7. Lyapunov Analysis

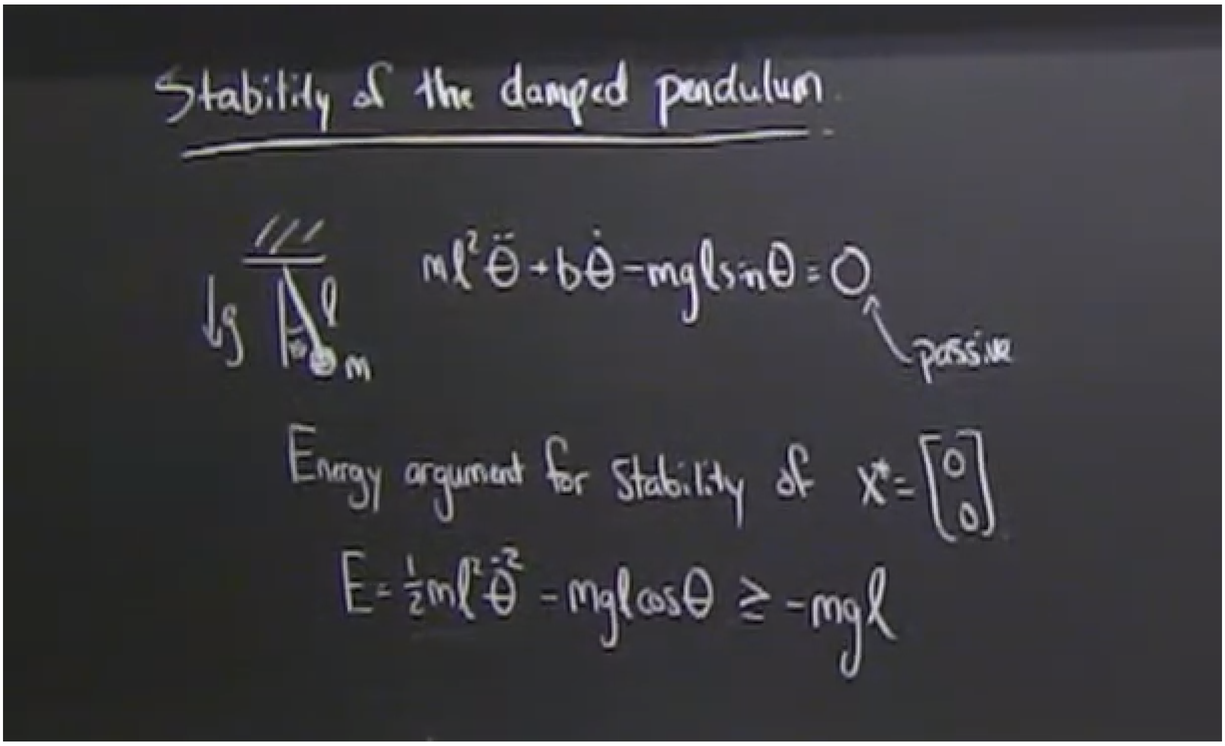
- Lyapunov functions
- Lots of examples
- Energy-based Swing-Up

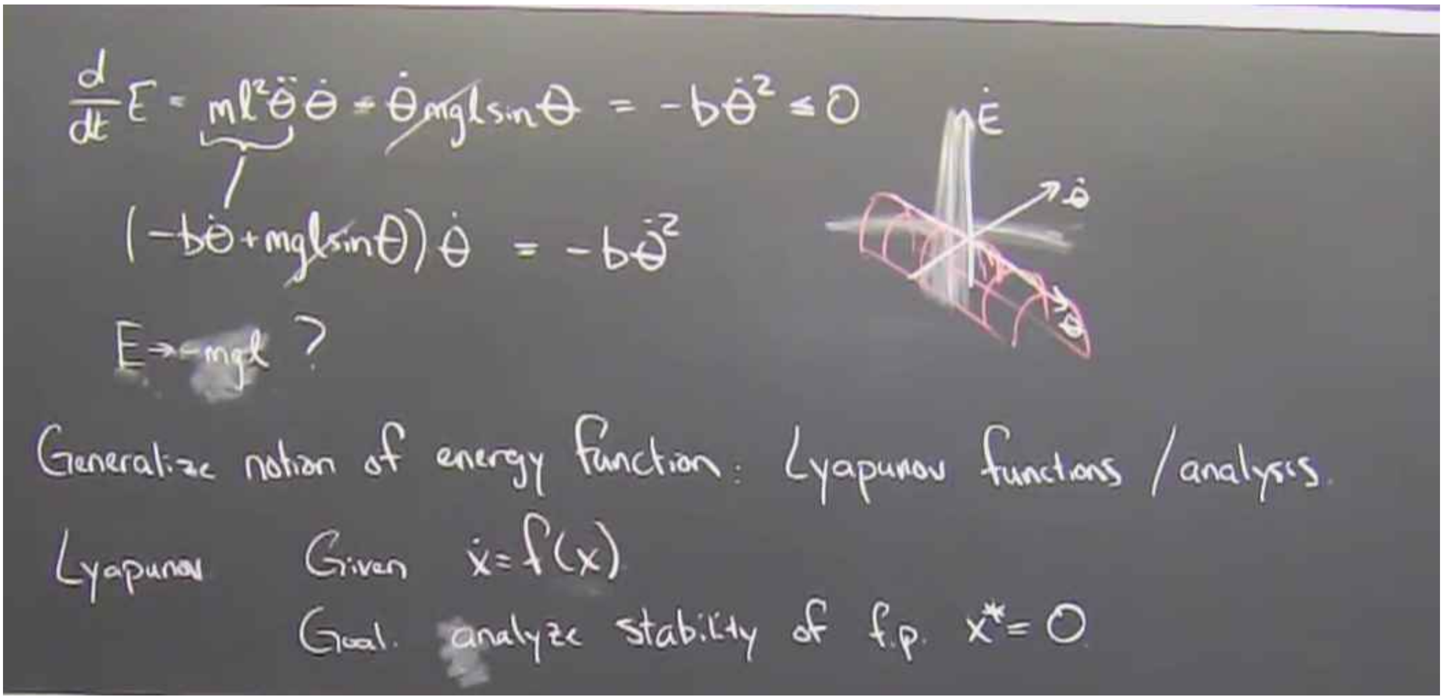
In last couple lectures, they are about optimal control Dynamic Programming, and in last lecture: Approximate DP with function approximation

Today: Sufficient conditions ask for less than full optimality

"Accomplish the task" is often formulated with tools from stability analysis.

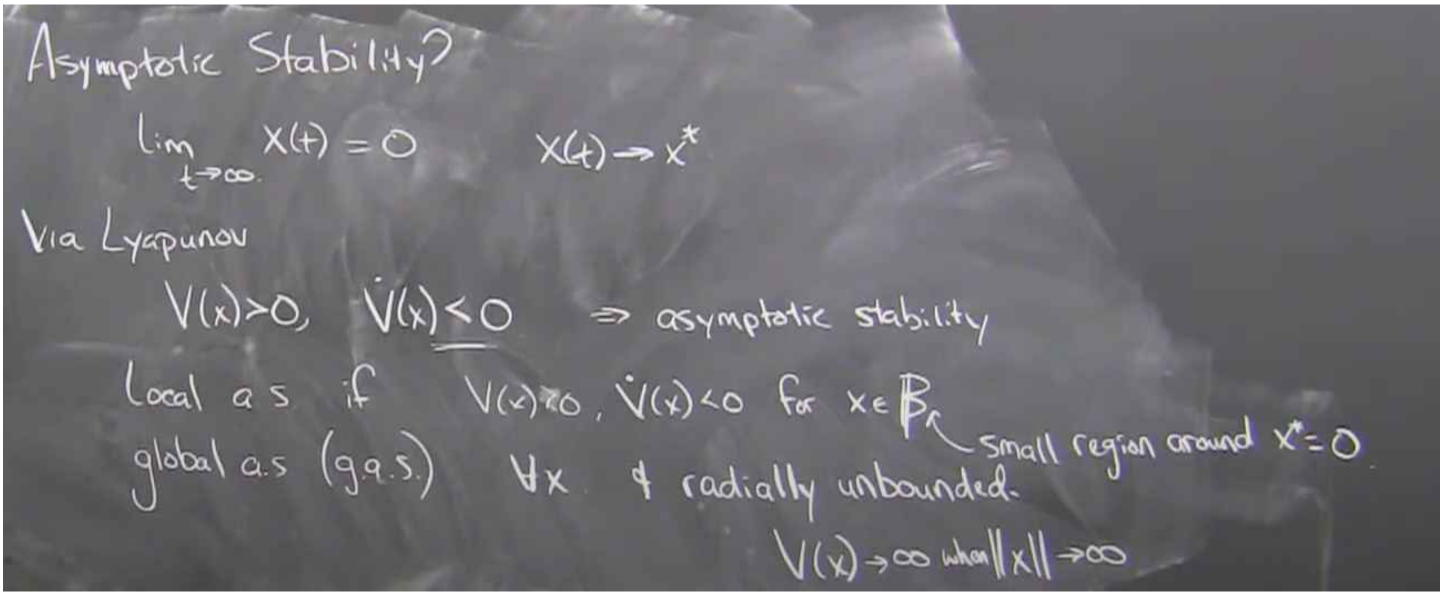
1. Example:





- Lyapunov stability

  "If it starts in that region, i will stay in that region.
- Asmptotic stability
  - Global A.S.
  - Local A.S.
  - Regional A.S.



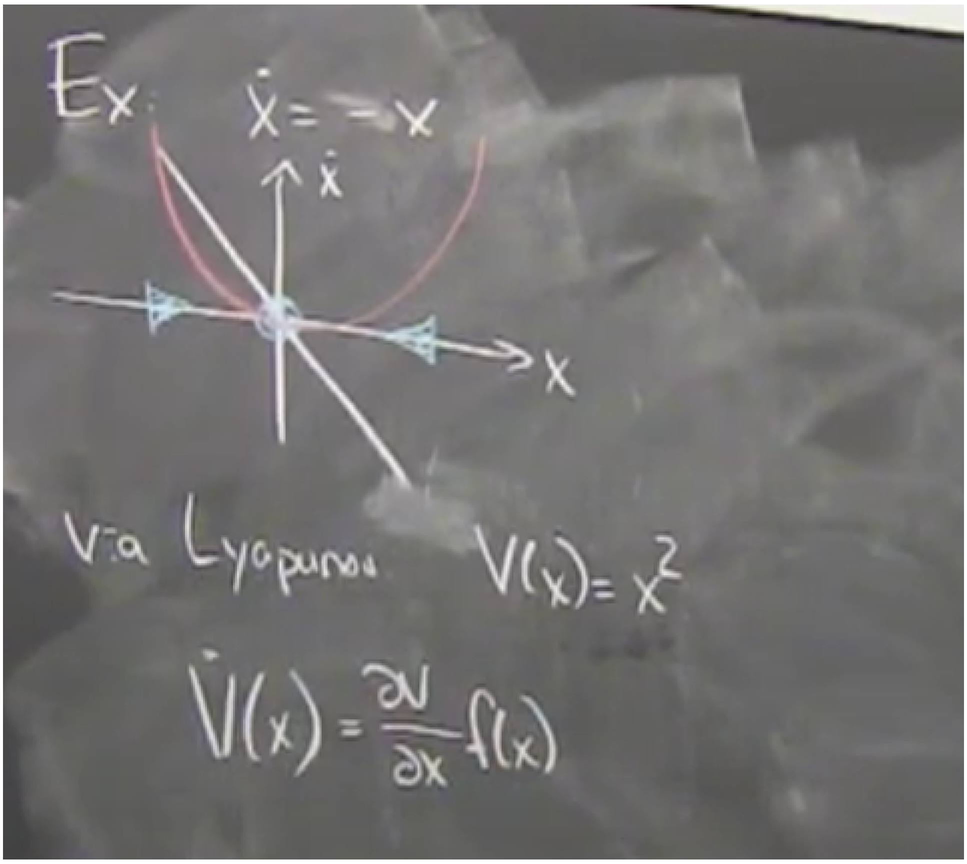
- Exponential stability

$$||x(t) - x^*|| < ce^{-\alpha t}$$

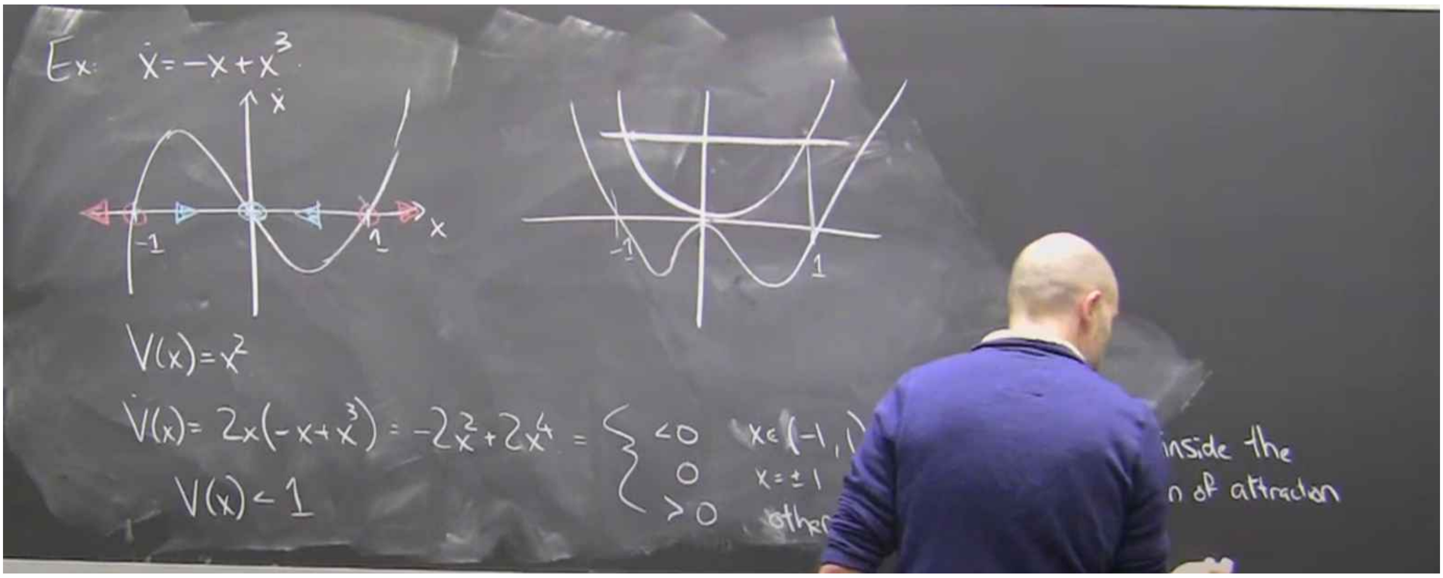
$$V(x) > 0, \quad \dot V(x) = -\alpha\, V(x),\ \alpha\ > 0$$

$$\Rightarrow V(x(t))\ \le\ V(x(0)) \cdot\ e^{-\alpha t}$$

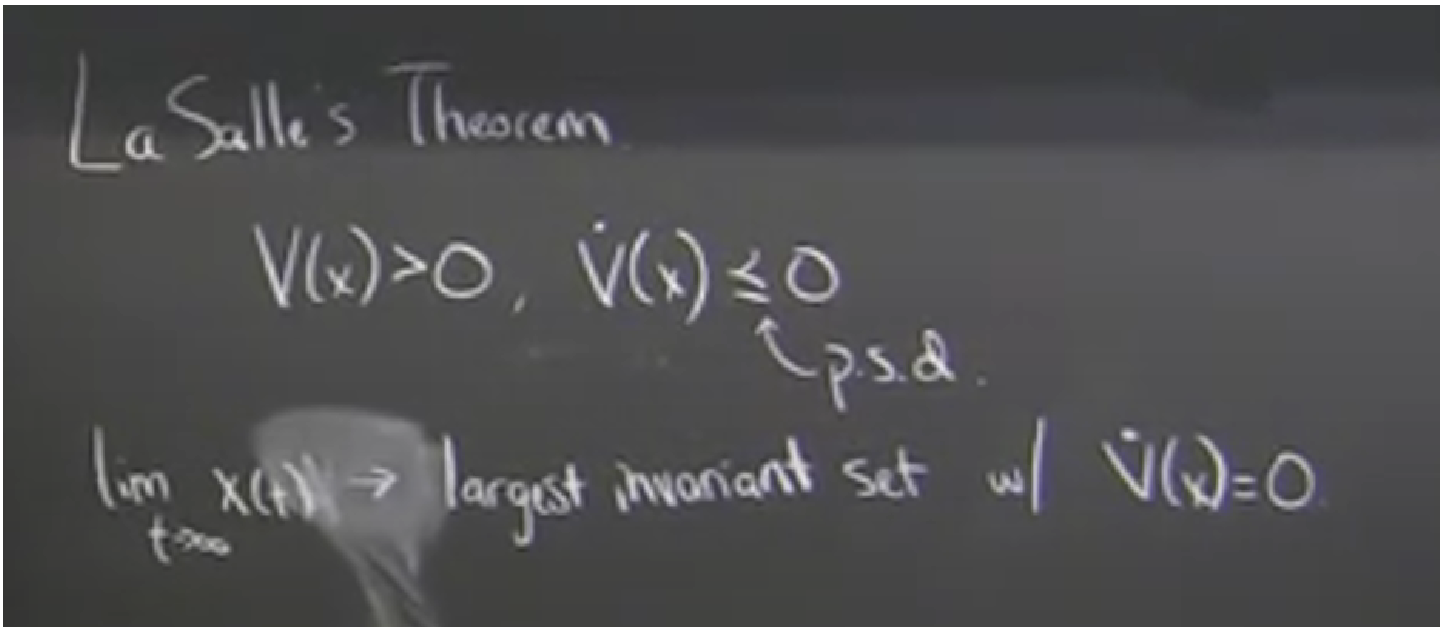
2. Example



3. Example



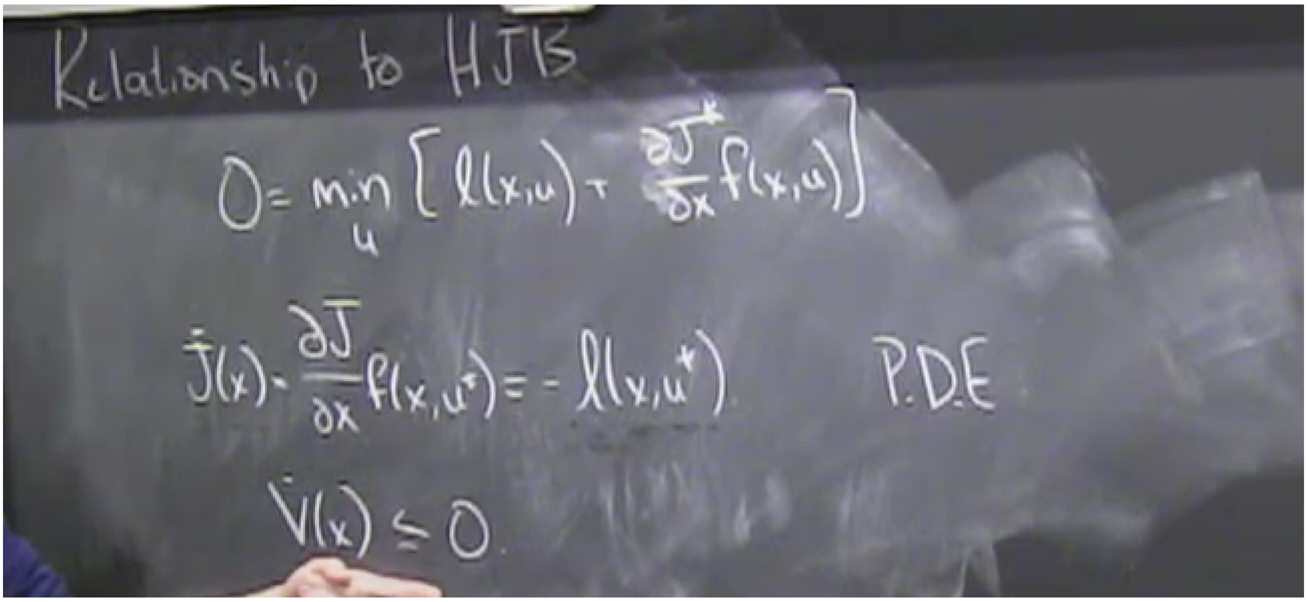
- LaSalle's Theorem



- Lyapunov Relationship to HJB

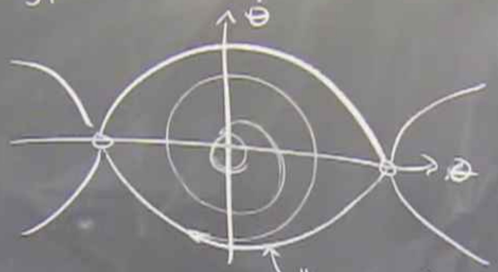
  Cost function is lyapunov function. As long as the cost function is positive, it is stable.

  As long as cost function is position, every step cost to go function is Lyapunov.



4. Example, Energy base Swing-Up

Energy-based swing-up.



"homoclinic" orbit

$E(x) = mgl$

$$mgl\ddot{\theta} + mgl\sin\theta = u$$

$$\frac{dE}{dt} = u\dot{\theta}$$

$$E^d = mgl$$

$$V(x) = \frac{1}{2}\left(E(x) - E^d\right)^2$$

$$\dot{V}(x) = \dot{E}(x)\underbrace{\left(E(x) - E^d\right)}_{\tilde{E}} = u\dot{\theta}\tilde{E}$$

$u = -k\dot{\theta}\tilde{E}$

$E(x) = mgl$

$$\dot{V}(x) = \dot{E}(x)\underbrace{\left(E(x) - E^d\right)}_{\tilde{E}} = u\dot{\theta}\tilde{E}$$

Choose $u = -k\dot{\theta}\tilde{E}$
$k > 0$.

$$\dot{V}(x) = -k\dot{\theta}^2\tilde{E}^2 \leq 0.$$

$$\tilde{E} \to 0.$$