Introduction

Analysis & Design of Algorithms

24 de marzo de 2024

Exercise 1. The *floor* of a real number x, denoted by $\lfloor x \rfloor$, is the maximum integer that is less or equal to x. The *ceil* of a real number x, denoted by $\lceil x \rceil$, is the minimum integer that is greater or equal to x.
Prove the following inequalities, given that x, y are arbitrary real numbers, n is an arbitrary integer number, and a, b are arbitrary nonzero integers

(*a*) $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$

(*b*) $\lceil x \rceil + \lceil y \rceil - 1 \leq \lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil$

(*c*) $(n-1)/2 \leq \lfloor n/2 \rfloor \leq n/2$

(*d*) n, $n/2 \leq \lceil n/2 \rceil \leq (n+1)/2$

(*e*) n, $n = \lfloor n/2 \rfloor + \lceil n/2 \rceil$

(*f*) $x \in \mathbb{R}, a, b \in \mathbb{Z}$, $\lceil \frac{\lceil x/a \rceil}{b} \rceil = \lceil \frac{x}{ab} \rceil$

Exercise 2. Prove by induction that

(*a*) $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$, por all positive integers n.

(*b*) $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$, for all positive integers n.

(*c*) $n \leq 2^{n/2}$, for all positive integers $n \geq 4$.

Exercise 3.

(*a*) Find a simple formula of the sum $\sum_{k=1}^{n}(2k-1)$

(*b*) Show that $\sum_{k=0}^{\infty}(kx^k) = x/(1-x)^2$ when $|x| < 1$.

(*c*) Show that $\sum_{k=0}^{\infty}(k^2x^k) = x(1+x)/(1-x)^3$ when $|x| < 1$.

(*d*) Show that $\sum_{k=0}^{\infty}(k-1)/2^k = 0$.

(*e*) Find a simple form of $\prod_{k=2}^{n}(1 - 1/k^2)$

Exercise 4.

(a) Prove by induction that, for some constant c we have that $\sum_{k=0}^{n}(3^k) \leq c3^n \quad \forall n \in \mathbb{Z}^+$

(b) Show that $\sum_{k=1}^{\infty}(k/3^k) \leq 1$.

(c) Show that $\sum_{k=1}^{n} k \geq (n/2)^2$.

(d) Show that for some constant c, it holds $\sum_{k=1}^{n}(1/k^2) \leq c \quad \forall n \in \mathbb{Z}^+$.

(e) Find an upper bound for the sum $\sum_{k=0}^{\lfloor \lg n \rfloor} \lceil n/2^k \rceil$

Exercise 5. How many leaves has a full binary tree with n nodes? Use induction as one of the proofs

Exercise 6. Show that the height of a complete binary tree with k leaves is$\lg k$.