

$$a) \quad n^2 - 10n + 2 = O(n^2)$$

$$\rightarrow n^2 - 10n + 2 \leq 3n^2$$

$$+ \frac{n^2}{2} \leq n^2 \quad \forall n \geq 1$$

$$\frac{3}{2}n^2 \leq n^2 \quad \forall n \geq \sqrt{2}$$

$$\frac{3}{2}n^2 \leq n^2 \quad \forall n \geq 2$$

$$n^2 + 2 \leq 2n^2 \quad \forall n \geq 2$$

$$n^2 - 10n + 2 \leq 2n^2 \quad \forall n \geq 2$$

Para $C=2$ y $n_0=2$

$$b) \quad \left\lceil \frac{n}{2} \right\rceil = O(n)$$

$$\exists c, n_0 > 0 \quad \forall n \geq n_0 \rightarrow \left\lceil \frac{n}{2} \right\rceil \leq cn$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \left\lceil \frac{3n}{2} \right\rceil = \lceil n \rceil \leq n \Rightarrow \left\lceil \frac{n}{2} \right\rceil \leq n \quad \forall n \geq 1$$

$C=1$

$$c) \quad \lg n = O(\lg_{10} n)$$

$$\exists c, n_0 > 0 \quad \forall n \geq n_0 \rightarrow \lg n \leq c \lg_{10} n$$

$$\cancel{\lg n} \leq c \frac{\cancel{\lg n}}{\lg_2 10}$$

Para $C=4$ y $n_0=1$

$$\forall n \geq n_0$$

$$\lg n \leq 4 \lg_{10} n \quad \forall n \geq n_0$$

$$\lg 10 \leq C$$

$C=4$

(d) $n = O(2^n)$

$\exists c, n_0 > 0 \text{ t.q. } \forall n \geq n_0 \rightarrow n \leq c 2^n$

Para $C=1$ y $n_0=1$ cumple que: $n \leq 2^n$

Porque $\lg(n) \leq \lg(2^n)$

$\lg(n) \leq n \rightarrow$ es trivial
true

e) $\lg n$ no es $\Omega(n)$

$\lg n$ es $\Omega(n)$

$\rightarrow \exists c, n_0 > 0 \text{ t.q. } \forall n \geq n_0 \rightarrow cn \leq \lg n$

$c \leq \frac{\lg n}{n}$

$\lim_{n \rightarrow \infty} \frac{\lg n}{n} = 0$

$c \leq 0$

\rightarrow

f) $\frac{n}{100}$ no es $O(1)$

$\rightarrow \frac{n}{100}$ es $O(1)$

$\rightarrow \exists c, n_0 > 0 \text{ t.q. } \forall n \geq n_0 \rightarrow \frac{n}{100} \leq c$

$\rightarrow c$ depende de n
No sera constante

$$(h) \frac{1}{3}(n+1)(n-2) - 5 = \Theta(n^2).$$

$$\Rightarrow \frac{1}{3}(n^2 - n - 2) - 5 = \Theta(n^2)$$

$$\frac{1}{3}n^2 - \frac{1}{3}n - \frac{2}{3} - 5 = \Theta(n^2)$$

$$\frac{1}{3}n^2 - \frac{1}{3}n - \frac{17}{3} = \Theta(n^2)$$

$$\Rightarrow \exists c_1, c_2, n_0 > 0 \quad \forall n \geq n_0 \rightarrow c_1 n^2 \leq \frac{1}{3}n^2 - \frac{1}{3}n - \frac{17}{3} \leq c_2 n^2$$

Para big-O

$$\frac{1}{3}n^2 - \frac{1}{3}n - \frac{17}{3} \leq c_2 n^2$$

si quito según siendo menor

$$\left(\frac{1}{3}n^2 \leq n^2 \quad \forall n \geq 1 \wedge c_2 = 1 \right)$$

$$\frac{1}{3}n^2 - \frac{1}{3}n - \frac{17}{3} \leq n^2 \quad \checkmark$$

Para Big-Ω:

$$c_1 n^2 \leq \underbrace{\frac{1}{3}n^2}_{\text{torta}} - \frac{1}{3}n - \frac{17}{3}$$

los limitamos a mitad de torta

$$\frac{1}{2}n \leq \frac{1}{4}\left(\frac{1}{3}n^2\right) \rightarrow 4 \leq n$$

$$\frac{17}{2} \leq \frac{1}{4}\left(\frac{1}{3}n^2\right) \rightarrow \sqrt{17 \times 4} \leq n$$

Para $n \geq 9$

$$8,24 \leq n$$

$$\hookrightarrow 9 \leq n$$

$$\frac{1}{3}n^2 - \frac{1}{3}n - \frac{17}{3} \geq \frac{1}{4}\left(\frac{1}{3}n^2\right) - \frac{1}{4}\left(\frac{1}{3}n^2\right) - \frac{1}{4}\left(\frac{1}{3}n^2\right)$$

$$\frac{1}{3}n^2 - \frac{1}{3}n - \frac{17}{3} \geq \frac{4n^2 - n^2 - n^2}{12} = \frac{2n^2}{12} = \left(\frac{1}{6}\right)n^2$$

o sea para $c_1 = \frac{1}{6}$ y $c_2 = 1$ y $n_0 = 9$

$$\Rightarrow \frac{1}{6}n^2 \leq \frac{1}{3}n^2 - \frac{1}{3}n - \frac{17}{3} \leq n^2 \Rightarrow$$

$$\frac{1}{3}(n+1)(n-2) - 5 = \Theta(n^2) \quad \square$$

(i) $n \lg n - \lceil 2n/3 \rceil - \lg n + 4$ es $\Omega(2n \lg n)$

$$\exists c, n_0 > 0 \text{ t.q. } \forall n \geq n_0 \rightarrow c 2n \lg n \leq n \lg n - \lceil \frac{2n}{3} \rceil - \lg n + 4$$

$$c 2n \lg n \leq \underbrace{n \lg n - \lceil \frac{2n}{3} \rceil - \lg n + 4}_{\text{torta}} \xrightarrow{\text{acotamos a mitad de torta}}$$

$$1^\circ \lceil \frac{2n}{3} \rceil \leq \lceil \frac{3n}{3} \rceil = n \leq \frac{1}{4} n \lg n$$

$$1^\circ \lceil \frac{2n}{3} \rceil \leq \frac{1}{4} n \lg n$$

$$2^\circ \lg n \leq \frac{1}{4} n \lg n$$

$$(3) \quad 4 \leq \lg n$$

$$2^4 \leq n$$

$$2^\circ \lg n \leq \frac{1}{4} n \lg n$$

$$4 \leq n$$

$$\Rightarrow n_0 = 2^4$$

$$n \lg n - \lceil \frac{2n}{3} \rceil - \lg n + 4 \geq n \lg n - \frac{1}{4} n \lg n - \frac{1}{4} n \lg n = \frac{1}{2} n \lg n$$

$$= \left(\frac{1}{4} \cdot 2 \right) n \lg n$$

Para $c = \frac{1}{4}$ y $n_0 = 2^4$

$$c \cdot 2n \lg n \Rightarrow n \lg n - \lceil \frac{2n}{3} \rceil - \lg n + 4 \geq \left(\frac{1}{4} \right) 2n \lg n$$

o.o

$$n \lg n - \lceil \frac{2n}{3} \rceil - \lg n + 4 = \Omega(2n \lg n)$$

(j) $\lg n!$ es $\Omega(n \lg n)$

$$\lg n! \Rightarrow \lg n + \lg(n-1) + \lg(n-2) + \dots + \lg(1)$$

$$\sum_{i=1}^n \lg(i)$$

$$\sum_{i=1}^n \lg(i) = \Omega(n \lg n)$$

$$\Rightarrow \exists c, n_0 > 0 \text{ t.q. } \forall n \geq n_0 \rightarrow c n \lg n \leq \sum_{i=1}^n \lg(i)$$

$$\sum_{i=1}^n \lg(i) = \sum_{i=1}^n \lg(i) \leq \sum_{i=1}^n \lg(i)$$

$$\sum_{i=1}^n \lg(i) \leq \sum_{i=1}^n \lg(n) \leq \sum_{i=1}^n \lg(i)$$

$$n - \frac{n}{2} + 1$$

$$\frac{n}{2} \lg n \leq \lg n + \lg(n-1)$$

$$\frac{n}{2} (\lg(n) - \lg n) \leq \lg(n-1)$$

$$\sum_{i=1}^n \lg(n) + \sum_{i=1}^{\frac{n}{2}} \lg(i) \leq$$

$$2^{\frac{n}{2} \lg n} \leq 2$$

$$\sqrt{2^{n \lg n}} \leq$$

Ejercicio 2. Demostrar

(a) $\lg \sqrt{n} = O(\lg n)$

$\Rightarrow \exists c, n_0 > 0, \forall n \geq n_0 \rightarrow (\lg \sqrt{n}) \leq c \lg n$

$$\frac{1}{2} \lg n \leq c \lg n$$

Es verdad para $c=1$ y $n_0=1$

ya que tendremos: $\frac{1}{2} \lg n \leq \lg n \rightarrow$ Esto es verdad

ya que la función
logarítmica es
creciente

(b) Si $f(n) = O(g(n))$ y $g(n) = O(h(n))$ entonces $f(n) = O(h(n))$

$$f(n) = O(g(n))$$

$$\hookrightarrow f(n) \leq C_1 g(n)$$

$$g(n) = O(h(n))$$

$$\hookrightarrow (g(n) \leq C_2(h(n))) \times C_1$$

$$C_1 g(n) \leq C_1 \times C_2 h(n)$$

$$f(n) \leq C_1 g(n) \leq C h(n)$$

$$\underbrace{\hspace{10em}}$$

$$f(n) \leq C h(n) \Rightarrow \underline{f(n) = O(h(n))}$$

(c) Si $f(n) = O(g(n))$ y $g(n) = \Theta(h(n))$ entonces $f(n) = \Theta(h(n))$

Si $f(n) = O(g(n))$

$$\hookrightarrow f(n) \leq C_1 g(n)$$

$\wedge g(n) = \Theta(h(n))$

$$C_2 h(n) \leq g(n) \leq C_3 h(n)$$

$$\Rightarrow \{ f(n) = \Theta(h(n)) \}$$

$$n^2 - 2n \leq n^2$$

$$\frac{1}{2}(n^2 + 2) \leq n^2 \leq n^2 + 2$$

$$n^3 \leq n \leq n^3$$

$$n^2 + 2 \leq n^2 - 2n$$

(d) Si $f(n) = O(g(n))$ entonces $2^{f(n)} = O(2^{g(n)})$

Esto
Va a cumplir
para algun $c \geq 2$

$$2 \lg n \leq c \lg n \rightarrow 2^{2 \lg n} \leq 2^{c \lg n} ?$$

$n^2 \leq n \quad (\rightarrow \leftarrow) \quad \text{no}$

(f) $\max\{f(n), g(n)\} = \Theta(f(n) + g(n))$, para funciones no negativas $f(n)$ y $g(n)$.

Es cierto que $f(n) \leq f(n) + g(n) \wedge g(n) \leq f(n) + g(n)$

$$\text{Oo} \quad \max(f(n), g(n)) \leq f(n) + g(n)$$

$$\hookrightarrow \max(f(n), g(n)) = O(f(n) + g(n))$$

Es cierto que $f(n) + g(n) \leq 2 \max(f(n), g(n))$

$$\frac{1}{2}(f(n) + g(n)) \leq \max(f(n), g(n))$$

$$\Rightarrow \max(f(n), g(n)) = \Omega(f(n) + g(n))$$

$$\text{Oo} \quad \max(f(n), g(n)) = \underline{\Theta(f(n) + g(n))} \quad \Delta$$

(g) $(n+a)^b = \Theta(n^b)$, donde $a, b \in \mathbb{R}$ y $b > 0$.

$$C_1 n^b \leq n^b + \dots + a^b \leq C_2 n^b$$

$C_1 = 1$ cumple

Big-O:

$$\underbrace{n^b + a^b + \dots}_{b+1 \text{ términos}} \leq C_2 n^b$$

$$+ \downarrow \quad b+1 \text{ términos} \left\{ \begin{array}{l} n^b \leq n^b \\ a^b \leq n^b \\ \vdots \\ \leq n^b \end{array} \right. \quad \left. \begin{array}{l} \forall n \geq 1 \\ \forall n \geq a \end{array} \right\}$$

Para algún $n \geq X$
siendo X el máximo
valor de cada cota que
puede tomar n

$$(n+a)^b \leq (b+1)n^b$$

Para un $C_2 = b+1$

$$\hookrightarrow (n+a)^b \leq (b+1)n^b$$

$$\text{od } (n+a)^b = \Theta(n^b)$$

$$(h) \sqrt{n} = O(\lg^2 n).$$

(F)

Contradicción:

$$\exists c, n_0 > 0 \rightarrow \nexists n \geq n_0$$

$$\left[\frac{(\sqrt{n})'}{(\lg n)'} = \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n \ln 2}} \right]$$

$$\lg n = \frac{1}{\ln 2^n} \cdot \frac{n \ln 2}{2\sqrt{n}}$$

$$\frac{\sqrt{n} \ln 2}{2} = \infty$$

$$2^{n+1} - 1$$

$$(i) \sum_{k=1}^n k^{99} = \Theta(n^{100})$$

$$\underbrace{1 + 2^{99} + \dots + n^{99}}_{n^{100}}$$

↪

$$\exists c_1, c_2, n_0 > 0 \text{ s.t. } \forall n \geq n_0 \Rightarrow c_1 n^{100} \leq \sum_{k=1}^n k^{99} \leq c_2 n^{100}$$

(j) Suponga que $\lg(g(n)) \geq 1$ y que $f(n) \geq 1$ para todo n suficientemente grande. En ese caso, si $f(n) = O(g(n))$ entonces $\lg(f(n)) = O(\lg(g(n)))$.

$$\left. \begin{array}{l} \lg(g(n)) \geq 1 \\ 2^{\lg(g(n))} \geq 2^1 \\ \textcircled{g(n) \geq 2} \end{array} \right\} \begin{array}{l} f(n) \geq 1 \\ \lg(f(n)) \geq 0 \end{array}$$

Si:

$$\Rightarrow f(n) = O(g(n)) \longrightarrow \text{¿} \lg(f(n)) = O(\lg(g(n)))?$$

$$\exists c, n_0 > 0, \forall n \geq n_0$$

$$\hookrightarrow f(n) \leq c g(n)$$

$$\lg()$$

$$\lg(f(n)) \leq \lg(c g(n))$$

$$\lg(f(n)) \leq \lg c + \lg(g(n))$$

Queremos que

$$c \leq \lg(g(n))$$

$$\lg(f(n)) \leq \overset{c}{(c)} + \lg(g(n))$$

$$2^c \leq g(n)$$

$$\text{o sea } g(n) \geq 2^c$$

$$\lg(f(n)) \leq c + \lg(g(n))$$

$$\leq \lg(g(n)) + \lg(g(n))$$

$$\lg(f(n)) \leq 2 \lg(g(n)) \longrightarrow \text{para un } g(n) \geq 2^c$$

$$2^n + n^{2^0} = O(3^n)$$

$$2^n \leq 3^n \quad \checkmark$$

$$n^{2^0} \leq 3^n$$

↪ Para um $n = 100$

$$10^{40} \leq 3^{100}$$

$$2n^2 + n \lg n = \Theta(n^2)$$

$$2n^2 \leq 2n^2 \quad \forall n \geq 1$$

$$+ \quad (\lg n \leq n) \times n$$

$$\hookrightarrow n \lg n \leq n^2 \quad \forall n \geq 1$$

$$2n^2 + n \lg n \leq 3n^2 \quad \forall n \geq 1 \quad \wedge \quad \underline{C = 3}$$

$$n^{1 + \frac{2}{\sqrt{\lg n}}} = O(n \lg n) \quad (V)$$

$$\exists c, n_0 > 0 \text{ tal que } \forall n \geq n_0 \longrightarrow n^{1 + \frac{2}{\sqrt{\lg n}}} \leq c n \lg n$$

$$(\sqrt{\lg n}) \geq 2$$

$$\lg n \geq 4$$

$$n \geq 16$$

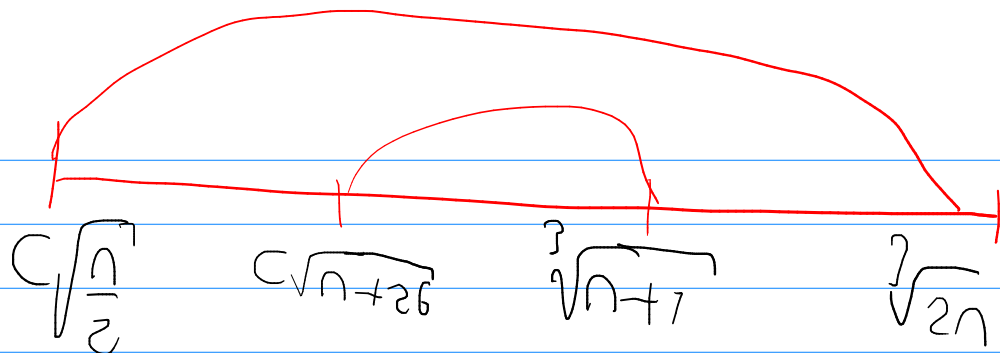
$$n^{1 + \frac{2}{\sqrt{\lg n}}} = O(n \lg n)$$

Para um $n \geq 2^{256}$

$$n^{1 + \frac{2}{2}}$$

$$n^{\frac{2}{\sqrt{\lg n}}} \leq \lg n$$

$$n^{0.3} < n^{0.5} \quad \checkmark$$



Por contradicción

$$\hookrightarrow \sqrt[3]{n+1} \leq \Omega(\sqrt{n-26})$$

$$\hookrightarrow \exists c, n_0 > 0 \text{ tq } \forall n \geq n_0 \longrightarrow \sqrt[3]{n+1} \geq c\sqrt{n-26} \geq 0$$

Enunciando el contraejemplo

$$\hookrightarrow \text{Sea } n = n_0 + 2c$$

la idea es la siguiente

$$C\frac{1}{2}\sqrt{n} \leq C\sqrt{\frac{1}{2}}\sqrt{n} = C\sqrt{\frac{n}{2}} \leq C\sqrt{n-26} \leq \sqrt[3]{n+1} \leq \sqrt[3]{2n} \leq 2\sqrt[3]{n}$$

$$\frac{n}{2} \leq n-26$$

$$n \leq 2n-52$$

$$\underline{n \geq 52}$$

$$C\frac{1}{2}\sqrt{n} \leq 2\sqrt[3]{n}$$

$$\left(\frac{C}{4} \sqrt{n} \leq \sqrt[3]{n} \right)^6$$

$$\left(\frac{C}{4} \right)^6 n^3 \leq n^2$$

$$n \leq \left(\frac{4}{C} \right)^6$$

$$n^{1.001} + n \lg n = O(n \lg n)$$

Por límites

$$\text{Si: } \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0 \quad \text{true}$$

Si no: FALSE

Segundo límites esto es falso

→ Por contradicción

$$\exists n_0, c > 0 \text{ t.q. } \forall n \geq n_0 \quad 0 < n^{1.001} + n \lg n \leq c n \lg n$$

$$\frac{n^{1.001}}{\lg n} + 1 \leq c$$

Para un

$$n = 2^{\frac{c}{0.001}}$$

$$\frac{2^{\frac{c}{0.001} \cdot 0.001}}{\lg 2^{\frac{c}{0.001}}} \Rightarrow \frac{2^c}{\frac{c}{0.001}} + 1 \leq c$$

$$\left(\frac{2^c}{10^3 c} + 1 \leq c \right) ?$$

$$\rightarrow (\sqrt{n+20})(\sqrt{n-24}) = \Theta(0.2024n)$$

$$\exists c_1, c_2, n_0 \text{ t.g. } \forall n \geq n_0$$

$$(\sqrt{n+20})(\sqrt{n-24}) \leq c_2(0.2024n)$$

$$\times \updownarrow \sqrt{n+20} \leq \frac{1}{0.2024} \cdot 0.2024 \sqrt{2n}$$

$$\sqrt{n-24} \leq \sqrt{2n} \quad \forall n \geq 25^2$$

$$(\sqrt{n+20})(\sqrt{n-24}) \leq \frac{1}{0.2024} \cdot 0.2024 \sqrt{4n^2}$$

$$(\sqrt{n+20})(\sqrt{n-24}) \leq \frac{2}{0.2024} \cdot 0.2024 n$$

↓
C

Big-Ω

$$c_1 0.2024n \leq (\sqrt{n+20})(\sqrt{n-24})$$

$$0.2024\sqrt{n} \leq \sqrt{n} \leq \sqrt{n+20} \quad \forall n \geq 1$$

$$\sqrt{n} \leq \sqrt{n-24} \quad \forall n \geq 1$$

$$0.2024\sqrt{n} \leq \sqrt{n+20} \quad \forall n \geq 1$$

$$\sqrt{n} \leq \sqrt{n-24} \quad \forall n \geq 25$$

$$0.2024n \leq (\sqrt{n+20})(\sqrt{n-24}) \quad \forall n \geq 1$$

$$2n^3 - 2023n^2 - 2023^2 n \lg n + 10^{10} \lg^2 n + 1 = O\left(\frac{1}{2} n^3\right)$$

$$- \overline{10^{10} \lg^2 n} \leq 10^{10} \cancel{n^2} \leq \cancel{n^3}$$

$$n \geq 10^{10}$$

$$n \geq 10^{10}$$

$$- 1 \leq n^2 \quad \forall n \geq 1$$

$$2n^3 - 2023n^2 - 2023^2 n \lg n + 10^{10} \lg^2 n + 1 \leq 2n^3 + n^3 + n^3$$

$$\leq 4n^3 = 4 \cdot 2^{2023} \cdot \frac{1}{2^{2023}} n^3$$

$$= 2^2 \cdot 2^{2023} \cdot \left(\frac{1}{2}\right)^{2023} n^3$$

$$2n^3 - 2023n^2 - 2023^2 n \lg n + 10^{10} \lg^2 n + 1 \leq 2^{2025} \left(\frac{1}{2}\right)^{2023} n^3$$

Para $n \geq C = 2^{2025}$ e $n_0 = 10^{10}$,

$$(a) \ n = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor + \left\lfloor \frac{n+2}{3} \right\rfloor \text{ for all } n \in \mathbb{Z}$$

1°

$$n = 3k \rightarrow k \in \mathbb{Z}$$

$$k + \left\lfloor \frac{1}{3} \right\rfloor$$

$$k + \left\lfloor \frac{2}{3} \right\rfloor$$

$$\left\lfloor \frac{3k}{3} \right\rfloor + \left\lfloor \frac{3k+1}{3} \right\rfloor + \left\lfloor \frac{3k+2}{3} \right\rfloor$$

$$k + k + k$$

$$3k = n \quad \checkmark$$

2°

$$n = 3k+1$$

$$\left\lfloor \frac{3k+1}{3} \right\rfloor + \left\lfloor \frac{3k+2}{3} \right\rfloor + \left\lfloor \frac{3k+3}{3} \right\rfloor$$

$$k+0 + k+0 + k+1$$

$$3k+1 = n \quad \checkmark$$

3°

$$n = 3k+2$$

$$\left\lfloor \frac{3k+2}{3} \right\rfloor + \left\lfloor \frac{3k+3}{3} \right\rfloor + \left\lfloor \frac{3k+4}{3} \right\rfloor$$

$$k+0 + k+1 + k+1$$

$$3k+2 = n$$

$$\square \quad n = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor + \left\lfloor \frac{n+2}{3} \right\rfloor \quad \square$$

