Probar que:

$$2^2 + 2^4 + 2^6 + \dots 2^{2n} = \left(\frac{4}{3}\right)(2^{2k} - 1) \quad \Big\} \; P(n)$$

Caso base:

$$P(1) = 2^{2(1)} = 2^2 = \left(\frac{4}{3}\right)\left(2^{2(1)} - 1\right) = 4 \quad \checkmark$$

H.I:

$P(k)$ es true $\longrightarrow$ ¿ $P(n+1)$ es true?

$\quad \hookrightarrow 2^2 + 2^4 + 2^6 + \dots + 2^{2k} = \frac{4}{3}(2^{2k} - 1) \longrightarrow 2^2 + 2^4 + 2^6 + \dots + 2^{2(k+1)}$

$$\hookrightarrow \Big\} \frac{4}{3}\left(2^{2(k+1)} - 1\right) \, ?$$

$$P(k+1) = \underbrace{2^2 + 2^4 + 2^6 + \dots + 2^{2k}} + 2^{2(k+1)}$$

$$\frac{4}{3}(2^{2k} - 1) + 2^{2k+2}$$

$$\frac{4}{3} 2^{2k} - \frac{4}{3} + 2^{2k} \cdot 2^2$$

$$\frac{2}{3} 2^{2k+1} - \frac{4}{3} + 2^{2k+1} \cdot 2$$

$$\frac{4}{3}\left(\frac{1}{2} 2^{2k+1} - 1 + \frac{3}{2} 2^{2k+1}\right)$$

$$\frac{4}{3}\left(\frac{4}{2} 2^{2k+1} - 1\right)$$

$$\therefore \; P(k+1) = \frac{4}{3}\left(2^{2k+2} - 1\right) = \frac{4}{3}\left(2^{2(k+1)} - 1\right) \quad \square$$

Probar que $\lg(n) \leq \sqrt{n}$ $\}\, P(n)$

$$\lim_{n\to\infty} \frac{\lg(n)}{\sqrt{n}} = \lim_{n\to\infty}$$

Caso Base $P(1)$

$$\lg(1) \leq \sqrt{1}$$
$$0 \leq 1 \quad \checkmark$$

$\lg(n) \leq \sqrt{n}$

$2^{\lg n} \leq 2^{\sqrt{n}}$

$\boxed{n \leq 2^{\sqrt{n}}}$

Paso Inductivo

$P(n)$ es true $\longrightarrow$ ¿$P(n+1)$ es true? $\boxed{n \leq 2^{\sqrt{n}}}$

$\lg(n) \leq \sqrt{n}$ $\qquad$ $\lg(n) \leq \sqrt{n} \leq \sqrt{n+1}$ $\forall n \geq 0$ $\quad n \leq$

$P(n+1) = \lg(n+1)$ $\qquad$ $\lg(n) \leq \sqrt{n+1}$

$n+1 \qquad 2^{\sqrt{n+1}}$ $\qquad$ $2 \leq 3$ $\qquad 2^{\lg n} \leq 2^{n}$

$2^{\sqrt{n}} + 1$ $\qquad$ $2 \leq 2^{?}$ $\qquad \boxed{n^2 \leq 2^n}$

$n \geq 4$

$n+1 \leq 2^{\sqrt{n}} + 1 \leq 2^{\sqrt{n+1}}$ $\qquad \dfrac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}\Big] = \dfrac{2\sqrt{x}}{x} = \dfrac{2\cdot\frac{1}{2}}{x}$ $\quad 3^2 \quad 2^3 \quad 64$

$\left(\lim_{n\to\infty} \dfrac{\lg n}{\sqrt{n}}\right)' \Rightarrow \lim_{n\to\infty} \dfrac{\frac{1}{n\ln(2)}}{\frac{1}{2\sqrt{n}}}\Big] = \dfrac{2\sqrt{n}}{n\ln(2)} = \boxed{\dfrac{2}{\sqrt{n}\,\ln(2)}}$

$\left(\lg n\right)' = \dfrac{1}{n\ln(2)}$ $\qquad$ $\left(\sqrt{n}\right)' = \dfrac{1}{2}\cdot n^{\frac{1}{2}-1} = \dfrac{1}{2}n^{-1/2} = \dfrac{1}{2\sqrt{n}}$

Probar que: $\lg n \leq \sqrt{n}$

$\lim_{n\to\infty} \dfrac{\lg n}{\sqrt{n}} = \lim_{n\to\infty} \dfrac{2\sqrt{n}}{n\ln(2)} = \lim_{n\to\infty} \dfrac{2}{\sqrt{n}\,\ln(2)} = \dfrac{2}{\infty} \approx 0$

$\infty\cdot\ln(2) = \infty$

$\boxed{\lg n \leq \sqrt{n}}$

Simplifique: $\left\lfloor \dfrac{n+m}{2} \right\rfloor + \left\lfloor \dfrac{n-m+1}{2} \right\rfloor$

Caso 1:

n es par y m es par

$\left.\begin{array}{l} n = 2a \\ m = 2b \end{array}\right\}$ $\left\lfloor \dfrac{2a+2b}{2} \right\rfloor$

$\left\lfloor \dfrac{\frac{2a}{2} - \frac{2b}{2} + \frac{1}{2}}{\underbrace{\qquad}_{\text{entero}}} \right\rfloor$

$a+b +$ $\left\lfloor \dfrac{2a-2b+1}{2} \right\rfloor$

$\dfrac{2a}{2} - \dfrac{2b}{2} + \left\lfloor \dfrac{1}{2} \right\rfloor$

$a + \cancel{b} + a - \cancel{b} + 0$

$2a$

$\left\lfloor \dfrac{n+m}{2} \right\rfloor + \left\lfloor \dfrac{n-m+1}{2} \right\rfloor = 2a = n$

Caso 2: n es par y m impar

$\left.\begin{array}{l} n = 2a \\ m = 2b+1 \end{array}\right\}$ $\left\lfloor \dfrac{2a+2b+1}{2} \right\rfloor + \left\lfloor \dfrac{2a-2b-1+1}{2} \right\rfloor$

$a + \cancel{b} + 0 + a - \cancel{b}$

$2a$

$\left\lfloor \dfrac{n+m}{2} \right\rfloor + \left\lfloor \dfrac{n-m+1}{2} \right\rfloor = 2a = n$

Caso 3: n es impar y m es par:

$\left.\begin{array}{l} n = 2a+1 \\ m = 2b \end{array}\right\}$ $\left\lfloor \dfrac{2a+1+2b}{2} \right\rfloor + \left\lfloor \dfrac{2a+1-2b+1}{2} \right\rfloor$

$a + \cancel{b} + a - b$

$$x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$$

X es real:

$$x = a + r \longrightarrow 0 \le r < 1 \qquad \lfloor r \rfloor = 0$$

$$\lfloor x \rfloor = \lfloor a + r \rfloor \quad ^{Z}$$

$$\lfloor x \rfloor = a + \lfloor r \rfloor \longrightarrow 0 \le r < 1 \to \lfloor r \rfloor = 0$$

$$\lfloor x \rfloor = a$$

$$x = a + r \qquad a \le a + r$$

$$\lfloor x \rfloor \le x$$

$$\lfloor x \rfloor = a$$

$$x = a + r$$

$$\lfloor x \rfloor = \lfloor a \rfloor + \lfloor r \rfloor \quad _{a+0}$$

$$\lfloor x \rfloor \le x$$

$$\lfloor x \rfloor = \lfloor a + r \rfloor$$

$$\lfloor x \rfloor = a + \lfloor r \rfloor$$

Obs: $\lfloor x + a \rfloor = \lfloor x \rfloor + a$ ^{entuo} (circled)

$$\lceil x + a \rceil = \lceil x \rceil + a$$

Como: $r < 1$

$$r - 1 < 0 \quad (+a)$$

$$a + r - 1 < a$$

$$x - 1 < \lfloor x \rfloor$$

$r < 1$

$$r - 1 < 0 \quad (+a)$$

$$\Longrightarrow \quad x - 1 < \lfloor x \rfloor \le x$$

Demostra $\quad x \le \lceil x \rceil$

Como $x = a + r \longrightarrow \lceil x \rceil = \lceil a + r \rceil \quad r \to 0 < r < 1 \Longrightarrow \boxed{\lceil r \rceil = 1}$ (circled)

$$\lceil x \rceil = a + \lceil r \rceil$$

$$\lceil x \rceil = a + 1$$

$r < 1 \quad (+a) \qquad$ si x es entvo

$$a + r < a + 1 \qquad \qquad x = \lceil x \rceil$$

$$x < \lceil x \rceil \qquad \qquad x \le \lceil x \rceil$$

Demostrar que $\quad \lceil x \rceil < x + 1 \quad$ $\boxed{\lceil x \rceil = a + 1}$ (circled)

Se cale. $\qquad \qquad \quad _{0 < r < 1}$ $\qquad$ si:

$$a < a + r \quad (+1) \qquad \qquad x \text{ es entvo}$$

$$a + 1 < a + r + 1 \qquad \boxed{\lceil x \rceil = x}$$ (circled) $\qquad \begin{cases} x < x + 1 \\ \lceil x \rceil < x + 1 \quad \checkmark \end{cases}$

$$\lceil x \rceil < x + 1 \quad \checkmark$$

Exr 1.43 Mostre que, para qualquer número inteiro $n \geq 1$,

$$\frac{n-1}{2} \leq \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} \qquad \text{e} \qquad \frac{n}{2} \leq \left\lceil \frac{n}{2} \right\rceil \leq \frac{n+1}{2}.$$

| > Demostrado plantilla ADA

Exr 1.44 Seja n um inteiro positivo. Prove que $\lfloor \lfloor n/2 \rfloor / 2 \rfloor = \lfloor n/4 \rfloor$.

$$\frac{n}{2} - 1 < \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} \quad \times \frac{1}{2}$$

$\lfloor x - y \rfloor$

Demostrar que:

$\lfloor x \rfloor - \lfloor y \rfloor \leq \lfloor x - y \rfloor$

$$\frac{n}{4} - \frac{1}{2} < \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \leq \frac{n}{4}$$

saco techo (

$\lfloor x \rfloor = m$ $\quad m \leq x < m+1$

$\lfloor y \rfloor = n$ $\quad n \leq y < n+1$

$$\left\lfloor \frac{n}{4} - \frac{1}{2} \right\rfloor < \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor \leq \left\lfloor \frac{n}{4} \right\rfloor$$

$$\left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{1}{2} \right\rfloor \leq \left\lfloor \frac{n}{4} - \frac{1}{2} \right\rfloor$$

$$\left\lfloor \frac{n}{4} \right\rfloor - 0$$

$m \leq x$ y

$m - n \leq x - y$

$\lfloor x \rfloor - \lfloor y \rfloor \leq \lfloor x - y \rfloor$

$$\therefore \left\lfloor \frac{n}{4} \right\rfloor < \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor \leq \left\lfloor \frac{n}{4} \right\rfloor$$

$$\left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor = \left\lfloor \frac{n}{4} \right\rfloor \quad \square$$

Otra solución:

$a = \left\lfloor \frac{n}{2} \right\rfloor$ $\qquad a \leq \frac{n}{2} < a+1$

$b = \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor = \left\lfloor \frac{a}{2} \right\rfloor$ $\qquad b \leq \frac{a}{2} < b+1$

$c = \left\lfloor \frac{n}{4} \right\rfloor$ $\qquad c \leq \frac{n}{4} < c+1$

¿ $b = c$?

$a \leq \frac{n}{2} \rightarrow \frac{a}{2} \leq \frac{n}{4}$

$\rightarrow b \leq \frac{a}{2} \leq \frac{n}{4} \rightarrow b \leq \frac{n}{4}$

$a < 2(b+1)$

$a \leq 2(b+1) - 1$

$a + 1 \leq 2(b+1)$

$\frac{a+1}{2} \leq b+1$

$$\left(\frac{n}{2} < a+1\right) \times \frac{1}{2} = \frac{n}{4} < \underbrace{\frac{a+1}{2} \leq b+1}$$

$\frac{n}{4} < b+1$

$\therefore b \leq \frac{n}{4} < b+1 \quad \therefore \left\lfloor \frac{n}{4} \right\rfloor = b$

$$\left\lfloor \frac{n}{4} \right\rfloor = \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor \quad \square$$

Probar que $\left\lceil \dfrac{\left\lceil \frac{x}{a} \right\rceil}{b} \right\rceil = \left\lceil \dfrac{x}{ab} \right\rceil$

$x \in \mathbb{R}, \; a,b \in \mathbb{Z}^+$

Sol:

$m = \left\lceil \dfrac{x}{a} \right\rceil \longrightarrow \quad m-1 < \boxed{\dfrac{x}{a} \leq m} \longrightarrow \left(m \geq \dfrac{x}{a} \right) \times \dfrac{1}{b}$

$\dfrac{m}{b} \geq \dfrac{x}{ab}$

transitividad

$n = \left\lceil \dfrac{\left\lceil \frac{x}{a} \right\rceil}{b} \right\rceil = \left\lceil \dfrac{m}{b} \right\rceil \longrightarrow ② \quad n-1 < \dfrac{m}{b} \leq n \implies \boxed{n \geq \dfrac{x}{ab}}$

$P = \left\lceil \dfrac{x}{ab} \right\rceil \longrightarrow ③ \quad P-1 < \dfrac{x}{ab} \leq P$

Probar que: $\boxed{P = n}$

1° Sabemos que: $n \geq \dfrac{m}{b} \geq \dfrac{x}{ab}$

Obs:

si $a, b \in \mathbb{Z}$

$a < b \iff a \leq b-1$

$a+1 \leq b$

2° $n - 1 < \dfrac{m}{b} \rightarrow (n-1)b < m$

$(n-1)b \leq m-1$

$n-1 \leq \dfrac{m-1}{b}$

$\left. \begin{array}{c} \\ \\ \end{array} \right\}$ De $\left(m-1 < \dfrac{x}{a} \right) \times \dfrac{1}{b}$

$n-1 < \dfrac{x}{ab} \xrightarrow[\text{transitividad}]{\text{por}} \dfrac{m-1}{b} < \dfrac{x}{ab}$

∴ $n-1 < \dfrac{x}{ab} \leq n$

∴ $\left\lceil \dfrac{x}{ab} \right\rceil = n \left. \begin{array}{c} \\ \end{array} \right\} \left\lceil \dfrac{x}{ab} \right\rceil = \left\lceil \dfrac{\left\lceil \frac{x}{a} \right\rceil}{b} \right\rceil$

$$\exists \mid \left\lceil \frac{n}{9} \right\rceil + \left\lfloor \frac{8n}{9} \right\rfloor = 1 \quad \forall n \in \mathbb{N} \; ; \; n \neq 0$$

Parafraseo

$$\exists \quad \forall n \quad n \neq 9$$

$$\left\lceil \frac{n}{9} \right\rceil = 1 \quad \wedge \quad \left\lfloor \frac{8n}{9} \right\rfloor = 0$$

$$\lceil m \rceil = a \implies a-1 < m \leq a$$
$$\lfloor m \rfloor = a \implies a \leq m < a+1$$

$$0 < \frac{n}{9} \leq 1 \qquad\qquad 0 \leq \frac{8n}{9} < 1$$

$$0 < \frac{n}{9} \leq 1$$
$$0 \leq \frac{8n}{9} < 1 \qquad +$$

$$0 < n < 2$$

$$\hookrightarrow \in \mathbb{N}$$

$$\boxed{n = 1}$$

$$\left\lceil \frac{n}{k} \right\rceil + \left\lfloor \frac{(k-1)n}{k} \right\rfloor = 1 \qquad \forall n \in \mathbb{N}$$

$$\boxed{n = 1}$$

$$\underset{\underset{0}{1}}{} \qquad \underset{\underset{1}{0}}{}$$

$$kn < 2k$$

$$\boxed{n < 2}$$

Si: $\left\lceil \dfrac{n}{k} \right\rceil = 1 \quad \wedge \quad \left\lfloor \dfrac{(k-1)n}{k} \right\rfloor = 0$ $\qquad n < k$

$\qquad\qquad (k-1)n \cancel{\in} k$

$$0 < \frac{n}{k} \leq 1 \qquad\qquad 0 \leq \frac{(k-1)n}{k} < 1$$

$$0 < \frac{n}{k} \leq 1$$
$$0 \leq \frac{(k-1)n}{k} < 1 \quad \Big\downarrow + \quad \Rightarrow \quad 0 < \frac{\cancel{n} + kn - \cancel{n}}{k} < 2$$

$$0 < n < 2$$

$$\boxed{n = 1}$$

Falta resolver

Sea X un numero real, $m, n \in \mathbb{Z}^+$

Mostrar que $\left\lfloor \dfrac{x+m}{n} \right\rfloor = \left\lfloor \dfrac{\lfloor x \rfloor + m}{n} \right\rfloor$

$m, n \in \mathbb{Z}^+$

$$\lfloor X + m \rfloor = \lfloor X \rfloor + m$$

$\dfrac{1}{n} \leq 1 \; ; \; \forall n \in \mathbb{Z}^+$

$\dfrac{x+m}{n} - 1 \leq \dfrac{x+m}{n} - \boxed{\dfrac{1}{n}}$

Por transitividad

$x - 1 < \lfloor X \rfloor \leq x \qquad (+m)$

$x + m - 1 < \lfloor X \rfloor + m \leq x + m \qquad (\times \frac{1}{n})$

$\dfrac{x+m}{n} - \dfrac{1}{n} \lll \dfrac{\lfloor X \rfloor + m}{n} \leq \dfrac{x+m}{n}$

$\dfrac{x+m}{n} - 1 < \dfrac{\lfloor X \rfloor + m}{n} \leq \dfrac{x+m}{n}$

saco $+\!\!\!\!\!/\!\!\!\!/$ piso

$\left\lfloor \dfrac{x+m}{n} - 1 \right\rfloor < \left\lfloor \dfrac{\lfloor X \rfloor + m}{n} \right\rfloor \leq \left\lfloor \dfrac{x+m}{n} \right\rfloor$

$\left\lfloor \dfrac{x+m}{n} \right\rfloor - 1 < \left\lfloor \dfrac{\lfloor X \rfloor + m}{n} \right\rfloor \leq \left\lfloor \dfrac{x+m}{n} \right\rfloor$

$\therefore \left\lfloor \dfrac{\lfloor X \rfloor + m}{n} \right\rfloor = \left\lfloor \dfrac{x+m}{n} \right\rfloor \quad \square$

$\lfloor x - a \rfloor = \lfloor x \rfloor - a$

$x - 1 < \lfloor X \rfloor \leq x \qquad (-a)$

$x - a$

$a \in \mathbb{Z}$
$m \in \mathbb{Z}$

$\lfloor x \rfloor = m \Rightarrow m \leq x < m+1$

saco piso $\begin{cases} m - a \leq x - a < m - a + 1 \\ m - a \leq \lfloor x - a \rfloor < m - a + 1 \end{cases}$

$\therefore \lfloor x - a \rfloor = \underline{m - a} = \lfloor x \rfloor - a \quad \square$

Convertir $\left\lceil \dfrac{a}{b} \right\rceil = \left\lfloor \dfrac{a+b-1}{b} \right\rfloor \longrightarrow$

$a, b \in \mathbb{Z}$

$\left\lceil \dfrac{a}{b} \right\rceil = m \longrightarrow \qquad m - 1 < \dfrac{a}{b} \leq m \qquad \underbrace{\left\lfloor \dfrac{a-1}{b} \right\rfloor}_{\text{Entvo}} < \underbrace{\left\lceil \dfrac{a}{b} \right\rceil}_{\text{Entvo}}$

$\left\lfloor \dfrac{a+b-1}{b} \right\rfloor = n \longrightarrow \qquad n \leq \dfrac{a+b-1}{b} < n+1 \qquad \left\lfloor \dfrac{a-1}{b} \right\rfloor + 1 \leq \left\lceil \dfrac{a}{b} \right\rceil$

$\dfrac{a}{b} + 1 - \dfrac{1}{b} \qquad\qquad\qquad \leq \left\lfloor \dfrac{a}{b} \right\rfloor \leq \left\lceil \dfrac{a}{b} \right\rceil$

$\left\lfloor \dfrac{a-1}{b} \right\rfloor + 1 \qquad \left\lfloor \dfrac{a-1}{b} \right\rfloor < \left\lfloor \dfrac{a}{b} \right\rfloor \qquad\qquad < \dfrac{a}{b} + 1$