Comparing Frequentist and Bayesian inference for a Bernoulli process

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Two different approaches

Frequentist paradigm

- it allows to perform inference about the parameter using probabilities calculated from the sampling distribution of the data

➜ the parameter is unknown, but fixed ➜ we cannot associate a probability to it

➜ the only probability is that of the random sample

➜ probabilities are not conditional on the actual data sample that has been measured and are interpreted as a long run relative frequency

- different types of inferences are possible:

1 - point estimation

2 - interval estimation

3 - hypothesis testing

Bayesian paradigm

- the posterior distribution is the key point

➜ it summarizes our belief about the parameter, after we have analyzed the data

➜ it allows to extract all the estimates on the parameter

1-Point Estimation

- a single statistic is calculated from the sample data and used to estimate the unknown parameter
- several theoretical approaches are possible: an example is the Maximum Likelihood Estimation (MLE)
- since the true value of the parameter is unknown, we can judge an estimator only on the sampling distribution of the estimator, i.e. the distribution of the estimator over all the possible random samples
- the expected value of an estimator measures the center of its distribution
- the *Bias* of an estimator is the difference from its expected value and the true value of the parameter

$$\mathrm{Bias}\left[\hat{\theta},\theta\right] = \mathrm{E}\left[\hat{\theta}\right] - \theta$$

- an estimator is *unbiased* if the mean of its sampling distribution is the true parameter value
- the *Mean Squared Error* of an estimator is

$$\begin{aligned}\mathrm{MSE}\left[\hat{\theta}\right] &= \mathrm{E}\left[\hat{\theta}-\theta\right]^2 \\ &= \int \left(\hat{\theta}-\theta\right)^2 f\left(\hat{\theta}\,\middle|\,\theta\right)\, d\hat{\theta}\end{aligned}$$

- it can be demonstrated that

$$\mathrm{MSE}\left[\hat{\theta}\right] = \mathrm{Bias}\left[\hat{\theta},\theta\right]^2 + \mathrm{Var}\left[\hat{\theta}\right]$$

Frequentist estimator

- in the Frequentist approach, an unbiased estimator for the Binomial distribution is

$$\hat{p}_F = \frac{y}{n}$$

- where y is the number of successes in n trials
- the properties of the estimator are:

$$\begin{aligned}\mathrm{E}\left[\hat{p}_F\right] &= p \\ \mathrm{Var}\left[\hat{p}_F\right] &= \frac{p(1-p)}{n} = \frac{pq}{n} \\ \mathrm{MSE}\left[\hat{p}_F\right] &= \mathrm{Bias}\left[p_F, p\right]^2 + \mathrm{Var}\left[p_F\right] \\ &= 0^2 + \frac{p(1-p)}{n}\end{aligned}$$

Bayesian estimator

- with the Bayesian approach, we use the posterior mean as an estimate for p
- let's assume we imposed a uniform prior, $\mathtt{Beta}(1, 1)$
- the posterior mean is

$$\hat{p}_B = m' = \frac{a'}{a' + b'}$$

- with $a' = 1 + y$ and $b' = 1 + n - y$
- therefore

$$\begin{aligned} \hat{p}_B &= \frac{1+y}{1+y+1+n-y} = \frac{y+1}{n+2} \\ &= \frac{y}{n+2} + \frac{1}{n+2} = \frac{np}{n+2} + \frac{1}{n+2} \end{aligned}$$

- the variance of the distribution is

$$\mathrm{Var}\left[\hat{p}_B\right] = \left(\frac{1}{n+2}\right)^2 n\,p\,(1-p)$$

- and the *Mean Square Error* becomes

$$\begin{aligned} \mathrm{MSE}\left[\hat{p}_B\right] &= \left[\frac{np}{n+2} + \frac{1}{n+2} - p\right]^2 + \left(\frac{1}{n+2}\right)^2 n\,p\,(1-p) \\ &= \left(\frac{1-2p}{n+2}\right)^2 + \left(\frac{1}{n+2}\right)^2 n\,p\,(1-p) \end{aligned}$$

Example: point estimation

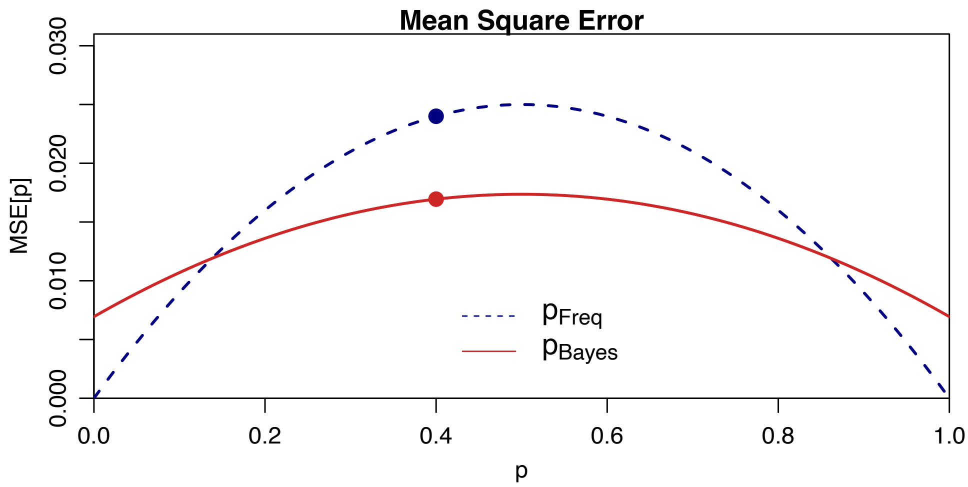
- let's suppose we have a Bernoulli process with $p = \frac{2}{5}$. We perform multiple samples from the distribution and the sample size is $n = 10$
- let's evaluate and compare the *Mean Square Error* for both Frequentist and Bayesian estimators
- we get

$$\mathrm{MSE}\left[\hat{p}_F\right] = \frac{0.4 \times 0.6}{10} = 0.024$$

$$\mathrm{MSE}\left[\hat{p}_B\right] = \left(\frac{1-0.8}{12}\right)^2 + \left(\frac{1}{12}\right)^2 \times 10 \times 0.4 \times 0.6 = 0.0169$$

- we can scan scan the estimator for different values of the true value domain

the Bayesian estimator is closer to the true value over most of the true value range



2-Interval Estimation

- we wish to find an interval ($low, high$) that has a predetermined probability of containing the parameter

Frequentist approach

- the parameter is fixed but unknown
- before the sample is taken, the interval endpoints are random
- once the data is known and the endpoints computed, there is nothing random anymore
- the interval is called a confidence interval for the parameter

- $(1-\alpha) \times 100\%$ confidence interval for a parameter θ is the interval ($low, high$) such that

$$P\left(\text{low} \leq \theta \leq \text{high}\right) = 1 - \alpha$$

- the most common criteria used to select the interval endpoints are

1. equal ordinates on the sampling distribution, $f(\text{low}) = f(\text{high})$
2. equal tail area on the sampling distribution

Frequentist Interval Estimation

once the interval is calculated, there is nothing left that is random

➜ the interval either contains the unknown fixed parameter or it does not

➜ the interval can no longer be regarded as a probability interval

The correct Frequentist paradigm is:

- $(1-\alpha) \times 100\%$ of the random intervals calculated in this way will contain the true value ➜ we have a $(1-\alpha) \times 100\%$ confidence that our interval does cointain it
- it is a misinterpretation to make a probability statement about the parameter θ from the calculated confidence interval

- very often the sampling distribution of the estimator can be approximated with a normal distribution, with the mean equal to the true value of the parameter

➜ the confidence interval gets the form

$$\text{estimator} \pm \text{critical value} \times \text{estimator standard deviation}$$

- if n is large:

$$\hat{p}_f = y/n \text{ is normal with mean } p \text{ and } \quad \sigma = \sqrt{p(1-p)/n}$$

- the approximate $(1-\alpha) \times 100\%$ equal area confidence interval for p is

$$\hat{p}_f \pm z_{\alpha/2} \sqrt{\frac{\hat{p}_f(1-\hat{p}_f)}{n}}$$

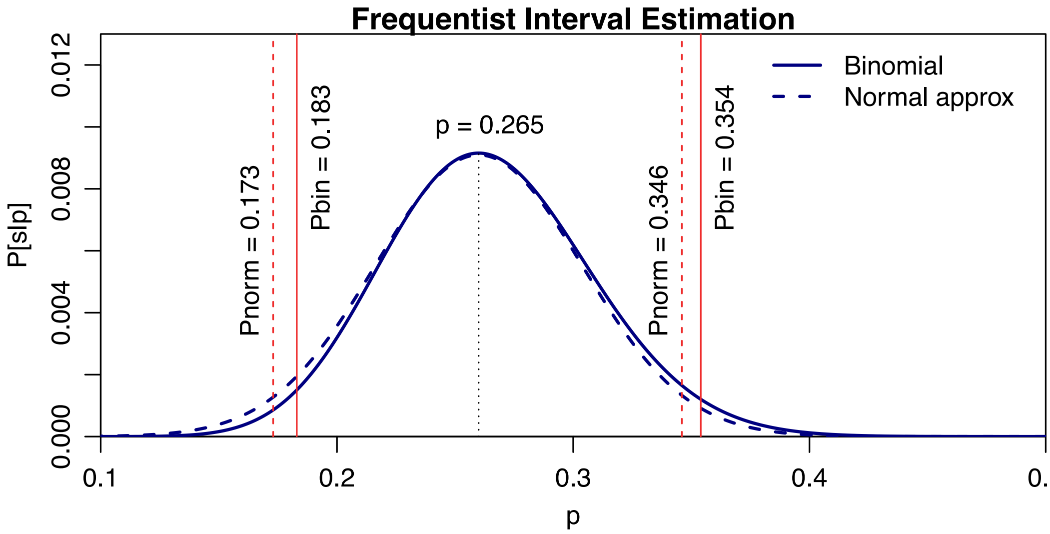
Example: interval estimation (F)

The problem

- a small town residents sample ($n = 100$) are interview about the construction of a new concert hall
- $y = 26$ express a positive opinion about it

Frequentist approach solution

- an unbiased estimator is $\hat{p}_F = y/n = 0.26$
- with standard deviation $\sigma = \sqrt{0.26 * (1 - 0.26)/100} = 0.0439$



Example: interval estimation (B)

Bayesian approach solution

1 - let's select a uniform prior, i.e. $\texttt{Beta}(1, 1)$, for our unknown parameter

- our posterior distribution is given by a $\texttt{Beta}$ distribution. since a $\texttt{Beta}$ prior is a conjugate function for the $\texttt{Binomial}$ distribution
- the posterior distribution is

$$\texttt{Beta}(a' = a + y, b' = b + n - y) = \texttt{Beta}(1 + 26, 1 + 74)$$

2 - as a second example, let's choose a $\texttt{Beta}$ prior with a mean value $m = 0.2$ and a standard deviation $\sigma = 0.08$. Since

$$m = \frac{a}{a+b} = p_\circ \quad \text{and} \quad \sigma_\circ^2 = \frac{ab}{(a+b)^2(a+b+1)} = np_\circ(1 - p_\circ)$$

- it can be rewritten giving:

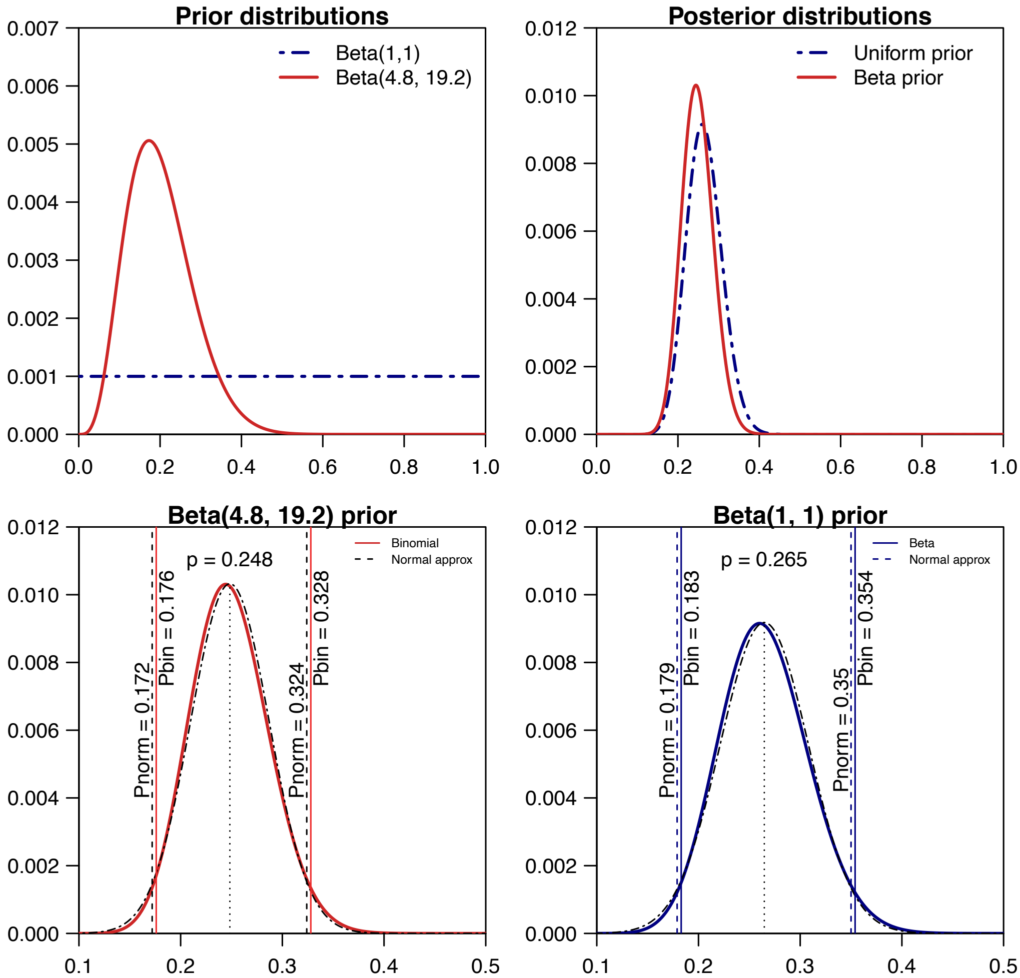
$$a + b + 1 = \frac{p_\circ(1 - p_\circ)}{\sigma_\circ^2} \quad \text{and} \quad a + b = \frac{a}{p_\circ}$$

- a $\texttt{Beta}(4.8, 19.2)$ prior gives a posterior distribution

$$\texttt{Beta}(a' = a + y, b' = b + n - y) = \texttt{Beta}(4.8 + 26, 19.2 + 74)$$

Example: interval estimation (B)

starting with different prior distribution, we get similar posteriors



with the posterior distribution we can calculate the credibility interval

3-Hypothesis Testing

Idea Behind

- researchers have some theory and want to know wheather or not the data actually support that theory
- scientists should not claim the discovery of a new effect if the discrepancy observed in the data could be due to chance alone
- Hypothesis Testing, also called Significance Testing, is the Frequentist statistical method used to check against claims unjustified in the data
- the nonexistence of the effect is set up as the null hypothesis
- when we accept the null hypothesis as true, it does not mean that we believe is is ’literally true’. Rather it means that chance alone remains a reasonable explanation for the observed discrepancy. Therefore we cannot discard chance as the sole explanation
- we distinguish

1 testing a one-side hypothesis when we are interested in detecting the effect in one direction

2 two-sided hypothesis when a test hypothesis is tested against two sided alternatives

3-Hypothesis Testing (HT)

ESP: Extrasensory perception experiment

- θ: probability of correctly choosing the colours
- if participants have paranormal abilities: $\theta > 0.5$

- the researchers has to formulate two distinct and alternative hypotheses:
  - the NULL Hypothesis, H_0: $\theta = 0.5$
  - and the alternative Hypothesis, H_1: $\theta > 0.5$

➜ the goal of HT is not to show that the alternative hypothesis is TRUE, but to show that the null hypothesis is FALSE

The TRIAL of NULL Hypothesis

- the NULL Hypothesis is the defendant
- the researcher is the persecutor
- the statistical test is the judge

presumption of innocence: the NULL Hypothesis is deemed to be TRUE unless you, the researcher, can prove beyound reasonable doubts that it is FALSE

Errors in HT

- the goal is not to eliminate errors, but to minimize them

	accept H_0	reject H_0
H_0 is TRUE	ok	error, type I
H_0 is FALSE	error, type II	ok

➜ important design principle: control the probability of type error I and keep it below some fixed probability α

➜ α is called the significance level of the test

the power of the test is the probability with which we reject the NULL Hypothesis when it is really FALSE

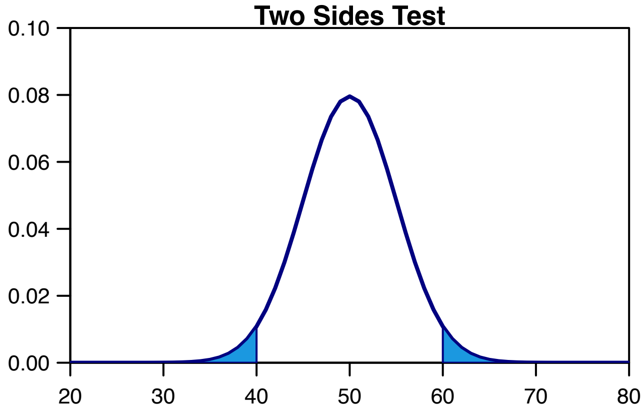
	accept H_0	reject H_0
H_0 is TRUE	$(1-\alpha)$ probability of correct retention	α type I error rate
H_0 is FALSE	β, type II error rate	$(1-\beta)$, power of the test

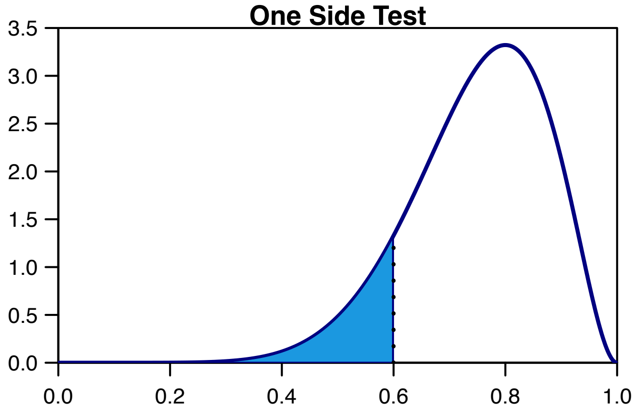
➜ a powerful HT has small values of β while keeping α fixed at some small desired level

α values used by convention among scientists: 0.05, 0.1 and 0.01

HT prescriptions

1) setup the NULL and alternative hypotheses
2) determine what the sampling distribution of the test statistic would be if the NULL hypothesis were TRUE
3) choose the level of significance, α and associate the critical regions to the distribution





4) calculate the value of the test statistic for the real data and compare to the critical value to make our decision : critical region ➜ values for which we would reject the NULL hypothesis
5) if we reject the NULL hypothesis, we say that the test has produced a significant result

Example: One-Side Hypothesis Test

The problem

- we wish to test the effect of a new treatment, to verify if it is better than the standard treatment as a parameter in the model
- $p =$ fraction of patients who benefit from the new treatment
- $p_\circ =$ fraction of patients who benefit from the standard treatment
- we know that $p_\circ = 0.6$
- 10 patients are given the new treatment and be observe that $y = 8$ patients benefit from the new treatment
- do we conclude that $p > 0.6$ at the 5% level of significance ?

Frequentist approach

1 - setup a null hypothesis

$$H_\circ : p \leq 0.6$$

2 - the alternative hypothesis (the new treatment is better) is

$$H_1 : p > 0.6$$

3 - the NULL distribution of the test statistic is the sampling distribution of the test statistic, given that the NULL hypothesis is true

$$\mathtt{Binom}(y \,|\, n, p = 0.6)$$

Example: One-Side Hypothesis Test (F)

4 - choose a level of significance

$$\alpha = 5\%$$

Note that since y has a discrete distribution, only some values of α are possible

5 - the rejection region is chosen so that it has a probability of α under the NULL distribution (Neyman and Pearson approach)

$y = 8$ lies in the acceptance region ➜ we do not reject $H_\circ$

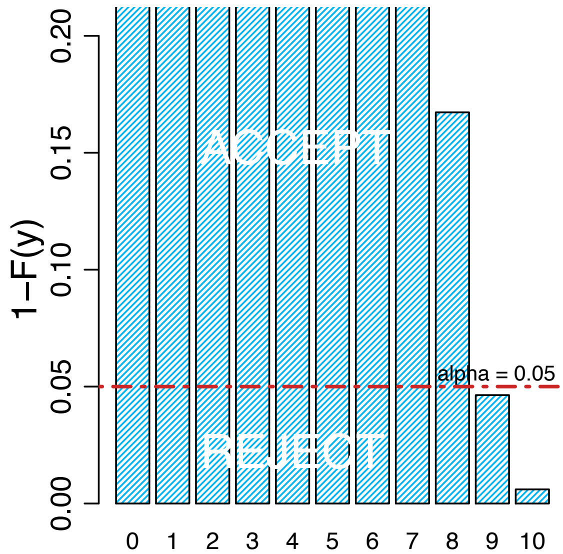
6 - the p-value is the probability of getting what we observed:

$$\mathtt{p-value} = \sum_{y_{obs}}^{n} f(y \mid p_\circ) = 0.1672$$

if $\mathtt{p\text{-}value} < \alpha$ ➜ the test statistic lies in the rejection region

α represents the long-run rate of rejecting a true null hypothesis

7 - an alternative way, due to Fisher, is to reject $H_\circ$ if $\mathtt{p\text{-}value} < \alpha$



Example: One-Side Hypothesis Test (B)

Bayesian approach

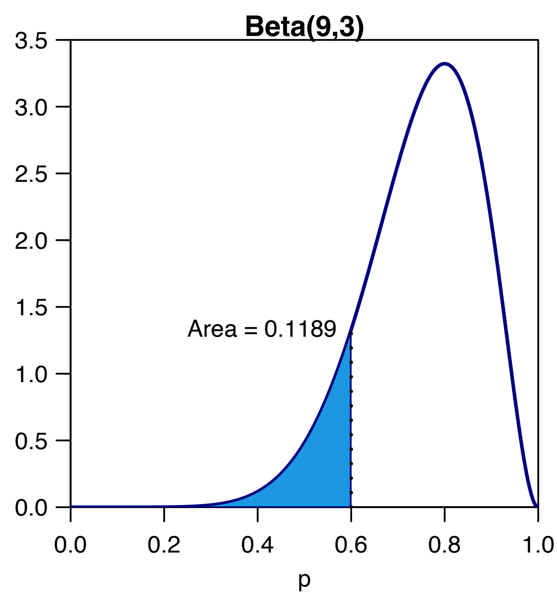
- we wish to test $H_\circ : p \leq p_\circ$ versus $H_1 : p > p_\circ$ at a level of significance α
- we evaluate the posterior probability of the null hypothesis, and integrate over the required region:

$$P\left(H_\circ : p \leq p_\circ \mid y\right) = \int_0^{p_\circ} g\left(p \mid y\right)\, dp$$

- we reject the null hypothesis if the posterior probability is less than α, the level of significance
- we use a uniform prior, $\mathtt{Beta}(1,1)$, for the parameter p
- given $y = 8$, the posterior density is $\mathtt{Beta}(9,3)$

$$\begin{aligned} P\left(p \leq 0.6 \middle| y = 8\right) &= \int_0^{0.6} \frac{\Gamma(12)}{\Gamma(3)\Gamma(9)} p^8 (1-p)^2 dp \\ &= 0.1189 \end{aligned}$$

- the result, 11.89%, is higher than $\alpha = 5\%$, therefore we cannot reject the null hypothesis at the 5% level of significance



Example: Two-Sides Hypothesis Test

- we want to detect any changes from the value $p_\circ$
- we setup the null hypothesis $H_\circ : p = p_\circ$ against the alternative hypothesis $H_1 : p \neq p_\circ$

The problem

- a coin is tossed $n = 15$ times
- we observe $y = 10$ heads

Q: Is the coin fair ?

Frequentist approach

1 - setup a null hypothesis

$$H_\circ : p = 0.5$$

2 - we want to test it against the alternative hypothesis

$$H_1 : p \neq 0.5$$

3 - the null distribution is the sampling distribution of y: $\mathtt{Bin}(y \,|\, n = 15, p = 0.5)$

Example: Two-Sides Hypothesis Test (F)

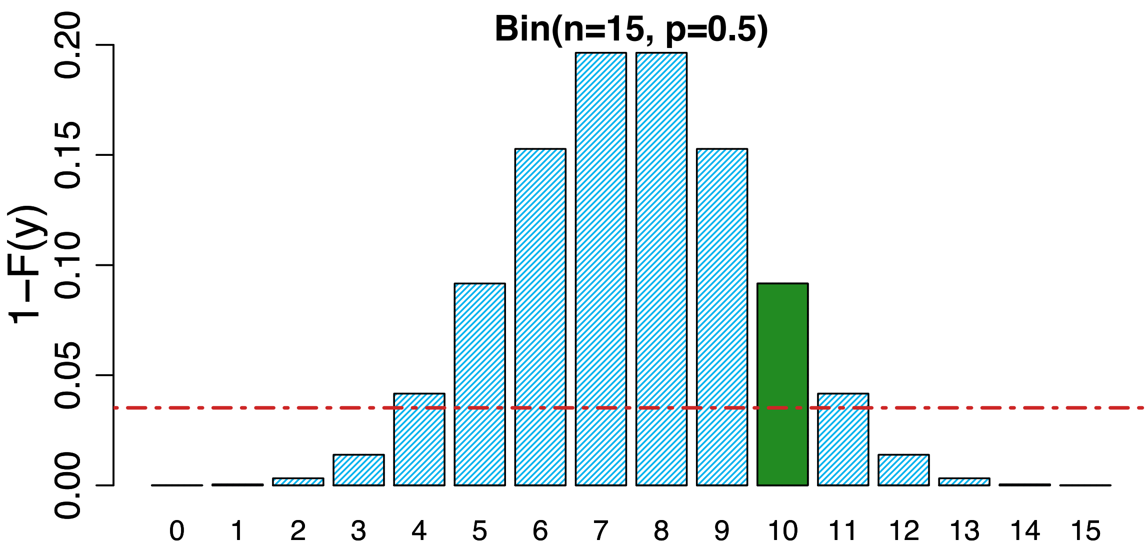
4 - in defining the rejection region, we take into account that y has a discrete distribution, and choose the level of significance as close to 5% as possible

$$\{y \leq 3\} \cup \{y \geq 12\} \text{ with } \alpha = 0.0352$$

5 - we observe $y = 10$, which lies inside the acceptance region

6 - we would have not rejected the null hypothesis also evaluating the `p-value`

$$P(y \geq 10) + P(y \leq 5) = 0.3018$$



Example: Two-Sides Hypothesis Test (B)

Bayesian approach

- the posterior distribution of the parameter, given the data, constraints our entire belief after getting the data
- but since the probability of an exact value represented by the point null hypothesis is zero

➜ need a correspondence similar to that of confidence intervals, using credible intervals

- we compute a $(1-\alpha) \times 100\%$ credible interval for p
- if $p_\circ$ lies inside the interval, we do not reject the null hypothesis, $H_\circ$; if it is outside, we reject $H_\circ$

The problem

- $n = 15$ coin tosses. We observe $y = 10$ heads

1 - set up a uniform prior $\texttt{Beta}(1, 1)$

2 - the posterior is $\texttt{Beta}(10+1, 5+1)$

3 - we calculate a 95% Bayesian credible interval

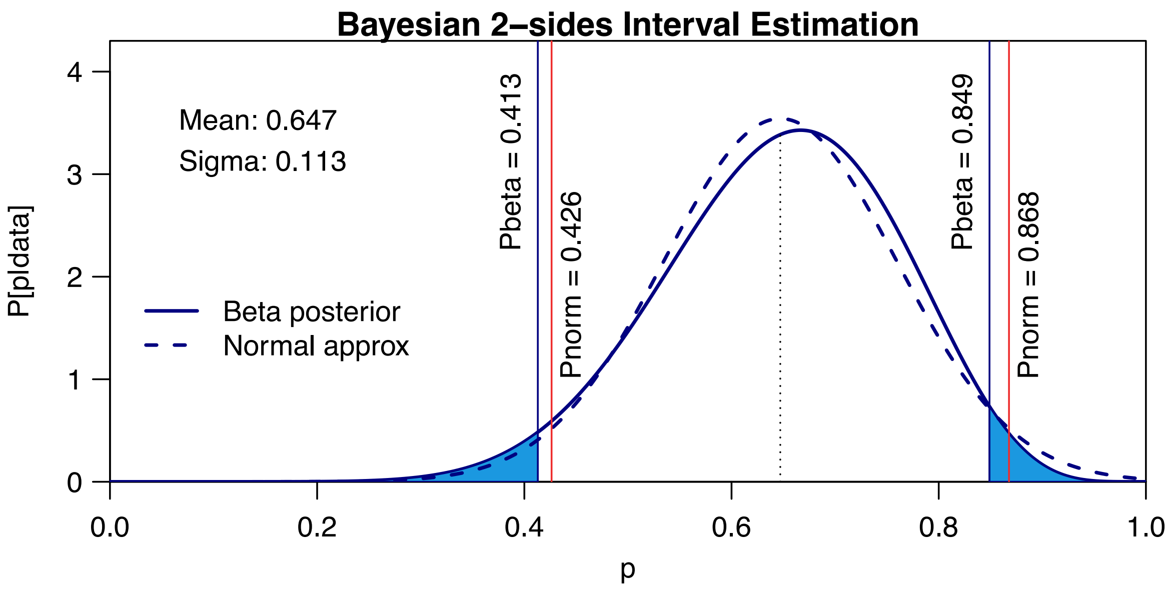
Example: Two-Sides Hypothesis Test (B)

4 - using a normal approximation we would get

$$\frac{11}{17} \pm 1.96 \times \sqrt{\frac{11 \times 6}{(11+6)^2(11+6+1)}} = 0.647 \pm 0.221$$

5 - our credibility interval is:

- $(0.413, 0.849)$, using a $\texttt{Beta}$ posterior
- $(0.426, 0.868)$, using a $\texttt{Normal}$ approximation



Some considerations on the *p*-value of the test

Neyman view

- the HT described does not make a distinction at all between a result that is barely significant and those highly significant
- let's run several HT on the same data:

Value of α	0.05	0.04	0.03	0.02	0.01
Reject H_0 ?	Y	Y	Y	N	N

- between 0.02 and 0.03 there is a value of α that would allow us to reject the NULL hypothesis

  the p-value is defined to be the smallest Type I error rate (α) that we are willing to tollerate if we want to reject the NULL hypothesis

- p summarizes all the possible hypothesis tests that we could have run:
  if $p \leq \alpha$ we would reject the NULL hypothesis

Some considerations on the *p*-value of the test

but

- the p value is not the probability that the NULL hypothesis is TRUE
- this statement is absolutely and completely wrong:

1) NULL Hypothesis tesing is a frequentist tool: we are not allowed to assign probability to a NULL hypothesis
   according to this view of probability, the NULL hypothesis is either TRUE or FALSE

Running HT in R

- R contains a whole lot of functions corresponding to different kinds of hypothesis test

```
binom.test(x=62, n=100, p=0.5)

        Exact binomial test

data:  62 and 100
number of successes = 62, number of trials = 100, p-value = 0.02098
alternative hypothesis: true probability of success is not equal to 0.5
95 percent confidence interval:
 0.5174607 0.7152325
sample estimates:
probability of success
                  0.62
```


Summary - global considerations

Frequentist paradigm

- it handles, separately, point estimation, confidence intervals and hypothesis tests
- the Frequentist statistics considers the parameter a fixed but unknown constant
- the sampling distribution of a statistic is its distribution over all the possible random samples, given the fixed parameter value
- the only probability allowed is a long-run relative frequency

Bayesian paradigm

- it bases all the estimates on the posterior distribution of the parameter

Summary - confidence/credibility intervals

Frequentist paradigm

- a $(1-\alpha) \times 100\%$ Frequentist interval for a parameter θ is an interval (θ_l, θ_h) such that
$$P(\theta_l \leq \theta \leq \theta_h) = 1 - \alpha$$
- $(1-\alpha) \times 100\%$ of the random intervals calculated this way do contain the true value ➜ we say we are $(1-\alpha) \times 100\%$ confident that the calculated interval contains the true parameter
- the `p-value` allows to reject the null hypothesis, at level α, if `p-value`$< \alpha$
- the `p-value` is not the probability the null hypothesis is true. It is the probability of observing what we observed given that the null hypothesis is true

Bayesian paradigm

- a $(1-\alpha) \times 100\%$ Bayesian credible interval for a parameter θ is a range of parameter values that has a posterior probability $(1-\alpha)$