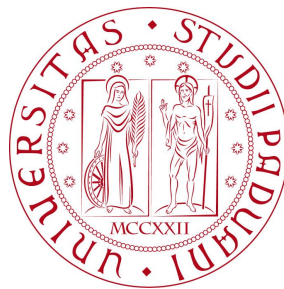


The 6 boxes toy model

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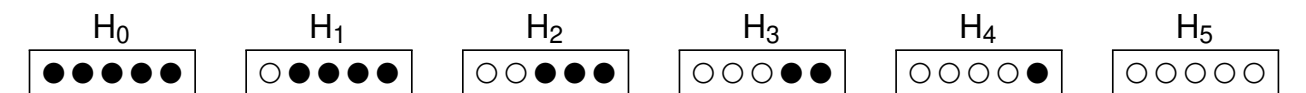
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The 6 Boxes Sampling Experiment

The Game

- 6 indistinguishable boxes are prepared with 5 black & white stone
- the composition differs for each box
- boxes are labeled H_j , according to the numbers of white stones in the box, with $j = 0, 1, \dots, 5$



The Rules of the Game

- we choose one box, randomly
- we try to infer the box content (i.e. the box id) by extracting at random one stone from the box
- the extracted stone is reinserted in the box (sampling with replacement)

The 6 Boxes Sampling Experiment

Our Background Information, I

- the following propositions are defined :

H_j : box j is selected ($j = 0, 1, \dots, 5$)

E_w : a white stone is extracted

E_b : a black stone is extracted

Our Quests

- 1) what is the probability of selecting one box ?
- 2) with the extraction of one stone, what is the probability of observing white, $P(E_w|I)$, or black, $P(E_b|I)$ on the next draw ?
- 3) how does the probability of the next extraction changes after the stone is extracted, and its color known ?

The space Ω of the events

- the following relations apply:

$$\bigcup_{j=0}^5 H_j = \Omega, \quad \text{and} \quad \bigcup_{k=b}^w E_k = \Omega$$

- in general, we are uncertain about all the combinations of E_k and H_j : the 12 constituents, $E_k \cap H_j$ do not share the same probability
- as an example:

$$P(E_w \cdot H_0|I) = 0, \quad P(E_w \cdot H_5|I) = 1$$

- E_k and H_j form a complete class of hypotheses, each event can be written as a logical sum of the constituents:

$$E_k = \bigcup_j (E_k \cap H_j), \quad \text{and} \quad H_j = \bigcup_k (E_k \cap H_j)$$

- since the events $E_k \cap H_j$ are mutually exclusive, by construction, we have:

$$P(E_k) = \sum_j P(E_k \cdot H_j|I) = \sum_j P(E_k|H_j I) P(H_j|I)$$

- and

$$P(H_j) = \sum_k P(H_j \cdot E_k|I) = \sum_k P(H_j|E_k I) P(E_k|I)$$

The Process of Knowledge

- E_k is an **observable effect**: we can experience it with our senses
- H_j is a **physical hypothesis**: it is not directly observable

Another rule of the game: we are not allowed to look inside the box !

→ H_j are the possible **causes of the effect**

- **Inference** : **guessing the causes from the effects**

Our experiment consists in

- 1 **extracting stones**, randomly and with replacement, **from an unknown box**
 - 2 **evaluating the probability** that the box is one of the six boxes
- aim of each measurement: **update our beliefs about each cause**, given all available information

and our calculations

- after the first extraction, $E^{(1)}$, we will compute:

$$P(H_j | E^{(1)} I)$$

- and, after the second extraction $E^{(2)}$:

$$P(H_j | E^{(1)} E^{(2)} I)$$

- and so forth
- what can be easily calculated is the probability of observing the different effects, giving each cause, $P(E_k | H_j I)$:

$$P(E_w | H_j I) = \frac{j}{5}, \quad \text{and} \quad P(E_b | H_j I) = 1 - P(E_w | H_j I) = \frac{5-j}{5}$$

and our calculations ...

- the product rule

$$\begin{aligned}P(E_k H_j | I) &= P(E_k | H_j I) P(H_j | I) \\ &= P(H_j | E_k I) P(E_k | I)\end{aligned}$$

- can be rewritten as

$$\frac{P(E_k | H_j I)}{P(E_k | I)} = \frac{P(H_j | E_k I)}{P(H_j | I)}$$

- we know $P(E_k | H_j I)$ and $P(E_k | I)$ can be evaluated as:

$$P(E_k | I) = \sum_j P(E_k | H_j I) P(H_j | I) = \frac{0 + 1 + 2 + 3 + 4 + 5}{5} \cdot \frac{1}{6} = \frac{1}{2}$$

- as we would expect

and our calculations

- we can rewrite the product rule as

$$\frac{P(H_j | E_k I)}{P(H_j | I)} = \frac{P(E_k | H_j I)}{P(E_k | I)} = 2 \cdot P(E_k | H_j I)$$

- in case of a white stone, $P(E_w | I) = 1$,

$$\frac{P(H_j | E_w I)}{P(H_j | I)} = 2 \cdot \frac{j}{5}$$

- while, for a black stone, $P(E_b | I) = 1$,

$$\frac{P(H_j | E_b I)}{P(H_j | I)} = 2 \cdot \frac{5-j}{5}$$

- putting all the ingredients together, we get Bayes' theorem

$$P(H_j | E_k I) = \frac{P(E_k | H_j I) P(H_j | I)}{\sum_j P(E_k | H_j I) P(H_j | I)}$$

- the denominator is just a normalization factor, and we can simply write:

$$P(H_j | E_k I) \propto P(E_k | H_j I) P(H_j | I)$$

- or, in clear text

$$\text{Posterior} \propto \text{Likelihood} \times \text{Prior}$$

- Bayes' theorem is simply a compact representation of what has been done in the previous steps.
- it is [a formal tool for updating beliefs using logic instead of only intuition](#)

Running the experiment

- we randomly select a box, and start to sample stones from the box
- after each extraction, we update the probabilities of each hypothesis, using Bayes' theorem:

$$P(H_j | I_n) = \frac{P(E^{(n)} | H_j I_{n-1}) P(H_j | I_{n-1})}{\sum_l P(E^{(n)} | H_l I_{n-1}) P(H_l | I_{n-1})}$$

- where $E^{(n)}$ refers to the n -th extraction,
- $P(E^{(n)} | H_j)$ have been computed before:

$$P(E_w^{(n)} | H_j) = \frac{j}{5}, \quad P(E_b^{(n)} | H_j) = \frac{5-j}{5}$$

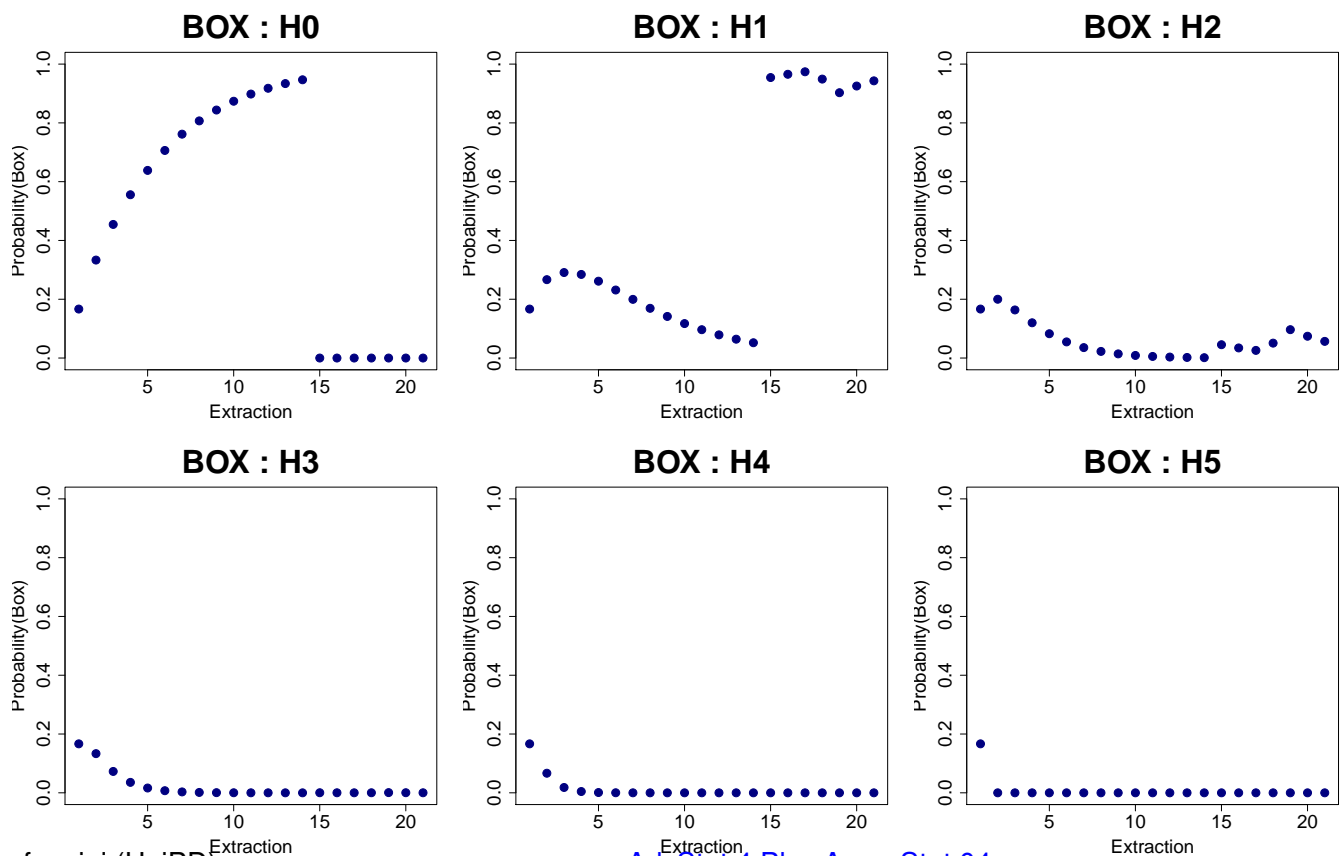
- and $P(H_j | I_{n-1})$ have been given by the calculations at extraction $(n-1)$ -th

Running the experiment

Trial	E	H_0	H_1	H_2	H_3	H_4	H_5	$P(E_w I_n)$
0	-	0.167	0.167	0.167	0.167	0.167	0.167	0.5
1	B	0.33	0.27	0.2	0.13	0.06	0	0.27
2	B	0.45	0.29	0.163	0.073	0.0182	0	0.18
3	B	0.55	0.28	0.12	0.036	0.004	0	0.13
4	B	0.64	0.26	0.08	0.016	0.001	0	0.096
5	B	0.71	0.23	0.05	0.007	2.2E-4	0	0.072
6	B	0.76	0.20	0.04	0.003	4.9e-5	0	0.056
7	B	0.81	0.17	0.02	0.001	1.0e-5	0	0.044
8	B	0.84	0.14	0.01	5.5e-4	2.2e-6	0	0.034
9	B	0.87	0.12	0.009	2.3e-4	4.5e-7	0	0.027
10	B	0.90	0.10	0.005	9.4e-5	9.2e-8	0	0.022
11	B	0.92	0.08	0.003	3.8e-5	1.9e-8	0	0.017
12	B	0.93	0.06	0.002	1.6e-5	3.8e-9	0	0.014
13	B	0.95	0.05	0.001	6.3e-6	7.8e-10	0	0.011
14	W	0	0.95	0.045	3.5e-4	5.7e-8	0	0.21
20	B	0	0.93	7.4e-2	3.8e-4	1.4e-8	0	0.21
40	W	0	0.998	1.4e-3	7.1e-9	8.7e-19	0	0.20

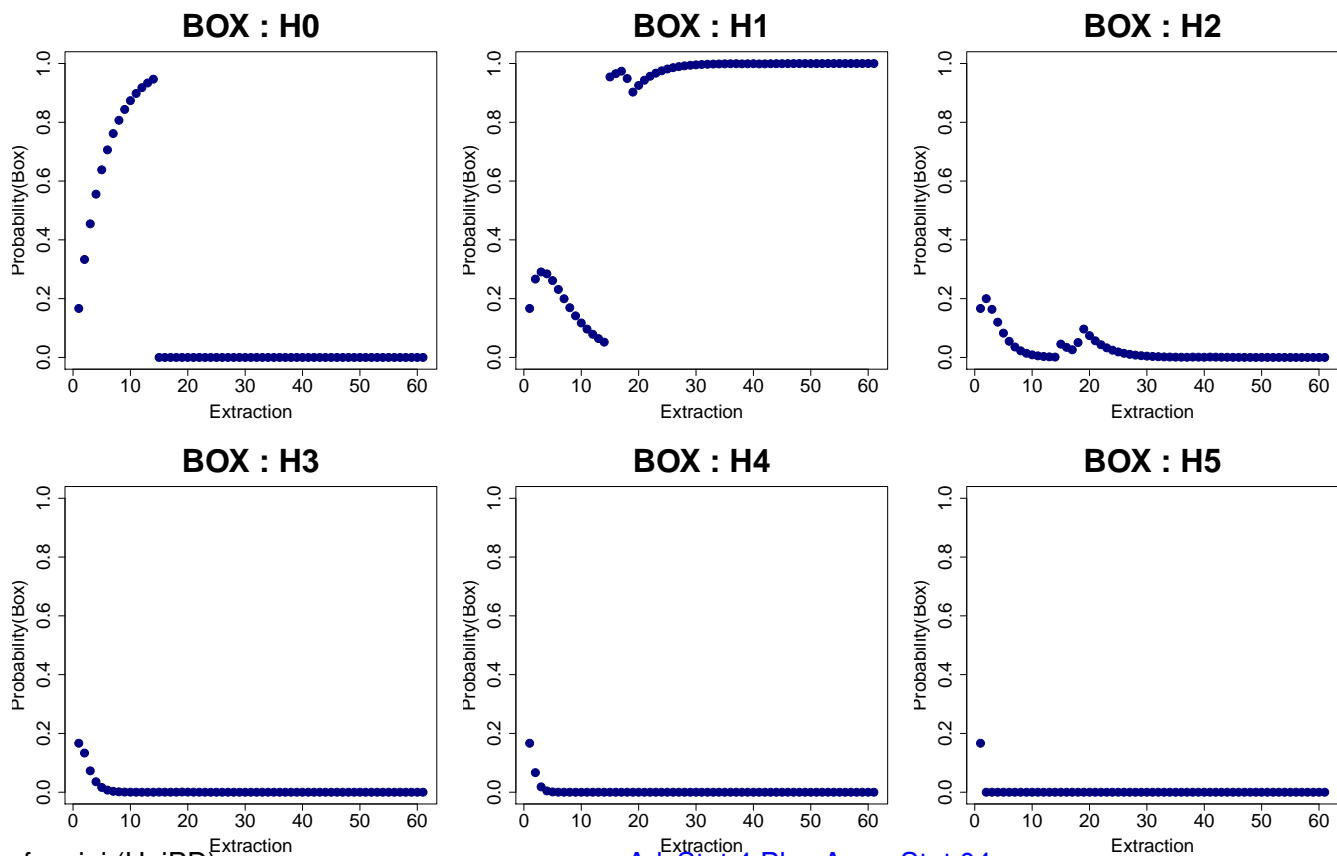
Run results : 20 samplings

- Run performed with `set.seed(89540)`
- important extraction at round 14



Run results : 60 samplings

- Box H_1 is the most probable : $\bigcirc \bullet \bullet \bullet \bullet$ $P(E_w|I_n) = 0.2$, as expected



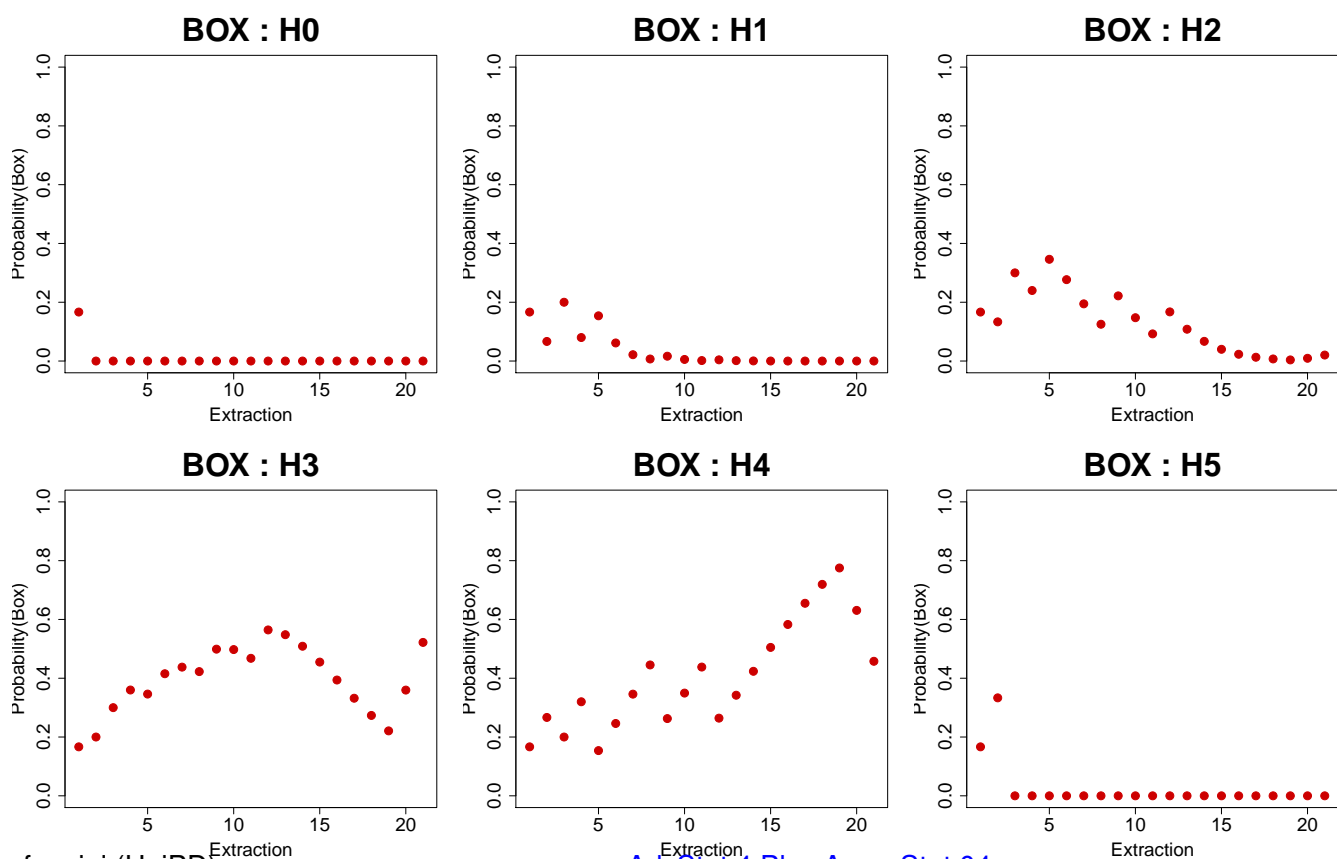
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New run results: 20 samplings

- Run performed with `set.seed(89540)`
- most flavored oscillates between H_3 and H_4



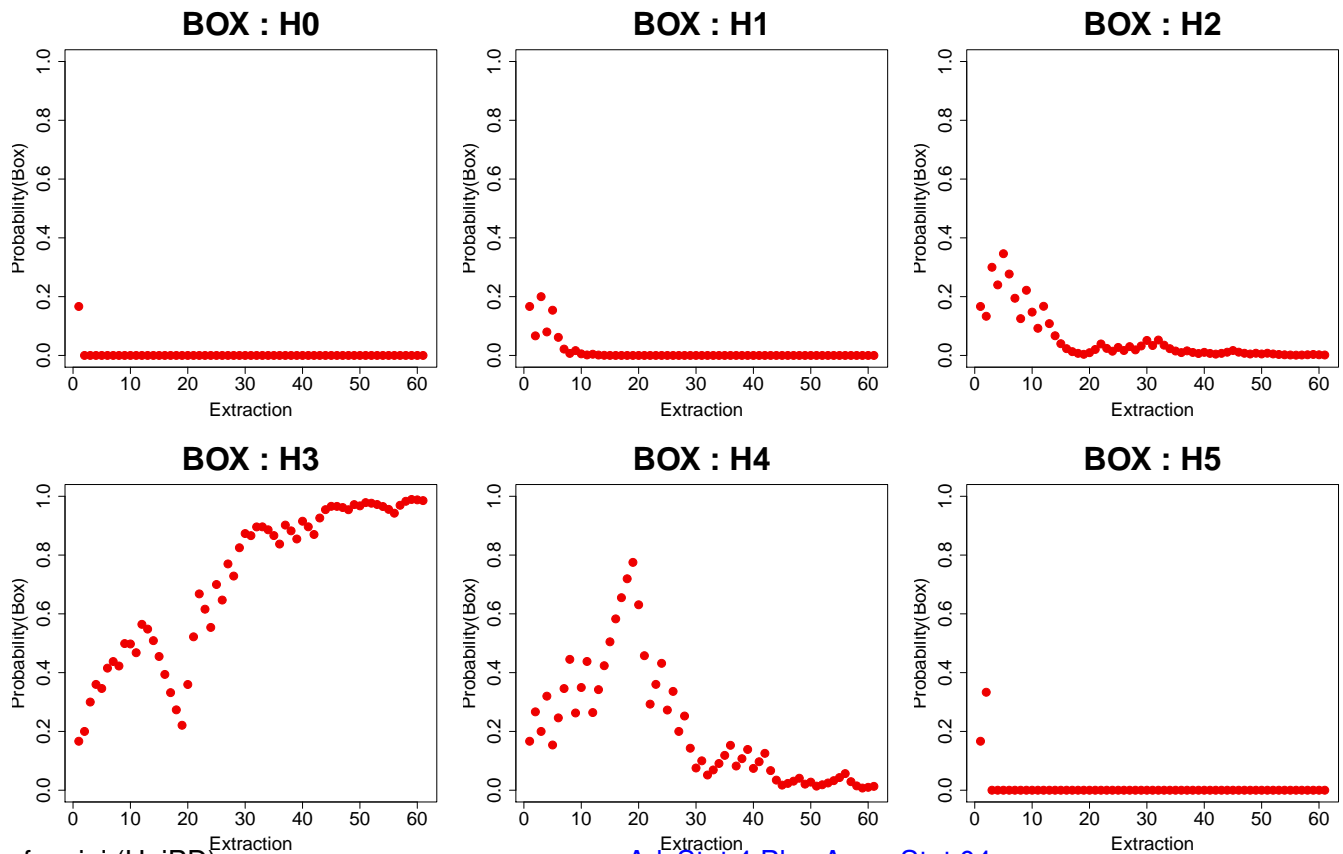
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New run results : 60 samplings

- Box H_3 is the most probable : ☐ ☐ ☐ ☒ ☒ $P(E_w|I_n) = 0.6$, as expected



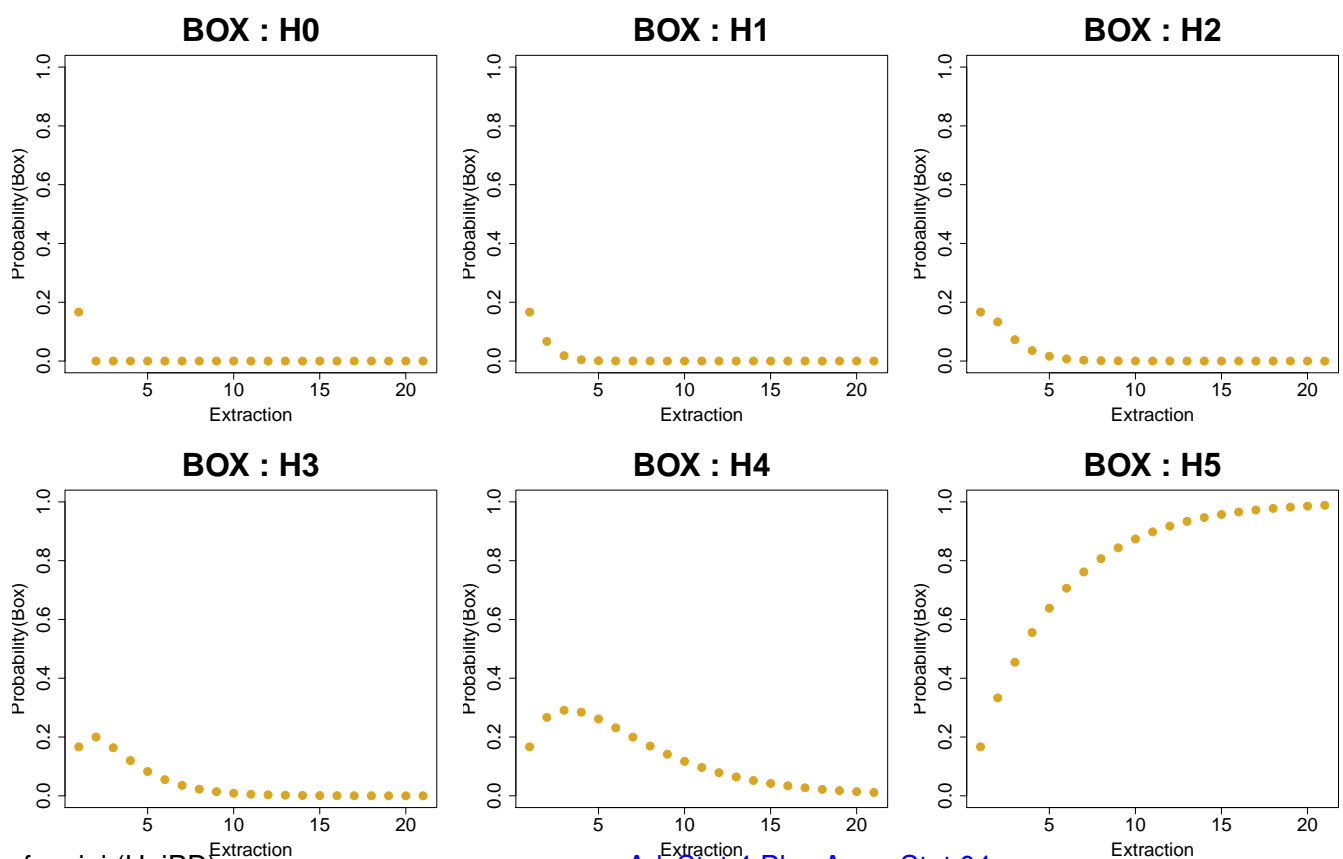
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Run with an extreme box

- Run performed with `set.seed(89540)` and box ☐ ☐ ☐ ☐ ☐



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Articles

- G. D'Agostini, *Teaching statistics in the physics curriculum: Unifying and clarifying role of subjective probability*, Am. Jour. Phys. 67, 1260 (1999), [arXiv:physics/9908014](https://arxiv.org/abs/physics/9908014)
- G. D'Agostini, *More lessons from the six box toy experiment*, [arXiv:1701.01143](https://arxiv.org/abs/1701.01143)
- G. D'Agostini, *Probability, propensity and probabilities of propensities (and of probabilities)*, [arXiv:1612.05292](https://arxiv.org/abs/1612.05292)

Additional Material

- G. D'Agostini Web Page at University of Rome, La Sapienza, <http://www.roma1.infn.it/~dagos/teaching.html>