**1.** *Robust powerline provisioning.* We have $m$ power plants, $i = 1, \ldots, m$, generate power to be delivered to $n$ destinations $j = 1, \ldots, n$. We wish to attach powerlines to minimize the cost of provisioning the lines while protecting against plant failures. We represent these lines as a bipartite graph between the $m$ plants and $n$ destination nodes. To connect a line between plant $i$ and destination $j$ requires we lay line of length $L_{ij}$, and given a radius $R_{ij}$ of the line, we assume the cost is proportional is the volume of metal required, that is, $L_{ij}R_{ij}^2$. We let $A_{ij} = R_{ij}^2$, which is proportional to the area of the line, and is also the cost per unit length of line with the given radius $R_{ij}$. The power available to node $j$ must be at least a usage level $u_j \in \mathbf{R}_+$, which is given. Plant $i$ has a total capacity $c_i$ on its power generation (also specified). If a there is a line between plant $i$ and destination $j$ and plant $i$ transmits power $P_{ij}^{\rm tr}$ on the line, the arrival power $P_{ij}^{\rm ar}$ satisfies

$$P_{ij}^{\rm tr} = P_{ij}^{\rm ar} + \alpha \frac{L_{ij}(P_{ij}^{\rm ar})^2}{R_{ij}^2} = P_{ij}^{\rm ar} + \alpha \frac{L_{ij}(P_{ij}^{\rm ar})^2}{A_{ij}},$$

where $\alpha > 0$ is a known constant related to the transmission losses. We require that the total arriving power to node $j$, $\sum_{i=1}^m P_{ij}^{\rm ar}$, is at least the usage level $u_j$ at $j$. Then a convex formulation of the minimum cost line allocation problem is to solve

$$\begin{array}{ll} \mbox{minimize} & \mathbf{tr}\, L^T A \\ \mbox{subject to} & P_{ij}^{\rm tr} \ge P_{ij}^{\rm ar} + \alpha \frac{L_{ij}(P_{ij}^{\rm ar})^2}{A_{ij}}, \quad i = 1, \ldots, m,\ j = 1, \ldots, n \\ & \sum_{i=1}^m P_{ij}^{\rm ar} \ge u_j, \quad j = 1, \ldots, n \\ & \sum_{j=1}^n P_{ij}^{\rm tr} \le c_i, \quad i = 1, \ldots, m, \end{array}$$

where the variables are the areas $A \in \mathbf{R}^{m \times n}$ and transmitted and arriving powers $P^{\rm tr}, P^{\rm ar} \in \mathbf{R}_+^{m \times n}$. (Note that $A$ is implicitly constrained to be positive in the above formulation; you should feel free to add explicit constraints on $P^{\rm tr}$ and $P^{\rm ar}$.)

The rest of the problem uses the data in `robust_power_data.*`.

(a) Using the given data, implement this problem in `CVX*`. For the resulting solution $A^\star$, give the number of entries $(i, j)$ with $|A_{ij}^\star| > 10^{-3}$ and the total cost $\mathbf{tr}\, L^T A^\star$ of your solution. (Here, we threshold at $10^{-3}$ as a proxy for zero entries in $A^\star$.)

It is unrealistic to assume that all powerstations and powerlines will remain up, so you would like to build robustness into your system. We would like to guarantee $m - k$ reliability: the system will provide the desired level of power $u_j$ at each destination node even if $k$ of the power plants fail.

(b) Let $O_i \in \{0, 1\}$ represent the event that plant $i$ is online, so $O_i = 1$ if plant $i$ is online and $O_i = 0$ if it is offline. Let $\mathcal{O}_k$ denote the set of such $O$ with at most $k$ offline plants. Reformulate the line allocation problem to include the robust power arrival constraint that

$$\sum_{i=1}^m O_i P_{ij}^{\rm ar} \ge u_j \quad \mbox{for all } O \in \mathcal{O}_k.$$

Give as explicit a form as you can, and solve the resulting robust optimization problem for values $k = 1, 2, 3, 4$ with the given data. For each $k$, give give the number of entries in your solution $A^\star$ with $|A_{ij}^\star| > 10^{-3}$ and the total cost $\mathbf{tr}\, L^T A^\star$. Give a one sentence explanation of your results. *Hint.* The `sumsmallest` or `sum_smallest` operation may be helpful.