Q1

(a)

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 10 \\ 16 \\ 15 \end{pmatrix}$$

$$x_B = B^{-1}b = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ 16 \\ 15 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 10 \end{pmatrix}$$

and $x_2 = x_3 = 0$

$$z_B = \begin{pmatrix} 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ 10 \end{pmatrix} = 15$$

(b)

Dual Vars:

$$y = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 0 \\ 0 \end{pmatrix}$$

Reduced Cost:

$$C_2{}' = 2 - \begin{pmatrix} \frac{3}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = 2 - \frac{3}{2} = \frac{1}{2} > 0$$

$$C_3{}' = 0 - \begin{pmatrix} \frac{3}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -\frac{3}{2} < 0$$

$\therefore$ Adding x_2 into basis would increase the objective

Leaving Var:

$$\alpha^2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{7}{2} \end{pmatrix}$$

Ratio:

- $x_1 = 5 \div \frac{1}{2} = 10$
- $x_4 = 1 \div \frac{1}{2} = 2$
- $x_5 = 10 \div \frac{7}{2} = \frac{20}{7}$

$\therefore x_4$ has the lowest ratio, x_4 leaves basis

(c)

New basis x_1, x_2, x_5

$B = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 1 \end{pmatrix}$

$b = \begin{pmatrix} 10 \\ 16 \\ 15 \end{pmatrix}$

$x_B = B^{-1} b = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & 0 \\ 10 & -7 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ 16 \\ 15 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$

$z_B = \begin{pmatrix} 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = 16$

Dual Vars:

$y = \begin{pmatrix} 2 & -3 & 10 \\ -1 & 2 & -7 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Reduced Cost:

$C_3{}' = 0 - \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$

$C_3{}' = 0 - \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -1 < 0$

$\therefore C_3{}' \leq 0 \land C_4{}' \leq 0$

$\therefore$ solution is optimal

$$x_1 = 4$$

$$x_2 = 2$$

$$x_3 = 0$$

$$x_4 = 0$$

$$x_5 = 3$$

$$z = 16$$

(d)

${C_3}' = 0 \longrightarrow$ Coefficient can be changed to any positive number.

(e)

$$\begin{aligned} z^{\Delta} &= y^T \left(b + \Delta e^j\right) \\ &= y^T b + y^T \Delta e^j \\ &= z + \Delta y_j \end{aligned}$$

Dual var is $n = 1$ in the second constraint $\rightarrow z = z + n\delta = z + \delta$

Q2

(a)

Sets
- S - Stores
- W - warehouse sites

Data
- C_w - cost for warehouse $w \in W$
- A_w - capacity of warehouse
- $d_{s,w}$ - cost of supplying $s \in S$ from $w \in W$
- t_s - tonnes of orders for $s \in S$

Variables
- $x_{s,w} \in \{0, 1\}$ assign store $s \in S$ to warehouse $w \in W$
- $y_w \in \{0, 1\}$ build warehouse $w \in W$ at site

Objective

$$\min \sum_{s \in S} \sum_{w \in W} d_{s,w} \times x_{s,w} + \sum_{w \in W} C_w \times y_w$$

Constraints
- Warehouse capacity $\sum_{s \in S} t_s \times x_{s,w} \leq A_w \times y_w, \forall w \in W$
- Assign each store to a warehouse

$$\sum_{w \in W} x_{s,w} = 1, \forall s \in S$$

check all the data and make sure they're used somewhere.

(b)

inherite everything from (a)

Sets

- $S' \subset S$ - set of chain stores

Variables

- $z_w \in \{0, 1\}$ - if $w \in W$ supplies a chain store

Constraints

- At least K warehouses

$$\sum_{w \in W} z_w \geq K$$

- Link z to x

$$z_w \leq \sum_{s \in S'} x_{s,w}, \forall w \in W$$

- Ar most 2 chain stores per warehouse

$$\sum_{s \in S'} x_{s,w} \leq 2, \forall w \in W$$

Q3

Part A

(a)

Data

- $P_{f,a}$ - probability of winning a race given fatigue f and action a
- d_a - change in fatigue from action a

Stages

- $t \in T$ - races

State

- f_t - fatigue at start of race t

Action

- $A_t = \{0, 1\}$, where
  - 0 - not enter the race
  - 1 - entering the race

Value Function

- $V_t(f_t)$ - total expected score when starting race t with fatigue f_t
- Base case

$$V_T(f_T) = P_{f_T,1}$$

- General case

$$V_t(f_t) = \max_{a_t \in A_t} \left\{P_{f_t,a_t} + V_{t+1}\left(\max\left(f_t + d_{a_t}, 0\right)\right)\right\}$$

(b)

$f_1 = 0$

$V_1(0) = \max\{\frac{1}{2} + V_2(3), V_2(0) = \frac{6}{5}$

$V_2(0) = \max\{\frac{1}{2} + V_2(3), V_3(0) = 1$

$V_2(3) = \max\{\frac{1}{5} + V_2(6), V_3(0) = \frac{7}{10}$

$V_3(0) = \max\{\frac{1}{2} + V_4(3), V_4(0) = \frac{7}{10}$

$V_3(3) = \max\{\frac{1}{5} + V_4(6), V_4(0) = \frac{1}{2}$

$V_3(5) = \max\{\frac{1}{8} + V_4(9), V_4(2) = \frac{1}{4}$

$V_4(0) = \frac{1}{2}, V_4(2) = \frac{1}{4}, V_4(3) = \frac{1}{5}, V_4(6) = \frac{1}{8}, V_4(9) = \frac{1}{11}$

$\therefore$ Race 1 - YES, 2 - NO, 3 - YES, 4 - YES.

Part B

Data
- $P_{f,a}$ - probability of winning a race given fatigue f and action a
- p_3 - probability of fatigue increase by 3 after a race

Stages
- $t \in T$ - races

State
- f_t - fatigue at start of race t
- W_t - number of races have been won at the start of race t

Action
- $A_t = \{0, 1\}$, where
  - 0 - not enter the race
  - 1 - entering the race

Value Function
- $V_t(f_t, W_t)$ - chance of wining ≥ 3 races when starting race t with fatigue f_t and current num of wins W_t

- Base case
  - $\forall 0 \leq t \leq T, W_t \geq 3 \longrightarrow V_{t(f_t, W_t)} = 1$
  - $\forall 0 \leq t \leq T, W_t + (T - t + 1) < 3 \longrightarrow V_{t(f_t, W_t)} = 0$
  - $t = T, V_T(f_T, W_T) = P_{f_T,1}$

- General case

$$V_t(f_t, W_t) = \max\{V_{t+1}^{\text{race}}, V_{t+1}^{\text{not race}}\}$$

where

- $V_{t+1}^{\text{not race}} = V_{t+1}(\max(f_t - 4, 0), W_t)$
- $V_{t+1}^{\text{race}} = P_{f_t,1} \times V_{t+1}^{\text{win}} + 1 - P_{f_t,1}) \times V_{t+1}^{\text{lose}}$, where
  - $V_{t+1}^{\text{win}} = p_3 \times V_{t+1}(f_t + 3, W_t + 1) + (1 - p_3) \times V_{t+1}(f_t + 2, W_t + 1)$
  - $V_{t+1}^{\text{lose}} = p_3 \times V_{t+1}(f_t + 3, W_t) + (1 - p_3) \times V_{t+1}(f_t + 2, W_t)$