AMATH 271 Final Project
Maple Worksheet
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```
restart :
with(DEtools) :
with(plots) :
with(plottools) :
with(DynamicSystems) :
```

Numerical solutions:
The parameters inthe equations:

$A := 1$

$$A := 1 \tag{1}$$

$B := 0.5$

$$B := 0.5 \tag{2}$$

$C := 9.81$

$$C := 9.81 \tag{3}$$

The differential equations:

$Eq1 := A\cdot\frac{d^2}{dt^2}(\varphi 1(t)) + B\cdot\frac{d^2}{dt^2}(\varphi 2(t))\cdot\cos(\varphi 1(t)-\varphi 2(t)) + B\cdot\left(\frac{d}{dt}(\varphi 2(t))\right)^2\cdot\sin(\varphi 1(t) - \varphi 2(t)) + C\cdot\sin(\varphi 1(t)) = 0$

$$Eq1 := \frac{d^2}{dt^2}\varphi 1(t) + 0.5\left(\frac{d^2}{dt^2}\varphi 2(t)\right)\cos(\varphi 1(t)-\varphi 2(t)) + 0.5\left(\frac{d}{dt}\varphi 2(t)\right)^2\sin(\varphi 1(t)-\varphi 2(t)) + 9.81\sin(\varphi 1(t)) = 0 \tag{4}$$

$Eq2 := \frac{d^2}{dt^2}(\varphi 2(t)) + A\cdot\frac{d^2}{dt^2}(\varphi 1(t))\cdot\cos(\varphi 1(t)-\varphi 2(t)) - A\cdot\left(\frac{d}{dt}(\varphi 1(t))\right)^2\cdot\sin(\varphi 1(t) - \varphi 2(t)) + C\cdot\sin(\varphi 2(t)) = 0$

$$Eq2 := \frac{d^2}{dt^2}\varphi 2(t) + \left(\frac{d^2}{dt^2}\varphi 1(t)\right)\cos(\varphi 1(t)-\varphi 2(t)) - \left(\frac{d}{dt}\varphi 1(t)\right)^2\sin(\varphi 1(t)-\varphi 2(t)) + 9.81\sin(\varphi 2(t)) = 0 \tag{5}$$

The initial conditions are:

$ICs := \varphi 1(0) = \frac{\mathrm{Pi}}{4},\ D(\varphi 1)(0) = 0,\ \varphi 2(0) = \frac{\mathrm{Pi}}{4},\ D(\varphi 2)(0) = 0$

$$ICs := \varphi 1(0) = \frac{\pi}{4},\ D(\varphi 1)(0) = 0,\ \varphi 2(0) = \frac{\pi}{4},\ D(\varphi 2)(0) = 0 \tag{6}$$

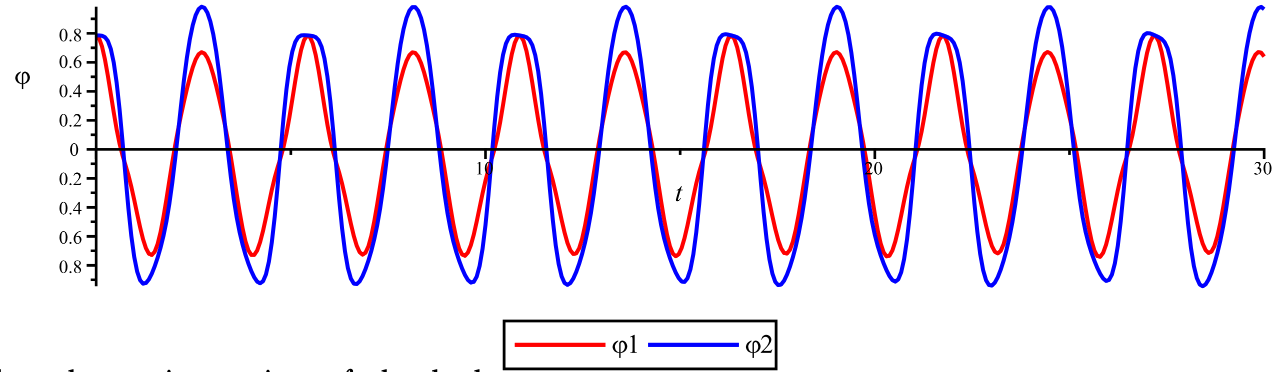
Compute the numerical solutions:

```
sol := dsolve({Eq1, Eq2, ICs}, numeric, output = listprocedure, maxfun = 0) :
```

Plot the angle-time plot:

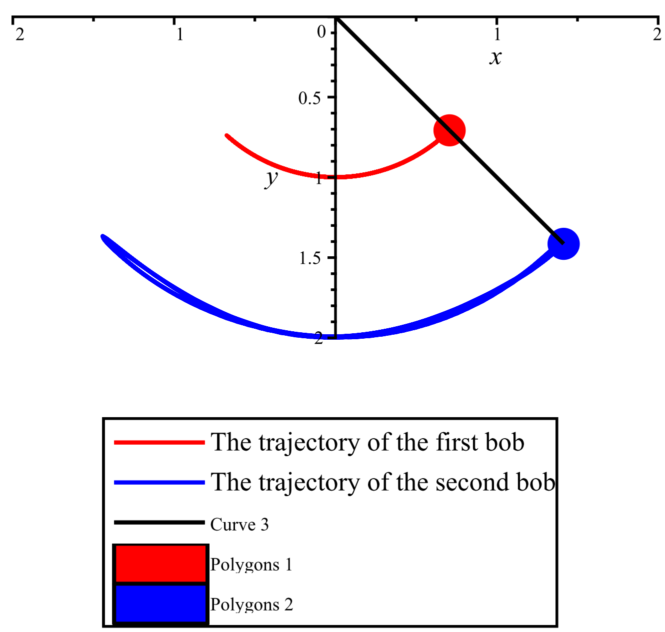
```
plot1 := odeplot(sol, [t, φ1(t)], t = 0..30, numpoints = 400, color = red) :
```

```
plot2 := odeplot(sol, [t, φ2(t)], t = 0..30, numpoints = 400, color = blue) :
display(plot1, plot2)
```



Plot the trajectories of the bobs:

```
trajectory1 := odeplot(sol, [sin(φ1(t)),  cos(φ1(t))], t = 0..30, numpoints = 400,
    color = red) :
trajectory2 := odeplot(sol, [sin(φ1(t)) + sin(φ2(t)),  cos(φ1(t))  cos(φ2(t))], t
    = 0..30, numpoints = 400, color = blue) :
d1 := disk([0.5·sqrt(2),  0.5·sqrt(2)], 0.1, color = red) :
d2 := disk([sqrt(2),  sqrt(2)], 0.1, color = blue) :
l1 := line([0, 0], [0.5·sqrt(2),  0.5·sqrt(2)], color = black, linestyle = solid) :
l2 := line([0.5·sqrt(2),  0.5·sqrt(2)], [sqrt(2),  sqrt(2)] , color = black,
    linestyle = solid) :
display(trajectory1, trajectory2, d1, d2, l1, l2, view = [ 2..2,  2..0], scaling
    = constrained)
```

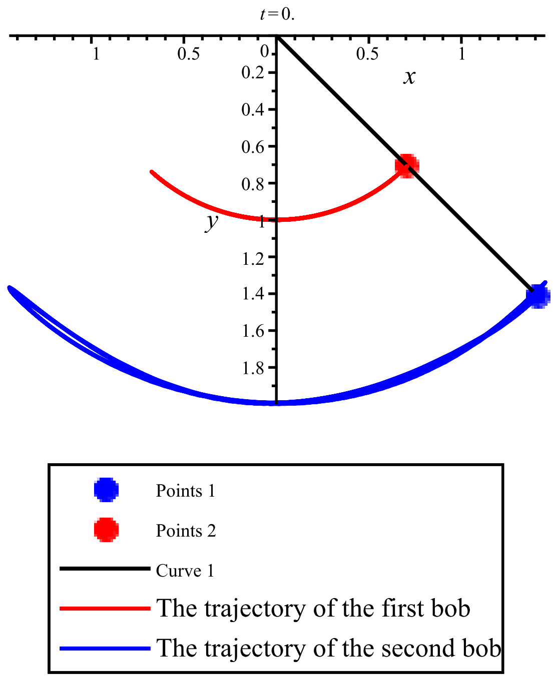


The animated trajectories:

```
TT1, TT2 := op(subs(sol, [φ1(t), φ2(t)])) :
fp := t → plots:-display(
        pointplot([sin(TT1(t)) + sin(TT2(t)),  cos(TT1(t))  cos(TT2(t))],
                color = blue, symbol = solidcircle, symbolsize = 25),
```

```
        pointplot([sin(TT1(t)), -cos(TT1(t))],
                  color = red, symbol = solidcircle, symbolsize = 25),
        plottools:-line([0, 0], [sin(TT1(t)), -cos(TT1(t))]),
        plottools:-line([sin(TT1(t)), -cos(TT1(t))],
                        [sin(TT1(t)) + sin(TT2(t)), -cos(TT1(t))-cos(TT2(t))]),
        scaling = constrained
      ):

animate(fp, [t], t = 0..30, frames = 200, background = [trajectory1, trajectory2]);
```



Let's try another set of initial conditions:

```
NewIC := φ1(0) = Pi/2, D(φ1)(0) = 0, φ2(0) = Pi, D(φ2)(0) = 0
```

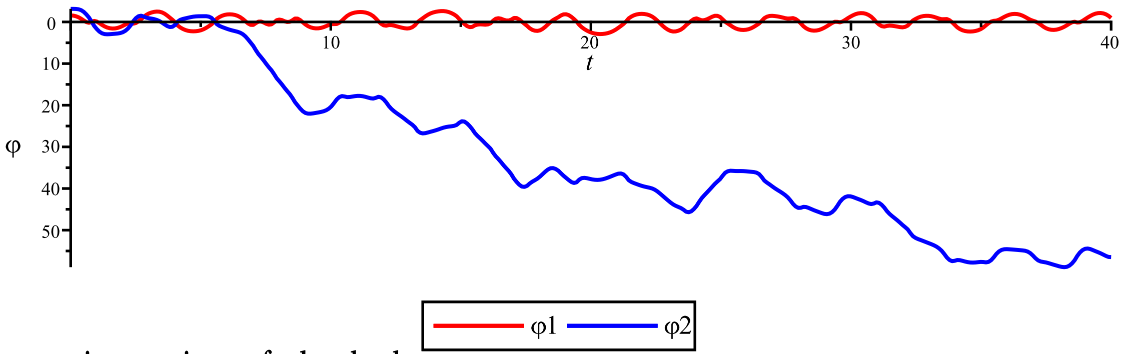
$$NewIC := \varphi 1(0) = \frac{\pi}{2},\ D(\varphi 1)(0) = 0,\ \varphi 2(0) = \pi,\ D(\varphi 2)(0) = 0 \tag{7}$$

Compute the numerical solutions:

```
Newsol := dsolve({Eq1, Eq2, NewIC}, numeric, output = listprocedure, maxfun = 0):
```

Plot the angle-time plot:

```
Newplot1 := odeplot(Newsol, [t, φ1(t)], t = 0..40, numpoints = 400, color = red):
Newplot2 := odeplot(Newsol, [t, φ2(t)], t = 0..40, numpoints = 400, color = blue):
display(Newplot1, Newplot2)
```



Plot the trajectories of the bobs:

Newtrajectory1 ≔ *odeplot*(*Newsol*, [sin(*φ1*(*t*)), cos(*φ1*(*t*))], *t* = 0 .. 40, *numpoints* = 2000, *color* = *red*) :

Newtrajectory2 ≔ *odeplot*(*Newsol*, [sin(*φ1*(*t*)) + sin(*φ2*(*t*)), cos(*φ1*(*t*)) cos(*φ2*(*t*))], *t* = 0 .. 40, *numpoints* = 2000, *color* = *blue*) :

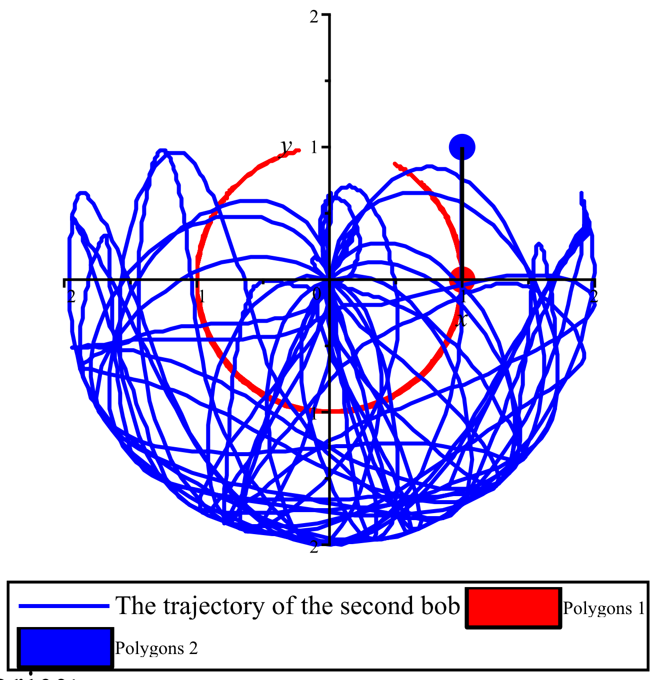
Newd1 ≔ *disk*([1, 0], 0.1, *color* = *red*) :

Newd2 ≔ *disk*([1, 1], 0.1, *color* = *blue*) :

Newl1 ≔ *line*([0, 0], [1, 0], *color* = *black*, *linestyle* = *solid*) :

Newl2 ≔ *line*([1, 0], [1, 1] , *color* = *black*, *linestyle* = *solid*) :

display(*Newtrajectory1*, *Newtrajectory2*, *Newd1*, *Newd2*, *Newl1*, *Newl2*, *view* = [2 .. 2, 2 .. 2], *scaling* = *constrained*)



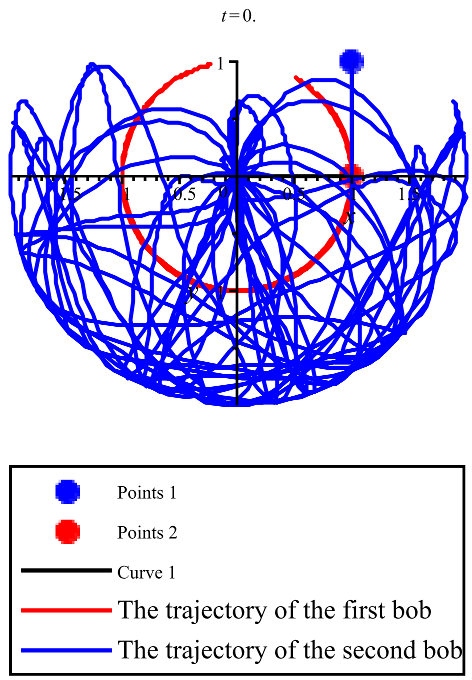
The animated trajectories:

NewTT1, *NewTT2* ≔ *op*(*subs*(*Newsol*, [*φ1*(*t*), *φ2*(*t*)])) :

fp2 ≔ *t* → *plots*:-*display*(
pointplot([sin(*NewTT1*(*t*)) + sin(*NewTT2*(*t*)), cos(*NewTT1*(*t*)) cos(*NewTT2*(*t*))],
color = *blue*, *symbol* = *solidcircle*, *symbolsize* = 25),
pointplot([sin(*NewTT1*(*t*)), cos(*NewTT1*(*t*))],
color = *red*, *symbol* = *solidcircle*, *symbolsize* = 25),
plottools:-*line*([0, 0], [sin(*NewTT1*(*t*)), cos(*NewTT1*(*t*))]),
plottools:-*line*([sin(*NewTT1*(*t*)), cos(*NewTT1*(*t*))],

$[\sin(NewTT1(t)) + \sin(NewTT2(t)),\ -\cos(NewTT1(t)) - \cos(NewTT2(t))]),$
$scaling = constrained$
$):$

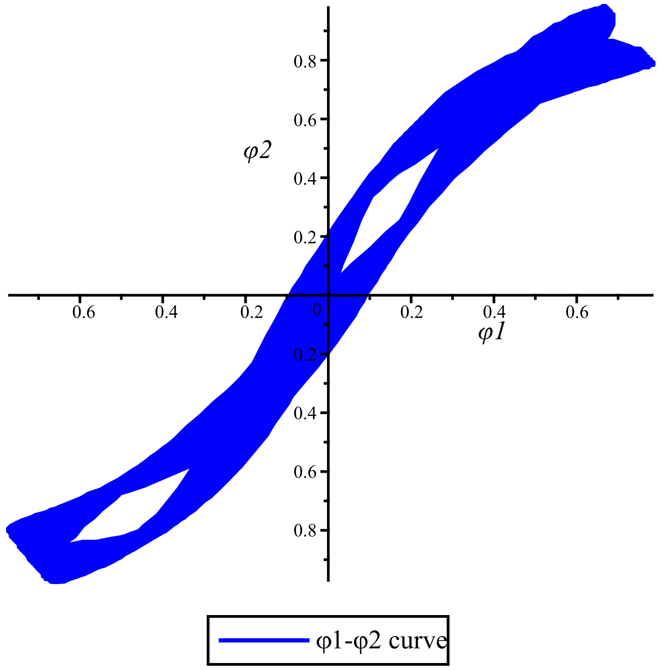
$animate(fp,\ [t],\ t = 0..40,\ frames = 200,\ background = [Newtrajectory1,\ Newtrajectory2]);$



Non-periodicity

The 41-42 plot for the above numerical solution:

$odeplot(sol,\ [\varphi 1(t),\ \varphi 2(t)],\ t = 0..100,\ numpoints = 4000,\ color = blue)$

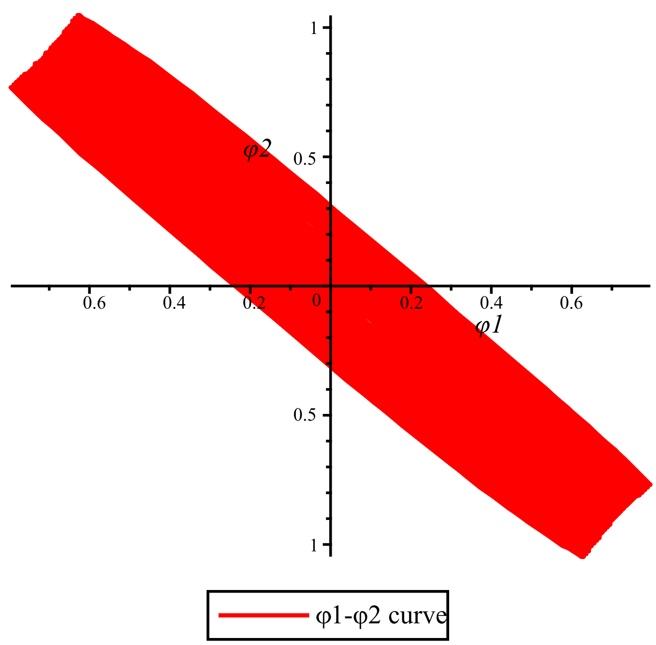


Let's try another set of initial conditions:

$IC2 := \varphi1(0) = \frac{\mathrm{Pi}}{5},\ \mathrm{D}(\varphi1)(0) = 0,\ \varphi2(0) = \frac{\mathrm{Pi}}{3},\ \mathrm{D}(\varphi2)(0) = 0$

$$IC2 := \varphi1(0) = \frac{\pi}{5},\ \mathrm{D}(\varphi1)(0) = 0,\ \varphi2(0) = \frac{\pi}{3},\ \mathrm{D}(\varphi2)(0) = 0 \tag{8}$$

```
sol2 := dsolve({Eq1, Eq2, IC2}, numeric, maxfun = 0) :
odeplot(sol2, [φ1(t), φ2(t)], t = 0..200, numpoints = 4000, color = red)
```



Sensitivity to initial conditions

These are two initial conditions that are very close to NewIC:

$IC3 := \varphi 1(0) = \frac{\mathrm{Pi}}{2},\ \mathrm{D}(\varphi 1)(0) = 0,\ \varphi 2(0) = \mathrm{Pi} + 0.001,\ \mathrm{D}(\varphi 2)(0) = 0$

$$IC3 := \varphi 1(0) = \frac{\pi}{2},\ \mathrm{D}(\varphi 1)(0) = 0,\ \varphi 2(0) = 3.142592654,\ \mathrm{D}(\varphi 2)(0) = 0 \tag{9}$$

$IC4 := \varphi 1(0) = \frac{\mathrm{Pi}}{2},\ \mathrm{D}(\varphi 1)(0) = 0,\ \varphi 2(0) = \mathrm{Pi} + 0.002,\ \mathrm{D}(\varphi 2)(0) = 0$

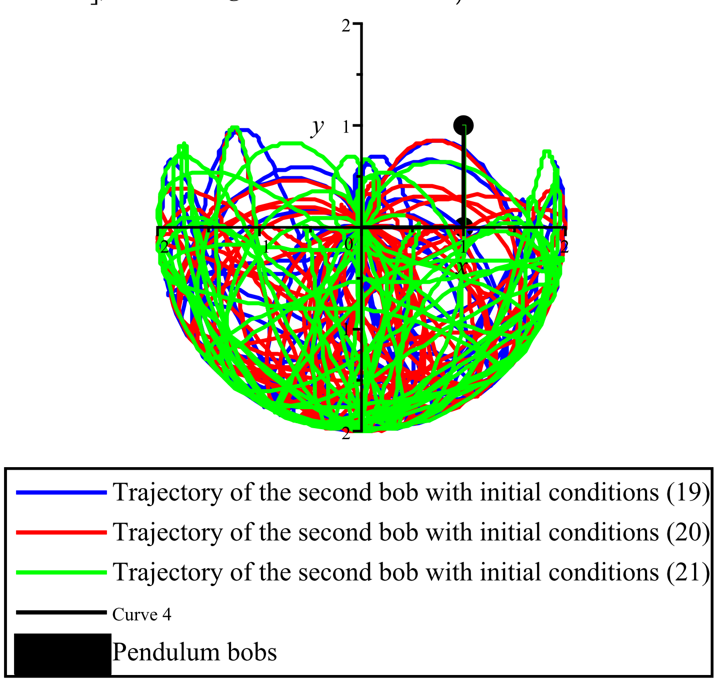
$$IC4 := \varphi 1(0) = \frac{\pi}{2},\ \mathrm{D}(\varphi 1)(0) = 0,\ \varphi 2(0) = 3.143592654,\ \mathrm{D}(\varphi 2)(0) = 0 \tag{10}$$

Compute the numerical solutions:

```
sol3 := dsolve({Eq1, Eq2, IC3}, numeric, output = listprocedure, maxfun = 0) :
sol4 := dsolve({Eq1, Eq2, IC4}, numeric, output = listprocedure, maxfun = 0) :
```

Plot the trajectories:

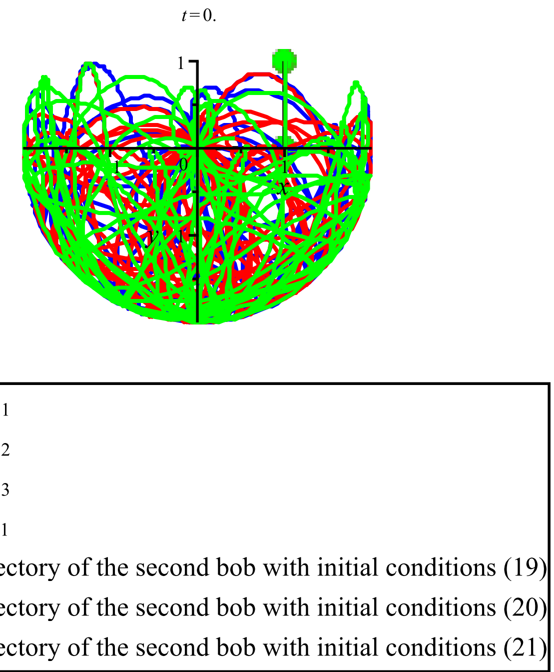
```
trajectory32 := odeplot(sol3, [sin(φ1(t)) + sin(φ2(t)), -cos(φ1(t)) - cos(φ2(t))], t
    = 0..40, numpoints = 2000, color = red) :
trajectory42 := odeplot(sol4, [sin(φ1(t)) + sin(φ2(t)), -cos(φ1(t)) - cos(φ2(t))], t
    = 0..40, numpoints = 2000, color = green) :
Newd13 := disk([1, 0], 0.1, color = black) :
Newd23 := disk([1, 1], 0.1, color = black) :
Newl13 := line([0, 0], [1, 0], color = black, linestyle = solid) :
Newl23 := line([1, 0], [1, 1] , color = black, linestyle = solid) :
display(Newtrajectory2, trajectory32, trajectory42, Newd13, Newd23, Newl13, Newl23,
    view = [-2..2, -2..2], scaling = constrained)
```



The animated plot:

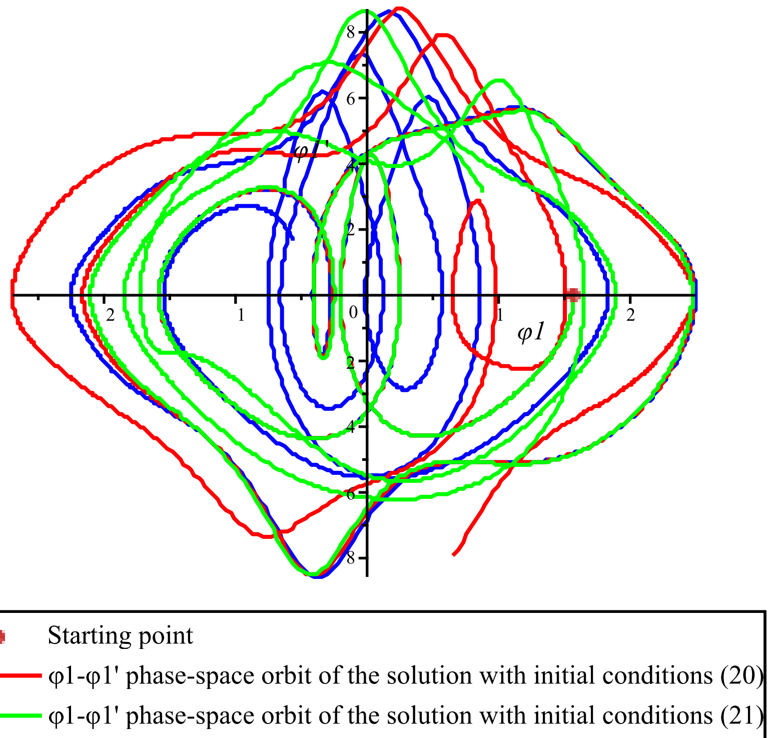
```
TT31, TT32 := op(subs(sol3, [φ1(t), φ2(t)])) :
TT41, TT42 := op(subs(sol4, [φ1(t), φ2(t)])) :
ini19 := t → plots:-display(
          pointplot([sin(NewTT1(t)) + sin(NewTT2(t)), -cos(NewTT1(t))
    -cos(NewTT2(t))],
                    color = blue, symbol = solidcircle, symbolsize = 25),
          pointplot([sin(NewTT1(t)), -cos(NewTT1(t))],
                    color = blue, symbol = solidcircle, symbolsize = 25),
          plottools:-line([0, 0], [sin(NewTT1(t)), -cos(NewTT1(t))]),
          plottools:-line([sin(NewTT1(t)), -cos(NewTT1(t))],
                         [sin(NewTT1(t)) + sin(NewTT2(t)), -cos(NewTT1(t))
    -cos(NewTT2(t))]),
          scaling = constrained
        ) :
 ini20 := t → plots:-display(
          pointplot([sin(TT31(t)) + sin(TT32(t)), -cos(TT31(t))-cos(TT32(t))],
                    color = red, symbol = solidcircle, symbolsize = 25),
          pointplot([sin(TT31(t)), -cos(TT31(t))],
                    color = red, symbol = solidcircle, symbolsize = 25),
          plottools:-line([0, 0], [sin(TT31(t)), -cos(TT31(t))]),
          plottools:-line([sin(TT31(t)), -cos(TT31(t))],
                         [sin(TT31(t)) + sin(TT32(t)), -cos(TT31(t))
    -cos(TT32(t))]),
          scaling = constrained
        ) :
 ini21 := t → plots:-display(
          pointplot([sin(TT41(t)) + sin(TT42(t)), -cos(TT41(t))-cos(TT42(t))],
                    color = green, symbol = solidcircle, symbolsize = 25),
          pointplot([sin(TT41(t)), -cos(TT41(t))],
                    color = green, symbol = solidcircle, symbolsize = 25),
          plottools:-line([0, 0], [sin(TT41(t)), -cos(TT41(t))]),
          plottools:-line([sin(TT41(t)), -cos(TT41(t))],
                         [sin(TT41(t)) + sin(TT42(t)), -cos(TT41(t))
    -cos(TT42(t))]),
          scaling = constrained
        ) :

animate([ini19, ini20, ini21], [t], t = 0..40, frames = 200, background
    = [Newtrajectory2, trajectory32, trajectory42]);
```

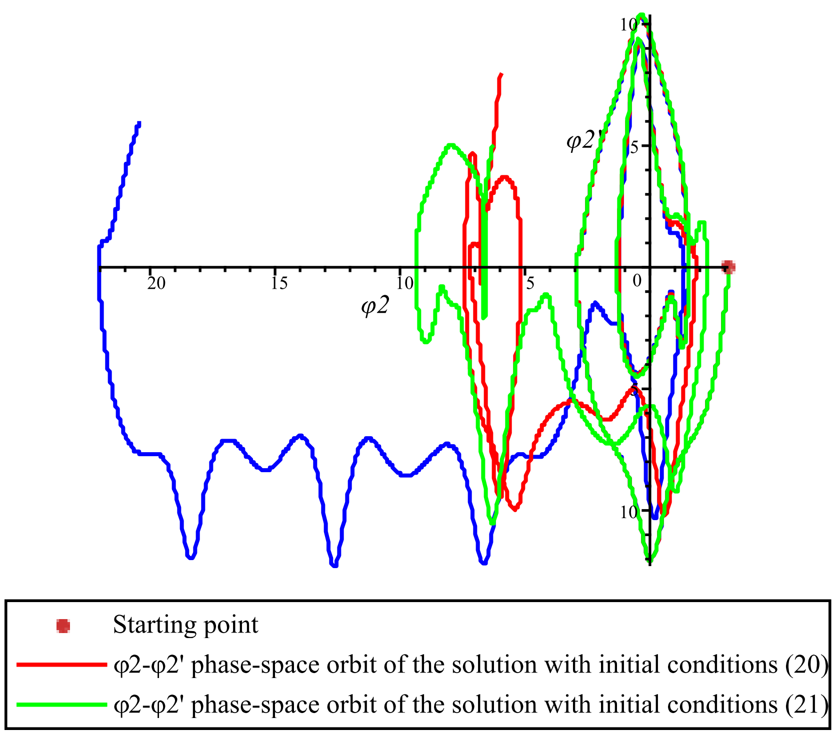


The phase plane plots:

```
phase11 := odeplot(Newsol, [φ1(t), diff(φ1(t), t)], t = 0..10, numpoints = 2000,
    color = blue) :
phase21 := odeplot(sol3, [φ1(t), diff(φ1(t), t)], t = 0..10, numpoints = 2000,
    color = red) :
phase31 := odeplot(sol4, [φ1(t), diff(φ1(t), t)], t = 0..10, numpoints = 2000,
    color = green) :
display(phase11, phase21, phase31, pointplot([Pi/2, 0],
    color = orange, symbol = solidcircle, symbolsize = 15))
```




```
phase12 := odeplot(Newsol, [φ2(t), diff(φ2(t), t)], t = 0..10, numpoints = 2000,
    color = blue) :
phase22 := odeplot(sol3, [φ2(t), diff(φ2(t), t)], t = 0..10, numpoints = 2000,
    color = red) :
phase32 := odeplot(sol4, [φ2(t), diff(φ2(t), t)], t = 0..10, numpoints = 2000,
    color = green) :
display(phase12, phase22, phase32, pointplot([Pi, 0],
                    color = orange, symbol = solidcircle, symbolsize = 15))
```



The Layapunov exponent

The Δφ2 - t curve:

$exponentplot := plot(\ln(\mathrm{abs}(TT32(t) - NewTT2(t))),\ t = 0..20,\ color = blue):$

This is an approximate fitting straight line:

$approxfitline := plot(1.1 \cdot t\ -6.5,\ t = 0..10,\ color = red):$

Display the plots:

$display(exponentplot,\ approxfitline)$