# Title - to be filled

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We introduce a framework for the study of 3-D electromagnetic wave propagation in smoothly-varying inhomogeneous, time-independent,linear media. Variation in dielectric permittivity and magnetic permeability are assumed to take place in the $z$ direction only.

The framework is consists of a system of three scalar, damped linear wave equations of variable coefficients. Scalar form is obtained by careful expansion of Maxwell's equations full curl-curl form. Full uncoupling of the system is achieved, and assumptions applied to acquire a more traceable 1-D variation - governed model are stated.

Profile-dependent dimensional analysis of the non-transverse field component equation allows qualitative analysis of solution's spatial behaviour in the time-harmonic form of the equations . Dimensional grouping yields valid comparisons of parameter magnitudes. Numerical simulations demonstrate variation of solution features corresponding to different asymptotic assumptions.

Localized qualitative change in spatial solution behaviour occur and can be tracked by the governing PDE signature. While sigmoidal dielectric profiles exhibit behaviour approximated by a limiting step-function profiles, Gaussian profiles may change the governing equation's signature, yielding abrupt change in the solution characterizing properties.

## I. INTRODUCTION

### I.1. Advantages of asymptotic methods

Uniformity
Simplicity; physical interpretation

### I.2. Selection of appropriate asymptotic expansion method

Left and right travelling waves required by system.
Regular perturbations - small $r$
WKB:

## II. SCALAR FORM OF THE GENERALIZED WAVE EQUATIONS IN SMOOTHLY VARYING MEDIA

### II.1. Constructing the generalized scalar wave equations

Let $\overrightarrow{E}(x,y,z,\omega) = \hat{E}(x,y,z)e^{i\omega t}$, $\overrightarrow{H}(x,y,z,\omega) = \hat{H}(x,y,z)e^{i\omega t}$ be time-harmonic electric and magnetic fields, respectively, satisfying Maxwell's equations

$$\nabla \times \overrightarrow{H} - i\omega \overrightarrow{D} = \overrightarrow{J} \tag{1a}$$

$$\nabla \times \overrightarrow{E} + i\omega \overrightarrow{B} = 0 \tag{1b}$$

$$\nabla \cdot \overrightarrow{D} = \rho \tag{1c}$$

$$\nabla \cdot \overrightarrow{B} = 0 \tag{1d}$$

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Assume a linear, isotropic material, so that the flux density vectors satisfy $\overrightarrow{D} = \epsilon \overrightarrow{E}$, and $\overrightarrow{B} = \mu \overrightarrow{H}$. We allow material inhomogeneity: $\mu = \mu(x,y,z)$, $\epsilon = \epsilon(x,y,z)$, where $\epsilon$ and $\mu$ are smooth functions of $x,y,z$.

For convenience we define $\mathcal{M} = \log(\mu)$, $\mathcal{E} = \log(\epsilon)$.

Maxwell's equations for inhomogeneous medium in vector form (compare [2], chap. 3)

$$\nabla(\nabla\cdot\overrightarrow{E}) - \nabla(\mathcal{M})\times\left(\nabla\times\overrightarrow{E}\right) - \nabla^2\overrightarrow{E} - \epsilon\mu\omega^2\overrightarrow{E} + i\omega\mu\overrightarrow{J} = 0 \tag{2}$$

are derived directly from Maxwell's equations using the curl-curl formulation. The first and second terms at the left hand side satisfy

$$\nabla\left(\nabla\cdot\overrightarrow{E}\right)_{x_j} = \frac{\partial}{\partial x_j}\left(\frac{\rho}{\epsilon}\right) - \left(\frac{\partial\overrightarrow{E}}{\partial x_j}\cdot\nabla\mathcal{E} + \overrightarrow{E}\cdot\frac{\partial}{\partial x_j}\nabla\mathcal{E}\right) \tag{3}$$

$$\left(\nabla\mathcal{M}\times\left(\nabla\times\overrightarrow{E}\right)\right)_{x_j} = \nabla\mathcal{M}\cdot\left(\frac{\partial\overrightarrow{E}}{\partial x_j} - \nabla E_{x_j}\right) \tag{4}$$

where $x_j = x, y, z$ for $j = 1, 2, 3$, respectively. Identity (3) is derived using the constitutive relation $\nabla\cdot\overrightarrow{D} = \rho$. Identity (4) is a non-trivial result of the Levi-Civita symbol calculus.

Note that if $\epsilon$ and $\mu$ are constant, equation (2) is the well-known system of uncouples standard wave equations.

### II.2. System uncoupling for x,y - independent medium

Assume that $\mu, \epsilon$ are $x, y$-independent. As in the constant $\epsilon$, $\mu$ case, the system (2) can be uncoupled in the

following sense: the $E_z$ equation is

$$\mathcal{E}''E_z + \mathcal{E}'E_z' + \nabla^2 E_z + \epsilon\mu\omega^2 E_z = \left(\frac{\rho}{\epsilon}\right)' + i\omega\mu J_z \quad (5)$$

where the primes denote $z$ derivatives. Note that term (4) contribution cancels due to $\mu$ strict $z$-dependency.
In such case, the $z$-component equation is a scalar PDE in $E_z$. Hence, if $\mu, \epsilon, \rho, \frac{\partial J_z}{\partial t}$ are known, (5) is decoupled from the system (2) and can be solved in a stand-alone manner.
For $x_1 = x$, $x_2 = y$, we get the equations

$$\mathcal{M}'\left(\frac{\partial E_{x_j}}{\partial z} - \frac{\partial E_z}{\partial x_j}\right) + \mathcal{E}'\frac{\partial E_z}{\partial x_j} + \nabla^2 E_{x_j} + \epsilon\mu\omega^2 E_{x_j} = i\omega\mu J_{x_j}$$
$$(6)$$

Equations (6) are coupled with the $E_z$ equation but not among themselves. Thus, once $E_z$ is obtained, it can be substituted into equations (6), each becoming a stand-alone scalar PDE in $E_{x_j}$, achieving total uncoupling of (2).

## III. SYMMETRY AND COHERENCE ASSUMPTIONS YIELD REDUCTION OF PDES TO ODES

Assume an $x, y$ spatially coherent source incident to an infinite slab $-L \leq z \leq L$, $-\infty < x, y < \infty$, at $z = -L$. The profiles $\epsilon = \epsilon(z) \in C^2(-L, L), \mu = \mu(z) \in C^1(-L, L)$ are assumed to have strictly positive real parts. We shall also assume sufficiently differentiable, $x, y$-independent $\rho, \overrightarrow{J}$.

Spatial coherence and $x, y$ independence of both equation (5) coefficients and inhomogeneous terms yield $x, y$ invariance, causing the $x, y$ derivatives in the PDE to vanish. The PDE is thus reduced to the ODE

$$\hat{E}_z'' + \mathcal{E}'\hat{E}_z' + \left(\mathcal{E}'' + \epsilon\mu\omega^2\right)\hat{E}_z = \left(\frac{\rho}{\epsilon}\right)' + i\omega\mu J_z \quad (7)$$

PDE (6) turns into an even simpler form, since $\hat{E}_z$ is $x, y$ independent, and therefore $\frac{\partial \hat{E}_z}{\partial x_j} = 0$. Thus the PDE is mapped to the ODE

$$\hat{E}_{x_j}'' + \mathcal{M}'\hat{E}_{x_j}' + \epsilon\mu\omega^2\hat{E}_{x_j} = i\omega\mu J_{x_j} \quad (8)$$

having no explicit $\hat{E}_z$ dependency.

## IV. REFLECTION-TRANSMISSION FORMULATION FOR AN INHOMOGENEOUS LAYER

### IV.1. Field components matching at the boundaries

Let the incidence wave be normal to the boundary surface $z = -L$. The media to the left and right of the layer

$-L \leq z \leq L$ have constant dielectric profile $\epsilon_1$, $\epsilon_2$ respectively. The layer's $z$-dependent dielectric profile is simply $\epsilon$. The electric and magnetic fields are denoted $\overrightarrow{E}_1$, $\overrightarrow{H}_1$, $\overrightarrow{E}$, $\overrightarrow{H}$, $\overrightarrow{E}_2$, $\overrightarrow{H}_2$ at $z < -L$, $-L < z < L$, $z > L$, respectively.
As long as analytic methods only are applied for solution assessment, no artificial boundary conditions are required to model 'free space interface'.
Boundary conditions at the interfaces $z = \pm L$ are determined classically, as their derivation is local in space. Further, by assumption, no net charge is accumulated on the boundary surfaces and no current flows along them. Thus

- the $\overrightarrow{E}$ tangential component is continuous across the interfaces. For simplicity, we discuss (from now on ) the $x$-polarized case,

$$E_{1x}|_{z=-L^-} = E_x|_{z=-L^+} \quad (9a)$$
$$E_x|_{z=-L^-} = E_{2x}|_{z=-L^+} \quad (9b)$$

- The $\overrightarrow{E}$ normal component satisfies

$$\epsilon_1 E_{1z}|_{z=-L^-} = \epsilon_{z=-L^+} E_z|_{z=-L^+} \quad (10a)$$
$$\epsilon_{z=L^-} E_z|_{z=L^-} = \epsilon_2 E_z|_{z=L^+} \quad (10b)$$

- The $\overrightarrow{H}$ tangential component is continuous (as the current $J_s = 0$)

$$H_{1y}|_{z=-L^-} = H_{1y}|_{z=-L^+} \quad (11a)$$
$$H_{1y}|_{z=L^-} = H_{1y}|_{z=L^+} \quad (11b)$$

- The $\overrightarrow{H}$ normal components satisfies, in the general case,

$$\mu_1 H_{1z}|_{z=-L^-} = \mu_{z=-L^+} H_z|_{z=-L^+} \quad (12a)$$
$$\mu_{z=L^-} H_z|_{z=L^-} = \mu_2 H_{2z}|_{z=L^+} \quad (12b)$$

Since $\mu(z)$ is assumed constant in the profiles studied in this work , these relations reduce to continuity.

### IV.2. The Boundary Value Problem Setting

Consider equation (8) for a tangentially-polarized wave, for which equation (21) is a dimensionally-grouped special case. Assume that the exists an analytic expression, exact or approximate, for left and right travelling wave solutions within the layer.
If a monochromatic source of frequency $\omega$ is located at $z \to -\infty$ and no sources exist for $z > L$, we have the decomposition

$$E_{1x} = E_i e^{-ik_1 z} + E_r e^{ik_1 z} \quad (13)$$

where $E_i e^{-ik_1}$ and $E_r e^{ik_1}$ are phasor forms of the incidence and reflected waves respectively, with $k_1 = \omega\sqrt{\mu\epsilon_1}$;

$$E_x(z) = E_L W_L(z) + E_R W_R(z) \qquad (14)$$

where $W_L(z)$, $W_R(z)$ are independent, left and right travelling wave solutions for the Helmholtz equation in the layer, and

$$E_{2x} = E_t e^{-ik_2 z} \qquad (15)$$

is the transmitted, pure right - travelling wave, with $k_2 = \omega\sqrt{\mu\epsilon_2}$.

The incidence wave constant multiplier $E_i$ is known. The rest of the constants: $E_r$, $E_L$, $E_R$, $E_t$ are complex values to be determined. Thus four equations are should be derived from the boundary conditions.

As in the classical theory, two equations are derived directly from the electric field tangent component boundary conditions at $z = \pm L$. Two extra equations are derived by converting the electric into magnetic field using Maxwell's equation (1b), and casting the corresponding boundary conditions. We get the system of equations

$$\begin{pmatrix} -e^{-iLk_1} & W_L(-L) & W_R(-L) & 0 \\ k_1 e^{-iLk_1} & iW_L'(-L) & iW_R'(-L) & 0 \\ 0 & W_L(L) & W_R(L) & -e^{-iLk_2} \\ 0 & iW_L'(L) & iW_R'(L) & -k_2 e^{-iLk_2} \end{pmatrix} \begin{pmatrix} E_r \\ E_L \\ E_R \\ E_t \end{pmatrix}$$
$$= \begin{pmatrix} e^{iLk_1} E_i \\ k_1 e^{iLk_1} E_i \\ 0 \\ 0 \end{pmatrix} \qquad (16)$$

### IV.3. Conclusions

While the solution to any second-order linear ODE can be written as a superposition of two independent solutions, From the matrix structure in system (16) we can deduce that the inner-layer solution must be written as a superposition of two independent solutions. Writing the solution as a sum of left and right travelling waves makes the physical meaning of the solution more tractable. As we shall see next, the approximation methods applied in this work - regular perturbations and WKB - have both properties.

Further, both the matrix and the input vector are complex valued. Hence we shall typically have a complex-valued solution vector. The computed components $E_r$, $E_L$, $E_R$, $E_t$ therefore yield both amplitude and phase information.

## V. DIMENSIONAL ANALYSIS IN A FOR A STRUCTURED, Z-DEPENDENT INHOMOGENEOUS LAYER

The formulations (2) is general enough to allows any twice-differentiable dielectric profile. Yet, to get qualitative insight about the solution behaviour using asymptotic analysis, dimensional analysis must be performed

for a given profile. In this sense, strict meaning can be attached to 'large' and 'small' parameters statements, allowing correct selection of asymptotic analysis technique.

### V.1. Construction of a Sigmoid dielectric Profile

As commonly assumed in optics, we consider a source-free (i.e. $\rho = 0$, $\overrightarrow{J} = 0$) layer $-L \leq z \leq L, -\infty < x, y < \infty$, in which $\mu$ is constant. The dielectric profile $\epsilon(z)$ interpolates smoothly and monotonically two values of the dielectric profile so that $\epsilon(-L) = \epsilon_1$ and $\epsilon(L) = \epsilon_2$. The sigmoid is a non-linear function, hence should have a dimensionless argument. To do so, we take some sigmoid-like function $c(\cdot)$ and normalize its argument by a constant $M$, yielding $c\left(\frac{z}{M}\right)$. The interpolating function is

$$\epsilon(z) = \frac{\epsilon_L c\left(\frac{L}{M}\right) - \epsilon_R c\left(-\frac{L}{M}\right) + (\epsilon_R - \epsilon_L)c\left(\frac{z}{M}\right)}{c\left(\frac{L}{M}\right) - c\left(-\frac{L}{M}\right)} \qquad (17)$$

which can be substituted into (7), (8). The resulting equations coefficients turn out to be cumbersome and not suggestive to interpretation. A more compact and meaningful form will now be presented next.

### V.2. Dimensional Analysis for a Sigmoidal Profile

Defining the dimensionless quantities

$$s = z/M \qquad (18a)$$
$$a = L/M \qquad (18b)$$
$$r = (\epsilon_R - \epsilon_L)/\epsilon_L \qquad (18c)$$
$$\Omega = M\omega\sqrt{\epsilon_L\mu} \qquad (18d)$$

In these terms, letting

$$q(s) = rc(s) + c(a) - (r+1)c(-a) \qquad (19)$$

yields

$$\frac{d}{dz} \rightarrow \frac{1}{M}\frac{d}{ds}$$
$$\frac{d^2}{dz^2} \rightarrow \frac{1}{M^2}\frac{d^2}{ds^2}$$
$$\mathcal{E}' \rightarrow \frac{rc'(s)}{Mq(s)}$$
$$\mathcal{E}'' \rightarrow \frac{rc''(s)}{M^2 q(s)} - \left(\frac{rc'(s)}{Mq(s)}\right)^2$$
$$\epsilon\mu\omega^2 \rightarrow \frac{q(s)\Omega^2}{M^2(c(a) - c(-a))}$$

The scaling of the $z$ coordinate by $M$ factors it out in the homogeneous case. Equation (7) is thus transformed

into

$$\frac{d^2\hat{E}_z}{ds^2} + P(s)\frac{d\hat{E}_z}{ds} + Q(s)\hat{E}_z = 0 \qquad (20)$$

$$P(s) = \frac{rc'(s)}{q(s)}$$

$$Q(s) = \frac{rc''(s)}{q(s)} - \left(\frac{rc'(s)}{q(s)}\right)^2 + \frac{q(s)\Omega^2}{c(a) - c(-a)}$$

and equation (8) becomes

$$\frac{d^2\hat{E}_{x_j}}{ds^2} + \frac{q(s)\Omega^2}{c(a) - c(-a)}\hat{E}_{x_j} = 0 \qquad (21)$$

### V.3.  Construction of the Gaussian Profile

We demonstrate complex solution behaviour patterns, inspired by linear, second order ODE theory with constant coefficients. To do so, a dielectric profile altering equation (7) coefficients signs in the is presented.

We consider again a source-free layer defined as in section V.1, with $\mu$ constant. This time we consider a Gaussian-shaped profile

$$\epsilon(z) = Ce^{-\left(\frac{z-m}{\sigma}\right)^2} \qquad (22)$$

Assume that the parameters $C$, $m$ are chosen to fit $\epsilon(-L) = \epsilon_1$, $\epsilon(L) = \epsilon_2$. Clearly, $C$ is positive.
For simplicity, we shall keep the profile in its current form, remembering that $\sigma$ is considered a tunable parameter.

### V.4.  Temporal Frequency-Dependent Spatial Solution behaviour

Consider ODE (7). Since constant $\mu$ and source-free layer are assumed, it has the form

$$\hat{E}_z'' - \frac{2(z-m)}{\sigma^2}\hat{E}_z' + \left(C\mu\omega^2 e^{-\frac{(z-m)^2}{\sigma^2}} - \frac{2}{\sigma^2}\right)\hat{E}_z = 0 \quad (23)$$

n Tuning $\sigma$ and $\omega$, The $\hat{E}_z$ coefficient $c_0(z)$ can be tuned to be uniformly positive or negative in the interval $-L < z < L$, or alter sign there. One should expect that the local solution behaviour would be determined by the equation's characteristic root, computed for a fixed value of $z$. This resulting qualitative behaviour should hold for neighbouring $z$ values due to the $\epsilon$ profile smoothness. Thus, the well-known repertoire of exponential / sinusoidal combinations would dominate the local solution shape.

A final note about the physical setting is in order. In a plane wave excitation situation, the incident wave $z$-component vanishes; Thus the interior $z$-component

also vanishes. Note, though, that other settings where the incident wave $z$-component may not vanish may be considered. For example, this is the case if the incident wave is carried by a wave guide.

## VI.  ASYMPTOTIC APPROXIMATION FOR THE SOLUTION IN A SIGMOIDAL DIELECTRIC PROFILE

We consider the sigmoid

$$c(z) = \frac{1}{1 + e^{-z}} \qquad (24)$$

### VI.1.  Thin Layer - Small Effective Width

A relatively simple case in that of a thin layer, $a \ll 1$. $s$ is small as well, since $-a < s < a$. Expanding The $E_{x_j}$ coefficient in equation (21) to series in both $a$, $s$ and dropping terms higher than 1,

$$\frac{q(s)}{c(a) - c(-a)} \approx \left(r\left(\frac{s}{2a} + \frac{1}{2}\right) + 1\right) = Q(s) \qquad (25)$$

yielding the approximation equation

$$\frac{d^2E_x}{ds^2} + \left(r\left(\frac{s}{2a} + \frac{1}{2}\right) + 1\right)\Omega^2 E_x = 0 \qquad (26)$$

Note that $Q(s)$ in (25) can become arbitrarily large only by increase of $r$.

#### VI.1.1.  Formal Regular Perturbation Expansion: low frequency, small effective dielectric dynamic range

We shall later show the meaning of $\Omega$ being $O(1)$, but for now let us assume that this is the case, and that $|r| \ll 1$.
Plugging the perturbation series

$$E_x(s) = E_x^0(s) + rE_x^1 + r^2E_x^2 + \dots \qquad (27)$$

where $j$ in $E_x^j$ is an index, into equation (26) and collecting $O(r^n)$, terms $n = 0, 1, \dots$ we get a sequence of equations

$$\frac{d^2E_x^0}{ds^2} + \Omega^2 E_x^0 = 0 \qquad (28a)$$

$$\frac{d^2E_x^1}{ds^2} + \Omega^2 E_x^1 = -\frac{\Omega^2(a+s)}{2a}E_x^0 \qquad (28b)$$

$$\dots$$

All equations are linear, of constant coefficients. The $O(r^0)$ equation is homogeneous, corresponding to inhomogeneous boundary condition represented by system (16). The $O(r^n)$, $n \geq 1$ equations are inhomogeneous

with homogeneous boundary conditions.
Equation (28a) has solutions

$$E_x^0(s) = e^{\pm i\Omega s} = e^{\pm i\omega\sqrt{\epsilon_L \mu} z} \qquad (29)$$

subject to the boundary conditions defined by (16). We thus define, in this case,

$$W_L(z) = e^{i\omega\sqrt{\epsilon_L \mu} z} \qquad (30a)$$

$$W_R(z) = e^{-i\omega\sqrt{\epsilon_L \mu}} \qquad (30b)$$

Differentiating $W_L(z)$, $W_R(z)$ and substituting into (16) and solving the system yields $E_r, E_L, E_R, E_t$.
Further improvement of the layer solution accuracy is achieved by solving equation (28b) and its successive $n \geq 2$ ones, each contributing a uniform $O(r^{n+1})$ accuracy.

### VI.1.2. Formal WKB expansion: high frequency, slowly varying medium

Assume that the term $Q(s)$ in (25) does not vary abruptly, and that $\Omega$ is large. We shall later demonstrate ways to measure the amount of allowed variation, and give meaning to the statement 'large $\Omega$'.
In such case, the physical optics approximations

$$E_x(s) \sim Q(s)^{-1/4} \exp\left(\pm i\Omega \int \sqrt{Q(s)} ds\right)$$

$$\approx \frac{\exp\left(\pm i\Omega \left(\frac{rs^2}{8a} + \frac{rs}{4} + s\right)\right)}{\sqrt[4]{\frac{r(a+s)}{4a} + 1}} \qquad (31)$$

are uniform approximations for the left and right travelling waves $W_L(s)$, $W_R(s)$ within the layer. As in the regular perturbations case, differentiating with respect to $z$, substituting in (16) and solving the linear system of equations yield the required constants $E_r, E_L, E_R, E_t$.

### VI.1.3. Parameter Magnitude Characterization - a Joined Scales Analysis

In order to select an appropriate asymptotic approximation method, a measure of terms magnitude. Since the parameters $r, a, \Omega$ are dimensionless, it is legitimate to let

$$r = \delta^\alpha \qquad (32a)$$

$$a = \delta^\beta \qquad (32b)$$

$$\Omega = \delta^\gamma \qquad (32c)$$

where $\delta$ is a small real number. Here we assume $r > 0$. The case $r < 0$ can be treated similarly. Since $-a \leq s \leq a$, we define also

$$s = \delta^\beta \xi \qquad (33)$$

Equation (21) is transformed into

$$\frac{d^2 E_x}{d\xi^2} + \left(\left(\frac{\xi}{2} + \frac{1}{2}\right)\delta^\alpha + 1\right)\delta^{2(\beta+\gamma)} E_x = 0 \qquad (34)$$

Regular perturbation holds when $\beta + \gamma \geq 0$, and $a > 0$, namely $a\Omega = O(1)$, and $r \ll 1$.

The WKB method requires a factored $E_x$ coefficient of the form $Q(s)\Omega^2$, where $Q(s)$ is slowly varying and $\Omega^2$ is large. Clearly, these conditions are satisfied for $\beta + \gamma < 0$, $\alpha > 0$, yet better insight can be obtained using by connecting $\alpha$ to $\beta$ and $\gamma$ through the WKB relative error term [1]. Letting

$$S_2(\xi) = \frac{\frac{d^2Q}{d\xi^2}}{8Q^{3/2}} - \frac{5\left(\frac{dQ}{d\xi}\right)^2}{32Q^{5/2}} \qquad (35)$$

the uniform relative error bound is

$$\exp\left(\frac{S_2(\xi)}{\Omega}\right) \approx 1 + \frac{S_2(\xi)}{\Omega}, \qquad \Omega \gg 1 \qquad (36)$$

Hence in order to verify that the WKB approximation holds, we should show that $\left|\frac{S_2}{\Omega}\right| \ll 1$ uniformly in the interval.
For $Q(\xi)$ being the $E_x(\xi)$ from equation

$$\frac{S_2}{\Omega} = \frac{5\delta^{\alpha+\gamma}}{24\sqrt{2}\left((\xi+1)\delta^\alpha + 2\right)^{3/2}} \qquad (37)$$

Hence further restriction, $\alpha + \gamma > 0$, is required for the WKB approximation to hold.

### VI.2. Thick Layer - Large Effective Width

## VII. ASYMPTOTIC APPROXIMATION FOR THE SOLUTIONS IN THE GAUSSIAN CASE

## VIII. NUMERICAL SOLVER THE PHYSICAL SETUP

### VIII.1. Absorbing Boundary Condition at the right boundary

A first order PDE formulation for the boundary condition is

$$\left[\frac{\partial E_{x_j}}{\partial t} + \frac{1}{\sqrt{\epsilon\mu}}\frac{\partial E_{x_j}}{\partial z}\right]_{z=L} = 0, \quad j = 1, 2, 3 \qquad (38)$$

Note that $\epsilon = \epsilon(L)$, $\mu = \mu(L)$. For monochromatic waves

$$\left[i\omega\hat{E}_{x_j} + \frac{1}{\sqrt{\epsilon\mu}}\frac{\partial \hat{E}_{x_j}}{\partial z}\right]_{z=L} = 0, \quad j = 1, 2, 3 \qquad (39)$$

## IX. UNIQUENESS OF THE ODE BOUNDARY VALUE SOLUTION

## X. AGREEMENT WITH THE CLASSICAL THEORY IN THE NARROW EFFECTIVE WIDTH LIMIT

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