

Probar que:

$$2^2 + 2^4 + 2^6 + \dots + 2^{2n} = \left(\frac{4}{3}\right)(2^{2n+1} - 1) \quad \} P(n)$$

Caso base:

$$P(1) = 2^{2(1)} = 2^2 = \left(\frac{4}{3}\right)(2^{2(1)+1} - 1) = 4 \quad \checkmark$$

H.I:

$P(k)$ es true \longrightarrow ¿ $P(k+1)$ es true?

$$\hookrightarrow 2^2 + 2^4 + 2^6 + \dots + 2^{2k} = \frac{4}{3}(2^{2k+1} - 1) \longrightarrow 2^2 + 2^4 + 2^6 + \dots + 2^{2(k+1)}$$

$$\hookrightarrow \frac{4}{3}(2^{2(k+1)+1} - 1) ?$$

$$P(k+1) = \underbrace{2^2 + 2^4 + 2^6 + \dots + 2^{2k}}_{\frac{4}{3}(2^{2k+1} - 1)} + 2^{2k+2}$$

$$\frac{4}{3}2^{2k+1} - \frac{4}{3} + 2^{2k+2}$$

$$\frac{2}{3}2^{2k+1} - \frac{4}{3} + 2^{2k+1} \cdot 2$$

$$\frac{4}{3} \left(\frac{1}{2} 2^{2k+1} - 1 + \frac{3}{2} 2^{2k+1} \right)$$

$$\frac{4}{3} \left(\frac{4}{2} 2^{2k+1} - 1 \right)$$

$$\text{o sea } P(k+1) = \frac{4}{3} (2^{2k+2} - 1) = \frac{4}{3} (2^{2(k+1)+1} - 1) \quad \text{D} \quad \checkmark$$

Probar que $\lg(n) \leq \sqrt{n}$ } $P(n)$

$$\lim_{n \rightarrow \infty} \frac{\lg(n)}{\sqrt{n}} = \lim_{n \rightarrow \infty}$$

Caso Base $P(1)$

$$\lg(1) \leq \sqrt{1}$$

$$0 \leq 1 \quad \checkmark$$

$$\lg(n) \leq \sqrt{n}$$

$$2^{\lg n} \leq 2^{\sqrt{n}}$$

$$n \leq 2^{\sqrt{n}}$$

Paso Inductivo

$P(n)$ es true

\rightarrow ¿ $P(n+1)$ es true?

$$n \leq 2^{\sqrt{n}}$$

$$\lg(n) \leq \sqrt{n}$$

$$\lg(n) \leq \sqrt{n} \leq \sqrt{n+1} \quad \forall n \geq 0$$

$$P(n+1) = \lg(n+1)$$

$$\lg(n) \leq \sqrt{n+1}$$

$$n+1 \leq 2^{\sqrt{n+1}}$$

$$2^{\sqrt{n}} + 1$$

$$2 \leq 3$$

$$2^2 \leq 2^3$$

$$2^{2^{\lg n}} \leq 2^n$$

$$n^2 \leq 2^n$$

$$n \geq 4$$

$$n+1 \leq 2^{\sqrt{n}} + 1 \leq 2^{\sqrt{n+1}}$$

$$\left[\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} \right] = \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}}$$

$$\left(\lim_{n \rightarrow \infty} \frac{\lg n}{\sqrt{n}} \right)' \Rightarrow \lim_{n \rightarrow \infty} \left[\frac{\frac{1}{n \ln(2)}}{\frac{1}{2\sqrt{n}}} \right] = \frac{2\sqrt{n}}{n \ln(2)} = \frac{2}{\sqrt{n} \ln(2)}$$

$$(\lg n)' = \frac{1}{n \ln(2)}$$

$$(\sqrt{n})' = \frac{1}{2} \cdot n^{\frac{1}{2}-1} = \frac{1}{2} n^{-1/2} = \frac{1}{2\sqrt{n}}$$

Probar que: $\lg n \leq \sqrt{n}$

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$$\lim_{n \rightarrow \infty} \frac{\lg n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{2\sqrt{n}}{n \ln(2)} = \lim_{n \rightarrow \infty} \frac{2}{\sqrt{n} \ln(2)} = \frac{2}{\infty} \approx 0$$

$$\lim_{n \rightarrow \infty} \lg n \leq \sqrt{n} \quad \square$$

Simplifique: $\left\lfloor \frac{n+m}{2} \right\rfloor + \left\lfloor \frac{n-m+1}{2} \right\rfloor$

Caso 1:

n es par y m es par

$$\left. \begin{array}{l} n = 2a \\ m = 2b \end{array} \right\} \left\lfloor \frac{2a+2b}{2} \right\rfloor$$

$$a+b + \left\lfloor \frac{2a-2b+1}{2} \right\rfloor$$

$$\frac{2a}{2} - \frac{2b}{2} + \left\lfloor \frac{1}{2} \right\rfloor$$

$$\cancel{a+b} + \cancel{a-b} + 0$$

$2a$

$$\left\lfloor \frac{n+m}{2} \right\rfloor + \left\lfloor \frac{n-m+1}{2} \right\rfloor = 2a = n$$

Caso

2: n es par y m impar

$$\left. \begin{array}{l} n = 2a \\ m = 2b+1 \end{array} \right\}$$

$$\left\lfloor \frac{2a+2b+1}{2} \right\rfloor + \left\lfloor \frac{2a-2b-1+1}{2} \right\rfloor$$

$$\cancel{a+b} + 0 + \cancel{a-b}$$

$2a$

$$\left\lfloor \frac{n+m}{2} \right\rfloor + \left\lfloor \frac{n-m+1}{2} \right\rfloor = 2a = n$$

Caso 3: n es impar y m es par:

$$\left. \begin{array}{l} n = 2a+1 \\ m = 2b \end{array} \right\} \left\lfloor \frac{2a+1+2b}{2} \right\rfloor + \left\lfloor \frac{2a+1-2b+1}{2} \right\rfloor$$

$$a+b + a-b$$

$$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$$

$$\lfloor x \rfloor = a$$

x es real:

$$x = a + r \rightarrow 0 \leq r < 1 \quad \lceil r \rceil = 0$$

$$\lfloor x \rfloor = \lfloor a + r \rfloor$$

$$\lfloor x \rfloor = a + \lfloor r \rfloor \rightarrow 0 \leq r < 1 \rightarrow \lfloor r \rfloor = 0$$

$$\lfloor x \rfloor = a$$

$$x = a + r$$

$$\lfloor x \rfloor = \lfloor a \rfloor + \lfloor r \rfloor$$

$$\lfloor x \rfloor \leq x$$

$$\lfloor x \rfloor = \lfloor a + r \rfloor$$

$$\lfloor x \rfloor = a + \lfloor r \rfloor$$

$$x = a + r \quad a \leq a + r$$

$$\lfloor x \rfloor \leq x$$

Obs: $\lfloor x+a \rfloor = \lfloor x \rfloor + a$

$$\lceil x+a \rceil = \lceil x \rceil + a$$

Como: $r < 1$

$$r-1 < 0 \quad (+a)$$

$$\lfloor a+r-1 \rfloor < a$$

$$x-1 < \lfloor x \rfloor$$

$$r < 1$$

$$r-1 < 0 \quad (+a)$$

$$\Rightarrow x-1 < \lfloor x \rfloor \leq x$$

Demuestra $x \leq \lceil x \rceil$

Como $x = a + r \rightarrow \lceil x \rceil = \lceil a + r \rceil \rightarrow r \rightarrow 0 \leq r < 1 \Rightarrow \lceil r \rceil = 1$

$$\lceil x \rceil = a + \lceil r \rceil$$

$$\lceil x \rceil = a + 1$$



$$r < 1 \quad (+a) \quad \text{si } x \text{ es entero}$$

$$a + r < a + 1$$

$$x < \lceil x \rceil$$

$$x = \lceil x \rceil$$

$$x \leq \lceil x \rceil$$

Demstrar que $\lceil x \rceil < x+1$ $\lceil x \rceil = a+1$

Se vale:

$$a < a + r \rightarrow 0 \leq r < 1 \quad (+1)$$

$$a+1 < a+r+1$$

$$\lceil x \rceil < x+1 \quad \checkmark$$

si:
 x es entero

$$\lceil x \rceil = x$$

$$x < x+1$$

$$\lceil x \rceil < x+1 \quad \checkmark$$

Exr 1.43 Mostre que, para qualquer número inteiro $n \geq 1$,

$$\frac{n-1}{2} \leq \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} \quad \text{e} \quad \frac{n}{2} \leq \left\lceil \frac{n}{2} \right\rceil \leq \frac{n+1}{2}.$$

> Demonstrado plantilla ADA

Exr 1.44 Seja n um inteiro positivo. Prove que $\lfloor \lfloor n/2 \rfloor / 2 \rfloor = \lfloor n/4 \rfloor$.

$$\frac{n}{2} - 1 < \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} \times \frac{1}{2}$$

$$\frac{n}{4} - \frac{1}{2} < \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor \leq \frac{n}{4}$$

será
+ pch

$$\left\lfloor \frac{n}{4} - \frac{1}{2} \right\rfloor < \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor \leq \left\lfloor \frac{n}{4} \right\rfloor$$

$$\left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{1}{2} \right\rfloor \leq \left\lfloor n - \frac{1}{2} \right\rfloor$$

$$\left\lfloor \frac{n}{4} \right\rfloor - 0$$

$$\text{ou} \left\lfloor \frac{n}{4} \right\rfloor < \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor \leq \left\lfloor \frac{n}{4} \right\rfloor$$

$$\lfloor x - y \rfloor$$

Provar que:

$$\lfloor x \rfloor - \lfloor y \rfloor \leq \lfloor x - y \rfloor$$

$$\lfloor x \rfloor = m$$

$$\lfloor y \rfloor = n$$

$$m \leq x < m+1$$

$$n \leq y < n+1$$

$$m \leq x$$

$$m - n \leq x - y$$

$$\lfloor x \rfloor - \lfloor y \rfloor \leq \lfloor x - y \rfloor$$

Outra solução:

$$a = \left\lfloor \frac{n}{2} \right\rfloor$$

$$b = \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor = \left\lfloor \frac{a}{2} \right\rfloor$$

$$c = \left\lfloor \frac{n}{4} \right\rfloor$$

$$b = c?$$

$$a \leq \frac{n}{2} < a+1$$

$$b \leq \frac{a}{2} < b+1$$

$$c \leq \frac{n}{4} < c+1$$

$$a \leq \frac{n}{2} \rightarrow \frac{a}{2} \leq \frac{n}{4}$$

$$\frac{a}{2} \leq \frac{n}{4}$$

$$b \leq \frac{a}{2} \leq \frac{n}{4} \rightarrow b \leq \frac{n}{4}$$

$$a < 2(b+1)$$

$$a \leq 2(b+1) - 1$$

$$a+1 \leq 2(b+1)$$

$$\frac{a+1}{2} \leq b+1$$

$$\left(\frac{n}{2} < a+1 \right) \times \frac{1}{2} = \frac{n}{4} < \frac{a+1}{2} \leq b+1$$

$$\text{ou} \quad b \leq \frac{n}{4} < b+1 \quad \text{ou} \quad \left\lfloor \frac{n}{4} \right\rfloor = b$$

$$\left\lfloor \frac{n}{4} \right\rfloor = \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor = c$$

$$\frac{n}{4} < b+1$$

Probar que $\left\lceil \frac{\lceil \frac{x}{a} \rceil}{b} \right\rceil = \left\lceil \frac{x}{ab} \right\rceil$
 $x \in \mathbb{R}, a, b \in \mathbb{Z}^+$

Sol:

$$m = \left\lceil \frac{x}{a} \right\rceil \rightarrow m-1 < \frac{x}{a} \leq m \rightarrow \left(m \geq \frac{x}{a} \right) \times \frac{1}{b}$$

$$n = \left\lceil \frac{\lceil \frac{x}{a} \rceil}{b} \right\rceil = \left\lceil \frac{m}{b} \right\rceil \rightarrow n-1 < \frac{m}{b} \leq n \Rightarrow n \geq \frac{x}{ab}$$

$$p = \left\lceil \frac{x}{ab} \right\rceil \rightarrow p-1 < \frac{x}{ab} \leq p$$

Probar que: $(p=n)$

1° Sabemos que, $n \geq \frac{m}{b} \geq \frac{x}{ab}$

2° $n-1 < \frac{m}{b} \rightarrow (n-1)b < m$

$$(n-1)b \leq m-1$$

$$n-1 \leq \frac{m-1}{b}$$

$$n-1 < \frac{x}{ab}$$

Obs:

si $a, b \in \mathbb{Z}$

$$a < b \leftrightarrow a \leq b-1$$

$$a+1 \leq b$$

De $\left(m-1 < \frac{x}{a} \right) \times \frac{1}{b}$

$$\frac{m-1}{b} < \frac{x}{ab}$$

por transitividad

0° $n-1 < \frac{x}{ab} \leq n$

0° $\left\lceil \frac{x}{ab} \right\rceil = n \} \left\lceil \frac{x}{ab} \right\rceil = \left\lceil \frac{\lceil \frac{x}{a} \rceil}{b} \right\rceil$

$$\left\lceil \frac{n}{q} \right\rceil + \left\lfloor \frac{8n}{q} \right\rfloor = 1 \quad \forall n \in \mathbb{N}; n \neq 0$$

Parafraseo

$$\forall n \quad n \neq q$$

$$\lceil m \rceil = a \rightarrow a-1 < m \leq a$$

$$\lfloor m \rfloor = a \rightarrow a \leq m < a+1$$

$$\left\lceil \frac{n}{q} \right\rceil = 1 \quad \wedge \quad \left\lfloor \frac{8n}{q} \right\rfloor = 0$$

$$0 < \frac{n}{q} \leq 1$$

$$0 \leq \frac{8n}{q} < 1$$

$$\left. \begin{array}{l} 0 < \frac{n}{q} \leq 1 \\ 0 \leq \frac{8n}{q} < 1 \end{array} \right\} \downarrow +$$

$$0 < n < 2$$

$$n \in \mathbb{N}$$

$$n=1$$

Sea k un n^o natural No nulo. Es claro que

$$\begin{matrix} \left\lceil \frac{n}{k} \right\rceil & + & \left\lfloor \frac{(k-1)n}{k} \right\rfloor & = & 1 & \forall n \in \mathbb{N} \\ \begin{matrix} 1 \\ 0 \end{matrix} & & \begin{matrix} 0 \\ 1 \end{matrix} \end{matrix}$$

$$\textcircled{n=1}$$

$$\begin{matrix} k n \leq 2k \\ \textcircled{n < 2} \end{matrix}$$

Si : $\left\lceil \frac{n}{k} \right\rceil = 1 \quad \wedge \quad \left\lfloor \frac{(k-1)n}{k} \right\rfloor = 0 \quad \begin{matrix} n < k \\ (k-1)n < k \end{matrix}$

$$0 < \frac{n}{k} \leq 1 \quad \quad \quad 0 \leq \frac{(k-1)n}{k} < 1$$

$$\begin{matrix} 0 < \frac{n}{k} \leq 1 \\ 0 \leq \frac{(k-1)n}{k} < 1 \end{matrix} \downarrow + \Rightarrow \begin{matrix} 0 < \frac{n + (k-1)n}{k} < 2 \\ 0 < n < 2 \\ \textcircled{n=1} \end{matrix}$$

Falta resolver

Ex. 151: Sea x un número real, $m, n \in \mathbb{Z}^+$

Mostrar que $\left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor$

$m, n \in \mathbb{Z}$

$$\lfloor x+m \rfloor = \lfloor x \rfloor + m$$

$\frac{1}{n} \leq 1, \forall n \in \mathbb{Z}^+$

$$x-1 < \lfloor x \rfloor \leq x \quad (+m)$$

$$x+m-1 < \lfloor x \rfloor + m \leq x+m \quad \left(\frac{1}{n} \right)$$

$$\frac{x+m}{n} - \frac{1}{n} < \frac{\lfloor x \rfloor + m}{n} \leq \frac{x+m}{n}$$

Por transitividad

$$\frac{x+m}{n} - 1 < \frac{\lfloor x \rfloor + m}{n} \leq \frac{x+m}{n}$$

saca ~~todo~~
 piso

$$\left\lfloor \frac{x+m}{n} - 1 \right\rfloor < \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor \leq \left\lfloor \frac{x+m}{n} \right\rfloor$$

$$\lfloor x-a \rfloor = \lfloor x \rfloor - a$$

$$x-1 < \lfloor x \rfloor \leq x \quad (-a)$$

$$\left\lfloor \frac{x+m}{n} \right\rfloor - 1 < \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor \leq \left\lfloor \frac{x+m}{n} \right\rfloor$$

$x-a$
 $a \in \mathbb{Z}$
 $m \in \mathbb{Z}$
 $\lfloor x \rfloor = m \Rightarrow m \leq x < m+1$

$\square \square \square$
 $\left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor = \left\lfloor \frac{x+m}{n} \right\rfloor \quad \square \square$

$m-a \leq x-a < m-a+1$

$m-a \leq \lfloor x-a \rfloor < m-a+1$

$\square \square \square \lfloor x-a \rfloor = m-a = \lfloor x \rfloor - a \quad \square \square$

$a, b \in \mathbb{Z}$

Converting $\left\lceil \frac{a}{b} \right\rceil = \left\lfloor \frac{a+b-1}{b} \right\rfloor \rightarrow$

$$\left\lceil \frac{a}{b} \right\rceil = n \rightarrow$$

$$n-1 < \frac{a}{b} \leq n$$

$$\underbrace{\left\lfloor \frac{a-1}{b} \right\rfloor}_{\in \mathbb{N}_0} < \underbrace{\left\lfloor \frac{a}{b} \right\rfloor}_{\in \mathbb{N}_0}$$

$$\left\lfloor \frac{a+b-1}{b} \right\rfloor = n \rightarrow$$

$$n \leq \frac{a+b-1}{b} < n+1$$

$$\left\lfloor \frac{a-1}{b} \right\rfloor + 1 \leq \left\lceil \frac{a}{b} \right\rceil$$

$$\frac{a}{b} + 1 - \frac{1}{b}$$

$$\leq \left\lfloor \frac{a}{b} \right\rfloor \leq \left\lceil \frac{a}{b} \right\rceil$$

$$\left\lfloor \frac{a-1}{b} \right\rfloor + 1$$

$$\left\lfloor \frac{a-1}{b} \right\rfloor < \left\lfloor \frac{a}{b} \right\rfloor$$

$$< \frac{a}{b} + 1$$

