The Correspondence Principle and Ehrenfest

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1. Introduction

This article is the result of a tangent I went on in my more immediate focus: a case study in quantum mechanics involving the Equivalence Principle. You may have actually been directed from that article to here. If not, that article will pair nicely with this one. In this post I aim to explain a very important result in Quantum Mechanics. Specifically Ehrenfest's Theorem.[1] This particular result, relating the rate of change of specific observable expectation values and the Hamiltonian of the system, provides an interface at which our classical intuitive world meets the more primitive world of quantum mechanics.

It may be the case, as I know it has for myself, that you have wondered the following: when exactly does the world become quantum? Gut feeling leads me to say, "when the world is small." But how small? Why is it that the various counterintuitive results of quantum mechanics never present themselves in ordinary life? Another way to phrase that last question is: Why is quantum mechanics not intuitive? My train of thought on that last question runs the following: The world is quantum mechanical. Humans live in the physical world. Humans, through evolution, developed cognitive faculties that permit the construction of mental models that map, at least roughly, to the physical world. It is useful, especially from an evolutionary perspective, for such mental models to be as accurate as possible. Intelligent organisms with more accurate mental models will be more likely to be the ones that pass on their genes to the next generation. Therefore, our intuition should be quantum.[2]

But that is not the case. Anecdotally, I've found that learning quantum mechanics leans more toward memorization than intuition—an unusual experience in the study of physics. Classical physics, despite its complexity, allows for a

[1]Nicholas Wheeler, "Remarks concerning the status some ramifications of Ehrenfest's Theorem" Reed College Physics Department, 1998. This is an excellent article and one I used for much of my motivation in the arguments.

[2]I wouldn't even consider myself an "armchair" biologist or cognitive scientist. In fact, I wouldn't even say I'm a hobbyist. For those of you more capable in these fields, I apologize for butchering it. Nonetheless I think the motivating point is made clear.

kind of "gut" understanding; even without all the empirical details and mathematical nuances, I have an instinct for how classical systems should behave. I can't say the same for quantum mechanics. While you do, to some extent, develop new intuitions as you study it, for me, it never evokes the same immediate, tangible sense as imagining a classical physics problem.

In this article I present, at least one element, in the answer as to why our intuition is classical and not quantum. I assert that it is the so-called Correspondence Principle that forms the basis of our classical intuition. The Correspondence Principle relates the classical and quantum mechanical formulations of the evolution of physical systems. The more general statement can be detailed through various case studies, providing quantitative measures of when the two regimes relate. Dirac noted that

> *"The value of classical analogy in the development of quantum mechanics depends on the fact that classical mechanics provides a valid description of dynamical systems under certain conditions, when the particles and bodies composing the systems are sufficiently massive for the disturbance accompanying an observation to be negligible. Classical mechanics must therefore be a limiting case of quantum mechanics"*[3]

In the following sections I intend to: a) derive the Ehrenfest Theorem from a derivation focused on the procedure of calculating expectation values in quantum mechanics; b) study a specific case of the theorem, the measurement of momentum; and c) provide a more general discussion of the Correspondence Principle.

2. Generalized Ehrenfest Theorem

Consider an observable $Q(x, p, t)$: we can determine the expectation value on a system in state by applying the generalized statistical interpretation. That is, apply the operator representing the observable in the state and compute its inner product:

$$\langle Q \rangle = \langle \Psi | \hat{Q} \Psi \rangle \tag{1}$$

Now what if we wish to determine the rate of change of this expectation value in a dynamical system? Take the derivative with respect to time and compute:

$$\frac{d}{dt}\langle Q \rangle = \frac{d}{dt}\langle \Psi | \hat{Q} \Psi \rangle \tag{2}$$

[3]P. A. M. Dirac, The Principles of Quantum Mechanics (Snowball Publishing, 2013), 84.

The derivative acting on the inner product is linear. So we can apply the product rule for the three separate terms

$$\frac{d}{dt}\langle\Psi|\hat{Q}\Psi\rangle = \langle\frac{\partial\Psi}{\partial t}|\hat{Q}\Psi\rangle + \langle\Psi|\frac{\partial\hat{Q}}{\partial t}\Psi\rangle + \langle\Psi|\hat{Q}\frac{\partial\Psi}{\partial t}\rangle \tag{3}$$

What is the partial derivative of with respect to time? Well recall the Schrödinger equation phrased as an eigen problem:

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi \tag{4}$$

With H being the Hamiltonian of the system. If we bring over the complex term we can restate the expectation value as:

$$\frac{d}{dt}\langle Q\rangle = -\frac{1}{i\hbar}\langle\hat{H}\Psi|\hat{Q}\Psi\rangle + \langle\Psi|\frac{\partial\hat{Q}}{\partial t}\Psi\rangle + \frac{1}{i\hbar}\langle\Psi|\hat{Q}\hat{H}\Psi\rangle \tag{5}$$

But the Hamiltonian, which is an observable, yields real eigenvalues (actual measurements) from a given measure on the state. These observables must be hermitian, so at least for the first term we can bring the operator over to the ket. Thus

$$\frac{d}{dt}\langle Q\rangle = -\frac{1}{i\hbar}\langle\Psi|\hat{Q}\hat{H}\Psi\rangle + \langle\Psi|\frac{\partial\hat{Q}}{\partial t}\Psi\rangle + \frac{1}{i\hbar}\langle\Psi|\hat{Q}\hat{H}\Psi\rangle \tag{6}$$

Now recall that the expectation value for an observable is calculated by applying the operator to the wave function and taking the inner product. In all three terms, that is what we've done. So we can write the above equation more cleanly as

$$\frac{d}{dt}\langle Q\rangle = -\frac{1}{i\hbar}\langle\hat{H}\hat{Q}\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle + \frac{1}{i\hbar}\langle\hat{Q}\hat{H}\rangle \tag{7}$$

If we flip the signs we notice the the expectation values listed for Q and H are equal to their commutator. That is

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle \tag{8}$$

What has been partially derived in the preceding lines is known as Ehrenfest's Theorem, or at least a general case of it.[4]

[4]The motivation for this derivation is taken from Griffiths and Schroeter. David J. Griffiths and Darrell F. Schroeter, Introduction to Quantum Mechanics (Cambridge: Cambridge University Press, 2018), 110.

3. The Equations of Motion

The result is far more clear when one inserts a specific operator in the place of Q. Consider the rate of change of momentum, where

$$\hat{p} = -i\hbar \frac{d}{dx} \tag{9}$$

Using the momentum operator as our observable, we wish to solve

$$\frac{d}{dt}\langle p\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{p}]\rangle + \langle\frac{\partial \hat{p}}{\partial t}\rangle \tag{10}$$

In this case the operator is explicitly independent of t, so the final term vanishes. To calculate the commutator we note that the Hamiltonian is[5]

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x) \tag{11}$$

The commutator is

$$[\frac{\hat{p}^2}{2m} + V(x), \hat{p}] \tag{12}$$

We note two identities of commutators that enable us to carry out this calculation. Firstly

$$[\hat{A}+\hat{B},\hat{C}] = (\hat{A}+\hat{B})(\hat{C}) - (\hat{C})(\hat{A}+\hat{B}) = \hat{A}\hat{C} - \hat{C}\hat{A} + \hat{B}\hat{C} - \hat{C}\hat{B} = [\hat{A},\hat{C}] + [\hat{B},\hat{C}] \tag{13}$$

Thus we can write

$$[\frac{\hat{p}^2}{2m} + V(x), \hat{p}] = [\frac{\hat{p}^2}{2m}, \hat{p}] + [V(x), \hat{p}] \tag{14}$$

The first term reduces to the commutator of the momentum with itself, which is zero. The last term can be rewritten by leveraging the second commutator identity[6]

[5]We assume here that the Hamiltonian is quadratic in the momenta and the potential of the system is independent of time. Not all systems adhere to this clean definition.

[6]A derivation for this identity was motivated by Griffiths and Schroeter problem 3.14 (108).The momentum operator is [
hat p = -i$\hbar \frac{d}{dx} = \frac{\hbar}{i}\frac{d}{dx}$ We then derive the result by allowing the commutator to act on a function

$$[f,\hat{p}]g = [f, \frac{\hbar}{i}\frac{d}{dx}]g = f\frac{\hbar}{i}\frac{dg}{dx} - \frac{\hbar}{i}\frac{d}{dx}(fg)$$

Apply the product rule to the second term and expand

$$[f,\hat{p}]g = f\frac{\hbar}{i}\frac{dg}{dx} - \frac{\hbar}{i}(\frac{df}{dx}g + \frac{dg}{dx}f)$$

Thus we finally state the change in the expectation value of momentum as

$$\frac{d}{dt}\langle p\rangle = -\langle\frac{\partial V}{\partial x}\rangle \tag{15}$$

A moment of reflection will identify this result as Newton's Second Law: the rate of change of momentum is equal to the negative gradient of the potential. But notice that this result indicates that the expectation value of the observable is that which obeys the classical law. This result is more aptly named **Ehrenfest's Theorem** and generates a succinct statement in the formalism of Quantum Mechanics: **expectation values obey classical laws**.

To expand this idea, perhaps another illustration would be useful. We should note that the above-stated equation can be written more generally when we admit that for the Hamiltonian it is true that

$$-\langle\frac{\partial V}{\partial x}\rangle = -\langle\frac{\partial H}{\partial x}\rangle \tag{16}$$

Thus

$$\langle\dot{p}\rangle = -\langle\frac{\partial H}{\partial x}\rangle \tag{17}$$

Similarly, let's now consider the position operator and commit it to the Ehrenfest Theorem. We have

$$\langle\dot{x}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{x}]\rangle \tag{18}$$

For the commutator we apply the same sum identity to produce

$$[\hat{H},\hat{x}] = [\frac{\hat{p}^2}{2m},\hat{x}] + [V(x),\hat{x}] \tag{19}$$

The potential and position do commute, so that term vanishes. What is the commutator of the momentum and position operators? We can expand the terms as

$$[\frac{\hat{p}^2}{2m},\hat{x}] = \frac{1}{2m}(p(px)-p(xp)+(px)p-(xp)p) = \frac{1}{2m}(p(px-xp)+(px-xp)p) \tag{20}$$

Then

$$[\frac{\hat{p}^2}{2m},\hat{x}] = \frac{1}{2m}(-p[x,p]-[x,pp]) = \frac{1}{2m}(-p(i\hbar)-(i\hbar)p) = -\frac{i\hbar}{m}p \tag{21}$$

Now the trick is to let g = 1. The derivatives of g die off and we are left with

$$[f,\hat{p}] = -\frac{\hbar}{i}\frac{df}{dx} = i\hbar\frac{df}{dx}$$

$$[f(x),\hat{p}] = i\hbar\frac{df}{dx}$$

The result itself is not particularly illuminating, but we can rephrase it in terms of the Hamiltonian to produce a more recognizable result. We note that

$$\frac{\partial H}{\partial p} = \frac{\partial}{\partial p}(\frac{\hat{p}^2}{2m}) = -2i\hbar\frac{\hat{p}}{2m} = -\frac{i\hbar}{m}p \tag{22}$$

Thus we can write

$$\langle \dot{x} \rangle = -\langle \frac{\partial H}{\partial p} \rangle \tag{23}$$

Applying Ehrenfest's theorem using the momentum and position operators has reproduced **Hamilton's equations**.

4. An Interesting Limit

One final extension of the above results is to consider the relations of canonical coordinates—the position and momenta—and the system's Hamiltonian in classical mechanics. We consider the Poisson Brackets on two functions of a phase space, u(q,p) and v(q,p), defined as[7]

$$\{u, v\} = \sum_{j=1}^{N}(\frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_j}) \tag{24}$$

[7] In the canonical coordinates, q and p, one can take the canonical relations as defined by the Lagrangian of the system:

$$\mathbf{p} = \nabla_{\dot{q}} L(\mathbf{q}, \dot{q}, t)$$

Where the Lagrangian is a function of the generalized coordinates, velocities, and time. The vectors p, q and its derivative are vectors in the phase space. Their components are the individual coordinates of the phase space, in this case the entries for j. There exists for a Lagrangian a Legendre transform which effectively inverts the equation above:

$$\mathbf{q} = \nabla_{\mathbf{p}} H(\mathbf{q}, \mathbf{p}, t)$$

Where H is the Hamiltonian. Functions who are Legendre transforms of each other satisfy the relation

$$H(\mathbf{q}, \mathbf{p}, t) = \dot{\boldsymbol{q}} \cdot \mathbf{p} - L(\mathbf{q}, \dot{q}, t)$$

You can use the same procedure in taking Lagrange's equations of

$$\dot{p} = \nabla_{\mathbf{q}} L(\mathbf{q}, \dot{q}, t)$$

And transforming them into the required Hamilton's equations:

$$\dot{p} = -\nabla_{\mathbf{q}} H(\mathbf{q}, \mathbf{p}, t)$$

In this case we iterate over the N degrees of freedom of the system (N canonical position coordinates and N canonical momenta coordinates). Now I will leave the elucidation of the details of the Poisson brackets and their meaning in classical mechanics to better derivations.[8] But an important result to note is the application of the Poisson bracket of a canonical position coordinate and the Hamiltonian (which we assume is a function of q and p). The result is:

$$\{q_j, H\} = \frac{\partial q_j}{\partial q_j}\frac{\partial H}{\partial p_j} = \frac{\partial H}{\partial p_j} = \dot{q}_j \tag{25}$$

In this case we are only considering the jth coordinate. Thus the Poisson brackets spit out the Hamiltonian equations of motion. You should be able to verify that the same occurs when you take the Poisson bracket of the momenta and Hamiltonian.

As I noticed myself when first studying the canonical commutation relations, it is the apparent similarity between the two operations. With a sign change and a switch in the order of q and H, we get the same result (in terms of expectation value) we got when calculating the commutator of the Hamiltonian and the position operator.

Thus it is noted that in the limiting case, as Planck's constant tends to zero, we have a relationship between the commutators and the classical Poisson brackets:

$$\lim_{\hbar\to 0} \frac{1}{i\hbar}[\hat{A}, \hat{B}] \ = \{A, B\} \tag{26}$$

We note that the "cookbook" for translating quantum to classical mechanics is to take the operators A and B and translate them to functions of the phase space A and B and take the limit as shown above. In the case of the commutator of the position and momentum operators we have:

$$\hat{A} = \hat{x}, \hat{B} = -i\hbar\frac{d}{dx} \tag{27}$$

Thus

$$[\hat{x}, \hat{p}] = i\hbar \tag{28}$$

[8]There are various useful texts for exploring the classical canonical relations. The texts which I have used are John R. Taylor, Classical Mechanics (University Science Books, 2005), 522-550. R. Douglas Gregory, Classical Mechanics (Cambridge: Cambridge University Press, 2006), 393-414. Leonard Susskind, The Theoretical Minimum (Basic Books, 2014), 145-189. H. C. Corben, Philip Stehle, Classical Mechanics, 2nd ed. (New York: Dover Publications, 1994), 154-162. I particularly liked Taylor's chapter (there is a reason this text is a standard). In undergrad I was assigned Gregory's however; I found the mathematical maturity more demanding, but having first taken notes from Gregory's I found Taylor's chapter more digestible (as I was able to focus purely on the heuristic arguments). Susskinds three chapters (on Hamiltons equations, Liouville Theorem, and Poisson Brackets) were also extremely helpful. Susskind's text in particular is very useful as he immediately relates the Poisson Brackets to their quantum counterparts. Shankar's summary of classical mechanics in chapter 2 (75-98, see full citation below) is also very helpful.

And applying the limit we have:

$$\lim_{\hbar \to 0} \frac{1}{i\hbar}[\hat{x}, \hat{p}] = \lim_{\hbar \to 0} \frac{1}{i\hbar} i\hbar = 1 \tag{29}$$

It should be clear that the Poisson bracket of the canonical position and momentum is equal to one. The key point here is that the correspondence between classical Poisson brackets and quantum commutators establishes a fundamental connection between the dynamics of classical and quantum systems. In particular, one can translate a classical dynamical equation into its quantum counterpart by replacing the Poisson bracket of two general dynamical variables with 1/i times their quantum commutator. Given this seemingly straightforward correspondence—where quantum canonical relations differ from their classical counterparts by only a proportionality factor—you might wonder why quantum mechanics leads to such counterintuitive results. The key distinction lies in the algebraic structure of quantum mechanics: while classical observables commute, quantum operators generally do not. This non-commutativity gives rise to fundamental results such as the uncertainty principle, where incompatible observables, like position and momentum, cannot be simultaneously determined with arbitrary precision.

5. Conclusions

In this article, I present a collection of my notes, drawn from various sources, on the classical-quantum correspondence. The more general correspondence principle, as discussed by Bohr, provides several ways to demonstrate the statistical emergence of classical physics from the fundamental quantum canonical relations. However, this correspondence is not universal. If quantum mechanics always reduced to classical mechanics, there would be no need for quantum theory at all. Instead, what is true—at least for certain systems—is that their average behavior tends to approximate classical evolution.

What conditions enable this correspondence? First, we require a state with a well-defined position and momentum wave function—one in which the uncertainties in both observables are small relative to the macroscopic scale. For example, if the uncertainty in a position-space wave function is on the order of a femtometer ($10^{-}15$ m), the corresponding momentum uncertainty is typically an order of magnitude smaller. A system prepared in such a state exhibits sharply peaked probability distributions around the classical values of position and momentum. As the system evolves, the expectation values of these observables approximately obey Hamilton's equations.

What about systems that do not fit this regime? These are states with significant uncertainty in both position and momentum distributions. Ehrenfest's theorem remains valid for such systems, as its derivation relies only on the hermiticity of the Hamiltonian. Thus, even in these cases, the expectation values of observables evolve according to the commutator of the observable with the Hamiltonian and any explicit time dependence of the observable itself. However, these systems are not ones we typically encounter in the macroscopic world, where objects exhibit well-defined momentum and position relative to their scale.

For certain systems—such as free particles—wave packets spread over time, maintaining expectation values that obey Ehrenfest's theorem but without producing statistical results that align with classical dynamics. These are the cases where the classical-quantum correspondence breaks down.

By contrast, in macroscopic systems—the ones humans have interacted with for millennia and studied for centuries—the large-scale, averaged behavior closely approximates the descriptions provided by Newton, Poisson, and Hamilton. This is where the correspondence principle is most apparent.