Module 5 Group Project
From Single Pendulum to the Quantum Field Theory

Section 2, Group 33
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Questions:
a) In high school (at least in the IB curriculum I did), we learned that the period of a single pendulum is:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

When the amplitude of swing is small. In this equation, l is the length of the pendulum and g is the gravitational acceleration. However, when deriving this formula, the mass of the string is ignored. Now, let's take the mass of the string into account. Use $I = \frac{1}{3}ml^2$ (m is the mass of the string and l is the length of the string) as the moment of inertia of the string. Write down the Newton's Second Law equation for rotation of the pendulum. Suppose we can still treat the bob as a point mass with a mass of M.

b) In the equation you derived in the last question, use $\ddot{\theta}$ (the second order derivative of θ with respect to time) for the angular acceleration α. And use θ to approximate $\sin\theta$ (this is valid when the amplitude of swing is small). Now you have derived a second-order differential equation about θ! Although you may not be able to solve it, use $\theta = A\cos(\omega t)$ (A and ω are constants to be determined) as a test solution. Substitute it into the equation and see what happens! Find the value of ω and use it to find the revised period of the pendulum. This small correction can help us improve the precision when using a single pendulum as a timer.

c) We hang a rigid body of arbitrary shape on a horizontal axle. When the center of mass of the rigid body is slightly displaced from the eqlibrium, the body will start to swing. This is called a compound pendulum or a physical pendulum. Suppose the moment of inertia of the body about the axle is I_0 and the mass of the body is m_0. The center of mass of the rigid body is h from the hanging point. Assume the amplitude of the swing is small. Make revisions to the expression of period you found in part b) to find the period of the compound pendulum. Then use the parallel axis theorem to find the moment of inertia of the body for a parallel axis through center of mass in terms of the period. The real world application of this is that we can use a compound pendulum to find the moment of inertia of a rigid body! This is an alternative approach to measure the moment of inertia. We have already seen one approach in Long Answer Question 2 of this module.

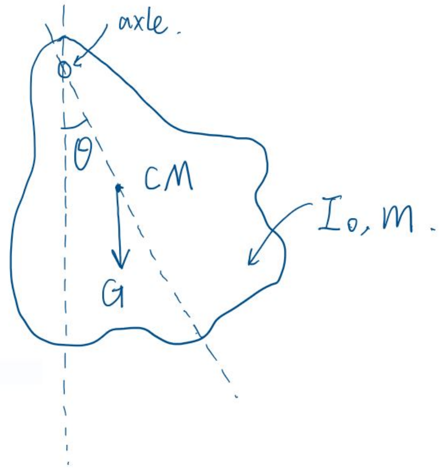

Figure 1: compound pendulum

d) The apparatus in the picture below is called a spring pendulum. It is supported by a spiral spring at the eqlibrium position. When the pendulum is displaced from the eqlibrium at an angle θ, the spiral spring will exert a torque of $-\kappa\theta$, where κ is a constant similar to the spring constant of an ordinary spring. The elastic potential energy of the spiral spring is $\frac{1}{2}\kappa\theta^2$. Write down the expression of the potential energy of the system (use the approximation $\cos\theta \approx 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$ when θ is small). Use $\lambda = \frac{\kappa}{m_1 g l}$ as a parameter. Then draw the potential diagrams when $\lambda > 1$ and $\lambda < 1$ and discuss the motion of the pendulum in these two cases. Assume that the arm of the spring pendulum is massless and the mass of the bob is m_1.

Note: What you will discover in this question is called symmetry breaking. The same phenomenon is also observed in the Quantum Field Theory. The free energy diagram of the Higgs Field has a similar shape as the potential diagram of the spring pendulum. The parameter in that case is temperature. The minimum of the free energy diagram of the Higgs Field corresponds to the state of vacuum. The symmetry breaking of the Higgs Field causes the cosmic inflation in the big bang. So, the spring pendulum can be viewed as a realistic toy model to illustrate a topic of cutting-edge physics.

To know more about symmetry breaking and the Higgs Mechanism, you can read Section 11.8 and 11.9 of *Introduction to Elementary Particles* by David Griffiths.

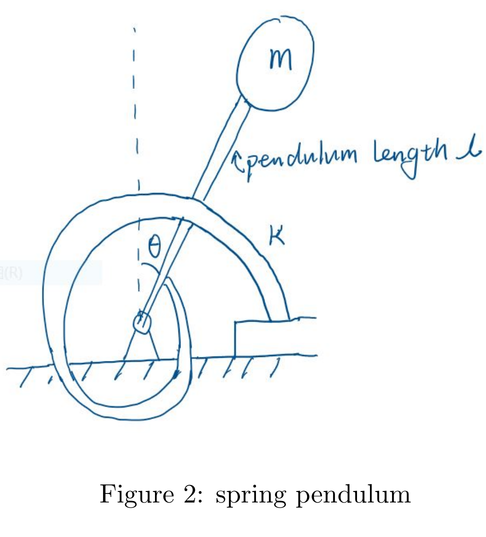

Figure 2: spring pendulum

Approach:

a) We choose the axis as the axis passing through the hanging point horizontally. First, we write down the moment of inertia of the pendulum about the axis. That is the sum of the moment of inertia of the bob and the moment of inertia of the string. Then, we need to find the net torque on the system. The net force on the system is exerted by gravity at the center of mass of the system. Find the moment arm by using the geometrical relations. Multiply the force and the moment arm to get the torque. After that, use the Newton's Second Law for rotation: the net torque on the system equals the product of the moment of inertia and the angular acceleration.

b) Follow the instructions of this question: substitute $\alpha = \ddot{\theta}$ and $\sin\theta \approx \theta$ into the equation. Then substitute $\theta = A\cos(\omega t)$ into the equation to verify that it is a solution. At the same time, find the value of the constant ω. The period of the pendulum equals $\frac{2\pi}{\omega}$.

c) In the expression of period found in the previous question, use the new moment of inertia I_0 instead of the old one. And use the new expression of torque to substitute the old one. This is how we get the period of this case. Then solve for I_0 in terms of T by using the expression of the period. After that, we use the parallel axis theorem to express I_{CM} in terms of I_0 and then in terms of T. This is the expression of I_{CM} we should use when measuring the moment of inertia by using a compound pendulum.

d) The potential energy of the system consists of two parts: the gravitational potential energy and the elastic potential energy. Choose the height of the equlibrium position of the pendulum as zero gravitational potential level. Then we can use the geometrical relations to find gravitational potential energy in terms of m, g, l and θ. The elastic potential energy of this case is just $\frac{1}{2}\kappa\theta^2$, similar to the elastic potential of an ordinary spring. Then we apply the approximation and the parameter mentioned in the question to the expression of potential energy. After that, we draw the potential energy diagram when $\lambda > 1$ and $\lambda < 1$. We use these diagrams to analyze the eqlibrium points (stable and unstable) and the symmetry of the system in each case.

Conceptual Components:
Moment of Inertia
Torque
Dynamics of Rotational Motion
Periodic Motion
Energy
Energy Diagrams

Target Quantities:
a) The Newton's Second Law equation for the rotation of the pendulum when taking the mass of string into account.
b) The value of ω and the revised period of the pendulum.
c) The period of the compound pendulum T and I_{CM} in terms of period T.
d) The expression of the potential energy of the system and the potential diagrams of each case. Analysis of the motion of the system.

Quantitive Relationships that May be Useful:
Gravity: $G = mg$
Moment of Inertia of a point mass: $I = mr^2$
Moment of Inertia of a Slender rod, axis through one end $I = \frac{1}{3}ml^2$ (we use this for the moment of inertia of the string)
Torque: $\tau = Fl$
Newton's Second Law for rotation: $\Sigma\tau = I\alpha$
Angular acceleration: $\alpha = \ddot{\theta}$
Period: $T = \frac{2\pi}{\omega}$
Center of mass: $r_{CM} = \frac{\Sigma mr}{\Sigma m}$
Parallel Axis theorem: $I_p = I_{CM} + Md^2$
Elastic potential energy of a spiral spring: $E_e = \frac{1}{2}\kappa\theta^2$
Gravitational potential energy: $E_g = mgh$
Force from potential energy: $F = -\frac{dU}{dx}$
Approximation: when θ is small, $\cos\theta \approx 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$

Solution:
a)

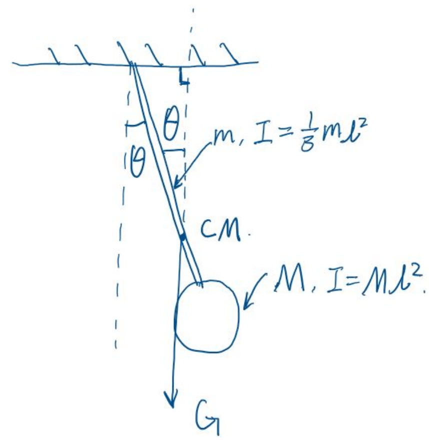

Figure 3: diagram for this part

We choose the axis as the axis passing through the hanging point horizontally. First we calculate the moment of inertia of the system. The moment of inertia of the system equals the sum of the moment of inertia of each part. In this case, it equals the sum of the moment of inertia of the string and the bob:

$$I = I_{string} + I_{bob} \tag{1}$$

According to the assumption of this question, I_{string} is:

$$I_{string} = \frac{1}{3}ml^2 \tag{2}$$

Since we treat the bob as a point mass, I_{bob} is:

$$I_{bob} = Ml^2 \tag{3}$$

Substitute (2) and (3) into (1), we get the moment of inertia of the system:

$$I = I_{string} + I_{bob} = \frac{1}{3}ml^2 + Ml^2 \tag{4}$$

Next, we need to find the net torque on this system. There is only one force that exerts a torque on the system with respect to the axis that we have chosen. That is the total gravity of the system. The gravity can be treated as acting on the center of mass of the system. So we need to find the center of mass of the system in order to find the moment arm. We choose the hanging point as $r = 0$. So:

$$r_{CM} = \frac{\Sigma mr}{\Sigma m} = \frac{m\frac{1}{2}l + Ml}{M + m} = \frac{ml + 2Ml}{2M + 2m} \tag{5}$$

From the diagram, we can see that the moment arm is:

$$l_g = r_{CM}\sin\theta = \frac{ml + 2Ml}{2M + 2m}\sin\theta \tag{6}$$

Therefore, the torque on the system is:

$$\tau = -l_g G = -\frac{ml + 2Ml}{2M + 2m}\sin\theta(m + M)g = -\frac{ml + 2Ml}{2}g\sin\theta \tag{7}$$

(The negative sign comes from the fact that the torque is always opposite to the direction of θ)
From the Newton's Second Law for rotation, we have

$$\Sigma\tau = I\alpha$$

$$-\frac{ml+2Ml}{2}g\sin\theta = (\frac{1}{3}ml^2 + Ml^2)\alpha \tag{8}$$

where α is the angular acceleration. This is the equation that we desire.

b)
By the instructions of the question, we substitute $\alpha = \ddot{\theta}$ and $\sin\theta \approx \theta$ into equation (8):

$$-\frac{ml+2Ml}{2}g\theta = (\frac{1}{3}ml^2 + Ml^2)\ddot{\theta} \tag{9}$$

Then, we substitute the test solution $\theta = A\cos(\omega t)$ into the right of the above equation:

$$(\frac{1}{3}ml^2 + Ml^2)\frac{d^2}{dt^2}A\cos(\omega t) = -A\omega^2(\frac{1}{3}ml^2 + Ml^2)\cos(\omega t)$$

Substitute the test solution into the left of equation (9), we get:

$$-A\frac{ml+2Ml}{2}g\cos(\omega t)$$

Compare these two results, we find out that when we take

$$\omega^2 = \frac{\frac{ml+2Ml}{2}g}{(\frac{1}{3}ml^2 + Ml^2)} = \frac{r_{CM}(m+M)g}{I} \tag{10}$$

$$\omega = \sqrt{\frac{(3ml+6Ml)g}{2ml^2+6Ml^2}} \tag{11}$$

The left side and the right side of the equation equals. This verifies that $\theta = A\cos(\omega t)$ is indeed a solution. Therefore, the period of this motion is:

$$T = \frac{2\pi}{\omega} = 2\pi/\sqrt{\frac{(3ml+6Ml)g}{2ml^2+6Ml^2}} = 2\pi\sqrt{\frac{2ml^2+6Ml^2}{(3ml+6Ml)g}} \tag{12}$$

When the mass of the string is negligible, $m \to 0$,

$$T \to 2\pi\sqrt{\frac{2\times 0\times l^2 + 6Ml^2}{(3\times 0\times l + 6Ml)g}} = 2\pi\sqrt{\frac{l}{g}} \tag{13}$$

We come back to the period of a simple pendulum. So, in realistic applications, when the mass of string is not negligible (when the pendulum is heavy and we have to use thick ropes instead of thin strings), we must use the revised formula if we still want precise results for the period.

c)

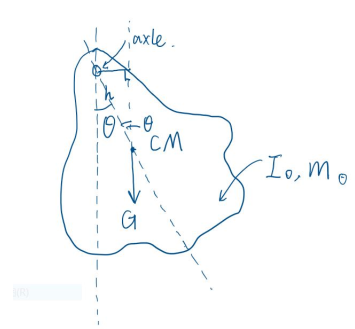

Figure 4: diagram for this part

In the case of compound pendulum, the distance of the center of mass from the axle is $r_{CM} = h$, the mass of the system is m_0 and the moment of inertia of the system is I_0. From equation (10), we can conjecture that the ω^2 for the compound pendulum is $\frac{hm_0 g}{I_0}$ (substitute h for r_{CM}, m_0 for $m + M$ and I_0 for I). Actually, this conjecture is valid, as we have already treated the single pendulum with a massive string as a compound pendulum for our derivations in part b). That is, we have considered the effect of the moment of inertia of the whole system, rather than treating the pendulum as a single point mass. It is also pretty clear that the single pendulum with a massive string is a specical condition of compound pendulum. As a result, we have:

$$\omega^2 = \frac{hm_0 g}{I_0}$$

$$\omega = \sqrt{\frac{hm_0 g}{I_0}} \tag{14}$$

$$T = \frac{2\pi}{\omega} = 2\pi / \sqrt{\frac{hm_0 g}{I_0}} = 2\pi \sqrt{\frac{I_0}{hm_0 g}} \tag{15}$$

This is the expression for the period of the compound pendulum.
Therefore, when T is known (for example, from experiment), we have I_0 as an expression of T:

$$I_0 = \left(\frac{T}{2\pi}\right)^2 hm_0 g \tag{16}$$

From the parallel axis theorem, we know that:

$$I_0 = I_{CM} + m_0 h^2 \tag{17}$$

Therefore, we have:

$$I_{CM} = I_0 - m_0 h^2 = \left(\frac{T}{2\pi}\right)^2 hm_0 g - m_0 h^2 \tag{18}$$

We have found the expression of I_{CM} in terms of the period T that we desire. This means we can measure the moment of inertia of an object by doing a compound pendulum experiment to measure the period. We have already seen another approach of measuring moment of inertia in Long Answer Question 2 of module 5.

d)

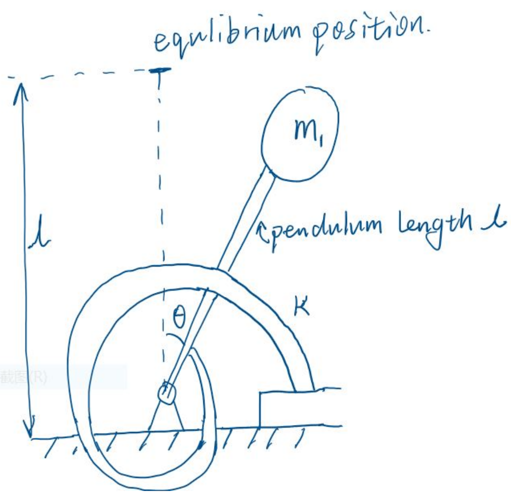

Figure 5: diagram for this part

We choose the height of the equlibrium position as zero gravitational potential level. From the diagram, we can see that the vertical distance of the bob from the zero potential level is:

$$\Delta h = l - l\cos\theta = l(1 - \cos\theta) \tag{19}$$

Since the bob is always below the zero potential level, its gravitational potential energy is negative:

$$E_g = -m_1 g \Delta h = m_1 g l (\cos\theta - 1) \tag{20}$$

(the negative sign comes from the fact that the bob is below the zero potential level.)
According to the assumption of the question, the elastic potential energy is:

$$E_e = \frac{1}{2}\kappa\theta^2 \tag{21}$$

The total potential energy of the system is:

$$E_p = E_g + E_e = m_1 g l(\cos\theta - 1) + \frac{1}{2}\kappa\theta^2 \tag{22}$$

Apply the approximation $\cos\theta \approx 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$ and the parameter $\lambda = \frac{\kappa}{m_1 g l}$ to this equation:

$$E_p = m_1 g l(\cos\theta - 1) + \frac{1}{2}\kappa\theta^2 \approx \frac{m_1 g l}{2}\left[\frac{\kappa}{m_1 g l}\theta^2 - \theta^2 + \frac{1}{12}\theta^4\right] = \frac{m_1 g l}{2}\left[(\lambda - 1)\theta^2 + \frac{1}{12}\theta^4\right] \tag{23}$$

Let's calculate $dE_p/d\theta$:

$$\frac{dE_p}{d\theta} = \frac{m_1 g l}{2}\frac{d}{d\theta}\left[(\lambda - 1)\theta^2 + \frac{1}{12}\theta^4\right] = \frac{m_1 g l}{2}\left[2(\lambda - 1)\theta + \frac{1}{3}\theta^3\right] \tag{24}$$

$dE_p/d\theta$ has three possible roots: $\theta = 0$, $\theta = \pm\sqrt{6(1-\lambda)}$. This means E_p can have at most three equlibrium points.

When $\lambda > 1$, $\theta = \pm\sqrt{6(1-\lambda)}$ are imaginary numbers, there is only one equlibrium position at $\theta = 0$:

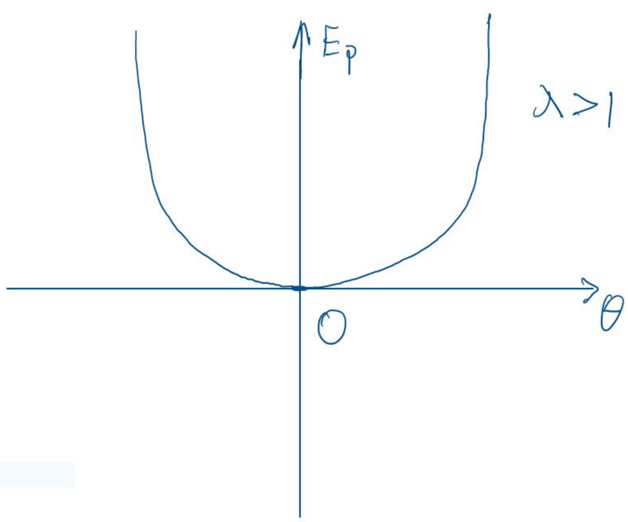

Figure 6: potential diagram when $\lambda > 1$

We can see that the equlibrium position at $\theta = 0$ is stable. This is because when the pendulum is displaced to either side of the equlibrium point, the "force" $F_\theta = -dE_p/d\theta$ (you can check right away this is in fact the torque caused by the potential energy) directs back to the equlibrium point. We can see that the equlibrium is symmetric in this case. The pendulum will vibrate around $\theta = 0$

When $\lambda < 1$, the three roots are all real numbers. Therefore, the system has three possible equlibrium points from the potential diagram:

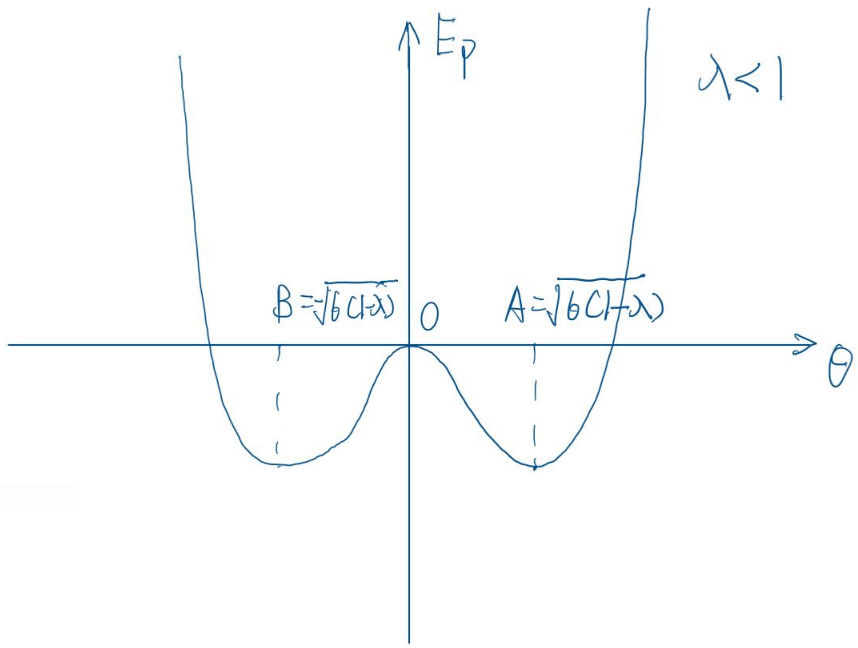

Figure 7: potential diagram when $\lambda < 1$

Similar as the previous case, A and B are stable equlibrium points, as when the pendulum is displaced to either side of A or B, the "force" $F_\theta = -dE_p/d\theta$ directs back to the equlibrium point. However, the equlibrium at $\theta = 0$ becomes unstable. From the diagram, it is clear that when the pendulum is displaced to either side of $\theta = 0$, the "force" $F_\theta = -dE_p/d\theta$ directs away from $\theta = 0$. If the pendulum starts to vibrate at $\theta = 0$, its E_p will soon fall to either A or B and the pendulum will vibrate around that point. The system turns into a less symmetric state. This is the reason why it is called symmetry breaking.

Evaluation:

a) Dimension check:

a) $[kg][m]\frac{[m]}{[s]^2} = [kg][m]^2\frac{1}{[s]^2}$

b) $\frac{1}{[s]} = \sqrt{\frac{[kg][m]\frac{[m]}{[s]^2}}{[kg][m]^2}}$

$[s] = \sqrt{\frac{[kg][m]^2}{[kg][m]\frac{[m]}{[s]^2}}}$

c) $[s] = \sqrt{\frac{[kg][m]^2}{[m][kg][m]/[s]^2}}$

$[kg][m]^2 = [s]^2[m][kg][m]/[s]^2 + [kg][m]^2$

d) $[J] = [kg][m]/[s]^2[m] + [N][m]$

b) Orders of magnitude

No numerical value is given in this question, so we cannot evaluate the reasonableness regarding the order of magnitude.

c) All questions have been answered.