

7. Lyapunov Analysis

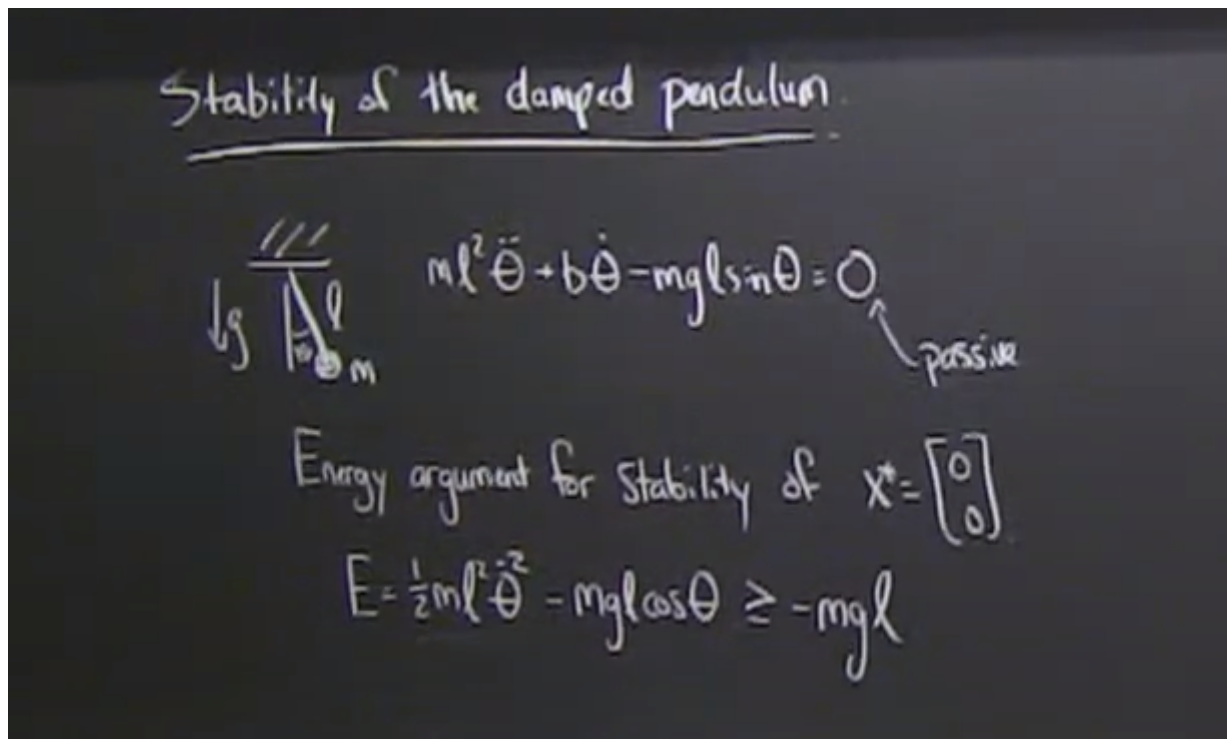
- Lyapunov functions
- Lots of examples
- Energy-based Swing-Up

In last couple lectures, they are about optimal control Dynamic Programming, and in last lecture: Approximate DP with function approximation

Today: Sufficient conditions ask for less than full optimality

"Accomplish the task" is often formulated with tools from stability analysis.

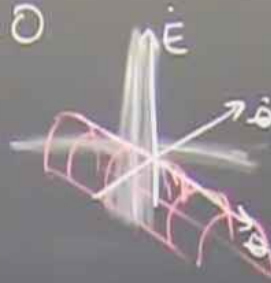
1. Example:



$$\frac{d}{dt} E = \underbrace{ml^2 \ddot{\theta} \dot{\theta}}_{(-b\dot{\theta} + mgl \sin \theta) \dot{\theta}} = \dot{\theta} mgl \sin \theta = -b \dot{\theta}^2 \leq 0$$

$$(-b\dot{\theta} + mgl \sin \theta) \dot{\theta} = -b \dot{\theta}^2$$

$$E \Rightarrow -mgl?$$



Generalize notion of energy function: Lyapunov functions / analysis.

Lyapunov Given $\dot{x} = f(x)$

Goal: analyze stability of f.p. $x^* = 0$

- Lyapunov stability

"If it starts in that region, i will stay in that region."

- Asymptotic stability

- Global A.S.
- Local A.S.
- Regional A.S.

Asymptotic Stability?

$$\lim_{t \rightarrow \infty} x(t) = 0 \quad x(t) \rightarrow x^*$$

Via Lyapunov

$$V(x) > 0, \quad \dot{V}(x) < 0 \Rightarrow \text{asymptotic stability}$$

Local a.s. if $V(x) > 0, \dot{V}(x) < 0$ for $x \in B_r$ small region around $x^* = 0$

global a.s. (g.a.s.) $\forall x$ & radially unbounded.

$$V(x) \rightarrow \infty \text{ when } \|x\| \rightarrow \infty$$

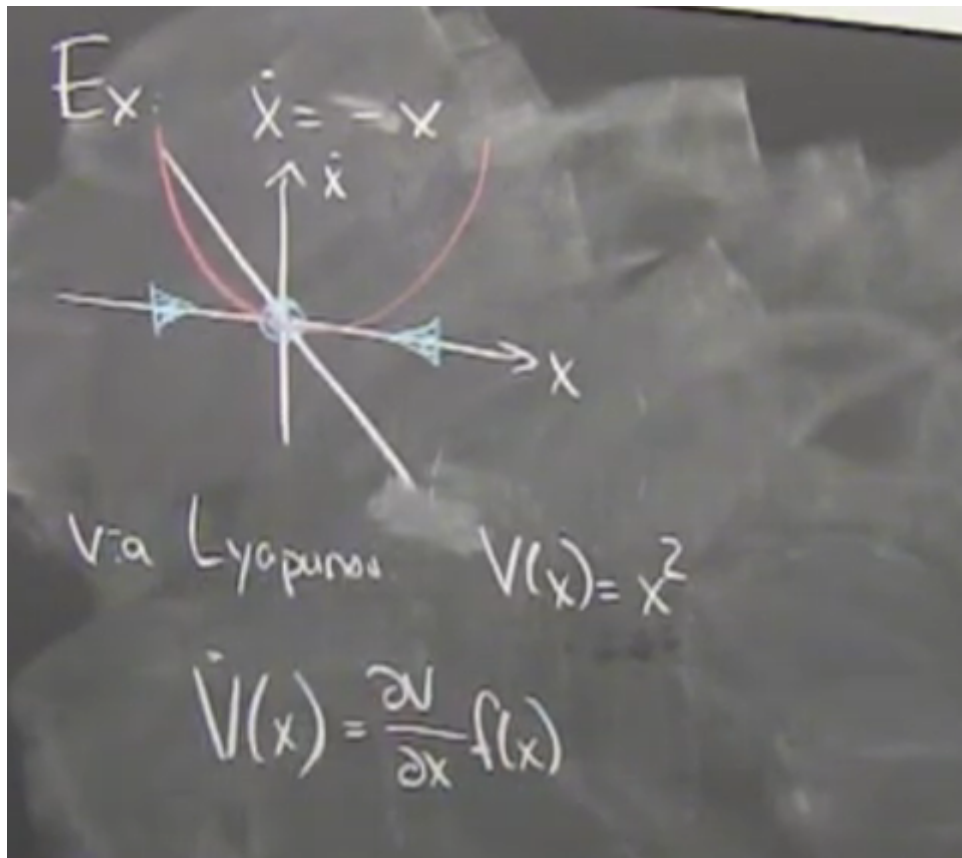
- Exponential stability

$$\|x(t) - x^*\| < ce^{-\alpha t}$$

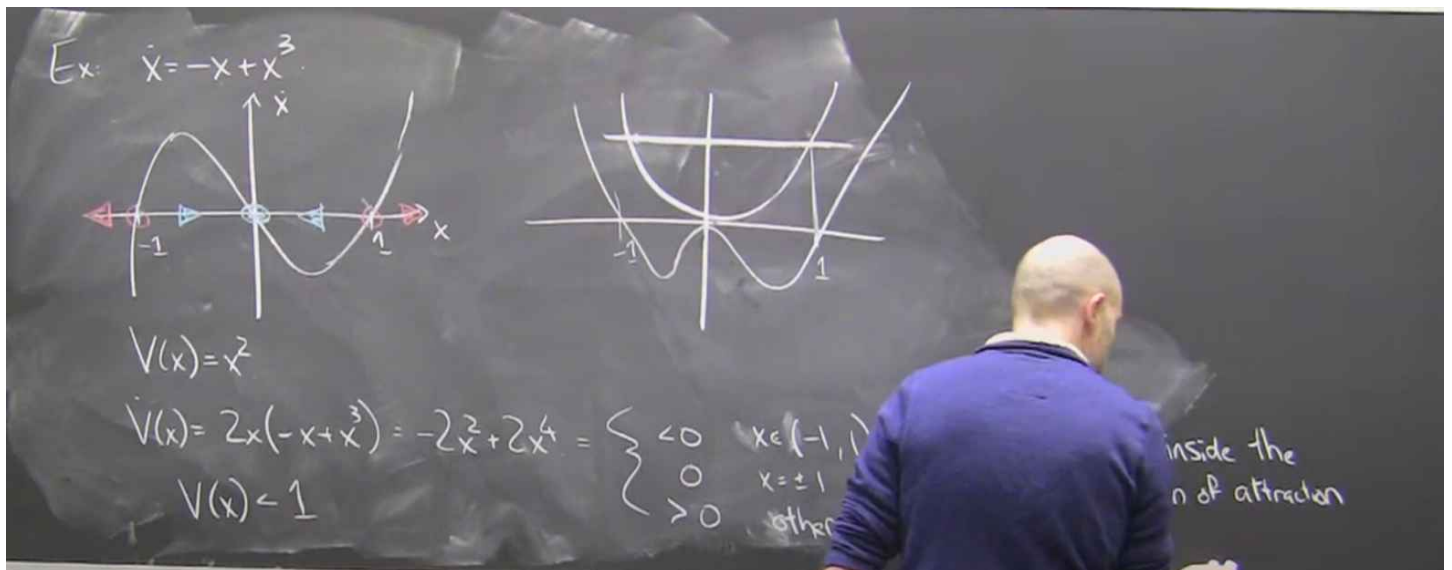
$$V(x) > 0, \quad \dot{V}(x) = -\alpha V(x), \quad \alpha > 0$$

$$\Rightarrow V(x(t)) \leq V(x(0)) \cdot e^{-\alpha t}$$

2. Example



3. Example



- LaSalle's Theorem

LaSalle's Theorem

$$V(x) > 0, \quad \dot{V}(x) \leq 0$$

↑ p.s.d.

$\lim_{t \rightarrow \infty} x(t) \rightarrow$ largest invariant set w/ $\dot{V}(x) = 0$

- Lyapunov Relationship to HJB

Cost function is Lyapunov function. As long as the cost function is positive, it is stable.

As long as cost function is position, every step cost to go function is Lyapunov.

Relationship to HJB

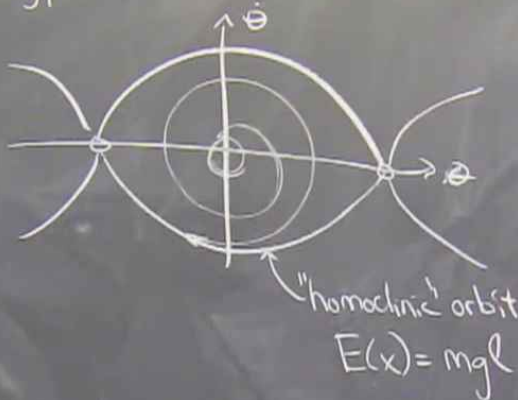
$$0 = \min_u \left[l(x, u) + \frac{\partial J^*}{\partial x} f(x, u) \right]$$

$$\dot{J}(x) \cdot \frac{\partial J}{\partial x} f(x, u^*) = -l(x, u^*) \quad \text{P.D.E.}$$

$$\dot{V}(x) \leq 0$$

4. Example, Energy based Swing-Up

Energy-based Swing-up.



$$mgl\dot{\theta} + mgl\sin\theta = u$$

$$\frac{dE}{dt} = u\dot{\theta}$$

$$E^d = mgl$$

$$V(x) = \frac{1}{2} (E(x) - E^d)^2$$

$$\dot{V}(x) = \dot{E}(x) (E(x) - E^d) = u\dot{\theta} \tilde{E}$$

$$u = -k\dot{\theta} \tilde{E}$$

$$E(x) = mgl$$

$$\dot{V}(x) = \dot{E}(x) (E(x) - E^d) = u\dot{\theta} \tilde{E}$$

$$\text{Choose } u = -k\dot{\theta} \tilde{E}$$

$$k > 0$$

$$\dot{V}(x) = -k\dot{\theta}^2 \tilde{E}^2 \leq 0$$

$$\tilde{E} \rightarrow 0$$