Convex Optimization

Homework 1

Winter 1399
Due date: 19th of Esfand

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1. For each of the following statements, either show that it is true, or give a (specific) counterexample.

   - If AB is full rank then A and B are full rank.
   - If A and B are full rank then AB is full rank.
   - If A and B have zero nullspace, then so does AB.
   - If A and B are onto, then so is AB.

   You can assume that $A \in R^{m\times n}$ and $B \in R^{n\times p}$. Some of the false statements above become true under certain assumptions on the dimensions of A and B. As a trivial example, all of the statements above are true when A and B are scalars, i.e., $n = m = p = 1$. For each of the statements above, find conditions on n, m, and p that make them true. Try to find the most general conditions you can. You can give your conditions as inequalities involving n, m, and p, or you can use more informal language such as "A and B are both skinny."

2. Determine if the following statements are true or false. No justification or discussion is needed for your answers. What we mean by "true" is that the statement is true for all values of the matrices and vectors given. You can't assume anything about the dimensions of the matrices (unless it's explicitly stated), but you can assume that the dimensions are such that all expressions make sense. For example, the statement "$A + B = B + A$" is true, because no matter what the dimensions of A and B (which must, however, be the same), and no matter what values A and B have, the statement holds. As another example, the statement $A^2 = A$ is false, because there are (square) matrices for which this doesn't hold. (There are also matrices for which it does hold, e.g., an identity matrix. But that doesn't make the statement true.)

   (a) If all coefficients (i.e., entries) of the matrix A are positive, then A is full rank.

   (b) If A and B are onto, then $A + B$ must be onto.

   (c) If A and B are onto, then so is the matrix $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$.

   (d) If A and B are onto, then so is the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$.

   (e) If the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is onto, then so are the matrices A and B.

   (f) If A is full rank and skinny, then so is the mtarix $\begin{bmatrix} A \\ B \end{bmatrix}$.

3. Is the set $\{a \in \mathbb{R}^k \mid p(0) = 1,\ |p(t)| \le 1 \text{ for } \alpha \le t \le \beta\}$, where

   $$p(t) = a_1 + a_2 t + \cdots + a_k t^{k-1},$$

   convex?

4. Describe the dual cone for each of the following cones.

   (a) $K = \{0\}$.

   (b) $K = \mathbb{R}^2$.

(c) $K = \{(x_1, x_2) \mid |x_1| \le x_2\}$.

(d) $K = \{(x_1, x_2) \mid x_1 + x_2 = 0\}$.

5. Determine if each set below is convex.

   (a) $\{(x, y) \in R^2_{++} \mid x/y \le 1\}$.

   (b) $\{(x, y) \in R^2_{++} \mid x/y \ge 1\}$.

   (c) $\{(x, y) \in R^2_{++} \mid xy \le 1\}$.

   (d) $\{(x, y) \in R^2_{++} \mid xy \ge 1\}$.

6. Let a and b be distinct points in $\mathbb{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b, i.e., $\{x \mid \|x - a\|_2 \le \|x - b\|_2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \le d$. Draw a picture.

7. Which of the following sets are convex?

   (a) A *slab, i.e.,* a set of the form $\{x \in \mathbb{R}^n \mid \alpha \le a^T x \le \beta\}$.

   (b) A *rectangle, i.e.,* a set of the form $\{x \in \mathbb{R}^n \mid \alpha_i \le x_i \le \beta_i, i = 1, 2, ..., n\}$. A rectangular is sometimes called a *hyperrectangle* when $n > 2$.

   (c) A *wedge, i.e.,* $\{x \in \mathbb{R}^n \mid a_1^T x \le b_1,\ a_2^T x \le b_2\}$.

   (d) The set of points closer to a given point than a given set, *i.e.,*

   $$\{x \mid \|x - x_0\|_2 \le \|x - y\|_2 \text{ for all } y \in S\}$$

   where $S \subseteq \mathbb{R}^n$.

   (e) The set of points closer to one set than another, *i.e.,*

   $$\{x \mid \mathbf{dist}(x, S) \le \mathbf{dist}(x, T)\},$$

   where $S, T \subseteq \mathbb{R}^n$, and

   $$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$

   (f) The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

   (g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b *i.e.,* the set $\{x \mid \|x - a\|_2 \le \theta\|x - b\|_2\}$. You can assume $a \ne b$ and $0 \le \theta \le 1$.

Good Luck!