MINKOWSKI PROBLEMS FOR GEOMETRIC MEASURES

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ABSTRACT. This paper describes the theory of Minkowski problems for geometric measures in convex geometric analysis. The theory goes back to Minkowski and Aleksandrov and has been developed extensively in recent years. The paper surveys classical and new Minkowski problems studied in convex geometry, PDEs, and harmonic analysis, and structured in a conceptual framework of the Brunn-Minkowski theory, its extensions, and related subjects.

CONTENTS

1. INTRODUCTION

Convex geometric analysis studies — from an analytic perspective — global geometric invariants and locally defined geometric measures of convex bodies in Euclidean space [196]. Global geometric invariants, such as volume and surface area, quantify the size of a convex body, while geometric measures characterize its shape. The geometric measures extend the concept of pointwise curvature invariants of smooth hypersurfaces to local invariants of arbitrary convex bodies. They are essential for studying nonsmooth convex bodies such as polytopes. The most important geometric measures arise as the differentials of global geometric invariants viewed as functionals on the space of convex bodies.

A geometric measure of a convex body is usually defined on or pushed to the unit sphere via either the radial or Gauss map. A basic example is the surface area measure which is

the differential of the volume of a convex body and is a measure on the unit sphere of outer unit normals of the convex body. A fundamental problem in convex geometric analysis is the characterization of geometric measures. In other words, given a measure on the unit sphere, does it arise as a specific geometric measure of a convex body? The characterization of surface area measure is known as the classical Minkowski problem, first studied by Minkowski for polytopes in 1897. For this reason, the characterization of a geometric measure is called the Minkowski problem for that measure.

In this article we survey old and new results on Minkowski problems for a range of geometric measures. Perhaps more importantly, we show that everything can be organized in a cohesive conceptual framework that arises from the Brunn-Minkowski theory, its extensions, and related subjects.

During the period 1890–1909, Minkowski [181, 182], building on the work of Brunn, created the Brunn-Minkowski theory of convex bodies in the Euclidean space, which lies at the core of convex geometry. A fundamental result of the theory is the Brunn-Minkowski inequality, which extends the classical isoperimetric inequality and demonstrates the concavity of volume with respect to the vector addition in the Euclidean space. In the 1930s, Aleksandrov and Fenchel [5, 72] studied more general geometric invariants and proved the Aleksandrov-Fenchel inequalities for mixed volumes. Since then, the study of sharp inequalities satisfied by geometric invariants of convex bodies has been a major focus in convex geometry. Beyond that, the geometric inequalities are closely linked to analytic inequalities for functions and measures on Euclidean space. The best known example of this is the equivalence between the isoperimetric and the L^1 Sobolev inequality. Gardner's Bulletin survey [86] explains the relationships between the Brunn-Minkowski inequality and many other geometric and analytic inequalities. Lutwak's survey paper [165] describes a number of affine geometric inequalities of affine geometric invariants. Schneider's book [196] is the definitive reference for the foundations of the Brunn-Minkowski theory.

Lacking the notion of a surface area measure, Minkowski formulated the Minkowski problem in terms of facet areas for polytopes and the Gauss curvature for smooth convex bodies [181, 182]. He solved the problem for these classes of bodies. Aleksandrov and Fenchel-Jessen unified Minkowski's results by defining surface area measure for all convex bodies. They resolved the Minkowski problem for arbitrary convex bodies.

However, the study of geometric measures of convex bodies and the corresponding Minkowski problems became a major focus of research in convex geometry only after Lutwak introduced L_p surface area measure and its L_p Minkowski problem in 1993 [166]. In 2016, dual curvature measures were constructed in the paper [119]. The study of geometric measures and Minkowski problems has greatly enriched the field of convex geometry.

The Minkowski problem for a geometric measure is simply the following: Given a measure on the unit sphere, what are the necessary and sufficient conditions that there exists a convex body whose geometric measure is equal to the given measure? If such a convex body exists, to what extent is it unique? In the setting of smooth convex bodies, this can be stated in terms of solving a fully nonlinear partial differential equation on the unit sphere. Significant advances in solving Minkowski problems have been made in recent years for not only geometric measures that arise in the classical Brunn-Minkowski theory, but also those in the dual Brunn-Minkowski theory, the L_p Brunn-Minkowski theory, integral geometry, and other subjects.

2. Elementary Minkowski Problems

Minkowski problems are natural geometric problems, related to the construction of convex bodies in the Euclidean space. To illustrate their geometric meaning, we describe three Minkowski problems for polygons in the plane. This helps us understand better the general Minkowski problems in any dimension.

2.1. Minkowski Problem for Polygons

More than a century ago, Minkowski [181] studied the problem of finding a convex polytope whose facets have prescribed surface areas and outer unit normals. In the plane, the problem can be stated as follows:

Given positive numbers $l_1, \ldots, l_m$ and unit vectors $v_1, \ldots, v_m$ in $\mathbb{R}^2$, construct a convex polygon P in $\mathbb{R}^2$ so that its sides have lengths $l_1, \ldots, l_m$ and corresponding outer normals $v_1, \ldots, v_m$.
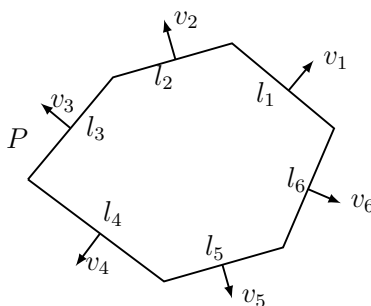
There is a discrete geometric measure on the unit circle associated with a convex polygon P:

$$S(P, \cdot) = \sum_{i=1}^{m} l_i \, \delta_{v_i},$$

where δ_{v_i} is the unit point mass (delta measure) at v_i. This measure, concentrated at the normal vectors and with weights of the side lengths of a polygon, has an analogue for an arbitrary convex body in higher dimension, called the *surface area measure* of the convex body.

The *Minkowski problem* for convex polygons reads: Given a discrete measure μ on the unit circle, is there a convex polygon P in $\mathbb{R}^2$ such that $S(P, \cdot) = \mu$?

This can be solved by a straightforward construction. One can assume that the unit vectors $v_1, \ldots, v_m$ are listed in counterclockwise order. Obviously, the outer unit normals of a convex polygon cannot lie on a half-circle. It is therefore necessary to assume this for the unit vectors $v_1, \ldots, v_m$. We can now try to construct a polygon as follows: Start with an oriented line segment N_1 of length l_1 that lies at a counterclockwise right angle to v_1. Next, draw an oriented line segment N_2 of length l_2 that starts at the endpoint of N_1, and is counterclockwise orthogonal to v_2. Repeat this for oriented line segments $N_3, \ldots, N_m$. These oriented line segments will bound a convex polygon if and only if the endpoint of N_m is the starting point of N_1. It is not hard to show that this holds if and only if $l_1 v_1 + \cdots + l_m v_m = 0$. Thus, the solution to the Minkowski problem for polygons is the following:

Let $l_1, \ldots, l_m$ be positive numbers and $v_1, \ldots, v_m$ be unit vectors in $\mathbb{R}^2$. Then there exists a convex polygon P in $\mathbb{R}^2$ with side lengths $l_1, \ldots, l_m$ and corresponding outer unit normals

$v_1, \ldots, v_m$ *if and only if* $l_1 v_1 + \cdots + l_m v_m = 0$ *and the unit vectors* $v_1, \ldots, v_m$ *do not all lie in a unit half-circle. Moreover, the polygon* P *is unique up to translation.*

For a detailed elementary proof of this result, see Klain [132].

2.2. Aleksandrov Problem for Polygons

Aleksandrov [7] went on to define a new geometric measure, called *integral curvature*, one of the curvature measures constructed later by Federer [71]. Aleksandrov proposed and solved the Minkowski problem for integral curvature, now called the *Aleksandrov problem*.

For a convex polygon in $\mathbb{R}^2$ that contains the origin in its interior, the unit vectors from the origin to the vertices are called the radial vectors of the polygon, and the angle at a vertex formed by the normals of two adjacent sides is called a normal angle of the polygon. The Aleksandrov problem can be stated as follows:

Given positive numbers $\alpha_1, \ldots, \alpha_m$ *and unit vectors* $u_1, \ldots, u_m$ *in* $\mathbb{R}^2$, *construct a convex polygon* P *in* $\mathbb{R}^2$ *that contains the origin in its interior with normal angles* $\alpha_1, \ldots, \alpha_m$ *and radial vectors* $u_1, \ldots, u_m$.
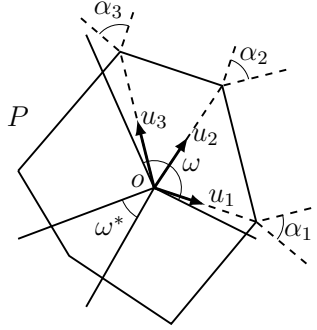
There is a discrete geometric measure on the unit circle associated with this convex polygon,

$$J(P, \cdot) = \sum_{i=1}^{m} \alpha_i \, \delta_{u_i}.$$

This measure, concentrated at the radial vectors and with weights of the normal angles of the polygon, was extended by Aleksandrov to an arbitrary convex body in any dimension, called the integral curvature of the convex body.

The *Aleksandrov problem* for convex polygons reads: Given a discrete measure μ on the unit circle, is there a convex polygon P in $\mathbb{R}^2$ that contains the origin in its interior such that $J(P, \cdot) = \mu$?

Since the sum of normal angles of a convex polygon is equal to 2π, there is the necessary condition, $\alpha_1 + \cdots + \alpha_m = 2\pi$. Also, the given unit vectors cannot all lie in a half circle. But these conditions are not sufficient.

Assume that the radial vectors $u_1, \ldots, u_m$ are given on the unit circle in a counterclockwise order. Let $\omega \subset S^1$ be an angular sector that contains only the radial vectors $u_i, u_{i+1}, \ldots, u_j$, where $1 \leq i \leq j \leq m$. Then for every normal vector v at a boundary point of P whose radial direction is inside ω, there exists $u \in \omega$ such that $u \cdot v > 0$, that is, v is outside of the polar angle,

$$\omega^* = \{v \in S^1 : v \cdot u \leq 0 \text{ for all } u \in \omega\}.$$

This implies that

$$\alpha_i + \cdots + \alpha_j + |\omega^*| < 2\pi, \quad \text{i.e.} \quad \alpha_i + \cdots + \alpha_j < \pi + |\omega|.$$

This condition has also to be satisfied by the normal angles. Aleksandrov showed that these necessary conditions are also sufficient. The solution to the Aleksandrov problem can be stated as follows:

If $\alpha_1, \ldots, \alpha_m$ are positive numbers and $u_1, \ldots, u_m$ are unit vectors in $\mathbb{R}^2$ in a counterclockwise order, then there exists a convex polygon P in $\mathbb{R}^2$ containing the origin in its interior so that it has normal angles $\alpha_1, \ldots, \alpha_m$ and radial vectors $u_1, \ldots, u_m$ if and only if the unit vectors $u_1, \ldots, u_m$ do not all lie in a half circle,

$$\alpha_1 + \cdots + \alpha_m = 2\pi, \quad and \quad \alpha_i + \cdots + \alpha_j < \pi + |\omega|, \quad 1 \le i \le j \le m,$$

for any angle ω that contains the radial vectors $u_i, \ldots, u_j$.

One could try to give a construction to the Aleksandrov problem in $\mathbb{R}^2$ similar to that of the Minkowski problem described in the previous section. But it does not seem easy to have one.

2.3. Logarithmic Minkowski Problem for Polygons

Given a convex polygon P and an interior point $o \in P$, each side ℓ of P uniquely determines a triangle that has ℓ as a side and o as a vertex. In addition to the side lengths and the normal angles of a convex polygon, the areas of these triangles are also natural measurements of the polygon. Prescribing the triangle areas of a polygon is called the *logarithmic Minkowski problem* for polygons, which can be stated as follows:

Given positive numbers $a_1, \ldots, a_m$ and unit vectors $v_1, \ldots, v_m$ in $\mathbb{R}^2$, construct a convex polygon P in $\mathbb{R}^2$ with triangle areas $a_1, \ldots, a_m$ and corresponding outer unit normals $v_1, \ldots, v_m$.
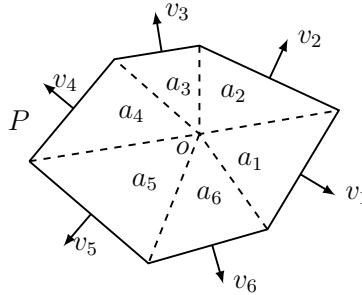
There is another discrete geometric measure on the unit circle associated with a polygon P,

$$V(P, \cdot) = \sum_{i=1}^{m} a_i \, \delta_{v_i},$$

which concentrates at the unit normal vectors and has weights of triangle areas. The total measure of $V(P, \cdot)$ is obviously the area of P, denoted by $V(P)$. This measure can be defined for an arbitrary convex body in higher dimensions, called the *cone-volume measure* of the convex body.

The logarithmic Minkowski problem for convex polygons reads: Given a discrete measure μ on the unit circle, is there a convex polygon P in $\mathbb{R}^2$ such that $V(P, \cdot) = \mu$?

The logarithmic Minkowski problem is much more difficult to solve than the Minkowski problem and the Aleksandrov problem. In fact, even the planar discrete case, as stated above, has not been solved completely.

If the prescribed measure is symmetric, then there can be an obstruction. The simplest example is if the prescribed unit normals are $v_1, v_2, v_3 = -v_1, v_4 = -v_2$ and the corresponding prescribed areas are $a_1, a_2, a_3 = a_1, a_4 = a_2$. It is clear that for such a measure, the polygon must be a parallelogram centered at the origin. This implies that $a_1 = a_2$. Therefore, if the prescribed measure has $a_1 \neq a_2$, then there is no solution. On the other hand, if $a_1 = a_2$ and P is a solution, then uniqueness does not hold, because the polygon ϕP, where ϕ is a linear transformation such that $\phi v_1 = b v_1$ and $\phi v_2 = b^{-1} v_2$, $b > 0$, is also a solution.

More generally, if P is an m-sided polygon such that, for some $3 \leq q \leq m - 1$, $v_q = -v_1$ and $a_q = a_1$ and Q is the convex hull of the edges normal to v_1, v_q, then it is not difficult to show that

$$a_1 + a_q = \frac{1}{2}V(Q) \leq \frac{1}{2}V(P),$$

where equality holds if and only if P is a parallelogram. This gives a necessary condition for the prescribed measure to be the cone-volume measure of a polygon.

For origin-symmetric polygons, Stancu [206] showed, using a heat flow argument, that this is also a sufficient condition and proved the following:

Let $a_1, \ldots, a_k$ be positive numbers and $v_1, \ldots, v_k, -v_1, \ldots, -v_k$ be unit vectors ordered counterclockwise in $\mathbb{R}^2$. Then there exists an origin-symmetric convex polygon P with triangle areas $a_1, \ldots, a_k, a_1, \ldots, a_k$ and corresponding unit normal vectors $v_1, \ldots, v_k, -v_1, \ldots, -v_k$ if and only if

$$a_i < \frac{1}{2}(a_1 + \cdots + a_k), \quad i = 1, \ldots, k, \quad \text{when } k > 2, \quad \text{or}$$
$$a_1 = a_2, \quad \text{when } k = 2.$$

Partial results were also obtained in the general case, which remains open in $\mathbb{R}^2$.

3. Basic Concepts of Convex Bodies

To make the article readable for people from different fields, we try to provide basic concepts and facts in this and later sections. Without affecting the reading, one can jump to the next section and easily look back for a concept if necessary.

3.0.1. Basic Notations. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and B^n the n-dimensional unit ball in $\mathbb{R}^n$ centered at the origin. Sometimes, we also write B^n as B. The boundary of B^n is the $(n-1)$-dimensional unit sphere denoted by S^{n-1}. Let $G_{n,i}$ denote the Grassmann manifold of i-dimensional subspaces in $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, let $x \cdot y$ be the Euclidean inner product and $|x|$ the Euclidean norm.

The integration with respect to the Lebesgue measure in $\mathbb{R}^n$ or on a hypersurface in $\mathbb{R}^n$ is denoted by dx. The integration with respect to the normalized Haar measure on $G_{n,i}$ is denoted by $d\xi$. Let $\mathcal{H}^k$ be the k-dimensional Hausdorff measure in $\mathbb{R}^n$.

Denote by $C^+(S^{n-1})$ the set of positive continuous functions on S^{n-1}, and by $C_e^+(S^{n-1})$ the set of positive continuous even functions on S^{n-1}. For a Borel measure μ on S^{n-1}, denote its total mass by $|\mu|$.

3.0.2. Convex bodies. A *convex body* K in $\mathbb{R}^n$ is a compact convex subset with nonempty interior. Denote by $\text{int}\,K$ the set of interior points of K, and ∂K the boundary of K. Let $\mathcal{K}_o^n$ be the set of all convex bodies that contain the origin in their interiors. We often call an origin-symmetric convex body a symmetric convex body.

The *polar body* of $K \in \mathcal{K}_o^n$ is defined to be

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

Denote by $V(K)$ the volume of K and by $S(K)$ the surface area of ∂K that is the Hausdorff measure $\mathcal{H}^{n-1}(\partial K)$. Let $\omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ be the volume of B^n. The surface area of S^{n-1} is $\frac{2\pi^{n/2}}{\Gamma(n/2)} = n\omega_n$.

A convex body P is an n-dimensional *polytope* if it is the convex hull of finitely many points $x_1, \ldots, x_m$ in $\mathbb{R}^n$.

3.0.3. Homogeneous Functions and Monge-Ampère Type Equations on S^{n-1}. For a twice-differentiable function h in $\mathbb{R}^n$, let ∇h and $\nabla^2 h$ be the gradient and Hessian in $\mathbb{R}^n$, respectively. If h is homogeneous of degree 1, then ∇h is homogeneous of degree 0, and

$$y \cdot \nabla h(y) = h(y), \quad \nabla^2 h(y)y = 0, \quad y \in \mathbb{R}^n.$$

Thus, any $v \in S^{n-1}$ is a unit eigenvector of $\nabla^2 h(v)$ corresponding to its zero eigenvalue. Then the restriction $\nabla^2 h(v)|_{v^\perp}$ is a linear transformation in $v^\perp$ whose matrix representation under an orthonormal basis of $v^\perp$ is $(\nabla_{ij} h + h\delta_{ij})$, where $(\nabla_{ij} h)$ is the spherical Hessian with respect to the orthonormal basis and δ_{ij} is the Kronecker delta. If h is also a convex function, then $(\nabla_{ij} h + h\delta_{ij})$ is positive semi-definite.

Most of the partial differential equations associated with Minkowski problems are Monge-Ampère type equations on S^{n-1} which can be stated as follows: Given a nonnegative continuous function f on S^{n-1}, find a function h on the unit sphere S^{n-1} such that

$$F(h, \nabla h) \det \left(\nabla_{ij} h + h\delta_{ij} \right) = f. \tag{3.1}$$

3.0.4. Support Function. A convex body has two basic functional descriptions, namely, its support and radial functions.

The *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ of a convex body K in $\mathbb{R}^n$ is defined by

$$h_K(y) = \max\{x \cdot y : x \in K\}, \quad y \in \mathbb{R}^n.$$

The support function is convex and homogeneous of degree 1.

For each $v \in S^{n-1}$, the hyperplane

$$H_K(v) = \{x \in \mathbb{R}^n : x \cdot v = h_K(v)\}$$

is called the *supporting hyperplane to K with unit normal v*, which extends the concept of a tangent hyperplane. Geometrically, the value $h_K(v)$, $v \in S^{n-1}$, is the signed distance from the origin to the support hyperplane $H_K(v)$.

3.0.5. Radial Function. The *radial function* $\rho_K : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ of a convex body K in $\mathcal{K}_o^n$ is defined by

$$\rho_K(x) = \max\{t > 0 : tx \in K\}, \quad x \in \mathbb{R}^n.$$

It is homogeneous of degree -1. If $u \in S^{n-1}$, then $\rho_K(u)$ is the radial distance from the origin to the boundary ∂K in direction u. When $x \in \partial K$, $\rho_K(x) = 1$.

support function	$h_K(v) = x \cdot v$
radial function	$\rho_K(u) = |x|$
radial map	$r_K(u) = x$
Gauss map	$\nu_K(x) = v$
radial Gauss map	$\alpha_K(u) = v$

The convexity of h_K implies that h_K is twice-differentiable almost everywhere in $\mathbb{R}^n$, that is, h_K has the Taylor expansion at almost all $y \in \mathbb{R}^n$,
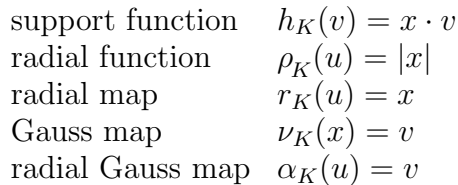
$$h_K(y + z) = h_K(y) + z \cdot \nabla h_K(y) + \frac{1}{2} z \cdot \left(\nabla^2 h_K(y) z \right) + o(|z|^2).$$

The Hessian $\nabla^2 h_K(y)$ is positive semi-definite.

3.0.6. Curvature Function. The convexity of h_K implies that the restriction $\nabla^2 h_K(y)|_{y^\perp}$ is a positive semi-definite linear transformation on $y^\perp$. If $v \in S^{n-1}$ and h_K is twice differentiable at v, then

$$f_K(v) = \det \left(\nabla^2 h_K(v)|_{v^\perp} \right) \tag{3.2}$$

is called the *curvature function of K*. If h_K is C^2, then

$$f_K(v) = \det \left(\nabla_{ij} h_K(v) + h_K(v) \delta_{ij} \right).$$

If $f_K(v) > 0$, then $f_K(v)$ is the reciprocal of the Gauss curvature at the boundary point of K with outer unit normal v.

3.0.7. Wulff Shape and Convex Hull. A basic construction of a convex body from a function is the Wulff shape. Given a positive continuous function h on S^{n-1}, a convex body $[h]$, called the *Wulff shape* of h, is defined by

$$[h] = \{ x \in \mathbb{R}^n : x \cdot v \le h(v) \text{ for all } v \in S^{n-1} \}.$$

It contains the origin in its interior. Intuitively, it is the largest convex body whose support function is nowhere greater than h. Obviously,

$$K = [h_K], \quad \text{for } K \in \mathcal{K}_o^n.$$

Another basic construction of a convex body from a function is the convex hull. Given a positive continuous function ρ on S^{n-1}, a convex body $\langle \rho \rangle$, called the *convex hull* of ρ, is defined as the convex hull of the points $\rho(u)u$, $u \in S^{n-1}$, i.e.,

$$\langle \rho \rangle = \text{conv}\{ \rho(u)u : u \in S^{n-1} \}. \tag{3.3}$$

It is also a convex body that contains the origin in its interior. It is the smallest convex body whose radial function is nowhere less than ρ, and

$$K = \langle \rho_K \rangle, \quad \text{for } K \in \mathcal{K}_o^n.$$

Wulff shape, convex hull, and polar are related by the following formula (see, for example, [119])

$$[1/\rho] = \langle \rho \rangle^*.$$

3.0.8. Hausdorff Metric. Convergence of sequences of convex bodies can be defined using the *Hausdorff distance*,

$$d_H(K, L) = \|h_K - h_L\|_\infty.$$

A basic fact of convergence of sequences of convex bodies is the Blaschke selection theorem: *If a sequence $\{K_i\}$ of convex bodies in $\mathbb{R}^n$ is uniformly bounded, then it has a subsequence that converges to a compact convex set in $\mathbb{R}^n$.* It is a version of the classical Bolzano-Weierstrass theorem in $\mathbb{R}^n$.

3.0.9. Radial and Gauss Maps. There are two fundamental maps between the boundary of a convex body and the unit sphere, known as the radial and Gauss maps.

The *radial map* $r_K : S^{n-1} \to \partial K$ of a convex body K in $\mathcal{K}_o^n$ is defined by

$$r_K(u) = \rho_K(u)u, \quad u \in S^{n-1}.$$

Geometrically, r_K maps a radial direction u to the boundary point $\rho_K(u)u$ in that direction. The radial map is a homeomorphism between the radial sphere and the boundary of K. The boundary ∂K is differentiable at $r_K(u)$ if and only if ρ_K is differentiable at u. The gradient $\nabla \rho_K(u)$ in $\mathbb{R}^n$ is an inward pointing normal vector of ∂K at $r_K(u)$.

A point $x \in \partial K$ is called a regular or smooth point if ∂K is differentiable at x. Denote by $\partial' K$ the set of regular points of ∂K. The boundary ∂K has a unique outer unit normal $\nu_K(x)$ at a regular point $x \in \partial' K$. This defines the *Gauss map*, $\nu_K : \partial' K \to S^{n-1}$. That is, the Gauss map ν_K maps a regular point to the unique outer unit normal at the point. The Gauss map can be expressed using the gradient of the radial function,

$$\nu_K(x) = -\frac{\nabla \rho_K(x)}{|\nabla \rho_K(x)|}, \quad x \in \partial' K.$$

Since the boundary ∂K is differentiable almost everywhere with respect to the Hausdorff measure $\mathcal{H}^{n-1}$, the Gauss map is defined almost everywhere on ∂K.

The *radial Gauss map* α_K, which maps a unit radial vector to the corresponding outer unit normal vector, is defined almost everywhere on the radial sphere S^{n-1} by

$$\alpha_K = \nu_K \circ r_K,$$

that is,

$$\alpha_K(u) = -\frac{\nabla \rho_K(u)}{|\nabla \rho_K(u)|} \text{ if } u \in r_K^{-1}(\partial' K).$$

radial map
$r_K(\omega) = \sigma$
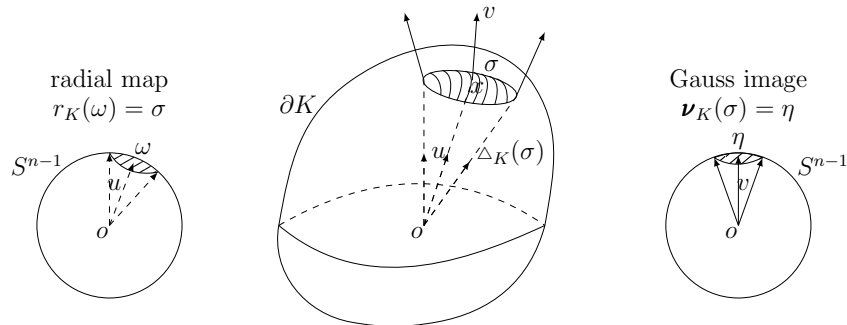
Gauss image
$\boldsymbol{\nu}_K(\sigma) = \eta$

3.0.10. Normal Cone and Gauss Image. If $K \in \mathcal{K}_o^n$, then the *normal cone* of K at $x \in \partial K$ is the set of all outer unit normals of supporting hyperplanes that contain x, i.e.,

$$N(K, x) = \{v \in S^{n-1} : (y - x) \cdot v \leq 0 \text{ for all } y \in K\} = \{v \in S^{n-1} : x \in H_K(v)\}.$$

The normal cone at x contains only one unit vector v if and only if ρ is differentiable at x.

For $\sigma \subset \partial K$, the *Gauss image of σ*, also called the *spherical image of σ*, is the union of the normal cones at each $x \in \sigma$, i.e.,

$$\boldsymbol{\nu}_K(\sigma) = \{v \in S^{n-1} : x \in H_K(v) \text{ for some } x \in \sigma\} = \bigcup_{x \in \sigma} N(K, x) \subset S^{n-1}.$$

If ∂K is differentiable at $x \in \partial K$, then there is a unique supporting hyperplane, which implies that the normal cone has only a single vector and therefore

$$\boldsymbol{\nu}_K(\{x\}) = N(K, x) = \{\nu_K(x)\}.$$

For a set $\eta \subset S^{n-1}$ of unit normal vectors, the *reverse Gauss image of η* is defined by

$$\boldsymbol{\nu}_K^*(\eta) = \{x \in \partial K : x \in H_K(v) \text{ for some } v \in \eta\} \subset \partial K. \tag{3.4}$$

When K contains the origin in its interior, it is more convenient to use the *radial Gauss image* of $\omega \subset S^{n-1}$, defined to be

$$\boldsymbol{\alpha}_K(\omega) = \boldsymbol{\nu}_K(r_K(\omega)), \tag{3.5}$$

or equivalently,

$$\boldsymbol{\alpha}_K(\omega) = \{v \in S^{n-1} : r_K(u) \in H_K(v) \text{ for some } u \in \omega\} = \bigcup_{u \in \omega} N(K, r_K(u)).$$

The radial Gauss image maps subsets of the radial sphere to subsets of the normal sphere.

The *reverse radial Gauss image* $\boldsymbol{\alpha}_K^*$ is defined by

$$\boldsymbol{\alpha}_K^*(\eta) = \{u \in S^{n-1} : r_K(u) \in H_K(v) \text{ for some } v \in \eta\}, \quad \eta \subset S^{n-1}, \tag{3.6}$$

that is,

$$\boldsymbol{\alpha}_K^*(\eta) = r_K^{-1}(\boldsymbol{\nu}_K^*(\eta)).$$

The reverse radial Gauss image maps subsets of the normal sphere to subsets of the radial sphere. For an arbitrary convex body K (that contains the origin in its interior), the reverse radial Gauss image of K and the radial Gauss image of K^* satisfy (see [119])

$$\boldsymbol{\alpha}_{K^*}(\eta) = \boldsymbol{\alpha}_K^*(\eta), \quad \eta \subset S^{n-1}.$$

4. Descriptions of Minkowski Problems and Variational Methods

4.0.1. Geometric Measures of Convex Bodies. The essential view is that an interesting geometric measure should be the differential of an important geometric invariant. Well-known geometric measures arise in such a way. For example, surface area measure is the differential of volume, area measures are differentials of quermassintegrals, and dual curvature measures are differentials of dual quermassintegrals. We give a description of geometric measures.

Let $\mathcal{K}$ be a class of convex bodies in $\mathbb{R}^n$ and $\mathcal{B}$ the set of finite Borel measures on S^{n-1}. A geometric measure of convex bodies over $\mathcal{K}$ is a measure-valued map,

$$M : \mathcal{K} \to \mathcal{B}, \quad K \mapsto M(K, \cdot),$$

which is the differential of a global geometric invariant W. That is, for each convex body K,

$$\frac{dW(K_t)}{dt}\Big|_{t=0^+} = \int_{S^{n-1}} f \, dM(K, \cdot), \tag{4.1}$$

where K_t, $0 \le t < \delta$, is a perturbation family of convex bodies generated by K, a geometric operation, such as the Wulff shape or convex hull, and a continuous function f on S^{n-1}. The class $\mathcal{K}$ is assumed to contain every such perturbation family K_t.

The essential difficulty in proving such a variational formula is that convex bodies are not smooth and the Wulff shape and convex hull are also not smooth as operations.

4.0.2. General Minkowski Problem. *Given a finite Borel measure μ on S^{n-1}, find necessary and sufficient conditions on the measure to guarantee the existence of a convex body K in $\mathbb{R}^n$ that is a solution to the geometric measure equation (called the Minkowski equation)*

$$M(K, \cdot) = \mu. \tag{4.2}$$

In the smooth case and when μ has a density f, the measure equation (4.2) often becomes a Monge-Ampère type equation (3.1). The associated uniqueness problem reads as follows.

4.0.3. Uniqueness Problem of Geometric Measure. *For two convex bodies K and L, determine how K and L are related if $M(K, \cdot) = M(L, \cdot)$.*

The main method to solve the measure equation (4.2) is a variational method. For different measure equations, different variational methods are needed which have similar steps but require different techniques and details to carry them out. We give a rough description of the steps.

4.0.4. Variational Method. (**1**) Construct an optimization problem of some geometric invariant that is related to the geometric measure. Note that the space of convex bodies is not an appropriate setting for a variational problem, since arbitrary deformations of a convex body are not necessarily convex. Instead, it is necessary to use a function space and the Wulff shape $[h]$ or convex hull $\langle h \rangle$ of a positive continuous function h. In view of the formula (4.1), a possible example is the maximization problem

$$\sup \left\{ W([h]) : \int_{S^{n-1}} \varphi(h) \, d\mu = 1, \ h \in C^+(S^{n-1}) \right\}, \tag{4.3}$$

where $\varphi : (0, \infty) \to \mathbb{R}$ is some continuous function, for instance, $\varphi(t) = t^p$ or $\varphi(t) = \log t$. The maximum may be achieved when h is the support function of a convex body.

(**2**) Prove the existence of solutions to the optimization problem. One needs to show that an optimizing sequence is convergent and its limit gives a non-degenerate convex body. To do this, one shows that it suffices to consider only optimizing sequences of support functions of convex bodies and to show that an optimizing sequence is uniformly bounded. Then the Blaschke selection theorem implies that the optimizing sequence converges to the support function of a compact convex set. Then one needs to show that the compact convex set has nonempty interior, and also often needs to show that the origin is inside the interior.

(**3**) Prove that a solution of the optimization problem gives a solution of the Minkowski problem, that is, the Euler-Lagrange equation of the optimization problem (4.3) is the measure equation (4.2). The key is to show that the geometric invariant W is differentiable and its differential is the geometric measure M. That is, one needs to show that the variational formula (4.1) also holds true for $t = 0^-$.

5. MINKOWSKI PROBLEMS IN THE BRUNN-MINKOWSKI THEORY

In this section, we describe Minkowski problems of geometric measures associated with mixed volumes.

5.1. Basics of the Brunn-Minkowski Theory

5.1.1. Mixed Volumes. The Brunn-Minkowski theory studies mixed volumes that arise from the interaction of volume, the Minkowski addition, and differentiation. If K, L are convex bodies in $\mathbb{R}^n$, the Minkowski sum $K + L$ and dilation tK, $t > 0$, are convex bodies defined by

$$K + L = \{x + y : x \in K, \ y \in L\}, \quad tK = \{tx : x \in K\}. \tag{5.1}$$

There are identities,

$$h_{K+L} = h_K + h_L, \quad h_{tK} = th_K.$$

The first mixed volume $V_1(K, L)$ of two convex bodies K and L is, up to a constant factor, the one-sided derivative of the volume with respect to the Minkowski addition,

$$V_1(K, L) = \frac{1}{n} \lim_{t \to 0^+} \frac{V(K + tL) - V(K)}{t}.$$

Higher order mixed volumes can be defined as follows. It can be shown (see, for example, Schneider [196, Theorem 5.1.7]) that $V(K + tL)$ is a polynomial with respect to t. The mixed volumes $V_i(K, L)$, $0 \leq i \leq n$, of K and L are defined in terms of the coefficients of the polynomial,

$$V(K + tL) = \sum_{i=0}^{n} \binom{n}{i} V_i(K, L) t^i, \quad t > 0.$$

More generally, one can define the mixed volume of m convex bodies, $K_1, \ldots, K_m$, to be the coefficients of the homogeneous polynomial,

$$V(t_1 K_1 + \cdots + t_m K_m) = \sum_{i_1, \cdots, i_n = 1}^{m} V(K_{i_1}, \ldots, K_{i_n}) \, t_{i_1} \cdots t_{i_n}, \quad t_1, \ldots, t_m > 0,$$

where the coefficients $V(K_{i_1}, \cdots, K_{i_n})$ are nonnegative, symmetric in the indices, and dependent only on the bodies $K_{i_1}, \cdots, K_{i_n}$. The invariant $V(K_{i_1}, \cdots, K_{i_n})$ is called the *mixed volume* of convex bodies $K_{i_1}, \cdots, K_{i_n}$, with the property $V(K_{i_1}, \cdots, K_{i_n}) = V(K_{i_1})$ when $i_1 = \cdots = i_n$.

The ith mixed volume of K and L is given by

$$V_i(K, L) = V(K, \ldots, K, L, \ldots, L),$$

where K appears $n - i$ times, and L appears i times.

Mixed volumes are multi-linear with respect to the Minkowski addition. By this and the Riesz representation theorem, it is easily shown that for convex bodies $K_2, \ldots, K_n$ in $\mathbb{R}^n$ there exists a unique finite Borel measure $S(K_2, \ldots, K_n, \cdot)$ on S^{n-1} such that

$$V(K_1, K_2, \cdots, K_n) = \frac{1}{n} \int_{S^{n-1}} h_{K_1}(v) \, dS(K_2, \ldots, K_n, v),$$

for any convex body K_1 in $\mathbb{R}^n$. The measure $S(K_2, \ldots, K_n, \cdot)$ is called the *mixed area measure* of $K_2, \ldots, K_n$. It is translation invariant. The centroid of a mixed area measure is at the

origin,

$$\int_{S^{n-1}} v\, dS(K_2, \ldots, K_2, v) = 0.$$

Mixed volumes satisfy the Aleksandrov-Fenchel inequality,

$$V(K_1, K_2, K_3, \ldots, K_n)^2 \geq V(K_1, K_1, K_3, \ldots, K_n)V(K_2, K_2, K_3, \ldots, K_n).$$

One corollary is the following:

$$V(K_1, K_2, \ldots, K_n)^n \geq V(K_1)V(K_2) \cdots V(K_n),$$

which implies Minkowski's first inequality for mixed volumes,

$$V_1(K, L)^n \geq V(K)^{n-1}V(L). \tag{5.2}$$

Equality holds if and only if K and L are homothetic. When L is the unit ball, (5.2) becomes the classical isoperimetric inequality.

The Minkowski inequality is equivalent to the Brunn-Minkowski inequality, which states that volume is a log-concave function with respect to Minkowski sum. Specifically, given convex bodies K and L in $\mathbb{R}^n$,

$$V((1-t)K + tL) \geq V(K)^{1-t}V(L)^t, \quad t > 0, \tag{5.3}$$

with equality if and only if K and L are homothetic. An equivalent form of the inequality is

$$V(K+L)^{\frac{1}{n}} \geq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}}. \tag{5.4}$$

There are by now many proofs of the Brunn-Minkowski inequality. See, for example, the book by Schneider [196] and the survey by Gardner [86].

5.1.2. Quermassintegrals. The most important mixed volumes are quermassintegrals, which include volume, surface area and mean width.

The first mixed volume $V_1(K, L)$ can be viewed as the surface area of K with respect to the (possibly asymmetric) norm on $\mathbb{R}^n$ whose unit ball is L. In particular, if B is the standard Euclidean unit ball,

$$nV_1(K, B) = \mathcal{H}^{n-1}(\partial K) = S(K) = \text{ Euclidean surface area of } \partial K.$$

More generally, the *ith quermassintegral of K* is defined to be

$$W_i(K) = V_i(K, B).$$

It turns out that

$$W_0(K) = V(K), \; W_1(K) = \frac{1}{n}S(K), \; \text{ and } W_n(K) = V(B) = \omega_n.$$

The surface area of a polytope is the sum of areas of the $(n-1)$-dimensional faces. More generally, the quermassintegral $W_{n-i}(K)$ is a weighted sum of the areas of the i-dimensional faces of a polytope.

When K has a smooth boundary, quermassintegrals are the integrals of intermediate curvatures of the hypersurface ∂K,

$$W_{i+1}(K) = \frac{1}{n}\int_{\partial K} H_i(x)\, dx, \quad i = 0, 1, \ldots, n-1,$$

where $H_0 = 1$ and H_i is the *ith* intermediate curvature of ∂K, that is, the *ith* normalized elementary symmetric function of the principal curvatures of ∂K.

For a general convex body K, the quermassintegrals can be defined as integrals of the areas of projections of K onto subspaces,

$$W_{n-i}(K) = \frac{\omega_n}{\omega_i} \int_{G_{n,i}} V_i(K|\xi) \, d\xi,$$

where $K|\xi$ denotes the orthogonal projection of K onto the subspace $\xi \in G_{n,i}$.

5.1.3. Area Measures. Geometric measures associated with mixed volumes are mixed area measures. The most important case is about two convex bodies. For convex bodies K and L in $\mathbb{R}^n$, the *ith mixed area measure* $S_i(K, L, \cdot)$ is defined by

$$S_i(K, L, \cdot) = S(K, \ldots, K, L, \ldots, L, \cdot), \quad i = 0, \ldots, n-1,$$

where K appears i times and L appears $n - i - 1$ times. When L is the unit ball, the area measure

$$S_i(K, \cdot) = S_i(K, B, \cdot), \quad i = 0, \ldots, n-1,$$

is called the *ith area measure of K*. The $(n-1)$th area measure $S_{n-1}(K, \cdot)$ is called the surface area measure of K, usually denoted by $S(K, \cdot)$ or S_K.

The total mass of the ith area measure is the $(n - i)$th quermassintegral with factor n,

$$S_i(K, S^{n-1}) = nW_{n-i}(K), \quad i = 0, \ldots, n-1. \tag{5.5}$$

When the boundary ∂K is C^2 and has positive curvature, then

$$dS_i(K, v) = \sigma_i \, dv, \quad i = 0, \ldots, n-1,$$

where

$$\sigma_i = \frac{H_{n-i-1}}{H_{n-1}}$$

is called the ith curvature function and is equal to the ith normalized elementary symmetric function of the principal radii (i.e., reciprocals of the principal curvatures). When $i = 0$, $S_0(K, \cdot)$ is spherical Lebesgue measure and $\sigma_0 = 1$. When $i = n - 1$, $S_{n-1}(K, \cdot) = S(K, \cdot)$ is the surface area measure of K and $\sigma_{n-1} = f_K$ is the curvature function of K.

Mixed area measures are one-sided derivatives of mixed volumes. Given convex bodies K, Q, L,

$$\frac{d}{dt} V_{n-i-1}(K + tQ, L) \Big|_{t=0^+} = \frac{i+1}{n} \int_{S^{n-1}} h_Q(v) \, dS_i(K, L, v), \quad i = 0, \ldots, n-1. \tag{5.6}$$

5.1.4. Uniqueness Problem of Area Measures. A convex body is uniquely determined, up to a translation, by its ith area measure. More generally, for convex bodies K, Q, and a smooth convex body L, $i = 1, \ldots, n-1$,

$$S_i(K, L, \cdot) = S_i(Q, L, \cdot) \implies K = Q + x_0, \quad \text{for some } x_0 \in \mathbb{R}^n.$$

When L is not smooth, the uniqueness problem of mixed area measures is still unsolved. It follows from mixed volume inequalities and their equality conditions (see Schneider [196], p. 448).

5.1.5. Surface Area Measure. The surface area measure $S(K, \cdot)$ is a fundamental geometric measure of convex bodies. It generalizes and interpolates between the extreme cases of a smooth positively curved convex body and a convex polytope. It has a more direct geometric definition. The idea is to use the reverse Gauss image (3.4) and define the surface area measure to be the pullback of $\mathcal{H}^{n-1}(\partial K, \cdot)$ by the reverse Gauss image,

$$S(K, \eta) = \mathcal{H}^{n-1}\big(\boldsymbol{\nu}_K^*(\eta)\big), \quad \eta \subset S^{n-1}, \tag{5.7}$$

where $\boldsymbol{\nu}_K^*(\eta)$ is the reverse Gauss image of a Borel set η defined in (3.4). That is, $S(K, \eta)$ is equal to the surface area of the subset $\boldsymbol{\nu}_K^*(\eta) \subset \partial K$ whose unit normals belong to η. Equivalently, it can be defined by

$$\int_{S^{n-1}} f(v)\, dS(K, v) = \int_{\partial' K} f(\nu_K(x))\, dx, \tag{5.8}$$

for any bounded Borel function f over S^{n-1}. The total mass of surface area measure is the surface area of K, i.e., $S(K, S^{n-1}) = S(K)$.

By the Lebesgue-Radon-Nikodym decomposition theorem, the surface area measure can be decomposed into the sum of two mutually singular measures λ_K and ζ_K,

$$S(K, \cdot) = \lambda_K + \zeta_K, \quad d\lambda_K = f_K(v)\, dv, \tag{5.9}$$

where λ_K is absolutely continuous with respect to the spherical Lebesgue measure with density f_K, the curvature function of K, see (3.2).

When K is a polytope with outer unit normals $v_1, \ldots, v_m$ to its facets and corresponding facet areas $a_1, \ldots, a_m$, then $S(K, \cdot)$ is the discrete measure,

$$S(K, \cdot) = \zeta_K = \sum_{i=1}^{m} a_i\, \delta_{v_i}, \tag{5.10}$$

where δ_{v_i} is the delta measure concentrated at v_i.

5.2. Classical Minkowski Problem for Surface Area Measure

In this section we present in detail the classical Minkowski problem first introduced by Minkowski. It asks what are the necessary and sufficient conditions for a Borel measure on the unit sphere to be the surface area measure of a convex body. The classical Minkowski problem is the fundamental one in the Brunn-Minkowski theory, as formulated by Fenchel and Jessen [72] and by Aleksandrov [6]:

The Minkowski Problem. *Given a finite Borel measure μ on S^{n-1}, find the necessary and sufficient conditions such that there exists a convex body K in $\mathbb{R}^n$ whose surface area measure is μ, that is, the measure equation $S(K, \cdot) = \mu$ has a solution K.*

When the surface area measure of a convex body is discrete, then the body is a polytope and the measure is given by (5.10). The Minkowski problem can therefore be viewed as that of finding a convex polytope with prescribed facet areas and corresponding outer unit normals. The discrete Minkowski problem in $\mathbb{R}^2$ is described in §2.1. In $\mathbb{R}^n$, it states:

Discrete Minkowski Problem. *Given positive numbers $f_1, \ldots, f_m$, and unit vectors $v_1, \ldots, v_m$ in $\mathbb{R}^n$, find the necessary and sufficient conditions such that there exists a convex polytope K in $\mathbb{R}^n$ whose facets have areas $f_1, \ldots, f_m$ and corresponding outer unit normals $v_1, \ldots, v_m$.*

When μ is absolutely continuous with respect to spherical Lebesgue measure, then, by (5.9), the Minkowski problem is equivalent to that of finding a convex body with prescribed curvature function (3.2).

Since the surface area measure of a convex body is invariant under translations, there are infinitely many solutions to the Minkowski problem for a given measure. Moreover, as we have already seen in §2.1, the translation invariance also implies that there is an obstruction to a measure being the surface area measure of a convex body. Below, we show that the Minkowski problem for a given finite measure has a solution if and only if the measure is not concentrated on a closed hemisphere and has its centroid at the origin, that is, it satisfies (5.11).

5.2.1. Solution to the Minkowski Problem. Minkowski solved this problem and also proved the uniqueness of the solutions for the discrete and continuous cases [181, 182]. Aleksandrov [5, 6] and Fenchel & Jessen [72] independently solved the problem for arbitrary convex bodies by introducing surface area measure and using a variational method that transforms the Minkowski problem into an optimization problem. They proved the following:

Minkowski Existence Theorem. *If μ is a finite Borel measure on S^{n-1}, then the measure equation $S(K, \cdot) = \mu$ has a solution if and only if μ is not concentrated on a closed hemisphere and*

$$\int_{S^{n-1}} v \, d\mu(v) = 0. \tag{5.11}$$

5.2.2. Outline of Proof for the Minkowski Existence Theorem. We give a sketch of Aleksandrov's variational approach. The main steps are as described in the previous section.

The optimization problem associated with the classical Minkowski problem is the following maximization problem of volume,

$$\sup \left\{ V([h]) : \int_{S^{n-1}} h \, d\mu = 1, \ h \in C^+(S^{n-1}) \right\}. \tag{5.12}$$

This maximization problem is motivated by the mixed volume inequality (5.2). If μ is the surface area of K, the maximum is achieved only when h is, up to a positive scalar factor, the support function of K.

To prove the existence of solutions to the maximization problem, one needs to show that an optimizing sequence is uniformly bounded. The uniform bound comes from the equation $\int_{S^{n-1}} h \, d\mu = 1$ quite easily. Then Blaschke's selection theorem implies that the optimizing sequence converges to the support function of a compact convex set. The non-degeneracy of the limiting compact convex set follows from the maximality of its volume.

To show the Euler-Lagrange equation of the maximization problem is the equation of the Minkowski problem, one needs to show that $V([h])$ is differentiable with respect to $h \in C^+(S^{n-1})$. This is a key nontrivial step in Aleksandrov's proof. Given $h \in C^+(S^{n-1})$, it suffices to consider variations of the form

$$h_t(v) = h(v) + t f(v),$$

where $f \in C(S^{n-1})$ and $t \in \mathbb{R}$ is small. Aleksandrov's formula is

$$\left. \frac{dV([h_t])}{dt} \right|_{t=0} = \int_{S^{n-1}} f(v) \, dS([h], v), \quad f \in C(S^{n-1}). \tag{5.13}$$

The proof of this formula uses properties of the surface area measure of the Wulff shape and Minkowski's mixed volume inequality (5.2) (see, for example, Lemma 7.5.3 in Schneider [196], or Lemma 1 in Haberl-LYZ [108]).

5.2.3. Continuous Minkowski Problem. Recall that if the surface area measure S_K is absolutely continuous with respect to the spherical Lebesgue measure,

$$dS_K = f_K \, dv,$$

then the density f_K is the curvature function of K. If the boundary of K is C^2, then f_K is continuous and satisfies

$$f_K(\nu_K(x)) = \frac{1}{\kappa_K(x)},$$

where $\nu_K : \partial K \to S^{n-1}$ is the Gauss map and $\kappa_K : \partial K \to (0, \infty)$ is the Gauss curvature.

If we assume that

$$d\mu = f \, dv,$$

where $f \in C^+(S^{n-1})$, then we can ask if there is a solution to the Minkowski problem that is a convex body K with C^2 boundary. If so, then f is the curvature function of K, i.e.,

$$f_K(\nu_K(x)) = f(\nu_K(x)), \quad \text{for all } x \in \partial K.$$

Since the curvature function is the reciprocal of Gauss curvature, this is equivalent to the following:

Continuous Minkowski Problem. *Given a positive continuous function g on S^{n-1}, find the necessary and sufficient conditions such that there exists a closed convex hypersurface Σ in $\mathbb{R}^n$ whose Gauss curvature κ and Gauss map ν satisfy $\kappa(\nu^{-1}) = g$.*

On one hand, any solution Σ to the continuous Minkowski problem gives a solution to the Minkowski problem

$$dS_K = \frac{1}{g(v)} \, dv, \tag{5.14}$$

where K is the convex body with $\partial K = \Sigma$. On the other hand, the Minkowski existence theorem shows that there is a solution K to the Minkowski problem (5.14) if and only if

$$\int_{S^{n-1}} \frac{v}{g(v)} \, dv = 0.$$

Therefore, if we can show that the body K has C^2 boundary, then it is a solution to the continuous Minkowski problem. This is equivalent to showing that the support function h_K is C^2 because the positivity of the curvature function $f_K = 1/g$.

A generalized continuous Minkowski problem in the differential geometry is to find an embedding $G : S^{n-1} \to \Sigma \subset \mathbb{R}^n$ such that $\kappa(G) = g$ for a given positive continuous function g on S^{n-1}. A solution was obtained by Gluck [91, 92].

5.2.4. Continuous Minkowski Problem as a PDE. The continuous Minkowski problem is equivalent to the Monge-Ampère equation

$$\det(\nabla_{ij} h + h\delta_{ij}) = f, \tag{5.15}$$

where f is a positive continuous function on S^{n-1}. The solution h satisfies that

$$(\nabla_{ij} h + h\delta_{ij}) \quad \text{is positive definite} \tag{5.16}$$

if and only if it is the support function of a convex body K whose curvature function is f. This is also equivalent to (5.15) being an elliptic partial differential equation.

Therefore, if h is a solution to (5.15) that satisfies (5.16), then it is the support function of a solution to the continuous Minkowski problem. In particular, the continuous Minkowski problem becomes a regularity question for solutions to elliptic Monge-Ampère equations on a sphere. This is a difficult question studied by many scholars. Major contributions were made by Lewy [140], Nirenberg [185], Cheng-Yau [51], Pogorolov [190], Caffarelli [33], culminating in the following result:

Regularity of the Minkowski Problem. *If $g \in C^\alpha(S^{n-1})$, $g > 0$, and*

$$\int_{S^{n-1}} \frac{v}{g(v)} \, dv = 0,$$

then there exists a convex body K in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary such that its Gauss curvature κ as a function of its outer unit normal is equal to g.

It is of interest to generalize the continuous Minkowski problem of prescribing Gauss curvature to intermediate curvatures, that is, replacing the Gauss curvature κ by the ith intermediate curvature H_i of ∂K. Guan-Guan [96] solved the symmetric case by proving:

Let $g \in C^{1,1}(S^{n-1})$ be positive and even and $i \in \{1, \ldots, n-2\}$. Then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ with $C^{3,\alpha}$ boundary and positive curvature so that its ith intermediate curvature H_i as a function of its outer unit normal is equal to g.

They also obtained results for the nonsymmetric case. Aleksandrov [8] and Chern [52,53] posed the uniqueness problem of prescribing intermediate curvatures.

5.3. Minkowski Problems of Area Measures

Which finite Borel measure on the unit sphere can be a mixed area measure of a convex body? This is the problem of prescribing mixed area measures, which states:

5.3.1. Minkowski Problem of Mixed Area Measures. *Given a finite Borel measure μ on S^{n-1}, a convex body L in $\mathbb{R}^n$, and $i \in \{1, \ldots, n-1\}$, find the necessary and sufficient conditions such that there exists a convex body K in $\mathbb{R}^n$ satisfying $S_i(K, L, \cdot) = \mu$.*

The case $i = n-1$ is the Minkowski problem. The general case has not been studied except when $L = B^n$ which is the following:

5.3.2. Christoffel-Minkowski Problem. *Given a finite Borel measure μ on S^{n-1} and $i \in \{1, \ldots, n-1\}$, find the necessary and sufficient conditions such that there exists a convex body K in $\mathbb{R}^n$ satisfying $S_i(K, \cdot) = \mu$.*

When μ has a density f, this problem is equivalent to the partial differential equation

$$\sigma_i\big(\nabla_{kj}h + h\delta_{kj}\big) = f,$$

where σ_i denotes the normalized ith elementary symmetric function of eigenvalues of the matrix, that is, the sum of ith principal minors of the matrix divided by $\binom{n-1}{i}$.

The case $i = 1$ is the Christoffel problem with corresponding partial differential equation

$$\Delta_{S^{n-1}}h + (n-1)h = f,$$

where $\Delta_{S^{n-1}}$ is the spherical Laplacian. The smooth case in $\mathbb{R}^3$ was first treated by Christoffel [59] as a problem of prescribing the sum of radii of curvature. The general case in all dimensions

was solved independently by Firey [77, 78] and Berg [15]. Schneider [197] gave more explicit necessary and sufficient conditions for the existence of solutions in the polytope case. The case of even measures and symmetric convex bodies was studied in connection with the spherical Radon transform by Goodey-Weil [93]. A short approach to the Christoffel problem was found by Grinberg and Zhang [95]. A Fourier transform approach was given by Goodey-Yaskin-Yaskina [94]. When f is a positive Lipschitz function, Li-Wan-Wang [147] gave simpler necessary and sufficient conditions for the existence of solutions.

For $1 < i < n-1$, the Christoffel-Minkowski problem has remained unsolved for measures. For sufficiently smooth convex bodies of revolution, Firey [80] gave necessary and sufficient conditions for the existence of solution. For centrally symmetric convex bodies with $C^{2,\alpha}$ boundary and positive curvature, Zhang [233] showed that the Christoffel-Minkowski problem is solvable in a neighborhood of f when it is solvable for a smooth f. A breakthrough was achieved by Guan-Ma [100] who found sufficient conditions for both existence and regularity of solutions to the Christoffel-Minkowski problem when f is assumed to be sufficiently smooth:

Let $i \in \{1, \dots, n-2\}$ and μ a finite Borel measure on S^{n-1} with positive density $f \in C^{1,1}(S^{n-1})$. If the centroid of μ is at the origin, i.e.,

$$\int_{S^{n-1}} v f(v) \, dv = 0,$$

and $f^{-\frac{1}{i}}$ is convex when it is extended to $\mathbb{R}^n \setminus \{0\}$ homogeneously of degree 1, then there exists a convex body K of class $C^{3,\alpha}$ and positive curvature so that $S_i(K, \cdot) = \mu$.

The proof in [100] required an additional technical assumption on the density f. This was shown to be unnecessary by Sheng-Trudinger-Wang [204] and also by Andrews and Ma in an unpublished note. Bryan-Ivaki-Scheuer [32] recently used a fully nonlinear curvature flow to derive the existence result under those conditions of f.

The continuity method was used to prove the results above. A natural question is whether Christoffel-Minkowski problems could be solved using a variational method similar to that used by Minkowski, Aleksandrov, and Fenchel for the classical Minkowski problem. The main obstacle for doing this is the lack of a differential formula for quermassintegrals like the one for volume (5.13). The one-sided differential formula (5.6) is not sufficient for transforming the Christoffel-Minkowski problem into an optimization problem.

5.4. Minkowski Problems of Curvature Measures

5.4.1. Curvature Measures of Convex Bodies. Given any $x \in \mathbb{R}^n$, let $d(K, x)$ be the distance from x to $K \in \mathcal{K}_o^n$, and, given $t > 0$, denote the annulus with thickness t by

$$K_t = \{x \notin K : d(K, x) \le t\} = (K + tB) \setminus K.$$

Consider the metric projection map

$$p_K : \mathbb{R}^n \setminus K \to \partial K,$$

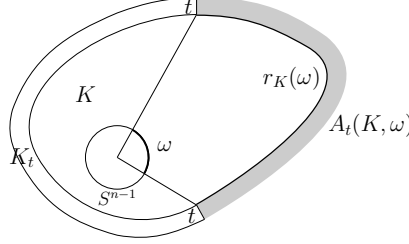where $p_K(x) \in \partial K$ is the unique point such that

$$|p_K(x) - x| = d(K, x).$$

Given a Borel set $\omega \subset S^{n-1}$, let

$$A_t(K, \omega) = K_t \cap p_K^{-1}(r_K(\omega)). \tag{5.17}$$

There is the Steiner-type formula,

$$V(A_t(K,\omega)) = \frac{1}{n} \sum_{i=0}^{n-1} \binom{n}{i} t^{n-i} C_i(K,\omega), \tag{5.18}$$

where $C_i(K,\cdot)$ is a Borel measure on S^{n-1}, called the *ith curvature measure* of K.

Federer [71] defined curvature measures as measures in $\mathbb{R}^n$ for sets of positive reach that are the sets such that the metric projection map can be defined. Convex bodies are sets of positive reach. Schneider [198] gave a unified treatment of curvature measures and area measures for convex bodies. As shown in [198], curvature measures of a convex body concentrate on its boundary. The curvature measures defined above for a convex body $K \in \mathcal{K}_o^n$ are pushforwards of Federer's curvature measures to the unit sphere S^{n-1} by the inverse radial map r_K^{-1}. Since the radial map r_K is bi-Lipschitz, one can define the curvature measures equivalently on either ∂K or S^{n-1}, although they are different measures.

The 0th curvature measure $C_0(K,\cdot)$ was defined and studied earlier by Aleksandrov, who called it the *integral curvature of K*. It can be defined, for Borel $\omega \subset S^{n-1}$, by

$$C_0(K,\omega) = \mathcal{H}^{n-1}(\boldsymbol{\alpha}_K(\omega)), \tag{5.19}$$

where $\boldsymbol{\alpha}_K$ is the radial Gauss image map defined by (3.5). In other words, $C_0(K,\omega)$ is the spherical Lebesgue measure of $\boldsymbol{\alpha}_K(\omega)$, which has total measure $C_0(K,S^{n-1})$ equal to the surface area of S^{n-1}. Its geometric meaning is more apparent when K is a polytope. In this case, $C_0(K,\cdot)$ is a discrete measure concentrated on the radial directions of the vertices, where the mass of the measure at a vertex is the solid angle formed by the unit normals.

On the other hand, the $(n-1)$th curvature measure $C_{n-1}(K,\cdot)$ on S^{n-1} can be defined, for each Borel $\omega \subset S^{n-1}$, by

$$C_{n-1}(K,\omega) = \mathcal{H}^{n-1}(r_K(\omega)),$$

that is, $C_{n-1}(K,\omega)$ is the surface area of the subset of ∂K whose radial directions lie in ω. Recall that the surface area measure $S(K,\eta) = S_{n-1}(K,\eta)$ is the surface area of the subset of ∂K corresponding to the set η of normal directions. Thus,

$$C_{n-1}(K,\boldsymbol{\alpha}_K^*(\eta)) = S_{n-1}(K,\eta), \tag{5.20}$$

for each Borel $\eta \subset S^{n-1}$, where $\boldsymbol{\alpha}_K^*$ is the reverse radial Gauss image map (3.6).

The total mass of the ith curvature measure gives the $(n-i)$th quermassintegral,

$$C_i(K,S^{n-1}) = nW_{n-i}(K), \quad i = 0, \ldots, n-1.$$

If ∂K is C^2 and has positive curvature, the curvature measure $C_i(K,\cdot)$ can be written as

$$C_i(K,\omega) = S_i(K,\boldsymbol{\alpha}_K(\omega)) = \int_{r_K(\omega)} H_{n-i-1} \, d\mathcal{H}^{n-1}, \quad \omega \subset S^{n-1},$$

where r_K is the radial map and H_{n-i-1} is the $(n-i-1)$th mean curvature. Since the surface area element of ∂K has the formula,

$$d\mathcal{H}^{n-1} = \rho_K^{n-2}(u)|\nabla \rho_K(u)|\, du,$$

we have

$$dC_i(K, u) = H_{n-i-1}(r_K(u))\rho_K^{n-2}(u)|\nabla \rho_K(u)|\, du, \quad u \in S^{n-1}, \quad i = 0, \ldots, n-1,$$

where du is the surface area element of S^{n-1}.

5.4.2. Uniqueness Problem of Curvature Measures. The uniqueness of a convex body with given curvature measure $C_i(K, \cdot)$ was shown by Aleksandrov [7] for $i = 0$, and by Schneider [198] for $i = 1, \ldots, n-1$. It states: *Let K and L be convex bodies in $\mathcal{K}_o^n$ and $i \in \{0, 1, \ldots, n-1\}$. If*

$$C_i(K, \cdot) = C_i(L, \cdot),$$

then $K = L$ when $i \in \{1, \ldots, n-1\}$ and K is a dilation of L when $i = 0$.

5.4.3. Minkowski Problems of Curvature Measures. This can be stated as follows:

Given a finite Borel measure μ on S^{n-1} and $i \in \{0, 1, \ldots, n-1\}$. What are the necessary and sufficient conditions such that there exists a convex body K in $\mathcal{K}_o^n$ satisfying $C_i(K, \cdot) = \mu$?

The cases of $i = 1, \ldots, n-1$ are called the *Aleksandrov-Minkowski* problems. The case $i = 0$ is the *Aleksandrov problem* of prescribing integral curvature. The discrete Aleksandrov problem has a clear geometric meaning. The case in $\mathbb{R}^2$ is described in §2.2. In $\mathbb{R}^n$, it states:

Given positive numbers $f_1, \ldots, f_m$ and unit vectors $u_1, \ldots, u_m$ in $\mathbb{R}^n$, find the necessary and sufficient conditions so that there exists a convex polytope P whose vertices have outer normal angles $f_1, \ldots, f_m$ and radial unit vectors $u_1, \ldots, u_m$.

Aleksandrov [7] gave a complete solution to the Aleksandrov problem.

5.4.4. Solution to the Aleksandrov Problem. Aleksandrov [7] proved the following:

If μ is a finite Borel measure on S^{n-1}, then there exists a convex body K in $\mathcal{K}_o^n$ such that $C_0(K, \cdot) = \mu$ if and only if μ satisfies the Aleksandrov condition,

$$|\mu| = n\omega_n > \mathcal{H}^{n-1}(\omega^*) + \mu(\omega), \tag{5.21}$$

for every compact and spherically convex set $\omega \in S^{n-1}$, where ω^* is the polar set of ω,

$$\omega^* = \{v \in S^{n-1} : v \cdot u \leq 0 \text{ for all } u \in \omega\}.$$

Aleksandrov proved the discrete case for polytopes first by a topological argument using what is now known as the Aleksandrov mapping lemma. He extended the solution to the general case by approximation. A mass transport approach to the Aleksandrov problem was given by Oliker [187] and later differently by Bertrand [16].

5.4.5. Solution to the Symmetric Aleksandrov Problem. A longstanding open problem was whether there exists a direct variational proof, similar to that of the Minkowski problem, demonstrating the existence of a solution to the Aleksandrov problem. Prior to the work [119], it was unknown whether there is a geometric invariant whose differential is the integral curvature. It was shown in [119] that the integral curvature is related to the differential of the *entropy of convex body K* in $\mathcal{K}_o^n$,

$$E(K) = \int_{S^{n-1}} \log \rho_K(u)\, du. \tag{5.22}$$

In particular, if ρ_t is a logarithmic variation of the radial function of a convex body K, i.e.,

$$\log \rho_t(v) = \log \rho_K(v) + tg(v) + o(t, v) \text{ and } \lim_{t \to 0} \frac{o(t, v)}{t} = 0 \text{ uniformly,}$$

and $\langle \rho_t \rangle$ is the convex hull of ρ_t, see (3.3), then

$$\frac{d}{dt} E(\langle \rho_t \rangle^*) \Big|_{t=0} = - \int_{S^{n-1}} g(v) \, dC_0(K, v). \tag{5.23}$$

Using the logarithmic variation of the radial function is to avoid an extra factor of radial function so that one gets the exact differential formula above. Using this formula, a direct variational proof of the solution to the symmetric case of the Aleksandrov problem was given in [120]. It was also observed in [120] that the Aleksandrov condition (5.21) holds for any even measure on S^{n-1}. Thus, for the symmetric case, we have

If μ is a finite even Borel measure on S^{n-1}, then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ such that $C_0(K, \cdot) = \mu$ if and only if μ is not concentrated on a great sub-sphere of S^{n-1} and $|\mu| = n\omega_n$.

For the general case, a direct variational proof of Aleksandrov's theorem was given in [27]. The associated maximization problem is the following,

$$\sup_{\rho \in C^+(S^{n-1})} \left\{ \frac{1}{|\mu|} \int_{S^{n-1}} \log \rho \, d\mu + \frac{1}{n\omega_n} E(\langle \rho \rangle^*) \right\}.$$

When the measure μ has a density function $f : S^{n-1} \to \mathbb{R}$, the partial differential equation of the Aleksandrov problem can be written as a Monge-Ampère type equation on S^{n-1}:

$$\det(\nabla_{ij} h + h \delta_{ij}) = \frac{|\nabla h|^n}{h} f, \tag{5.24}$$

where h is the unknown function on S^{n-1} to be found. A solution h to this equation is the support function of K^*, where K is a solution to the Aleksandrov problem.

Regularity of solutions to the Aleksandrov problem was investigated by Pogorelov [189], Guan-Li [97] and Oliker [188].

5.4.6. Existence and Regularity of the Aleksandrov-Minkowski Problem. Guan-Lin-Ma [98] and Guan-Li-Li [99] studied the Aleksandrov-Minkowski problem for the cases $i = 1, \ldots, n-2$ when the given measure has an everywhere positive density. Guan-Li-Li [99] proved the following existence and regularity theorem:

Let $i \in \{1, \ldots, n-2\}$ and μ be a finite Borel measure on S^{n-1} with density $f \in C^2(S^{n-1})$ and $f > 0$. If $f^{-\frac{1}{i}}$ is convex when f is extended to $\mathbb{R}^n \setminus \{0\}$ homogeneously of degree $-n$, then there exists a convex body K of $C^{3,\alpha}$ and positive curvature so that $C_i(K, \cdot) = \mu$.

Guan-Lin-Ma [98] had proved the same theorem earlier but with a stronger assumption, namely that $f^{-\frac{1}{i}}$ is strictly convex when f is extended to $\mathbb{R}^n \setminus \{0\}$ homogeneously of degree $-n$.

6. Dual Minkowski Problems in the Dual Brunn-Minkowski Theory

The dual Brunn-Minkowski theory of star bodies was introduced by Lutwak in 1970s [163]. It is based on a remarkable duality in convex geometry, related to but distinct from the

standard duality between a convex body and its polar. The duality is a heuristic concept that is used as a guiding principle. An excellent explanation of this conceptual duality is given in Schneider [196] (p. 507). The dual Brunn-Minkowski theory plays a central role in the study of intersection bodies and, in the late 1980s and 1990s, was a crucial element in the solution to the Busemann-Petty problem ([84, 85, 133, 134, 164, 235]). For convex bodies, problems in the dual Brunn-Minkowski theory are often deeper and significantly more challenging than their analogues in the classical Brunn-Minkowski theory. This duality between projections of convex bodies and central intersections of star bodies is thoroughly discussed in Gardner [83].

Central to the dual Brunn-Minkowski theory are dual mixed volumes and the newly discovered dual curvature measures. Until recently, it was not known how to derive the dual curvature measures using variations of dual mixed volumes.

The characterization of dual curvature measures is called the dual Minkowski problem. The fundamental case, namely the logarithmic Minkowski problem, is the characterization of the cone-volume measure. The general dual Minkowski problem exhibits more subtleties and is more challenging than the classical Minkowski problem in the Brunn-Minkowski theory.

6.1. Basics of the Dual Brunn-Minkowski Theory

6.1.1. Dual Mixed Volumes. A *star set* in $\mathbb{R}^n$ is a compact set K, where if $x \in K$, then the line segment from 0 to x also lies in K. Given $u \in S^{n-1}$, let $\mathbb{R}^+ u$ denote the ray from o in the direction u.

If M, N are star sets, then $M \cap (\mathbb{R}^+ u)$ and $N \cap (\mathbb{R}^+ u)$ are overlapping line segments and their Minkowski sum is a longer line segment in the same direction. The *radial sum* of M and N is defined to be the star set $M \widetilde{+} N$, where for each $u \in S^{n-1}$,

$$(M \widetilde{+} N) \cap \mathbb{R}^+ u = M \cap (\mathbb{R}^+ u) + N \cap (\mathbb{R}^+ u).$$

There is the formula,

$$\rho_{M \widetilde{+} N}(x) = \rho_M(x) + \rho_N(x), \quad x \in \mathbb{R}^n \setminus \{0\},$$

where the radial function of a star set M, $\rho_M : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$, is defined to be

$$\rho_M(x) = \max\{t > 0 : tx \in M\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

If a star set M contains the origin in its interior and the radial function ρ_M is continuous, M is called a *star body*.

Let $M_1, \ldots, M_m$ be star bodies in $\mathbb{R}^n$. The volume of $t_1 M_1 \widetilde{+} \cdots \widetilde{+} t_m M_m$, $t_1, \ldots, t_m > 0$,

$$V(t_1 M_1 \widetilde{+} \cdots \widetilde{+} t_m M_m) = \frac{1}{n} \int_{S^{n-1}} \left(t_1 \rho_{M_1}(u) + \cdots + t_m \rho_{M_m}(u) \right)^n du, \tag{6.1}$$

is a homogeneous nth-order polynomial in $t_1, \ldots, t_m$. It can therefore be written as

$$V(t_1 M_1 \widetilde{+} \cdots \widetilde{+} t_m M_m) = \sum_{i_1, \ldots, i_n = 1}^{m} \widetilde{V}(M_{i_1}, \cdots, M_{i_n}) t_{i_1} \cdots t_{i_n},$$

where the coefficients $\widetilde{V}(M_{i_1}, \cdots, M_{i_n})$ are nonnegative, symmetric functions of the bodies $M_1, \cdots, M_m$. They are called *dual mixed volumes*. By (6.1), they have the integral representation

$$\widetilde{V}(M_{i_1}, \cdots, M_{i_n}) = \frac{1}{n} \int_{S^{n-1}} \rho_{M_{i_1}}(u) \cdots \rho_{M_{i_n}}(u) \, du.$$

For $q \in \mathbb{R}$, the *qth dual mixed volume* of star bodies M and N is defined to be

$$\widetilde{V}_q(M, N) = \frac{1}{n} \int_{S^{n-1}} \rho_M^{n-q}(u) \rho_N^q(u) \, du,$$

and the *qth dual quermassintegral* $\widetilde{W}_q(M)$ of a star body M is

$$\widetilde{W}_q(M) = \widetilde{V}_q(M, B) = \frac{1}{n} \int_{S^{n-1}} \rho_M^{n-q}(u) \, du,$$

where B is the unit ball in $\mathbb{R}^n$.

Let

$$\widetilde{V}_q(K) = \widetilde{W}_{n-q}(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^q(u) \, du. \tag{6.2}$$

When $q \neq 0$, the dual quermassintegral $\widetilde{V}_q(K)$ can be written as

$$\widetilde{V}_q(K) = \begin{cases} \dfrac{q}{n} \displaystyle\int_K |x|^{q-n} dx & q > 0 \\ \dfrac{|q|}{n} \displaystyle\int_{\mathbb{R}^n \setminus K} |x|^{q-n} dx & q < 0, \end{cases}$$

which is related to the Riesz potential of the characteristic functions $\mathbb{1}_K$ and $\mathbb{1}_{\mathbb{R}^n \setminus K}$.

The integer cases $\widetilde{V}_1(K), \ldots, \widetilde{V}_n(K)$ have a clear geometric meaning. They are integrals of lower dimensional volumes of intersections of K with subspaces,

$$\widetilde{V}_i(K) = \frac{\omega_n}{\omega_i} \int_{G_{n,i}} V_i(K \cap \xi) \, d\xi, \quad i = 1, \ldots, n. \tag{6.3}$$

The normalized dual quermassintegral $\overline{V}_q(K)$ is defined to be

$$\overline{V}_q(K) = \begin{cases} \left(\dfrac{\widetilde{V}_q(K)}{\omega_n} \right)^{\frac{1}{q}} & \text{if } q \neq 0 \\ \exp \left(\dfrac{1}{n\omega_n} \displaystyle\int_{S^{n-1}} \log \rho_K(u) \, du \right) & \text{if } q = 0. \end{cases}$$

6.1.2. Dual Curvature Measures. Huang-LYZ [119] discovered the geometric measures associated with dual quermassintegrals, called *dual curvature measures*. For a convex body K that contains the origin in its interior and $q \in \mathbb{R}$, the *qth dual curvature measure* $\widetilde{C}_q(K, \cdot)$ is defined by

$$\widetilde{C}_q(K, \eta) = \frac{1}{n} \int_{\boldsymbol{\alpha}_K^*(\eta)} \rho_K^q(u) \, du = \frac{1}{n} \int_{S^{n-1}} \mathbb{1}_{\boldsymbol{\alpha}_K^*(\eta)}(u) \rho_K^q(u) \, du, \tag{6.4}$$

for each Borel set $\eta \subset S^{n-1}$. It can also be defined by an integral over the boundary ∂K,

$$\widetilde{C}_q(K, \eta) = \frac{1}{n} \int_{\nu_K^{-1}(\eta)} (x \cdot \nu_K(x)) |x|^{q-n} \, dx,$$

or as a linear functional,

$$\int_{S^{n-1}} f(v) \, d\widetilde{C}_q(K, v) = \frac{1}{n} \int_{S^{n-1}} f(\alpha_K(u)) \rho_K^q(u) \, du.$$

For a polytope P with outer unit normals $v_1, \ldots, v_m$, the dual curvature measure of P is a discrete measure given by

$$\widetilde{C}_q(P, \cdot) = \sum_{i=1}^{m} c_i \delta_{v_i}, \text{ where } c_i = \int_{\boldsymbol{\alpha}_P^*(v_i)} \rho_K^q(u) \, du.$$

If K is strictly convex, then the dual curvature measure can be expressed in terms of the surface area measure and support function,

$$d\widetilde{C}_q(K, \cdot) = \frac{1}{n} h_K |\nabla h_K|^{q-n} dS(K, \cdot).$$

When $q = 0$, $\widetilde{C}_0(K, \cdot)$ is called the *radial angle measure*. For a polytope P,

$$\widetilde{C}_0(P, \cdot) = \sum_{i=1}^{m} c_i \delta_{v_i}, \quad c_i = \mathcal{H}^{n-1}(\boldsymbol{\alpha}_P^*(v_i)),$$

where the weights c_i are the radial angles of the cones formed by the origin and facets of P.

Up to a constant factor, the radial angle measure of a convex body is the integral curvature of the polar body,

$$\widetilde{C}_0(K, \cdot) = \frac{1}{n} C_0(K^*, \cdot).$$

When $q = n$, $\widetilde{C}_n(K, \cdot)$ is the cone-volume measure of K, denoted by $V(K, \cdot)$ or V_K,

$$d\widetilde{C}_n(K, \cdot) = \frac{1}{n} h_K dS(K, \cdot) = dV(K, \cdot).$$

The fact that cone-volume measure and integral curvature are related to each other was not known until it was discovered that both measures are dual curvature measures.

The concept of dual curvature measures in the dual Brunn-Minkowski theory corresponds to the concept of curvature measures in the Brunn-Minkowski theory, see [119] for detailed explanations. Here, we mention the duality of geometric constructions of curvature measures and dual curvature measures.

Let K be a convex body that contains the origin in its interior in $\mathbb{R}^n$. Define the *radial projection map* $\widetilde{p}_K : \mathbb{R}^n \setminus K \to \partial K$ by

$$\widetilde{p}_K(x) = \rho_K(x) x,$$

for $x \in \mathbb{R}^n \setminus K$. For $x \in \mathbb{R}^n$, the *radial distance* $\widetilde{d}(K, x)$ of x to K, is defined by

$$\widetilde{d}(K, x) = \begin{cases} |x - \widetilde{p}_K(x)| & x \notin K \\ 0 & x \in K. \end{cases}$$

For $t > 0$, consider the radial annulus with thickness t,

$$\widetilde{K}_t = \{x \notin K : \widetilde{d}(K, x) \le t\} = (K \widetilde{+} tB) \setminus K.$$

For $t > 0$ and a Borel set $\eta \subset S^{n-1}$, define the local radial annulus,

$$\widetilde{A}_t(K, \eta) = \widetilde{K}_t \cap \widetilde{p}_K^{-1}(\boldsymbol{\nu}_K^*(\eta)). \tag{6.5}$$
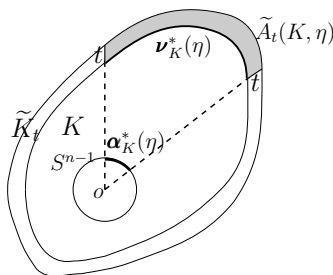
Then there is the dual Steiner-type formula,

$$V(\widetilde{A}_t(K,\eta)) = \sum_{i=0}^{n} \binom{n}{i} t^{n-i} \widetilde{C}_i(K,\eta). \tag{6.6}$$

Formulas (6.5) and (6.6) are dual to formulas (5.17) and (5.18).

Dual curvature measures are differentials of dual quermassintegrals, as shown by the following differential formulas,

$$\frac{d}{dt} \log \overline{V}_q(\langle \rho_t \rangle^*) \Big|_{t=0} = -\frac{1}{\widetilde{V}_q(\langle \rho_0 \rangle^*)} \int_{S^{n-1}} g(u)\, d\widetilde{C}_q(\langle \rho_0 \rangle^*, u), \tag{6.7}$$

where $\langle \rho_t \rangle$ is the convex hull of the logarithmic family of variations ρ_t defined by

$$\log \rho_t(u) = \log \rho_0(u) + tg(u) + o(t,u), \quad \lim_{t \to 0} \frac{o(t,u)}{t} = 0 \ \text{ uniformly.}$$

The term $o(t,u)$ is for generality which makes it easy to reformulate in the dual case. That is, it can also be formulated as differential formulas of dual quermassintegrals of Wulff shapes,

$$\frac{d}{dt} \log \overline{V}_q([h_t]) \Big|_{t=0} = \frac{1}{\widetilde{V}_q([h_0])} \int_{S^{n-1}} f(v)\, d\widetilde{C}_q([h_0], v), \tag{6.8}$$

where $[h_t]$ is the Wulff shape of the logarithmic family of variations h_t defined by

$$\log h_t(v) = \log h_0(v) + tf(v) + o(t,v), \quad \lim_{t \to 0} \frac{o(t,v)}{t} = 0 \ \text{ uniformly.}$$

When $q = n$, (6.8) is equivalent to (5.13). Similar to the role of Aleksandrov's differential formula of volume in the classical Minkowski problem, differential formulas (6.7) and (6.8) are critical for solving the Minkowski problems of dual curvature measures. The proofs of (6.7) and (6.8) were given by Huang-LYZ [119, Thereoms 4.4 and 4.5].

6.1.3. Cone-Volume Measure. Cone-volume measure $V(K, \cdot)$ of a convex body $K \in \mathcal{K}_o^n$ can be defined more geometrically. For a Borel set $\sigma \subset \partial K$, let the *radial cone of σ* be the convex hull of σ and the origin (see the figure in §3.0.9),

$$\triangle_K(\sigma) = \{sx : 0 \le s \le 1, \ x \in \sigma\}. \tag{6.9}$$

If K is a polytope, we will call the radial cone formed by the origin and a facet of K a *facet radial cone* of the polytope.

Recall from (3.4) that the reverse Gauss image of a subset $\eta \subset S^{n-1}$, denoted $\boldsymbol{\nu}_K^*(\eta)$, is the set of points in ∂K whose normal cones intersect η. The cone-volume measure of a Borel subset $\eta \subset S^{n-1}$ is defined to be

$$V(K, \eta) = \mathcal{H}^n(\triangle_K(\boldsymbol{\nu}_K^*(\eta)), \quad \eta \subset S^{n-1}.$$

For example, the cone-volume measure of a polytope P, whose facets have radial cones $\triangle_1, \ldots, \triangle_m$ and corresponding outer unit normals $v_1, \ldots, v_m$, is the discrete measure,

$$V(P, \cdot) = \sum_{i=1}^{m} \mathcal{H}^n(\triangle_i) \, \delta_{v_i}, \qquad (6.10)$$

where $\mathcal{H}^n(\triangle_i)$ is the volume of $\triangle_i$ and δ_{v_i} is the delta measure at $v_i \in S^{n-1}$.

The volume of the radial cone can be written as an integral either on the unit sphere or on the boundary of the convex body using the radial function or support function,

$$V(K, \eta) = \int_{\triangle_K(\boldsymbol{\nu}_K^*(\eta))} dx = \frac{1}{n} \int_{\boldsymbol{\alpha}_K^*(\eta)} \rho_K^n(u) \, du = \frac{1}{n} \int_{\nu_K^{-1}(\eta)} (x \cdot \nu_K(x)) \, dx.$$

The last equality above shows that cone-volume measure and surface area measure satisfy the formula,

$$dV(K, \cdot) = \frac{1}{n} h_K \, dS(K, \cdot). \qquad (6.11)$$

Given an affine transformation $\phi \in SL(n)$, the cone-volume measure of ϕK satisfies

$$V(\phi K, \eta) = V(K, \overline{\phi^{-t} \eta}), \quad \eta \subset S^{n-1}, \quad \phi \in SL(n),$$

where

$$\overline{\phi^{-t} \eta} = \left\{ \frac{\phi^{-t} v}{|\phi^{-t} v|} : v \in \eta \right\}$$

and ϕ^{-t} is the inverse transpose of ϕ.

When the boundary of K is C^2 and has positive Gauss curvature κ, the cone-volume measure of K has a positive density,

$$dV(K, v) = \frac{h_K}{n\kappa} \, dv.$$

6.2. Logarithmic Minkowski Problem of Cone-Volume Measure

The Minkowski problem for cone-volume measure is the following.

Logarithmic Minkowski Problem. *Given a finite Borel measure μ on S^{n-1}, find the necessary and sufficient conditions such that there exists a convex body K in $\mathbb{R}^n$ whose cone-volume measure is μ, that is, the measure equation $V(K, \cdot) = \mu$ has a solution K.*

When the cone-volume measure of a convex body is discrete, then the body is a polytope and the measure is given by (6.10). The logarithmic Minkowski problem can therefore be viewed as a problem of finding a convex polytope with prescribed outer unit normals and cone volumes of the corresponding facet radial cones. The discrete logarithmic Minkowski problem in $\mathbb{R}^2$ is described in §2.3. In $\mathbb{R}^n$, it states:

Discrete Logarithmic Minkowski Problem. *Given positive numbers $f_1, \ldots, f_m$, and unit vectors $v_1, \ldots, v_m$ in $\mathbb{R}^n$, find the necessary and sufficient conditions such that there exists a convex polytope K in $\mathbb{R}^n$ whose facet radial cones have volumes $f_1, \ldots, f_m$ with corresponding facet outer unit normals $v_1, \ldots, v_m$.*

When the measure μ has a positive density f, the question can be formulated as an elliptic Monge-Ampère equation,

$$\det(\nabla_{ij} h + h \delta_{ij}) = \frac{f}{h}. \qquad (6.12)$$

The first results in 2-dimension were given by Gage and Li [82] for measures with smooth densities and by Stancu [206] for discrete measures. Chou and Wang [56] extended Gage and Li's result to higher dimensions. Böröczky-LYZ [25] then solved the problem for arbitrary even measures in all dimensions. There are only partial results for the logarithmic Minkowski problem for arbitrary measures, because the obstructions to a solution are not fully understood.

The necessary and sufficient conditions for an even measure to be the cone-volume measure of a symmetric convex body are known as the subspace concentration condition.

6.2.1. Subspace Concentration Condition. For an origin-symmetric convex body $K \subset \mathbb{R}^n$, we first look at how large the cone-volume measure $V_K(\xi \cap S^{n-1})$ can be for a 1-dimensional subspace ξ. If ξ is spanned by unit vector v, then

$$V_K(\xi \cap S^{n-1}) = V_K(\{v, -v\}) = \frac{2}{n} hF, \tag{6.13}$$

where $h = h_K(v)$, $F = \mathcal{H}^{n-1}(K_v)$, and $K_v = \partial K \cap H_K(v) = \boldsymbol{\nu}_K^*(\{v\})$.

Note that the volume of the convex hull of K_v and K_{-v} is at least $2hF$. This is because the intersection of a hyperplane (parallel to $H_K(v)$) with the convex hull is $tK_v + (1-t)K_{-v}$ for some $0 \le t \le 1$. By the Brunn-Minkowski inequality,

$$\mathcal{H}^{n-1}(tK_v + (1-t)K_{-v}) \ge (tF^{\frac{1}{n-1}} + (1-t)F^{\frac{1}{n-1}})^{n-1} = F.$$

Since the convex hull is contained in K, we have $2hF \le V(K)$. By this and (6.13), it follows that

$$V_K(\xi \cap S^{n-1}) \le \frac{1}{n} V(K). \tag{6.14}$$

This motivates a general condition for cone-volume measure:

A finite Borel measure μ on S^{n-1} is said to satisfy the subspace concentration condition if, for any subspace ξ of $\mathbb{R}^n$,

$$\mu(\xi \cap S^{n-1}) \le \frac{\dim \xi}{n} \mu(S^{n-1}), \tag{6.15}$$

and equality holds if and only if there exists a linear subspace $\xi' \subset \mathbb{R}^n$ complementary to ξ such that the support of μ lies in $S^{n-1} \cap (\xi \cup \xi')$.

The measure μ on S^{n-1} is said to satisfy the *strict subspace concentration inequality* if the inequality in (6.15) is strict for each subspace $\xi \subset \mathbb{R}^n$ such that $0 < \dim \xi < n$.

He-Leng-Li [110] and Henk-Schürmann-Wills [112] independently showed that the cone-volume measure of a symmetric polytope satisfies the subspace concentration condition, with an alternate proof given by Xiong [229]. Böröczky-LYZ [25] showed that the cone-volume measure of any symmetric convex body satisfies the subspace concentration condition.

If K is not necessarily symmetric but its centroid is at the origin, then its cone-volume measure satisfies the subspace concentration condition. This was proved for polytopes in $\mathbb{R}^3$ by Xiong [229], for polytopes in $\mathbb{R}^n$ by Henk-Linke [113], and for all convex bodies by Böröczky-Henk [22].

Applications of the subspace concentration condition to reverse affine isoperimetric inequalities were given in [22, 110, 113, 229]. Connections of it to the so-called K-stability property in algebraic geometry [45–47] was observed by Hering-Nill-Süss [114].

6.2.2. Solution to the Symmetric Logarithmic Minkowski Problem.   If the logarithmic Minkowski problem is restricted to even measures and symmetric convex bodies, it is called the symmetric logarithmic Minkowski problem.

The symmetric logarithmic Minkowski problem was solved for polygons in $\mathbb{R}^2$ by Stancu [206]. For arbitrary convex bodies in all dimensions, a complete solution to the symmetric logarithmic Minkowski problem was established by Böröczky-LYZ [25]. They proved the following:

A non-zero finite even Borel measure on the unit sphere S^{n-1} is the cone-volume measure of an origin-symmetric convex body in $\mathbb{R}^n$ if and only if it satisfies the subspace concentration condition.

The proof in [25] uses a variational method. A Wulff shape construction is needed, as in the proof of the Minkowski problem, to define a differentiable functional whose minimum gives a solution to the logarithmic Minkowski problem. In particular, given a measure μ on S^{n-1}, if there is a function $h \in C_e^+(S^{n-1})$ that minimizes the functional

$$\Phi_\mu[h] = V([h])^{-\frac{1}{n}} \exp \int_{S^{n-1}} \log h \, d\mu, \tag{6.16}$$

then there exists a convex body K whose support function h_K also minimizes this functional. That h_K satisfies the Euler-Lagrange equation for the functional implies that the cone-volume measure of K is a scalar multiple of μ. Rescaling K then proves the theorem.

We now need to show that a minimizer exists. Since the functional of the minimization problem is monotone, it is easily shown that there exists a minimizing sequence, where the functions are all support functions of convex bodies. Since the functional is invariant under rescaling of the bodies, one can assume that the convex bodies have bounded diameter. It follows that there exists a subsequence that converges to a convex set. The difficult part is proving that the limit is in fact a convex body, i.e, it has a nonempty interior. The proof of this uses sharp estimates for both the volume $V([h])$ of the Wulff shape and the integral $\int_{S^{n-1}} \log h \, d\mu$. The main obstacle is to deal with the concentration of μ in subspaces. To overcome it, the crucial idea is to construct a spherical partition that can be used to express the concentrations of μ in subspaces of all dimensions. Then the subspace concentration condition is invoked to obtain sharp estimates for the integral in the minimization functional. This technique is also important in solving other Minkowski problems.

6.2.3. Solution to the Non-Symmetric Logarithmic Minkowski Problem.   When a convex body is not origin-symmetric, its cone-volume measure may not satisfy the subspace concentration condition. However, Zhu [242], Böröczky-Hegedűs-Zhu [21], and Chen-Li-Zhu [44] showed that the subspace concentration condition is sufficient for prescribing cone-volume measure.

A linear subspace ξ in $\mathbb{R}^n$, $0 < \dim \xi < n$, is called an *essential subspace* with respect to a Borel measure μ on S^{n-1} if $\xi \cap \text{supp}\mu$ is not contained in a closed hemisphere of the subsphere $\xi \cap S^{n-1}$. If μ is discrete and the vectors in its support are in general position, i.e., any k vectors in $\text{supp}\xi$, $1 \le k \le n$, are linearly independent, then it is easily seen there are no essential subspaces with respect to μ.

For arbitrary convex polytopes, the following solution to the logarithmic Minkowski problem was proved:

If μ is a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere and satisfies the subspace concentration condition for all essential subspaces, then μ is the cone-volume measure of a polytope in $\mathbb{R}^n$ containing the origin in its interior.

For polytopes whose outer unit normals are in general position, this was first proved by Zhu [242]. Böröczky-Hegedűs-Zhu [21] proved it for general convex polytopes. Chen-Li-Zhu [44] then used an approximation argument to prove:

If μ is a non-zero finite Borel measure on S^{n-1} that satisfies the subspace concentration condition, then there exists a convex body K containing the origin in $\mathbb{R}^n$ such that $V(K, \cdot) = \mu$.

The necessary and sufficient condition for the non-symmetric case is still unknown. Liu-Lu-Sun-Xiong [153] found a necessary condition for a convex body in $\mathbb{R}^2$ with centroid at the origin and solved the logarithmic Minkowski problem for quadrilaterals.

6.2.4. Minkowski Problems and Self-Similar Solutions of Gauss Curvature Flows. Another approach to solving Minkowski problems for smooth solutions is using anisotropic Gauss curvature flow equations. One considers an anisotropic Gauss curvature flow equation related to the partial differential equation of a Minkowski problem, and normalize the flow equation by rescaling so that the self-similar solution of the flow equation satisfies the PDE.

Let $X(\cdot, t) : S^{n-1} \to \mathbb{R}^n$ be a parametrization of the boundaries of a family of smooth convex bodies $K_t \in \mathcal{K}_o^n$ with positive Gauss curvature $\kappa(\cdot, t)$ and outer unit normal $\nu(\cdot, t)$ at $X(\cdot, t)$, and let f be a positive smooth function on S^{n-1}. For the logarithmic Minkowski problems, the anisotropic Gauss curvature flow equation is

$$\frac{\partial X}{\partial t}(u, t) = -\kappa(u, t) f(\nu(u, t)) \nu(u, t). \tag{6.17}$$

When $f = 1$, this is the Gauss curvature flow first studied by Firey [79], subsequently by Tso [211], Chow [58], Andrews [9], Andrews-Chen [11], and Huang-Liu-Xu [122]. The existence and regularity aspects of the Gauss curvature flow were resolved by Tso [211]. The smooth convergence to a ball in $\mathbb{R}^3$ was first proved by Andrews [9]. For higher dimensions, Guan-Ni [101] proved the smooth convergence to a self-similar solution, namely a soliton. Brendle-Choi-Daskalopoulos [30] proved that the soliton is a ball, also proved in Choi [54]. The general case was studied by Chou-Zhu [57], Andrews [10], and others.

The main technical part of the approach is to solve the normalized anisotropic Gauss curvature flow equation for all $t > 0$ and prove that the solution converges to a smooth self-similar solution as $t \to \infty$. This requires finding a priori estimates of solutions to the flow equation.

This approach has been applied to finding smooth solutions of many other Minkowski problems of geometric measures. For example, Chou-Wang [55] introduced the following logarithmic Gauss curvature flow

$$\frac{\partial X}{\partial t}(u, t) = -\log \frac{\kappa(u, t)}{f(\nu(u, t))} \nu(u, t), \tag{6.18}$$

and showed that its solution converges to the smooth solution of the classical Minkowski problem.

6.3. Logarithmic Brunn-Minkowski Conjecture

6.3.1. Uniqueness Conjecture of Cone-Volume Measure. Solutions to the logarithmic Minkowski problem need not be unique. In particular, if the cone-volume measure of a symmetric convex

body K is supported on a union of linear subspaces $\xi_1, \ldots, \xi_m \subset \mathbb{R}^n$ of positive dimension such that $\mathbb{R}^n = \xi_1 \oplus \cdots \oplus \xi_m$, then

$$K = (K \cap \xi_1) + \cdots + (K \cap \xi_m).$$

In that case, given any $a_1, \ldots, a_m > 0$ such that $a_1 \cdots a_m = 1$, the convex body

$$K' = (a_1 K \cap \xi_1) + \cdots + (a_m K \cap \xi_m) \tag{6.19}$$

has the same cone-volume measure as K. This leads to the following conjecture:

Uniqueness Conjecture of Cone-Volume Measure. *If $K, L \subset \mathbb{R}^n$ are symmetric convex bodies such that $V(K, \cdot) = V(L, \cdot)$, then there exists a direct sum decomposition $\mathbb{R}^n = \xi_1 \oplus \cdots \oplus \xi_m$ such that*

$$K = (K \cap \xi_1) + \cdots + (K \cap \xi_m), \quad L = (a_1 K \cap \xi_1) + \cdots + (a_m K \cap \xi_m),$$

where $a_1 \cdots a_m = 1$, $a_1, \ldots, a_m > 0$.

Note that a direct sum decomposition with $m > 1$ is not a smooth convex body. Thus, if K and L are smooth, then the decomposition is trivial with $m = 1$ and $\xi_1 = \mathbb{R}^n$, and therefore the question becomes whether the following holds:

$$V(K, \cdot) = V(L, \cdot) \quad \implies \quad K = L. \tag{6.20}$$

When K and L are symmetric convex bodies in $\mathbb{R}^2$, the conjecture was proved by Böröczky-LYZ [24]. Ma [178] gave a different proof, and Xi-Leng [217] extended this to a larger class of not necessarily symmetric 2-dimensional bodies. In earlier work, Gage [81] proved the conjecture for smooth symmetric bodies with positive curvature in $\mathbb{R}^2$, and Stancu [207] did it for symmetric polygons. If K and L are convex bodies that are assumed to be symmetric about coordinates hyperplanes, the conjecture was established by Saroglou [195].

When the bodies are assumed to be smooth, there are many partial results. In 1974 Firey [79] proved (6.20) when K is smooth and symmetric and L is a ball centered at the origin. In other words, a smooth symmetric convex body K with constant cone-volume measure is a ball. Firey conjectured that this holds even if K is not assumed to be symmetric:

If the ratio of the support function and Gauss curvature of a convex body K in $\mathbb{R}^n$ is constant, $h_K / \kappa = constant$, then K is a ball.

Firey's conjecture for 3-dimensional convex bodies was proved by Andrews [9]. Andrews-Guan-Ni [12] obtained partial results in $\mathbb{R}^n$. The conjecture was fully solved by Choi [54]. Further results were proved by Brendle-Choi-Daskalopoulos [30].

Kolesnikov and Milman [135] proved the uniqueness conjecture of cone-volume measure in a range of cases that includes when K and L are positively curved C^2 symmetric convex bodies lying in a sufficiently small C^2-neighborhood of the unit ball in the space ℓ_q^n, $q \geq 2$, when the dimension n is sufficiently large. Chen-Huang-Li-Liu [41] proved it when K is a $C^{2,\alpha}$ symmetric body that lies in a sufficiently small neighborhood of the ball B^n and L is any symmetric convex body in $\mathbb{R}^n$.

6.3.2. Logarithmic Minkowski Inequality. Böröczky-LYZ [24, 25] observed that an affirmative answer to the uniqueness conjecture of cone-volume measure implies the following logarithmic Minkowski inequality:

Given symmetric convex bodies $K, L \subset \mathbb{R}^n$,

$$\int_{S^{n-1}} \log \frac{h_L}{h_K} \, dV(K, \cdot) \geq \frac{V(K)}{n} \log \frac{V(L)}{V(K)}. \tag{6.21}$$

The implication can be shown as follows. Given a symmetric convex body K, note that (6.21) is equivalent to

$$\Phi_\mu[h_K] \leq \Phi_\mu[h_L], \tag{6.22}$$

where Φ_μ is the functional defined by (6.16) and $\mu = V_K$. On the other hand, the proof of Böröczky-LYZ [25] shows that there exists a symmetric convex body K' such that $V_{K'} = V_K$ and, for any symmetric convex body L,

$$\Phi_\mu[h_{K'}] \leq \Phi_\mu[h_L]. \tag{6.23}$$

If the uniqueness conjecture holds, then K and K' satisfy (6.19), which in turn implies that $\Phi_\mu[h_K] = \Phi_\mu[h_{K'}]$. Therefore, (6.22) holds for any symmetric convex body L.

Thus, the uniqueness results cited in the previous section all imply the logarithmic Minkowski inequality under the assumptions described. Precisely, inequality (6.21) is true if K is an origin-symmetric convex body in a $C^{2,\alpha}$ neighborhood of the Euclidean unit ball [41], K is an origin-symmetric convex body in $\mathbb{R}^2$ [24], or K is a convex body in $\mathbb{R}^n$ that is symmetric about coordinates hyperplanes [195].

6.3.3. Logarithmic Brunn-Minkowski Inequality. Given symmetric convex bodies K, L and $0 < t < 1$, define the geometric mean of K and L to be the body $K^{1-t}L^t$ to be the Wulff shape of the geometric mean $h_K^{1-t}h_L^t$.

The logarithmic Minkowski inequality (6.21) is equivalent to the following conjecture [24]:

If $K, L \subset \mathbb{R}^n$ are symmetric convex bodies, then

$$V(K^{1-t}L^t) \geq V(K)^{1-t}V(L)^t, \quad 0 < t < 1. \tag{6.24}$$

Since $(1-t)K + tL \supseteq K^{1-t}L^t$, the logarithmic Brunn-Minkowski inequality (6.24) implies the Brunn-Minkowski inequality (5.3) for symmetric convex bodies.

Since the logarithmic Minkowski inequality is equivalent to the logarithmic Brunn-Minkowski inequality, the results described in the previous section also imply the logarithmic Brunn-Minkowski inequality.

Colesanti-Livshyts-Marsiglietti [63] and Colesanti-Livshyts [62] proved (6.24) when K and L are in a small C^2 neighborhood of the Euclidean unit ball B^n. Kolesnikov-Milman [135] showed (6.24) when K and L are in a small C^2 neighborhood of the unit ball of the space ℓ_q^n, $q \geq 2$ when the dimension n is sufficiently large.

6.4. Minkowski Problems of Dual Curvature Measures

In [119], Huang-LYZ posed the following problem for dual curvature measures.

6.4.1. Dual Minkowski Problem. *Given a finite Borel measure μ on S^{n-1} and $q \in \mathbb{R}$, what are necessary and sufficient conditions for the existence of a convex body $K \in \mathcal{K}_o^n$ solving the measure equation,*

$$\widetilde{C}_q(K, \cdot) = \mu \,?$$

Since the nth dual curvature measure $\widetilde{C}_n(K, \cdot)$ is the cone-volume measure $V(K, \cdot)$, the case of $q = n$ of the dual Minkowski problem is the logarithmic Minkowski problem for cone-volume measure. Since the 0th dual curvature measure $\widetilde{C}_0(K, \cdot)$ is Aleksandrov's integral curvature (the 0th curvature measure on S^{n-1}) of the polar body K^* (with the constant factor $1/n$), the case of $q = 0$ of the dual Minkowski problem is equivalent to the Aleksandrov problem. The logarithmic Minkowski problem and the Aleksandrov problem were thought to be two entirely different problems. They are now unified as special cases of the dual Minkowski problem.

Because of the geometric meaning of dual quermassintegrals about sections of convex bodies, the other integer cases, $q = 1, \ldots, n-1$, of the dual Minkowski problem, are of special significance.

When the measure μ has a density function $f : S^{n-1} \to \mathbb{R}$, the partial differential equation of the dual Minkowski problem can be written as the following Monge-Ampère type equation on S^{n-1}:

$$\det(\nabla_{ij} h + h\delta_{ij}) = \frac{|\nabla h|^{n-q}}{h} f. \tag{6.25}$$

To solve the dual Minkowski problem, the paper [119] used a new variational method and considered the maximization problem,

$$\max_{g \in C_e^+(S^{n-1})} \left\{ \frac{1}{|\mu|} \int_{S^{n-1}} \log g \, d\mu + \frac{1}{q} \log \left(\frac{1}{n\omega_n} \int_{S^{n-1}} h_{\langle g \rangle}^{-q} \, dv \right) \right\}, \tag{6.26}$$

where $\langle g \rangle$ is the convex hull of g, see (3.3), while the classical variational method for the Minkowski problem uses the Wulff shape. The use of convex hull for the dual Minkowski problem is quite natural because the convex hull and the Wulff shape are dual concepts (see [119]). The variational formulas (6.7) and (6.8) are the crucial tools to show that a solution to the maximization problem gives a solution to the dual Minkowski problem.

Sufficient conditions, which are also necessary conditions for dual curvature measures, for the existence of a solution to the maximization problem (6.26) are crucial for obtaining sharp estimates for the two integrals in the maximizing functional — the entropy-type integral and the dual quermassintegral. The entropy-type integral with respect to the given measure μ was dealt with earlier in the paper [25]. The estimates obtained in that paper for the logarithmic Minkowski problem are not enough for the dual Minkowski problem and more delicate estimates are needed, but the technique of spherical partition can still be used. Estimates for the dual quermassintegrals of degree $q > 0$ are much more difficult to obtain than the known case of $q = n$, where the dual quermassintegral is just volume. Sharp estimates for the dual quermassintegral (the second integral in the maximizing functional) are obtained by the application of generalized spherical coordinates. The sharp estimates for both the entropy-type integral and the dual quermassintegral are then applied to analyzing the convergence of a maximizing sequence, and this shows the existence of a solution to the maximization problem.

For the symmetric case, the necessary and sufficient conditions for dual curvature measures are the conditions that show the concentration of the measures on great subspheres.

6.4.2. Subspace Mass Inequality. For $0 < q < n$, a finite Borel measure μ on S^{n-1} is said to satisfy *the qth subspace mass inequality* if

$$\frac{\mu(\xi \cap S^{n-1})}{\mu(S^{n-1})} < \begin{cases} i/q & i < q \\ 1 & i \geq q \end{cases} \tag{6.27}$$

for any proper subspace ξ of dimension i in $\mathbb{R}^n$.

When $0 < q \leq 1$, the qth subspace mass inequality simply says that the measure μ is not concentrated on a great subsphere.

6.4.3. Existence of Solution to the Symmetric Dual Minkowski Problem. *Let $0 < q < n$ and μ be a non-zero even finite Borel measure on S^{n-1}. Then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ such that $\widetilde{C}_q(K, \cdot) = \mu$ if and only if μ satisfies the qth subspace mass inequality (6.27).*

The case of $0 < q \leq 1$ was proved by Huang-LYZ [119]. For the case of $1 < q < n$, with a stronger sufficient condition than (6.27), the existence of solution was also proved in [119]. That sufficient condition was refined as (6.27) by Zhao [237] for integer $q = 2, \ldots, n-1$, and by Böröczky, Henk & Pollehn [23] for any $1 < q < n$, independently. That is, the qth subspace mass inequality is a necessary condition for the qth dual curvature measure to satisfy. Zhao [237] also showed that for integer $q = 2, \ldots, n-1$, the qth subspace mass inequality is a sufficient condition for a finite Borel measure to be the qth dual curvature measure of a symmetric convex body. Böröczky-LYZ-Zhao [26] generalized Zhao's result to any real $1 < q < n$. The dual Minkowski problem for even data and $q = 0$ is equivalent to the Aleksandrov problem for even data, solved by Aleksandrov himself. A variational proof appeared in [120]. When $q = n$, the dual Minkowski problem for even data is the logarithmic Minkowski problem for even data, solved in [25]. Thus, the dual Minkowski problem is completely solved for the symmetric case when $0 \leq q \leq n$.

When the given measure μ on S^{n-1} is absolutely continuous with respect to the spherical Lebesgue measure, the measure μ has no concentration on a great subsphere. Thus, the subspace mass inequality is true. In this case, the maximization problem (6.26) can be solved quite easily similar to solving the classical Minkowski problem. In fact, if μ has density f that is even and bounded below by a positive constant, then for any $q > 0$ the dual Minkowski problem has a symmetric convex body as a solution. However, it is difficult to see how to solve the dual Minkowski problem for a general measure by using approximation and the solution to the case of the measure with a density.

The dual Minkowski problem for $q < 0$, similar to the classical Minkowski problem, does not require any non-trivial measure concentration condition and was solved by Zhao [238].

6.4.4. Existence of Solution to the Dual Minkowski Problem with $q < 0$. *Suppose $q < 0$ and μ is a non-zero finite Borel measure on S^{n-1}. Then there exists a convex body K in $\mathbb{R}^n$ that contains the origin in its interior such that $\widetilde{C}_q(K, \cdot) = \mu$ if and only if μ is not concentrated in any closed hemisphere.*

6.4.5. Uniqueness of the Dual Minkowski Problem. The uniqueness problem of dual curvature measures was shown for $q \leq 0$. The case when $q = 0$ is classical and follows from the uniqueness result of integral curvature shown by Aleksandrov. For convex bodies K, L in $\mathbb{R}^n$ that contain the origin in their interiors, $\widetilde{C}_0(K, \cdot) = \widetilde{C}_0(L, \cdot)$ implies that $K = cL$ for $c > 0$, and for $q < 0$, $\widetilde{C}_q(K, \cdot) = \widetilde{C}_q(L, \cdot)$ implies that $K = L$. This was proved by Zhao [238]. The case when $q > 0$ has not yet been settled.

The uniqueness together with the variational solution to the dual Minkowski problem would imply unproven Brunn-Minkowski type inequalities of dual quermassintegrals, for example,

$$\widetilde{V}_q(K + L)^{\frac{1}{q}} \geq \widetilde{V}_q(K)^{\frac{1}{q}} + \widetilde{V}_q(L)^{\frac{1}{q}},$$

for origin-symmetric convex bodies K, L in $\mathbb{R}^n$ and $1 < q \neq n$. This inequality was conjectured by Lutwak in an unpublished paper shown to Yang and Zhang in 1990s. The case of $q = n$ is the classical Brunn-Minkowski inequality and the case of $q \leq 1$ can be easily shown by using the inclusion $K \widetilde{+} L \subseteq K + L$.

6.4.6. Regularity of Solution to the Dual Minkowski Problem and q-Gauss Curvature Flow. Motivated by the dual Minkowski problem, Li-Sheng-Wang [145] introduced the q-Gauss curvature flow,

$$\frac{\partial X}{\partial t}(u,t) = -|X(u,t)|^{n-q}\kappa(u,t)f(\nu(u,t))\nu(u,t), \quad q \in \mathbb{R}, \tag{6.28}$$

where $X(\cdot,t) : S^{n-1} \to \mathbb{R}^n$ is a parametrization of the boundaries of a family of smooth convex bodies $K_t \in \mathcal{K}_o^n$ with positive Gauss curvature $\kappa(\cdot,t)$ and outer unit normal $\nu(\cdot,t)$ at $X(\cdot,t)$, and f is a given positive smooth function f on S^{n-1}. When $q = n$, this flow equation is the anisotropic Gauss curvature flow (6.17).

The self-similar solution of a normalized flow equation (6.28) satisfies the Monge-Ampère type equation (6.25) of the dual Minkowski problem. Li-Sheng-Wang [145] gave solutions to the q-Gauss curvature flow which yield regularity results to the dual Minkowski problem:

Suppose a Borel measure μ on S^{n-1} has a density f that is positive and C^∞. When $q < 0$, the solution K to the dual Minkowski problem is C^∞ and uniformly convex (i.e., of positive principal curvatures). When $q > 0$ and f is even, the solution K is C^∞ and uniformly convex.

When $n = 2$ and $q > 0$, similar regularity results, but without assuming evenness, were shown by Chen-Li [42]. Again in the plane when $q \geq 2$ is an even integer, C^2 regularity of the solution K was proved by Huang-Jiang [121] under a weaker assumption on f. When $f = 1$ and the Gauss curvature is replaced by an intermediate curvature in (6.28), convergence results were proved by Li-Sheng-Wang [146].

6.4.7. Affine Dual Minkowski Problem. In the formula (6.3) of dual quermassintergals of a convex body in $\mathbb{R}^n$, if the intersection volume is replaced by its nth power, one gets the so-called *affine dual quermassintegrals* of the convex body (up to a normalization). Recently, Cai-Leng-Wu-Xi [35] were able to construct their *affine dual curvature measures* and studied the associated *affine dual Minkowski problem*.

7. Chord Minkowski Problems from Integral Geometry

While quermassintegrals and dual quermassintegrals are the two families of fundamental geometric invariants in the Brunn-Minkowski theory and its dual theory, a third family of geometric invariants of convex bodies called *chord integrals* arises from integral geometry and geometric probability. These are moments of lengths of intersections of random lines with convex bodies. They also give the second moment of areas of intersections of random planes with convex bodies [234]. Chord integrals are closely related to X-rays in geometric tomography [83] and Riesz potentials of characteristic functions of convex bodies.

In the recent paper [169], the authors introduce the geometric measures derived from chord integrals, called *chord measures*. They are the second family of translation invariant geometric measures of convex bodies besides Aleksandrov-Fenchel-Jessen's area measures. We describe Minkowski problems associated with these measures.

7.1. Chord Integrals and Measures

7.1.1. Chord Integrals. Let $\mathscr{L}^n$ be the affine Grassmannian of lines in $\mathbb{R}^n$. Lines in $\mathscr{L}^n$ are determined by their rotations (directions) and translations. The Haar measure on $\mathscr{L}^n$ is

normalized to be a probability measure when restricted to rotations of lines and to be $(n-1)$-dimensional Lebesgue measure when restricted to translations of lines. The *chord integral* $I_q(K)$ of convex body K in $\mathbb{R}^n$ is defined by

$$I_q(K) = \int_{\mathscr{L}^n} |K \cap \ell|^q \, d\ell, \qquad q \geq 0, \tag{7.1}$$

where $|K \cap \ell|$ denotes the length of the chord $K \cap \ell$, $\ell \in \mathscr{L}^n$, and $d\ell$ is the normalized Haar measure on $\mathscr{L}^n$.

Volume and surface area are chord integrals. Indeed, there are integral formulas,

$$I_1(K) = V(K), \quad I_0(K) = \frac{\omega_{n-1}}{n\omega_n} S(K), \quad I_{n+1}(K) = \frac{n+1}{\omega_n} V(K)^2,$$

which are Crofton's volume formula, Cauchy's integral formula for surface area, and the Poincaré-Hadwiger formula, respectively (see [193, 194]).

The relationship between chord integrals and the Riesz potentials of characteristic functions of convex bodies is given by

$$I_q(K) = \frac{q(q-1)}{n\omega_n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\mathbf{1}_K(x)\mathbf{1}_K(y)}{|x-y|^{n-q+1}} \, dxdy, \quad q > 1, \tag{7.2}$$

(see [193, 194]). There is also the formula,

$$I_q(K) = \frac{q(1-q)}{n\omega_n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\mathbf{1}_K(x)\mathbf{1}_{K^c}(y)}{|x-y|^{n-q+1}} \, dxdy, \quad 0 < q < 1, \tag{7.3}$$

where K^c is the complement of K. Formula (7.3) was shown by Qin [191], and implicitly by Ludwig [160].

The integral on the right-hand side of (7.3) is called a fractional integral of K. Formula (7.3) and Cauchy's integral formula for $I_0(K)$ show that the fractional integral tends to the surface area of K, up to a constant factor, as $q \to 0^+$, which is a fact proved by Bourgain-Brezis-Mironescu [29] for any bounded Borel set in $\mathbb{R}^n$. A generalization of the fractional integral, called anisotropic fractional integral, was given by Ludwig [160] when the Euclidean norm is replaced by a non-Euclidean norm. Results in [29] were extended in [160].

The relationship between chord integrals and dual quermassintegrals is given by

$$I_q(K) = \frac{q}{\omega_n} \int_K \widetilde{V}_{q-1}(K, z) \, dz, \quad q > 0,$$

(see [90, 234]), where $\widetilde{V}_{q-1}(K, z)$ is the dual quermassintegral of K with respect to a point z in K,

$$\widetilde{V}_{q-1}(K, z) = \frac{1}{n} \int_{S_z^+} \rho_{K-z}^{q-1}(u) \, du, \quad S_z^+ = \{u \in S^{n-1} : \rho_{K-z}(u) > 0\}.$$

As a function of $z \in K$, $\widetilde{V}_{q-1}(K, z)$ is continuous when $q > 1$, but only lower semi-continuous when $0 < q \leq 1$. When $z \in \partial K$ and $0 < q < 1$, $\widetilde{V}_{q-1}(K, z)$ may be infinite.

7.1.2. Chord Measures. In the paper [169], differential formulas of chord integrals were derived. Then a new family of geometric measures was defined. For each $q > 0$, the *qth chord measure* $F_q(K, \cdot)$ of a convex body K in $\mathbb{R}^n$ is defined by

$$F_q(K, \eta) = \frac{2q}{\omega_n} \int_{\nu_K^{-1}(\eta)} \widetilde{V}_{q-1}(K, z) \, dz, \quad \text{for any Borel set } \eta \subset S^{n-1}. \tag{7.4}$$

Each $F_q(K, \cdot)$ is a finite Borel measure on S^{n-1} and is the differential of I_q at K; i.e.,

$$\frac{d}{dt}\Big|_{t=0^+} I_q(K + tL) = \int_{S^{n-1}} h_L(v)\, dF_q(K, v), \tag{7.5}$$

for each convex body L. The case of $q = 1$ corresponds to the classical formula (5.13) and the measure $F_1(K, \cdot)$ is the classical surface area measure $S(K, \cdot)$, i.e., $F_1(K, \cdot) = S(K, \cdot)$.

The extremal case $q \to 0^+$ is very interesting. It exhibits a surprising connection between dual quermassintegrals and mean curvature: For each sufficiently smooth convex body K and each $z \in \partial K$,

$$\lim_{q \to 0^+} q\widetilde{V}_{q-1}(K, z) = \frac{(n-1)\omega_{n-1}}{2n} H(z),$$

where $H(z)$ is the mean curvature of ∂K at z. This implies that for a sufficiently smooth convex body the area measure $S_{n-2}(K, \cdot)$ is a limiting case of chord measures,

$$\lim_{q \to 0^+} F_q(K, \eta) = \frac{(n-1)\omega_{n-1}}{n\omega_n} S_{n-2}(K, \eta), \tag{7.6}$$

for each Borel set $\eta \subset S^{n-1}$. Thus, one can define

$$F_0(K, \cdot) = \frac{(n-1)\omega_{n-1}}{n\omega_n} S_{n-2}(K, \cdot).$$

However, it is not clear what the limit in (7.6) is for non-smooth convex bodies.

The chord measures $F_q(K, \cdot)$, $q \geq 0$, are translation invariant. This implies that

$$\int_{S^{n-1}} v\, dF_q(K, v) = 0. \tag{7.7}$$

The relation between surface area measure and cone-volume measure can be extended to chord measures and cone-chord measures. The *qth cone-chord measure* $G_q(K, \cdot)$ is defined by

$$dG_q(K, \cdot) = \frac{1}{n + q - 1} h_K\, dF_q(K, \cdot). \tag{7.8}$$

When $q = 1$, $G_1(K, \cdot)$ is the cone volume measure $V(K, \cdot)$. In this case, (7.8) corresponds to (6.11). The chord integral, the total mass of cone chord measure, and the chord measure are related by

$$I_q(K) = G_q(K, S^{n-1}) = \frac{1}{n + q - 1} \int_{S^{n-1}} h_K(v)\, dF_q(K, v).$$

7.2. Minkowski Problems of Chord Measures

The Minkowski problem of prescribing chord measures was posed in [169]:

7.2.1. Chord Minkowski Problem. *Let $q \geq 0$. If μ is a finite Borel measure on S^{n-1}, what are necessary and sufficient conditions on μ to guarantee the existence of a convex body $K \subset \mathbb{R}^n$ that solves the equation,*

$$F_q(K, \cdot) = \mu? \tag{7.9}$$

When the given finite measure μ has a density, equation (7.9) becomes a new Monge-Ampère type equation on S^{n-1},

$$\det\left(\nabla_{ij} h(v) + h(v)\delta_{ij}\right) = \frac{f(v)}{\widetilde{V}_{q-1}([h], \nabla h(v))}, \tag{7.10}$$

where $\widetilde{V}_{q-1}([h], \nabla h(v))$ denotes the dual quermassintegral of the Wulff shape $[h]$ with respect to the point $\nabla h(v) \in \mathbb{R}^n$, which is a function of v. Note that the dual quermassintegral is expressed by the radial function and depends implicitly on h because h corresponds to the support function. Thus, $\widetilde{V}_{q-1}([h], \nabla h(v))$ is a function of h and $\nabla h(v)$ only in the implicit sense and is not locally defined. Also, since $\nabla h(v)$ is a boundary point of $[h]$, $\widetilde{V}_{q-1}([h], \nabla h(v))$ may be infinite when $0 < q < 1$.

This chord Minkowski problem was solved in [169] when $q > 0$:

7.2.2. Solution to the Chord Minkowski Problem. *Let $q > 0$. If μ is a finite Borel measure on S^{n-1}, then there exists a convex body K in $\mathbb{R}^n$ satisfying $F_q(K, \cdot) = \mu$ if and only if μ is not concentrated on a closed hemisphere and*

$$\int_{S^{n-1}} v \, d\mu(v) = 0.$$

This solution extends that of the classical Minkowski problem in §5.2.1. The proof is variational, similar to the classical case. It involves solving a maximization problem for the functional $\Phi(h)$ over $C^+(S^{n-1})$,

$$\Phi(h) = I_q([h])^{\frac{1}{n+q-1}} \left(\int_{S^{n-1}} h \, d\mu \right)^{-1}.$$

This is shown by using the variational formula (7.5).

The log-Minkowski problem of cone-volume measure in §6.2 was extended to the following:

7.2.3. The Chord Log-Minkowski Problem. *Let $q \geq 0$. If μ is a finite Borel measure on S^{n-1}, what are necessary and sufficient conditions on μ to guarantee the existence of a convex body K in $\mathcal{K}_o^n$ that solves the equation,*

$$G_q(K, \cdot) = \mu? \tag{7.11}$$

When $q = 1$, the chord log-Minkowski problem is just the log-Minkowski problem, When $q = 0$, it is a log-Christoffel-Minkowski problem, which may well be even more difficult to solve than the still open Christoffel-Minkowski problem.

The partial differential equation associated with (7.11) is the following Monge-Ampère equation on S^{n-1},

$$\det \left(\nabla_{ij} h + h \delta_{ij} \right) = \frac{f}{h \, \widetilde{V}_{q-1}([h], \nabla h)}. \tag{7.12}$$

Similar to the situation with the log-Minkowski problem versus the classical Minkowski problem, the chord log-Minkowski problem is a harder problem than the chord Minkowski problem. The following solution is obtained in [169] and [191] for the symmetric case.

7.2.4. Solution to the Symmetric Chord Log-Minkowski Problem. *Let μ be an even finite Borel measure on S^{n-1} and $0 < q \leq n+1$. Then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ that solves the equation,*

$$G_q(K, \cdot) = \mu,$$

if μ satisfies the subspace mass inequality:

$$\frac{\mu(\xi_k \cap S^{n-1})}{|\mu|} < \frac{k + \min\{k, q-1\}}{n+q-1}, \tag{7.13}$$

for each k-dimensional subspace $\xi_k \subset \mathbb{R}^n$ and each $k = 1, \ldots, n-1$.

The above solution of the chord log-Minkowski problem for the case where $q = 1$ is the solution to the log-Minkowski problem obtained in [25]. The case of $1 < q \leq n + 1$ was proved in [169]. By using a different approach via formula (7.3), Qin [191] proved the case of $0 < q < 1$. For more recent studies on chord Minkowski problems, see [105, 123, 149, 192].

7.2.5. Problems. Some important open problems related to chord measures need to be studied. The uniqueness problem of chord measures is a major problem:

If $F_q(K, \cdot) = F_q(L, \cdot)$ for a positive q, is K a translation of L?

The cases of $q = 1, n + 1$ are known, which are the uniqueness result of surface area measure. An affirmative answer to this uniqueness question together with the variational solution to the chord Minkowski problem would imply Brunn-Minkowski type geometric inequalities for chord integrals,

$$I_q(K + L)^{\frac{1}{n+q-1}} \geq I_q(K)^{\frac{1}{n+q-1}} + I_q(L)^{\frac{1}{n+q-1}}, \quad q > 0,$$

with equality if and only if K and L are homothetic. It is the classical Brunn-Minkowski inequality when $q = 1, n + 1$. The case of $q = 0$ is the classical Brunn-Minkowski inequality of surface area which is a consequence of the Aleksandrov-Fenchel inequality (see [196]). Other cases have not been studied.

Regularity in the chord Minkowski problems is also highly interesting. It is natural to solve the chord Minkowski problems by using the continuity method or an anistropic Gauss curvature flow so that smooth solutions are obtained. The nonlocal quantity $\widetilde{V}_{q-1}(\llbracket h \rrbracket, \nabla h)$ in the PDE (7.10) may create some obstacles. Recently, Hu-Huang-Lu [117] obtained regularity results for Riesz potential and applied them to the regularity of the chord log-Minkowski problem.

While chord integrals come from intersections of convex bodies with straight lines in $\mathbb{R}^n$, it is natural to consider intersections of convex bodies with flat planes.

8. L_p Minkowski Problems in the L_p Brunn-Minkowski Theory

The basic algebraic operation in the Brunn-Minkowski theory is the Minkowski addition of convex bodies (see (5.1)). By the motivation of interactions between convex geometry and functional analysis, extending the Minkowski addition to the L_p Minkowski addition leads to the L_p Brunn-Minkowski theory which generalizes the classical Brunn-Minkowski theory. The L_p Brunn-Minkowski theory has found many connections and applications in functional analysis and other subjects. The L_p Minkowski addition was first studied by Firey [76] for $p > 1$ in early 1960s. The L_p Brunn-Minkowski theory gained its status only after the celebrated work of Lutwak [166, 167] in 1990s. Subsequent studies on affine isoperimetric inequalities [36, 106, 107, 170, 171], valuations [155, 156, 158, 159, 161], affine surface areas [179, 201, 212–214], and Minkowski problems [25, 56, 126, 168, 172, 173] have made it a significant theory with rich and deep results.

The critical concept introduced by Lutwak is the L_p surface area measure that extends the classical surface area measure of Aleksandrov, Fenchel and Jessen. The problem of prescribing L_p surface area measure is called the L_p Minkowski problem.

8.0.1. Basics of L_p Brunn-Minkowski Theory. For convex bodies K and L in $\mathcal{K}_o^n$ and $p \geq 1$, $(h_K^p + h_L^p)^{\frac{1}{p}}$ is a positive convex function of homogeneous of degree 1, and hence it is the support function of a convex body in $\mathcal{K}_o^n$, denoted by $K +_p L$ and called the L_p *sum* of K and

L. Thus,

$$h_{K +_p L} = (h_K^p + h_L^p)^{\frac{1}{p}}.$$

The L_1 sum is the usual Minkowski sum.

The L_p sum can be extended to the case of $p < 1$ by using Wulff shape. For $p, s, t \in \mathbb{R}$, suppose $sh_K^p + th_L^p$ is positive. The L_p sum $s \cdot K +_p t \cdot L$ is defined as the Wulff shape generated by the function $(sh_K^p + th_L^p)^{\frac{1}{p}}$ when $p \neq 0$ and by $h_K^s h_L^t$ when $p = 0$, i.e.,

$$s \cdot K +_p t \cdot L = [(sh_K^p + th_L^p)^{\frac{1}{p}}], \quad s \cdot K +_0 t \cdot L = [h_K^s h_L^t].$$

Note that the geometric mean $K^{1-t} L^t$ is the L_0 sum $(1-t) \cdot K +_0 t \cdot L$.

We remark that the L_p sum $s \cdot K +_p t \cdot L$ is defined for $K, L \in \mathcal{K}_o^n$, i.e., when the origin is inside the interiors of K and L. But it is still well defined even if the origin is on the boundary of K or L as long as $sh_K^p + th_L^p$ is positive.

As demonstrated by Lutwak [166], area measures can be extended to L_p area measures. For a convex body K in $\mathcal{K}_o^n$ and $p \in \mathbb{R}$, its L_p *ith area measure* is the finite Borel measure on S^{n-1} defined by

$$dS_{p,i}(K, \cdot) = h_K^{1-p} dS_i(K, \cdot), \quad i = 0, 1, \ldots, n - 1. \tag{8.1}$$

The case of $i = n - 1$, $S_{p,n-1}(K, \cdot)$, is called the L_p *surface area measure* and is often denoted by $S_p(K, \cdot)$ when there is no confusion. L_p area measures arise naturally as the right-side differentials of quermassintegrals with respect to the L_p sum,

$$\frac{d}{dt} W_{n-i-1}(K +_p t \cdot L)\Big|_{t=0^+} = \frac{i+1}{np} \int_{S^{n-1}} h_L^p(v) \, dS_{p,i}(K, v), \quad i = 0, 1, \ldots, n - 1, \tag{8.2}$$

for $p \neq 0$. There is a similar formula for the case of $p = 0$.

A more geometric definition of the L_p surface area measure is

$$S_p(K, \eta) = \int_{x \in \nu_K^{-1}(\eta)} (x \cdot \nu_K(x))^{1-p} \, dx, \quad \text{for any Borel set } \eta \subset S^{n-1}, \tag{8.3}$$

where for $p > 1$, K contains the origin in its interior, for $p < 1$, K contains the origin (which may be on the boundary of K), and for $p = 1$, K is any convex body. The L_1 surface area measure is obviously the surface area measure $S(K, \cdot)$. When $p = 0$, the L_0 surface area measure is n times the cone-volume measure $V(K, \cdot)$.

The L_p surface area measure is not only the right-side differential of the volume functional at a convex body with respect to the L_p Minkowski sum, but also the left-side differential. In fact, the differential formula (5.13) implies that

$$\frac{d}{dt} V(K +_p t \cdot L)\Big|_{t=0} = \frac{1}{p} \int_{S^{n-1}} h_L^p(v) \, dS_p(K, v). \tag{8.4}$$

This formula is critical for solving the L_p Minkowski problem. The case of $p = 0$ corresponds to the geometric mean and cone-volume measure, discussed in the previous section.

The integral above defines the L_p mixed volume $V_p(K, L)$ of convex bodies K and L,

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(v) \, dS_p(K, v).$$

When $K = L$, $V_p(K, K) = V(K)$. When L is the unit ball B^n, $V_p(K, B^n)$, denoted by $S_p(K)$, is called the L_p surface area of K. There is the L_p mixed volume inequality,

$$V_p(K, L)^n \geq V(K)^{n-p} V(L)^p, \quad p > 1, \tag{8.5}$$

with equality if and only if K is a dilate of L.

The L_p mixed volume inequality follows from Minkowski's first mixed volume inequality (5.2) and Hölder's inequality. The equality conditions of (8.5) implies the following uniqueness result:

Let K and L be convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors and $p > 1$. If $S_p(K, \cdot) = S_p(L, \cdot)$, then $K = L$ when $p \neq n$, and K is a dilation of L when $p = n$.

When $p < 1$, L_p surface area measure is not in general unique. It was conjectured that for $0 < p < 1$ the L_p mixed volume inequality still holds for origin-symmetric convex bodies. This would be true if the logarithmic Minkowski inequality (6.21) is true, see Böröczky-LYZ [24]. See Chen-Li-Zhu [43, 44], Jian-Lu-Wang [130] and Li-Liu-Lu [144] for results about the case of $p < 1$. Recently, Milman [180] gave a new strong non-uniqueness result for the even L_p-Minkowski problem when $-n < p < 0$.

Suppose that p is less than and close to 1. Kolesnikov-Milman [135] showed that $K = L$ if K and L have positive curvature and are close in C^2 norm. Chen-Huang-Li-Liu [41] strengthened this local result as a global one by using the Schauder estimates in PDEs to show that $K = L$ if K and L have positive curvature and are of class $C^{2,\alpha}$.

Geometric measures in the Brunn-Minkowski theory and the dual Brunn-Minkowski theory have L_p analogues. Therefore, Minkowski problems discussed before have their L_p versions. Here, we describe the L_p Minkowski problem, the L_p Aleksandrov problem, the L_p dual Minkowski problem, and the L_p chord Minkowski problem. The L_p Minkowski problem has been studied most.

8.0.2. L_p Minkowski Problem. The classical Minkowski problem of prescribing surface area measure was generalized to the L_p Minkowski problem of prescribing L_p surface area measure by Lutwak [166]:

Given a finite Borel measure μ on S^{n-1} and $p \in \mathbb{R}$, what are the necessary and sufficient conditions for the existence of a convex body $K \in \mathcal{K}_o^n$ satisfying

$$S_p(K, \cdot) = \mu ? \tag{8.6}$$

The associated partial differential equation of the L_p Minkowski problem is the Monge-Ampère equation

$$\det(\nabla_{ij} h + h\delta_{ij}) = h^{p-1} f. \tag{8.7}$$

The problem contains three special cases which have important geometric meanings. When $p = 1$, it is the classical Minkowski problem discussed in Section 5.2. When $p = 0$, it is the logarithmic Minkowski problem described in Section 6.2. When $p = -n$, it is called the *centro-affine Minkowski problem*. In this case, the absolutely continuous part of the L_{-n} surface area measure with respect to the spherical Lebesgue measure is $h_K^{1+n} f_K$ whose $(n+1)$-th root is the affine distance in affine differential geometry (see [186], pp. 62-63). Prescribing the affine distance is a long standing unsolved problem, which is then a special case of the L_p Minkowski problem and the partial differential equation is

$$\det(\nabla_{ij} h + h\delta_{ij}) = \frac{f}{h^{n+1}}. \tag{8.8}$$

The L_p Minkowski problem has been solved for $p \geq 1$ and also solved for $0 \leq p < 1$ in the symmetric case.

Existence of Solutions to the Symmetric L_p **Minkowski Problem.** *Let* μ *be a finite even Borel measure that is not concentrated on a great subsphere. Then there exists an origin-symmetric convex body* K *such that* $S_p(K, \cdot) = \mu$ *when* $0 < p \neq n$ *and* $S_n(K, \cdot) = V(K)\mu$.

This was proved for $p > 1$ by Lutwak [166] and for $0 < p < 1$ by Haberl-LYZ [108]. A different proof and the volume normalized formulation were given by LYZ [172].

Existence of Solutions to the Asymmetric L_p **Minkowski Problem.** *Let* μ *be a finite Borel measure on* S^{n-1} *that is not concentrated in a closed hemisphere. Then, for each* $p > 1$, *there exists a unique convex body* K *that contains the origin such that*

$$dS(K, \cdot) = h_K^{p-1} d\mu, \quad 1 < p \neq n, \quad and \quad S_n(K, \cdot) = V(K)\mu. \tag{8.9}$$

Moreover, K *contains the origin in its interior if either* $p \geq n$ *or* μ *is discrete.*

This result was first proved by Chou-Wang [56], while the fact that the solution contains the origin in its interior if either $p \geq n$ or μ is discrete was shown by Hug-LYZ [126]. Equation (8.9) is a volume normalized formulation of (8.6), first given by LYZ [172]. When $p \neq n$, the volume factor in (8.9) can always be absorbed by a dilation. When the origin is contained in the interior of K, equations (8.6) and (8.9) are equivalent. However, there exist examples which show that the origin can be on the boundary of K when $1 < p < n$, see [126]. In this case, (8.9) does not imply (8.6).

Chou-Wang [56] proved the result for measures with smooth densities first by using the continuity method, which also gives regularity of solutions. Then the general case of measures was shown by approximation. Hug-LYZ [126] used the variational method to prove the result for discrete measures first and followed by an approximation argument. The optimization problem associated with the L_p Minkowski problem is

$$\inf_{h \in C^+(S^{n-1})} \left\{ V([h])^{-\frac{1}{n}} \left(\int_{S^{n-1}} h^p \, d\mu \right)^{\frac{1}{p}} \right\}.$$

If this minimization problem has a positive solution, then its Wulff shape gives a solution to the L_p Minkowski problem. This can easily be seen by a proof similar to the classical case. When μ is discrete or $p \geq n$, the minimization problem has a positive solution when $p > 1$. However, when $1 < p < n$, the minimization problem, in general, has only a nonnegative solution whose Wulff shape may not contain the origin in its interior even if the given measure has a positive smooth density. Thus, additional techniques are used to finish the proof.

Existence of Solutions When $0 < p < 1$**.**

If $0 < p < 1$, *and* μ *is a finite Borel measure on* S^{n-1} *not concentrated on a great subsphere, then there exists a convex body* K *containing the origin so that* $S_p(K, \cdot) = \mu$.

The polytope case was proved by Zhu [242] and then the general case was shown by Chen-Li-Zhu [43] by an approximation argument.

Note that for $p < 1$ the L_p surface area measure $S_p(K, \cdot)$ may concentrate on a great subsphere when the origin is on the boundary of K. Indeed, if K is a cone whose vertex is the origin, then $S_p(K, \cdot)$ is a single point mass. It was conjectured [18]:

Let $0 < p < 1$, *and let* μ *be a non-trivial finite Borel measure on* S^{n-1}. *Then there exists a convex body* K *containing the origin so that* $S_p(K, \cdot) = \mu$ *if and only if* μ *does not concentrate on a pair of antipodal points.*

The planar case $n = 2$ was proved independently by Böröczky-Trinh [28] and Chen-Li-Zhu [43]. Partial results on higher dimensions were obtained in [18, 43].

Existence of Solutions When $p < 0$. When $p < 0$, the L_p Minkowski problem has been studied when the given Borel measure on S^{n-1} is either discrete or absolutely continuous. For the discrete case, there is the following solution:

If $p < 0$ and μ is a discrete measure on S^{n-1}, then there exists a polytope P in $\mathbb{R}^n$ whose outer unit normals are in general position so that $S_p(P, \cdot) = \mu$ if and only if the support of μ is in general position and not concentrated on a closed hemisphere.

When $p = -n$, this result gives a solution to the centro-affine Minkowski problem for polytopes whose outer unit normals are in general position, proved by Zhu [243]. Then he [244] generalized his result to any $p < 0$ and to a larger class of polytopes. Jian-Lu-Zhu [131] gave a solution to the centro-affine Minkowski problem for the class of convex bodies symmetric about coordinates hyperplanes. Li [143] showed that there may exist infinitely many solutions which are not affine equivalent.

For the absolutely continuous case, there is the following result:

If $-n < p < 0$, and μ is a non-trivial Borel measure on S^{n-1} with a density function in $L_{\frac{n}{n+p}}(S^{n-1})$, then there exists a convex body K containing the origin so that $S_p(K, \cdot) = \mu$.

The result was proved by Chow-Wang [56] when the density function of μ is bounded and positive. The planar case was proved by Chen [48] when the density function μ is continuous and even. The general case was shown by Bianchi-Böröczky-Colesanti-Yang [18] by an approximation argument.

For the smooth symmetric case, Bryan-Ivaki-Scheuer [31] gave a unified approach for all $p > -n$ by using curvature flow.

For $p \leq 0$, the L_p Minkowski problem has multiple solutions for a given measure. This was proved by Jian-Lu-Wang [130] for $p \in (-n, 0)$, by He-Li-Wang [111] for $p \in (-\infty, -n + 1)$, and by Chen-Li-Zhu [44] for $p = 0$.

For $p \leq -2$, Do-Zhu [68] showed that the L_p Minkowski problem in $\mathbb{R}^2$ has a solution when the density function μ is continuous and even for each coordinate. For $p < -n$, Guang-Li-Wang [104] proved that the L_p Minkowski problem in $\mathbb{R}^n$ has a $C^{3,\alpha}$, $0 < \alpha < 1$, solution if μ has a positive $C^{1,1}$ density function.

There are applications to Sobolev-type inequalities of functions, in particular, the affine Sobolev inequalities [171, 236]. Geometric measures are defined for functions, and convex bodies are constructed from such geometric measures by using solutions of the L_p Minkowski problem. See [60, 107, 109, 162, 171, 173, 220, 236].

While the L_p Minkowski problem has been extensively studied, the following L_p *Christoffel-Minkowski problem* needs more attention.

Given a finite Borel measure μ on S^{n-1}, a $p \in \mathbb{R}$, and an $i \in \{1, \ldots, n - 1\}$, what are the necessary and sufficient conditions for the existence of a convex body $K \in \mathcal{K}_o^n$ satisfying

$$S_{p,i}(K, \cdot) = \mu\,? \tag{8.10}$$

When $p \geq i + 1$ and μ has a smooth density which satisfies some convexity conditions, existence results were proved by Hu-Ma-Sheng [116], and the even case of $1 < p < i + 1$ was studied by Guan-Xia [102]. The uniqueness of solutions was proved by Lutwak [166] for $p > 1$ and for $1 - i < p < 1$ was partially proved by Chen [39].

8.0.3. L_p Aleksandrov Problem. In [120], the L_p integral curvature was defined and the L_p Aleksandrov problem was studied.

L_p **Curvature Measures.** For convex body K in $\mathcal{K}_o^n$, the L_p *ith curvature measure* $C_{p,i}(K, \cdot)$ is defined by

$$dC_{p,i}(K, \cdot) = \rho_K^p dC_i(K, \cdot).$$

The case of $i = 0$, $C_{p,0}(K, \cdot)$ is called the L_p *integral curvature* of K, which can also be defined by

$$C_{p,0}(K, \omega) = \int_{\alpha_K(\omega)} \rho_K^p \left(\alpha_K^{-1}(v) \right) dv.$$

It can be viewed as the differential of the entropy of convex body with respect to L_p sum,

$$\frac{d}{dt} E(K^* +_p t \cdot L^*) \Big|_{t=0} = \frac{1}{p} \int_{S^{n-1}} \rho_L(u)^{-p} \, dC_{p,0}(K, u), \quad p \neq 0, \qquad (8.11)$$

and for $p = 0$,

$$\frac{d}{dt} E(K^* +_0 t \cdot L^*) \Big|_{t=0} = - \int_{S^{n-1}} \log \rho_L(u) \, dC_0(K, u), \quad p \neq 0. \qquad (8.12)$$

These formulas are consequences of the differential formula (5.23) together with the formula $\langle \rho \rangle^* = [\rho^{-1}]$.

L_p **Aleksandrov-Minkowski Problem.** *Given a finite Borel measure μ on S^{n-1}, a $p \in \mathbb{R}$, and an $i \in \{0, 1, \ldots, n-1\}$, what are the necessary and sufficient conditions for the existence of a convex body $K \in \mathcal{K}_o^n$ satisfying*

$$C_{p,i}(K, \cdot) = \mu \,? \qquad (8.13)$$

The case of $i = 0$ is called the L_p *Aleksandrov Problem.*

When the given measure μ has a density f, the partial differential equation associated with the L_p Aleksandrov problem can be written as the following Monge-Ampère equation on S^{n-1},

$$\det(\nabla_{ij} h + h \delta_{ij}) = h^{p-1} |\nabla h|^n f. \qquad (8.14)$$

A solution of the equation would be the support function h_{K^*} of the polar body of K.

The L_0 Aleksandrov problem is just the classical Aleksandrov problem described in Section 5.4.3. Huang-LYZ [120] solved the L_p Aleksandrov problem when $p > 0$.

Existence of a Solution to the L_p Aleksandrov Problem. *Let μ be a finite Borel measure on S^{n-1} and $p > 0$. Then there exists a convex body $K \in \mathcal{K}_o^n$ so that μ is the L_p-integral curvature of K if and only if μ is not concentrated in any closed hemisphere.*

When $p < 0$, the symmetric case was studied in [120] and the following result was proved.

Existence of a Solution to the Symmetric L_p Aleksandrov Problem. *Let μ be a non-zero finite even measure on S^{n-1}. Then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ so that μ is the L_p-integral curvature of K if μ vanishes on great sub-spheres.*

These results are proved by using variational arguments. The associated optimization problem is the following,

$$\sup_{\rho \in C^+(S^{n-1})} \left\{ \frac{E(\langle \rho \rangle^*)}{n \omega_n} - \frac{1}{p} \log \int_{S^{n-1}} \rho^{-p} \, d\mu \right\},$$

where $E(\langle \rho \rangle^*)$ is the entropy of convex body $\langle \rho \rangle^*$, see (5.22).

When $-1 < p < 0$, the symmetric L_p Aleksandrov problem has the following solution:

Let μ be a finite even Borel measure on S^{n-1} and $p \in (-1, 0)$. Then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ such that $C_{p,0}(K, \cdot) = \mu$ if and only if μ is not concentrated on a great subsphere.

This result was first proved by Zhao [239] for the discrete case and was recently proved by Mui [183] for the general case. Although the approaches are still variational, their proofs are different from that in [120].

For $p \leq -1$, a sufficient measure concentration condition for the existence of solution to the symmetric L_p Aleksandrov problem was given by Mui [183].

8.0.4. L_p Dual Minkowski Problem. In [176], LYZ constructed L_p dual curvature measures which include the L_p surface area measure, the L_p integral curvature, and dual curvature measures. This shows that those geometric measures in the Brunn-Minkowski theory, in the dual Brunn-Minkowski theory, and in the L_p Brunn-Minkowski theory can be unified. Therefore, the Minkowski problems associated with the geometric measures can also be unified.
L_p **Dual Curvature Measures.** Let K be a convex body in $\mathcal{K}_o^n$, Q a star body in $\mathbb{R}^n$, and $p, q \in \mathbb{R}$. The (p, q)th *dual curvature measure of K with respect to Q*, $\widetilde{C}_{p,q}(K, Q, \cdot)$, is a finite Borel measure on S^{n-1} defined by

$$\int_{S^{n-1}} g(v) \, \widetilde{C}_{p,q}(K, Q, v) = \frac{1}{n} \int_{S^{n-1}} g(\alpha_K(u)) h_K(\alpha_K(u))^{-p} \rho_K(u)^q \rho_Q(u)^{n-q} \, du, \qquad (8.15)$$

for each continuous $g : S^{n-1} \to \mathbb{R}$. When Q is the unit ball B, denote $\widetilde{C}_{p,q}(K, B, \cdot)$ by $\widetilde{C}_{p,q}(K, \cdot)$. The L_p dual curvature measures have the following three important cases:

(1) L_p surface area measure,

$$\widetilde{C}_{p,n}(K, Q, \cdot) = \frac{1}{n} S_p(K, \cdot);$$

(2) dual curvature measure,

$$\widetilde{C}_{0,q}(K, B, \cdot) = \widetilde{C}_q(K, \cdot);$$

(3) L_p integral curvature,

$$\widetilde{C}_{p,0}(K, B, \cdot) = \frac{1}{n} C_{p,0}(K^*, 0).$$

The L_p dual curvature measures are differentials of dual quermassintegrals with respect to the L_p Minkowski sum. To state the differential formulas in a unified way, we define the *normalized power function*, $t^{\bar{q}}$, $t > 0$, $q \in \mathbb{R}$,

$$t^{\bar{q}} = \begin{cases} \frac{1}{q} t^q & q \neq 0 \\ \log t & q = 0 \end{cases},$$

and the *normalized dual mixed volume* $\widetilde{V}_{\bar{q}}(K, Q)$ is defined by,

$$\widetilde{V}_{\bar{q}}(K, Q) = \frac{1}{n} \int_{S^{n-1}} \left(\frac{\rho_K}{\rho_Q} \right)^{\bar{q}} (u) \rho_Q^n(u) \, du. \qquad (8.16)$$

Obviously, when $q \neq 0$, $q\widetilde{V}_{\bar{q}}(K, Q)$ is just the dual mixed volume $\widetilde{V}_q(K, Q)$; and when $q = 0$, $\widetilde{V}_{\bar{0}}(K, B)$ is the dual entropy $\widetilde{E}(K)$. Then, we have the differential formula,

$$\frac{d}{dt} \widetilde{V}_{\bar{q}}(K +_p t \cdot L, Q) \Big|_{t=0} = \int_{S^{n-1}} h_L^{\bar{p}}(v) \, d\widetilde{C}_{p,q}(K, Q, v).$$

L_p **Dual Minkowski Problem.** *Let Q be a star body in $\mathbb{R}^n$ and $p, q \in \mathbb{R}$. If μ is a finite Borel measure on S^{n-1}, what are the necessary and sufficient conditions for the existence of a convex body $K \in \mathcal{K}_o^n$ satisfying*

$$\widetilde{C}_{p,q}(K, Q, \cdot) = \mu ? \tag{8.17}$$

When the given measure μ has a density f, the partial differential equation associated with the L_p dual Minkowski problem can be written as the following Monge-Ampère equation on S^{n-1},

$$\det(\nabla_{ij} h + h\delta_{ij}) = h^{p-1}\rho_Q^{q-n}(\nabla h)f. \tag{8.18}$$

The partial differential equations (5.15), (5.24), (6.12), (6.25), (8.7), (8.8), and (8.14) are all special cases of (8.18).

When $p = q = 0$, the L_0 dual Minkowski problem is the absolutely continuous case of the Gauss image problem which will be described in the next section. It was solved by Böröczky-LYZ-Zhao [27] (see the next section), and for the smooth case it was also solved by Li-Wang [148] with a mass transport approach.

For $q > n - 1$ and some negative values of p, Guang-Li-Wang [103] showed existence of solutions to the L_p dual Minkowski problems when μ has a positive smooth density and Q has a smooth boundary.

When Q is the unit ball, the L_p dual Minkowski problem has been studied in a number of papers. Huang and Zhao [125] proved that it has a solution if and only if μ is not concentrated in any closed hemisphere of S^{n-1} when $p > 0$ and $q < 0$. When $p \neq q$, $p, q > 0$, and μ is even, they also showed that there is a symmetric solution if and only if μ is not concentrated on any great subsphere of S^{n-1}. The solution to the symmetric case was extended to the general case by Böröczky-Fodor [20] when $p > 1$, $q > 0$, and $p \neq q$. These results were proved by Chen-Li [38] for the larger range of $p > 0$ and $q \neq n$ by an expanding Gauss curvature flow,

$$\frac{\partial X}{\partial t}(u, t) = |X|^{\frac{n-q}{1-p}} \kappa^{\frac{1}{1-p}} f(\nu)^{-1}\nu.$$

Chen-Huang-Zhao [37] obtained regularity results for symmetric solutions when $pq \geq 0$ by a (rescaled) contracting Gauss curvature flow,

$$\frac{\partial X}{\partial t}(u, t) = -|X|^{n-q}(X \cdot \nu)^p \kappa f(\nu)\nu,$$

which extends (6.28).

Huang-Zhao [125] used a variational method and Böröczky-Fodor [20] dealt with the discrete case first by a variational method followed by an approximation. The optimization problem for the L_p dual Minkowski problem is

$$\inf_{h \in C^+(S^{n-1})} \left\{ \widetilde{V}_q([h])^{-\frac{1}{q}} \left(\int_{S^{n-1}} h^p \, d\mu \right)^{\frac{1}{p}} \right\}.$$

The harder case of the L_p dual Minkowski problem is when $p \leq 0$ and $q > 0$. Both the logarithmic Minkowski problem ($p = 0$ and $q = n$) and the centro-affine Minkowski problem ($p = -n$ and $q = n$) belong to this case.

When the given measure μ is absolutely continuous, uniqueness results were proved by Huang-Zhao [125] when $p > q$, and nonuniqueness results were proved by Li-Liu-Lu [144] when $p < 0 < q$.

8.0.5. L_p Chord Minkowski Problem. Chord measures were extended to L_p *chord measures* [169],

$$F_{p,q}(K, \eta) = \frac{2q}{\omega_n} \int_{\nu_K^{-1}(\eta)} (z \cdot \nu_K(z))^{1-p} \widetilde{V}_{q-1}(K, z) \, dz, \quad \eta \subset S^{n-1}, \tag{8.19}$$

for $q > 0$ and any $p \in \mathbb{R}$ if K contains the origin in its interior. There is the formula,

$$dF_{p,q}(K, \cdot) = h_K^{1-p} dF_q(K, \cdot).$$

It was shown in [169] that the differential of the chord integral I_q with respect to L_p Minkowski combinations leads to the L_p chord measure: for $p \neq 0$,

$$\left. \frac{d}{dt} \right|_{t=0} I_q(K +_p t \cdot L) = \frac{1}{p} \int_{S^{n-1}} h_L^p(v) dF_{p,q}(K, v).$$

The following L_p **chord Minkowski problem** was studied in [219]:

Let μ be a finite Borel measure on S^{n-1}, $p \in \mathbb{R}$ and $q \geq 0$. What are the necessary and sufficient conditions for the existence of a convex body $K \in \mathcal{K}_o^n$ that solves the equation,

$$F_{p,q}(K, \cdot) = \mu ? \tag{8.20}$$

When $p = 1$, this is the chord Minkowski problem, and the $q = 1$ case is the L_p Minkowski problem.

When the given measure μ has a density f that is an integrable nonnegative function on S^{n-1}, equation (8.20) becomes a new Monge-Ampère type partial differential equation,

$$\det \left(\nabla_{ij} h + h \delta_{ij} \right) = \frac{h^{p-1} f}{\widetilde{V}_{q-1}([h], \nabla h)}, \quad \text{on } S^{n-1}. \tag{8.21}$$

The symmetric case of the L_p chord Minkowski problem was solved in [219] when $p, q > 0$:

Let $p, q > 0$. If μ is an even finite Borel measure on S^{n-1} that is not concentrated on a great subsphere, then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ such that

$$F_{p,q}(K, \cdot) = \mu, \quad when \ p \neq n + q - 1,$$
$$\frac{F_{p,q}(K, \cdot)}{V(K)} = \mu, \quad when \ p = n + q - 1.$$

When $q = 1$, this result is a solution to the symmetric L_p Minkowski problem, see [108, 166, 172]. When $p > 1$, the symmetry condition can be dropped. Based on the techniques of [126], [219] gave the following solution:

Let $p > 1, q > 0$. If μ is a finite Borel measure on S^{n-1} that is not concentrated in any closed hemisphere, then there exists a convex body K with nonnegative support function $h_K \geq 0$ so that

$$dF_q(K, \cdot) = h_K^{p-1} d\mu, \quad when \ p \neq n + q - 1,$$
$$\frac{dF_q(K, \cdot)}{V(K)} = h_K^{p-1} d\mu, \quad when \ p = n + q - 1.$$

Moreover, $h_K > 0$ if μ is discrete or if $p \geq n$.

Again, when $q = 1$, this result is a solution to the L_p Minkowski problem, see [56, 126].

8.0.6. Orlicz Minkowski Problems. Extending the power function in the L_p Brunn-Minkowski theory to a more general function is the natural next step to further develop the theory. This has led to the so-called Orlicz Brunn-Minkowski theory, see the papers [87, 108, 158, 174, 175, 218, 241]. The Orlicz Minkowski problem for surface area measure can be stated as follows [56, 108]:

If $\varphi : (0, \infty) \to (0, \infty)$ is a fixed continuous function and μ is a finite Borel measure on S^{n-1} which is not concentrated on a closed hemisphere of S^{n-1}, then does there exist a convex body K in $\mathbb{R}^n$ such that

$$\varphi(h_K)\, dS(K, \cdot) = d\mu?$$

This equation arises when the L_p norm in the optimization problems associated with L_p Minkowwski problems is replaced by the Orlicz norm

$$\|f\|_\phi = \inf\left\{t > 0 : \frac{1}{|\mu|}\int_{S^{n-1}} \phi\left(\frac{f}{t}\right) d\mu \le \phi(1)\right\}, \quad \text{where } \phi(t) = \int_0^t \frac{1}{\varphi(s)}\, ds.$$

Because of the nonhomogeneous nature of the equation, it is of interest to consider the equation

$$c\,\varphi(h_K)\, dS(K, \cdot) = d\mu,$$

for some positive constant c that may be dependent on K. A natural choice of the constant is a power of volume, $c = V(K)^\alpha$, for some $\alpha \in \mathbb{R}$. The Orlicz Minkowski problem has been studied in [17, 108, 118, 129, 141, 152, 208–210, 215, 216, 226–228].

It is also of interest to generalize the Orlicz Minkowski problem to geometric measures other than surface area measure. For example, the equation for the Orlicz Minkowski problem for dual curvature measure, called the Orlicz dual Minkowski problem, can be written,

$$c\,\varphi(h_K)\, d\widetilde{C}_q(K, \cdot) = d\mu,$$

with $c = V(K)^\alpha$, for some $\alpha \in \mathbb{R}$. See [73,88,89,240]. Gardner et.al. [88,89] gave a construction of general dual curvature measures and posed their related Orlicz Minkowski problems.

8.0.7. Anisotropic Gauss curvature flows. The solvability of L_p and Orlicz Minkowski problems, except the L_p chord Minkowski problem, described in this section amounts to solving the following Monge-Ampère type equation,

$$\varphi(h)G(\nabla h)\det(\nabla_{ij}h + h\delta_{ij}) = f. \tag{8.22}$$

This equation is associated with an anistropic Gauss curvature flow, see for example Liu-Lu [152],

$$\frac{\partial X(x,t)}{\partial t} = -\frac{(X \cdot \nu)\kappa}{\varphi(X \cdot \nu)G(X)}f(\nu)\nu + X, \tag{8.23}$$

where G is a positive continuous function in $\mathbb{R}^n$. For different cases of φ and G, these problems have been studied in [37, 56, 145, 146, 152].

9. GAUSS IMAGE PROBLEM AND SPHERICAL MASS TRANSPORT

Recall that the radial Gauss image $\boldsymbol{\alpha}_K$ of a convex body K in $\mathbb{R}^n$ defined by (3.5) maps a Borel set $\omega \subset S^{n-1}$ to a Lebesgue measurable set $\boldsymbol{\alpha}_K(\omega)$.

9.0.1. Gauss Image Measure. Suppose λ is a measure defined on Lebesgue measurable subsets of S^{n-1}, and $K \in \mathcal{K}_o^n$. The *Gauss image measure* $\lambda(K, \cdot)$ associated with λ and K is defined by

$$\lambda(K, \omega) = \lambda(\boldsymbol{\alpha}_K(\omega)), \quad \text{Borel } \omega \subset S^{n-1}.$$

In general, $\lambda(K, \cdot)$ is only an outer measure. If λ is absolutely continuous with respect to the Lebesgue measure on S^{n-1} and $K \in \mathcal{K}_o^n$, then $\lambda(K, \cdot)$ is a Borel measure on S^{n-1}.

The concept of the Gauss image measure underlies the definitions of many geometric measures presented here. For example, the integral curvature of K is the Gauss image measure of spherical Lebesgue measure via K, see (5.19). The surface area measure $S(K, \cdot)$ is the Gauss image measure of the $(n-1)$th curvature $C_{n-1}(K, \cdot)$ via K^*, see (5.20). Dual curvature measures are also Gauss image measures, see (6.4).

9.0.2. Gauss Image Problem. Motivated by this, the following problem was posed in [27]:

Suppose λ is a measure defined on the Lebesgue measurable subsets of S^{n-1} and μ is a Borel measure on S^{n-1}. What are the necessary and sufficient conditions, on λ and μ, so that there exists a convex body $K \in \mathcal{K}_o^n$ such that

$$\lambda(K, \cdot) = \mu \,? \tag{9.1}$$

When λ is spherical Lebesgue measure, the Gauss image problem is the Aleksandrov problem. The essence of the Gauss image problem is to understand how measures on S^{n-1} are transformed by the Gauss image map of a convex body. Thus, the Gauss image problem is essentially a problem of spherical mass transport.

When μ has a smooth density f and λ has a smooth density g, the geometric equation (9.1) can be written as a Monge-Ampère type equation,

$$g\left(\frac{\nabla h}{|\nabla h|}\right) \det\left(\nabla_{ij}h + h\delta_{ij}\right) = \frac{|\nabla h|^n}{h} f. \tag{9.2}$$

Note that this equation is a special case of the equation (8.18) when g is the nth power of the radial function of a star body and $p = q = 0$. Thus, the Gauss image problem becomes the Aleksandrov problem.

9.0.3. Aleksandrov Related Measures. There is an obvious necessary condition required for the Gauss image problem to have a solution for two given measures λ and μ. First, recall that if $\omega \subset S^{n-1}$ is contained in a closed hemisphere, then its *polar set* ω^* is given by

$$\omega^* = \{v \in S^{n-1} : u \cdot v \leq 0, \text{ for all } u \in \omega\} = \bigcap_{u \in \omega} \{v \in S^{n-1} : u \cdot v \leq 0\}.$$

A subset $\omega \subset S^{n-1}$ is called spherically convex if the cone $\{tu : t \geq 0, u \in \omega\}$ is a nonempty convex proper subset of $\mathbb{R}^n$. Given a convex body $K \subset \mathbb{R}^n$ and a spherically convex $\omega \subset S^{n-1}$, it is not hard to show that (see [27, Lemma 3.2])

$$\boldsymbol{\alpha}_K(\omega) \subset S^{n-1} \setminus \omega^*,$$

and the set $(S^{n-1} \setminus \omega^*) \setminus \boldsymbol{\alpha}_K(\omega)$ has interior points. It follows that given any spherically convex set $\omega \subset S^{n-1}$,

$$\lambda(K, \omega) = \lambda(\boldsymbol{\alpha}_K(\omega)) \leq \lambda(S^{n-1}) - \lambda(\omega^*).$$

Moreover, if λ is assumed to be positive for every nonempty open subset of S^{n-1}, then strict inequality must hold. This necessary condition motivates the following:

Two Borel measures λ and μ are said to be Aleksandrov related, if for any spherically convex compact set $\omega \subset S^{n-1}$,

$$\lambda(\omega^*) + \mu(\omega) < \lambda(S^{n-1}) = \mu(S^{n-1}).$$

Since $\omega^{**} = \omega$, this is a symmetric relationship. By the discussion above, if λ is positive on open subsets of S^{n-1}, then this is a necessary condition for μ to be the Gauss image measure of λ. Below we will see that it is also a sufficient condition.

9.0.4. Solutions to the Gauss Image Problem. In [27] the following theorem was proved:

Let μ be a finite Borel measure on S^{n-1} and λ be a measure on S^{n-1} that is absolutely continuous with respect to the spherical Lebesgue measure. If μ and λ are Aleksandrov related, then there exists a convex body K in $\mathcal{K}_o^n$ such that $\mu = \lambda(K, \cdot)$.

When both μ and λ have positive densities, the result was proved also by Li-Wang [148] by the approach of spherical mass transport. When μ is discrete and λ is absolutely continuous, the result was strengthened by Semenov [203] who introduced the so-called *weakly Aleksandrov related* condition.

It is important to study the Gauss image problem when λ is a discrete measure, due to connections with discrete mass transport and combinatorics. Significant progress to this end was made by Semenov [202].

10. CAPACITARY MINKOWSKI PROBLEMS FROM HARMONIC ANALYSIS

In his remarkable work [127], Jerison introduced the electrostatic capacitary measure of a convex body as the differential of electrostatic (Newtonian, or L_2) capacity and solved its Minkowski problem completely; see also [128] for a direct proof. This establishes amazing connections between convex geometry and harmonic analysis related to partial differential equations. Important connections of convex geometry to harmonic analysis, functional analysis and probability theory related to geometric and analytic inequalities were well described by Ball [14]. In the paper [64], Colesanti-Nyström-Salani-Xiao-Yang-Zhang extended the concept of electrostatic capacitary measure to L_p capacitary measure which is the differential of the L_p capacity. They solved the L_p capacitary Minkowski problem in $\mathbb{R}^n$ when $1 < p < 2$. Then, Akman-Gong-Hineman-Lewis-Vogel [1] solved the L_p capacitary Minkowski problem when $2 < p < n$.

10.0.1. L_p Capacitary Measure. Let K be a convex body in $\mathbb{R}^n$ and $1 < p < n$. The L_p capacity $\mathrm{Cap}_p(K)$ of K is defined by

$$\mathrm{Cap}_p(K) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^p dx : \ u \in C_c^\infty(\mathbb{R}^n) \text{ and } u \geq 1 \text{ on } K \right\}.$$

The L_2 case $\mathrm{Cap}_2(K)$ is the electrostatic capacity.

The p-equilibrium potential U_K of K is the unique solution to the boundary value problem

$$\begin{cases} \Delta_p U = 0 & \text{in } \mathbb{R}^n \setminus K, \\ \\ U = 1 & \text{on } \partial K \quad \text{and} \quad \lim_{|x| \to \infty} U(x) = 0, \end{cases}$$

where Δ_p is the p-Laplace operator,

$$\Delta_p U = \nabla \cdot (|\nabla U|^{p-2} \nabla U).$$

A proof of the existence and uniqueness of U_K can be found in [138]. When approaching the boundary of $\mathbb{R}^n \setminus K$ non-tangentially, the limit of ∇U_K exists $\mathcal{H}^{n-1}$-almost everywhere on ∂K and its norm $|\nabla U_K|$ is L_p integrable over ∂K, i.e.,

$$\int_{\partial K} |\nabla U_K|^p \, dx < \infty.$$

This is a well-known result of Dahlberg [67] when $p = 2$, and the general case was shown by Lewis and Nyström [139]. Therefore, a measure $\mu_p(K, \cdot)$ of K, called the p-*capacitary measure of K*, can be defined by

$$\mu_p(K, \eta) = \int_{\nu_K^{-1}(\eta)} |\nabla U_K|^p \, dx, \quad \text{for any Borel set } \eta \subset S^{n-1}.$$

An important fact is that the p-capacitary measure is the differential of the p-capacity, that is,

$$\frac{d}{dt} \mathrm{Cap}_p(K + tL) \Big|_{t=0} = (p-1) \int_{S^{n-1}} h_L \, d\mu_p(K, \cdot). \tag{10.1}$$

The proof of this differential formula is not easy. The case of $p = 2$ was proved by Jerison [127] and the general case $1 < p < n$ was proved in [64]. It is crucial for solving the following capacitary Minkowski problem.

10.0.2. The p-Capacitary Minkowski Problem. *Let μ be a finite Borel measure on S^{n-1} and $1 < p < n$. Under what necessary and sufficient conditions does there exist a convex body K in $\mathbb{R}^n$ such that $\mu_p(K, \cdot) = \mu$?*

If the measure μ has a density f, the equation associated with this Minkowski problem is a Monge-Ampère type equation on S^{n-1}

$$\det \left(\nabla_{ij} h + h \delta_{ij} \right) = \frac{f}{|\nabla U_{[h]}|^p},$$

where h is the unknown positive function on S^{n-1} to be found, $U_{[h]}$ is the p-equilibrium potential of the Wulff shape $[h]$, and $|\nabla U_{[h]}|$ is viewed as a function on S^{n-1} under the inverse Gauss map $\nu_{[h]}^{-1}$.

10.0.3. Solution to the p-Capacitary Minkowski Problem. *Let μ be a finite Borel measure on S^{n-1} and $1 < p < n$. Then there exists a convex body K in $\mathbb{R}^n$ such that $\mu_p(K, \cdot) = \mu$ if and only if μ is not concentrated on a closed hemisphere and*

$$\int_{S^{n-1}} v \, d\mu(v) = 0.$$

This was first proved by Jerison [127] when $p = 2$, then Colesanti, et al. [64] proved the case of $1 < p < 2$ under an extra technical condition, and finally Akman, et al. [1] proved the general case.

10.0.4. Uniqueness Problem of the p-Capacitary Measure. *Let K_0, K_1 be convex bodies in $\mathbb{R}^n$ and $1 < p < n$. If*

$$\mu_p(K_0, \cdot) = \mu_p(K_1, \cdot),$$

then K_0 is a translate of K_1 when $p \neq n - 1$, and K_0, K_1 are homothetic when $p = n - 1$.

This was first proved by Caffarelli-Jerison-Lieb in [34] when $p = 2$, and the general case was proved by Colesanti et al. [64] by using the differential formula (10.1) and the capacitary Brunn-Minkowski inequality of Colesanti-Salani [65].

10.0.5. (p,q)-Capacitary Minkowski Problem. In [245], Zou and Xiong defined the (p,q)-*capacitary measure* of a convex body K,

$$d\mu_{p,q}(K,\cdot) = h_K^{1-q} d\mu_p(K,\cdot), \quad 1 < p < n, \ q \in \mathbb{R},$$

which is the differential of the p-capacity with respect to L_q Minkowski addition,

$$\frac{d\mathrm{Cap}_p(K +_q t \cdot L)}{dt}\Big|_{t=0} = \frac{p-1}{q} \int_{S^{n-1}} h_L^q(v)\, d\mu_{p,q}(K,v), \quad 1 < p < n, \ q \geq 1.$$

They solved the associated (p,q)-Capacitary Minkowski problem when $1 < p < n$ and $q > 1$. In particular, for the discrete case, the following result was proved:

If μ is a discrete measure that is not concentrated on any closed hemisphere, then for $1 < p < n$ and $q > 1$ with $p + q \neq n$, there exists a unique convex polytope K containing the origin in its interior such that

$$\mu_{p,q}(K,\cdot) = \mu.$$

Existence proofs for capacitary Minkowski problems take a variational approach, relying on (10.1) and its analogues. It is of interest to use continuity methods or geometric flows to find smooth solutions to the capacitary Minkowski problems.

Generalizations of the capacitary Minkowski problems and other Minkowski-type problems for various capacities have been studied in [2, 3, 49, 50, 61, 115, 177, 221–225, 230, 245].

11. Gaussian Minkowski Problems from Probability Theory

Geometric measures of convex bodies that are related to probability measures are also of great interest. Huang-Xi-Zhao [124] constructed the *Gaussian surface measure* of convex bodies from the Gaussian probability measure and studied its *Gaussian Minkowski problem*. A general construction of a geometric measure of a convex body from a probability measure was given by Livshyts [154]. This provides new connections among convex geometry, probability theory, and partial differential equations.

11.0.1. Gaussian Surface Area Measure. Let $K \in \mathcal{K}_o^n$. The Gaussian measure $\gamma_n(K)$ of K is defined by

$$\gamma_n(K) = \frac{1}{(2\pi)^{n/2}} \int_K e^{-\frac{|x|^2}{2}}\, dx.$$

The *Gaussian surface area measure* $S_{\gamma_n}(K,\cdot)$ *of* K is a Borel measure on S^{n-1} defined by

$$S_{\gamma_n}(K,\eta) = \frac{1}{(2\pi)^{n/2}} \int_{\nu_K^{-1}(\eta)} e^{-\frac{|x|^2}{2}}\, dx, \quad \text{for any Borel set } \eta \subset S^{n-1}. \tag{11.1}$$

The measure $S_{\gamma_n}(K,\cdot)$ is the differential of the Gaussian measure of K:

$$\frac{d}{dt}\gamma_n(K + tL)\Big|_{t=0} = \int_{S^{n-1}} h_L\, dS_{\gamma_n}(K,\cdot). \tag{11.2}$$

The total measure $S_{\gamma_n}(K, S^{n-1})$ is a known invariant, called the *Gaussian surface area (or Gaussian perimeter)* of K. It was first studied by Borell [19] for proving his Gaussian isoperimetric inequality, see also Latała [137]. Then as a different invariant, it was studied by Ball [13] for a reverse Gaussian isoperimetric inequality, see also Nazarov [184]. The formula (11.2) proved by Huang-Xi-Zhao [124] clarifies the connections among these studies

and shows a striking similarity between the Brunn-Minkowski theory in the Euclidean space and its analog in the Gaussian space.

11.0.2. Uniqueness Problem of Gaussian Surface Area Measure. Let K, L be convex bodies in $\mathcal{K}_o^n$ with $\gamma_n(K), \gamma_n(L) \geq \frac{1}{2}$. It is shown in [124] that

$$S_{\gamma_n}(K, \cdot) = S_{\gamma_n}(L, \cdot) \implies K = L.$$

Huang-Xi-Zhao [124] showed this uniqueness result by using the Ehrhard inequality, which is a Brunn-Minkowski type inequality for the Gauss measure, see [69, 70].

11.0.3. Gaussian Minkowski problem. *Given a finite Borel measure μ on S^{n-1}, what are the necessary and sufficient conditions on μ so that there exists a convex body $K \in \mathcal{K}_o^n$ satisfying*

$$S_{\gamma_n}(K, \cdot) = \mu\,?$$

When the given measure has a density, the partial differential equation associated with the Gaussian Minkowski problem can be written as the Monge-Ampère type equation on S^{n-1},

$$\det\left(\nabla_{ij} h + h \delta_{ij}\right) = e^{\frac{|\nabla h|^2}{2}} f. \tag{11.3}$$

11.0.4. Existence of Solutions to the Symmetric Gaussian Minkowski Problem. *Let μ be a finite even Borel measure on S^{n-1} that is not concentrated on any great subsphere. Then for any $0 < \alpha < \frac{1}{n}$, there exists an origin-symmetric convex body K such that*

$$\mu = \frac{1}{\gamma_n(K)^{1-\alpha}} S_{\gamma_n}(K, \cdot).$$

A variational proof of this result is given in [124], which is similar to the proof given in [108]. By using the degree theory for second-order nonlinear elliptic operators developed in Li [150] for the smooth case, followed by an approximation argument, Huang-Xi-Zhao [124] also proved the following non-normalized solution:

Let μ be an even measure on S^{n-1} that is not concentrated on any great subsphere and $|\mu| < \frac{1}{\sqrt{2\pi}}$. Then there exists a unique origin-symmetric K with $\gamma_n(K) > \frac{1}{2}$ such that $S_{\gamma_n}(K, \cdot) = \mu$.

A similar solution to the non-symmetric case was obtained by Feng-Liu-Xu [75].

Related studies on the Gaussian Minkowski problem and its L_p variant can be found in the papers [40, 74, 151, 205].

A more general construction is to replace the Gaussian density by any density function g of a Borel measure in $\mathbb{R}^n$,

$$S_g(K, \eta) = \int_{\nu_K^{-1}(\eta)} g(x)\, dx, \quad \text{for any Borel set } \eta \subset S^{n-1}.$$

This was defined by Livshyts [154]. When g is even, homogeneous, and of certain concavity, the corresponding Minkowski problem was solved in [154], extending the solution of the classical Minkowski problem for symmetric convex bodies. The results in [154] were generalized by Kryvonos-Langharst [136].

While convex geometry primarily focuses on convex bodies in $\mathbb{R}^n$, one can often replace convex bodies by functions in $\mathbb{R}^n$, usually with certain convexity properties. It is interesting to construct geometric measures from functions and solve the corresponding Minkowski problems. In [173], the L_p surface area measure $S_p(f, \cdot)$, $p \geq 1$, a Borel measure on S^{n-1}, was defined

for $f : \mathbb{R}^n \to \mathbb{R}$ with L_p weak derivative,

$$\int_{S^{n-1}} \varphi(u)^p dS_p(f, u) = \int_{\mathbb{R}^n} \varphi(-\nabla f(x))^p \, dx,$$

where φ is a nonnegative continuous homogeneous function of degree 1 in $\mathbb{R}^n$. A similar concept in $\mathbb{R}^n$, called the moment measure $\mu(f, \cdot)$, was defined in [66] for a nonnegative integrable log-concave function f in $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \varphi(x) \, d\mu(f, x) = \int_{\mathbb{R}^n} \varphi(-\nabla \log f(x)) f(x) \, dx$$

where φ is a nonnegative Borel function in $\mathbb{R}^n$.

Another direction of study is to replace convex bodies by pseudo-cones. They are unbounded closed convex sets not containing the origin and viewed as the counterpart to convex bodies containing the origin. Schneider [199] defined the surface area measure for pseudo-cones and studied its Minkowski problem. Further studies were carried out in [4, 142, 200, 231, 232].

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