Module 1 Group Project

The Application of Projectile Motion in Military

Section 2, Group 60
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Note: Assume that the air-resistence is negligible for all parts.

Background Introduction:
For the first Physics group project, group 60 decided to go with a military theme and created questions involving projectile motion. In modern warfare, with the use of missiles and ICBM's, projectile motion becomes extremely useful for determining different aspects of the military's mission, such as making sure they will hit the intended target and avoiding getting hit themselves. Part A of our project is a question including constant velocity and determining how fast a plane must be moving when a bomb is dropped in order to hit enemy targets. This knowledge becomes extremely useful for fighters in order to get one step closer to achieving their goal and ensuring they don't waste resources. Part B includes what height a plane must be flying in order to avoid getting hit by gunfire. This module could also be used for a drone doing surveillance or a bomber plane, like the one mentioned in Part A, so that they can determine the correct height to fly at while keeping their men and equipment alive and safe. Finally, Part C aims to derive an equation to figure out how to hit an enemy who is behind cover. This can be applied in many different areas such as areas involving mountains, dense urban areas, and even in an attempted siege of a city. Group 60 thought that given all of the conflict in the world and the interests in military advancement, projectile motion questions based off of the military were the most relevant to everyone around the world during these modern times.

Figure 1: Artillery fire. From https://www.defenceconnect.com.au/land-amphibious/6360-live-fire-training-returns-for-1-regt

Part A:

Question:

A bomber plane drops a bomb in order to hit a target. When it drops the bomb, it is flying horizontally at a constant velocity v. What would be the equation of the bomb's trajectory (you can set up the coordinate system in any way you like, assume that v is known)? Suppose that the plane is at a height h when it drops the bomb. How long does it take for the bomb to hit the ground? What is v if the bomb hits the target that is at a horizontal distance d from the position of dropping the bomb?

Approach:

1. Set up a Cartesian coordinate system. Choose the position of the plane when it drops the bomb as the origin, x axis as horizontally to the right and y axis as vertically upward. This would be convenient for writing down the equations of motion.

2. Write down the equations of motion for the x component and y component of the bomb's motion. The x component would be uniform linear motion and the y component would be free fall. Then manipulate the equations to get the equation of the trajectory (an equation that does not contain time).

3. When the bomb hits the ground, its y coordinate would be $-h$ in our coordinate system. We can use the y component equation to work out time t, then substitute it into the x component equation to find the distance it travels horizontally before hitting the ground. This distance equals d. We can use this relation to find the v we desire.

Target Quantity:

Equation of the trajectory: y as a function of x (assume that v is known)
The time it takes for the bomb to hit the ground: t_h
The speed of the plane: v

Quantitive relationships that may be useful:

For uniform linear motion: $x(t) = vt$
For free fall (choose the launch point as origin): $y(t) = v_y t - \frac{1}{2}gt^2$

Diagram:

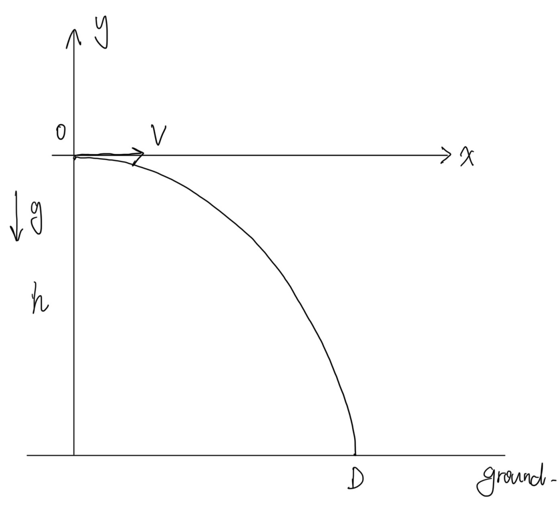

Figure 2: Diagram for this problem

Solution:

Set up a Cartesian coordinate system. Choose the position of the plane when it drops the bomb as the origin, x axis as horizontally to the right and y axis as vertically upward.

Write down the equations of motion for the x component and y component. Since the x component is uniform linear motion and y component is free fall, we have:

$$x = vt \tag{1}$$

$$y = v_y t - \frac{1}{2}gt^2 = -\frac{1}{2}gt^2 \tag{2}$$

Since $v_y = 0$. From equation (1), we can work out t as a function of x by dividing both sides of that equation by v:

$$t = \frac{x}{v}$$

Substituting this into equation (2), we get the equation of trajectory we desire:

$$y = -\frac{1}{2}g(\frac{x}{v})^2 = -\frac{gx^2}{2v^2} \tag{3}$$

When the bomb hits the ground, $y = -h$. Substitute this into equation (2):

$$-\frac{1}{2}gt_h^2 = -h$$

$$t_h = \sqrt{\frac{2h}{g}} \tag{4}$$

This is the time it takes for the bomb to hit the ground.

When the bomb hits the ground, the horizontal distance it travels would be:

$$x = vt_h = v\sqrt{\frac{2h}{g}}$$

In order to hit the target, this distance equals d:

$$d = v\sqrt{\frac{2h}{g}}$$

Therefore, the speed v must be:

$$v = d\sqrt{\frac{g}{2h}} \tag{5}$$

Evaluation:

$$[m] = \frac{[m/s^2][m]^2}{[m/s]^2}$$

$$[s] = \sqrt{\frac{[m]}{[m/s^2]}}$$

$$[m] = [m/s]\sqrt{\frac{[m]}{[m/s^2]}}$$

Therefore, the answers have proper dimensions.
No numerical value is given, so we cannot determine whether the answer is reasonable regarding its order of magnitude.
All original questions have been answered.

Part B:

Question:

Enemies are firing at the bomber plane by using an anti-aircraft gun. Its shells have a speed v_0 when they leave the gun. The gun can fire at any angle between $0°$ and $90°$. Which region should the plane not fly into in order to avoid being hit?

Approach:

1. Set up a Cartesian coordinate system. Choosing the gun as the origin, x axis as horizontally to the right and y axis as vertically upward would be convenient for us to write down the equation of motion.

2. Write down the equation of motion for a particular gun shell. This would just be the equation of motion for projectiles. Then manipulate the equations to get the equation of the trajectory (An equation about x, y and other parameters only. It does not contain time t).

3. In order to solve this problem, we need to find the maximum height the shell can reach at each x coordinate. That is at a particular x coordinate, the shell can reach anywhere below this height but nowhere above this height. In order to do that, we need to fix an x first, then manipulate the equation to find the angle of firing θ that makes y maximum. The way to do that depends on the form of the equation of trajectory. We shall see how to do that when we actually derive that equation.

4. When x is varying, the maximum height equation would give the maximum height that the shell can reach at each x coordinate. The curve given by this equation would be the boundary of the region that the shell can reach. So the shell cannot reach anywhere above this curve. Our plane should always fly above this curve to avoid being hit.

Target Quantity:

y_{max} as a function of x. This is the boundary of the region that the shell can reach.

Quantitive relationships that may be useful:

For projectile motion (choose the launch position as the origin):

$$x(t) = v_0 \cos\theta t$$

$$y(t) = v_0 \sin\theta t - \frac{1}{2}gt^2$$

Diagram:

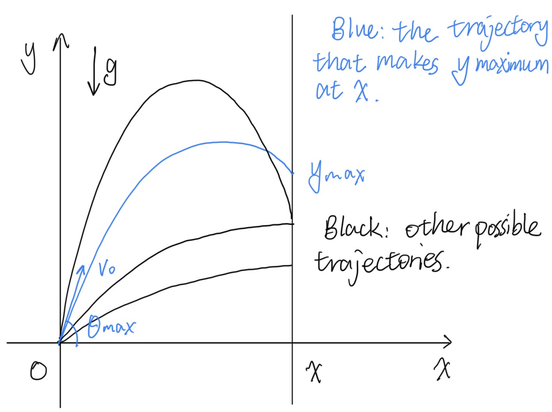

Figure 3: Diagram for this problem

Solution:

Set up a Cartesian coordinate system and choose the anti-aircraft gun as the origin. Let the x axis be horizontally to the right and y axis be vertically upward.

First of all, write down the equation of motion for a shell. Assume it is fired at an angle of θ relative to the x axis:

$$x = v_0 \cos\theta t \tag{6}$$

$$y = v_0 \sin\theta t - \frac{1}{2}gt^2 \tag{7}$$

From equation (6), we can work out time t as a function of x by dividing both sides of equation (6) by $v_0 \cos\theta$: $t = \frac{x}{v_0 \cos\theta}$. Substitute t into equation (7) and we can get:

$$\begin{aligned} y &= v_0 \sin\theta \frac{x}{v_0 \cos\theta} - \frac{1}{2}g(\frac{x}{v_0 \cos\theta})^2 \\ &= x\tan\theta - \frac{gx^2}{2v_0^2}(\frac{1}{\cos^2\theta}) \\ &= x\tan\theta - \frac{gx^2}{2v_0^2}(1 + \tan^2\theta) \end{aligned} \tag{8}$$

This is the equation of the trajectory of a shell.

Then we fix x in equation (8). We can rearrange equation (8) into this form, in order to determine y_{max}:

$$\begin{aligned} y &= -\frac{gx^2}{2v_0^2}\tan^2\theta + x\tan\theta - \frac{gx^2}{2v_0^2} \\ &= -\frac{gx^2}{2v_0^2}[\tan^2\theta - \frac{2v_0^2}{gx}\tan\theta + (\frac{v_0^2}{gx})^2] + \frac{gx^2}{2v_0^2}(\frac{v_0^2}{gx})^2 - \frac{gx^2}{2v_0^2} \\ &= -\frac{gx^2}{2v_0^2}(\tan\theta - \frac{v_0^2}{gx})^2 + \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} \end{aligned} \tag{9}$$

Since $-(\tan\theta - \frac{v_0^2}{gx})^2$ is non-positive, when it equals zero, y would obtain its maximum value y_{max}. That is when $\tan\theta = \frac{v_0^2}{gx}$:

$$y = y_{max} = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} \tag{10}$$

This is the maximum height that the shell can reach at the coordinate x when x is fixed. When x is varying, this equation gives the maximum height that the shell can reach at each x coordinate. Notice that the curve given by this equation is a parabola. The shell can reach anywhere below this parabola, but cannot reach the area above this parabola. So our plane should avoid flying into the region below the parabola given by equation (10).

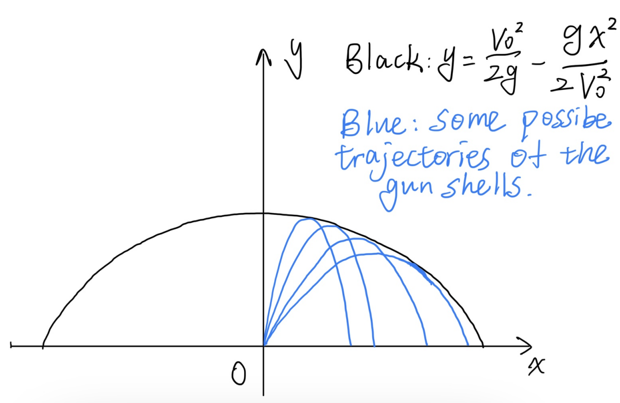

Figure 4: All possible trajectories of the shells are enclosed by the parabola given by equation (10). The shells cannot reach the area above that parabola.

Evaluation:

$$[m] = \frac{[m/s]^2}{[m/s^2]} - \frac{[m/s^2][m]^2}{[m/s]^2}$$

Therefore, the answer has a proper dimension.
No numerical value is given, so we cannot determine whether the answer is reasonable regarding its order of magnitude.
All original questions have been answered.

Part C:

Question:
In order to hit our enemies, we can use an artillery piece instead of a bomber plane. A howitzer is a kind of artillery characterized by its curved trajectories. It enables us to attack enemies hiding behind covers (for example, behind mountains). Suppose we are at the foot of a mountain and our enemies are at the foot of the mountain on the other side. So in order to hit the target by using our howitzer, we need to let our shell fly over the mountain. Suppose the mountain is a triangle with base angles θ and φ. In order to let our shell just fly over the mountain top, what would be our firing angle α (in terms of θ and φ)?

Approach:
1. Set up a Cartesian coordinate system. Choose our position as the origin, x axis as horizontally to the right and y axis as vertically upward. This would be convenient for wring down the equation of motion for the shells.

2. Assume that the initial speed of the shells are v_0 and the firing angle is α. Write down the equation of motion for gun shells. This would just be the equation of motion for projectiles. Similarly to what we did in Part A, we manipulate the equations to get the equation of the trajectory (does not contain time t).

3. Assume that the enemies are at $(x_A, 0)$, the mountain top is at (x_1, y_1) and the mountain top is x_2 away (horizontally) from the enemies. Since the mountain is a triangle, we can find geomertical relations for x_A, x_1, y_1 and x_2 in terms of θ and φ. Then we can substitute these coordinates into the equation of the trajectory, since they are all on the trajectory. After that, we manipulte the equations and see what relation we can get for α, θ and φ.

Target Quantity:
Firing angle α in terms of θ and φ.

Quantitive relationships that may be useful:
For projectile motion (choose the launch position as the origin):

$$x(t) = v_0 \cos\theta t$$

$$y(t) = v_0 \sin\theta t - \frac{1}{2}gt^2$$

Diagram:

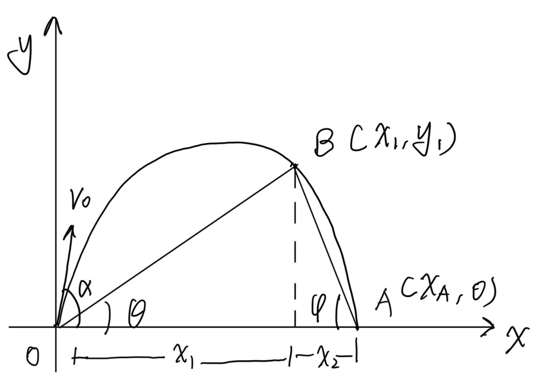

Figure 5: Diagram for this problem

Solution:
Set up the Cartesian coordinate system first. Choose our position as the origin, x axis as horizontally to the right and y axis as vertically upward.

We write down the equation of motion for our shell first. Assume its initial speed is v_0 and is fired at an angle α relative to the x axis:

$$x = v_0 \cos\alpha t \tag{11}$$

$$y = v_0 \sin \alpha t - \frac{1}{2} g t^2 \tag{12}$$

Similarly to what we did in Part A, we can get time t as a function of x by dividing $v_0 \cos \alpha$ on both sides of equation (11) : $t = \frac{x}{v_0 \cos \alpha}$. Then we substitute this to equation (12) to get the equation of the trajectory:

$$\begin{aligned} y &= v_0 \sin \alpha \frac{x}{v_0 \cos \alpha} - \frac{1}{2} g (\frac{x}{v_0 \cos \alpha})^2 \\ &= x \tan \alpha - \frac{g x^2}{2 v_0^2 \cos^2 \alpha} \end{aligned} \tag{13}$$

Substitute the coordinate of the enemies $(x_A, 0)$ into this equation. Since the enemies are on the trajectory of our shell, we can get:

$$0 = x_A \tan \alpha - \frac{g x_A^2}{2 v_0^2 \cos^2 \alpha}$$

$$x_A = \frac{2 v_0^2 \sin \alpha \cos \alpha}{g} \tag{14}$$

From the diagram, we can find the following geometrical relations for x_1, x_2, y_1, x_A, since they are the legs of right triangles:

$$x_1 = \frac{y_1}{\tan \theta}, \quad x_2 = \frac{y_1}{\tan \varphi}, \quad x_1 + x_2 = x_A$$

Combining these three equations, we can get:

$$x_A = x_1 + x_2 = y_1 (\frac{1}{\tan \theta} + \frac{1}{\tan \varphi})$$

$$y_1 = \frac{x_A}{\frac{1}{\tan \theta} + \frac{1}{\tan \varphi}} = x_A \frac{\tan \theta \tan \varphi}{\tan \theta + \tan \varphi} \tag{15}$$

$$x_1 = \frac{y_1}{\tan \theta} = x_A \frac{\tan \varphi}{\tan \theta + \tan \varphi} \tag{16}$$

Substituting y_1 given by equation (15) and x_1 given by equation (16) into equation (13) (this is the coordinate of the mountain top, and the mountain top is on the trajectory), we can get:

$$x_A \frac{\tan \theta \tan \varphi}{\tan \theta + \tan \varphi} = x_A \frac{\tan \varphi}{\tan \theta + \tan \varphi} \tan \alpha - \frac{g}{2 v_0^2 \cos^2 \alpha} (x_A \frac{\tan \varphi}{\tan \theta + \tan \varphi})^2$$

Rearranging this equation, we can get:

$$\frac{g \tan \varphi x_A}{2 v_0^2 \cos^2 \alpha (\tan \theta + \tan \varphi)} = \tan \alpha - \tan \theta$$

Substitute x_A given by equation (14) into the above equation:

$$\frac{g \tan \varphi}{2 v_0^2 \cos^2 \alpha (\tan \theta + \tan \varphi)} \frac{2 v_0^2 \sin \alpha \cos \alpha}{g} = \tan \alpha - \tan \theta$$

Rearrange this equation. After some derivation, we can get the result we desire:

$$\tan \alpha = \tan \theta + \tan \varphi \tag{17}$$

Therefore:

$$\alpha = \arctan(\tan \theta + \tan \varphi)$$

Evaluation:
The answer is dimensionless.
No numerical value is given, so we cannot determine whether the answer is reasonable regarding its order of magnitude.
All original questions have been answered.