Convex Optimization
Homework 2

Winter 1399
Due date: 29th of Esfand

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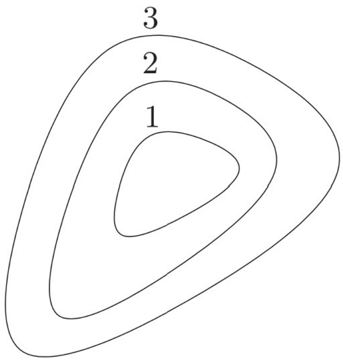
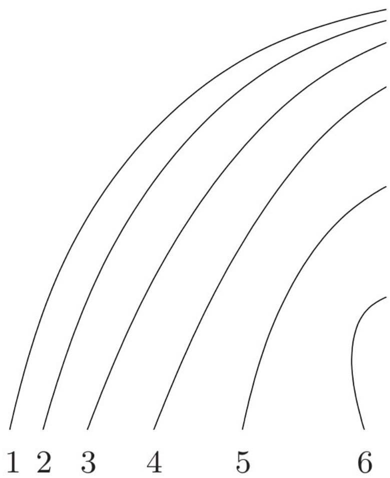
1. Some level sets of a function f are shown below. The curve labeled 1 shows $\{x|\ f(x) = 1\}$, etc.



Could f be convex (concave, quasiconvex, quasiconcave)? Explain your answer. Repeat for the level curves shown below.



2. For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

   (a) $f(x) = e^x - 1$ *on* $\mathbb{R}$.
   (b) $f(x_1, x_2) = x_1 x_2$ *on* $\mathbb{R}^2_{++}$.
   (c) $f(x_1, x_2) = 1/(x_1 x_2)$ *on* $\mathbb{R}^2_{++}$.
   (d) $f(x_1, x_2) = x_1/x_2$ *on* $\mathbb{R}^2_{++}$.
   (e) $f(x_1, x_2) = x_1^2/x_2$ *on* $\mathbb{R} \times \mathbb{R}_{++}$.
   (f) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, *where* $0 \leq \alpha \leq 1$, *on* $\mathbb{R}^2_{++}$.

3. Let $\lambda_1(X) \geq \lambda_2(X) \geq \ldots \geq \lambda_n(X)$ denote the eigenvalues of a matrix $X \in S_n$. We have already seen several functions of the eigenvalues that are convex or concave functions of X.

- The maximum eigenvalue $\lambda_1(X)$ is convex. The minimum eigenvalue $\lambda_n(X)$ is concave.
- The sum of the eigenvalues (or trace), $\mathrm{tr}\, X = \lambda_1(X) + ... + \lambda_n(X)$, is linear.
- The sum of the inverses of the eigenvalues (or trace of the inverse), $tr(X^{-1}) = \sum_{i=1}^n 1/\lambda_i(X)$, is convex on S_n^{++}.
- The geometric mean of the eigenvalues, $(\det X)^{1/n} = \left(\prod_{i=1}^n \lambda_i(X)\right)^{1/n}$, and the logarithm of the product of the eigenvalues, $X = \sum_{i=1}^n \log \lambda_i(X)$ are concave on $X \in S_n^{++}$

In this problem we explore some more functions of eigenvalues, by exploiting variational characterizations.

(a) Show that $\sum_{i=1}^k \lambda_i(X)$ is convex on S_n . *Hint.* Use the variational characterization

$$\sum_{i=1}^k \lambda_i(X) = \sup \left\{ \mathrm{tr} \left(V^T X V \right) \mid V \in \mathbf{R}^{n \times k}, V^T V = I \right\}$$

(b) show that $\left(\prod_{i=n-k+1}^n \lambda_i(X)\right)^{1/k}$ is concave on S_n^{++}. *Hint.* For $X \succ 0$ we have

$$\left(\prod_{i=n-k+1}^n \lambda_i(X)\right)^{1/k} = \tfrac{1}{k} \inf \left\{ \mathrm{tr} \left(V^T X V \right) \mid V \in \mathbf{R}^{n \times k}, \det V^T V = 1 \right\}$$

(c) show that $\sum_{i=n-k+1}^n \log \lambda_i(X)$ is concave on S_n^{++}.

$$\prod_{i=n-k+1}^n \lambda_i(X) = \inf \left\{ \prod_{i=1}^k \left(V^T X V \right)_{ii} \mid V \in \mathbf{R}^{n \times k}, V^T V = I \right\}$$

4. *Properties of conjugate functions.*

(a) *Conjugate of convex plus affine function.* Define $g(x) = f(x) + c^T x + d$, where f is convex. Express g^* in terms of f^* (and c, d).

(b) *Conjugate of perspective.* Express the conjugate of the perspective of a convex function f in terms of f^*.

(c) *Conjugate and minimization.* Let $f(x, z)$ be convex in (x, z) and define $g(x) = inf_z f(x, z)$. Express the conjugate g^* in terms of f^*.
As an application, express the conjugate of $g(x) = inf_z \{h(z) | Az + b = x\}$, where h is convex, in terms of h^*, A, and b.

(d) *Conjugate of conjugate.* Show that the conjugate of the conjugate of a closed convex function is itself: $f = f^{**}$ if f is closed and convex. (A function is closed if its epigraph is closed) *Hint.* Show that f^{**} is the pointwise supremum of all affine global underestimators of f. Then apply the result of exercise 3.28.

5. show that the function

$$f(x) = \frac{\|Ax - b\|_2^2}{1 - x^T x}$$

is convex on $\{x \mid \|x\|_2 < 1\}$

6. For $x \in \mathbf{R}^n$, with $n > 1, x_{[k]}$ denotes the kth largest entry of x, for $k = 1, \ldots, n$, so, for example, $x_{[1]} = \max_{i=1,\ldots,n} x_i$ and $x_{[n]} = \min_{i=1,\ldots,n} x_i$. Functions that depend on these sorted values are called order statistics or order functions. Determine the curvature of the order statistics below, from the choices convex, concave, or neither. For each function, explain why the function has the curvature you claim. If you say it is neither convex nor concave, give a counterexample showing it is not convex, and a counterexample showing it is not concave. All functions below have domain $\mathbf{R}^n$

(a) $\mathrm{median}(x) = x_{[(n+1)/2]}$. (You can assume that n is odd.)

(b) The range of values, $x_{[1]} - x_{[n]}$

(c) The midpoint of the range, $\left(x_{[1]} + x_{[n]}\right)/2$.

(d) Interquartile range, defined as $x_{[n/4]} - x_{[3n/4]}$. (You can assume that $n/4$ is an integer.)

(e) Symmetric trimmed mean, defined as

$$\frac{x_{[n/10]} + x_{[n/10+1]} + \cdots + x_{[9n/10]}}{0.8n + 1},$$

the mean of the values between the 10th and 90th percentiles. (You can assume that $n/10$ is an integer.)

(f) Lower trimmed mean, defined as

$$\frac{x_{[1]} + x_{[2]} + \cdots + x_{[9n/10]}}{0.9n + 1},$$

Remark. For the functions defined in (d)–(f), you might find slightly different definitions in the literature. Please use the formulas above to answer each question.