Module 6 Group Project
the Earth-Moon system

Section 2, Group 52
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December 4, 2020

Question:
As we all know, one day equals 24 hours. However, it is not always the case. Based on paleontological remains, scientists have discovered that the length of days in ancient times is shorter than that of today. The actions of tides cause the Earth's rotational angular velocity to decrease slowly. As a result of this, the distance between the Earth and the Moon and the total energy of the Earth-Moon system is changing slowly. In this problem we will explore the mechanism behind this phenomenon.

In this problem we will assume the mass of the Earth is M_E and the mass of the Moon is M_m. The distance between the centers of those two is r. We assume that the Moon can be treated as a point mass, and the Earth's rotation axis and the moon's axis of circular motion around the earth both pass through the center of the Earth. The Earth's and the Moon's angular speed are ω_E and ω_M respectively.

i) Prove that the relation between the rate of change of the distance between the Earth and the Moon, $\dot{r}$, and the rate of change of the Earth's rotational angular speed, $\dot{\omega}_E$, is:

$$\dot{r} = -\frac{2\sqrt{r}I_E}{M_m\sqrt{GM_E}}\dot{\omega}_E$$

where I_E is the moment of inertia of the Earth relative to its axis of rotation, and G is the gravitational constant. Conclude that since ω_E is decreasing, the Moon is moving away from the Earth.

ii) Derive the expression of the rate of change of the total mechanical energy of the Earth-Moon system.

iii) Assume that the distance between the Earth and the Moon now is $r(0)$ and the rotatioal angular speed of the Earth now is $\omega_E(0)$. Because of the actions of tides, after the time t, the distance becomes $r(t)$ and the angular speed becomes $\omega_E(t)$. Assume the Moon is still doing circular motion around the Earth. Prove that:

$$\omega_E(t) = \omega_E(0) - \frac{M_m\sqrt{GM_E}}{I_E}(\sqrt{r(t)} - \sqrt{r(0)})$$

Then, calculate the length of a day on the Earth when r increases by 2%.

Note: you can use the data: $G = 6.672 \times 10^{-11} N \cdot m^2/kg^2$, $M_E = 5.89 \times 10^{24} kg$, $M_m = 7.34 \times 10^{22} kg$, $r = 3.84 \times 10^8 m$, $R_E = 6.37 \times 10^6 m$ and $I_E = 9.56 \times 10^{37} kg \cdot m^2$ for your derivation.

Diagram of this problem:

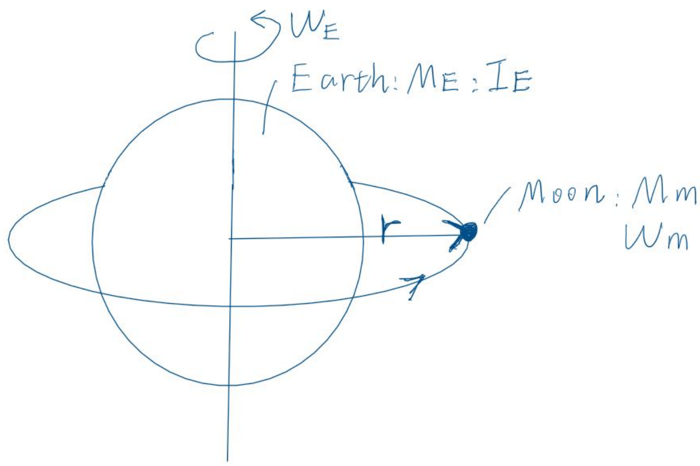

Figure 1: the Earth-Moon system

Approach:
i) Since there is no external torque on the Earth-Moon system, its angular momentum is conserved. The expression of the total angular momentum of the Earth-Moon system would contain ω_E, ω_M and r. Also, the centripetal force for the Moon's circular motion is provided by the gravitational force exerted by the Earth. This will give the relation between ω_M and r. Substitute this into the equation of the conservation of angular momentum to derive the relation between ω_E and r.

ii) The mechanical energy of the Earth-Moon system consists of the rotatioal kinetic energy of the Earth (about ω_E), the kinetic energy of the Moon (about ω_M) and the gravitational potential energy of the system (about r). The relations between these variables and r or $\dot{r}$ will have been derived in part (i). Substitute the relations derived in part (i) into the expression of the total energy to find relation between $\dot{E}$ and $\dot{r}$.

iii) This relation can be derived by manipulating the expressions we get during the process of deriving the result of part (i). This is because this relation is also a relation between ω_E and r, so we use the same conditions as part (i) to derive it. After deriving this, substitute the data into this equation to calculate $\frac{\omega_E(t)}{\omega_E(0)}$. Use the formula $T = \frac{2\pi}{\omega}$ to caculate the new length of days on the Earth.

Conceptual Components:
Rotational motion
Energy
Conservation of angular momentum
Moment of inertia
Gravity

Target Quantities:
i) Prove the given equation.
ii) The rate of change of the total mechanical energy of the Earth-Moon System $\dot{E}$.
iii) Prove the given equation. The new length of days on the Earth when r increases by 2%.

Quantitive Relationships that May be Useful:
Angular momentum of the Earth's rotation: $L_E = I_E\omega_E$
Angular momentum of the Moon's rotation around the Earth: $L = M_m r^2 \omega_M$
Conservation of angular momentum: $\frac{\mathrm{d}L_{total}}{\mathrm{d}t} = 0$
Linear speed: $v = \omega r$

Centripetal force: $F_c = \dfrac{mv^2}{r} = \dfrac{m(\omega r)^2}{r} = m\omega^2 r$

Gravity: $F_G = \dfrac{Gm_1m_2}{r_{12}^2}$

Rotational kinetic energy: $E_r = \dfrac{1}{2}I\omega^2$

Kinetic energy: $E_k = \dfrac{1}{2}mv^2$

Gravitational potential energy: $E_p = -\dfrac{Gm_1m_2}{r_{12}}$

Period: $T = \dfrac{2\pi}{\omega}$

Solution:
i)
The total angular momentum of the Earth-Moon system is:

$$L_{total} = L_E + L_M = I_E\omega_E + M_m r^2\omega_M \tag{1}$$

Since there is no external torque on the Earth-Moon system, the angular momentum is conserved. This means:

$$\frac{\mathrm{d}L_{total}}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(I_E\omega_E + M_m r^2\omega_M) = I_E\dot{\omega}_E + M_m\frac{\mathrm{d}}{\mathrm{d}t}(r^2\omega_M) = 0 \tag{2}$$

The centripetal force of the Moon's circular motion around the Earth is provided by the gravitational force exerted by the Earth on the Moon. This means:

$$F_c = M_m\omega_M^2 r = F_G = \frac{GM_EM_m}{r^2} \tag{3}$$

From this equation, we can solve for the value of ω_M:

$$\omega_M = \sqrt{\frac{GM_E}{r^3}} \tag{4}$$

Substitute this into equation (2):

$$I_E\dot{\omega}_E + M_m\frac{\mathrm{d}}{\mathrm{d}t}(r^2\cdot\sqrt{\frac{GM_E}{r^3}}) = 0$$

Rearrange this equation:

$$I_E\dot{\omega}_E + M_m\sqrt{GM_E}\frac{\mathrm{d}}{\mathrm{d}t}(r^{\frac{1}{2}}) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^{\frac{1}{2}}) = -\frac{I_E\dot{\omega}_E}{M_m\sqrt{GM_E}} \tag{5}$$

Calculate the derivative of $r^{\frac{1}{2}}$ with respect to time, we have:

$$\frac{1}{2\sqrt{r}}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{2\sqrt{r}}\dot{r} = -\frac{I_E\dot{\omega}_E}{M_m\sqrt{GM_E}} \tag{6}$$

Multiply both sides of the equation by $2\sqrt{r}$, we get the result we need prove:

$$\dot{r} = -\frac{2\sqrt{r}I_E}{M_m\sqrt{GM_E}}\dot{\omega}_E \tag{7}$$

Because of the actions of tides, ω_E is decreasing. This means $\dot{\omega}_E$ is negative and according to the above expression, $\dot{r}$ is positive. This means r is increasing, the Moon is moving away from the Earth.

ii)
The total energy of the Earth-Moon system consists of three parts: the rotatioal kinetic energy of the Earth, E_{er}, the kinetic energy of the Moon, E_{km}, and the gravitational potential energy, E_p:

$$E = E_{er} + E_{km} + E_p \tag{8}$$

The expression of E_{er} is:

$$E_{er} = \frac{1}{2} I_E \omega_E^2 \tag{9}$$

The expression of E_{km} is:

$$E_{km} = \frac{1}{2} M_m v_M^2 \tag{10}$$

where v_M is the linear speed of the Moon. Since the Moon is rotating about the axis that passes through the center of the Earth, we have the following relation between its linear speed and its angular speed:

$$v_M = \omega_M r$$

Substitute this into equation (10), we have:

$$E_{km} = \frac{1}{2} M_m \omega_M^2 r^2 \tag{11}$$

The gravitational potential energy E_p is:

$$E_p = -\frac{GM_E M_m}{r} \tag{12}$$

Substitute equation (9), (11), (12) into equation (8), we get the expression of the total energy E:

$$E = \frac{1}{2} I_E \omega_E^2 + \frac{1}{2} M_m \omega_M^2 r^2 - \frac{GM_E M_m}{r} \tag{13}$$

Substitute equation (3) into the above expression, we get:

$$\begin{aligned} E &= \frac{1}{2} I_E \omega_E^2 + \frac{1}{2}(M_m \omega_M^2 r) r - \frac{GM_E M_m}{r} = \frac{1}{2} I_E \omega_E^2 + \frac{1}{2}(\frac{GM_E M_m}{r^2}) r - \frac{GM_E M_m}{r} \\ &= \frac{1}{2} I_E \omega_E^2 + \frac{1}{2}\frac{GM_E M_m}{r} - \frac{GM_E M_m}{r} = \frac{1}{2} I_E \omega_E^2 - \frac{GM_E M_m}{2r} \end{aligned} \tag{14}$$

In order to find the rate of change of energy, we differentiate both sides of the above equation with respect to time:

$$\dot{E} = \frac{\mathrm{d}E}{\mathrm{d}t} = I_E \omega_E \frac{\mathrm{d}\omega_E}{\mathrm{d}t} + \frac{GM_E M_m}{2r^2}\frac{\mathrm{d}r}{\mathrm{d}t} = I_E \omega_E \dot{\omega}_E + \frac{GM_E M_m}{2r^2} \dot{r} \tag{15}$$

Substitute the expression of $\dot{r}$ in equation (7) into equation (15), we have:

$$\begin{aligned} \dot{E} &= I_E \omega_E \dot{\omega}_E - \frac{GM_E M_m}{2r^2} \frac{2\sqrt{r} I_E}{M_m \sqrt{GM_E}} \dot{\omega}_E \\ &= I_E \omega_E \dot{\omega}_E - \frac{\sqrt{GM_E}}{r\sqrt{r}} I_E \dot{\omega}_E = I_E \dot{\omega}_E (\omega_E - \frac{\sqrt{GM_E}}{r\sqrt{r}}) \end{aligned} \tag{16}$$

Although it is not required, we do some quantitive analysis here. Consider the expression in the parenthesis $(\omega_E - \frac{\sqrt{GM_E}}{r\sqrt{r}})$. Substitute $\omega_E = \frac{2\pi}{24h \cdot 3600s/h} = 7.27 \times 10^{-5} rad/s$, $G = 6.672 \times 10^{-11} N \cdot m^2/kg^2$, $M_E = 5.89 \times 10^{24} kg$ and $r = 3.84 \times 10^8 m$ into this expression, we get:

$$\omega_E - \frac{\sqrt{GM_E}}{r\sqrt{r}} = 7.27 \times 10^{-5} rad/s - \frac{\sqrt{6.672 \times 10^{-11} N \cdot m^2/kg^2 \times 5.89 \times 10^{24} kg}}{(3.84 \times 10^8 m)^{\frac{3}{2}}} = 7.01 \times 10^{-5}/s$$

which is positive. Since we also know that $\dot{\omega}_E$ is negative and I_E is positive, $\dot{E}$ is negative overall. This is consistent with the fact that the actions of tides cause the total energy of the Earth-Moon system to decrease.

iii)
Multiply both sides of equation (5) by dt, we can get:

$$\mathrm{d}(r^{\frac{1}{2}}) = -\frac{I_E}{M_m\sqrt{GM_E}}\mathrm{d}\omega_E$$

$$\mathrm{d}\omega_E = -\frac{M_m}{I_E}\sqrt{GM_E}\mathrm{d}(\sqrt{r}) \tag{17}$$

When ω_E changes from $\omega_E(0)$ to $\omega_E(t)$, $\sqrt{r}$ changes from $\sqrt{r(0)}$ to $\sqrt{r(t)}$. Therefore, we can integrate both sides of the above equation over the same time period $0 \sim t$:

$$\int_{\omega_E(0)}^{\omega_E(t)} \mathrm{d}\omega_E = -\int_{\sqrt{r(0)}}^{\sqrt{r(t)}} \frac{M_m}{I_E}\sqrt{GM_E}\mathrm{d}(\sqrt{r})$$

Evaluate this integral, we get:

$$\omega_E(t) - \omega_E(0) = -\frac{M_m}{I_E}\sqrt{GM_E}(\sqrt{r(t)} - \sqrt{r(0)}) \tag{18}$$

Add $\omega_E(0)$ on both sides of this equation, we get the result that we need to prove:

$$\omega_E(t) = \omega_E(0) - \frac{M_m\sqrt{GM_E}}{I_E}(\sqrt{r(t)} - \sqrt{r(0)}) \tag{19}$$

Rearrange equation (19), we have:

$$\frac{\omega_E(t)}{\omega_E(0)} = 1 - \frac{M_m\sqrt{GM_E r(0)}}{I_E\omega_E(0)}(\sqrt{\frac{r(t)}{r(0)}} - 1) \tag{20}$$

Substitute the data $G = 6.672 \times 10^{-11}N \cdot m^2/kg^2$, $M_E = 5.89 \times 10^{24}kg$, $M_m = 7.34 \times 10^{22}kg$, $r(0) = 3.84 \times 10^8 m$, $R_E = 6.37 \times 10^6 m$, $\omega_E(0) = 7.27 \times 10^{-5}rad/s$ and $I_E = 9.56 \times 10^{37}kg \cdot m^2$ into this equation. When r increases by 2%, $r(t) = 1.02r(0)$, we have:

$$\begin{aligned}\frac{\omega_E(t)}{\omega_E(0)} &= 1 - \frac{(7.34 \times 10^{22}kg) \cdot \sqrt{(6.672 \times 10^{-11}N \cdot m^2/kg^2) \cdot (5.89 \times 10^{24}kg) \cdot (3.84 \times 10^8 m)}}{(9.56 \times 10^{37}kg \cdot m^2) \cdot (7.27 \times 10^{-5}rad/s)}(\sqrt{1.02} - 1) \\ &= 0.96 \end{aligned} \tag{21}$$

Consider the period of the Earth's rotation T:

$$\frac{T(t)}{T(0)} = \frac{\frac{2\pi}{\omega_E(t)}}{\frac{2\pi}{\omega_E(0)}} = \frac{\omega_E(0)}{\omega_E(t)} = \frac{1}{0.96} = 1.04 \tag{22}$$

Since $T(0)$ is the period of the Earth's rotation now, which is $24h$, $T(t) = T(0) \cdot 1.04 = 1.04 \times 24h = 25h$. This means the length of a day on the Earth when r increases by 2% is $25h$.

Evaluation:

a) Dimension check:

i) $\frac{[m]}{[s]} = \frac{\sqrt{[m]}[kg][m]^2}{[kg]\sqrt{[m]^3[kg]^{-1}[s]^{-2}[kg]}}\frac{1}{[s]^2}$ (equation 7)

ii) $\frac{[kg][m]^2}{[s]^3} = [kg][m]^2\frac{1}{[s]^2}(\frac{1}{[s]} + \frac{\sqrt{[m]^3[kg]^{-1}[s]^{-2}[kg]}}{[m]\sqrt{[m]}})$ (equation 16)

iii) $\frac{1}{[s]} = \frac{1}{[s]} + \frac{[kg]\sqrt{[m]^3[kg]^{-1}[s]^{-2}[kg]}}{[kg][m]^2}\sqrt{[m]}$ (equation 19)

b) Orders of magnitude

i) No numerical value.

ii) No numerical value in the required answer.

iii) The answer is reasonable regarding its order of magnitude.

c) All questions have been answered.