

# Statistical Models and Inference - Part I

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AA 2022/2023 - Stat Lect. 5



## Data Modeling

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- we perform **experiments** and make **observations** to **learn about a phenomenon**
- to interpret data, we have to model them

### Inference

- make general statements about a phenomenon **through a model**, using **noisy and incomplete data**
  - must **describe** both the **Phenomenon** (i.e. Model) and the **Measurement Process**
- ▷ Key to Data Modeling: use data together with generative model (theory) and measurement model (experimental practice) to derive consistent probabilistic inferences

- given some data,  $D$ , we usually want to perform three actions:
- ▷ **parameter estimation**:  
for a specific Model  $H$ , with parameters  $\theta$ , infer the values of model parameters, i.e.  $P(\theta | D, H)$ , the **parameter posterior pdf**
- ▷ **model comparison**:  
given a set of models  $\{H_j\}$ , find out which one is best supported by data. This means finding  $P(H_j | D)$ , the **model posterior probability**
- ▷ **prediction**:  
given a model  $H$ , inferred from the data, **predict new data at some new location** (in the parameter space or time)

## Bayesian Model Comparison

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- we start by looking at **model comparison** for the simple case of models with no parameters
- ▷ using our data  $D$ , we look for  $P(H | D)$
- since  $H \cdot \bar{H} = 0$  and  $H + \bar{H} = \Omega$ , we can write

$$\begin{aligned} P(D) &= P(DH) + P(D\bar{H}) \\ &= P(D | H) P(H) + P(D | \bar{H}) P(\bar{H}) \end{aligned}$$

- our quantity of interest,  $P(H | D)$ , is related to Bayes' theorem by

$$\begin{aligned} P(H | D) &= \frac{P(D | H) P(H)}{P(D)} = \frac{P(D | H) P(H)}{P(D | H) P(H) + P(D | \bar{H}) P(\bar{H})} \\ &= \frac{1}{1 + \frac{P(D | \bar{H}) P(\bar{H})}{P(D | H) P(H)}} = \frac{1}{1 + \frac{1}{R}} \end{aligned}$$

- with  $R = \frac{P(D | H) P(H)}{P(D | \bar{H}) P(\bar{H})}$  the **posterior odd ratio** of the models

# Bayesian Model Comparison

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- it is easy to demonstrate that

$$\frac{P(H | D)}{P(\bar{H} | D)} = R = \frac{P(D | H) P(H)}{P(D | \bar{H}) P(\bar{H})}$$

- in order to determine  $P(H | D)$ , we need three quantities:

▷  $P(D | H)$  : the probability of measuring  $D$  when  $H$  is true

▷  $P(D | \bar{H})$  : the probability of measuring  $D$  when  $H$  is not true (i.e. false)

▷  $P(H)$  : the probability that  $H$  is true, independently of the data (and, of course,  $P(\bar{H}) = 1 - P(H) \Rightarrow P(H)$  tells us how probable the model is

- but, shouldn't we have information to tell us that  $H$  is more likely than  $\bar{H}$ , we could set

$$P(H) = P(\bar{H})$$

- and  $R$  becomes the Bayes factor

$$BF = \frac{P(D | H)}{P(D | \bar{H})}$$

- i.e. the ratio of the probability of the data under each model

# Bayesian Model Comparison

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- should we have more models,  $\{H_j\}$ , with  $\sum P(H_j) = 1$ , the probability of data becomes

$$P(D) = \sum_j P(D | H_j) P(H_j)$$

- and the posterior probability of model # 1,  $H_1$ , becomes

$$P(H_1 | D) = \frac{P(D | H_1) P(H_1)}{P(D)}$$

- if we do not have a complete set of models, we cannot compute the posterior probabilities, but we can still compute the odds ratio or Bayes factor between any two models

$$BF = \frac{P(D | H_1)}{P(D | H_2)} \quad \text{and} \quad R = \frac{P(D | H_1) P(H_1)}{P(D | H_2) P(H_2)}$$

# Example

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## Problem

- a test for a disease is 90% reliable
- the probability of testing positive, in absence of the disease, is 0.07
- we know that among people aged 40 to 50 with no symptoms 8 in 1000 have the disease

Q: if a person in his/her 40 tests positive, what is the probability that he/she has the disease ?

## Background information

- we build the following propositions:
  - $D$ : a person is tested positive
  - $H$ : a person has the disease
- and probabilities
  - $P(D | H) = 0.9$
  - $P(D | \bar{H}) = 0.07$
  - $P(H) = 0.008$

## Example - analytical solution

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- we build

$$R = \frac{P(D | H) P(H)}{P(D | \bar{H}) P(\bar{H})} = \frac{9 \cdot 10^{-1} \times 8 \cdot 10^{-3}}{7 \cdot 10^{-2} \times (1 - 8 \cdot 10^{-3})} = 0.1035$$

- therefore

$$P(H | D) = \frac{1}{1 + 1/R} = 0.094$$

- even though a positive test result is quite probable (assuming the person has the disease), it is very unlikely that he/she has the disease
- what is decisive in the computation of  $P(H | D)$  is the ratio between

$$P(D | H) = P(D | H) P(H) = 7.2 \cdot 10^{-3}$$

(positive result, assuming the disease is present)

- and

$$P(D | \bar{H}) = P(D | \bar{H}) P(\bar{H}) = 7 \cdot 10^{-2}$$

(positive result, assuming the disease is absent)

# Example - R solution

```
post <- function(p.d.m, p.d.notm, p.m) {
  p.notm <- 1 - p.m
  odds.ratio <- (p.d.m * p.m) /
    (p.d.notm * p.notm)
  p.m.d <- 1/(1 + 1/odds.ratio)
}

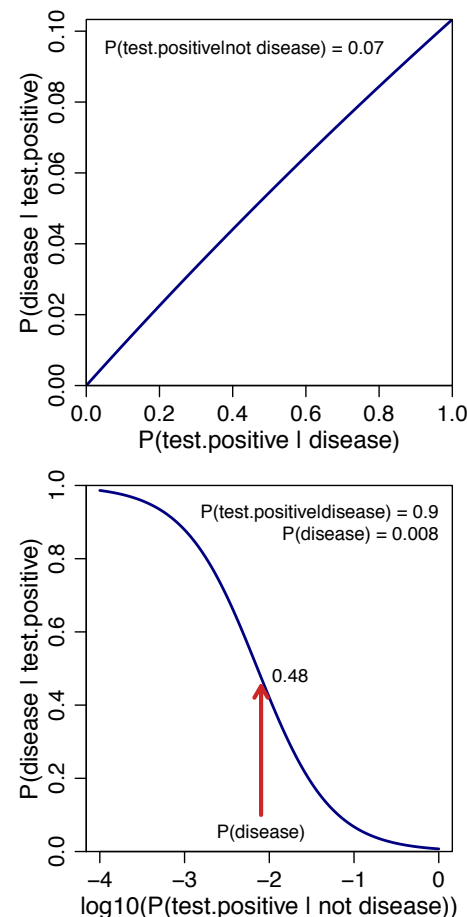
p.d.m <- seq(0, 1, 0.01) # True positive
p.d.notm <- 0.07         # False positive
p.m <- 0.008             # Disease Prior

p.m.d <- post(p.d.m, p.d.notm, p.m)
plot(p.d.m, p.m.d, type='l', lwd=2, col='navy')

p.d.m <- 0.9             # True positive
p.d.notm <- 10^seq(-4,0, 0.02) # False positive
p.m <- 0.008             # Disease Prior

p.m.d <- post(p.d.m, p.d.notm, p.m)
plot(log10(p.d.notm), p.m.d, type='l', col='navy')
```

- only once the false positive rate drops below the base rate ( $P(H)$ ) does the test starts to be useful



## Occam's Razor

### Look at the picture

- $Q_1$  How many boxes are in the picture ?
- $Q_2$  How many boxes are in the vicinity of the tree ?

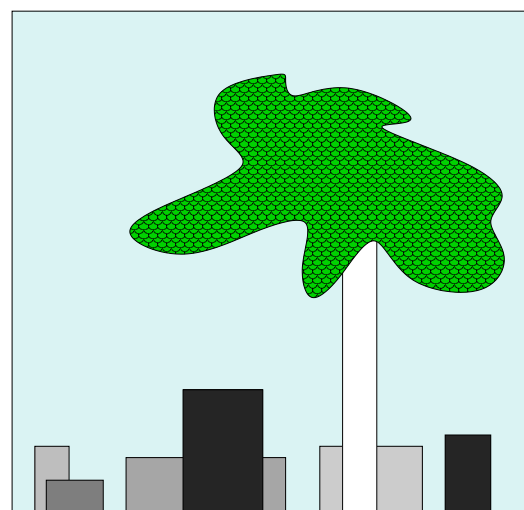
### What is Occam's Razor

- If several explanations are compatible with a set of observation, Occam's razor advises us to buy the simplest.

→ Accept the simplest explanation that fits the data

But why ?

- a theory with mathematical beauty is more likely to be correct than an ugly one that fit some experimental data (P.A.M. Dirac)
- choerent inference, embodied by bayesian probability, automatically embodies Occam's razor, quantitatively



# Model comparison and Occam's Razor

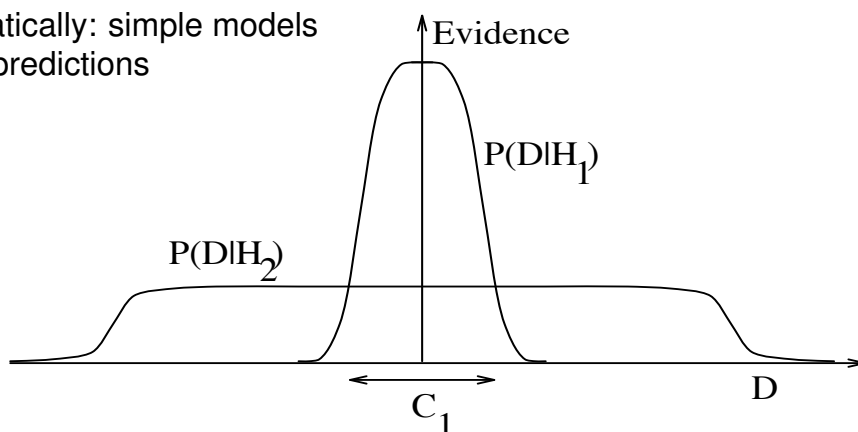
- we evaluate the plausibility of two alternative theories:  $H_1$  and  $H_2$
- we derived the probability ratio between the two theories as

$$\frac{P(H_1 | D)}{P(H_2 | D)} = \frac{P(D | H_1) P(H_1)}{P(D | H_2) P(H_2)}$$

- how is this related to Occam's razor when  $H_1$  is a simpler model than  $H_2$  ?  
the ratio

$$\frac{P(D | H_1)}{P(D | H_2)}$$

which depends on the data embodies  
Occam's razor automatically: simple models  
tend to make precise predictions



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## Example

(1)

- we have a sequence of numbers:  
-1, 3, 7, 11
- we want to predict the next two numbers in the sequence

Answer<sub>1</sub>: 15 and 19

i.e. add 4 to the previous number

→ the sequence is an arithmetic progression

Answer<sub>2</sub>: -19.9 and 1043.8

i.e. we apply  $-x^3/11 + 9x^2/11 + 23/11$  to the previous number

→ the sequence is generated by a cubic function

- let's compare the two models:  $H_g$  (geometric) versus  $H_c$  (cubic)
- in general, an arithmetic progression is more frequent than a cubic, but since this would out a bias in the ratio  $P(H_g)/P(H_c)$ , we assume they have equal probabilities
- let's compute  $P(D \mid H_g)$ :

$$P(D \mid H_g) = \frac{1}{101} \frac{1}{101} = 10^{-4}$$

(we have assumed that these number could be anywhere in the interval  $[-50, 50]$ )

- to compute  $P(D \mid H_c)$ , we use the same probability distribution for evaluating the coefficients

$$P(D \mid H_c) = \left(\frac{1}{101}\right) \left(\frac{4}{101} \frac{1}{50}\right) \left(\frac{4}{101} \frac{1}{5}\right) \left(\frac{2}{101} \frac{1}{50}\right) = 2.5 \cdot 10^{-12}$$

- the ratio favours the geometric hypothesis  $H_g$

$$P(D \mid H_g)/P(D \mid H_c) = 40 \cdot 10^6/1$$

## Occam's Razor and the *box/boxes behind the tree*

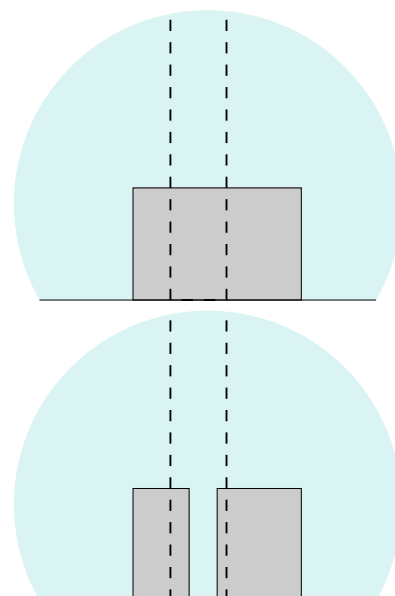
- let's go back to our example:

$Q_1$  Are there one or two boxes behind the tree ?

- $H_1$  says there is one box with four free parameters (three coordinates and colour)
- $H_2$  says there are two boxes (eight free parameters)
- putting some constraint on the colour (any of 16 values) and of the possible height (20 values in pixels), we get:

$$\frac{P(D \mid H_1)P(H_1)}{P(D \mid H_2)P(H_2)} = \frac{\frac{1}{20} \frac{1}{20} \frac{1}{20} \frac{1}{16}}{\frac{1}{20} \frac{1}{20} \frac{10}{20} \frac{1}{16} \frac{1}{20} \frac{1}{20} \frac{10}{20} \frac{1}{16}} = \frac{1000}{1}$$

the more complex model has four extra parameters for size and colours: it has to pay two big Occam factors ( $1/20$  and  $1/6$ ) for the suspicious coincidence that the two boxes have exactly the same colour and exact height



# Data Modeling with Parametric Models

- **generative model** : theory predicting observable data from model parameters
  - the model just studied did not have any parameter: it was either **true** or **false**
- the simplest generative model is a straight line

$$f(x; a, b) = a + b \cdot x$$



- but our measurements will differ from the model due to noise

$$y = f(x; a, b) + \epsilon$$

- and the noise model - we call it the **measurement model** - has also parameters
  - given our set of data  $D = \{y_j\}$  at specified values  $\{x_j\}$ , we want to infer the values of the parameters for the generative model
  - in some cases we want to find the best set of parameters that predicts the data
  - but data are noisy  $\rightarrow$  there is no unique solution
- we look for the probability distributions of the parameters,  $P(\theta \mid D H)$ , also called **parameter posterior pdf**. Thanks to Bayes' theorem

$$P(\theta \mid D H) = \frac{P(D \mid \theta H) P(\theta \mid H)}{P(D \mid H)}$$

## The Likelihood

- $P(D \mid \theta H)$  is the **Likelihood** probability
  - it is a key function since it describes both the phenomenon and the data
  - it tells us the probability of getting the data we measured, given some value of the parameters
- $H$  specifies:
  - a generative model  **the equation for the straight line  $f(x; a, b)$**
  - a measurement model  **how the measurement of  $y$  at a given  $x$  differs from  $f(x; a, b)$  due to noise**
- the measurement model describes  $\epsilon$  in  $y = f(x; a, b) + \epsilon$ 
  - example: Gaussian distribution with variance  $\sigma^2$ . The Likelihood for any measurement is

$$P(y \mid \theta H) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(y - f(x; a, b))^2}{2\sigma^2}\right)$$

- telling us that the measurement has a Gaussian distribution about the true value
- $\theta = \theta(a, b; \sigma)$  is the union of the generative and measurements models



- $P(\theta | H)$  is the Prior probability
  - it encapsulates all the information we have, independent of the data
- it is called Prior because is the background information we have before obtaining the Data
- different people may have different information, or different opinion on what prior information is important
- this is not a weakness of inference
- it just reflects reality: we do not only use our immediate measurements to reach scientific conclusions

## The Posterior

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- $P(\theta | D H)$  is the Posterior probability
  - it is the pdf over the model parameters, given data and background information
- from Bayes' theorem

$$\text{Posterior} \propto \text{Likelihood} \times \text{Prior}$$

- the proportionality is through  $P(D | H)$ , a normalization factor which is independent of  $\theta$ . Therefore:

$$P(\theta | D H) = \frac{1}{Z} P(D | \theta H) P(\theta | H)$$

- with  $Z = P(D | H)$
- from a conceptual point of view, inference is really that straightforward
- Bayesian inference is the process of improving our knowledge of the model parameters by using the data

▷ we update the Prior using the Likelihood to obtain the Posterior

- $P(D | H)$  is the Evidence
  - is the denominator of Bayes's equation and it gives the probability of observing the Data  $D$ , assuming the model  $H$  to be true, for any values of  $\theta$

$$P(D | H) = \int P(D | \theta H) P(\theta | H) d\theta$$

- evidence plays a key role in model comparison
- as a normalization constant, it is very important if we want to compute certain quantities from the posterior
- sometimes the integral can be calculated analytically, but for many real-world problems, we have to resort to numerical integration → [Markov Chain Monte Carlo](#)

## Bayesian Inference of repeated Bernoulli trials

# Bayesian analysis of coin tossing

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## Problem

- we have a coin and we toss it  $n$  times
- the coin lands heads in  $r$  of them
- $Q$  is the coin fair ? (i.e.  $\pi = \frac{1}{2}$ )

## Comment

- no definitive answer exists
- only a probabilistic answer can be provided
- we are looking for

$$P(\pi \mid n, r, H)$$

- from Bayes' theorem

$$P(\pi \mid n, r, H) = \frac{P(r \mid \pi, n, H) P(\pi \mid H)}{P(r \mid n, H)}$$

**Comment:**  $n$  is not part of the Prior since it is independent of the number of coin tosses

## Coin tossing model and probabilities

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### Our Measurement Model

- $\pi$  : probability of getting heads in one toss
- $\pi$  is constant in all the tosses
- all tosses are independent

### The Likelihood

- the appropriate Likelihood is the binomial distribution

$$P(r \mid \pi, n, H) = \binom{n}{r} \pi^r (1 - \pi)^{n-r} \quad \text{with } r \leq n$$

**Comment:**  $n$  is part of the data, but it is on the right side since it is fixed before starting to collect data

# Coin tossing : a uniform Prior

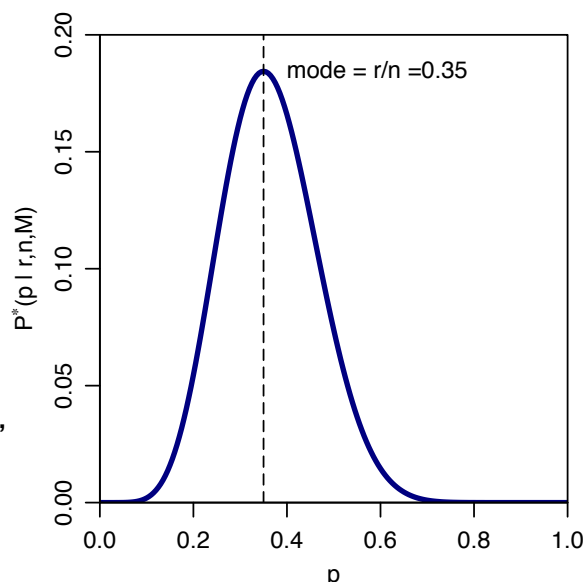
- let's adopt a uniform prior,  $P(\pi | H) \sim \mathcal{U}(0, 1)$
- the Posterior pdf is simply proportional to the Likelihood

$$P(\pi | r, n, H) = \frac{1}{Z} \pi^r (1 - \pi)^{n-r} = \frac{1}{Z} P^*(\pi | r, n, H)$$

- the normalization factor  $Z$  (i.e. the evidence  $P(r | n, H)$ ) does not depend on  $\pi$
- the mode is at  $r/n$

```
n <- 20
r <- 7
p <- seq(0, 1, length.out = 201)
p.post <- dbinom(x=r, size=n, prob=p)

plot(p, p.post,
     xaxs='i', yaxs='i', col='navy',
     type='l', lty=1, lwd = 3,
     ylim=c(0, 0.2),
     xlab="p",
     ylab=expression(paste(P~symbol("pi"),
                           "(p|r,n,H)")))
```



## Uniform Prior

### Comments

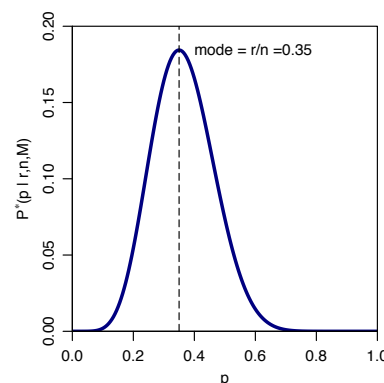
- the curve is not binomial in  $\pi$ , but it is binomial in  $r$
- the posterior is not-normalized: the integral over  $\pi$  is not unity
- we need the normalization factor only if we want to calculate expected values: i.e. mean and variance
- given the un-normalized posterior pdf,  $P^*(\pi | r, n, H)$ ,

$$E[\pi] = \int_0^1 \pi \cdot P(\pi | r, n, H) d\pi = \frac{1}{Z} \int_0^1 \pi \cdot \pi^r (1 - \pi)^{n-r} d\pi$$

- with

$$Z = \int_0^1 P^*(\pi | r, n, H) d\pi \approx \sum_j P^*(\pi_j | r, n, H) \Delta\pi_j$$

- estimated using numerical integration

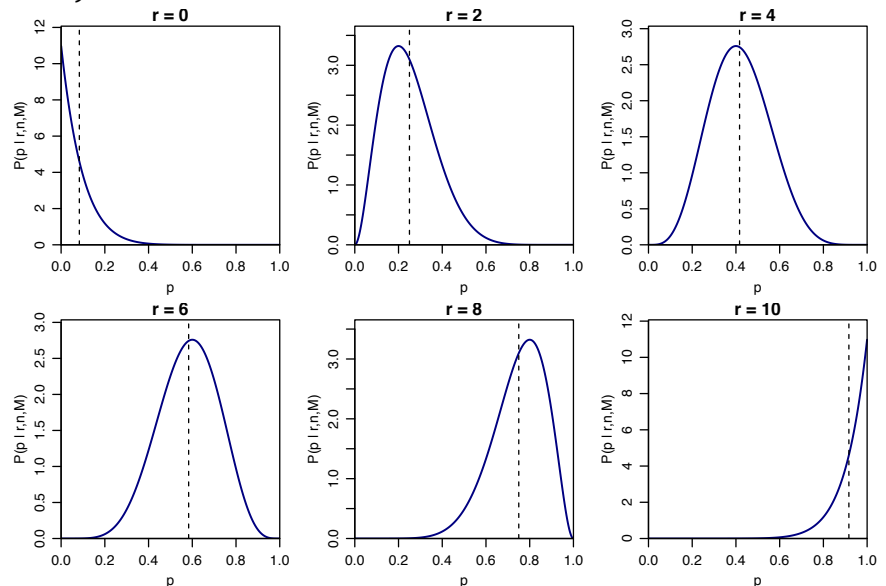


# Uniform Prior

```
n <- 10; n.sample <- 2000; delta.p <- 1/n.sample
p <- seq(from=1/(2*n.sample), by=1/n.sample, length.out=n.sample)

for(r in seq(from=0, to=10, by=2)) {
  p.star <- dbinom(x=r, size=n, prob=p)
  p.norm <- p.star/(delta.p*sum(p.star))
  plot(p, p.norm, type="l", lwd=1.5, col='navy',
       xlim=c(0,1), ylim=c(0,1.1*max(p.norm)),
       xaxs="i", yaxs="i", xlab="p", ylab="P(p|_r,n,H)")
  title(main=paste("r=",r), line=0.3, cex.main=1.2)
  p.mean <- delta.p*sum(p*p.norm)
  abline(v=p.mean, lty=2)
}
```

- interval  $[0, 1]$  is divided into `n.sample` intervals
- un-normalized pdf is evaluated at the center of each point
- a grid of probability is created
- with the normalized posterior, the expected value is computed



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## Coin tossing : a Beta Prior

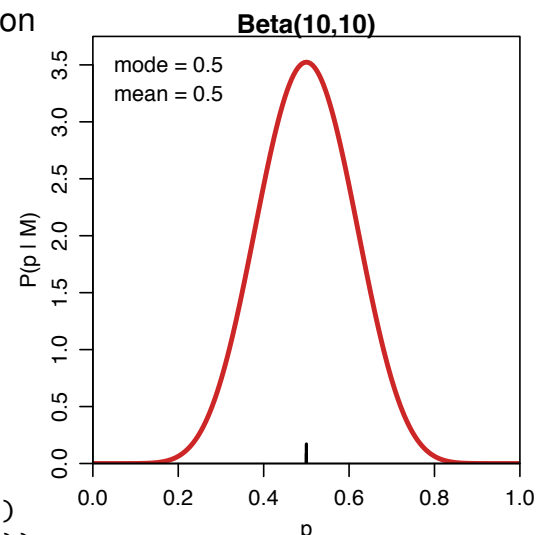
- given a random coin, we may believe the coin is fair, or close to fair
- an appropriate probability density function is the Beta distribution

$$P(\pi | r, n, H) = \frac{1}{B(\alpha, \beta)} \pi^{\alpha-1} (1 - \pi)^{\beta-1} \quad \text{with } \alpha > 0, \beta > 0$$

Note: for  $\alpha = \beta = 1$  we get a uniform distribution

- if  $\alpha = \beta$  the function is symmetric, and the mean and mode are 0.5
- the larger  $\alpha$  (when  $\alpha \geq 1$ ), the narrower the distribution

```
alpha <- 10; beta <- 10
p <- seq(0, 1, length.out = 201)
p.prior <- dbeta(p, alpha, beta)
plot(p, p.prior, xaxs='i', yaxs='i',
     col='navy', type='l', lty=1, lwd = 3,
     ylim=c(0,3.75),
     xlab="p", ylab=paste("P(p|_H)"),
     main=paste("Beta(",alpha,"",beta,")"))
mode <- (alpha - 1)/(alpha + beta - 2)
lines(c(mode, mode), c(0, 0.2), lty=5, lwd=2)
mean <- alpha/(alpha + beta)
lines(c(mean, mean), c(0, 0.2), lty=2, lwd=2)
text(0.05, 3.5, adj=0, paste("mode=", mode))
text(0.05, 3.25, adj=0, paste("mean=", mean))
```



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- multiplying the Prior by the likelihood, and absorbing the terms not depending on  $\pi$  in the constant term  $Z$ , we get

$$\begin{aligned}P(\pi \mid r, n, H) &= \frac{1}{Z} \pi^r (1 - \pi)^{n-r} \times \pi^{\alpha-1} (1 - \pi)^{\beta-1} \\&= \frac{1}{Z} \pi^{r+\alpha-1} (1 - \pi)^{n-r+\beta-1}\end{aligned}$$

- multiplying the Posterior with this Likelihood, we get the same form for the Posterior (another Beta distribution)
- the normalization constant is

$$Z = B(r + \alpha, n - r + \beta)$$

- we say the Prior and Posterior are [conjugate distributions](#)
- ▷ [the Prior is the conjugate Prior for this Likelihood function](#)

## Beta Prior

---

- if we start with a Beta Prior with parameters  $\alpha_p$  and  $\beta_p$ , and then measure  $r$  heads in  $n$  tosses, the Posterior is a Beta functions with parameters

$$\alpha = \alpha_p + r \quad \text{and} \quad \beta = \beta_p + n - r$$

- mean and mode for the Posterior are

$$\text{mean} = \frac{\alpha_p + r}{\alpha_p + \beta_p + n} \quad \text{and} \quad \text{mode} = \frac{\alpha_p + r - 1}{\alpha_p + \beta_p + n - 2}$$

- if we compare the result with that obtained with a Uniform Prior ( $\mathcal{U}(0, 1) \sim \text{Beta}(\alpha = 1, \beta = 1)$ ), we get

$$\text{mean} = \frac{1 + r}{2 + n} \quad \text{and} \quad \text{mode} = \frac{r}{n}$$

# Beta Prior vs Uniform Prior

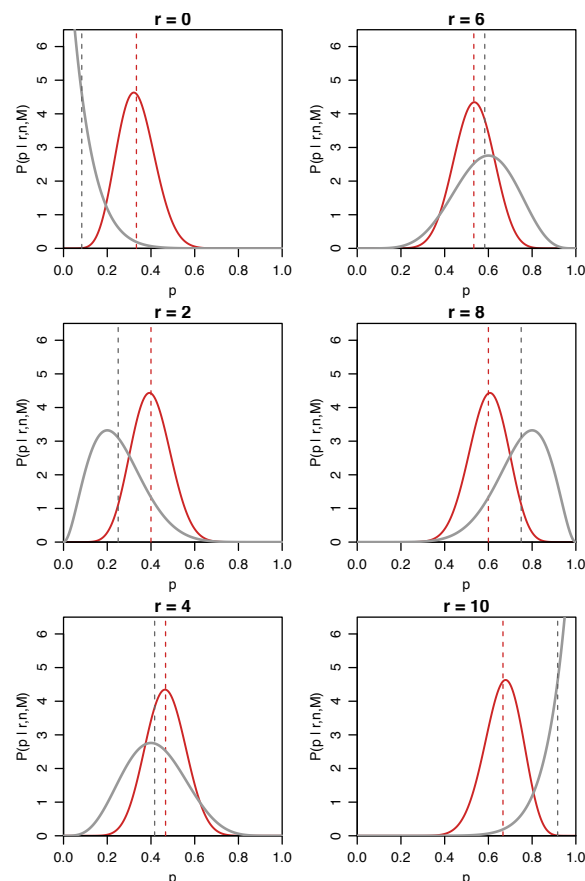
```
n <- 10;
alpha.prior <- 10; beta.prior <- 10
n.sample <- 2000; delta.p <- 1/n.sample

p <- seq(from=1/(2*n.sample),
         by=1/n.sample, length.out=n.sample)

par(mfrow=c(3,3))

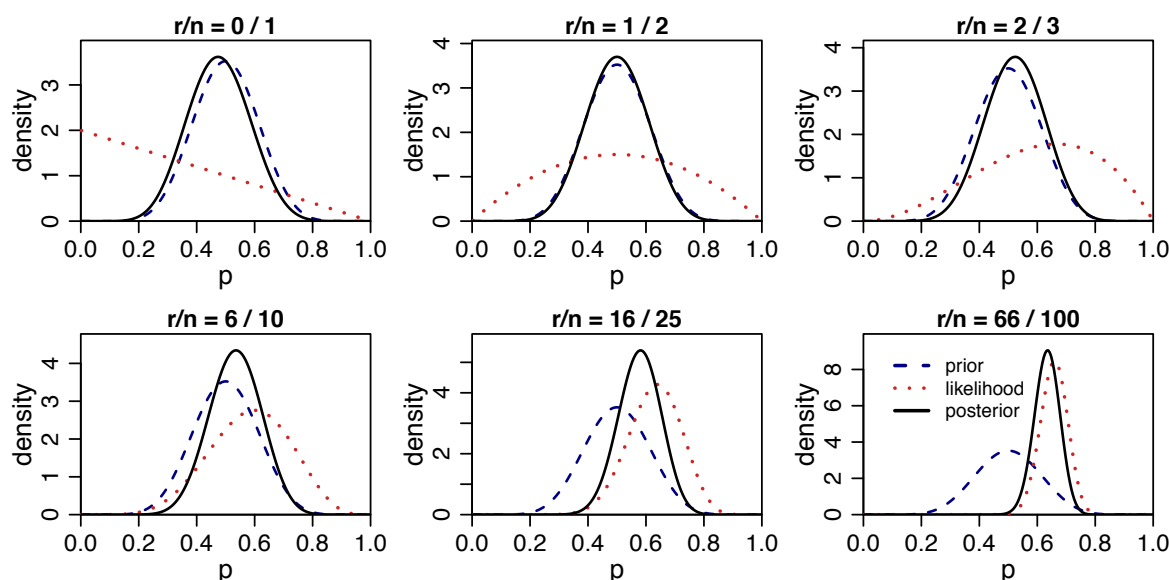
for(r in seq(from=0, to=10, by=2)) {
  post.beta <- dbeta(x=p,
                    alpha.prior+r,
                    beta.prior+n-r)
  plot(p, post.beta, type="l", lwd=1.5,
       col='firebrick3', ...)
  p.mean.b <- delta.p*sum(p*post.beta)
  abline(v=p.mean.b,
        col='firebrick3', lty=2)

  # overplot posterior with Unif Prior
  post.unif <- dbinom(x=r, size=n, prob=p)
  lines(p,
        post.unif/(delta.p*sum(post.unif)))
  p.norm.u <- post.unif/
    (delta.p*sum(post.unif))
  p.mean.u <- delta.p*sum(p*p.norm.u)
  abline(v=p.mean.u, col="grey60", lty=2)
}
```



## Posterior evolution with data size

- the outcome of only few coin flips tells us little about the fairness of a coin. Our state of knowledge after the analysis of the data is strongly dependent on what we knew or assumed a priori
- as the evidence grows, we are eventually brought to the same conclusions irrespective of our initial beliefs
- the **posterior** pdf is then **dominated** by the **likelihood** function
- the **choice of the prior** becomes **largely irrelevant**



# Posterior Evolution, R code

```
alpha.prior <- 10; beta.prior <- 10
Nsamp <- 200

delta.p <- 1/Nsamp
p <- seq(from=1/(2*Nsamp),
         by=1/Nsamp,
         length.out=Nsamp)
p.prior <- dbeta(x=p,
                 alpha.prior,
                 beta.prior)

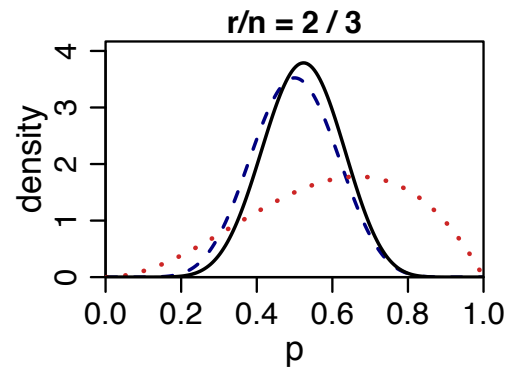
n.str <- readline("Enter n extractions: ")
n.seq <- as.numeric(unlist(strsplit(n.str, ",")))

# Loop over the vector
for (n in n.seq) {
  r <- as.integer((2/3) * n)

  p.like <- dbinom(x=r, size=n, prob=p)
  p.like <- p.like/(delta.p*sum(p.like))
  p.post <- dbeta(x=p, shape1=alpha.prior+r, shape2=beta.prior+n-r)

  plot(p, p.prior, type="l", xlim=c(0,1), ...)

  lines(p, p.like, col='firebrick3',lwd=2, lty=3)
  lines(p, p.post, lwd=1.5)
  title(main=paste("r/n=",r,"/",n), line=0.3, cex.main=1.2)
  ...
}
```



## Parameters best estimates and reliability

- once the posterior is determined, we wish to summarize our inference on a parameter with two numbers:
  - the best estimates
  - and a measure of its reliability
- probability distribution associated with the parameter  $\Rightarrow$  a measure of how much we believe the result lies in the neighborhood of that point
- **Best estimate**  $\rightarrow$  maximum of the posterior pdf

$$\theta_o = \text{MAX} \{P(\theta \mid D, H)\}$$

- which means

$$\left. \frac{dP}{d\theta} \right|_{\theta_o} = 0 \quad \text{and} \quad \left. \frac{d^2P}{d\theta^2} \right|_{\theta_o} < 0$$

- to get a **measurement of the reliability** of our 'best estimate', we need to look at the **spread of the posterior pdf around  $\theta_o$** .



# Parameters best estimates and reliability

---

- let's consider a **Taylor expansion** of the posterior pdf **around**  $\theta_0$ .
- rather than working with the pdf, the calculations will be done with the natural logarithm

$$\begin{aligned} L &= \ln P(\theta \mid D, H) \\ &= L(\theta_0) + \frac{1}{2} \left. \frac{d^2 P}{d\theta^2} \right|_{\theta_0} (\theta - \theta_0)^2 + \dots \end{aligned}$$

## Comments

- $L(\theta_0)$  is a constant and tells us nothing about the slope of the posterior pdf
- the linear term in  $(\theta - \theta_0)$  is missing since we are expanding about a maximum
- the quadratic term is the dominant factor and it determines the width of the pdf
- ignoring higher order contributions and taking the exponential of the Taylor expansion

$$P(\theta \mid D, H) \sim A \exp \left[ \frac{1}{2} \left. \frac{d^2 P}{d\theta^2} \right|_{\theta_0} (\theta - \theta_0)^2 \right]$$

with  $A$ , a normalization constant

# Parameters best estimates and reliability

---

- we have approximated our posterior pdf by a Gaussian distribution

$$P(\theta \mid \theta_0, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \frac{(\theta - \theta_0)^2}{\sigma^2} \right]$$

- comparing the two functions, we get

$$\left. \frac{d^2 L}{d\theta^2} \right|_{\theta_0} = -\frac{1}{\sigma^2} \quad \Rightarrow \quad \sigma = \left( - \left. \frac{d^2 L}{d\theta^2} \right|_{\theta_0} \right)^{-1/2}$$

- our inference about the quantity of interest is

$$\theta = \theta_0 \pm \sigma$$

- with:
  - $\theta_0$  our **best estimate** for  $\theta$
  - $\sigma$  a **measurement of its reliability**
- for a Gaussian distribution

$$P(|\theta - \theta_0| \leq \sigma \mid DH) \sim 0.67$$

$$P(|\theta - \theta_0| \leq 2\sigma \mid DH) \sim 0.95$$

- the Posterior is

$$P(\pi | r, n, H) \propto \pi^r (1 - \pi)^{n-r}$$

- taking the natural logarithm

$$L = \text{const} + r \ln \pi + (n - r) \ln (1 - \pi)$$

$$\frac{dL}{d\pi} = \frac{r}{\pi} - \frac{n-r}{1-\pi} \quad \text{and} \quad \frac{d^2L}{d\pi^2} = -\frac{r}{\pi^2} - \frac{n-r}{(1-\pi)^2}$$

- from the request of a maximum

$$\frac{dL}{d\pi} = 0 \quad \Rightarrow \quad \pi_o = \frac{r}{n}$$

- the reliability is given by the second derivative

$$\left. \frac{d^2L}{d\pi^2} \right|_{\pi_o} = -\frac{r}{\pi_o^2} - \frac{n-r}{(1-\pi_o)^2} = -\frac{n}{\pi_o(1-\pi_o)}$$

- therefore

$$\sigma = \left( - \left. \frac{d^2L}{d\theta^2} \right|_{\theta_o} \right)^{-1/2} = \sqrt{\frac{\pi_o(1-\pi_o)}{n}} = \frac{1}{n} \sqrt{r(n-r)}$$

# Parameters estimates, coin example, Beta Prior

- the Posterior is

$$P(\pi | r, n, H) \propto \pi^{r+\alpha-1} (1 - \pi)^{n-r+\beta-1}$$

- taking the natural logarithm

$$L = \text{const} + (r + \alpha - 1) \ln \pi + (n - r + \beta - 1) \ln (1 - \pi)$$

$$\frac{dL}{d\pi} = \frac{r + \alpha - 1}{\pi} - \frac{n - r + \beta - 1}{1 - \pi} \quad \text{and} \quad \frac{d^2L}{d\pi^2} = -\frac{r + \alpha - 1}{\pi^2} - \frac{n - r + \beta - 1}{(1 - \pi)^2}$$

- from the request of a maximum

$$\frac{dL}{d\pi} = 0 \quad \Rightarrow \quad \pi_o = \frac{r + \alpha - 1}{n + \alpha + \beta - 2}$$

- the reliability is given by the second derivative

$$\left. \frac{d^2L}{d\pi^2} \right|_{\pi_o} = -\frac{r + \alpha - 1}{\pi_o^2} - \frac{n - r + \beta - 1}{(1 - \pi_o)^2} = -(\alpha + \beta + n - 2) \frac{\alpha + r}{\alpha + r - 1}$$

- therefore

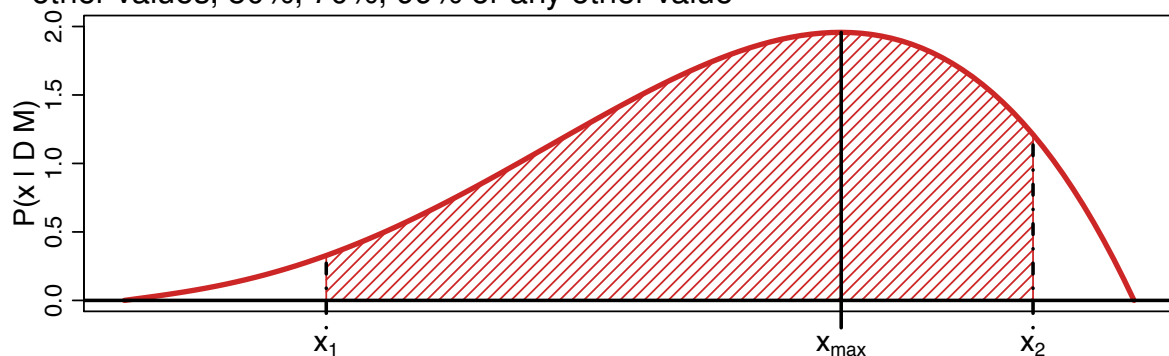
$$\sigma = \left( - \left. \frac{d^2L}{d\theta^2} \right|_{\theta_o} \right)^{-1/2} = \frac{1}{\alpha + \beta + n - 2} \sqrt{\frac{\alpha + r - 1}{\alpha + r}}$$

# Asymmetric Posterior pdfs

- our derivation of the reliability of the parameter estimate (i.e. the error) relies on the validity of the quadratic expansion
- this is usually a reasonable approximation
- however there are times when the posterior pdf is markedly asymmetric
- while the maximum of the posterior can still be regarded as giving the best estimate, the concept of symmetric error bars does not seem appropriate
- a good way to express the reliability is through a confidence interval

$$P(x_1 \leq x < x_2 \mid D, H) = \int_{x_1}^{x_2} P(x \mid D, H) dx \sim 0.95$$

- Why 95% confidence level ?
- it is traditionally seen as a reasonable value, but nothing stops us from quoting other values, 50%, 70%, 99% or any other value



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## Assigning Priors

- probabilistic inference provides answers to well-posed problems but
- it **does not define** our **models**
- it **does not define** the **priors**
- or tell us which data to collect and how
- with the coin example we learned how the posterior pdf depends on both the prior and the likelihood
  - when data are poor, the prior plays a more dominant role

### How do we assign a Prior ?

- 1) a prior should incorporate any relevant information we have about the problem (→ we implicitly use priors all the time in every day life)
- 2) some principles can help us to adopt an appropriate prior

### Principle of insufficient reason

- also called the **principle of indifference**
- if we have a set of mutually exclusive outcomes, and we do not expect any one of them more likely, we should assign equal probabilities

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# Assigning Priors

## Maximum Entropy

- it is based on the idea of finding the least informative (most entropic) distribution, given certain information
- example:  
if only mean and variance are known, it shows that the Gaussian is the least informative distribution

## Empirical Bayes

- priors are estimated from some general properties of the data
- we can take the posterior from one analysis to be the prior of the next analysis, if they involve independent data
- the final posterior will be identical to having combined the two data sets together with the original prior
- let  $D_1$  and  $D_2$  be two independent data sets

$$\begin{aligned}P(\theta \mid D_1 D_2) &\propto P(D_1 D_2 \mid \theta) P(\theta) \\&\propto \underbrace{P(D_2 \mid \theta)}_{\text{likelihood for } D_2} \underbrace{P(D_1 \mid \theta)}_{\text{posterior from } D_1} \times P(\theta)\end{aligned}$$

## Exercise : a survey for the next Uni elections

### The Problem

- In proximity of the elections for student's representatives in some University board, [Anna](#), [Chris](#) and [Maggie](#) decide to perform a survey among their classmates to check how strong is their candidate friend
- the aim is to infer the probability that she gets elected

### Step 1: choosing the Priors

- Before starting the interviews, they have different opinions about the results of the elections:
  - [Anna](#) thinks that there will be a 20% chances that their friend will be elected, and moreover, the probability has a standard deviation of 0.08. She therefore assumes a Beta prior such that:

$$E[x] = \frac{a}{a+b} = 0.2 \quad 1 - E[x] = \frac{b}{a+b} = 0.8 \quad \frac{0.2 \times 0.8}{a+b+1} = 0.08^2,$$

which means  $a = 4.8$  and  $b = 19.2$

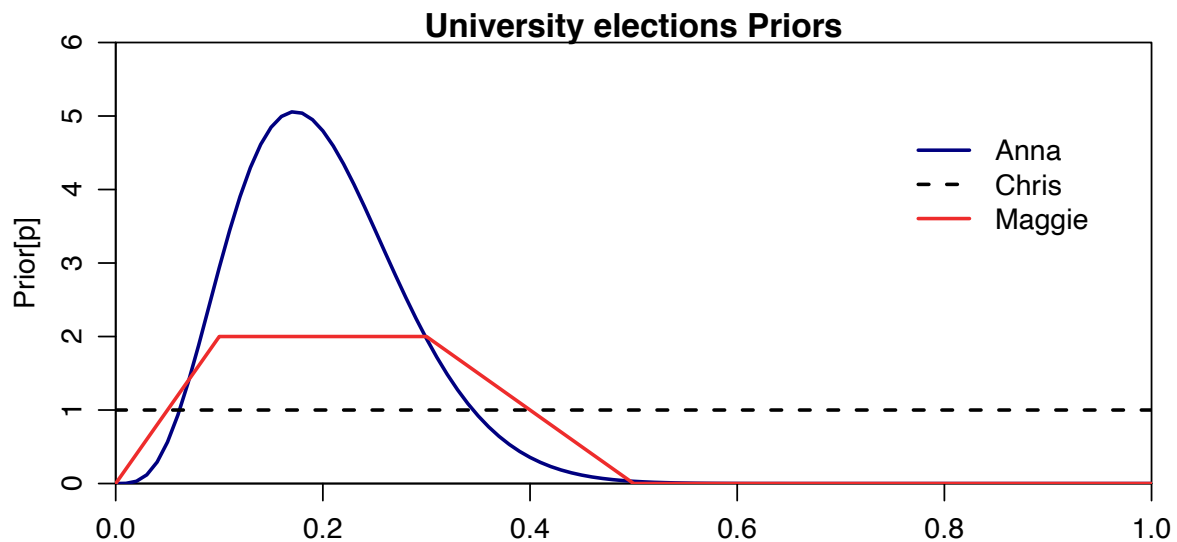
- [Chris](#) is a new student and he does not have any feeling how popular their candidate is, therefore he assumes a Uniform prior distribution. For him  $a = b = 1$

# Exercise : a survey for the next Uni elections (2)

## Step 1: choosing the Priors (cont'd)

Before starting the interviews, they have different opinions about the results of the elections:

- **Maggie** thinks that the probability distribution is flat, but not over the whole domain. Therefore she assumes a trapezoidal distribution which is flat between 0.1 and 0.3, and goes to zero outside that domain



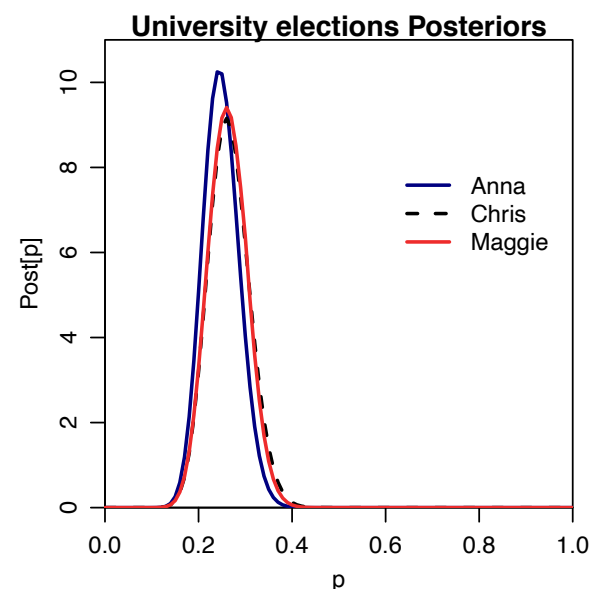
# Exercise : a survey for the next Uni elections (3)

## Step 2: getting the data

- now they start the survey and decide to interview  $n = 100$  students regularly coming to the University canteen but they do not personally know
- out of the interviewed students,  $x = 26$  claim they will support and vote the candidate

## Step 3: computing the Posterior

- Anna and Chris use a Beta prior  $\rightarrow$  they get a conjugate prior  $\text{Beta}(\alpha = a + x, \beta = b + n - x)$ 
  - **Anna** has  $\text{Beta}(\alpha = 4.8 + 26, \beta = 19.2 + 74)$
  - **Chris** gets  $\text{Beta}(\alpha = 1 + 26, \beta = 1 + 74)$
  - **Maggie** has to perform a numerical computation of the posterior, given her user-defined Prior



# Exercise : a survey for the next Uni elections (4)

## Step 4: computing Credibility Intervals

- given the Posterior distributions, we can compute the mean value and the variance
- by integrating the Posterior distribution, it is possible to compute the Credibility Interval, 95%, as the area between the 2.5% and 97.5%
- Maggie's estimate must be done by numerical integration

	$\text{Post}(\alpha, \beta)$	mean	sigma	95% Cr. Int.
Anna	$\text{Beta}(\alpha = 30.8, \beta = 93.2)$	0.248	0.039	0.177 - 0.328
Chris	$\text{Beta}(\alpha = 27, \beta = 75)$	0.265	0.043	0.184 - 0.354
Maggie	numerical	0.262	0.042	0.183 - 0.346

