Probar que:

$$2^2 + 2^4 + 2^6 + \dots 2^{2k} = \left(\frac{4}{3}\right)(2^{2k}-1) \quad \Big\} P(n)$$

Caso base:

$$P(1) = 2^{2(1)} = 2^2 = \left(\frac{4}{3}\right)\left(2^{2(1)}-1\right) = 4 \checkmark$$

H.I:

$P(k)$ es true $\longrightarrow$ ¿ $P(n+1)$ es true?

$$2^2+2^4+2^6+\dots+2^{2k} = \frac{4}{3}(2^{2k}-1) \longrightarrow 2^2+2^4+2^6+\dots+2^{2(k+1)}$$

$$¿\frac{4}{3}\left(2^{2(k+1)}-1\right)?$$

$$P(k+1) = \underbrace{2^2+2^4+2^6+\dots+2^{2k}}+2^{2(k+1)}$$

$$\frac{4}{3}(2^{2k}-1) + 2^{2k+2}$$

$$\frac{4}{3}2^{2k} - \frac{4}{3} + 2^{2k}\cdot 2^2$$

$$\frac{2}{3}2^{2k+1} - \frac{4}{3} + 2^{2k+1}\cdot 2$$

$$\frac{4}{3}\left(\frac{1}{2}2^{2k+1} - 1 + \frac{3}{2}2^{2k+1}\right)$$

$$\frac{4}{3}\left(\frac{4}{2}2^{2k+1} - 1\right)$$

$$\therefore P(k+1) = \frac{4}{3}\left(2^{2k+2}-1\right) = \underline{\frac{4}{3}\left(2^{2(k+1)}-1\right)} \quad D$$

Probar que $\lg(n) \leq \sqrt{n}$ $\}$ $P(n)$

$\lim_{n \to \infty} \frac{\lg(n)}{\sqrt{n}} = \lim_{n \to \infty}$

Caso Base $P(1)$

$\lg(1) \leq \sqrt{1}$

$0 \leq 1$ ✓

$\lg(n) \leq \sqrt{n}$

$2^{\lg n} \leq 2^{\sqrt{n}}$

$n \leq 2^{\sqrt{n}}$

Paso Inductivo

$P(n)$ es true $\longrightarrow$ ¿$P(n+1)$ es true?

$n \leq 2^{\sqrt{n}}$

$\lg(n) \leq \sqrt{n}$ $\qquad$ $\lg(n) \leq \sqrt{n} \leq \sqrt{n+1}$ $\forall n \geq 0$ $\qquad$ $n \leq$

$P(n+1) = \lg(n+1)$ $\qquad$ $\lg(n) \leq \sqrt{n+1}$

$2^{2\lg n} \leq 2^{n}$

$n+1$ $\qquad$ $2^{\sqrt{n+1}}$ $\qquad$ $2 \leq 3$

$2^{\sqrt{n}} + 1$ $\qquad$ $2^2 \leq 2^3$

$n^2 \leq 2^n$

$n > 4$

5^2 $\quad$ 2^5

$n+1 \leq 2^{\sqrt{n}} + 1 \leq 2^{\sqrt{n+1}}$

$\left[\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}\right] = \frac{2\sqrt{x}}{x} = \frac{2x^{1/2}}{x^1}$ $\quad$ 26 $\quad$ 64

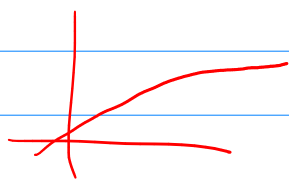
$\left(\lim_{n \to \infty} \frac{\lg n}{\sqrt{n}}\right)' \Rightarrow \lim_{n \to \infty} \left[\frac{\frac{1}{n\ln(2)}}{\frac{1}{2\sqrt{n}}}\right] = \frac{2\sqrt{n}}{n\ln(2)} = \frac{2}{\sqrt{n}\ln(2)}$

$(\lg n)' = \frac{1}{n\ln(2)}$ $\qquad$ $(\sqrt{n})' = \frac{1}{2} \cdot n^{\frac{1}{2}-1} = \frac{1}{2} n^{-1/2} = \frac{1}{2\sqrt{n}}$

Probar que: $\lg n \leq \sqrt{n}$

$\lim_{n \to \infty} \frac{\lg n}{\sqrt{n}} = \lim_{n \to \infty} \frac{2\sqrt{n}}{n\ln(2)} = \lim_{n \to \infty} \frac{2}{\sqrt{n}\ln(2)} = \frac{2}{\infty} \approx 0$

$\infty \cdot \ln(2) = \infty$



$\lg n \leq \sqrt{n}$ $\quad$ n

Simplifique: $\left\lfloor \frac{n+m}{2} \right\rfloor + \left\lfloor \frac{n-m+1}{2} \right\rfloor$

Caso 1:

n es par y m es par

$$\left.\begin{array}{l} n = 2a \\ m = 2b \end{array}\right\} \quad \left\lfloor \frac{\cancel{2}a + \cancel{2}b}{\cancel{2}} \right\rfloor$$

$$\left\lfloor \underbrace{\frac{2a}{2} - \frac{2b}{2}}_{\text{Entero}} + \frac{1}{2} \right\rfloor$$

$$a+b \quad + \quad \left\lfloor \frac{2a-2b+1}{2} \right\rfloor$$

$$\frac{2a}{2} - \frac{2b}{2} + \left\lfloor \frac{1}{2} \right\rfloor$$

$$a + \cancel{b} + a - \cancel{b} + 0$$

$$2a$$

$$\left\lfloor \frac{n+m}{2} \right\rfloor + \left\lfloor \frac{n-m+1}{2} \right\rfloor = 2a = n$$

Caso 2: n es par y m impar

$$\left.\begin{array}{l} n = 2a \\ m = 2b+1 \end{array}\right\} \quad \left\lfloor \frac{2a+2b+1}{2} \right\rfloor + \left\lfloor \frac{2a-2b-\cancel{1}+1}{2} \right\rfloor$$

$$a + \cancel{b} + 0 + a - \cancel{b}$$

$$2a$$

$$\left\lfloor \frac{n+m}{2} \right\rfloor + \left\lfloor \frac{n-m+1}{2} \right\rfloor = 2a = n$$

Caso 3: n es impar y m es par.

$$\left.\begin{array}{l} n = 2a+1 \\ m = 2b \end{array}\right\} \quad \left\lfloor \frac{2a+1+2b}{2} \right\rfloor + \left\lfloor \frac{2a+1-2b+1}{2} \right\rfloor$$

$$a + b + a - b$$

$$x-1<\lfloor x\rfloor \leq x \leq \lceil x\rceil < x+1$$

$\lfloor x\rfloor = a$

x es real:

$x = a + r \longrightarrow 0 \leq r < 1 \qquad \lfloor r\rfloor = 0$

$x = a + r$

$\lfloor x\rfloor = \lfloor a + r\rfloor$ ($a \in \mathbb{Z}$)

$\lfloor x\rfloor = \lfloor a\rfloor + \lfloor r\rfloor$ (a + 0)

$\lfloor x\rfloor = a + \lfloor r\rfloor \quad 0 \leq r \leq 1 \to \lfloor r\rfloor = 0$

$\lfloor x\rfloor \leq x$

$\lfloor x\rfloor = a$

$\lfloor x\rfloor = \lfloor a + r\rfloor$

$\lfloor x\rfloor = a + \lfloor r\rfloor$

$x = a + r \qquad a \leq a + r$

$\lfloor x\rfloor \leq x$

Obs: $\lfloor x + a\rfloor = \lfloor x\rfloor + a$ (entero)

$\lceil x + a\rceil = \lceil x\rceil + a$

Como: $r < 1$

$r - 1 < 0 \quad (+a)$

$\underbrace{a + r}_{} - 1 < a$

$x - 1 < \lfloor x\rfloor$

$r < 1$

$r - 1 < 0 \quad (+a)$

$\Rightarrow \quad x - 1 < \lfloor x\rfloor \leq x$

Demostra $\quad x \leq \lceil x\rceil$

Como $x = a + r \longrightarrow \lceil x\rceil = \lceil a + r\rceil \qquad r \to 0 < r < 1 \Rightarrow \lceil r\rceil = 1$

$\lceil x\rceil = a + \lceil r\rceil$

$\lceil x\rceil = a + 1$

$r < 1 \quad (+a) \qquad$ si x es entero

$\underbrace{a + r}_{} < a + 1 \qquad x = \lceil x\rceil$

$x < \lceil x\rceil$

$x \leq \lceil x\rceil$

Demostrar que $\quad \lceil x\rceil < x + 1 \qquad \lceil x\rceil = a + 1$

Se sabe: $\quad 0 < r < 1$

$a < a + r \quad (+1) \qquad$ Si: x es entero $\qquad x < x + 1$

$a + 1 < a + r + 1 \qquad \lceil x\rceil = x \qquad \lceil x\rceil < x + 1$ ✓

$\lceil x\rceil < x + 1$ ✓

Exr 1.43 Mostre que, para qualquer número inteiro $n \geq 1$,

$$\frac{n-1}{2} \leq \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} \quad \text{e} \quad \frac{n}{2} \leq \left\lceil \frac{n}{2} \right\rceil \leq \frac{n+1}{2}.$$

1 > Demostrado plantilla ADA

Exr 1.44 Seja n um inteiro positivo. Prove que $\lfloor \lfloor n/2 \rfloor / 2 \rfloor = \lfloor n/4 \rfloor$.

$$\frac{n}{2} - 1 < \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} \times \frac{1}{2}$$

$\lfloor x - y \rfloor$

Demostrar que:

$\lfloor x \rfloor - \lfloor y \rfloor \leq \lfloor x - y \rfloor$

$$\frac{n}{4} - \frac{1}{2} < \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \leq \frac{n}{4}$$

$\lfloor x \rfloor = m$ $\quad m \leq x < m+1$

$\lfloor y \rfloor = n$ $\quad n \leq y < n+1$

saco techo

$$\left\lfloor \frac{n}{4} - \frac{1}{2} \right\rfloor < \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor \leq \left\lfloor \frac{n}{4} \right\rfloor$$

$m \leq x$

$n \leq y$

$$\left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{1}{2} \right\rfloor \leq \left\lfloor \frac{n}{4} - \frac{1}{2} \right\rfloor$$

$\left\lfloor \frac{n}{4} \right\rfloor - 0$

$m - n \leq x - y$

$\lfloor x \rfloor - \lfloor y \rfloor \leq \lfloor x - y \rfloor$

$$\therefore \left\lfloor \frac{n}{4} \right\rfloor < \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor \leq \left\lfloor \frac{n}{4} \right\rfloor$$

Otra solución:

$$\left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor = \left\lfloor \frac{n}{4} \right\rfloor \quad \square$$

$a = \left\lfloor \frac{n}{2} \right\rfloor$ $\quad a \leq \frac{n}{2} < a+1$

$a \leq \frac{n}{2} \rightarrow \frac{a}{2} \leq \frac{n}{4}$

$b = \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor = \left\lfloor \frac{a}{2} \right\rfloor$ $\quad b \leq \frac{a}{2} < b+1$

$\rightarrow b \leq \frac{a}{2} \leq \frac{n}{4} \rightarrow b \leq \frac{n}{4}$

$a < 2(b+1)$

$c = \left\lfloor \frac{n}{4} \right\rfloor$ $\quad c \leq \frac{n}{4} < c+1$

$a \leq 2(b+1) - 1$

$a+1 \leq 2(b+1)$

¿ $b = c$?

$\frac{a+1}{2} \leq b+1$

$$\left(\frac{n}{2} < a+1\right) \times \frac{1}{2} = \frac{n}{4} < \frac{a+1}{2} \leq b+1$$

$\frac{n}{4} < b+1$

$\therefore b \leq \frac{n}{4} < b+1 \quad \therefore \left\lfloor \frac{n}{4} \right\rfloor = b$

$$\left\lfloor \frac{n}{4} \right\rfloor = \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor \quad \square$$

Probar que $\left\lceil \frac{\lceil \frac{x}{a} \rceil}{b} \right\rceil = \left\lceil \frac{x}{ab} \right\rceil$

$x \in \mathbb{R}$, $a, b \in \mathbb{Z}^+$

Sol:

$m = \lceil \frac{x}{a} \rceil \rightarrow m-1 < \frac{x}{a} \leq m \rightarrow \left(m \geq \frac{x}{a}\right) \times \frac{1}{b}$

$\frac{m}{b} \geq \frac{x}{ab}$

transitividad

$n = \left\lceil \frac{\lceil \frac{x}{a} \rceil}{b} \right\rceil = \lceil \frac{m}{b} \rceil \rightarrow n-1 < \frac{m}{b} \leq n \Rightarrow n \geq \frac{x}{ab}$

$p = \lceil \frac{x}{ab} \rceil \rightarrow p-1 < \frac{x}{ab} \leq p$

Probar que: $p = n$

1° Sabemos que: $n \geq \frac{m}{b} \geq \frac{x}{ab}$

Obs: si $a, b \in \mathbb{Z}$

$a < b \Leftrightarrow a \leq b-1$

$a+1 \leq b$

2° $n-1 < \frac{m}{b} \rightarrow (n-1)b < m$

$(n-1)b \leq m-1$

$n-1 \leq \frac{m-1}{b}$

De $\left(m-1 < \frac{x}{a}\right) \times \frac{1}{b}$

$\frac{m-1}{b} < \frac{x}{ab}$

por transitividad

$n-1 < \frac{x}{ab}$

o.d. $n-1 < \frac{x}{ab} \leq n$

d.d. $\lceil \frac{x}{ab} \rceil = n$ $\Big\}$ $\lceil \frac{x}{ab} \rceil = \left\lceil \frac{\lceil \frac{x}{a} \rceil}{b} \right\rceil$

$$\nexists \left\lceil \frac{n}{9} \right\rceil + \left\lfloor \frac{8n}{9} \right\rfloor = 1 \quad \forall n \in \mathbb{N} \,;\, n \neq 0$$

Parafraseo

$$\exists\, \forall n \quad n \neq q$$

$$\left\lceil \frac{n}{9} \right\rceil = 1 \quad \wedge \quad \left\lfloor \frac{8n}{9} \right\rfloor = 0$$

$\lceil m \rceil = a \rightarrow a-1 < m \leq a$

$\lfloor m \rfloor = a \longrightarrow a \leq m < a+1$

$$0 < \frac{n}{9} \leq 1 \qquad 0 \leq \frac{8n}{9} < 1$$

$$0 < \frac{n}{9} \leq 1$$

$$0 \leq \frac{8n}{9} < 1$$

$+$

$$0 < n < 2$$

$\hookrightarrow \in \mathbb{N}$

$n = 1$

Sea k un nº natural No nulo. Es cierto que

$$\left\lceil \frac{n}{k} \right\rceil + \left\lfloor \frac{(k-1)n}{k} \right\rfloor = 1 \quad \forall n \in \mathbb{N}$$

$\underset{0}{1} \qquad \underset{1}{0}$

$n=1$

$\cancel{k}n < 2\cancel{k}$

$n < 2$

Si: $\left\lceil \frac{n}{k} \right\rceil = 1 \wedge \left\lfloor \frac{(k-1)n}{k} \right\rfloor = 0$

$n < k$

$(k-1)n < k$

$$\underbrace{0 < \frac{n}{k} \leq 1 \qquad 0 \leq \frac{(k-1)n}{k} < 1}$$

$$\left.\begin{array}{l} 0 < \frac{n}{k} \leq 1 \\ 0 \leq \frac{(k-1)n}{k} < 1 \end{array}\right\downarrow + \Rightarrow 0 < \frac{\cancel{n} + kn - \cancel{n}}{k} < 2$$

$0 < n < 2$

$n=1$

Falta resolver

Exr. 151: Sea x un numero real, $m,n \in \mathbb{Z}^+$

Mostrar que $\left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor$

$m,n \in \mathbb{Z}^+$

$$\lfloor x+m \rfloor = \lfloor x \rfloor + m$$

$\frac{1}{n} \leq 1 ; \forall n \in \mathbb{Z}^+$

$$x-1 < \lfloor x \rfloor \leq x \quad (+m)$$

$$x+m-1 < \lfloor x \rfloor + m \leq x+m \quad (\times \tfrac{1}{n})$$

$\frac{x+m}{n} - 1 \leq \frac{x+m}{n} - \frac{1}{n}$

$$\frac{x+m}{n} - \frac{1}{n} < \frac{\lfloor x \rfloor + m}{n} \leq \frac{x+m}{n}$$

Por transitividad

$$\frac{x+m}{n} - 1 < \frac{\lfloor x \rfloor + m}{n} \leq \frac{x+m}{n}$$

Saco ~~techo~~ piso

$$\left\lfloor \frac{x+m}{n} - 1 \right\rfloor < \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor \leq \left\lfloor \frac{x+m}{n} \right\rfloor$$

$\lfloor x-a \rfloor = \lfloor x \rfloor - a$

$$\left\lfloor \frac{x+m}{n} \right\rfloor - 1 < \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor \leq \left\lfloor \frac{x+m}{n} \right\rfloor$$

$x-1 < \lfloor x \rfloor \leq x \quad (-a)$

$$\therefore \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor = \left\lfloor \frac{x+m}{n} \right\rfloor \quad \square$$

$x-a$

$a \in \mathbb{Z}$

$m \in \mathbb{Z}$

$\lfloor x \rfloor = m \Rightarrow m \leq x < m+1$

$m-a \leq x-a < m-a+1$

Saco piso

$m-a \leq \lfloor x-a \rfloor < m-a+1$

$\therefore \lfloor x-a \rfloor = m-a = \lfloor x \rfloor - a \quad \square$

Convertir $\left\lceil \frac{a}{b} \right\rceil = \left\lfloor \frac{a+b-1}{b} \right\rfloor \longrightarrow$

$a, b \in \mathbb{Z}$

$\left\lceil \frac{a}{b} \right\rceil = m \longrightarrow m-1 < \frac{a}{b} \leq m$

$\underbrace{\left\lfloor \frac{a-1}{b} \right\rfloor}_{\text{Entero}} < \underbrace{\left\lceil \frac{a}{b} \right\rceil}_{\text{Entero}}$

$\left\lfloor \frac{a+b-1}{b} \right\rfloor = n \longrightarrow n \leq \frac{a+b-1}{b} < n+1$

$\left\lfloor \frac{a-1}{b} \right\rfloor + 1 \leq \left\lceil \frac{a}{b} \right\rceil$

$\frac{a}{b} + 1 - \frac{1}{b}$

$\leq \left\lfloor \frac{a}{b} \right\rfloor \leq \left\lceil \frac{a}{b} \right\rceil$

$\left\lfloor \frac{a-1}{b} \right\rfloor + 1$

$\left\lfloor \frac{a-1}{b} \right\rfloor < \left\lfloor \frac{a}{b} \right\rfloor$

$< \frac{a}{b} + 1$