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We introduce a framework for the study of 3-D electromagnetic wave propagation in smoothly-varying inhomogeneous, time-independent,linear media. Variation in dielectric permittivity and magnetic permeability are assumed to take place in the z direction only.

The framework is consists of a system of three scalar, damped linear wave equations of variable coefficients. Scalar form is obtained by careful expansion of Maxwell's equations full curl-curl form. Full uncoupling of the system is achieved, and assumptions applied to acquire a more traceable 1-D variation - governed model are stated.

Profile-dependent dimensional analysis of the non-transverse field component equation allows qualitative analysis of solution's spatial behaviour in the time-harmonic form of the equations . Dimensional grouping yields valid comparisons of parameter magnitudes. Numerical simulations demonstrate variation of solution features corresponding to different asymptotic assumptions.

Localized qualitative change in spatial solution behaviour occur and can be tracked by the governing PDE signature. While sigmoidal dielectric profiles exhibit behaviour approximated by a limiting step-function profiles, Gaussian profiles may change the governing equation's signature, yielding abrupt change in the solution characterizing properties.

I. INTRODUCTION

I.1. Problem formulation

The problem of electromagnetic waves propagation in inhomogeneous media is a the subject of a vast variety of books, research monographs and papers. In its most basic form, the problem is treated in electrodynamics textbooks [4, 6] in the context of plane wave propagation in a medium having one or more jump discontinuities in its dielectric permittivity or its magnetic permeability. The classical problem approach for solution is based on applying electromagnetic boundary conditions to stitch phasor-formed solution for Maxwell's equations generated at either side of the jump discontinuity. A peer into the literature shows that this is the dominant point of view taken in more advanced studies as well. Typically, material inhomogeneity is approximated by a stratified media of many layers.

I.2. Advantages of asymptotic methods

Uniformity
Simplicity; physical interpretation

I.3. Literature

Key points for comparison:

1. treatment of the equations - direct derivation of the electric field equations? or through through magnetic vector potential?
2. is there dimensional analysis? to what extent?
3. What types of boundary conditions are treated? and how?
4. closed form solution vs. perturbation methods, dependency on dimensional analysis.
5. Easily interpreted physical meaning:
   (a) Ability to treat arbitrary dielectric / magnetic profiles, subject to smoothness constraints only.
   (b) Solution formed as left and right travelling waves.
   (c) Small number of parameters.
6. Treated use cases.

Books:

1. Brekhovskikh's monograph [3], chapter 3, is a standard reference for our subject matter.
   (a) Terms generated by inhomogeneity in Maxwell's equations appear only for the constant magnetic permeability case. Some terms that appear by varying magnetic permeability are dropped. See also Goodman [5], for a similar formulation closer to the one presented in our work.
   (b) While elements of deriving second order equations for the electric field appear, an approach applying the calculation of vector potential is favoured.

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(c) In some cases, assumption regarding slow variation is stated, but results in the introduction of a 'small parameter', lacking a comprehensive dimensional analysis.

(d) The WKB approach is introduced and derived from first principle, but no use case presented. Might be due to the vague definition of slow varying equation coefficients make its utility unclear.

(e) Use-cases are derived by mapping of the hypergeomeric equation to variable harmonic oscillator, to get closed-from solutions. The problem is that there is no direct relation between a pre-set dielectric profile and a solution method.

2. Ishimuru: 3-13 waves in inhomogeneous media, 3-14 WKB method, 3-15 Bremmer series

   (a) The generalized wave equation derived assuming constant magnetic permeability

   (b) Error bound for WKB given, but without dimensional analysis.

   (c) No treatment of boundary conditions

   (d) Bremmer series: replacing WKB integration constants by z-varying functions. Bootstraping process generates series representation, converging under assumption on wave number variation. Point being: get all the next terms to control error. Contrasting asymptotic theory spirit - optimize performance, even for divergent series. Further, in real-life physical scenario, high order terms may be compatible in size to noise.

3. Marcuse chap. 7 - Light propagation in square law media - graded in dex optics in the case of axially-symmetric materials, chap 8 (Maxwell's equns in inhomogeneous media) - 'imperfection' is z-variation in the medium. Since 'hard' to cope, approximation by step functions is recommended.

4. Frederick Wooten chap 2 - wave eqn for absorbing medium

5. Landau electrodynamics of continuous media - modelling the frequency dependence of the refractive index

6. Dressel-Gruner - Electrodynamics of solids - wave eqn in inhomogeneous media - spatial variations

7. Antonov-Harmon-Yaresko

8. Felsen - Marcuvitz

I.4. Applications

1. Bass - Handbook of optics - vol 2. inside: Duncan T Moore - gradient index optics. Axial gradients - correction of aberrations. Referring to Sands: any aspheric surface can be converted to a graded index profile. Difference - dependence of the refractive index in frequency. Thus the convertion can be applied only in the monochromatic case.

2. Brekhovskikh [3] chapter 2.

I.5. Novelty

1. Full derivation of electric field equations for both z-varying dielectric permittivity and magnetic permeability.

2. Uncoupled, scalar PDEs for all both non-constant electric permittivity and magnetic permeability.

3. Dimensional analysis: strict definition of 'slow' and 'fast' variation of parameters.

4. Classification of parameter configurations where certain asymptotic methods are applicable.

5. Straight-forward definition of reflection-transmission relation into and out of an inhomogeneous layer.

6. Easily physical interpretation due to approximate phasor, left and right travelling wave solutions.

7. Characterization of highly complex profiles using a small number of parameters.1

II. SCALAR FORM OF THE GENERALIZED WAVE EQUATIONS IN SMOOTHLY VARYING MEDIA

II.1. Constructing the generalized scalar wave equations

Let $\overrightarrow{E}(x,y,z,\omega) = \hat{E}(x,y,z)e^{i\omega t}$, $\overrightarrow{H}(x,y,z,\omega) = \hat{H}(x,y,z)e^{i\omega t}$ be time-harmonic electric and magnetic fields, respectively, satisfying Maxwell's equations

$$\nabla \times \overrightarrow{H} - i\omega\overrightarrow{D} = \overrightarrow{J} \tag{1a}$$

$$\nabla \times \overrightarrow{E} + i\omega\overrightarrow{B} = 0 \tag{1b}$$

$$\nabla \cdot \overrightarrow{D} = \rho \tag{1c}$$

$$\nabla \cdot \overrightarrow{B} = 0 \tag{1d}$$

Assume a linear, isotropic material, so that the flux density vectors satisfy

$$\overrightarrow{D} = \epsilon \overrightarrow{E} \tag{2a}$$

$$\overrightarrow{B} = \mu \overrightarrow{H} \tag{2b}$$

We allow material inhomogeneity: $\mu = \mu(x, y, z)$, $\epsilon = \epsilon(x, y, z)$, where ϵ and μ are smooth functions of x, y, z. For convenience we define $\mathcal{M} = \log(\mu)$, $\mathcal{E} = \log(\epsilon)$.
Maxwell's equations for inhomogeneous medium in vector form (compare [5], chap. 3)

$$\nabla(\nabla \cdot \overrightarrow{E}) - \nabla(\mathcal{M}) \times \left(\nabla \times \overrightarrow{E}\right) - \nabla^2 \overrightarrow{E} - \epsilon\mu\omega^2 \overrightarrow{E} + i\omega\mu \overrightarrow{J} = 0 \tag{3}$$

are derived directly from Maxwell's equations using the curl-curl formulation. The first and second terms at the left hand side satisfy

$$\nabla \left(\nabla \cdot \overrightarrow{E}\right)_{x_j} = \frac{\partial}{\partial x_j} \left(\frac{\rho}{\epsilon}\right) - \left(\frac{\partial \overrightarrow{E}}{\partial x_j} \cdot \nabla \mathcal{E} + \overrightarrow{E} \cdot \frac{\partial}{\partial x_j} \nabla \mathcal{E}\right) \tag{4}$$

$$\left(\nabla \mathcal{M} \times \left(\nabla \times \overrightarrow{E}\right)\right)_{x_j} = \nabla \mathcal{M} \cdot \left(\frac{\partial \overrightarrow{E}}{\partial x_j} - \nabla E_{x_j}\right) \tag{5}$$

where $x_j = x, y, z$ for $j = 1, 2, 3$, respectively. Identity (4) is derived using the constitutive relation $\nabla \cdot \overrightarrow{D} = \rho$. Identity (5) is a non-trivial result of the Levi-Civita symbol calculus.
Note that if ϵ and μ are constant, equation (3) is the well-known system of uncouples standard wave equations.

II.2. System uncoupling for x,y - independent medium

Assume that μ, ϵ are x, y-independent. As in the constant ϵ, μ case, the system (3) can be uncoupled in the following sense: the E_z equation is

$$\mathcal{E}'' E_z + \mathcal{E}' E_z' + \nabla^2 E_z + \epsilon\mu\omega^2 E_z = \left(\frac{\rho}{\epsilon}\right)' + i\omega\mu J_z \tag{6}$$

where the primes denote z derivatives. Note that term (5) contribution cancels due to μ strict z-dependency.
In such case, the z-component equation is a scalar PDE in E_z. Hence, if $\mu, \epsilon, \rho, \frac{\partial J_z}{\partial t}$ are known, (6) is decoupled from the system (3) and can be solved in a stand-alone manner.
For $x_1 = x$, $x_2 = y$, we get the equations

$$\mathcal{M}' \left(\frac{\partial E_{x_j}}{\partial z} - \frac{\partial E_z}{\partial x_j}\right) + \mathcal{E}' \frac{\partial E_z}{\partial x_j} + \nabla^2 E_{x_j} + \epsilon\mu\omega^2 E_{x_j} = i\omega\mu J_{x_j} \tag{7}$$

Equations (7) are coupled with the E_z equation but not among themselves. Thus, once E_z is obtained, it can be substituted into equations (7), each becoming a stand-alone scalar PDE in E_{x_j}, achieving total uncoupling of (3).

III. SYMMETRY AND COHERENCE ASSUMPTIONS YIELD REDUCTION OF PDES TO ODES

Assume an x, y spatially coherent source incident to an infinite slab $-L \leq z \leq L$, $-\infty < x, y < \infty$, at $z = -L$. The profiles $\epsilon = \epsilon(z) \in C^2(-L, L), \mu = \mu(z) \in C^1(-L, L)$ are assumed to have strictly positive real parts. We shall also assume sufficiently differentiable, x, y-independent ρ, $\overrightarrow{J}$.

Spatial coherence and x, y independence of both equation (6) coefficients and inhomogeneous terms yield x, y invariance, causing the x, y derivatives in the PDE to vanish. The PDE is thus reduced to the ODE

$$\hat{E}_z'' + \mathcal{E}' \hat{E}_z' + \left(\mathcal{E}'' + \epsilon\mu\omega^2\right) \hat{E}_z = \left(\frac{\rho}{\epsilon}\right)' + i\omega\mu J_z \tag{8}$$

PDE (7) turns into an even simpler form, since $\hat{E}_z$ is x, y independent, and therefore $\frac{\partial \hat{E}_z}{\partial x_j} = 0$. Thus the PDE is mapped to the ODE

$$\hat{E}_{x_j}'' + \mathcal{M}' \hat{E}_{x_j}' + \epsilon\mu\omega^2 \hat{E}_{x_j} = i\omega\mu J_{x_j} \tag{9}$$

having no explicit $\hat{E}_z$ dependency.

IV. REFLECTION-TRANSMISSION FORMULATION FOR AN INHOMOGENEOUS LAYER

IV.1. Field components matching at the boundaries

Let the incidence wave be normal to the boundary surface $z = -L$. The media to the left and right of the layer $-L \leq z \leq L$ have constant dielectric profile ϵ_1, ϵ_2 respectively. The layer's z-dependent dielectric profile is simply ϵ. The electric and magnetic fields are denoted $\overrightarrow{E}_1$, $\overrightarrow{H}_1$, $\overrightarrow{E}$, $\overrightarrow{H}$, $\overrightarrow{E}_2$, $\overrightarrow{H}_2$ at $z < -L$, $-L < z < L$, $z > L$, respectively.
As long as analytic methods only are applied for solution assessment, no artificial boundary conditions are required to model 'free space interface'.
Boundary conditions at the interfaces $z = \pm L$ are determined classically, as their derivation is local in space. Further, by assumption, no net charge is accumulated on the boundary surfaces and no current flows along them. Thus

- the $\overrightarrow{E}$ tangential component is continuous across the interfaces. For simplicity, we discuss (from now on) the x-polarized case,

$$E_{1x}|_{z=-L^-} = E_x|_{z=-L^+} \tag{10a}$$

$$E_x|_{z=-L^-} = E_{2x}|_{z=-L^+} \tag{10b}$$

- The $\overrightarrow{E}$ normal component satisfies

$$\epsilon_1 E_{1z}|_{z=-L^-} = \epsilon_{z=-L^+} E_z|_{z=-L^+} \tag{11a}$$
$$\epsilon_{z=L^-} E_z|_{z=L^-} = \epsilon_2 E_z|_{z=L^+} \tag{11b}$$

- The $\overrightarrow{H}$ tangential component is continuous (as the current $J_s = 0$)

$$H_{1y}|_{z=-L^-} = H_{1y}|_{z=-L^+} \tag{12a}$$
$$H_{1y}|_{z=L^-} = H_{1y}|_{z=L^+} \tag{12b}$$

- The $\overrightarrow{H}$ normal components satisfies, in the general case,

$$\mu_1 H_{1z}|_{z=-L^-} = \mu_{z=-L^+} H_z|_{z=-L^+} \tag{13a}$$
$$\mu_{z=L^-} H_z|_{z=L^-} = \mu_2 H_{2z}|_{z=L^+} \tag{13b}$$

  Since $\mu(z)$ is assumed constant in the profiles studied in this work , these relations reduce to continuity.

IV.2. The Boundary Value Problem Setting

Consider equation (9) for a tangentially-polarized wave, for which equation (22) is a dimensionally-grouped special case. Assume that the exists an analytic expression, exact or approximate, for left and right travelling wave solutions within the layer.
If a monochromatic source of frequency ω is located at $z \to -\infty$ and no sources exist for $z > L$, we have the decomposition

$$E_{1x} = E_i e^{-ik_1 z} + E_r e^{ik_1 z} \tag{14}$$

where $E_i e^{-ik_1}$ and $E_r e^{ik_1}$ are phasor forms of the incidence and reflected waves respectively, with $k_1 = \omega\sqrt{\mu\epsilon_1}$;

$$E_x(z) = E_L W_L(z) + E_R W_R(z) \tag{15}$$

where $W_L(z)$, $W_R(z)$ are independent, left and right travelling wave solutions for the Helmholtz equation in the layer, and

$$E_{2x} = E_t e^{-ik_2 z} \tag{16}$$

is the transmitted, pure right - travelling wave, with $k_2 = \omega\sqrt{\mu\epsilon_2}$.
The incidence wave constant multiplier E_i is known. The rest of the constants: E_r, E_L, E_R, E_t are complex values to be determined. Thus four equations are should be derived from the boundary conditions.
As in the classical theory, two equations are derived directly from the electric field tangent component boundary conditions at $z = \pm L$. Two extra equations are derived by converting the electric into magnetic field using Maxwell's equation (1b), and casting the corresponding boundary conditions. We get the system of equations

$$\begin{pmatrix} -e^{-iLk_1} & W_L(-L) & W_R(-L) & 0 \\ k_1 e^{-iLk_1} & iW_L'(-L) & iW_R'(-L) & 0 \\ 0 & W_L(L) & W_R(L) & -e^{-iLk_2} \\ 0 & iW_L'(L) & iW_R'(L) & -k_2 e^{-iLk_2} \end{pmatrix} \begin{pmatrix} E_r \\ E_L \\ E_R \\ E_t \end{pmatrix} = \begin{pmatrix} e^{iLk_1} E_i \\ k_1 e^{iLk_1} E_i \\ 0 \\ 0 \end{pmatrix} \tag{17}$$

Note that both the matrix and the input vector are complex valued. Hence we shall typically have a complex-valued solution vector. The computed components E_r, E_L, E_R, E_t therefore yield both amplitude and phase information.

IV.3. Summarizing Remarks

While the inner-layer solution may be written as a superposition of two any two linearly independent solutions for equation (8), the special form of left and right travelling waves simplifies the analysis of electromagnetic boundary conditions.
This idea is clearly not new, as it appears in all piecewise-constant media 1-D electrodynamic boundary value problems textbook examples. In our case, conceptual adequacy of selecting the ''proper" solution pair yields a simple, physically tractable matrix representation of the solution matching at the boundaries.
Notice that from this point of view, approximate solution pairs having the opposing travelling wave property are more beneficial than exact, closed-form solution given in terms of special functions, that may not have this property.
Indeed, we shall see that several classes such approximate solution pairs do exist, and are valid in a wide range of physically applicable parameter ranges.

V. ASYMPTOTIC ANALYSIS IN A FOR A SIGMOIDAL, Z-DEPENDENT INHOMOGENEOUS LAYER

The formulations (3) is general enough to handle any twice-differentiable dielectric profile. Yet, to get qualitative insight about the solution behaviour using asymptotic analysis, dimensional analysis must be performed for a given profile. In this sense, strict meaning can be attached to 'large' and 'small' parameters statements, allowing correct selection of asymptotic analysis technique.
A sigmoidal structure was selected since analysis results may, for some parameter settings, be compared to the simple single-jump step function case.

V.1. Construction of a Sigmoid dielectric Profile

As commonly assumed in optics, we consider a source-free (i.e. $\rho = 0, \overrightarrow{J} = 0$) layer $-L \leq z \leq L, -\infty < x, y < \infty$, in which μ is constant. The dielectric profile $\epsilon(z)$ interpolates smoothly and monotonically two values of the dielectric profile so that $\epsilon(-L) = \epsilon_1$ and $\epsilon(L) = \epsilon_2$. The sigmoid is a non-linear function, hence should have a dimensionless argument. To do so, we take some sigmoid-like function $c(\cdot)$ and normalize its argument by a constant M, yielding $c\left(\frac{z}{M}\right)$. The interpolating function is

$$\epsilon(z) = \frac{\epsilon_L \mathrm{c}\left(\frac{L}{M}\right) - \epsilon_R \mathrm{c}\left(-\frac{L}{M}\right) + (\epsilon_R - \epsilon_L)\mathrm{c}\left(\frac{z}{M}\right)}{\mathrm{c}\left(\frac{L}{M}\right) - \mathrm{c}\left(-\frac{L}{M}\right)} \tag{18}$$

which can be substituted into (8), (9). The resulting equations coefficients turn out to be cumbersome and not suggestive to interpretation. A more compact and meaningful form will now be presented next.

V.2. Dimensional Analysis for a Sigmoidal Profile

Defining the dimensionless quantities

$$s = z/M \tag{19a}$$
$$a = L/M \tag{19b}$$
$$r = (\epsilon_R - \epsilon_L)/\epsilon_L \tag{19c}$$
$$\Omega = M\omega\sqrt{\epsilon_L \mu} \tag{19d}$$

In these terms, letting

$$q(s) = rc(s) + c(a) - (r+1)c(-a) \tag{20}$$

yields

$$\begin{aligned}
\frac{d}{dz} &\rightarrow \frac{1}{M}\frac{d}{ds} \\
\frac{d^2}{dz^2} &\rightarrow \frac{1}{M^2}\frac{d^2}{ds^2} \\
\mathcal{E}' &\rightarrow \frac{rc'(s)}{Mq(s)} \\
\mathcal{E}'' &\rightarrow \frac{rc''(s)}{M^2 q(s)} - \left(\frac{rc'(s)}{Mq(s)}\right)^2 \\
\epsilon\mu\omega^2 &\rightarrow \frac{q(s)\Omega^2}{M^2(c(a) - c(-a))}
\end{aligned}$$

The scaling of the z coordinate by M factors it out in the homogeneous case. Equation (8) is thus transformed into

$$\frac{d^2\hat{E}_z}{ds^2} + P(s)\frac{d\hat{E}_z}{ds} + Q(s)\hat{E}_z = 0 \tag{21}$$
$$P(s) = \frac{rc'(s)}{q(s)}$$
$$Q(s) = \frac{rc''(s)}{q(s)} - \left(\frac{rc'(s)}{q(s)}\right)^2 + \frac{q(s)\Omega^2}{c(a) - c(-a)}$$

and equation (9) becomes

$$\frac{d^2\hat{E}_{x_j}}{ds^2} + \frac{q(s)\Omega^2}{c(a) - c(-a)}\hat{E}_{x_j} = 0 \tag{22}$$

VI. ASYMPTOTIC APPROXIMATION FOR THE SOLUTION IN A SIGMOIDAL DIELECTRIC PROFILE

We consider the sigmoid

$$c(z) = \frac{1}{1 + e^{-z}} \tag{23}$$

VI.1. Thin Layer - Small Effective Width

A relatively simple case in that of a thin layer, $a \ll 1$. s is small as well, since $-a < s < a$. Expanding The E_{x_j} coefficient in equation (22) to series in both a, s and dropping terms of order higher than 1,

$$\frac{q(s)}{c(a) - c(-a)} \approx \left(r\left(\frac{s}{2a} + \frac{1}{2}\right) + 1\right) = Q(s) \tag{24}$$

yielding the approximation equation

$$\frac{d^2E_x}{ds^2} + \left(r\left(\frac{s}{2a} + \frac{1}{2}\right) + 1\right)\Omega^2 E_x = 0 \tag{25}$$

Note that $Q(s)$ in (24) can become arbitrarily large only by increase of r.

VI.2. Parameter Magnitude Characterization - a Joined Scales Analysis

In order to select an appropriate asymptotic approximation method, a measure of terms magnitude. Since the parameters r,a, Ω are dimensionless, it is legitimate to let

$$r = \delta^\alpha \tag{26a}$$
$$a = \delta^\beta \tag{26b}$$
$$\Omega = \delta^\gamma \tag{26c}$$

where δ is a small real number.Here we assume $r > 0$. The case $r < 0$ can be treated similarly. Since $-a \leq s \leq a$, we define also

$$s = \delta^\beta \xi \tag{27}$$

Equation (22) is transformed into

$$\frac{d^2E_x}{d\xi^2} + \left(\left(\frac{\xi}{2} + \frac{1}{2}\right)\delta^\alpha + 1\right)\delta^{2(\beta+\gamma)}E_x = 0 \tag{28}$$

Regular perturbation holds when $\beta+\gamma \geq 0$, and $a > 0$, namely $a\Omega = O(1)$, and $r \ll 1$.

The WKB method requires a factored E_x coefficient of the form $Q(s)\Omega^2$, where $Q(s)$ is slowly varying and Ω^2 is large. Clearly, these conditions are satisfied for $\beta+\gamma < 0$, $\alpha > 0$, yet better insight can be obtained using by connecting α to β and γ through the WKB relative error term [2]. Letting

$$S_2(\xi) = \frac{\frac{d^2Q}{d\xi^2}}{8Q^{3/2}} - \frac{5\left(\frac{dQ}{d\xi}\right)^2}{32Q^{5/2}} \tag{29}$$

the uniform relative error bound is

$$\exp\left(\frac{S_2(\xi)}{\Omega}\right) \approx 1 + \frac{S_2(\xi)}{\Omega}, \qquad \Omega \gg 1 \tag{30}$$

Hence in order to verify that the WKB approximation holds, we should show that $\left|\frac{S_2}{\Omega}\right| \ll 1$ uniformly in the interval.
For $Q(\xi)$ being the $E_x(\xi)$ from equation

$$\frac{S_2}{\Omega} = \frac{5\delta^{\alpha+\gamma}}{24\sqrt{2}\left((\xi+1)\delta^{\alpha}+2\right)^{3/2}} \tag{31}$$

Hence further restriction, $\alpha+\gamma > 0$, is required for the WKB approximation to hold.

VI.3. Formal Regular Perturbation Expansion: low frequency, small effective dielectric dynamic range

We shall later show the meaning of Ω being $O(1)$, but for now let us assume that this is the case, and that $|r| \ll 1$.
Plugging the perturbation series

$$E_x(s) = E_x^0(s) + rE_x^1 + r^2E_x^2 + \dots \tag{32}$$

where j in E_x^j is an index, into equation (25) and collecting $O(r^n)$, terms $n = 0, 1, \dots$ we get a sequence of equations

$$\frac{d^2E_x^0}{ds^2} + \Omega^2E_x^0 = 0 \tag{33a}$$

$$\frac{d^2E_x^1}{ds^2} + \Omega^2E_x^1 = -\frac{\Omega^2(a+s)}{2a}E_x^0 \tag{33b}$$

$$\dots$$

All equations are linear, of constant coefficients. The $O(r^0)$ equation is homogeneous, corresponding to inhomogeneous boundary condition represented by system (17). The $O(r^n)$, $n \geq 1$ equations are inhomogeneous with homogeneous boundary conditions.
Equation (33a) has solutions

$$E_x^0(s) = e^{\pm i\Omega s} = e^{\pm i\omega\sqrt{\epsilon_L\mu}z} \tag{34}$$

subject to the boundary conditions defined by (17). We thus define, in this case,

$$W_L(z) = e^{i\omega\sqrt{\epsilon_L\mu}z} \tag{35a}$$

$$W_R(z) = e^{-i\omega\sqrt{\epsilon_L\mu}} \tag{35b}$$

Differentiating $W_L(z)$, $W_R(z)$ and substituting into (17) and solving the system yields E_r,E_L,E_R,E_t.
Further improvement of the layer solution accuracy is achieved by solving equation (33b) and its successive $n \geq 2$ ones, each contributing a uniform $O(r^{n+1})$ accuracy.

VI.4. Formal WKB expansion: high frequency, slowly varying medium

Assume that the term $Q(s)$ in (24) does not vary abruptly, and that Ω is large. We shall later demonstrate ways to measure the amount of allowed variation, and give meaning to the statement 'large Ω'.
In such case, the physical optics approximations

$$\begin{aligned} E_x(s) &\sim Q(s)^{-1/4}\exp\left(\pm i\Omega\int\sqrt{Q(s)}ds\right) \\ &\approx \frac{\exp\left(\pm i\Omega\left(\frac{rs^2}{8a}+\frac{rs}{4}+s\right)\right)}{\sqrt[4]{\frac{r(a+s)}{4a}+1}} \end{aligned} \tag{36}$$

are uniform approximations for the left and right travelling waves $W_L(s)$, $W_R(s)$ within the layer. As in the regular perturbations case, differentiating with respect to z, substituting in (17) and solving the linear system of equations yield the required constants E_r,E_L,E_R,E_t.

Appendix A: Derivation of the generalized wave equations from Maxwell's equations

In order to transform Maxwell's equation into a system of wave equations, we need the following set of vector calculus identities:

$$\nabla \times (\psi \vec{a}) = \nabla\psi \times \vec{a} + \psi \nabla \times \vec{a} \tag{A1a}$$

$$\nabla \times \nabla \times \vec{a} = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a} \tag{A1b}$$

$$\nabla \cdot (\psi \vec{a}) = \psi \nabla \cdot \vec{a} + \vec{a} \cdot \nabla\psi \tag{A1c}$$

We start from equation (1b). Substituting the relation (2b) between magnetic fields in conjunction and dividing by μ we get

$$\frac{1}{\mu}\nabla \times \vec{E} + i\omega\vec{H} = 0$$

Taking the curl of both sides

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \vec{E}\right) + i\omega\nabla \times \vec{H} = 0 \tag{A2}$$

We'll apply thew vector calculus identities (A1) to decompose the summed terms in (A2).
Using identity (A1a) we get

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \vec{E}\right) = \nabla\left(\frac{1}{\mu}\right) \times \left(\nabla \times \vec{E}\right) + \frac{1}{\mu}\nabla \times \nabla \times \vec{E} \tag{A3}$$

Now we apply (A1b) to (A3) and get

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \vec{E}\right) = \nabla\left(\frac{1}{\mu}\right) \times \left(\nabla \times \vec{E}\right) + \frac{1}{\mu}\left(\nabla(\nabla \cdot \vec{E}) - \nabla^2\vec{E}\right) \tag{A4}$$

The $\vec{H}$ - term in (A2) can be replaced using equation (1a):

$$\nabla \times \vec{H} = i\omega D + J \tag{A5}$$

and using the constitutive relation (2a), equation (A5) becomes

$$i\omega\nabla \times \vec{H} = -\omega^2\epsilon\vec{E} + i\omega J \tag{A6}$$

Substituting (A4), (A6) into (A2) and multiplying by μ we get the generalized wave equations system (3).

Appendix B: Component-wise expansion of the term $\left(\mu\nabla\left(\frac{1}{\mu}\right) \times \left(\nabla \times \vec{E}\right)\right)$

To handle the repeated cross product we'll use Levi-Civita's permutation symbols ϵ_{ijk} [1] defined by

$$\begin{gathered}\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1 \\ \text{all other } \epsilon_{ijk} = 0\end{gathered} \tag{B1}$$

The following lemma is presented as an exercise in [1], p. 150.

Lemma .1 *Let ϵ_{ijk} be defined by (B1). Then*

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \tag{B2}$$

where δ_{ij} is Kronecker's delta.

Proof .1 *Noticed that each term in the k-sum $\epsilon_{ijk}\epsilon_{pqk}$ is non-vanishing only if $i, j, p, q \neq k$, $i \neq j$, $p \neq q$. For example, for the $k = 1$ term we need only to examine the possibilities*

$$\begin{array}{llll} i = 2, & j = 3, & p = 2, & q = 3 \\ i = 3, & j = 2, & p = 2, & q = 3 \\ i = 2, & j = 3, & p = 3, & q = 2 \\ i = 3, & j = 2, & p = 3, & q = 2 \end{array}$$

Substituting and using (B1) we get that in all cases $\epsilon_{ij1}\epsilon_{pq1} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$.
Similar results can be shown for $k = 2, 3$. Finally, the summation (B2) holds because i, j, p, q substitutions that yield a non-vanishing for a specific value of k cause the other k-terms to vanish.

Lemma .2 *Let $\vec{a}$ be a vector of x, y, z-differentiable entries. Then*

$$\left(\vec{a} \times (\nabla \times \vec{E})\right)_i = \vec{a} \cdot \left(\frac{\partial\vec{E}}{\partial x_i} - \nabla E_i\right) \tag{B3}$$

Proof .2 *Using the Levi-Civita symbol (B1),*

$$\left(\vec{a} \times (\nabla \times \vec{E})\right)_i = \epsilon_{ijk} a_j \left(\epsilon_{kpq}\frac{\partial E_q}{\partial x_p}\right) = \epsilon_{ijk}\epsilon_{kpq} a_j \frac{\partial E_q}{\partial x_p}$$

By lemma (.1)

$$\left(\vec{a} \times (\nabla \times \vec{E})\right)_i = \left(\delta_{ip}\delta jq - \delta_{iq}\delta_{jp}\right) a_j \frac{\partial E_q}{\partial x_p}$$

but

$$\delta_{ip}\delta_{jq} a_j \frac{\partial E_q}{\partial x_p} = a_j \frac{\partial E_j}{\partial x_i}$$

$$\delta_{iq}\delta_{jp} a_j \frac{\partial E_q}{\partial x_p} = a_j \frac{\partial E_i}{\partial x_j}$$

Hence

$$\left(\vec{a} \times (\nabla \times \vec{E})\right)_i = a_j \left(\frac{\partial E_j}{\partial x_i} - \frac{\partial E_i}{\partial x_j}\right)$$

Explicit summation yields

$$\left(\vec{a} \times (\nabla \times \vec{E})\right)_i = a_1\left(\frac{\partial E_1}{\partial x_i} - \frac{\partial E_i}{\partial x_1}\right) + a_2\left(\frac{\partial E_2}{\partial x_i} - \frac{\partial E_i}{\partial x_2}\right) + a_3\left(\frac{\partial E_3}{\partial x_i} - \frac{\partial E_i}{\partial x_3}\right) = \vec{a} \cdot \left(\frac{\partial\vec{E}}{\partial x_i} - \nabla E_i\right)$$

as required.

Corollary .1 *Relation (5) holds:*
Substitute $\vec{a} = \nabla\mathcal{M}$ into (B3).

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