Quantum Mechanics and Free Fall

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1 Introduction

In this article, we consider an interesting problem, which, as noted by Nauenberg, is often overlooked: a formulation of the equivalence principle in non-relativistic quantum mechanics.[1]

First, we provide a brief summary of the history of the equivalence principle, and provide a more technically scoped definition for our uses in the paper. We specifically define the weak equivalence principle, which establishes the local indistinguishability between freely falling and inertial frames. In section three, we derive a solution to the one-dimensional non-relativistic Schrödinger equation for an uncharged particle falling in Earth's uniform gravitational field. We do so by considering a coordinate transformation to a freely falling frame; and then apply physical constraints to determine the phase of the wave function. In section four we determine the physical observables of the system, such as the probability distributions for the wave packet in both coordinate systems, and the classical expectation value for the position of the particle. Our discussion in section four provides an intuitive exposition of the observations made by both observers in the distinct frames. To illustrate the equivalence principle, we attempt to elucidate the physical explanation for the wave packet's evolution in both coordinate systems. We illustrate Ehrenfest's Theorem using a simulation, depicting the evolution of the probability distributions for the wave packet in each coordinate system. In section six, we offer a summary of our results, specifically stating a technical description of the equivalence principle in quantum mechanics. We provide a brief summary of the implications of the principle of stationary action in quantum mechanics, demonstrating the wave function's phase dependence on the classical action of the system. Using this result, we then discuss, briefly, a thought-experiment, indicating a potential divergence between the equivalence principle and quantum mechanics. In particular, we note that the phase of a wave function has a local dependence on the gravitational field strength, and as such an interferometry experiment, measuring wave

[1]Nauenberg, Michael. "Einstein's Equivalence Principle in Quantum Mechanics Revisited." American Journal of Physics 84, no. 11 (November 1, 2016): 879–82. https://doi.org/10.1119/1.4962981. I was first introduced to this paper by Griffiths, David. Schroeter, Darrell. Introduction to Quantum Mechanics. (Cambridge, United Kingdom: Cambridge University Press, 2018). 80-81, Problem 2.51.

interference, should be able to discriminate between two locally inertial frames of reference with distinct gravitational field strengths.

2 The Equivalence Principle

A consequence of the division between classical and quantum physics, at least anecdotally, one often thinks “classically” or “quantumly.” This confusion is mostly due to necessity; quantum mechanics provides radically different interpretations for the physical evolution of systems. This division, however, from my experience, has led me to not to consider the implications of a result. More generally, physicists and students often divide physics into tightly guarded regimes, where the laws of physics in this domain become rigid and insular. I should perhaps scope this statement more directly to the experience as a physics amateur and student. We often think of certain problems in physics as a problem of electrodynamics, or thermodynamics, or classical mechanics, or quantum mechanics, without admitting that the true physical description of the system is one that is governed by the underlying laws of nature, which certainly is democratic to all regimes of physics.

Einstein, during his development of a general theory of relativity, stated the equivalence principle, a principle that related the physics of systems in gravitational fields and those in uniformly accelerated fields, in the following way:

> *if we assume that the systems K and K' are physically equivalent, that is, if we assume that we may just as well regard the system K as being in a space free from gravitational fields, if we then regard K as uniformly accelerated. This assumption of exact physical equivalence makes it impossible for us to speak of the absolute acceleration of the system of reference, just as the usual theory of relativity forbids us to talk of the absolute velocity of the system; and it makes the equal falling of all bodies in a gravitational field seem a matter of course.*[2]

The equivalence principle had its roots in the writings of Galileo, who in 1638 broke with the scholastic Aristotelian view of the acceleration of bodies on Earth being proportional to their mass. He instead postulated the universality of free fall, but he did not comment on the source of such acceleration.[3] Newton, in the *Principia*, reinforced Galileo's universality of free fall and, in fact, proposed an early formulation of the equivalence principle and the equivalence of inertial motion and free fall[4] Einstein was driven by his belief in the equivalence principle, a heuristic that would make both accelerated motion and inertial equivalent.

[2]Albert Einstein, On the Influence of Gravitation on the Propagation of Light, 1911. Quoted from Stephen Hawking, On the Shoulders of Giants: The Great Works of Physics and Astronomy, (Philadelphia: Running Press, 2002), 1194.

[3]Darrigol, Olivier. Relativity principles and theories from Galileo to Einstein. (Oxford, United Kingdom: Oxford University Press, 2022.), 6-7.

[4]Darrigol, Relativity, 17.

Special relativity had equated those reference frames undergoing uniform motion. But Galileo's discovery of the constancy of free fall indicated that a more generalized principle could be extended, which in effect would dissolve the privilege between uniform and accelerated motion.[5]We will not discuss the general theory of relativity here but only state its core principle of covariance: that the laws of physics are the same in **all** frames of reference—accelerated and uniform.

For many physics students, the equality of inertial mass, the mass that is found in Newton's Second Law, and gravitational mass, which is used in the classical gravitational action, is taken as an axiom. But there is no reason for this to be true, and one could imagine an inequality between the two masses. However, experimental evidence, such as the Eötvös experiment[6], has provided justification for the equality to high precision, though minor discrepancies have led some to consider alternative "fifth forces."[7]

The equivalence principle is usually stated in the following way: "All freely falling, non-rotating (frames of reference) are equivalent for the performance of all physical experiments."[8] This asserts that a frame of reference, undergoing uniform acceleration due to gravity, is equivalent to an inertial frame of reference, where Newton's Second Law holds true. These frames, such as those like the falling elevator of Einstein, we call locally inertial frames of reference. It is local in the sense that the principle only holds for some finite region of the frame of reference. For instance, we know that the uniform acceleration due to gravity near the Earth's surface is, in fact, not entirely constant. Even in the pre-Einsteinian theory of gravity, the inverse square law dictates that the acceleration of a body will decrease as it moves away radially from the Earth's surface. However, because of the huge mass of the Earth, this difference is minuscule, and for all effective purposes, we can treat such experiments as though they existed in a uniform gravitational field. A more explicit definition of the equivalence principle is given in the formulation of the general theory of rela-

[5]Gutfreund, Hanoch. Renn, Jurgen. The Road to Relativity: The History and Meaning of Einstein's "The Foundation of General Relativity". (Oxford, United Kingdom: Princeton University Press, 2015.), 15-16.

[6]The Eötvös experiment, an interesting story in the annals of physics, was a relatively simple experiment aimed to falsify the equality of gravitational and inertial mass. Two masses, connected by a rod, were suspended by a fibre such that the rod could rotate freely. Attached to it was a mirror, and the apparatus was setup such that minor rotations in the torsion balance could be detected through the deflection of light on the mirror. In the non-inertial lab frame (the Earth), the forces acting on the system include the fibre tension, gravity, and a centrifugal force due to the Earth's rotation. The former forces are calculated using the gravitational mass, and the latter non-inertial force would use the inertial mass. If the rod is at rest in the lab frame, the net force must be zero. For this to be true, the gravitational and inertial masses must be equal. For an interesting historical sketch of the equivalence of gravitational and inertial masses, see Boniolo, Giovanni. "Theory and Experiment. The Case of Eötvös' Experiments." The British Journal for the Philosophy of Science 43, no. 4 (1992): 459–86. https://doi.org/10.1093/bjps/43.4.459.

[7]Schwarz, John. Schwarz, Patricia. Special Relativity: From Einstein to Strings. (Cambridge, United Kingdom: Cambridge University Press, 2004), 190.

[8]Rindler, Wolfgang. Relativity: Special, General, and Cosmological. (Oxford, United Kingdom: Oxford University Press, 2006), 19. I changed cabins to frames of reference for the usage in this paper.

tivity: that all events in spacetime can be regarded as a 4-dimensional manifold with four coordinates, and the equivalence principle states that for an event in spacetime one can define a locally inertial coordinate system, whereby the metric of the coordinate system becomes Lorentzian, and thus the laws of special relativity hold.[9]

The equivalence principle is often divided into two statements, a weak and a strong form. The weak form is the most common manifestation of the statement, which only equivocates gravity with inertial motion. Differing from this is the strong equivalence principle, which states that any law of physics that can be stated in a specific form in special relativity has an identical form in a locally inertial curved spacetime.[10] In this paper we will mostly be concerned with the weak equivalence principle.

Before we begin our calculations, it is important to summarize some of the ideas we have discussed. If the weak equivalence principle is true, then there should be no discrimination between freely falling frames and inertial frames. If there were a physical difference between such frames, then the equivalence principle, even in its weak form, would be contradicted. Similarly, we should be able to translate between inertial frames, which are permeated by a uniform gravitational field, and these freely falling frames of reference. As such, we should expect to recover, for a freely falling frame, some laws of physics that are identical to a frame we consider to be globally inertial.

3 Solving the Wave Function

We will consider the problem in one dimension, with the particle falling in the direction of the negative z-axis.[11] The Schrödinger equation is

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m_i}\frac{\partial^2\psi}{\partial z^2} + V\psi. \tag{1}$$

Here V is the gravitational potential

[9]In general relativity a set of spacetime coordinates, x^μ parameterize the spacetime manifold, which has a metric $g_{\mu\nu}(x)$. The spacetime interval is defined using the metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

The equivalence principle can be stated mathematically as: for a given event is spacetime, p, there exists a coordinate system, x^μ, such that p corresponds to the origin of the coordinate system, and the metric at the origin takes the form $g_{\mu\nu}(0) = \eta_{\mu\nu}$, which is the Lorentz metric. Thus the observer at p will recover the equations of motion dictated by special relativity. The coordinate system which satisfies this condition is said to be a tangent space to the spacetime manifold, incident at p. This tangent space is a Minkowski spacetime. A "strong" gravitational field corresponds to strong curvature on the manifold. For more exposition see Schwarz, Schwarz, Special Relativity, 313-314.

[10]Schutz, Bernard. A First Course in General Relativity. (Cambridge, United Kingdom: Cambridge University Press, 1990), 184. Schutz states the strong form of the equivalence principle as: "Any physical law which can be expressed in tensor notation in SR (special relativity) has exactly the same form in a locally inertial frame of curved spacetime."

[11]This calculation is carried out in a similar vein as Nauenberg.

$$V = m_g g z. \tag{2}$$

Now we consider the thought experiment of Einstein. We perform a coordinate transformation from the inertial lab frame to a non-inertial frame undergoing uniform acceleration. The new primed coordinate system is defined as follows:

$$z' = z + \frac{at^2}{2} \tag{3}$$

$$t' = t \tag{4}$$

We will write (1) in the new coordinate system, performing the required transformations on the unprimed coordinates. The potential (2) simply becomes

$$V = m_g g \left(z' - \frac{at'^2}{2} \right). \tag{5}$$

The partial derivatives undergo respective transformations in accordance with the chain (and product) rule. Explicitly, the temporal derivatives transform as

$$\frac{\partial \psi(z',t')}{\partial t} = \frac{\partial \psi}{\partial z'}\frac{\partial z'}{\partial t} + \frac{\partial \psi}{\partial t'}\frac{\partial t'}{\partial t} = \frac{\partial \psi}{\partial t'} + at'\frac{\partial \psi}{\partial z'}. \tag{6}$$

This reduces to

$$\frac{\partial \psi(z',t')}{\partial t} = \frac{\partial \psi}{\partial t'} + at'\frac{\partial \psi}{\partial z'}. \tag{7}$$

Likewise, the spatial derivatives are

$$\frac{\partial \psi(z',t')}{\partial z} = \frac{\partial \psi}{\partial z'}\frac{\partial z'}{\partial z} + \frac{\partial \psi}{\partial t'}\frac{\partial t'}{\partial z} = \frac{\partial \psi}{\partial z'} \tag{8}$$

since the primed temporal coordinate is independent of the unprimed spatial coordinate. It follows that the second derivative is simply

$$\frac{\partial^2 \psi(z',t')}{\partial z^2} = \frac{\partial^2 \psi}{\partial z'^2}. \tag{9}$$

Thus, the Schrodinger equation in (1) becomes

$$i\hbar \left(\frac{\partial \psi}{\partial t'} + at'\frac{\partial \psi}{\partial z'} \right) = -\frac{\hbar^2}{2m_i}\frac{\partial^2 \psi}{\partial z'^2} + m_g g \left(z' - \frac{at'^2}{2} \right) \psi. \tag{10}$$

This is the Schrödinger equation written in the primed coordinates. As we shall see shortly, this is the effective statement of the equivalence principle: that the laws of physics are covariant with regards to coordinate transformations. In this case, the evolution of the wave function, governed by the Schrödinger equation, is the "law of physics." So for this to be true, we should be able to produce a solution to (10), which we can then insert back into (1) under the correct coordinate transformations.

We assume an ansatz for the wave function of the form[12]

$$\psi(z', t') = \phi(z', t')e^{iS(z',t')}. \tag{11}$$

Here ϕ is the wave function for a free particle that is determined by the Schrödinger equation with zero potential:

$$i\hbar\frac{\partial \phi}{\partial t'} = -\frac{\hbar^2}{2m_i}\frac{\partial^2 \phi}{\partial z'^2}. \tag{12}$$

$S(z', t')$ is a real function of the primed coordinates. We will discuss this ansatz more in section five and the appendices, but for now we assume that this species of solution will hold. We will use the ansatz in (11) and insert it into (10). The time derivative is

$$\frac{\partial \psi}{\partial t'} = \frac{\partial}{\partial t'}\left(\phi(z', t')e^{iS(z',t')}\right) = \frac{\partial \phi}{\partial t'}e^{iS(z',t')} + i\frac{\partial S}{\partial t'}\phi e^{iS(z',t')}. \tag{13}$$

The spatial derivatives are

$$\begin{aligned}\frac{\partial \psi}{\partial z'} &= \frac{\partial}{\partial z'}\left(\phi(z', t')e^{iS(z',t')}\right)\\ &= \frac{\partial \phi}{\partial z'}e^{iS(z',t')} + i\frac{\partial S}{\partial z'}\phi e^{iS(z',t')}.\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\partial^2 \psi}{\partial z'^2} &= \frac{\partial}{\partial z'}\left(\frac{\partial \phi}{\partial z'}e^{iS(z',t')} + i\frac{\partial S}{\partial z'}\phi e^{iS(z',t')}\right)\\ &= \left(\frac{\partial^2 \phi}{\partial z'^2} + i\frac{\partial \phi}{\partial z'}\frac{\partial S}{\partial z'} + i\frac{\partial^2 S}{\partial z'^2}\phi + i\frac{\partial \phi}{\partial z'}\frac{\partial S}{\partial z'} - \left(\frac{\partial S}{\partial z'}\right)^2\phi\right)e^{iS(z',t')}\\ &= \left(\frac{\partial^2 \phi}{\partial z'^2} + 2i\frac{\partial \phi}{\partial z'}\frac{\partial S}{\partial z'} - \left(\frac{\partial S}{\partial z'}\right)^2\phi + i\frac{\partial^2 S}{\partial z'^2}\phi\right)e^{iS(z',t')}\end{aligned} \tag{15}$$

Using the derivatives in (10) we have

$$\begin{aligned}i\hbar\left(\frac{\partial \phi}{\partial t'} + i\frac{\partial S}{\partial t'}\phi + at'\left(\frac{\partial \phi}{\partial z'} + i\frac{\partial S}{\partial z'}\phi\right)\right)e^{iS(z',t')} = -\frac{\hbar^2}{2m_i}\left(\frac{\partial^2 \phi}{\partial z'^2} + 2i\frac{\partial \phi}{\partial z'}\frac{\partial S}{\partial z'} - \left(\frac{\partial S}{\partial z'}\right)^2\phi + i\frac{\partial^2 S}{\partial z'^2}\phi\right)e^{iS(z',t')}\\ + m_g g\left(z' - \frac{at'^2}{2}\right)\phi e^{iS(z',t')}.\end{aligned} \tag{16}$$

[12]For a full justification of this assumption see the discussion in section 5 regarding the impact of classical action in quantum mechanics.

The phase factor can be eliminated on both sides, reducing the equation to

$$i\hbar\left(\frac{\partial\phi}{\partial t'}+i\frac{\partial S}{\partial t'}\phi+at'\left(\frac{\partial\phi}{\partial z'}+i\frac{\partial S}{\partial z'}\phi\right)\right)=-\frac{\hbar^2}{2m_i}\left(\frac{\partial^2\phi}{\partial z'^2}+2i\frac{\partial\phi}{\partial z'}\frac{\partial S}{\partial z'}-\left(\frac{\partial S}{\partial z'}\right)^2\phi+i\frac{\partial^2 S}{\partial z'^2}\phi\right)$$
$$+m_g g\left(z'-\frac{at'^2}{2}\right)\phi. \tag{17}$$

If we subtract the left-hand side by the right-hand we impose a vanishing constraint on the equation. Since the free particle wave function, ϕ, and its derivatives, $\frac{\partial\phi}{\partial z'}$, are obviously non-vanishing, their coefficients must vanish. Therefore, we collect by coefficients of ϕ and $\frac{\partial\phi}{\partial z'}$

$$\begin{aligned}\left(-\hbar\frac{\partial S}{\partial t'}-at'\hbar\frac{\partial S}{\partial z'}-\frac{\hbar^2}{2m_i}\left(\frac{\partial S}{\partial z'}\right)^2+\frac{i\hbar^2}{2m_i}\frac{\partial^2 S}{\partial z'^2}-m_g g\left(z'-\frac{at'^2}{2}\right)\right)\phi\\+\left(i\hbar at'+\frac{i\hbar^2}{m_i}\frac{\partial S}{\partial z'}\right)\frac{\partial\phi}{\partial z'}+i\hbar\frac{\partial\phi}{\partial t'}+\frac{\hbar^2}{2m_i}\frac{\partial^2\phi}{\partial z'^2}=0.\end{aligned} \tag{18}$$

The equation can be simplified since the final two terms vanish in light of (12). Equation (18) imposes vanishing constraints on the coefficients of ϕ and $\frac{\partial\phi}{\partial z'}$. Take the latter into consideration first, which requires that

$$i\hbar at'+\frac{i\hbar^2}{m_i}\frac{\partial S}{\partial z'}=0. \tag{19}$$

The differential equation is separable as

$$\partial S=-\frac{m_i at'}{\hbar}\partial z'. \tag{20}$$

Thus we have $S(z',t')$ determined up to a constant function in z'

$$S(z',t')=-\frac{m_i at'z'}{\hbar}+f(t'). \tag{21}$$

Where $f(t')$ is a function of the t' coordinates only. The partial solution can be used to explicate the partial derivatives of the phase factor

$$\frac{\partial S}{\partial t'}=-\frac{m_i az'}{\hbar}+\frac{df}{dt'}. \tag{22}$$

$$\frac{\partial S}{\partial z'}=-\frac{m_i at'}{\hbar}. \tag{23}$$

$$\frac{\partial^2 S}{\partial z'^2}=0. \tag{24}$$

Therefore, the vanishing coefficient condition of ϕ from (18) simplifies to

$$\hbar\left(\frac{m_i a z'}{\hbar}-\frac{df}{dt'}\right)+at'\hbar\left(\frac{m_i a t'}{\hbar}\right)-\frac{\hbar^2}{2m_i}\left(\frac{m_i a t'}{\hbar}\right)^2-m_g g\left(z'-\frac{at'^2}{2}\right)=0 \tag{25}$$

$$m_i a z'-\hbar\frac{df}{dt'}+m_i a^2 t'^2-\frac{m_i a^2 t'^2}{2}-m_g g z'+\frac{m_g g^2 t'^2}{2}=0 \tag{26}$$

Now we impose the condition of the freely falling frame whereby the non inertial frame is accelerated at the same rate as uniform acceleration due to gravity. This imposes $a = g$

$$m_i g z'-\hbar\frac{df}{dt'}+m_i g^2 t'^2-\frac{m_i g^2 t'^2}{2}-m_g g z'+\frac{m_g g^2 t'^2}{2}=0 \tag{27}$$

For the left-hand side we are forced to equate like terms in the t' variable. This forces a recovery of Einstein's weak equivalence principle, namely

$$\begin{aligned} m_i g z' &= m_g g z' \\ m_i &= m_g \end{aligned} \tag{28}$$

The equivalence of inertial and gravitational mass. For simplicity we will write $m = m_i = m_g$. The final vanishing condition gives a differential equation

$$\frac{df}{dt'}=-\frac{mg^2t'^2}{\hbar}. \tag{29}$$

Thus

$$f(t')=-\frac{mg^2t'^3}{3\hbar}+constant. \tag{30}$$

The initial conditions of $f(0) = 0$ vanishes the constant. Thus our phase factor is

$$S(z',t')=-\frac{mgt'}{\hbar}\left(z'-\frac{gt'^2}{3}\right) \tag{31}$$

and the wave function for our non inertial frame using (11) is

$$\psi(z',t')=\phi(z',t')\exp\left[-\frac{imgt'}{\hbar}\left(z'-\frac{gt'^2}{3}\right)\right]. \tag{32}$$

The coordinate transformations between the non inertial and inertial frames are given by (3) and trivially (4). Thus the wave function in the original coordinate system is

$$\psi(z,t)=\phi\left(z+\frac{gt^2}{2},t\right)\exp\left[-\frac{imgt'}{\hbar}\left(z+\frac{gt^2}{6}\right)\right]. \tag{33}$$

4 Observables

We will use, for the sake of an explicit demonstration, a Gaussian wave packet in one-dimension as the solution to the free particle wave function. For full details, see Appendix I. The one-dimensional wave function is

$$\phi(z', t') = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\gamma} e^{-a\frac{x^2}{\gamma^2}}. \tag{34}$$

We impose an initial condition on the wave packet

$$\phi(z', 0) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}. \tag{35}$$

The positive (real) constant a corresponds to the initial spatial width of the wave packet. In particular, we can set an initial value for a so long as we ensure consistency with the uncertainty principle. Strictly speaking we have

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{36}$$

The uncertainty in x is related to the initial width of the wave packet by

$$\Delta x = \frac{1}{\sqrt{2a}}. \tag{37}$$

Thus we state the uncertainty principle as

$$\frac{1}{\sqrt{2a}} \hbar \sqrt{2a} \geq \frac{\hbar}{2}. \tag{38}$$

We can then set $a = 1$ without violation. So (34) becomes

$$\phi(z', t') = \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\gamma} e^{-\frac{x^2}{\gamma^2}}. \tag{39}$$

From (32) the complete wave function is then

$$\psi(z', t') = \left(\frac{4}{\pi}\right)^{1/4} \frac{1}{\gamma} e^{-\frac{x^2}{\gamma^2}} \exp\left[-\frac{mgt'}{\hbar}\left(z' - \frac{gt'^2}{3}\right)\right]. \tag{40}$$

The probability distribution in a spatial basis is determined by multiplying (40) by its complex conjugate. It should be clear that the phase factor vanishes and we are left with

$$|\psi(z', t')|^2 = \sqrt{\frac{2}{\pi}} w e^{-2w^2 z'^2}. \tag{41}$$

Where w is defined as

$$w = \sqrt{\frac{1}{1 + (2\hbar t'/m)^2}}. \tag{42}$$

This is the one-dimensional probability distribution for the particle in non inertial frame. We can comment briefly on its form when comparing it to the general function of a Gaussian distrubtion

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{43}$$

Here μ is the mean and σ is the standard deviation of the probability distribution. It should be clear that a conversion between these two forms indicates that

$$\sigma = \frac{1}{2w} = \frac{1}{2}\sqrt{1 + (\frac{2\hbar t}{m})^2}. \tag{44}$$

The width of the distribution grows with time. We will comment more on this in section 5. The probability distribution can be determined for the observer in the inertial frame by performing the coordinate transformations given by (3) and (4).[13] The probability distribution function is

$$|\psi(z,t)|^2 = \sqrt{\frac{2}{\pi}} w e^{-2w^2(z+\frac{gt^2}{2})^2}. \tag{45}$$

w is still given by (42) under the transformation $t' \to t$. So we note that the standard deviation grows at the same rate, but the probability distribution is centered on an evolving mean given by $\frac{gt^2}{2}$. If it is not yet obvious consider the expectation values for the position in both frames. In the non inertial frame we have

$$\langle z' \rangle = \langle \psi(z',t')|z'\psi(z',t')\rangle = \int_{-\infty}^{\infty} z'|\phi(z',t')|^2 \exp\left[-\frac{imgt'}{\hbar}\left(z' - \frac{gt'^2}{3}\right)\right] \exp\left[\frac{imgt'}{\hbar}\left(z' - \frac{gt'^2}{3}\right)\right] dz'. \tag{46}$$

As usual the phase factor portion cancels. The square of the ϕ solution is simply (41), so the integral is

$$\langle z' \rangle = \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} z' e^{-2w^2 z'^2} dz' = 0. \tag{47}$$

$|\phi|^2$ is an even function. Its product with z' produces an odd function and evaluated on the interval $[-\infty, \infty]$ gives zero. Similarly, the expectation value of the position in the inertial frame is

$$\langle z \rangle = \langle \psi(z,t)|z\psi(z,t)\rangle = \int_{-\infty}^{\infty} z\left|\phi\left(z + \frac{gt^2}{2}, t\right)\right|^2 dz = \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} z e^{-2w^2\left(z+\frac{gt^2}{2}\right)^2}. \tag{48}$$

[13] Or one could perform the coordinate transformations directly at (33) and take the square modulus. The phase factor vanishes in both calculations when using the complex conjugate. Some tedious algebra will recover the above stated result.

Perform a change of variables letting $u = z + \frac{gt^2}{2}$ such that

$$\langle z \rangle = \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} \left(u - \frac{gt^2}{2} \right) e^{-2w^2u^2} du = \sqrt{\frac{2}{\pi}} w \left(\int_{-\infty}^{\infty} u e^{-2w^2u^2} du - \frac{gt^2}{2} \int_{-\infty}^{\infty} e^{-2w^2u^2} du \right). \tag{49}$$

The first integral is odd so it vanishes; the latter integral evaluates to

$$\int_{-\infty}^{\infty} e^{-2w^2u^2} du = \sqrt{\frac{\pi}{2w^2}}. \tag{50}$$

Which cancels out with the constants in (49). Thus we are left with a familiar result

$$\langle z \rangle = -\frac{gt^2}{2} \tag{51}$$

This is the manifestation of **Ehrenfest's Theorem**: expectation values obey classical laws. The expectation value for the particle is the classical trajectory of a falling body discovered by Galileo.

5 Discussion

The concept of equivalence is easier to understand in classical mechanics but more difficult to grasp in non-relativistic quantum mechanics. We began with an inertial system that existed in a uniform gravitational field and attempted to describe the quantum mechanical dynamics of a falling particle. To do so, we transformed our coordinate system from an observer, who is at rest, watching the particle fall, to another observer who falls alongside the particle. In this freely falling frame of reference, we recovered in equation (32) the wave function of a free particle in one dimension. We found that the physical observables, namely the probability distribution (41) and the classical position expectation value (47), are identical to those of an observer in a global inertial system who observes a free particle.

Let us explore a qualitative description of the system. We imagine, unfortunate as it is for the poor soul who is doing the free fall, repeating this experiment over thousands and millions of trials. Each observer has a particular device that can detect the position of the particle. Over the course of these trials, both observers will recover histograms similar to the ones simulated below.

As seen in Fig. 1, the observers will eventually, over the course of a large number of trials, recover histograms with smooth distributions, centering on a mean. In this case the mean is about $z' = 0$ (and by extension $z = 0$). It is clear from the histograms that an obvious divergence from the classical theory is present. In classical mechanics, the experiment at $t = 0$ is quite boring: the observer will always measure the particle at $z = 0$. But in the quantum problem, observers will detect a spread of the initial positions of the particle. Even in the large n case, the observer will detect hundreds of measurements where the

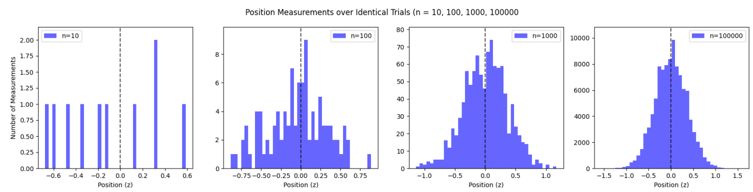

Figure 1: A histogram of position measurements (z', z) for a particle at t = 0. The histogram is given by using numpy random.choice() using the inertial probability distribution as the sample boundary. There are four histograms given by number of trials, 10, 100, 1000, 100000

particle is over 1 unit to the right and left (above and below the measurement platform). This is, of course, due to the standard deviation of the wave packet, which is given by (44). We had set the initial spread to $a = 1$, which as a result at $t = 0$ gives an initial standard deviation of $\sigma = \frac{1}{2}$. You may be motivated to ask, why not just set $a = 0$ and assume we recover classical mechanics from the quantum mechanical formulation? But we can quickly see why this is an aphysical solution. Firstly, the Gaussian wave packet would vanish in (35) and thus be un-normalized. Also, an initial width of the wave packet vanishing to a single measurement would violate the uncertainty principle. As

$$\frac{1}{\sqrt{2a}}\hbar\sqrt{2a} \geq \frac{\hbar}{2}. \tag{52}$$

Now let us consider what happens as we begin to take successive measurements in time. We will consider the probability distributions given by (41) and (45). At $t = 0$ we recover an unsurprising result. Both observers will calculate the same probability distributions. The probability distributions have the same width that is given by (44) at $t = t' = 0$. The mean for both distributions is centered on $z = z' = 0$ and attenuates both above and below the particle in accordance with the Gaussian distribution. This is demonstrated in Fig. 2.

Now the particle begins to fall, and we repeat the process described above. Both observers take large amounts of measurements at $t = t' = 0.5$. On the left of Fig. 3, the inertial observer standing on the platform recovers a probability distribution. Its width has now increased, governed by (44), which is a function of time. The spread of the wave function has increased, and the observer will notice more measurements further from the mean. Meanwhile, the mean has evolved, as shown by the trailing line. This is the classical trajectory of the particle. This is once again a demonstration of Ehrenfest's Theorem: the mean (expectation value of z) evolves in accordance with the classical laws of physics. In this case, the uniform acceleration is due to gravity. On the right, the non-inertial freely falling observer will still have their mean centered on $z' = 0$, as expected by (47). However, both observers will agree on the spread of the wave packet.

A useful heuristic, one that poignantly demonstrates the equivalence principle, is to imagine how the two observers would reconcile their physical observa-

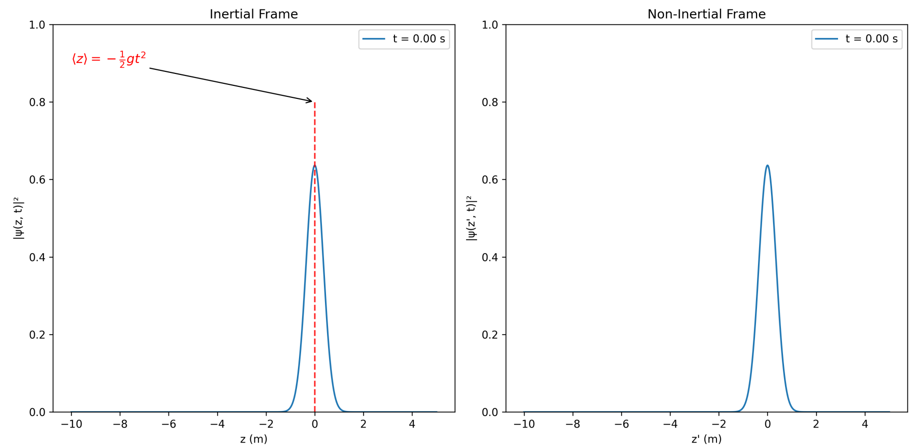

Figure 2: A plot comparing the probability distributions $|\psi(z,t)|^2$ at $t = t' = 0$ for two reference frames: the inertial frame (left) and the locally inertial, freely falling frame (right). The wave functions used to calculate the probability distributions are given by equations (32) and (33), representing solutions to the time-dependent Schrödinger equation for a particle in a gravitational field. The probability density is computed using the np.abs() method in Python. The inertial frame plot includes an arrow indicating the mean position value, corresponding to the classical trajectory of the particle. Physical constants are simplified for ease of graphing, with $\hbar = g = m_e = 0.5$. The horizontal axis represents position z (in arbitrary units), and the vertical axis represents the probability density $|\psi(z,t)|^2$. The plot highlights the differences in the probability distributions between the two frames, illustrating the effects of the gravitational field on the quantum dynamics of the particle. The figure was generated using Python with Matplotlib.

tions. Let us consider first the inertial observer. In their frame of reference, the particle, initially at rest at $z = 0$ in a uniform gravitational field, is released and begins to accelerate. In terms of the dynamics of the Schrödinger equation, the gravitational potential begins to decrease as the particle accelerates towards the surface of the Earth. So, for the inertial observer, they explain the widening of their wave packet as a function of the particle falling in a potential well. As it tends towards the surface of the Earth, the particle dynamics become dominated by the free particle equation.

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial z^2} \tag{53}$$

as $V \to 0$. Its potential energy is converted to kinetic energy. The position wave function spreads over time, and its inverse Fourier transform, the momentum-space wave function, localizes as the kinetic energy of the particle becomes defined.

Now consider the non-inertial observer, who falls alongside the particle. From their frame of reference, one that is uniformly accelerated, around them

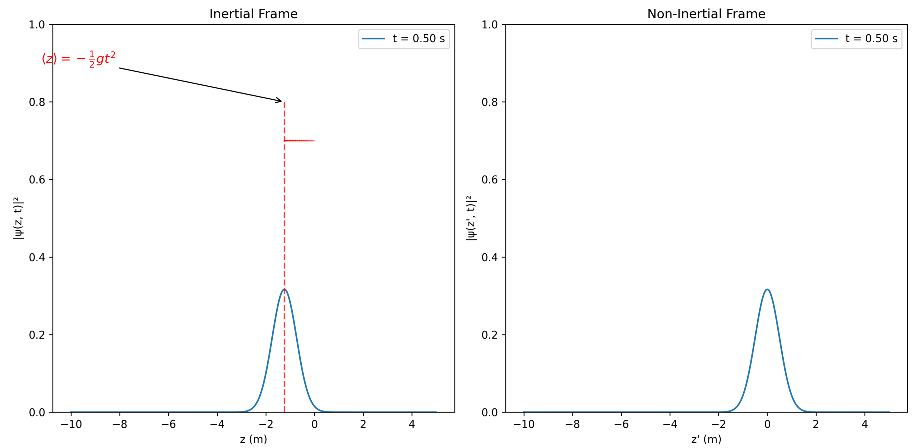

Figure 3: A plot comparing the probability distributions $|\psi(z,t)|^2$ for the inertial frame under uniform gravity (left) and the locally inertial, freely falling frame (right) after $t = t' = 0.5$. The probability distribution is calculated using the same function as Fig. 2, np.abs(). The physical constants are simplified as $\hbar = g = m_e = 0.5$. The mean of the left plot has followed the classical particle trajectory under uniform gravity. Both wave packets have increased in width according to (42), which in this case is approximately 0.447. The standard deviation from (44) is these simplified units is $\sigma = 1.118$.

their experiments indicate an inertial system. The particle, defined by some initial conditions, evolved according to (53). For this observer, the particle was and is always a free particle, which evolves without an external potential energy. As the particle is not bounded by any potential, it evolves just as any free particle will, with its position-space wave function widening over time.

We can summarize in this discussion the quantum mechanical application of the equivalence principle. Originally, we are provided with a particle in one dimension, which, initially at rest, begins to fall under uniform acceleration due to gravity. To determine the equations of motion of the particle, we performed a coordinate transformation using equations (3) to a frame of reference falling uniformly alongside the particle. In this frame of reference, the Schrödinger equation is one of a free particle. We then determined the wave function of the particle in (32) and its probability distribution in (41). We could then invert the coordinate transformations to recover the wave function in the original coordinate system. This is a restatement of the equivalence principle: the local observer in the primed coordinate system cannot, through any local experiment, determine whether they are truly in an inertial frame of reference in deep space or whether they are freely falling in a uniform gravitational field. More specifically, since the wave functions for both cases differ only by a phase factor, which vanishes in any observable calculation, there is no way to discriminate between the two. So, in accordance with the classical theory, we state once again that a local gravitational field can be eliminated by an appropriate coordinate transformation, one that corresponds to an observer who moves under uniform acceleration in the field. We will note in the final section what the "physicality" of the phase factor is and whether it has any effects on the equivalence principle.

6 Conclusion

We should summarize some of the results of the calculations performed in this paper. Namely, the derivation of a solution to the Schrödinger equation with a uniform gravitational potential energy. We did this by performing a coordinate transformation to a uniformly accelerating frame, the lab frame of the observer freely falling. In the case of the observer's frame acceleration, relative to the inertial frame, a being equal to acceleration due to gravity, g, we recover the forced equivalence of inertial and gravitational mass. In addition to this, we demonstrate what exactly the equivalence principle means in this case: specifically, that the wave packet, observed over repetitive experiments by both observers, increases in width as time increases. We discussed how the two observers reconcile this fact: the inertial observer cites the particles decreasing gravitational potential energy as they accelerate uniformly towards the surface of the Earth, and the non-inertial observer watches the time evolution of a free particle whose expectation value remains at some constant observation point. To further reiterate the equivalence principle, we found in our freely falling frame that we recover, up to a phase factor, the wave function of a free particle in an inertial frame of reference. Thus, we can corroborate the equivalence principle: that quantum mechanics seemingly agrees with the statement that its laws of physics permit a coordinate transformation whereby the effects of gravity can be eliminated by transforming to a system that falls freely in this frame. We then discover that this new frame is seemingly identical to that of an inertial frame, with a free particle remaining at rest deep in space.

There are some interesting extensions of this problem I aim to tackle in the future. Firstly, we noted in our solution to the wave equation that our ansatz contained a phase factor, which we determined to be a function of the observer's coordinates (31). But when we calculated the physical observables, such as classical expectation values and the probability distribution functions (41) and (45), we showed an equivalence between the two. So the question remains: what is the physical meaning of this phase factor?

For the purposes of this paper, we will take it as an assumption that the wave functions in two frames of reference can be related by a phase shift. Thus, we can write the wave function in the primed frame in terms of the unprimed one, or

$$\psi' = e^{i\theta}\psi. \tag{54}$$

θ in this case is the phase of the wave function, and is determined using the classical action

$$\theta = \frac{S}{\hbar}. \tag{55}$$

The action is defined as expected

$$S = \int_{t_1}^{t_2} L dt \tag{56}$$

where L is the Lagrangian of the system, considered between two times, t_1 and t_2. In our case the Lagrangian in the inertial system is

$$L = \frac{1}{2} m\dot{z}^2 - mgz. \tag{57}$$

But the system is uniformly accelerated so we can write the velocity using the classical trajectory

$$\dot{z} = -gt. \tag{58}$$

Thus, our Lagrangian becomes

$$L = -\frac{1}{2} mg^2t^2 - mgz. \tag{59}$$

The action is then

$$S = \int_0^t \left(-\frac{1}{2} mg^2t^2 - mgz\right) dt = -\frac{1}{6} g^2t^3 - mgzt. \tag{60}$$

Now we invoke the inverse coordinate transformations given by (3). The action becomes

$$S' = -\frac{1}{6} g^2t'^3 - mg(z' - \frac{gt'^2}{2})t' = \frac{1}{3} mg^2t'^3 - mgz't'. \tag{61}$$

If we use these actions to write the phases we have

$$\theta' = -mgt' \left(z' - \frac{gt'^2}{3}\right). \tag{62}$$

Which is the same argument that was used in the exponential argument in (31). Likewise, in the inertial frame of reference, we recover from (60) the same phase argument used in (33). This is, in fact, what justifies the usage of the ansatz in (11).[14]

The relevance here is that locally, phase does not impart any physical meaning to measurements. That is why the probability distributions recovered from the wave functions are invariant (other than the inertial frame having a mean that follows the particles classical trajectory). But globally we would expect to

[14]For a more detailed justification and theory of action in quantum mechanics see Shankar, R. Principles of Quantum Mechanics (New Haven, USA: Springer, 2014). Specifically Chapter 8, The Path Integral Formulation of Quantum Theory, 223-235.

see an effect of phase. Wave interference is determined by the phase difference of super-positioned waves. This fact motivates an interesting thought experiment: let us imagine a truly privileged frame of reference, one that is globally inertial, and another frame that, like the free-falling scientist, is locally inertial. The equivalence principle should mean that there are no physical experiments that can discriminate between the two. Both observers will claim that they have a frame of reference whose laws of physics are inertial. But, as demonstrated in the calculations carried out in section 2, the freely falling frame has a phase factor, which is a function in part of the gravitational field.

Though I will not inquire too deeply into the experiment here, we will discuss, briefly, a hypothetical experiment regarding interferometry. We imagine two particles, which are initially prepared in identical states, and then we split the particles with a beam splitter, sending one particle into a uniform gravitational field and the other into the perfectly inertial frame of reference. Let us imagine too that each frame has an observer; in the frame with the uniform gravitational field, the observer freely falls under gravity with the particle. In the perfectly inertial frame, the observer sits perfectly at rest with the particle. Of course the probability distributions for both functions would be identical, and so too would their expectation values. But, in the truly inertial frame, the wave function of the particle would have no classical action, whereas the particle in free fall would pick up a phase shift in accordance with (61). Now, if we could somehow recombine the beams, we should detect some type of interference pattern governed in part by the properties of the local gravitational field. This result seemingly contradicts the strong equivalence principle, in which we lose global equivalence of all frames. The locally inertial frame, when compared to a globally inertial frame, has an observable difference, which could be measured in terms of the interference pattern.

Furthermore, (62) indicates that the phase of the wave function is dependent on the strength of the gravitational field.[15] Thus, we could extend the thought experiment to include a beam splitter that takes two particles, prepared in the same state, and sends them into regions of space with distinct gravitational fields. In each of these regions, an observer falls alongside the particle; both

[15]If we consider an arbitrary uniform gravitational field with uniform acceleration a, the carrying out a similar coordinate transformation as done in the text above would produce from (60)

$$S = -\frac{1}{6}a^2t^3 - mazt.$$

We would expect then in an experiment where one particle is sent into Earth's gravitational field, and the other into the arbitrary field, each particle would pick up a different phase. For the arbitrary particle its phase would be

$$\theta'' = -mat''\left(z'' - \frac{at''^2}{3}\right).$$

The double prime is meant to indicate that the coordinates (z'', t'') are for the observer who falls under uniform gravity in this new field. The phase difference would manifest as $\Delta\theta = \theta' - \theta''$. There would be an interference pattern due to constructive and destructive superposition of the two waves. This indicates, at least in the scope of this completely hypothetical situation, non-relativistic quantum mechanics could discriminate between locally inertial frames.

observers seemingly would agree on the evolution of the probability distribution. However, when recombining the beams, each wave function would pick up a phase given by (62). Thus, once the beams recombine, we should notice an interference pattern, whose phase would be given by the difference in the phases. In theory, this experiment would then be able to distinguish between locally inertial frames of reference, which is at odds with the concept of global covariance. We should note a commonly cited caveat to the equivalence principle, which may be important for framing this discussion. Even before quantum mechanics, there had been indications of potential violations of the weak equivalence principle. An exemplar of this issue is an electric charge falling in a uniform gravitational field. In accordance with classical electrodynamics, the charge should generate a radiation field that siphons kinetic energy from the particle. But for the observer who falls alongside the charge, the charge will not radiate, as inertial charges do not generate radiation fields. A resolution to this problem is to "restrict the class of experiments covered by the EP to those that are isolated from bodies or fields outside the (inertial frame of reference)."[16] Whether we should add a similar caveat for this problem in quantum mechanics, I will leave to the consideration of experts and the reader.

We will leave the discussion above for either a future inquiry or, more appropriately, to subject matter experts. However, this potential divergence between quantum mechanics and classical field theory is but one manifestation of the current incapability of a reconciliation of the two physical theories. The general theory of relativity indicates that there seemingly is no way to differentiate between a locally inertial frame of reference, such as the freely falling frame, and a truly globally inertial frame of reference. However, as noted in the final comments above, quantum mechanics seemingly offers potential for physical experiments to discriminate between global and local inertial systems and further discriminate between local inertial fields depending on the strength of their local gravitational fields.

There are some additional extensions to this paper that we think are worthwhile in exploring. Firstly, it would be useful to attempt a drawn out inquiry into the calculation carried out in this section regarding relative phase differences. It would be useful to plan a thought experiment in interferometry, which in practice could measure relative phase differences and discriminate inertial frames of reference. Another extension of this paper is to include relativistic corrections. Notice that the coordinate transformations in (3) and the potential energy calculation are classical. It would be a worthwhile activity to amend this paper with relativistic corrections.

In summary, this paper provided a solution to the one-dimensional Schrödinger equation for an uncharged particle experiencing free fall. We considered a coor-

[16]Rindler, Relativity, 22-23. I was first introduced to this idea in Griffiths text on Electrodynamics (Griffiths, David. Introduction to Electrodynamics. (Cambridge, United Kingdom: Cambridge University Press, 2017), 501, Problem 11.34.) In the problem you can use the hyperbolic motion of a particle to calculate the radiation reaction force. The calculation produces $F_{rad} = 0$, indicating that the particle does not radiate. Thus an observer, who falls with a particle will not calculate a radiation field for the particle.

dinate transformation to a freely falling frame, and then used the solution the wave function function in this locally inertial frame, to determine the solution to the originally posed problem. We calculated the probability distributions and classical position expectation value for booth coordinate systems. We then illustrated the evolution of the wave packets in both frames using a Python simulation. Our discussion considered some of the physical implications of our derivation, particularly noting that the phase factor of the wave function, determined by the classical action of the system, indicates that the phase of wave functions in locally inertial frames with distinct gravitational fields would be different. This, as described in the thought experiment, should lead to interference patterns for particles whose wave functions evolved in locally inertial fields with distinct gravitational fields. This suggests a possible divergence between classical field theory and quantum mechanics, specifically challenging the notion of global covariance. We should note that our non-relativistic derivation is only a sketch and approximate form of the complete solution; a relativistic quantum field theory would need to be used to further inquire into the potential violations of global covariance.[17] This article provides a simple example of the unique challenge posed to physicists in the twenty-first century: reconciling two physical theories, relativity and quantum mechanics, whose physical formulation remains inconsistent with each other.

7 Appendix I: Derivation of a Solution to the Free Particle Schrödinger Equation

We begin with a free particle given by an initial wave function

$$\Psi(z,0) = Ae^{-az^2}, \tag{63}$$

where A and a are positive, real constants. We impose normalization on the initial wave function to determine the coefficient of the wave packet

$$|A|^2 \int_{-\infty}^{\infty} e^{-2az^2}\, dz = |A|^2 \sqrt{\frac{\pi}{2a}}, \tag{64}$$

thus

[17] Two interesting articles expand on this concept: Maciej Trzetrzelewski, "On the Equivalence Principle and Relativistic Quantum Mechanics," Foundations of Physics 50 (2020): 1253–1269, https://doi.org/10.1007/s10701-020-00388-8; Elias Okon and Craig Callender, "Does Quantum Mechanics Clash with the Equivalence Principle—And Does It Matter?" European Journal for Philosophy of Science 1 (2011): 133–145, https://doi.org/10.1007/s13194-010-0009-z. Trzetrzelewski provides a quantum field theoretic derivation which concludes that certain quantum constraints are irreconcilable with the equivalence principle. Okon and Callender provide a more general discussion regarding this incapability, expanding on some potential caveats noted in the main text. In their words, the quantum mechanical equivalence principle states that "all the dynamical matter fields "feel" the same metric. Quantum mechanics won't challenge this, but maybe quantum gravity will."

$$A = \left(\frac{2a}{\pi}\right)^{1/4}. \tag{65}$$

The normalized initial wave function is then

$$\Psi(z,0) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-az^2}. \tag{66}$$

The free particle admits a momentum space wave function, who is related to the initial spatial wave function by the inverse Fourier transform

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(z,0) e^{-ikz}\, dz.^{18} \tag{67}$$

Therefore we can determine this wave function, and then use the Fourier transform to find the spatial wave function. The integral is

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-az^2} e^{-ikz}\, dz \; = \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-i(az^2+kz)}\, dz \tag{68}$$

We consider an integral of the form

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)}\, dx. \tag{69}$$

We let

$$y = \sqrt{a}(x + \frac{b}{2a}) \rightarrow dy = \sqrt{a}dx. \tag{70}$$

and

$$(ax^2 + bx) = y^2 - \left(\frac{b^2}{4a}\right). \tag{71}$$

The integral can then be solved as

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-(ax^2+bx)}\, dx &= \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-\left(y^2 - \left(\frac{b^2}{4a}\right)\right)}\, dy \\ &= \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-y^2} e^{\left(\frac{b^2}{4a}\right)} dy \\ &= \frac{e^{\left(\frac{b^2}{4a}\right)}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-y^2} dy \\ &= \sqrt{\frac{\pi}{a}} e^{\left(\frac{b^2}{4a}\right)}. \end{aligned} \tag{72}$$

[18] I should note here that $\phi(k)$ is the momentum space representation of the quantum state, not the free particle spatial wave solution as noted in (11) and carried out throughout the main body of the paper.

Thus the momentum space representation becomes

$$\phi(k) = \frac{1}{(2\pi a)^{1/4}} e^{-\frac{k^2}{4a}}. \tag{73}$$

The spatial wave function is related to the momentum space wave function by the Fourier transform

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i\left(kz - \frac{hk^2}{2m}t\right)}\, dk. \tag{74}$$

The integral becomes

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{4a}} e^{i\left(kz - \frac{hk^2}{2m}t\right)}\, dk \\
&= \frac{1}{\sqrt{2\pi}(2\pi a)^{1/4}} \int_{-\infty}^{\infty} e^{-\left(k^2\left(\frac{1}{4a} - i\frac{\hbar}{2m}t\right) + ikz\right)}\, dk.
\end{aligned} \tag{75}$$

We reduce the integral by writing

$$\begin{aligned}
a &= \frac{1}{4a} - \frac{i\hbar}{2m}t \\
b &= iz
\end{aligned} \tag{76}$$

The integral is the same as the one evaluated (68). Therefore our wave function, after simplification becomes

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-az^2/(1+2i\hbar at/m)}}{\sqrt{(1+2i\hbar at/m)}}. \tag{77}$$

We can introduce a notation simplification letting

$$\gamma = \sqrt{(1+2i\hbar at/m)}, \tag{78}$$

which then gives a final wave function

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\gamma} e^{-az^2/\gamma}. \tag{79}$$

8 Appendix II: Python implementation

Visual simulations of the probability distributions and measurements trials were implemented using a Python script. The source code for the experiment can be found at my GitHub.[19] The code provides a framework which permits the calculation of the Gaussian wave packet, whose formula is given in the text by (41), given a set of physical constants defined in the configuration and parameters section. A code snippet is given below.

```
def gaussian_wave_packet(z, t, z_0, sigma_0):
    # Calculates a Gaussian wave packet using the
    # simplified physical constants.
    # If you want to use the actual
    # physical constants use:
    # h_d -> hbar, m_d -> m_e, g_d -> g
    w = np.sqrt(1 / (1 + ((2 * h_d * t) / m_d) ** 2))
    return np.sqrt(2 / np.pi) * w * np.exp(-2 * (w ** 2) * (z ** 2))
```

There are two additional functions which calculate the wave functions for the inertial and freely falling observers, governed by equations (32) and (33). A strength of this implementation is that the general form of a coordinate transformation permits the use of equation (54). This means that a user leveraging this simulation can calculate their own phase factor using the method described in the appendix. They can also call the Gaussian wave packet which is invariant. This permits them to implement a new wave packet solution, based on the local gravitational field properties. The wave functions are:

```
def inertial_wave_function(z, t, z_0, sigma_0):
    # Calculates the wave function for the inertial coordinate system.
    xi = z + 0.5 * g * t ** 2
    psi0 = gaussian_wave_packet(xi, t, z_0, sigma_0)
    phase_factor = np.exp(-1j * m_d * g_d * t *
    (z + (1 / 6) * g * t ** 2) / hbar)
    return psi0 * phase_factor

def noninertial_wave_function(z_prime, t, z_0, sigma_0):
    # Computes the wave function for the freely falling coordinate system
    psi0 = gaussian_wave_packet(z_prime, t, z_0, sigma_0)
    phase_factor = np.exp(-1j * m_d * g_d * t *
    (z_prime - (1 / 3) * g * t ** 2) / hbar)
    return psi0 * phase_factor
```

The probability distributions are calculated using the numpy np.abs() method. The simulations are implemented is the MatPlot library. In particular the code provides four simulations: a probability distribution plot of the inertial frame versus the freely falling frame; a three-dimensional plot of the inertial probability distribution; a simulation plotting the inertial probability distribution and a classical particle falling at the same rate, indicating Ehrenfest's Theorem for the mean following the classical trajectory; and finally a simulation of repetitive experiments plotted as a histogram.

[19] `https://github.com/Eharriman/physics/blob/main/schrodinger_equivalence_principle.py`