Soft-Margin SVM

$$\min \ \frac{1}{2}\|w\|_2^2 + \frac{c}{n}\sum_{i=1}^{n} s_i$$

$$y_i(w^T x_i + b) \geq 1 - s_i \quad \forall i$$

$$s_i \geq 0 \quad \forall i$$

$$w^* = \sum_{i=1}^{n} \alpha_i^* y_i \underline{x_i}$$

Dual Problem

$$\max_{\alpha,\beta} \ -\frac{1}{2}\sum_{ij} \alpha_i \alpha_j y_i y_j x_i^T x_j + \sum \alpha_i$$

$$\sum_i \alpha_i y_i = 0$$

$$\alpha_i + \beta_i = \frac{c}{n}$$

$$\alpha_i \geq 0 \quad \beta_i \geq 0 \quad \forall i$$

primal problem

$$\min f(x)$$

$$g_i(x) \leq 0 \quad \forall i$$

Lagrangian

$$L(x,\lambda) = f(x) + \sum_i \lambda_i g_i(x)$$

Dual

$$\max_{\lambda \geq 0} L_D(\lambda)$$

Lagrange Dual function $L_D(\lambda) = \min_x L(x,\lambda)$

Eliminate $\beta_i \Rightarrow \quad \beta_i = \frac{c}{n} - \alpha_i, \quad \beta_i \geq 0 \Rightarrow 0 \leq \alpha_i \leq \frac{c}{n}$

Define $K_{ij} = \langle x_i, x_j \rangle \qquad \mathrm{Diag}(y) = \begin{bmatrix} y_1 & & \\ & \ddots & \\ & & y_n \end{bmatrix}$

Simplify the dual:

$$\max \ -\frac{1}{2}\alpha^T \mathrm{Diag}(y) \cdot K \ \mathrm{Diag}(y) \cdot \alpha + \sum_{i=1}^{n} \alpha_i$$

$$0 \leq \alpha_i \leq \frac{c}{n} \quad \forall i$$

$$\sum \alpha_i y_i = 0$$

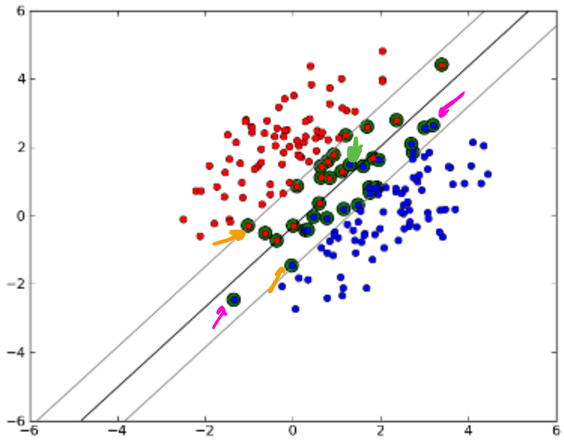
Test sample x

$$\hat{y} \rightarrow \mathrm{sign}(w^{*T} x + b^*) = \mathrm{sign}\left(\sum_i \alpha_i^* y_i \langle x_i, x \rangle + b^*\right)$$

Complementary slackness: $\quad \left(g_i(x^*) \cdot \lambda_i^* = 0 \right)$

$$\alpha_i^* \cdot \left(1 - s_i^* - y_i(w^{*T} x_i + b^*)\right) = 0$$

If $y_i(w^{*T}x_i + b^*) + s_i^* = 1$, we call x_i a <u>support vector</u>.



If x_i is not a support vector, we have $\alpha_i^* = 0$

$$w^* = \sum_{i=1}^{n} \alpha_i^* y_i \underline{x_i} = \sum_{\substack{\text{support} \\ \text{vectors}}} \alpha_i^* y_i \underline{x_i}$$

(a) If $y_i(w^{*T}x_i + b_i) > 1$, then $s_i^* = 0$ and x_i is not a SV.

- (b) If $y_i(w^{*T}x_i + b_i) = 1$, then $s_i^* = 0$ and x_i is a SV
- (c) If $0 \le y_i(w^{*}x_i + b_i) < 1$, then $s_i^* > 0$ and x_i is a SV
- (d) If $y_i(w^{*T}x + b_i) < 0$ then $s_i^* > 0$ and x_i is a SV

How to find b^*?

Find an i s.t. $0 < \alpha_i^* < \frac{c}{n}$ which means $\beta_i = \frac{c}{n} - \alpha_i > 0$

Comp. Slackness $s_i^* = 0$ and $y_i(w^{*T}x_i + b^*) = 1 - s_i^* = 1$

Solve for $b^* = \frac{1}{y_i} - w^{*T}x_i = y_i - w^{*T}x_i$

since $y_i \in \{-1, +1\}$