$$

$$\begin{cases} \vec{B} \sim 3\vec{B}_0 \end{cases} \quad (22.28b)$$

$$\begin{cases} \vec{H} \sim \vec{0} \end{cases} \quad (22.28c)$$

In this case, the total field  $\vec{B}$  inside the sphere is stronger than outside as shown in Fig. 22.2.

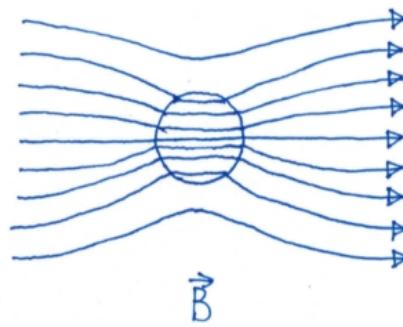


Figure 22.2

In the beginning of our discussion about magnetic materials, we mentioned that non-linear materials (e.g., ferromagnets) are of most interest. These are harder to study because the linear relationship (22.23) does not suffice. In order to study these materials, we choose a point of interest along the  $B(H)$  characteristics (see, e.g., Fig. 20.2b)

and linearize the characteristics around that point by derivation. We can then use this new local linear characteristics to find  $\vec{M}$  as a function of  $\vec{B}$ .

As for the case of the sphere, we can find the total  $\vec{B}$  and  $\vec{H}$ -fields also for an infinite cylindrical magnet and a magnetic slab. For the former, at each point inside the magnet

$$\begin{cases} \vec{B} = \vec{B}_0 + \mu_0 \vec{M} \end{cases} \quad (22.29a)$$

$$\begin{cases} \vec{H} = \vec{H}_0 \end{cases} \quad (22.29b)$$

Using again (22.23) and the definition of  $\vec{H}_0$ , we find

$$M_r \mu_0 \frac{\vec{B}_0}{\mu_0} = \vec{B}_0 + \mu_0 \vec{M} \quad (22.30)$$

and, thus,

$$\vec{M} = \frac{\vec{B}_0}{\mu_0} (\mu_r - 1) \quad (22.31)$$

Obviously  $\vec{M} = \vec{B} = \vec{H} = \vec{0}$  outside the magnet. For the latter, at each point inside the slab we find

$$\begin{cases} \vec{B} = \vec{B}_0 \end{cases} \quad (22.32a)$$

$$\begin{cases} \vec{H} = \vec{H}_0 - \vec{M} \end{cases} \quad (22.32b)$$

Using again (22.23) and the definition of  $\vec{H}_0$ , we find

$$\vec{M} = \frac{\vec{B}_0}{\mu_0} \frac{\mu_r - 1}{\mu_r} \quad (22.33)$$

We can summarize the behavior at each point inside a permanent magnet of arbitrary shape by defining a magnetization efficiency coefficient  $\gamma$  and a demagnetization factor  $S$  and write

$$\begin{cases} \vec{B} = \vec{B}_0 + \gamma \mu_0 \vec{M} \end{cases} \quad (22.34a)$$

$$\begin{cases} \vec{H} = \vec{H}_0 - S \vec{M} \end{cases} \quad (22.34b)$$

Using the definition of  $\vec{H}$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \quad (22.35)$$

as well as

$$\vec{H}_0 = \frac{\vec{B}_0}{\mu_0} \quad (22.36)$$

we obtain

$$\frac{\vec{B}}{\mu_0} - \vec{M} = \vec{H}_0 - S\vec{M} \quad (22.37)$$

and then

$$\vec{B} = \mu_0 \vec{H}_0 - \mu_0 S \vec{M} + \mu_0 \vec{M} \quad (22.38)$$

By inserting (22.38) into (22.34a) we finally find  $S = 1 - \gamma$ .

In summary, for a uniformly magnetized sphere  $\gamma = 2/3$  and  $S = 1/3$ , for a uniformly magnetized infinite cylindrical magnet  $\gamma = 1$  and  $S = 0$  (ideal magnetization), and for uniformly magnetized slab  $\gamma = 0$  and  $S = 1$  (ideal demagnetization).

## 22.2 Ampère's Equivalence Theorem

Historically, it was attempted to treat magnetism as electricity by defining a magnetic charge. While we also defined magnetic charges, we added the condition that such charges cannot be split into positive and negative charges (a permanent magnet with a north and a south pole when broken into two pieces, both pieces maintain a north and south pole). As it turns out magnetism cannot be described without assuming current distributions. It was Ampère to state the principle according to which given a magnetic dipole  $\vec{m}$ ,

$$\vec{m} = AI\vec{n} \quad (22.39)$$

where  $I$  is a current on a loop of area  $A$  and oriented according to  $\vec{n}$ .

# Chapter 23

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## 23.1 The Fundamental Laws of the Electromagnetic Field in Vacuum - Integral Form

The following laws are defined for lines and surfaces that are fixed with respect to an inertial reference frame.

So far, we have studied the laws of the electric and magnetic field only under static (stationary or steady) conditions. We will now consider the case where both fields vary with time. As it will be shown, in this case the two fields are not independent. On the contrary, they are intimately connected, forming a single entity that is usually referred to as the electromagnetic field.

As we know, the force acting on a charge  $q$  moving with velocity  $\vec{v}$  in an inertial reference frame is

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (23.1)$$

where  $\vec{E}$  and  $\vec{B}$  are the electric and magnetic field, respectively. This relationship makes it possible to define operatively (i.e., measure)  $\vec{E}$  and  $\vec{B}$ .

It is clear that the laws describing  $\vec{E}$  and  $\vec{B}$  under time varying conditions must be more general than those found in the stationary case and have to include the latter as special cases. The discovery of such laws derived from the analysis of a large body of experimental facts that eventually led to Maxwell's electromagnetic synthesis. To date, Maxwell's theory is a complete description of all macroscopic electromagnetic phenomena.

Following Maxwell's synthesis, the laws of the electromagnetic field in vacuum can be reduced to five fundamental equations. We already encountered the first three laws in several occasions,

$$\iint_{\Sigma} \vec{E} \cdot \vec{n} dS = \frac{1}{\epsilon_0} \iiint_{\Omega} \rho dV \quad (23.2)$$

Gauss' theorem,

$$\iint_{\Sigma} \vec{B} \cdot \vec{n} dA = 0 \quad (23.3)$$

solenoidal property of  $\vec{B}$ , and

$$\iint_{\Sigma} \vec{J} \cdot \vec{n} dA = - \iiint_{\Omega} \frac{\partial}{\partial t} \rho \cdot dV \quad (23.4)$$

charge conservation (continuity equation). In Eqs. (23.2)-(23.4),  $\Sigma$  is a generic closed surface, at rest in the inertial reference frame where the fields are defined, and  $\Omega$  is the region of space enclosed by  $\Sigma$ .

It is an experimental fact that Gauss' theorem (which, more properly, hereafter we will refer to as Gauss' law; this is because the properties we are about to discuss were proven by experiments, rather than being demonstrated theoretically) continues to be valid even when the charges move inside  $\Sigma$ . Also the solenoidal property of  $\vec{B}$  continues to be valid when the field current sources vary in time and, eventually, move in space. Similarly for the charge conservation.

Note that all other symbols in Eqs. (23.2)-(23.4) have the usual meaning.

The remaining two laws are the circulation of  $\vec{E}$  and  $\vec{B}$  along a generic closed line  $\gamma$ , considered to be at rest in the given reference frame. In integral form these laws are

$$\oint_{\gamma} \vec{E} \cdot \vec{t} d\ell = - \iint_{\Sigma_{\gamma}} \frac{\partial}{\partial t} \vec{B} \cdot \vec{n} dA \quad (23.5)$$

which goes under the name of Faraday-Neumann law, and

$$\oint_{\gamma} \vec{B} \cdot \vec{t} d\ell = \mu_0 \iint_{\Sigma_{\gamma}} \vec{J} \cdot \vec{n} dA + \mu_0 \iint_{\Sigma_{\gamma}} \epsilon_0 \frac{\partial}{\partial t} \vec{E} \cdot \vec{n} dA \quad (23.6)$$

which is called Ampère-Maxwell law. In these two equations,  $\Sigma_{\gamma}$  is a generic open surface, at rest in the given inertial reference frame, having  $\gamma$  as a border and oriented consistently with the orientation on  $\gamma$ . Note that, in the stationary case, (23.5) and (23.6) become the irrotational property of  $\vec{E}$  and Ampère's law, respectively.

### 23.1.1 Time-Derivative Swap

We note that the domain of integration in Eqs. (23.4)-(23.6) does not vary in time. Hence, we can move the time derivative outside the sign of integral, obtaining

$$\iint_{\Sigma} \vec{J} \cdot \vec{n} dA = - \frac{d}{dt} \iiint_{\Omega} \rho dV \quad (23.4')$$

$$\oint_{\gamma} \vec{E} \cdot \vec{t} d\ell = - \frac{d}{dt} \iint_{\Sigma_{\gamma}} \vec{B} \cdot \vec{n} dA \quad (23.5')$$

$$\oint_{\gamma} \vec{B} \cdot \vec{t} d\ell = \mu_0 \iint_{\Sigma_{\gamma}} \vec{J} \cdot \vec{n} dA + \mu_0 \frac{d}{dt} \iint_{\Sigma_{\gamma}} \epsilon_0 \vec{E} \cdot \vec{n} dA \quad (23.6')$$

Note that, in Eqs. (23.4)-(23.6) the time derivative is denoted as a partial derivative. In fact, in general, the integrand functions depend both on time and position. In Eqs. (23.4')-(23.6'), however, the time derivative is denoted as an ordinary derivative because the double integrals only depend on time.

### 23.1.2 Solenoidal Properties

The value taken by the second term in Eqs. (23.5) and (23.6) is independent from the particular surface  $\Sigma_\gamma$  having  $\gamma$  as border. As a consequence, the fields  $\partial\vec{B}/\partial t$  and  $\vec{G} = \vec{J} + \epsilon_0\partial\vec{E}/\partial t$  must be solenoidal. In fact, given two generic open surfaces  $\Sigma_{\gamma_1}$  and  $\Sigma_{\gamma_2}$  that share  $\gamma$  as border, by definition their union is a closed surface  $\Sigma = \Sigma_{\gamma_1} \cup \Sigma_{\gamma_2}$  (see Fig. 23.1). The standard orientation of a closed surface is such that the normal unit vector  $\vec{n}$  at each point of the surface is directed outside the surface. However, choosing the orientations  $\vec{n}_1$  and  $\vec{n}_2$  of each separate open surface to be consistent with the orientation of the closed line  $\gamma$  (e.g., counterclockwise in the figure), it results that  $\vec{n}_1 = -\vec{n}$  and  $\vec{n}_2 = \vec{n}$ . Assuming a generic vector field  $\vec{F}$  is defined in the region of space occupied by  $\Sigma$ , if the field is solenoidal, it must be

$$\Phi_\Sigma = \iint_{\Sigma} \vec{F} \cdot \vec{n} dA = 0 \quad (23.7)$$

Furthermore, from the very construction of  $\Sigma$ , it must also be

$$\begin{aligned} \iint_{\Sigma} \vec{F} \cdot \vec{n} dA &= \iint_{\Sigma_{\gamma_1} \cup \Sigma_{\gamma_2}} \vec{F} \cdot \vec{n} dA \\ &= \iint_{\Sigma_{\gamma_1}} \vec{F} \cdot \vec{n} dA + \iint_{\Sigma_{\gamma_2}} \vec{F} \cdot \vec{n} dA \\ &= \iint_{\Sigma_{\gamma_1}} \vec{F} \cdot (-\vec{n}_1) dA + \iint_{\Sigma_{\gamma_2}} \vec{F} \cdot \vec{n}_2 dA \\ &= \iint_{\Sigma_{\gamma_2}} \vec{F} \cdot \vec{n}_2 dA - \iint_{\Sigma_{\gamma_1}} \vec{F} \cdot \vec{n}_1 dA = 0 \end{aligned} \quad (23.8)$$

from which it finally follows that

$$\iint_{\Sigma_{\gamma_1}} \vec{F} \cdot \vec{n}_1 dA = \iint_{\Sigma_{\gamma_2}} \vec{F} \cdot \vec{n}_2 dA \quad (23.9)$$

In other words, if a vector field is solenoidal its flux through any closed surface is zero or, equivalently, its flux through all open surfaces with equal border is constant. Due to the arbitrariness of the open surface to calculate the flux, in this case, the flux is said to be associated (or linked) with the border (assumed to be closed) common to all considered surfaces.

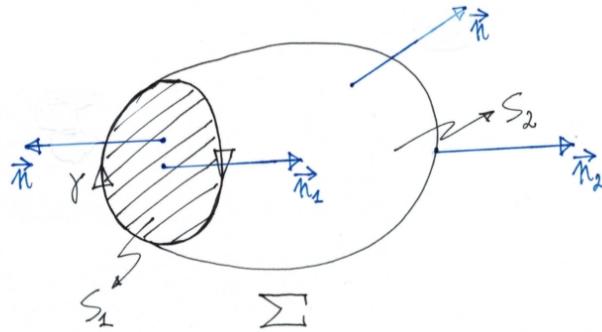


Figure 23.1

Going back to Maxwell's equations, given a closed surface  $\Sigma$ , it must be

$$\iint_{\Sigma} \frac{\partial}{\partial t} \vec{B} \cdot \vec{n} dA = 0 \quad (23.10)$$

and

$$\iint_{\Sigma} \vec{G} \cdot \vec{n} dA = \iint_{\Sigma} (\vec{J} + \epsilon_0 \frac{\partial}{\partial t} \vec{E}) \cdot \vec{n} dA = 0 \quad (23.11)$$

Equation (23.10) implies that

$$\frac{d}{dt} \iint_{\Sigma} \vec{B} \cdot \vec{n} dA = 0 \quad (23.12)$$

As a consequence,

$$\iint_{\Sigma} \vec{B} \cdot \vec{n} dA = k \quad (\text{constant}) \quad (23.13)$$

for each time  $t$ . Similarly, from Eqs. (23.11) we obtain

$$\iint_{\Sigma} \vec{J} \cdot \vec{n} dA = - \iint_{\Sigma} \epsilon_0 \frac{\partial}{\partial t} \vec{E} \cdot \vec{n} dA \quad (23.14)$$

Substituting the first term of (23.14) with the second term of (23.4), we then obtain

$$-\iiint_{\Omega} \frac{\partial}{\partial t} \rho dV = - \iint_{\Sigma} \epsilon_0 \frac{\partial}{\partial t} \vec{E} \cdot \vec{n} dA, \quad (23.15)$$

from which

$$\frac{d}{dt} \iiint_{\Omega} \rho dV = \frac{d}{dt} \iint_{\Sigma} \epsilon_0 \vec{E} \cdot \vec{n} dA$$

or, equivalently,

$$\frac{d}{dt} \left( \iiint_{\Omega} \rho dV - \iint_{\Sigma} \epsilon_0 \vec{E} \cdot \vec{n} dA \right) = 0 \quad (23.16)$$

As a consequence,

$$\iint_{\Sigma} \epsilon_0 \vec{E} \cdot \vec{n} dA - \iiint_{\Omega} \rho dV = k' \quad (\text{constant}) \quad (23.17)$$

for each time  $t$ .

Note that Eqs. (23.13) and (23.17) have been derived from Eqs. (23.4)-(23.6), without using Eqs. (23.2) and (23.3). These two equations add the further information that, at a certain time  $t$ , the constants  $k$  and  $k'$  must be equal to zero. Since they must be constant for each time, this implies that  $k = k' = 0$  for each time  $t$ .

For these reasons, Eqs. (23.4)-(23.6) are often considered the fundamental laws of the electromagnetic field in vacuum, and Eqs. (23.2) and (23.3), i.e., Gauss' law and the solenoidal principle of  $\vec{B}$ , as simple initial conditions.



# Chapter 24

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## 24.1 Electromagnetic Induction

As shown in the previous chapter, the equation expressing the fact that  $\vec{B}$  must be solenoidal and Gauss' law can be regarded as initial conditions for the three fundamental laws (23.4)-(23.6). Among these fundamental equations, the first, which represents the charge conservation principle, has already been studied in PHYS 242 and, thus, does not require any further comments.

The second law, Eq. (23.5), represents the fact that the circulation of the electric field is, in general, different than zero in presence of time-varying magnetic fields. This is a new phenomenon. In fact, so far we assumed the electric field to be irrotational. This is correct both in electrostatics and in presence of stationary currents.

A direct consequence of (23.5) is that the integral of  $\vec{E}$  along an open line  $\gamma_{AB}$  (the integral is called electric tension or potential difference along  $\gamma_{AB}$ ) depends on the integration limits  $A$  and  $B$  and also on the integration path in presence of a variable magnetic field.

In PHYS 242, we introduced a physical quantity called the electromotive force (emf), the value of which is given by the circulation of the electromotive field along a closed line  $\gamma$ . It is common to introduce an emf, acting on a generic closed line  $\gamma$ , also for the electric field. This is a delicate extension. In fact, it is important to notice that, so far, the concepts of electromotive field and emf were associated only with force fields of nonelectric origin. Hereafter, the term emf shall also indicate line integrals associated with electric fields.

By naming such an emf  $\mathcal{E}_\gamma$ , we define

$$\mathcal{E}_\gamma = \oint_\gamma \vec{E} \cdot \vec{td}\ell \quad (24.1)$$

Therefore, (23.5') reads

$$\mathcal{E}_\gamma = - \frac{d}{dt} \iint_{\Sigma_\gamma} \vec{B} \cdot \vec{n} dA = - \frac{d}{dt} \Phi_\gamma \quad (24.2)$$

where  $\Phi_\gamma$  is the flux of the magnetic field linked with  $\gamma$ . This relationship is called Faraday-Neumann's law (same name as the associated Maxwell's equation) and states

that the emf acting along a closed line is equal (aside from the sign) to the time derivative of the magnetic flux linked with the line itself. This phenomenon is typically called electromagnetic induction and the emf  $\mathcal{E}_\gamma$  is referred to as the emf induced by the magnetic field variation. This is one of the most important phenomenon in electromagnetism. Every time-varying magnetic field is associated with a rotational electric field, the emf of which (along a closed line) is given by (24.2).

### 24.1.1 Faraday-Neumann's Law for Two Circuits

Consider a filiform circuit  $C_1$  as shown in Fig. 24.1. Assume the circuit is connected to a source  $G$  that generates a current  $i_1$ , flowing through the circuit and variable in time. This current generates a time-varying magnetic field  $\vec{B}_1(t)$ . Consider a generic closed and oriented line  $\gamma$ , external to the initial circuit and not linked to it. The flux  $\Phi_{\gamma 1}$  due to the magnetic field  $\vec{B}_1(t)$  linked with  $\gamma$  also varies with time. As a consequence, an emf of type (24.2) originates along  $\gamma$ .

Assume now to introduce a filiform conductor  $C_2$  in correspondence of  $\gamma$ . The emf due to  $\vec{B}_1(t)$  will generate an induced current  $i_2(t)$  circulating in  $C_2$ .

The sequence of causes and effects taking place in the two circuits is actually more complex. In fact, the induced current  $i_2$  generates a time-varying magnetic field on its own. As a consequence, the total flux linked with  $\gamma$  (assumed to be the longitudinal axis of  $C_2$ ) is due to both  $i_1(t)$  and  $i_2(t)$ .

Moreover, the current  $i_1$  itself is generated by both the source  $G$  and the emf induced by  $i_2$ .

Nevertheless, according to Faraday-Neumann's law the emf acting along  $\gamma$  (i.e.,  $C_2$ ) is given (aside from the sign) by the time derivative of the total magnetic field linked with  $\gamma$ .

In general, this means that the flux  $\Phi_\gamma$  in (24.2) should be regarded as the effective flux of the total time-varying magnetic field (due to all possible causes) acting in the region of space where  $\gamma$  is defined.

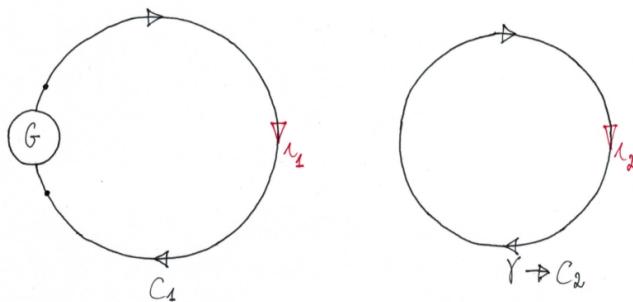


Figure 24.1

### 24.1.2 Lenz's Law

Consider a closed conducting loop  $\gamma$  at rest in a given inertial reference frame. A permanent magnet is brought closer and closer to the loop, as shown in Fig. 24.2. The magnetic field generated by the magnet is constant in time (with respect to the reference frame of the magnet). However, the magnetic field in the region of space

occupied by the loop varies in time due to the motion of the magnet. In particular, the field increases when the magnet gets closer to the loop. Hence, the flux linked with  $\gamma$  varies in time, generating an emf along  $\gamma$  that then generates a current flowing through the loop. Also in this case, the current is generated by the variation of the total flux linked with the loop.

Assume the magnet gets closer to the loop. In order to calculate the induced emf we begin by orienting  $\gamma$  arbitrarily, i.e., clockwise in Fig. 24.2. Consider then an open surface  $\Sigma_\gamma$  having  $\gamma$  as a border and oriented consistently with the orientation chosen for  $\gamma$ . In the figure, the normal unit vector is directed to the right at each point on  $\Sigma_\gamma$ . If the magnet is positioned as indicated in the figure, the flux  $\Phi_\gamma$  linked with  $\gamma$  is positive because the magnet field  $\vec{B}$  always forms an acute angle with  $\vec{n}$  (i.e., at each point on  $\Sigma_\gamma$ ,  $\cos \alpha > 0$ , where  $\alpha = \angle(\vec{B}, \vec{n})$ ). As a consequence, the function  $\Phi_\gamma(t)$  increases monotonically with time and, thus, its time derivative is positive. Therefore, due to the negative sign in (24.2), the emf induced along  $\gamma$  is negative. This means that the electric field induced along  $\gamma$  acts in the opposite direction with respect to the orientation chosen for  $\gamma$ . The current induced along  $\gamma$  must have the same sign as the induced emf (and electric field). Thus, the magnetic field generated by such a current contributes negatively to the total flux linked with  $\gamma$ . The induced current generates a magnetic field that tends to oppose to the cause (field  $\vec{B}$  variation) that generates it. This statement is referred to as Lenz's law and represents a conservation of energy principle. In fact, if the negative sign was absent in (24.2) a perpetual motion of the first kind would be possible, where the magnet (constant) field generates a field due to the current induction on  $\gamma$ , which strengthens the magnet field and so forth. Note that, this would be possible in absence of any acceleration (and, thus, force) on the magnet, which only has to move at a constant velocity in an inertial reference frame (with  $\gamma$  at rest).

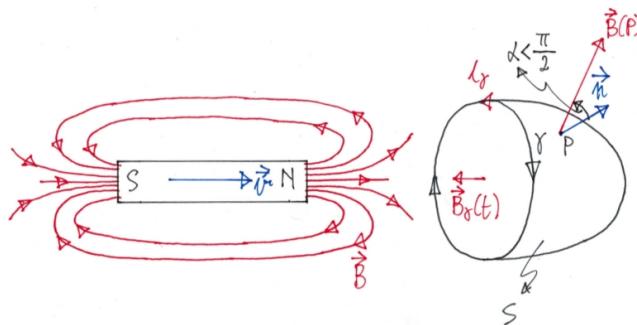


Figure 24.2: The magnet has two poles, North (N) and South (S). The field  $\vec{B}$  is directed from N to S outside the magnet.

In the above examples, we have seen that the emf induced in a loop (e.g., a conducting filiform circuit) acts as a source connected to the loop. Similarly to a source, it generates a current in the loop,

$$i = \frac{\mathcal{E}_\gamma}{R_\gamma} \quad (24.3)$$

where  $R_\gamma$  is the loop resistance. The main difference with respect to the cases studied

in PHYS 242 is in that back then the electromotive field was confined uniquely within the source. Now, instead, the induced electric field, which acts as an electromotive field, acts at each point of the circuit. The induced emf can be regarded as a source “distributed” along the entire circuit. In other words, it is as if there were many elementary sources along the circuit, connected in series and with total emf given by the sum of the emf of each elementary source,

$$\mathcal{E}_\gamma = - \frac{d}{dt} \Phi_\gamma$$

### 24.1.3 Electromagnetic Induction in an Open Circuit

Consider a system as in Fig. 24.2. However, assume the conducting loop with axis  $\gamma$  is now interrupted (open at at least in one point). With respect to the reference frame where the loop is at rest, when the magnet moves, an electric field is induced in the loop, acting on the conductor charge free carriers and putting them in motion. In this case, however, the current cannot get closed around the broken loop. The scenario is similar to that of a source with open terminals (see PHYS 242). We can imagine that each infinitesimal loop element is associated with an elementary source and all elementary sources are in series. Thus, the entire conductor can be thought as a single generator along which is distributed an emf equal to the sum of the emf for each elementary source. The resulting emf is the line integral along the conductor of the induced electromotive field.

Also in this case, a charge distribution gets localized at the limiting surface of the open conductor, i.e., at the interface between the conductor and the external environment. Such a charge distribution generates an electric field that gets superimposed to the electromotive field. If the physical configuration of the system is such that the electric field induced in the conductor is constant in time (this assumption serves to guarantee that there are no displacement currents in the system; see next chapter), no current can flow in the circuit and the scenario is equivalent to the open circuit source studied in PHYS 242. The electric field generated by the charge distribution in the conductor counter-balances the action of the induced electric field at each point of the conductor. While in the case of a source, the electromotive field acts only within the source, in the case of induced electromotive fields, the field acts at each point in the conductor. Hence, at each point in the conductor there is a superposition between the induced (rotational) electromotive field and the reaction (irrotational) field. We will study the case of displacement current in the next section.

# Chapter 25

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## 25.1 Barlow's Wheel

Consider a conducting disk with radius  $R$  that rotates about its central axis with angular velocity  $\vec{\omega}$ ; Assume the absolute value  $|\vec{\omega}| = \text{const}$ . Two conducting pads touch the lateral surface of the disk and of the disks' axis. The two pads are connected electrically to two electrodes  $A$  and  $B$ . Suppose the device is located in a region of space with a uniform magnetic field  $\vec{B}$  directed normally to the plain of the disk and constant in time, see Fig. 25.1a. As shown in the figure we choose the cylindrical coordinate system  $Or\phi z$ . With respect to this system, the tangential velocity of a point  $P$  on the disk is given by

$$\vec{u} = \omega r \vec{u}_\phi \quad (25.1)$$

and

$$\vec{B} = -B \vec{u}_z \quad (25.2)$$

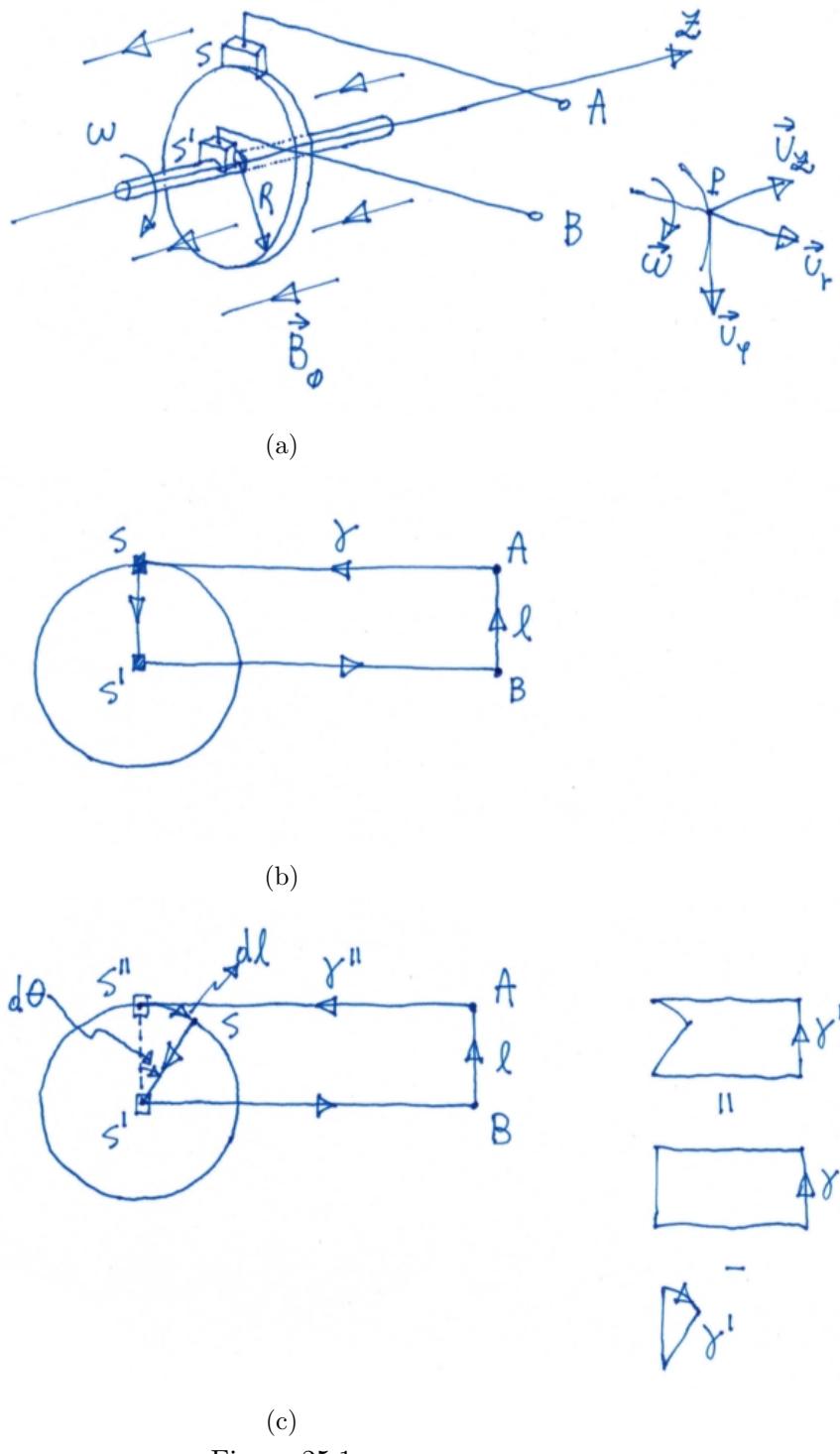


Figure 25.1

In the laboratory reference frame, there are no electromagnetic induction phenomena because the magnetic field is time independent. However, the electrons inside the disk

are acted upon by both a centripetal force

$$\vec{F}_c = m_e \omega^2 \vec{r} \quad (25.3)$$

where  $m_e$  is the electron mass and a Lorentz's force

$$\vec{F}_L = -e\vec{u} \times \vec{B} \quad (25.4)$$

where  $-e$  is the electron charge. Note that the centripetal force is the same as in the so-called Nichols' disk. In that case, the effects of  $\vec{F}_c$  are negligible (we remind that the effect of  $\vec{F}_c$  is transmitted to the electrons by the positive charges that form the metallic reticle; i.e., if the electrons were a free gas inside the disk there would be no effect on them due to  $\vec{F}_c$ ).

Under the action of all these forces the electrons move inside the disk, perturbing the local electric neutrality and, thus, generating a reaction field  $\vec{E}$ . When the electrodes  $A$  and  $B$  are open an equilibrium condition takes place among all forces acting on the electrons:

$$m_e \omega^2 \vec{r} - e\vec{u} \times \vec{B} - e\vec{E} = \vec{0} \quad (25.5)$$

The reaction field  $\vec{E}$  has two components each of which counterbalances  $\vec{F}_c$  and  $\vec{F}_L$ . Assuming  $\vec{F}_c \sim \vec{0}$  and keep calling  $\vec{E}$  the component of  $\vec{E}$  that balances  $\vec{F}_L$ , we obtain

$$e\vec{u} \times \vec{B} + e\vec{E} = \vec{0} \quad (25.6)$$

from which we can define a motional field

$$\vec{E}_{\text{mot}} = \vec{u} \times \vec{B} = -\vec{E} \quad (25.7)$$

Referring to Fig. 25.1b, the closed line  $\gamma$  comprises a segment  $SS'$ , which moves together with the disk and is oriented radially with respect to  $O$ , two conductive segments  $AS$  and  $S'B$ , and a non-conductive segment  $BA$  of length  $\ell$ . Since

$$\vec{u} \times \vec{B} = \omega r \vec{u}_\varphi \times (-\vec{u}_z) B = -\omega r B \vec{u}_r \quad (25.8)$$

the emf  $\mathcal{E}_{\gamma \text{mot}}$  associated with  $\vec{E}_{\text{mot}}$  along  $\gamma$  is given by

$$\begin{aligned} \mathcal{E}_{\gamma \text{mot}} &= \oint_{\gamma} \vec{E}_{\text{mot}} \cdot \vec{t} d\ell = \int_R^0 -\omega r B \vec{u}_r \cdot \vec{u}_r dr \\ &= \int_0^R \omega B r dr = \frac{1}{2} \omega B R^2 \end{aligned} \quad (25.9)$$

We remind that when integrating from  $R$  to 0,  $\vec{t} d\ell = +\vec{u}_r dr$ . Such an emf is present only when the disk rotates and segment  $SS'$  rotates with it.

Consider now the new position of the disk after an infinitesimal time  $dt$ , as shown in Fig. 25.1c. When the ray  $SS'$  has rotated by an angle  $d\alpha = \omega dt$ , the total flux of  $\vec{B}$

through the deformed  $\gamma$  is equal to the original flux minus the flux of  $\vec{B}$  through the infinitesimal triangle with base  $d\ell$  and height  $R$ ,

$$d\Phi_\gamma \approx -\frac{1}{2} R B d\ell = -\frac{1}{2} R B R d\alpha = -\frac{1}{2} R^2 B \omega dt \quad (25.10)$$

Thus

$$\mathcal{E}_{\gamma\text{mot}} = -\frac{d\Phi_\gamma}{dt} = \frac{1}{2} \omega B R^2 \quad (25.11)$$

which is equivalent to (25.9). This result represents Faraday-Neumann-Lenz's law for deformable lines.

## 25.2 Displacement Current

We will now study the last law of electromagnetism in vacuum, given by Eqs. (23.6) and (23.6'). This law is concerned with the circulation of the magnetic field  $\vec{B}$  and represents a generalization of Ampère's law given in magnetostatics.

The main difference between Ampère's law in magnetostatics in the case of stationary currents and its generalization to time-varying currents and fields is in that, in the first case, the circulation of  $\vec{B}$  is proportional to the flux of  $\vec{J}$  (total current density) linked with the line  $\gamma$  along which the circulation is evaluated; in the second case, a new term given by the flux of vector  $\epsilon_0 \partial \vec{E} / \partial t$  must be added to the flux of  $\vec{J}$ . As a consequence, in magnetostatics the field of  $\vec{J}$  is solenoidal, whereas in the time-varying case the field

$$\vec{G} = \vec{J} + \epsilon_0 \frac{\partial}{\partial t} \vec{E} \quad (25.12)$$

is solenoidal.

With respect to  $\vec{B}$ , the vector fields  $\vec{J}$  and  $\epsilon_0 \partial \vec{E} / \partial t$  play similar roles: Each of them contributes independently to the circulation of  $\vec{B}$ . In particular, if in a region of space  $\vec{J} = \vec{0}$  at each point in the region (i.e., the region does not contain any conductor where currents can circulate or any magnetic material with magnetization currents), but a time-varying electric field  $\vec{E}$  exists in the region, in such a region exists a magnetic field  $\vec{B}$ . In fact, the circulation of  $\vec{B}$  must necessarily be different than zero according to (23.6). Hence, any time-varying electric field is associated with a magnetic field that fulfills (23.6). This phenomenon, which is referred to magnetoelectric induction, is the "dual" of the electromagnetic induction.

In the light of the previous considerations, a time-varying electric field can be regarded as a current because, as a current, it generates a magnetic field. In order to distinguish the conduction current density  $\vec{J}$  from  $\epsilon_0 \partial \vec{E} / \partial t$ , the latter is called displacement current density.

### 25.2.1 Charging Capacitor

Consider a circuit comprising a stationary emf source  $\mathcal{E}$ , a capacitor with capacitance  $C$ , and a conductor with resistance  $R$  that connects  $C$  to the source (see Fig. 25.2). Assume that at the time instant when the source is connected to the capacitor, the latter is uncharged. Then, the charge free carriers in the conductor and source begin to move under the action of the electromotive field in the source. However, assuming the

capacitor plates to be separated by vacuum, no charge can move between the plates. As a consequence, positive charges start to accumulate on one wall of the capacitor and negative charges accumulate on the other wall (or, equivalently, a reduction of positive charges takes place on the second wall). A direct consequence of such time-varying charges on the capacitor walls is the generation of a time-varying electric field  $\vec{E}$  between the walls. In other words, while a conduction current flows in the conductor, a displacement current takes place between the walls of the capacitor.

Consider a closed surface  $\Sigma$  that encloses one wall of the capacitor only (see Fig. 25.2). The flux of vector  $\vec{G}$  outside such a closed surface must be zero. Hence,

$$0 = \iint_{\Sigma} \vec{G} \cdot \vec{n} dA = \iint_{\Sigma} \vec{J} \cdot \vec{n} dA + \iint_{\Sigma} \epsilon_0 \frac{\partial}{\partial t} \vec{E} \cdot \vec{n} dA \quad (25.13)$$

where  $\vec{n} dA$  is an oriented infinitesimal surface element on  $\Sigma$ .

Since there are no charge free carriers in vacuum, it must be  $\vec{J} = \vec{0}$  at each point on  $\Sigma$ , except for the points on the intersection  $\Sigma_c$  between  $\Sigma$  and the conductor. Thus,

$$\iint_{\Sigma} \vec{J} \cdot \vec{n} dA = \iint_{\Sigma_c} \vec{J} \cdot \vec{n} dA = i_c \quad (25.14)$$

where  $i_c$  is the conduction current intensity circulating in the conductor. From (25.13), we find

$$\iint_{\Sigma} \epsilon_0 \frac{\partial}{\partial t} \vec{E} \cdot \vec{n} dA = -i_d \quad (25.15)$$

where  $i_d$  is the displacement current intensity in the capacitor. Assuming the electric field outside the capacitor to be negligible, the conduction current in the conductor is equal to the flux of the displacement current density inside the capacitor.

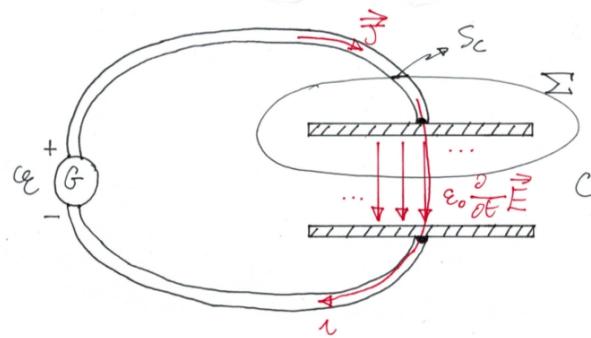


Figure 25.2

Note that, the minus sign in (25.15) is due to the fact that  $\vec{J}$  enters into  $\Sigma_c$ , while  $\partial \vec{E} / \partial t$  exits from  $\Sigma$ . Also note that, under stationary conditions for  $\vec{J}$ , after an initial transient, any time-varying phenomena stop upon reaching a steady-state regime. The scenario would be different if  $\vec{J}$  was also a time-varying vector. In fact, under stationary

conditions when the capacitor is fully “charged” (steady-state regime), there is no displacement current between its walls anymore.

Note that the displacement current can be thought as an electromagnetic wave within the walls of the capacitor.

# Chapter 26

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## 26.1 Magnetic Energy Associated with a Current

We here introduce the concept of magnetic energy associated with a current by means of a simple example.

It is worth mentioning upfront that the electromagnetic induction phenomena are key to write a mathematical expression for the magnetic energy. This is the reason why such a topic was not considered earlier in the course.

### 26.1.1 Single Circuit with Current

Consider a conducting loop with longitudinal axis  $\gamma$ . Assume a current  $i$  flows in the loop. If there are no other circuits with a current in the region of space of  $\gamma$ , the only magnetic field in the region is that generated by  $i$ . Under these conditions, assuming  $L$  is the self-inductance coefficients associated with  $\gamma$ , the magnetic flux linked through  $\gamma$  is given by

$$\Phi_\gamma = L_i \quad (26.1)$$

If  $i$  varies in time, from (26.1) also  $\Phi_\gamma$  varies in time. Then, because of Faraday-Neumann law, the loop is characterized by an induced emf

$$\mathcal{E} = -\frac{d}{dt} \Phi_\gamma = -\frac{d}{dt} (L_i) = -L \frac{d}{dt} i \quad (26.2)$$

This emf is usually called self-induction emf. Because of Lenz's law, its sign is such that the self-induction emf opposes to the current  $i$  that generates it. On this respect, we observe that when writing (26.2) we assumed that the orientation on  $\gamma$  used to evaluate  $i$  is the same as that used to evaluate  $\mathcal{E}$ . In fact, assuming  $L > 0$ , we oriented the closed line  $\gamma$  (for evaluating  $i$ ) and the normal unit vector  $\vec{n}$  to the generic surface (open) with border  $\gamma$  (for evaluating  $\Phi_\gamma$ ) consistently in (26.1). Moreover, the same criterion is used when writing the magnetic induction laws in the form (26.2). In fact, the positive direction on  $\gamma$  used to evaluate  $\mathcal{E}$  is consistent with the direction of  $\vec{n}$  in the electromagnetic induction given by (26.2).

We can now calculate the magnetic energy associated with a single circuit with current. To this purpose, consider a quasi filiform loop with cross-section  $S$ , length  $\ell$ , and resistivity  $\nu$ . The loop is assumed to be connected to an emf source,  $\mathcal{E}_g$ , which is constant in time (e.g., a battery). Figure 26.1 shows such a circuit. Before closing the

switch  $T$ , no current can flow in the loop. When the switch is closed at time  $t = 0$ , a current  $i$  can start to flow through the circuit.

Considered now a closed line  $\gamma$  comprising the loop and the conducting leads that connect the loop to the source, the total emf acting along  $\gamma$  at a generic instant  $t > 0$  is

$$\mathcal{E}_\gamma = \oint_\gamma (\vec{E}_m + \vec{E}) \cdot \vec{t} d\ell = \mathcal{E}_g - \frac{d}{dt} \Phi_\gamma \quad (26.3)$$

where  $\Phi_\gamma$  indicates the flux linked with  $\gamma$  at a generic time  $t$  and  $\vec{E}_m$  is the electromotive field acting in the source. Furthermore, at each point in the conducting loop  $\vec{J} = \sigma \vec{E}$  (Ohm's law) and at each point inside the source  $\vec{J} = \sigma_g (\vec{E} + \vec{E}_m)$ , we have

$$\int_{\gamma_\ell} \vec{E} \cdot \vec{t} d\ell = \int_{\gamma_\ell} \frac{1}{\sigma} \vec{J} \cdot \vec{t} d\ell \quad (26.4a)$$

and

$$\int_{\gamma_g} (\vec{E} + \vec{E}_m) \cdot \vec{t} d\ell = \int_{\gamma_g} \frac{1}{\sigma_g} \vec{J} \cdot \vec{t} d\ell \quad (26.4b)$$

where  $\gamma_\ell$  and  $\gamma_g$  are the portions of  $\gamma$  in the loop and in the source, respectively, and  $\sigma$  and  $\sigma_g$  the corresponding conductivity values. Thus,

$$\begin{aligned} \mathcal{E}_\gamma &= \oint_\gamma (\vec{E}_m + \vec{E}) \cdot \vec{t} d\ell \\ &= \int_{\gamma_g} (\vec{E}_m + \vec{E}) \cdot \vec{t} d\ell + \int_{\gamma_\ell} \vec{E} \cdot \vec{t} d\ell \\ &= \int_{\gamma_g} \frac{1}{\sigma_g} \vec{J} \cdot \vec{t} d\ell + \int_{\gamma_\ell} \frac{1}{\sigma} \vec{J} \cdot \vec{t} d\ell \end{aligned} \quad (26.5)$$

By multiplying the last equation by the conductor cross-section  $S$

$$\mathcal{E}_\gamma = \int_{\gamma_g} \frac{1}{\sigma_g} \vec{J} \cdot \frac{S}{S} \cdot \vec{t} d\ell + \int_{\gamma_\ell} \frac{1}{\sigma} \vec{J} \cdot \frac{S}{S} \cdot \vec{t} d\ell \quad (26.6)$$

By noting that  $\vec{J}$  is directed as  $\vec{t} d\ell$  and that  $JS$  is the current  $i$ , constant on each circuit cross-section (in the loop, the displacement current has been neglected and only the conduction current has been considered), we have

$$\begin{aligned} \mathcal{E}_\gamma &= i \int_{\gamma_\ell} \frac{1}{\sigma} \frac{d\ell}{S} + i \int_{\gamma_g} \frac{1}{\sigma_g} \frac{d\ell}{S} \\ &= i(R_\ell + R_g) \end{aligned} \quad (26.7)$$

where  $R_\ell$  and  $R_g$  are the resistances associated with the loop and source, respectively.

Finally, by substituting (26.7) into (26.3), we obtain

$$\mathcal{E}_g - \frac{d}{dt} \Phi_\gamma = (R_\ell + R_g) i \quad (26.8)$$

From (26.2), it then follows that

$$\mathcal{E}_g = (R_\ell + R_g) i + L \frac{d}{dt} i \quad (26.9)$$

We thus obtain a linear ordinary differential equation with constant coefficients. This equation describes the time evolution of the current  $i$  in the circuit. Assuming

$$i(0) = 0 \quad (26.10)$$

as initial condition, the solution of (26.9) is

$$i(t) = I_0(1 - e^{-t/\tau}) \quad (26.11)$$

where

$$\tau = \frac{L}{R_\ell + R_g} \quad (26.12a)$$

and

$$I_0 = \frac{\mathcal{E}_g}{R_\ell + R_g} \quad (26.12b)$$

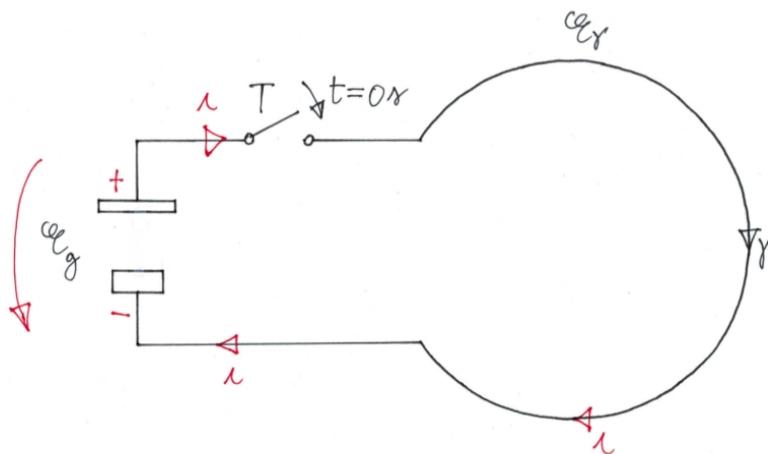


Figure 26.1

Note that, the general solution of the homogeneous equation associated with (26.9) is of the type

$$i' = A e^{-t/\tau} \quad (26.13)$$

where  $A$  is an arbitrary constant. A special solution of the complete equation is

$$i_j = \frac{\mathcal{E}_g}{R_\ell + R_g} \quad (26.14)$$

It is easy to be convinced that (26.13) is a general solution by substituting it into (26.9) for  $\mathcal{E}_g = 0$ ,

$$(R_\ell + R_g)i + L \frac{d}{dt} i = 0 \quad (26.15)$$

from which

$$i + \frac{L}{R_\ell + R_g} i = i + \tau i = 0 \quad (26.16)$$

where  $\tau$  is given by (26.12a). Substituting (26.13),

$$A_\ell^{-t/\tau} - \cancel{\tau} \frac{A}{\cancel{\tau}} i^{-t/\tau} = 0 \quad (26.17)$$

Note that in (26.16),  $i = di/dt$ . It is also easy to be convinced that (26.14) is a special solution of (26.9),

$$\mathcal{E}_g = (R_\ell + R_g) \frac{\mathcal{E}_g}{R_\ell + R_g}$$

Hence, the general solution of the complete equation (26.9) is given by the superposition of (26.13) and (26.14),

$$i = A e^{-t/\tau} + \frac{\mathcal{E}_g}{R_\ell + R_g} \quad (26.18)$$

Finally, by choosing  $A$  such that  $i(0) = 0$ , we obtain

$$\begin{aligned} i(0) = 0 &= A e^{-0/\tau} + \frac{\mathcal{E}_g}{R_\ell + R_g} \\ &= A + \frac{\mathcal{E}_g}{R_\ell + R_g} \end{aligned} \quad (26.19)$$

from which

$$A = -\frac{\mathcal{E}_g}{R_\ell + R_g} \quad (26.20)$$

It is clear that  $[A] = A$ , thus, we call  $A = -I_0$  and obtain

$$\begin{aligned} i(t) &= -I_0 e^{-t/\tau} + I_0 \\ &= I_0(1 - e^{-t/\tau}) \end{aligned}$$

A plot of (26.10) as a function of time is shown in Fig. 26.2. For  $t > 0$ , the current  $i$  increases monotonically, tending asymptotically to  $I_0$ .

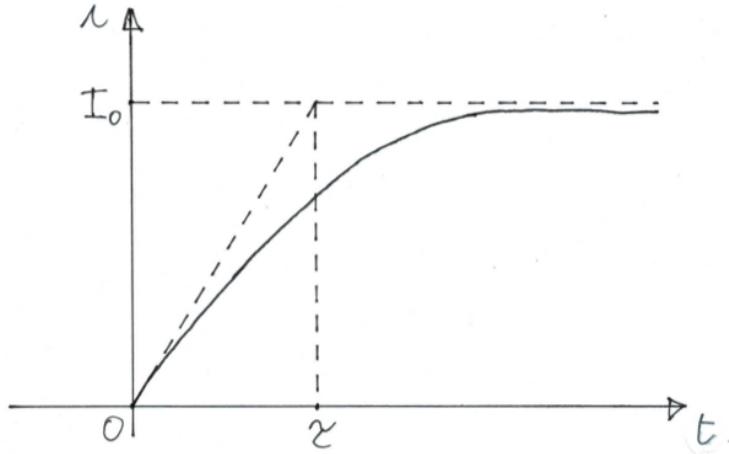


Figure 26.2

We can use (26.9) to calculate the magnetic energy associated with the circuit. By multiplying both sides of (26.9) by  $i$ , we obtain

$$\mathcal{E}_g i = (R_\ell + R_g) i^2 + L i \frac{d}{dt} i \quad (26.21)$$

The first term of (26.21) is the power generated by the emf source  $\mathcal{E}_g$  at a generic time instant  $t$  (see PHYS 242). The second term is the sum of the power dissipated as heat due to the Joule effect in the resistances  $R_\ell$  and  $R_g$  and the term  $L i d i / dt$ . By observing that

$$L i \frac{d}{dt} i = \frac{d}{dt} \left( \frac{1}{2} L i^2 \right) \quad (26.22)$$

and by integrating (26.22) between  $t = 0$  and the generic instant  $t$ , we have

$$\int_0^t \mathcal{E}_g i \, dt = \int_0^t (R_\ell + R_g) i^2 \, dt + \int_0^t \frac{d}{dt} \left( \frac{1}{2} L i^2 \right) \, dt \quad (26.23)$$

This result shows that the total energy (integrals of the power in time) generated by the source in the time interval  $[0, t]$  is equal to the sum of the energy dissipated in the circuit because if the Joule effect and the energy

$$u_m = \int_0^t \frac{d}{dt} \left( \frac{1}{2} L i^2 \right) \, dt = \frac{1}{2} L i^2 \quad (26.24)$$

The latter is by definition the magnetic energy associated with the current  $i$ . Independently from the procedure used to calculate  $u_m$ , the magnetic energy depends only on the value of  $L$  and  $i$ . In particular, when  $i$  is constant in time, so is  $u_m$ .

In order to illustrate the physical meaning of  $u_m$ , assume to connect a circuit with a current  $I$  to a conductor with resistance  $R$ . By setting  $\mathcal{E}_g = 0$  in (26.9), we obtain

$$0 = R\iota + L \frac{d}{dt} \iota \quad (26.25)$$

By integration, we find  $\iota(t) = Ae^{-t/\tau'}$ , with  $\tau' = L/R$  and  $A$  an arbitrary constant. By choosing  $A$  so that  $\iota(0) = I$ , we have

$$\iota = \iota(t) = Ie^{-t/\tau'} \quad (26.26)$$

A plot of (26.16) is shown in Fig. 26.3. The current  $\iota$  tends asymptotically to zero for  $t \rightarrow +\infty$ .

The energy dissipated in  $R$  because of the Joule effect in the time window  $[0, +\infty)$  must derive from the magnetic energy initially stored in the circuits. In fact,

$$\begin{aligned} \int_0^{+\infty} R\iota^2 dt &= \int_0^{+\infty} R(Ie^{-t/\tau'})^2 dt \\ &= \int_0^{+\infty} RI^2 e^{-2t/\tau'} dt \\ &= RI^2 \left[ -\frac{\tau'}{2} e^{-2t/\tau'} \right]_0^{+\infty} \\ &= \frac{1}{2} LI^2 \end{aligned} \quad (26.27)$$

In conclusion, the energy associated with a circuit with self inductance  $L$  and current  $\iota$  is given by

$$u_m = \frac{1}{2} LI^2 \quad (26.28)$$

This energy is stored in the circuit during the time window when the current increases from 0 to  $\iota$  and can be given back by the circuit itself when the current decreases from the initial value to 0.

When  $R = 0$ , the time constant  $\tau' = L/R$  becomes unlimited. In this case, the charging process of the inductor (we indicate with this term a circuit with a nonzero self inductance) never ends and the current increases linearly according to

$$\iota = \frac{\mathcal{E}_g}{L} t \quad (26.29)$$

Similarly, when trying to discharge the inductor on a zero resistance, the current remains constant in time because the stored energy (in the inductor) cannot be dissipated in the circuit.

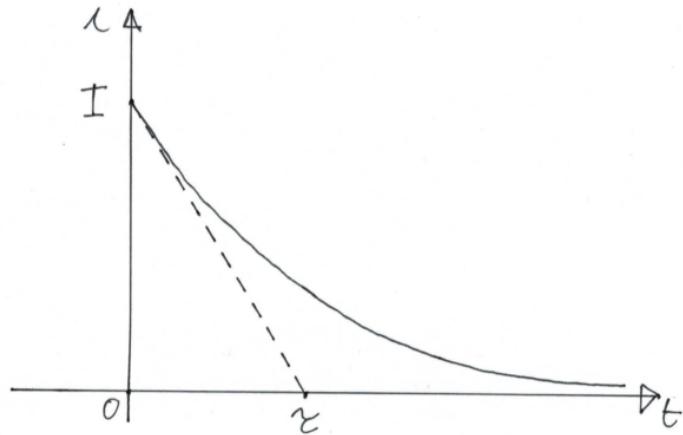


Figure 26.3

## 26.2 Two Circuits with Current

Consider two filiform circuits  $\gamma_1$  and  $\gamma_2$  carrying, in general, the time dependent currents  $\iota_1$  and  $\iota_2$  respectively (see 26.4). We intend to calculate the magnetic energy associated with this pair of circuits. To this end, we suppose to connect the circuits with the emf sources  $\mathcal{E}_1$  and  $\mathcal{E}_2$  as shown in the figure. Following the same argument as for a single circuit, for  $\gamma_1$  we have

$$\mathcal{E}_1 - \frac{d}{dt} \Phi_{\gamma_1} = R_1 \iota_1 \quad (26.30)$$

where  $\Phi_{\gamma_1}$  is the flux linked with the circuit,  $R_1$  is its total resistance (i.e., the sum of the source resistance  $R_{g1}$  and the loop resistance  $R_{\ell 1}$ ), and  $\iota_1$  is the current flowing through  $\gamma_1$ .

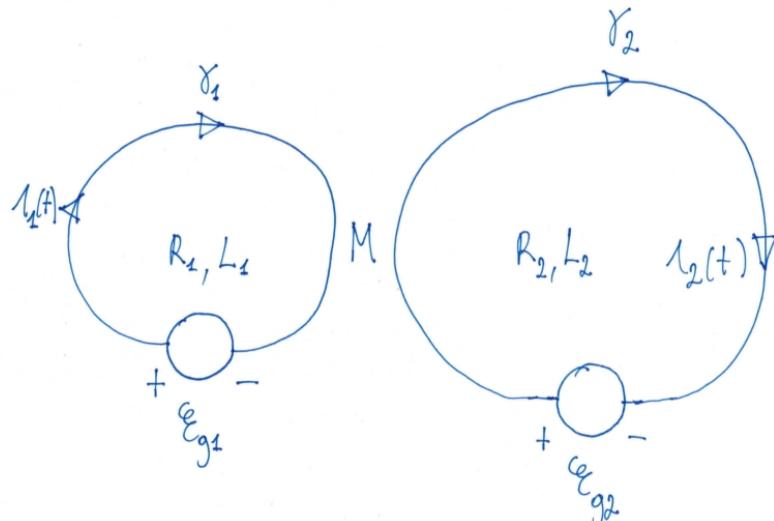


Figure 26.4

Similarly, for  $\gamma_2$

$$\mathcal{E}_2 - \frac{d}{dt} \Phi_{\gamma_1} = R_2 \iota_2 \quad (26.31)$$

with obvious meaning of the symbols.

From the definition of inductance, we have  $\Phi_{\gamma_1}$

$$\left\{ \begin{array}{l} \Phi_1 = L_1 \iota_1 + M \iota_2 \\ \Phi_2 = L_2 \iota_2 + M \iota_1 \end{array} \right. \quad (26.32a)$$

$$\left\{ \begin{array}{l} \Phi_1 = L_1 \iota_1 + M \iota_2 \\ \Phi_2 = L_2 \iota_2 + M \iota_1 \end{array} \right. \quad (26.32b)$$

where  $M$  is the mutual inductance between  $\gamma_1$  and  $\gamma_2$ . By substituting (26.32a) and (26.32b) into (26.30) and (26.31), we obtain

$$\left\{ \begin{array}{l} \mathcal{E}_1 = R_1 \iota_1 + L_1 \frac{d}{dt} \iota_1 + M \frac{d}{dt} \iota_2 \\ \mathcal{E}_2 = R_2 \iota_2 + L_2 \frac{d}{dt} \iota_2 + M \frac{d}{dt} \iota_1 \end{array} \right. \quad (26.33a)$$

$$\left\{ \begin{array}{l} \mathcal{E}_1 = R_1 \iota_1^2 + L_1 \iota_1 \frac{d}{dt} \iota_1 + M \iota_1 \frac{d}{dt} \iota_2 \\ \mathcal{E}_2 = R_2 \iota_2^2 + L_2 \iota_2 \frac{d}{dt} \iota_2 + M \iota_2 \frac{d}{dt} \iota_1 \end{array} \right. \quad (26.33b)$$

By multiplying (26.33a) by  $\iota_1$  and (26.33b) by  $\iota_2$ , we find the power associated with each circuit.

$$\left\{ \begin{array}{l} \mathcal{E}_1 \iota_1 = R_1 \iota_1^2 + L_1 \iota_1 \frac{d}{dt} \iota_1 + M \iota_1 \frac{d}{dt} \iota_2 \\ \mathcal{E}_2 \iota_2 = R_2 \iota_2^2 + L_2 \iota_2 \frac{d}{dt} \iota_2 + M \iota_2 \frac{d}{dt} \iota_1 \end{array} \right. \quad (26.34a)$$

$$\left\{ \begin{array}{l} \mathcal{E}_1 \iota_1 = R_1 \iota_1^2 + L_1 \iota_1 \frac{d}{dt} \iota_1 + M \iota_1 \frac{d}{dt} \iota_2 \\ \mathcal{E}_2 \iota_2 = R_2 \iota_2^2 + L_2 \iota_2 \frac{d}{dt} \iota_2 + M \iota_2 \frac{d}{dt} \iota_1 \end{array} \right. \quad (26.34b)$$

By integrating (26.34a) and (26.34b) between  $t = 0$  and a generic time instant  $t$  and summing them term by term, we have

$$\begin{aligned} \int_0^t (\mathcal{E}_1 \iota_1 + \mathcal{E}_2 \iota_2) dt &= \int_0^t (R_1 \iota_1^2 + R_2 \iota_2^2) dt \\ &+ \int_0^t L_1 \iota_1 \frac{d}{dt} \iota_1 dt + \int_0^t L_2 \iota_2 \frac{d}{dt} \iota_2 dt \\ &+ \int_0^t \left( M \iota_1 \frac{d}{dt} \iota_2 + M \iota_2 \frac{d}{dt} \iota_1 \right) dt \end{aligned} \quad (26.35)$$

and, finally,

$$\begin{aligned} \int_0^t (\mathcal{E}_1 \iota_1 + \mathcal{E}_2 \iota_2) dt &= \int_0^t (R_1 \iota_1^2 + R_2 \iota_2^2) dt \\ &+ \int_0^t \frac{d}{dt} \left( \frac{1}{2} L_1 \iota_1^2 + \frac{1}{2} L_2 \iota_2^2 \right) dt \\ &+ \int_0^t \frac{d}{dt} (M \iota_1 \iota_2) dt \end{aligned} \quad (26.36)$$

The integral on the left-hand side of (26.36) is the total energy generated by the two sources in the time interval  $[0, t]$ . The first integral on the right-hand side is the energy dissipated in the circuits because of Joule's effect in the same time interval. Thus, from the second fundamental theorem of calculus, the magnetic energy associated with two circuits is given by

$$\begin{aligned} U_m(t) &= \frac{1}{2} L_1 i_1^2(t) + \frac{1}{2} L_2 i_2^2(t) + M i_1(t) i_2(t) \\ &\quad - \frac{1}{2} L_1 i_1^2(0) - \frac{1}{2} L_2 i_2^2(0) - M i_1(0) i_2(0) \end{aligned} \quad (26.37)$$

Equation (26.37) shows that in order to “charge” the two circuits when  $M \neq 0$ , in addition to the self-inductance energy terms  $(1/2)L_1 i_1^2$  and  $(1/2)L_2 i_2^2$  there must be a mutual inductance term given by  $M i_1 i_2$ . The sign of the self-inductance energy terms is always positive because by definition  $L_1 > 0$  and  $L_2 > 0$ ; However, the sign of the mutual inductance energy term can be either positive or negative, depending on the sign of  $M$ ,  $i_1$ , and  $i_2$  (see PHYS 242). As a consequence, the total magnetic energy of the two circuits  $U_m$  can be larger or smaller than the sum of the two self-inductance energy terms. For example assume to have two loops of current that are on top of each other, almost touching each other (i.e., the two loops practically coincide). Assume each loop to carry with equal intensity, but opposite sign. Since the two loops practically coincide, we can regard them as a single loop with zero current. In this case, we expect  $U_m = 0$ . In fact, for this system

$$L_1 = L_2 = M \quad \text{and} \quad i_1 = -i_2 = i$$

and thus

$$U_m = \frac{1}{2} M i^2 + \frac{1}{2} M i^2 - M i^2 = 0 \quad (26.38)$$

where we assumed  $i(0) = 0$ .

It is possible to obtain the magnetic energy (26.37) by means of the results in the chapter on forces on circuits with currents. To this end, we first assume that only circuit  $\gamma_1$  is present. If the circuit “charged” with the current  $i_1$ , its energy is

$$U_{m_1} = \frac{1}{2} L_1 i_1^2 \quad (26.39)$$

Note that here and hereafter we assume all initial currents (i.e., at  $t = 0$ ) to be zero and we hide the time dependence of the currents. Then suppose to “charge” circuit  $\gamma_2$  while keeping it at a very large distance (in the limit infinite) from  $\gamma_1$ . Under these conditions, the influence of  $\gamma_1$  on  $\gamma_2$  can be neglected and

$$U_{m_2} = \frac{1}{2} L_2 i_2^2 \quad (26.40)$$

Finally, suppose to translate  $\gamma_2$  from infinite to a position nearby  $\gamma_1$ , while making sure that the functional dependence of  $i_1$  and  $i_2$  on time remains unchanged during the translation. For example, in order to maintain  $i_1$  unchanged, we must compensate the

emf induced by the variation of the flux  $\Phi_{12}$  due to  $\iota_2$  and linked with  $\gamma_1$ . This can be achieved by inserting a source in  $\gamma_1$  with emf  $\mathcal{E} = +d\Phi_{12}/dt$ . In order to translate  $\gamma_2$  to its final position it is necessary to perform a mechanical work due to the interaction between the currents and an electrical work to keep the currents unchanged during the translation. The mechanical work  $W$ , which by definition is the external work against the field forces, is given by

$$W = -\iota_2(\Phi'_2 - \Phi_2) \quad (26.41)$$

where  $\Phi_2$  and  $\Phi'_2$  are the fluxes linked with  $\gamma_2$  when this is at infinite distance and nearby  $\gamma_1$ , respectively. By definition

$$\left\{ \begin{array}{l} \Phi_2 = L_2 \iota_2 \\ \Phi'_2 = L_2 \iota_2 + M \iota_1 \end{array} \right. \quad (26.42a)$$

$$\left\{ \begin{array}{l} \Phi_2 = L_2 \iota_2 \\ \Phi'_2 = L_2 \iota_2 + M \iota_1 \end{array} \right. \quad (26.42b)$$

Therefore

$$W = -\iota_2 M \iota_1 = -M \iota_1 \iota_2 \quad (26.43)$$

The electrical works that must be given to the two circuits in order to maintain the currents unchanged during the translation of  $\gamma_2$  are, thus, those associated with two emfs inserted in each circuit. The electrical work  $W_1$  associated with the emf inserted in  $\gamma_1$  is

$$W_1 = \int_0^{t'} \iota_1 \frac{d}{dt} \Phi_{12} dt = \iota_1 \Phi_{12} = M \iota_1 \iota_2 \quad (26.44)$$

where  $t'$  is the time required to translate  $\gamma_2$  to its final position. Similarly, the electrical work  $W_2$  associates with the emf inserted in  $\gamma_2$  is

$$W_2 = \int_0^T \iota_2 \frac{d\Phi_{21}}{dt} dt = \iota_2 \Phi_{21} = M \iota_1 \iota_2 \quad (26.45)$$

By summing all work terms and adding them to the magnetic energies  $U_{m_1}$  and  $U_{m_2}$ , we have

$$\begin{aligned} U_m &= U_{m_1} + U_{m_2} + W + W_1 + W_2 \\ &= \frac{1}{2} L_1 \iota_1^2 + \frac{1}{2} L_2 \iota_2^2 + M \iota_1 \iota_2 \end{aligned} \quad (26.46)$$

In this sense the term  $M \iota_1 \iota_2$  represents the electrical work generated by each emf  $\mathcal{E}_1$  and  $\mathcal{E}_2$  that was inserted in the circuits to maintain the currents  $\iota_1$  and  $\iota_2$  unchanged during the translation of  $\gamma_2$ . Note that the two emfs generate in total a work equal to  $2M \iota_1 \iota_2$ , half of which becomes mechanical work and the other half gets stored as magnetic energy.

The magnetic energy  $U_m$  associated with the system of two circuits can be rewritten as

$$U_m = \frac{1}{2} \Phi_{11} \iota_1 + \frac{1}{2} \Phi_{22} \iota_2 + \frac{1}{2} \Phi_{12} \iota_1 + \frac{1}{2} \Phi_{21} \iota_2 \quad (26.47)$$

where

$$\left\{ \begin{array}{l} \Phi_{11} = L_1 \iota_1 \\ \Phi_{12} = M \iota_2 \\ \Phi_{22} = L_2 \iota_2 \\ \Phi_{21} = M \iota_1 \end{array} \right. \quad (26.48a)$$

$$\left\{ \begin{array}{l} \Phi_{11} = L_1 \iota_1 \\ \Phi_{12} = M \iota_2 \\ \Phi_{22} = L_2 \iota_2 \\ \Phi_{21} = M \iota_1 \end{array} \right. \quad (26.48b)$$

$$\left\{ \begin{array}{l} \Phi_{11} = L_1 \iota_1 \\ \Phi_{12} = M \iota_2 \\ \Phi_{22} = L_2 \iota_2 \\ \Phi_{21} = M \iota_1 \end{array} \right. \quad (26.48c)$$

$$\left\{ \begin{array}{l} \Phi_{11} = L_1 \iota_1 \\ \Phi_{12} = M \iota_2 \\ \Phi_{22} = L_2 \iota_2 \\ \Phi_{21} = M \iota_1 \end{array} \right. \quad (26.48d)$$

By grouping together the terms with  $\iota_1$  and  $\iota_2$ , we have

$$U_m = \frac{1}{2} \iota_1 (\Phi_{11} + \Phi_{12}) + \frac{1}{2} \iota_2 (\Phi_{22} + \Phi_{21}) \quad (26.49)$$

By naming  $\Phi_1$  and  $\Phi_2$  the fluxes linked with  $\gamma_1$  and  $\gamma_2$ , respectively, due to both currents simultaneously,

$$\left\{ \begin{array}{l} \Phi_1 = \Phi_{11} + \Phi_{12} \\ \Phi_2 = \Phi_{22} + \Phi_{21} \end{array} \right. \quad (26.50a)$$

$$\left\{ \begin{array}{l} \Phi_1 = \Phi_{11} + \Phi_{12} \\ \Phi_2 = \Phi_{22} + \Phi_{21} \end{array} \right. \quad (26.50b)$$

we finally have

$$U_m = \frac{1}{2} (\Phi_1 \iota_1 + \Phi_2 \iota_2) \quad (26.51)$$

### 26.2.1 $N$ Circuits with Currents

We can now extend the previous results to a general system of  $N$  circuits  $\gamma_1, \gamma_2, \dots, \gamma_N$ , carrying the currents  $\iota_1, \iota_2, \dots, \iota_N$ . Following the derivation for the case of two circuits suppose to insert  $N$  emfs  $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_N$ . We have

$$\left\{ \begin{array}{l} \mathcal{E}_1 = R_1 \iota_1 + \frac{d}{dt} \Phi_1 = R_1 \iota_1 + \frac{d}{dt} (L_1 \iota_1 + M_{12} \iota_2 + \dots + M_{1N} \iota_N) \\ \vdots \quad \vdots \\ \mathcal{E}_N = R_N \iota_N + \frac{d}{dt} \Phi_N = R_N \iota_N + \frac{d}{dt} (M_N \iota_1 + M_{N2} \iota_2 + \dots + L_N \iota_N) \end{array} \right.$$

By multiplying the first equation by  $\iota_1$ , the second equation by  $\iota_2$ , and so on and so forth, summing all equation term by term, and integrating between  $t = 0$  and  $t$  (assume all currents to be equal to zero), we find

$$U_m = \frac{1}{2} \sum_{k=1}^N \iota_k \Phi_k \quad (26.52)$$

where  $\iota_k$  is the current carried by the circuit  $\gamma_k$  at time  $t$  and  $\Phi_k$  is the flux due to all the  $N$  currents linked with  $\gamma_k$  at the same time  $t$ . Note the analogy between this result and the electrostatic energy associated with a system of charged conductors.



## Chapter 27

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### 27.1 The Electromagnetic Field Laws in Presence of Materials

The general integral form of the laws of the electromagnetic field when dielectric and/or magnetic materials are present are

$$\iint_{\Sigma} \vec{D} \cdot \vec{n} dA = q_f \quad (\text{Gauss' law}) \quad (27.1)$$

$$\iint_{\Sigma} B \cdot \vec{n} dA = 0 \quad (\text{solenoidal property of } B) \quad (27.2)$$

$$\iint_{\Sigma} \vec{J}_c \cdot \vec{n} dA = - \frac{d}{dt} q_f \quad (\text{charge conservation}) \quad (27.3)$$

$$\oint_{\gamma} \vec{E} \cdot t d\ell = - \iint_{\Sigma_{\gamma}} \frac{\partial}{\partial t} \vec{B} \cdot \vec{n} dA \quad (\text{Faraday-Neumann-Lenz's law}) \quad (27.4)$$

$$\oint_{\gamma} \vec{H} \cdot t d\ell = \iint_{\Sigma_{\gamma}} \left( \vec{J}_c + \frac{\partial}{\partial t} \vec{D} \right) \cdot \vec{n} dA \quad (\text{Ampère-Maxwell's law}) \quad (27.5)$$

where  $\vec{J}_c$  is the conduction current,  $q_f$  is the free charge in the considered region of space, and  $\Sigma$ ,  $\gamma$ , and  $\Sigma_{\gamma}$  are the usual surfaces and line at rest in the inertial reference frame of the fields. On the one hand, solenoidal property of  $\vec{B}$  as well as Faraday-Neumann-Lenz's law are the same in vacuum and in the case of materials. On the other hand, Gauss' law and Ampère-Maxwell's law get modified in presence of materials, as in the static case. In fact, in (27.1) on the left-hand side we have flux of the displacement vector  $\vec{D}$ , and on the right-hand side the total free charge  $q_f$  within the region of space limited by  $\Sigma$ ; In (27.5), on the left-hand side we have the circulation of the vector  $\vec{H}$ , and on the right-hand side the algebraic sum of the conduction current density  $\vec{J}_c$  and of the displacement current  $\partial \vec{D} / \partial t$  linked with  $\gamma$ . Note that, in the case of materials the displacement current depends on the displacement vector  $\vec{D}$ . This is the reason

## 27.1. THE ELECTROMAGNETIC FIELD LAWS IN PRESENCE OF MATERIALS

why this current is called displacement current (it was Maxwell to coin this name when conducting experiments on the dielectric in presence of a time-varying electric field: The electric dipoles were “displaced” by the field, from which the name of the associated current).

At last, even in the presence of materials, only the conduction current  $\vec{J}_c$  and the free charge  $q_f$  appear in Eq. (27.3).

The constitutive relations for the dielectric and magnetic materials in the case of time-dependant fields are more complex than in the static case. In fact, when the fields vary very rapidly with time these relations must be changed compared to the static case (see PHYS 442). However, so long the fields vary in time sufficiently slowly (we will give a more quantitative argument in PHYS 442), the relations are

$$\left\{ \begin{array}{ll} \vec{D} = \vec{D}(\vec{E}) & \text{for dielectrics} \end{array} \right. \quad (27.6a)$$

$$\left\{ \begin{array}{ll} \vec{B} = \vec{B}(\vec{H}) & \text{for magnetic materials} \end{array} \right. \quad (27.6b)$$

$$\left\{ \begin{array}{ll} \vec{J}_c = \vec{J}_c(\vec{E}, \vec{E}_m) & \text{for conductors} \end{array} \right. \quad (27.6c)$$

Note that  $\vec{E}_m$  is the electromotive field of non-electric origin. In the case of linear materials, the constitutive relations become

$$\left\{ \begin{array}{ll} \vec{D} = \epsilon \vec{E} & \end{array} \right. \quad (27.7a)$$

$$\left\{ \begin{array}{ll} \vec{B} = \mu \vec{H} & \end{array} \right. \quad (27.7b)$$

$$\left\{ \begin{array}{ll} \vec{J}_c = \sigma(\vec{E} + \vec{E}_m) & \end{array} \right. \quad (27.7c)$$

where the values of  $\epsilon$ ,  $\mu$ , and  $\sigma$  can be assumed to be equal to the values in the static case.

A further complication appears when the materials move in the inertial reference frame where the fields are defined, particularly when their velocities are very high (i.e., comparable to the velocity of light in vacuum; see PHYS 442). In these cases, in the constitutive relations vector  $\vec{D}$  depend not only on vector  $\vec{E}$ , but also on vector  $\vec{B}$ . Similarly, vector  $\vec{B}$  depends on both  $\vec{E}$  and  $\vec{H}$ . These cases require a relativistic theory. A simple non-relativistic case is that of a conductor moving with velocity  $u \ll c$ . This scenario takes place, for example, in most electric machines (e.g., engines, generators, washing machines, . . .). In these cases the constitutive relations for  $\vec{J}_c$  is well approximated by

$$\vec{J}_c = \sigma(\vec{E} + \vec{E}_m + \vec{u} \times \vec{B}) \quad (27.8)$$

Note that the quantities in these equations are supposed to be measured with respect to the inertial reference system of the laboratory (i.e., not with respect to the conductor system). Moreover, it is assumed that the electrical conductivity  $\sigma$  is equal to the static value. As a consequence, inside the moving conductor acts a motional field  $\vec{u} \times \vec{B}$  that has a similar role as electromotive field  $\vec{E}_m$ . Thus, in the case of a conductor moving at velocity  $\vec{u}$ , an emf is induced in the conductor. For filiform conductors the induced

emf is

$$\mathcal{E}_{\gamma\text{mot}} = \int_{\gamma} \vec{u} \times \vec{B} \cdot d\ell \quad (27.9)$$

where  $\gamma$  represents the conductor's axis. The local form of (27.9) can be written as

$$d\mathcal{E}_{\gamma\text{mot}} = \vec{u} \times \vec{B} \cdot t d\ell \quad (27.10)$$

We proved Eq. (27.9) when studying the Barlow's wheel. Thus, in the case of a conductor the longitudinal axis of which moves at a velocity  $\vec{u}$ ,  $\gamma(t)$ , Faraday-Neumann-Lenz's law can be written as

$$\oint_{\gamma(t)} (\vec{E} + \vec{u} \times \vec{B}) \cdot \vec{t} d\ell = - \frac{d}{dt} \iint_{\Sigma_{\gamma(t)}} \vec{B} \cdot \vec{n} dA \quad (27.11)$$

By substituting (27.8) into the (27.11), we obtain

$$\oint_{\gamma(t)} \frac{\vec{J}_c}{\sigma} \cdot \vec{t} d\ell = - \frac{d}{dt} \Phi_{\gamma} \quad (27.12)$$

Multiplying and dividing within the sign of integral by the area  $A_c$  of the normal cross-section of the conductor, we have

$$\oint_{\gamma(t)} \frac{\vec{J}_c A_c}{\sigma A_c} \cdot \vec{t} d\ell = - \frac{d}{dt} \Phi_{\gamma} \quad (27.13)$$

The product  $\vec{J}_c A_c \cdot \vec{t} = \iota$  is the current intensity circulating through the conductor. Therefore,

$$\oint_{\gamma(t)} \frac{\iota}{\sigma A_c} d\ell = - \frac{d}{dt} \Phi_{\gamma} \quad (27.14)$$

When the field vary slowly enough so that the displacement current around the conductor can be neglected (this is the case in the electrical machines), the current intensity  $\iota$  can be assumed to be constant. Hence,

$$\iota \oint_{\gamma(t)} \frac{1}{\sigma A_c} d\ell = R\iota = - \frac{d}{dt} \Phi_{\gamma} \quad (27.15)$$

This relationship shows that in order to calculate the induced current through the moving loop, we can consider an emf  $\mathcal{E}_{\gamma}$  acts on the conductor,

$$\mathcal{E}_{\gamma} = \mathcal{E}_{\gamma E} + \mathcal{E}_{\gamma\text{mot}} = - \frac{d}{dt} \Phi_{\gamma} \quad (27.16)$$

where  $\mathcal{E}_{\gamma}$  is associated with field  $\vec{E}_m$  and  $\mathcal{E}_{\gamma\text{mot}}$  with  $\vec{u} \times \vec{B}$ .

## 27.2 Maxwell's Laws in Local Form

### 27.2.1 Case 1

Consider a region of space  $\Omega$  in the three-dimensional Euclidean space, where  $\vec{E} \in C^1(\Omega)$  and  $\vec{B} \in C^1(\Omega)$ . Typically this means all charge and current densities are  $C^0(\Omega)$  and therein limited. By means of Stokes' theorem for the terms on the left-hand side of Eqs. (27.4) and (27.5), we find

$$\oint_{\gamma} \vec{E} \cdot \vec{t} d\ell = \iint_{\Sigma_{\gamma}} \vec{\nabla} \times \vec{E} \cdot \vec{n} dA \quad (27.17)$$

$$\oint_{\gamma} \vec{H} \cdot \vec{t} d\ell = \iint_{\Sigma_{\gamma}} \vec{\nabla} \times \vec{H} \cdot \vec{n} dA \quad (27.18)$$

As a consequence, Faraday-Neumann-Lenz's and Ampère-Maxwell's laws become

$$\iint_{\Sigma_{\gamma}} \vec{\nabla} \times \vec{E} \cdot \vec{n} dA = \iint_{\Sigma_{\gamma}} \frac{\partial}{\partial t} \vec{B} \cdot \vec{n} dA \quad (27.19)$$

$$\iint_{\Sigma_{\gamma}} \vec{\nabla} \times \vec{H} \cdot \vec{n} dA = \iint_{\Sigma_{\gamma}} \left( \vec{J}_c + \frac{\partial}{\partial t} \vec{D} \right) \cdot \vec{n} dA \quad (27.20)$$

Due to the arbitrariness of  $\Sigma_{\gamma}$ , we have

$$\vec{\nabla} \times \vec{E} = - \frac{\partial}{\partial t} \vec{B} \quad (27.21)$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_c + \frac{\partial}{\partial t} \vec{D} \quad (27.22)$$

These two equations are Maxwell's equations in differential form. To this we must add the continuity equation

$$\vec{\nabla} \cdot \vec{J}_c + \frac{\partial}{\partial t} \rho_f = 0 \quad (27.23)$$

as well as the two initial conditions

$$\vec{\nabla} \cdot \vec{D} = \rho_f \quad (27.24)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (27.25)$$

In vacuum, i.e., in absence of free charges and conduction currents, Maxwell's equa-

tions and the initial conditions become

$$\vec{\nabla} \times \vec{E} = - \frac{\partial}{\partial t} \vec{B} \quad (27.26)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{E} \quad (27.27)$$

$$\vec{\nabla} \cdot \vec{E} = 0 \quad (27.28)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (27.29)$$

This set of equations clearly shows the dual behavior of the fields  $\vec{E}$  and  $\vec{B}$ . Additionally, they show the intimate interconnection between  $\vec{E}(t)$  and  $\vec{B}(t)$ .

### 27.2.2 Case 2

We now study the case of charge and current densities with the discontinuity at the points of the surface  $\Sigma$  separating two regions of space 1 and 2, i.e., of surface charge and current densities. Gauss' Law for  $\vec{D}$  and the solenoidal property of  $\vec{B}$  follow from the static case,

$$(\vec{D}_2 - \vec{D}_1) \cdot \vec{n} = \sigma_f \quad (27.30)$$

$$(\vec{B}_1 - \vec{B}_2) \cdot \vec{n} = 0 \quad (27.31)$$

where  $\vec{D}_1$ ,  $\vec{D}_2$ ,  $\vec{B}_1$ , and  $\vec{B}_2$  are the values of the displacement and the magnetic induction fields in the neighborhood of a point on  $\Sigma$  in region 1 and 2, respectively.

The arguments for the electromagnetic and magnetoelectric laws are more complex. We begin by considering cut-type curve  $\gamma_c$  at rest in the inertial reference frame of the fields. As always, half of  $\gamma_c$  resides below  $\Sigma$  in region 1 and the other half above  $\Sigma$  in region 2. The normal unit vector  $\vec{n}$  at each point on  $\Sigma$  is directed from 1 to 2. The two lateral sides of the curve have length  $h$ , whereas the bottom and top sides have length  $d\ell$ . By applying the electromagnetic induction law to  $\gamma_c$  and in the limit  $h \rightarrow 0^+$ , we obtain

$$\oint_{\gamma} \vec{E} \cdot \vec{t} d\ell = E_{t1} d\ell - E_{t2} d\ell = - \frac{d}{dt} \Phi_{\gamma_c} \quad (27.32)$$

Where  $\Phi_{\gamma_c}$  is the flux due to the  $\vec{B}$  and linked with  $\gamma_c$ . It is clear that this flux goes to 0 when  $h \rightarrow 0^+$ . Hence,

$$(E_{t1} - E_{t2}) d\ell = 0 \quad (27.33)$$

or, equivalently,

$$E_{t1} = E_{t2} \quad (27.34)$$

and in vector form

$$(\vec{E}_2 - \vec{E}_1) \times \vec{n} = 0 \quad (27.35)$$

Therefore, even in presence of time-varying fields, the components of  $\vec{E}$  along any tangent direction in the neighborhood of  $\Sigma$  are continuous.

Similarly, applying the magneto-electric law to  $\gamma_c$  in the limit  $h \rightarrow 0^+$ , we have

$$\begin{aligned} \oint_{\gamma_c} \vec{H} \cdot \vec{t} d\ell &= H''_{t1} d\ell - H''_{t2} d\ell \\ &= \mu_0 J'_{cs} d\ell + \mu_0 \frac{d}{dt} \Psi_{\gamma_c} \\ &= \mu_0 J'_{cs} d\ell \end{aligned} \quad (27.36)$$

where the meaning of the symbols is the same as in Fig. (19.2) and the flux due to  $\epsilon_0 \vec{E}$  through  $\gamma_c$  is equal to zero in the limit  $h \rightarrow 0^+$ ,  $\Psi_{\gamma_c} = 0$ . From (27.36)

$$B''_{t1} - B''_{t2} = \mu_0 J'_{cs} \quad (27.37)$$

The same argument applies to  $J''_{cs}$ . Therefore,

$$(\vec{H}_1 - \vec{H}_2) \times \vec{n} = \vec{J}_{cs} \quad (27.38)$$

In summary, for case 2 we have

$$(\vec{E}_2 - \vec{E}_1) \times \vec{n} = 0 \quad (27.39)$$

$$(\vec{H}_1 - \vec{H}_2) \times \vec{n} = \vec{J}_{cs} \quad (27.40)$$

$$(\vec{D}_2 - \vec{D}_1) \cdot \vec{n} = \sigma_f \quad (27.41)$$

$$(\vec{B}_1 - \vec{B}_2) \cdot \vec{n} = 0 \quad (27.42)$$

An interesting case where to apply Eqs. (27.39)-(27.42) is that of an electromagnetic field at the external surface of a perfect conductor (i.e., a conductor with zero resistivity  $\eta = 0$ ). In this case from the local form of Ohm's law

$$\eta \vec{J}_c = \vec{E} \quad (27.43)$$

we conclude that  $\vec{E} = 0$ . In fact, if this was not the case, for  $\eta = 0$  we would obtain an non-physical  $\vec{J} \rightarrow \infty$ . Hence, for a perfect conductor  $\vec{E} = \vec{0}$  at each point inside the conductor both in the electrostatic case and in the electrodynamic case. Since Faraday-Neumann-Lenz's law must also be valid at each point inside the conductor, we find

$$\vec{\nabla} \times \vec{E} = \vec{0} = - \frac{\partial}{\partial t} \vec{B} \quad (27.44)$$

This means that  $\vec{B}$  must be constant at each point inside the perfect conductor. In particular, when  $\vec{B} = 0$  at one instant in time, it must be zero at all times. In this case, at each point inside the conductor we have

$$\vec{E} = 0 \quad (27.45)$$

$$\vec{B} = 0 \quad (27.46)$$

Hence, in the case of a perfect conductor Eqs. (27.39)-(27.42) give

$$\vec{n} \cdot \vec{E} = \frac{\sigma}{\epsilon_0} \quad (27.47)$$

$$\vec{n} \times \vec{E} = 0 \quad (27.48)$$

$$\vec{n} \cdot \vec{B} = 0 \quad (27.49)$$

$$\vec{n} \times \vec{B} = \vec{J}_{cs} \quad (27.50)$$

where  $\vec{E}$  and  $\vec{B}$  are the electric and the magnetic fields at each point on the external surface of the conductor:  $\vec{E}$  is normal to the surface and  $\vec{B}$  tangent to it.



# Chapter 28

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## 28.1 Electromagnetic Potentials

We will now present the laws of electromagnetism by means of the field potentials.

As in magnetostatics, the solenoidal principle of  $\vec{B}$  implies that the field has infinite vector potentials. By indicating with  $\vec{A}$  one of such potentials, we have

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad (28.1)$$

Given an arbitrary scalar function  $\psi$ , the vector potentials of  $\vec{B}$  are related by the condition

$$\vec{A}' = \vec{A} + \vec{\nabla} \cdot \psi \quad (28.2)$$

By substituting (28.1) in the Faraday-Neumann law in local form, we obtain

$$\vec{\nabla} \times \vec{E} = - \frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} \quad (28.3)$$

By swapping the time and space derivatives in (28.3) we find

$$\vec{\nabla} \times \vec{E} = - \vec{\nabla} \times \left( \frac{\partial}{\partial t} \vec{A} \right) \quad (28.4)$$

and so

$$\vec{\nabla} \times \left( \vec{E} + \frac{\partial}{\partial t} \vec{A} \right) = 0 \quad (28.5)$$

As a consequence, the field  $(\vec{E} + \partial \vec{A} / \partial t)$  is irrotational.

Assuming the region of space where the field  $(\vec{E} + \partial \vec{A} / \partial t)$  is defined to be simply connected (Poincaré's Lemma), (28.5) is equivalent to

$$\vec{E} + \frac{\partial}{\partial t} \vec{A} = \vec{\nabla} \varphi \quad (28.6)$$

where  $\varphi$  is one possible scalar potential (defined, as always, up to an arbitrary constant) of the field  $(\vec{E} + \partial \vec{A} / \partial t)$ . For simplicity, we now define  $V = -\varphi$  and obtain

$$\vec{E} = - \frac{\partial}{\partial t} \vec{A} - \vec{\nabla} V \quad (28.7)$$

It is worth stressing the fact that the potential function  $V$  introduced here is not the potential of the electric field  $\vec{E}$  (which does not have a scalar potential). The function  $V$  is the potential of  $(\vec{E} + \partial \vec{A}/\partial t)$ , which is irrotational.

By substituting (28.7) into Gauss' law, we obtain

$$\vec{\nabla} \cdot \left( -\frac{\partial}{\partial t} \vec{A} - \vec{\nabla} V \right) = \frac{1}{\epsilon_0} \rho \quad (28.8)$$

and, thus,

$$\vec{\nabla}^2 V + \vec{\nabla} \cdot \left( \frac{\partial}{\partial t} \vec{A} \right) = -\frac{1}{\epsilon_0} \rho \quad (28.9)$$

Similarly, by substituting (28.1) and (28.7) into Ampère-Maxwell law in local form, we find

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} + \mu_0 \epsilon_0 \left( -\vec{\nabla} \left( \frac{\partial}{\partial t} V \right) - \frac{\partial^2}{\partial t^2} \vec{A} \right) \quad (28.10)$$

In summary,

$$\vec{\nabla}^2 V + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = -\frac{1}{\epsilon_0} \rho \quad (28.11a)$$

and

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) + \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{A} + \mu_0 \epsilon_0 \vec{\nabla} \left( \frac{\partial}{\partial t} V \right) = \mu_0 \vec{J} \quad (28.11b)$$

Note that, in both Eqs. (28.11a) and (28.11b) the electric and magnetic fields  $\vec{E}$  and  $\vec{B}$  do not appear explicitly. In fact, only the scalar and vector potentials  $V$  and  $\vec{A}$  appear in those equations. However, given a charge and current distribution, (28.11a) and (28.11b) are insufficient to find a unique solution for  $V$  and  $\vec{A}$ .

Consider the volume charge density  $\rho$  and electric current density  $\vec{J}$  are the sources of an electromagnetic field. Assume  $\rho$  and  $\vec{J}$  fulfil the continuity equation (???). Furthermore, assume a scalar potential  $V(P, t)$  and a vector potential  $\vec{A}(P, t)$  verify (28.11a) and (28.11b) at each point  $P$  and time  $t$ . We now intend to address the question whether two new functions  $V'(P, t)$  and  $\vec{A}'(P, t)$  can verify (28.11a) and (28.11b) for the same  $\rho$  and  $\vec{J}$ . To answer this question we impose

$$\vec{A}' = \vec{A} + \vec{\nabla} \psi \quad (28.12a)$$

and

$$V' = V + \varphi \quad (28.12b)$$

where  $\psi$  and  $\varphi$  are generic scalar functions of  $P$  and  $t$ .

We now want to verify whether (28.11a) and (28.11b) are still fulfilled when substituting  $\vec{A}$  and  $V$  with  $\vec{A}'$  and  $V'$ . For the first terms we have

$$\vec{\nabla}^2 V' + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}') = \vec{\nabla}^2 V + \vec{\nabla}^2 \varphi + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) + \frac{\partial}{\partial t} (\vec{\nabla}^2 \psi) \quad (28.13a)$$

and

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{A}') + \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{A}' + \mu_0 \epsilon_0 \vec{\nabla} \left( \frac{\partial}{\partial t} V' \right) \\ = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) + \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{A} + \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{\nabla} \psi + \mu_0 \epsilon_0 \vec{\nabla} \left( \frac{\partial}{\partial t} \varphi \right) \end{aligned} \quad (28.13b)$$

The function  $V$  and  $\vec{A}$  satisfy (28.11a) and (28.11b) by assumption. Therefore,  $V'$  and  $\vec{A}'$  satisfy the same equations (for the same  $\rho$  and  $\vec{J}$ ), so long

$$\vec{\nabla}^2 \varphi + \frac{\partial}{\partial t} \vec{\nabla}^2 \psi = 0 \quad (28.14a)$$

and

$$\frac{\partial^2}{\partial t^2} \vec{\nabla} \psi + \vec{\nabla} \left( \frac{\partial}{\partial t} \varphi \right) = 0 \quad (28.14b)$$

By imposing

$$\varphi = - \frac{\partial}{\partial t} \psi \quad (28.15)$$

Eqs. (28.14a) and (28.14b) are satisfied at each point  $P$  and time  $t$ . As a consequence, chosen an arbitrary scalar function  $\psi(P, t)$  and constructed the functions

$$\vec{A}' = \vec{A} + \vec{\nabla} \psi \quad (28.16a)$$

and

$$V' = V - \frac{\partial}{\partial t} \psi \quad (28.16b)$$

these functions satisfy (28.11a) and (28.11b). Hence,  $V'$  and  $\vec{A}'$  are new potentials of the electromagnetic field. The fields  $\vec{E}$  and  $\vec{B}$  that can be calculated from (28.7) and (28.1) are identical for both pairs  $V$  and  $\vec{A}$  and  $V'$  and  $\vec{A}'$ .

The conditions (28.16a) and (28.16b) that must be satisfied by  $V'$  and  $\vec{A}'$  to be potentials of the field are called gauge conditions.

Given the arbitrariness of the function  $\psi$ , we now intend to calculate an expression of the potential that is easy to calculate and carries interesting physical meaning. To this end, we remind that it is always possible to choose a function  $\psi_0$  such that the corresponding vector potential

$$\vec{A}_0 = \vec{A} + \vec{\nabla} \psi_0 \quad (28.17)$$

which is calculated starting from a generic vector potential  $\vec{A}$ , is solenoidal. In fact, it is sufficient to choose the function  $\psi_0$  such that

$$\vec{\nabla}^2 \psi_0 = - \vec{\nabla} \cdot \vec{A} \quad (28.18a)$$

In fact,

$$\vec{\nabla} \cdot \vec{A}_0 = \vec{\nabla} \cdot \vec{A} + \vec{\nabla}^2 \psi_0 = 0 \quad (28.18b)$$

for each point  $P$  and time  $t$ .

After choosing  $\psi_0$  so that (28.18a) is satisfied, the field potentials assume the form

$$\vec{A}_0 = \vec{A} + \vec{\nabla}\psi_0 \quad (28.19a)$$

and

$$V_0 = V - \frac{\partial}{\partial t}\psi_0 \quad (28.19b)$$

Remembering the vector calculus identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} \quad (28.20)$$

and substituting (28.19a) and (28.19b) into (28.11a) and (28.11b), we obtain

$$\vec{\nabla}^2 V_0 = -\frac{1}{\epsilon_0} \rho \quad (28.21a)$$

and

$$-\vec{\nabla}^2 \vec{A}_0 + \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{A}_0 + \mu_0 \epsilon_0 \vec{\nabla} \left( \frac{\partial}{\partial t} V_0 \right) = \mu_0 \vec{J} \quad (28.21b)$$

Thus, the choice of the function  $\psi_0$  (and consequently  $\vec{A}_0$ ) makes it possible to find a scalar potential  $V_0$  that fulfils Poisson equation (28.21a) and a vector potential  $\vec{A}_0$  that fulfils equation (28.21b). The expression for  $V_0$  is particularly simple and coincides with the potential of the electrostatic field generated by a charge distribution  $\rho(P, t)$ , with the charges at rest in their position and charge values corresponding to the considered time  $t$ .

When the charge distribution  $\rho(P, t)$  is known at each point  $P$  in space and time  $t$ , and assuming the region where  $\rho$  is nonzero to be limited, the scalar potential  $V_0(P_0, t)$  assumes the coulombian form

$$V_0(P_0, t) = \frac{1}{4\pi\epsilon_0} \iiint_{\tau} \frac{\rho(P, t)}{r_{PP_0}} d\tau \quad (28.22)$$

where  $\tau$  is the region of space where  $\rho$  is defined and  $r_{PP_0}$  the absolute value of the distance between points  $P$  and  $P_0$ . For this reason, the condition (28.18b), which is necessary for  $V_0$  to be of Coulomb type, is called the Coulomb condition.

Note that, according to (28.22) the scalar potential  $V_0$  at a generic point  $P_0$  and time  $t$  is determined by the charge distribution  $\rho$  at the same time  $t$ . This fact may lead to think to an instantaneous information transfer between  $P$  and  $P_0$ . However, this is not the case because the function  $V_0$  alone is not sufficient to determine the vector field  $\vec{E}$ , which is the only physical quantity to have an operative meaning.

Another important gauge condition is the so-called Lorentz condition, which consists to impose that the two potentials, here called  $V_1$  and  $\vec{A}_1$ , verify the equation

$$\vec{\nabla} \cdot \vec{A}_1 + \mu_0 \epsilon_0 \frac{\partial}{\partial t} V_1 = 0 \quad (28.23)$$

For (28.23) to be verified it is sufficient to choose a function  $\psi$  such that

$$\vec{\nabla}^2 \psi - \frac{\partial^2}{\partial t^2} \psi = -\vec{\nabla} \cdot \vec{A} - \mu_0 \epsilon_0 \frac{\partial}{\partial t} V \quad (28.24)$$

This can be demonstrated by substituting the gauge conditions (28.16a) and (28.16b) into (28.23). This choice must be interpreted in the following way: Given a generic pair of potentials  $V$  and  $\vec{A}$  that, in general, does not satisfy the Lorentz condition, in order to build a new pair of potentials  $V_1$  and  $\vec{A}_1$  that satisfies the condition it is sufficient to choose a function  $\psi(P, t)$  that satisfies (28.24) and, then, construct  $V_1$  and  $\vec{A}_1$  so that they verify (28.16a) and (28.16b). By substituting (28.23) into (28.11a) and (28.11b), we obtain

$$\vec{\nabla}^2 V_1 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} V_1 = -\frac{1}{\epsilon_0} \rho \quad (28.25a)$$

and

$$\vec{\nabla}^2 \vec{A}_1 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{A}_1 = -\mu_0 \vec{J} \quad (28.25b)$$

These equations, remarkably symmetric, are the fundamental wave equations of the electromagnetic field. Therein lies the velocity of light  $c = 1/\sqrt{\mu_0 \epsilon_0}$ .



# Chapter 29

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## 29.1 Magnetic Field Energy

We will now extend the result of Eq. (26.52) to the case of non-filiform circuits. We will try to give an expression in terms of the field  $\vec{B}$  itself.

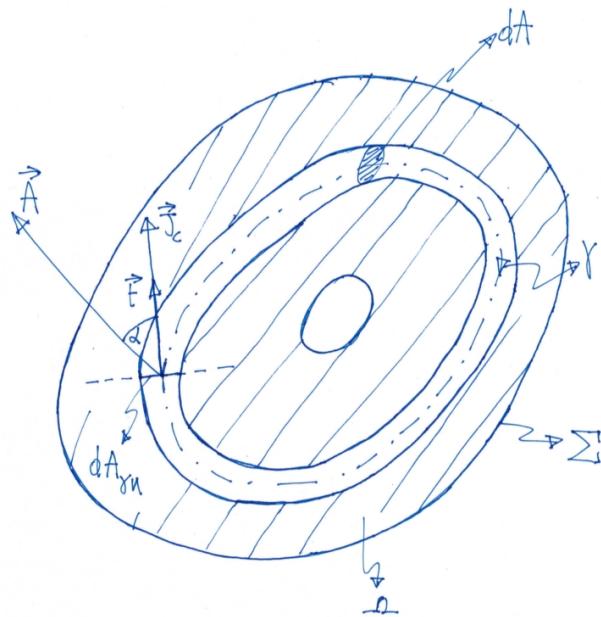


Figure 29.1

Consider a generic conductor occupying a region of space  $\Omega$  and carrying a current with volume density  $\vec{J}_c$ , which varies from point to point (see Fig. 29.1). Suppose to divide the entire conductor into a set of flux tubes  $\mathcal{F}$  for vector  $\vec{J}_c$ ; each tube is characterised by an infinitesimal cross-section  $dA$  and longitudinal axis  $\gamma$ ; further suppose that  $\Sigma_\gamma$  is a generic open surface having  $\gamma$  as a border and  $d\ell$  is the elementary current along the tube. By using Eq. (26.52) to the case under consideration and by

changing the sum into a volume integral, we have

$$U_m = \frac{1}{2} \int_{\mathcal{F}} d\iota_{\gamma} \iint_{\Sigma_{\gamma}} \vec{B} \cdot \vec{n} dA \quad (29.1)$$

where the first integral represents a “sum” relative to the set of elementary flux tubes of  $\vec{J}$  and the second (double) integral gives the flux associated with each flux tube. By defining a vector potential  $\vec{A}$  for the field  $\vec{B}$ , we have

$$\iint_{\Sigma_{\gamma}} \vec{B} \cdot \vec{n} dA = \oint_{\gamma} \vec{A} \cdot \vec{t} d\ell \quad (29.2)$$

and, thus,

$$U_m = \frac{1}{2} \int_{\mathcal{F}} d\iota_{\gamma} \oint_{\gamma} \vec{A} \cdot \vec{t} d\ell \quad (29.3)$$

By calling  $dA_{\gamma n}$  the area of the normal cross-section for the flux tube associated with  $\gamma$ , we have

$$d\iota_{\gamma} = \vec{J}_c \cdot \vec{n} dA_{\gamma n} \quad (29.4)$$

where  $\vec{n}$  is the normal unit vector associated with the normal cross-section of the flux tube. Hence,  $\vec{n} = \vec{t}$ , where  $\vec{t}$  is the tangent unit vector used in (29.3). As a consequence

$$U_m = \frac{1}{2} \int_{\mathcal{F}} J_t dA_{\gamma n} \oint_{\gamma} A_t d\ell \quad (29.5)$$

where  $J_t$  and  $A_t$  are the components of  $\vec{J}$  and  $\vec{A}$  along the small  $\vec{t}$ . This means

$$\vec{A} \cdot \vec{t} = A \cos \alpha = A_t \quad (29.6)$$

and

$$\vec{J} \cdot \vec{t} = J \cos \beta = J = J_t \quad (29.7)$$

where  $\alpha$  is shown in Fig. 29.1 and, because of the definition of flux tube,  $\beta = 0$ . Thus,

$$\begin{aligned} \vec{A} \cdot \vec{J} &= AJ \cos(\alpha - \beta) \\ &= AJ(\cos \alpha) \end{aligned} \quad (29.8)$$

and

$$U_m = \frac{1}{2} \iiint_{\Omega} J_t A_t dV = \frac{1}{2} \iiint_{\Omega} \vec{J} \cdot \vec{A} dV \quad (29.9)$$

This expression reminds that for the electrostatic energy in terms of a volume charge density and the associated electrostatic potential (see PHYS 242).

In the static case,

$$\vec{\nabla} \times \vec{H} = \vec{J}_c \quad (29.10)$$

and, thus,

$$U_m = \frac{1}{2} \iiint_{\Omega} \vec{\nabla} \times \vec{H} \cdot \vec{A} dV \quad (29.11)$$

Using a known vector calculus identity

$$\vec{\nabla} \times \vec{H} \cdot \vec{A} = \vec{\nabla} \cdot (\vec{H} \times \vec{A}) + \vec{H} \cdot \vec{\nabla} \times \vec{A} = \vec{\nabla} \cdot (\vec{H} \times \vec{A}) + \vec{H} \cdot \vec{B} \quad (29.12)$$

Hence,

$$U_m = \frac{1}{2} \iiint_{\Omega} \vec{\nabla} \cdot (\vec{H} \times \vec{A}) dV + \frac{1}{2} \iiint_{\Omega} \vec{H} \cdot \vec{B} dV \quad (29.13)$$

By means of the divergence theorem applying to the first integral, we find

$$U_m = \frac{1}{2} \iint_{\Sigma} (\vec{H} \times \vec{A}) \cdot \vec{n} dA + \frac{1}{2} \iiint_{\Omega} \vec{H} \cdot \vec{B} dV \quad (29.14)$$

where  $\Sigma$  is the surface enclosing  $\Omega$ . At each point outside  $\Omega$ ,  $\vec{J} = \vec{0}$ . Therefore, outside  $\Omega$

$$\vec{\nabla} \times \vec{H} = \vec{0} \quad (29.15)$$

The integral (29.11) can be thus extended to the entire space. In (29.14), the first integral gets extended to an infinite surface  $\Sigma_{\infty}$ , which can be thought as a sphere with infinite radius; the second integral gets extended to  $\Omega_{\infty}$ . Since,  $H \rightarrow 1/r^3$  and  $1/r^2$  for  $r \rightarrow +\infty$ , we conclude that the first integral can be neglected. Thus,

$$U_m = \frac{1}{2} \iiint_{\Omega} \vec{H} \cdot \vec{B} dV \quad (29.16)$$

We have thus obtained an expression for the magnetic energy in terms of  $\vec{H}$  and  $\vec{B}$ . As in the electrostatic case, the energy can be thought to be distributed in space with a volume density (energy per unit volume)

$$u_m = \frac{1}{2} \vec{H} \cdot \vec{B} \quad (29.17)$$

In the case of linear materials, we have

$$u_m = \frac{1}{2} \mu H^2 = \frac{1}{2} \frac{B^2}{\mu} \quad (29.18)$$

which, in the case of vacuum, becomes

$$u_m = \frac{1}{2} \mu_0 H^2 = \frac{1}{2} \frac{B^2}{\mu_0} \quad (29.19)$$



# Chapter 30

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## 30.1 Electromagnetic Field Energy

We will now state a fundamental property of the electromagnetic field that makes it possible to extend the results obtained for the electric and magnetic energies. We will make use of Maxwell's equations in vacuum, i.e.,

$$\left\{ \vec{\nabla} \times \vec{E} = - \frac{\partial}{\partial t} \vec{B} \right. \quad (30.1a)$$

$$\left. \vec{\nabla} \times \vec{H} = \vec{J}_c + \frac{\partial}{\partial t} \vec{D} \right. \quad (30.1b)$$

We multiply scalarly (dot product) the first equation by  $\vec{H}$  and the second by  $\vec{E}$ ,

$$\left\{ \vec{H} \cdot \vec{\nabla} \times \vec{E} = - \vec{H} \cdot \frac{\partial}{\partial t} \vec{B} \right. \quad (30.2a)$$

$$\left. \vec{E} \cdot \vec{H} = \vec{E} \cdot \vec{J}_c + \vec{E} \cdot \frac{\partial}{\partial t} \vec{D} \right. \quad (30.2b)$$

From the vector calculus identity

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \vec{\nabla} \times \vec{A} - \vec{A} \cdot \vec{\nabla} \times \vec{B} \quad (30.3)$$

by subtracting (30.2b) from (30.2a), we have

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = - \vec{H} \cdot \frac{\partial}{\partial t} \vec{B} - \vec{E} \cdot \frac{\partial}{\partial t} \vec{D} - E \vec{J}_c \quad (30.4)$$

For linear materials,

$$\left\{ \vec{H} \cdot \frac{\partial}{\partial t} \vec{B} = \frac{1}{2\mu} \frac{\partial}{\partial t} (B^2) \right. \quad (30.5a)$$

$$\left. \vec{E} \cdot \frac{\partial}{\partial t} \vec{D} = \frac{1}{2} \epsilon \frac{\partial}{\partial t} (E^2) \right. \quad (30.5b)$$

and, thus,

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = - \frac{\partial}{\partial t} \left( \frac{B^2}{2\mu} \right) - \frac{\partial}{\partial t} \left( \frac{1}{2} \epsilon E^2 \right) - \vec{E} \cdot \vec{J}_c \quad (30.6)$$

By integrating this equation with respect to a domain  $\Omega$ , we obtain

$$\begin{aligned} \iiint_{\Omega} \vec{\nabla} \cdot (\vec{E} \times \vec{H}) dV &= - \iiint_{\Omega} \frac{\partial}{\partial t} \left( \frac{B^2}{2\mu} + \frac{1}{2} \epsilon E^2 \right) dV - \iiint_{\Omega} \vec{E} \cdot \vec{J}_c dV \\ &= - \frac{d}{dt} \iiint_{\Omega} \left( \frac{B^2}{2\mu} + \frac{1}{2} \epsilon E^2 \right) dV - \iiint_{\Omega} \vec{E} \cdot \vec{J}_c dV \end{aligned} \quad (30.7)$$

where  $\Omega$  is fixed with respect to the inertial reference frame of the electromagnetic field. By means of the the divergence theorem for integral on the left-hand side of (30.7), we have

$$\iint_{\Sigma} (\vec{E} \times \vec{H}) \cdot \vec{n} dA = - \frac{d}{dt} \iiint_{\Omega} \left( \frac{B^2}{2\mu} + \frac{1}{2} \epsilon E^2 \right) dV - \iiint_{\Omega} \vec{E} \cdot \vec{J}_c dV \quad (30.8)$$

where  $\Sigma$  is the surface enclosing  $\Omega$ . Suppose that  $\Omega$  contains a conducting region characterized by an electromotive field (or fields)  $\vec{E}_m$  (e.g., a battery), from Ohm's law in local form we have

$$\vec{J}_c = \sigma (\vec{E} + \vec{E}_m) \quad (30.9)$$

where  $\sigma$  is the electrical conductivity of the conducting region. Inserting (30.9) into (30.8), we have

$$\iiint_{\Omega} \vec{E} \cdot \vec{J}_c dV = \iiint_{\Omega} \eta J_c^2 dV - \iiint_{\Omega} \vec{E}_m \cdot \vec{J}_c dV \quad (30.10)$$

where  $\eta = 1/\sigma$ . By substituting (30.10) into (30.8), we finally obtain

$$\begin{aligned} \iiint_{\Omega} \vec{E}_m \cdot \vec{J}_c dV &= \iiint_{\Omega} \eta J_c^2 dV + \frac{d}{dt} \iiint_{\Omega} \left( \frac{1}{2} \epsilon E^2 + \frac{1}{2} \frac{B^2}{\mu} \right) \\ &\quad + \iint_{\Sigma} (\vec{E} \times \vec{H}) \cdot \vec{n} dA \end{aligned} \quad (30.11)$$

This result is known as Poynting theorem. According to this theorem, the work due to the electromotive field  $\vec{E}_m$  per unit time on the electrical charges is equal to the sum of three terms: 1) The energy dissipated in the conductor contained within  $\Omega$  because of the Joule effect; 2) the time derivative of the sum of the electric and magnetic energies; 3) the outgoing flux through  $\Sigma$  of the vector field

$$\vec{\mathcal{P}} = \vec{E} \times \vec{H} \quad (30.12)$$

which is called Poynting vector. The quantity

$$u_{\text{em}} = \frac{1}{2} \epsilon E^2 + \frac{1}{2} \frac{B^2}{\mu} \quad (30.13)$$

is the electromagnetic energy volume density. This is clearly the sum of the electric energy volume density (PHYS 232) and the magnetic volume density.

The physical meaning of (30.8) can be better understood from an energetic point of view, using the first principle of thermodynamics. To this end, consider, for simplicity, the case in which all materials contained within  $\Omega$  are not polarizable and not magnetizable (i.e.,  $\epsilon = \epsilon_0$  and  $\mu = \mu_0$ ). In this case, the term

$$\begin{aligned} \iiint_{\Omega} \vec{E} \cdot \vec{J}_c dV &= \int_{\gamma} \vec{E} \cdot \vec{t} d\ell \iint_{\Sigma_t} J_c \vec{t} \cdot \vec{n} dA_n \\ &= \int_{\gamma} \vec{E} \cdot \vec{t} d\ell = \int_{\gamma} \vec{E} \cdot \vec{t} d\ell \frac{d}{dt} q_f \end{aligned} \quad (30.14)$$

where  $\gamma$  is a generic open line within  $\Omega$ ,  $\Sigma_t$  is the normal cross-section of a flux tube associated with  $\vec{J}_c$ ,  $dA_n$  is the area associated with  $\Sigma_t$ , and  $\vec{t} = \vec{n}$ . Equation (30.14) can be interpreted as the work per unit time due to electric field  $\vec{E}$  on the free charges in  $\Omega$ . Thus, multiplying by  $dt$  each term of (30.8), we obtain

$$\Phi_{\Sigma} dt + \delta W + dU_{em} = 0 \quad (30.15)$$

where  $\Phi_{\Sigma}$  is the flux of the  $\vec{\mathcal{P}}$  outside the closed surface  $\Sigma$ ,  $\delta W$  is the work in the time  $dt$  due to the electric field (note that, in general, this is an inexact differential because  $\vec{E}$  could be partially due to  $\vec{E}_m$ ) on the free charges in  $\Omega$ , and the  $dU_{em}$  is the corresponding differential energy associated with the electric and magnetic fields in  $\Omega$ . By applying the first principle of thermodynamics to the matter contained in  $\Omega$ , we can write

$$dW + \delta W_g = \delta Q + dU \quad (30.16)$$

where  $\delta W = dW + \delta W_g$ ,  $dW$  is the work in the time  $dt$  due to the irrotational electric field on the free charges on  $\Omega$ ,  $\delta W_g$  is the work in the time  $dt$  due to the rotational non-electric field on the free charges in  $\Omega$ ,  $\delta Q$  is the heat exiting  $\Omega$  in the time  $dt$ , and  $dU$  is the differential internal energy associated with the matter and, thus, with all particles in  $\Omega$ . Solving for  $\delta W$  in (30.15) and substituting the result into (30.16), we find

$$-\Phi_{\Sigma} dt - dU_{em} + \delta W_g = \delta Q + dU \quad (30.17)$$

By calling

$$dU_{\text{tot}} = dU + dU_{em} \quad (30.18)$$

the total differential energy of the system, we have

$$\delta W_g = \delta Q + dU_{\text{tot}} + \Phi_{\Sigma} dt \quad (30.19)$$

This relationship shows that the work due to the electromotive field in the time  $dt$  on the free charges in  $\Omega$  is given by: 1) A quantity of heat exiting  $\Omega$  in the time  $dt$ ; 2) a differential total energy equal to the sum of the internal energy of the system  $U$  and the internal energy of the electromagnetic field  $U_m$ ; 3) a term proportional (with

proportionality coefficient  $dt$ ) to the flux of the Poynting vector through  $\Sigma$ . The flux of the Poynting vector thus corresponds to an actual energy term, which, in the unit time exists (in an algebraic sense) the region  $\Omega$  through  $\Sigma$ .

Consider a simple static case in which a battery is connected to a resistor by means of perfectly conducting wires. Suppose the resistor is a cylindrical conductor of length  $\ell$  and cross-section with radius  $a$  and resistivity  $\eta$ . Equation (30.19) for the resistor gives

$$0 = \delta Q + dU_{\text{tot}} + \Phi_{\Sigma} dt \quad (30.20)$$

The electric field  $\vec{E}$  in the resistor is directed along its central axis and is given by

$$\vec{E} = \eta \vec{J}_c \quad (30.21)$$

where  $\vec{J}_c$  is the current density (suppose to be uniform) in the conductor.

For symmetry reasons, the magnetic field  $\vec{H}$  is tangent to the lateral surface of the conductor and, at each point on this surface, its intensity is given by

$$B(a) = \mu_0 \frac{J_c \pi a^2}{2\pi a} = \mu_0 \frac{J_c a}{2} \quad (30.22)$$

thus, the Poynting vector is directed radially into the conductor and its intensity is given by

$$\mathcal{P} = E \frac{B}{\mu_0} = \eta J_c^2 \frac{a}{2} \quad (30.23)$$

By integrating  $\vec{\mathcal{P}}$  on the entire surface of the cylinder  $\Sigma_{\text{cyl}}$ , we have

$$\begin{aligned} \Phi_{\Sigma_{\text{cyl}}} &= \iint_{\Sigma_{\text{cyl}}} (\vec{E} \times \vec{H}) \cdot \vec{n} dA = -\eta J_c^2 \frac{a}{2} 2\pi a \ell \\ &= -\eta \frac{\ell}{\pi a^2} J_c^2 (\pi a^2)^2 \\ &= -RI^2 \end{aligned} \quad (30.24)$$

where  $R$  is the resistance of the conductor and  $I$  the current through it.

By substituting (30.24) into (30.20) we have

$$dt \iint_{\Sigma_{\text{cyl}}} \vec{\mathcal{P}} \cdot \vec{n} dA = RI^2 dt = \delta Q \quad (30.25)$$

The energetic balance is verified considering that the field  $\vec{E}$  provides energy to the conductor by means of the flux of the Poynting vector through  $\Sigma_{\text{cyl}}$ . In this case, the flux of  $\vec{\mathcal{P}}$  through  $\Sigma$  (oriented outside  $\Sigma$ ) is negative. This is because the field  $\vec{E}$  transfers power into  $\Sigma$ .

When the flux of  $\vec{\mathcal{P}}$  through  $\Sigma_{\text{cyl}}$  (outside  $\Sigma_{\text{cyl}}$ ) is positive, power is irradiated away the conductor contained within  $\Sigma_{\text{cyl}}$ .

Note that, in the general non-linear cases, Eq. (30.8) becomes

$$\iint_{\Sigma} (\vec{E} \times \vec{H}) \cdot \vec{n} dA = - \iiint_{\Omega} \left( \vec{H} \cdot \frac{\partial}{\partial t} \vec{B} + \vec{E} \cdot \frac{\partial}{\partial t} \vec{D} \right) dV - \iiint_{\Omega} \vec{E} \cdot \vec{J}_c dV \quad (30.26)$$

# Chapter 31

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## 31.1 An Oscillating Circuit

Consider a circuit comprising an inductor with inductance  $L$  and a capacitor with capacitance  $C$  connected to each other by means of two zero resistance wires, one of which is interrupted by a switch (see Fig. 31.1). Suppose that at the time  $t = 0$  when the switch is closed the capacitor is charged with potential difference  $V_0$  and the inductor is completely discharged. We intend to study the circuit dynamics at all times after the switch has been closed. Assume that any displacement current outside the capacitor is negligible and, similarly, the magnetic field outside the inductor is also negligible.

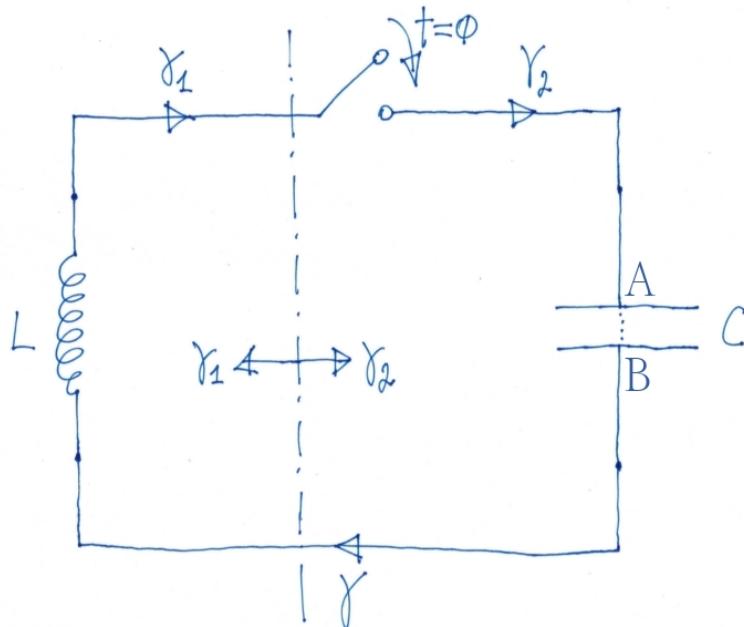


Figure 31.1

As soon as the switch is closed, a dynamic current  $i(t)$  circulates in the circuit. At any given instant in time, this current has the same value at the points of each section of the circuit. This current is a conduction current in the inductor and connection

wires, whereas it is a displacement current in the vacuum space between the capacitor's walls.

As shown in Fig. 31.1, line  $\gamma$  is an oriented and closed line associated with the entire circuit; in particular, this line comprises a segment between the two walls of the capacitor (which, otherwise, would represent a discontinuity of the line). By means of Faraday-Neumann-Lenz's law for line  $\gamma$ , we have

$$\oint_{\gamma} \vec{E} \cdot \vec{t} d\ell = - \frac{d}{dt} \Phi_{\gamma} \quad (31.1)$$

where  $\vec{E}$  is the electric field associated with the circuit and  $\Phi_{\gamma}$  the flux linked with entire circuit (represented geometrically by  $\gamma$ ). In general, the field  $\vec{E} = \vec{E}_m + \vec{E}_q$ , where  $\vec{E}_m$  is a rotational field due to the electromagnetic induction phenomena (corresponding to  $\partial \vec{A} / \partial t$ ) and  $\vec{E}_q$  is an irrotational field due to the distribution of the electrical charges (corresponding to  $\vec{\nabla} \psi$ ). By definition

$$\oint_{\gamma} \vec{E}_m \cdot \vec{t} d\ell = - \frac{d}{dt} \Phi_{\gamma} \quad (31.2)$$

and

$$\oint_{\gamma} \vec{E}_q \cdot \vec{t} d\ell = 0 \quad (31.3)$$

We now analyse in detail the phenomena taking place in inside the inductor. Under the assumption that the inductor is ideal (i.e., that it has zero resistivity), the local form of Ohm's law at each point inside the inductor is

$$0 = \eta \vec{J}_c = \vec{E} = \vec{E}_m + \vec{E}_q \quad (31.4)$$

where  $\vec{J}_c$  is the conduction current volume density. In order for  $\vec{J}_c$  to be finite (which is an obvious physical requirement), it must then be  $\vec{E} = \vec{0}$  at each point inside the inductor. As a consequence, at each points inside the inductor we have

$$\vec{E}_m = -\vec{E}_q \quad (31.5)$$

Suppose to divide the line  $\gamma$  in two parts  $\gamma_1$  and  $\gamma_2$ . The first part ( $\gamma_1$ ) comprises the inductor and part of the connection wires; the second part ( $\gamma_2$ ) comprises the capacitor, part of the connection wires, and the switch (supposed to be closed; see Fig. 31.1). We have,

$$\oint_{\gamma} \vec{E} \cdot \vec{t} d\ell = \int_{\gamma_1} \vec{E} \cdot \vec{t} d\ell + \int_{\gamma_2} \vec{E} \cdot \vec{t} d\ell \quad (31.6)$$

Because of condition (31.4),

$$\oint_{\gamma} \vec{E} \cdot \vec{t} d\ell = \int_{\gamma_2} \vec{E} \cdot \vec{t} d\ell \quad (31.7)$$

Inside the capacitor, the induced electromotive field  $\vec{E}_m \sim \vec{0}$  and, thus,

$$\oint_{\gamma} \vec{E} \cdot \vec{t} d\ell = \int_{\gamma_2} \vec{E} \cdot \vec{t} d\ell = \int_{\gamma_2} \vec{E}_q \cdot \vec{t} d\ell = \phi(A) - \phi(B) \quad (31.8)$$

where  $\phi(A)$  and  $\phi(B)$  are the values of the scalar potential associated with  $\vec{E}_q$  in correspondence of the capacitor's walls  $A$  and  $B$  (see Fig. 31.1). By calling

$$v_c = \phi(A) - \phi(B) \quad (31.9)$$

and by substituting (31.8) into (31.1), we find

$$v_c = - \frac{d}{dt} \Phi_{\gamma} \quad (31.10)$$

Since  $\phi_{\gamma}$  is due to the magnetic field generated by the current in the inductor, we have

$$\Phi_{\gamma} = L\iota \quad (31.11)$$

and, thus,

$$v_c = -L \frac{d}{dt} \iota \quad (31.12)$$

Calling  $q_f(t)$  the electrical charge on one capacitor's wall at time  $t$ , from the continuity equation for a generic closed surface encompassing the wall, we have

$$\iota(t) = \frac{d}{dt} q_f(t) \quad (31.13)$$

From the definition of capacitance, we obtain

$$C = \frac{dq_f}{d\phi} = \frac{dq_f}{dt} \frac{dt}{d\phi} = \iota(t) \frac{dt}{d\phi} \quad (31.14)$$

Thus, for  $\phi = v_c$ , we obtain

$$\iota(t) = C \frac{d}{dt} v_c \quad (31.15)$$

By substituting this expression by (31.12), we find

$$v_c(t) = -LC \frac{d^2}{dt^2} v_c \quad (31.16)$$

This equation describes the time dependence of the potential difference between the capacitor's walls after the switch has been closed. This is the classical equation of the harmonic oscillator in the case of zero damping. From calculus the general integral of this equation can be written as

$$v_c(t) = K_1 \sin(\omega t) + K_2 \cos(\omega t) \quad (31.17)$$

where  $K_1$  and  $K_2$  are the arbitrary constants to be determined from the initial conditions of the problem and

$$\omega = \frac{1}{\sqrt{LC}} \quad (31.18)$$

We can find the expression for the current  $i(t)$  associated with (31.17) from (31.15),

$$i(t) = C \frac{d}{dt} v_c = \omega C K_1 \cos(\omega t) - \omega C K_2 \sin(\omega t) \quad (31.19)$$

By means of the initial conditions

$$\begin{cases} v_c(0) = V_0 \end{cases} \quad (31.20a)$$

$$\begin{cases} i(0) = 0 \end{cases} \quad (31.20b)$$

We have

$$\begin{cases} v_c(0) = v_c(0) = K_2 = V_0 \end{cases} \quad (31.21a)$$

$$\begin{cases} i(0) = \omega C K_1 = 0 \end{cases} \quad (31.21b)$$

and, thus,

$$\begin{cases} K_1 = 0 \end{cases} \quad (31.22a)$$

$$\begin{cases} K_2 = V_0 \end{cases} \quad (31.22b)$$

Finally, by substituting the so-obtained value of  $K_1$  and  $K_2$  into (31.17) and (31.19), we obtain

$$\begin{cases} v_c(t) = V_0 \cos(\omega t) \end{cases} \quad (31.23a)$$

$$\begin{cases} i(t) = -\omega C V_0 \sin(\omega t) \end{cases} \quad (31.23b)$$

Both the current and potential difference at the capacitor's walls are characterized by a periodic dynamics without dissipation: The system represents an ideal oscillating circuit, also referred to as a resonator.

Considering that the energy dissipated in the conductors has been neglected (and the energy radiated by the system has also been neglected; see PHYS 442), the electrostatic energy initially stored in the capacitor,

$$U_c = \frac{1}{2} C V_0^2 \quad (31.24)$$

must be conserved in the circuit. Hence, at the instance in time when  $v_c = 0$  (in which case  $U_c = 0$ ), the energy initially stored in the capacitor must be fully stored inside the inductor. Similarly, at the instance in time when  $i = 0$  (in which case  $U_\ell = 0$ ) the energy must be fully stored in the capacitor. The energy (31.24) gets transferred continuously between the capacitor and the inductor at the rate inversely proportional to the period

$$T = 2\pi\sqrt{LC} \quad (31.25)$$