MATH3405 Examples

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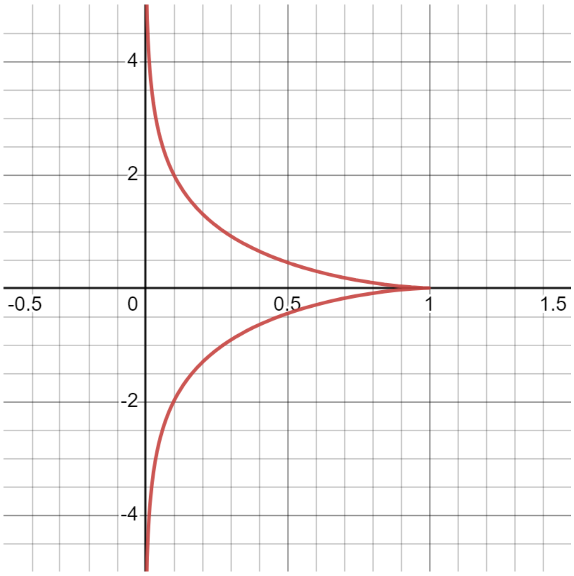
27/10/2023 - end of course

Contents

1 Curves

1.1 Tractrix

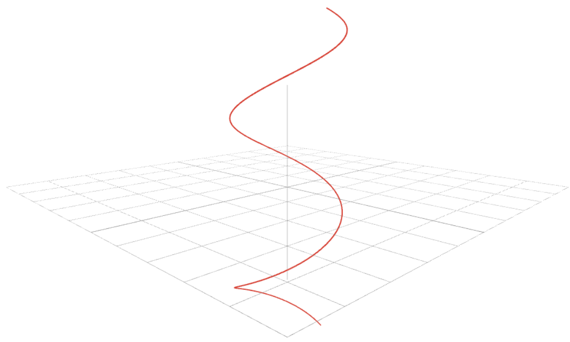


$$\begin{aligned}
\text{Paramaterisation:} \quad & \alpha(t) = (\sin(t), \cos(t) + \log(\tan(t/2))) \\
\text{(Signed) Curvature:} \quad & \kappa(t) = -|\tan(t)| \\
\text{Torsion:} \quad & \tau(t) = 0
\end{aligned}$$

Trivia:

- Length of segment of tangent line at $\alpha(t)$ to the y-axis is 1.

1.2 Helix



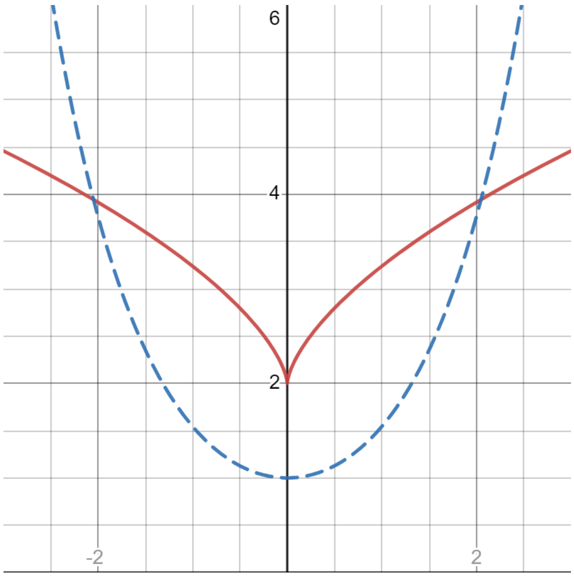
$$\begin{aligned} \text{Paramaterisation:} \quad & \alpha(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), b \cdot \frac{s}{c}\right) \\ \text{Curvature:} \quad & \kappa(s) = \frac{a}{c^2} \\ \text{Torsion:} \quad & \tau(s) = -\frac{b}{c^2} \end{aligned}$$

for $a, b, c > 0$.

Trivia:

- These are all the curves having constant curvature and constant torsion.
  - Torsion 0 degenerates to a circle.

1.3 Evolute



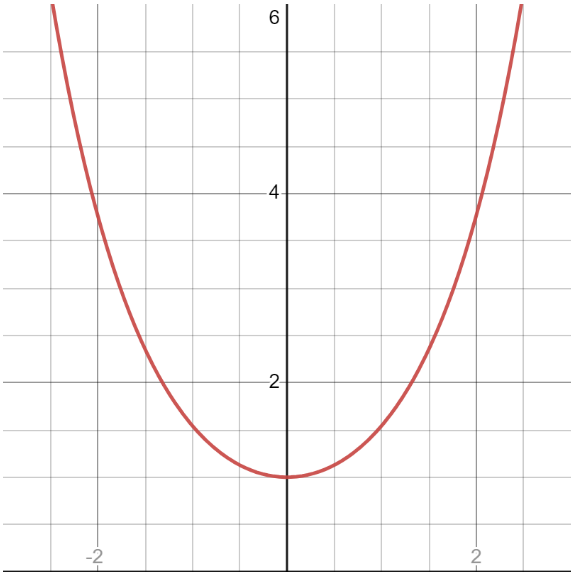
$$\text{Paramaterisation:} \quad \alpha(s) = \beta(s) + \frac{1}{\kappa_\beta(s)} n_\beta(s)$$

is the evolute of β.

Trivia:

- Tangent of α is normal to β everywhere

1.4 Catenary



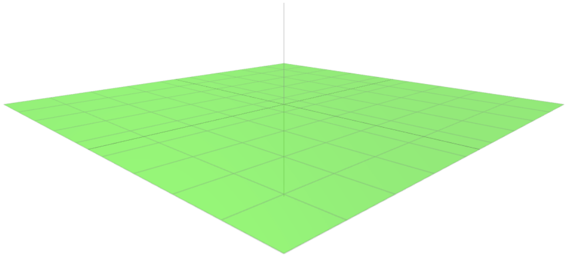
$$\begin{aligned} \text{Paramaterisation:} \quad & \alpha(t) = (t, \cosh(t)) \\ \text{(Signed) Curvature:} \quad & \kappa(t) = \operatorname{sech}^2(t) \\ \text{Torsion:} \quad & \tau(s) = 0 \end{aligned}$$

Trivia:

- something something hanging chains

2 Surfaces

2.1 Plane



With no further parameters,

$$\begin{aligned}
\text{Param:} \quad & \mathbf{x}(u,v) = (u,v,0) \\
\text{Curvature:} \quad & K = 0 \\
& H = 0 \\
\text{Orientable?:} \quad & \text{YES} \\
\text{Geodesics:} \quad & \gamma(t) = \mathbf{x}(a+bt, c+dt) \quad \text{(straight lines)} \\
\text{Exponential map:} \quad & \exp_p(V) = p + V
\end{aligned}$$

for $a, b, c, d \in \mathbb{R}$ with $(b,d) \neq (0,0)$.

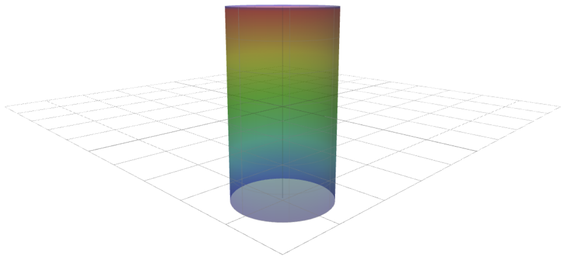
Relative to the above paramaterisation,

$$\begin{aligned}
\text{1ff:} \quad & E = 1 & F = 0 \quad & G = 1 \\
\text{2ff:} \quad & e = 0 & f = 0 \quad & g = 0 \\
\text{Christoffel Symbols:} \quad & \Gamma^1_{11} = 0 & \Gamma^1_{12} = 0 \quad & \Gamma^1_{22} = 0 \\
& \Gamma^2_{11} = 0 & \Gamma^2_{12} = 0 \quad & \Gamma^2_{22} = 0
\end{aligned}$$

Trivia:

- **Constant** 0 **Gauss curvature**.
- **Minimal surface**.
- Straight line geodesics.

2.2 Cylinder



With radius $R > 0$,

$$
\begin{aligned}
\text{Param:} \quad & \mathbf{x}(u,v) = (R\cos(u), R\sin(u), v) \\
\text{Curvature:} \quad & K = 0 \\
& H = -R/2 \\
\text{Orientable?:} \quad & \text{YES} \\
\text{Geodesics:} \quad & \gamma(t) = \mathbf{x}\,(a+bt, c+dt) \quad \text{(helices, circles and meridians)} \\
\text{Exponential map:} \quad & \exp_{\mathbf{x}(a,c)}(b\mathbf{x}_u + d\mathbf{x}_v) = \mathbf{x}\,(a+b, c+d)
\end{aligned}
$$

for $a, b, c, d \in \mathbb{R}$ with $(b,d) \neq (0,0)$

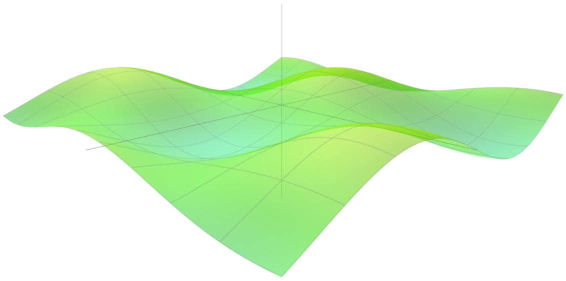
Relative to the above paramaterisation,

$$
\begin{aligned}
\text{1ff:} \quad & E = R^2 & F &= 0 & G &= 1 \\
\text{2ff:} \quad & e = -R & f &= 0 & g &= 0 \\
\text{Christoffel Symbols:} \quad & \Gamma^1_{11} = 0 & \Gamma^1_{12} &= 0 & \Gamma^1_{22} &= 0 \\
& \Gamma^2_{11} = 0 & \Gamma^2_{12} &= 0 & \Gamma^2_{22} &= 0
\end{aligned}
$$

Trivia:

- Surface of revolution, revolving $(\varphi(v), \psi(v)) = (R, v)$.
- With $R = 1$, this is locally isometric to the plane.
- Locally diffeomorphic to the plane.

2.3 Graph of a smooth function



With $z(u,v)$ the function to graph and $s = 1 + z_u^2 + z_v^2$,

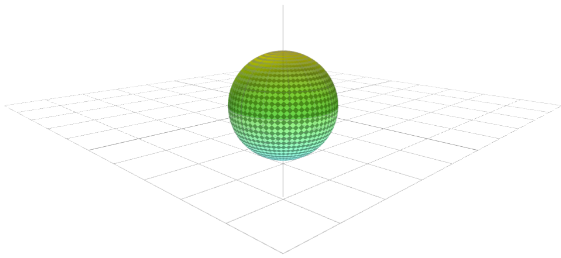
$$\begin{aligned}
\text{Param:} \quad & \mathbf{x}(u,v) = (u, v, z(u,v)) \\
\text{Curvature:} \quad & K = \frac{z_{uu}z_{vv} - z_{uv}^2}{s^2} \\
& H = s^{-3/2} \cdot \left(z_{uu} + z_{vv} + z_u^2 z_{vv} + z_v^2 z_{uu} - z_u z_v z_{uv}\right) \\
\text{Orientable?:} \quad & \text{YES} \\
\text{Geodesics:} \quad & \gamma(t) = (\text{please no, the equations are horrible}) \\
\text{Exponential map:} \quad & \exp_p(V) = (\text{no})
\end{aligned}$$

Relative to the above paramaterisation,

$$\begin{aligned}
\text{1ff:} \quad & E = 1 + z_u^2 & F &= z_u z_v & G &= 1 + z_v^2 \\
\text{2ff:} \quad & e = \frac{1}{\sqrt{s}} z_{uu} & f &= \frac{1}{\sqrt{s}} z_{uv} & g &= \frac{1}{\sqrt{s}} z_{vv} \\
\text{Christoffel Symbols:} \quad & \Gamma_{11}^1 = \frac{1}{s} z_u z_{uu} & \Gamma_{12}^1 &= \frac{1}{s} z_u z_{uv} & \Gamma_{22}^1 &= \frac{1}{s} z_u z_{vv} \\
& \Gamma_{11}^2 = \frac{1}{s} z_v z_{uu} & \Gamma_{12}^2 &= \frac{1}{s} z_v z_{uv} & \Gamma_{22}^2 &= \frac{1}{s} z_v z_{vv}
\end{aligned}$$

Trivia:
- Locally, every surface is one of these (perhaps graphed along a different axis).

2.4 Sphere



With radius $R > 0$,

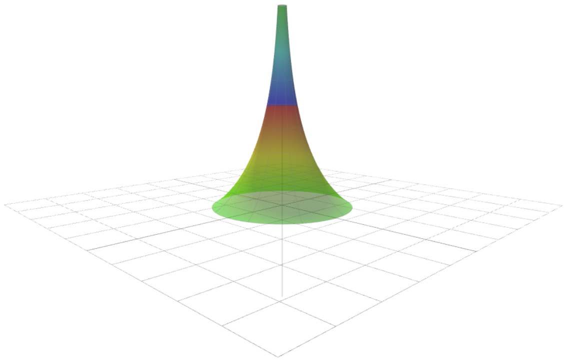
$$
\begin{aligned}
\text{Param:} \quad & \mathbf{x}(u,v) = (R\cos(v)\cos(u), R\cos(v)\sin(u), R\sin(v)) \\
\text{Curvature:} \quad & K = 1/R^2 \\
& H = -1/R \\
\text{Orientable?:} \quad & \text{YES} \\
\text{Geodesics:} \quad & \gamma(t) = (\text{great circles}) \\
\text{Exponential map:} \quad & \exp_p(V) = (\text{walk from } p \text{ until angle changes by } \|V\|/R)
\end{aligned}
$$

Relative to the above paramaterisation,

$$
\begin{aligned}
\text{1ff:} \quad & E = R^2\cos(v)^2 & F = 0 \qquad & G = R^2 \\
\text{2ff:} \quad & e = -R\cos(v)^2 & f = 0 \qquad & g = -R \\
\text{Christoffel Symbols:} \quad & \Gamma^1_{11} = 0 & \Gamma^1_{12} = -\tan(v) \qquad & \Gamma^1_{22} = 0 \\
& \Gamma^2_{11} = \cos(v)\sin(v) & \Gamma^2_{12} = 0 \qquad & \Gamma^2_{22} = 0
\end{aligned}
$$

Trivia:
- Surface of revolution, revolving $(\varphi(v), \psi(v)) = R(\cos(v), \sin(v))$.
- **Constant positive curvature**

2.5 Pseudosphere



With "radius" $R > 0$,

$$\text{Param:} \quad \mathbf{x}(u,v) = \left(Re^{-v/R}\cos(u), Re^{-v/R}\sin(u), \psi(v)\right)$$
$$\psi(v) = \ln\left(1 + \sqrt{1 - R^2 e^{-2v/R}}\right) + 2v/R - \sqrt{1 - R^2 e^{-2v/R}}$$
$$\text{Curvature:} \quad K = -1/R^2$$
$$H = \frac{1 - 2e^{-2v/R}}{2e^{-v/R}\sqrt{1 - e^{-2v/R}}}$$

Orientable?: YES

Geodesics: $\gamma(t) =$ (cursed. Great "pseudo-circles"?)

Exponential map: $\exp_p(V) =$ (cursed.)

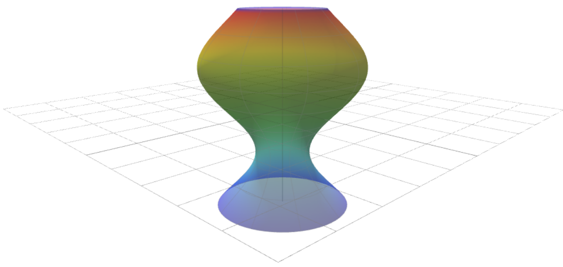
Relative to the above paramaterisation,

1ff:	$E = R^2 e^{-2v/R}$	$F = 0$	$G = 1$
2ff:	$e = Re^{-v/R}\sqrt{1 - e^{-2v/R}}$	$f = 0$	$g = -\dfrac{e^{-v/R}}{R\sqrt{1 - e^{-2v/R}}}$
Christoffel Symbols:	$\Gamma^1_{11} = 0$	$\Gamma^1_{12} = -1/R$	$\Gamma^1_{22} = 0$
	$\Gamma^2_{11} = Re^{-2v/R}$	$\Gamma^2_{12} = 0$	$\Gamma^2_{22} = 0$

Trivia:

- Surface of revolution, revolving the tractrix (φ, ψ) where $\varphi(v) = Re^{-v/R}$ and ψ is as before.
- **Constant negative curvature**

2.6 French Revolution



With $(\varphi(v), \psi(v))$ the curve to be revolved, assumed to be unit speed, and sometimes suppressing (v) in φ and ψ,

$$\begin{aligned}
\text{Param:} \quad & \mathbf{x}(u,v) = (\varphi(v)\cos(u), \varphi(v)\sin(u), \psi(v)) \\
\text{Curvature:} \quad & K = -\frac{\ddot{\varphi}}{\varphi} \\
& H = \frac{\varphi\ddot{\varphi} - \dot{\psi}^2}{2\varphi\dot{\psi}} \\
\text{Orientable?:} \quad & \text{YES} \\
\text{Geodesics:} \quad & \gamma(t) = (\text{things satisfying Clairaut's relation}) \\
\text{Exponential map:} \quad & \exp_p(V) = (\text{something lol})
\end{aligned}$$

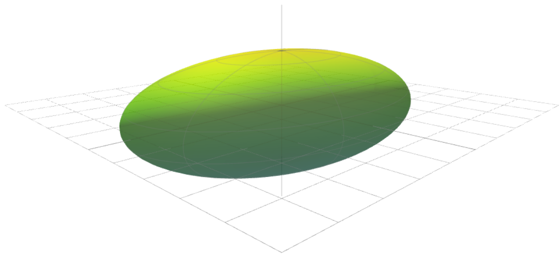
Relative to the above paramaterisation, writing φ and ψ for $\varphi(v)$ and $\psi(v)$,

$$\begin{array}{llll}
\text{1ff:} & E = \varphi^2 & F = 0 & G = 1 \\
\text{2ff:} & e = \varphi\dot{\psi} & f = 0 & g = -\ddot{\varphi}/\dot{\psi} \\
\text{Christoffel Symbols:} & \Gamma^1_{11} = 0 & \Gamma^1_{12} = \dot{\varphi}/\varphi & \Gamma^1_{22} = 0 \\
& \Gamma^2_{11} = -\varphi\dot{\varphi} & \Gamma^2_{12} = 0 & \Gamma^2_{22} = 0
\end{array}$$

Trivia:

- Good source of examples.

2.7 Ellipsoid



With $a, b, c > 0$ and $(x, y, z) = \mathbf{x}(u, v)$,

$$\begin{aligned}
\text{Param:} \quad & \mathbf{x}(u,v) = (a\sin(u)\cos(v), b\sin(u)\sin(v), c\cos(u)) \\
\text{Curvature:} \quad & K = \left(abc\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\right)^{-2} \\
& H = \text{lmao idk} \\
\text{Orientable?:} \quad & \text{YES} \\
\text{Geodesics:} \quad & \gamma(t) = \text{please no} \\
\text{Exponential map:} \quad & \exp_p(V) = \text{please no}
\end{aligned}$$

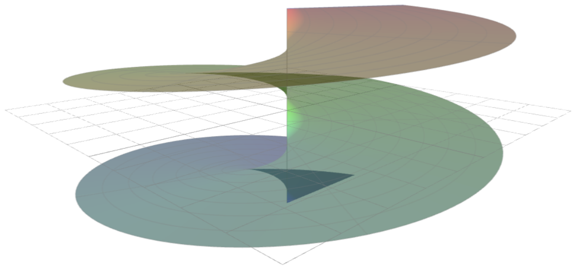
Relative to the above paramaterisation,

$$\begin{aligned}
\text{1ff:} \quad & E = \left(a^2\cos(v)^2 + b^2\sin(v)^2\right)\cos(u)^2 + c^2\sin(u)^2 \\
& F = \left(b^2 - a^2\right)\cos(u)\sin(u)\cos(v)\sin(v) \\
& G = \left(a^2\sin(v)^2 + b^2\cos(v)^2\right)\sin(u)^2 \\
\text{2ff:} \quad & = \text{I just... don't wanna...} \\
\text{Christoffel Symbols:} \quad & = \text{I just... don't wanna...}
\end{aligned}$$

Trivia:
- Surface of revolution (around some axis) only when $a = b$ or $a = c$ or $b = c$.

Credits:
- Gauss curvature: https://math.stackexchange.com/a/1484866

2.8 Helicoid



With $c \in \mathbb{R}$,

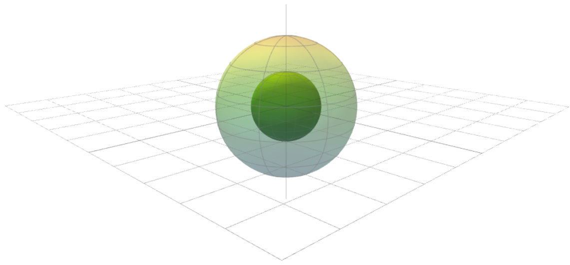
$$\text{Param:} \quad \mathbf{x}(u,v) = (v\cos(u), v\sin(u), cu)$$
$$\text{Curvature:} \quad K = -\frac{c^2}{v^2+c^2}$$
$$H = 0$$
$$\text{Orientable?:} \quad \text{YES}$$
$$\text{Geodesics:} \quad \gamma(t) = \text{(no)}$$
$$\text{Exponential map:} \quad \exp_p(V) = \text{(no)}$$

Relative to the above paramaterisation,

$$\begin{aligned}
&\text{1ff:} && E = v^2 + c^2 && F = 0 && G = 1 \\
&\text{2ff:} && e = 0 && f = \frac{c}{\sqrt{v^2+c^2}} && g = 0 \\
&\text{Christoffel Symbols:} && \Gamma^1_{11} = 0 && \Gamma^1_{12} = \frac{v}{v^2+c^2} && \Gamma^1_{22} = 0 \\
& && \Gamma^2_{11} = -v && \Gamma^2_{12} = 0 && \Gamma^2_{22} = 0
\end{aligned}$$

Trivia:
- **Minimal surface**.

2.9 Scaling



With $c > 0$, $\mathbf{x}(u,v)$ a paramaterisation of a surface S, and $\tilde{S} = cS$ the surface we are cataloguing,

$$
\begin{aligned}
\text{Param:} \quad & \tilde{\mathbf{x}}(u,v) = c\mathbf{x}(u,v) \\
\text{Curvature:} \quad & \tilde{K} = \frac{1}{c^2}K \\
& \tilde{H} = \frac{1}{c}H \\
\text{Orientable?:} \quad & \text{YES} \\
\text{Geodesics:} \quad & \tilde{\gamma}(t) = c\gamma(t) \\
\text{Exponential map:} \quad & \widetilde{\exp}_p(V) = c\exp_p(V)
\end{aligned}
$$

Relative to the above paramaterisation,

$$
\begin{aligned}
\text{1ff:} \quad & \tilde{E} = c^2E & \tilde{F} = c^2F & \quad \tilde{G} = c^2G \\
\text{2ff:} \quad & \tilde{e} = ce & \tilde{f} = cf & \quad \tilde{g} = cg \\
\text{Christoffel Symbols:} \quad & \tilde{\Gamma}^1_{11} = \Gamma^1_{11} & \tilde{\Gamma}^1_{12} = \Gamma^1_{12} & \quad \tilde{\Gamma}^1_{22} = \Gamma^1_{22} \\
& \tilde{\Gamma}^2_{11} = \Gamma^2_{11} & \tilde{\Gamma}^2_{12} = \Gamma^2_{12} & \quad \tilde{\Gamma}^2_{22} = \Gamma^2_{22}
\end{aligned}
$$

Trivia:

- It works exactly how you expect lol