Intro/Review of one- and two-qubit state tomography

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Quantum state tomography

Quantum state tomography (QST):

The procedure of experimentally determining a quantum state (from multiple copies of it)

State tomography is often used to determine performance metrics such as the fidelity of a prepared state to a target pure state:

$$F_{\psi_{\mathrm{t}}}(\rho) = \langle \psi_{\mathrm{t}} | \rho | \psi_{\mathrm{t}} \rangle$$

density matrix of the prepared state

The density matrix

ρ

Properties:
1. $\mathrm{Tr}[\rho]=1$
2. $\rho=\rho^{\dagger}$ (Hermitian)
3. Positive semidefinite: non-negative eigenvalues

For a pure state, $\rho=|\psi\rangle\langle\psi|$

A single-qubit density matrix can always be expressed in the form:

$$\rho=\frac{1}{2}\left(I+\alpha X+\beta Y+\gamma Z\right) \quad \alpha,\beta,\gamma\in R$$

$$|\alpha|^2,|\beta|^2,|\gamma|^2\leq 1$$

Property 3 is satisfied $\Leftrightarrow |\alpha|^2+|\beta|^2+|\gamma|^2\leq 1$

The density matrix

An n-qubit density matrix can always be expressed in the form:

$$\rho = \frac{1}{2^n}\left(\sum_i \alpha_i P_i\right)$$

P_i is the set of n-qubit Pauli operators

α_i are real-valued coefficients

Example:

The 2-qubit Pauli operators are

$$II, IX, IY, IZ, XI, YI, ZI, XX, XY, XZ, YX, YY, YZ, ZX, XY, ZZ$$

here, $IX = I \otimes X$ and so on.

For n-qubits, there are 4^n total n-qubit Pauli operators

Performing state tomography requires determining the coefficients α_i

So the 'cost' of performing tomography scales exponentially with the number of qubits.

Single-qubit state tomography

$$\rho = \frac{1}{2}\left(I + \alpha X + \beta Y + \gamma Z\right)$$

$$\alpha = \mathrm{Tr}\left[\rho X\right] = \left\langle X \right\rangle$$

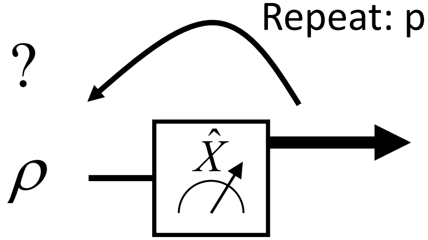
$$\beta = \mathrm{Tr}\left[\rho Y\right] = \left\langle Y \right\rangle$$

$$\gamma = \mathrm{Tr}\left[\rho Z\right] = \left\langle Z \right\rangle$$

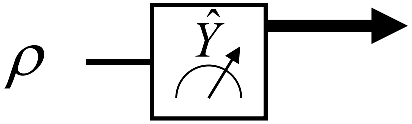
So $\rho = \frac{1}{2}\left(I + \left\langle X \right\rangle X + \left\langle Y \right\rangle Y + \left\langle Z \right\rangle Z\right)$

Experimental 1-Q state tomography

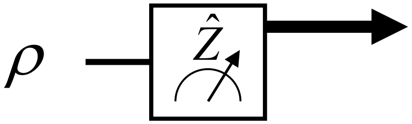
Repeat: prepare state and measure



+1,-1,-1,+1,-1,.....+1, +1 $\longrightarrow$ estimate: $\overline{m} = \left\langle \hat{X} \right\rangle_{\text{est}}$



+1,+1,-1,-1,+1,.....-1, +1 $\longrightarrow$ estimate: $\overline{m} = \left\langle \hat{Y} \right\rangle_{\text{est}}$



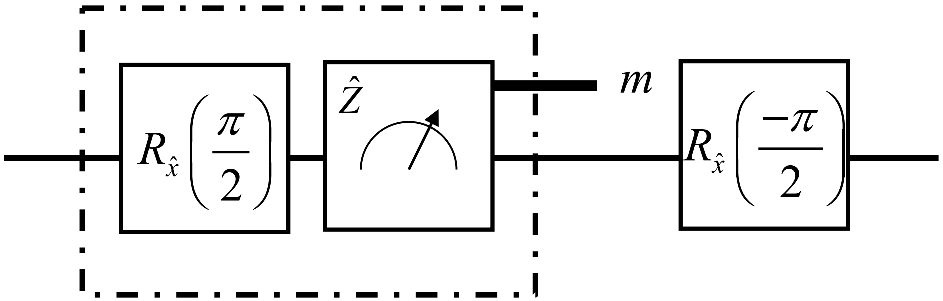
-1,+1,+1,+1,-1,.....-1, -1 $\longrightarrow$ estimate: $\overline{m} = \left\langle \hat{Z} \right\rangle_{\text{est}}$

$$\rho_{\text{est}} = \frac{1}{2}\left(I + \langle X \rangle_{\text{est}} X + \langle Y \rangle_{\text{est}} Y + \langle Z \rangle_{\text{est}} Z \right)$$

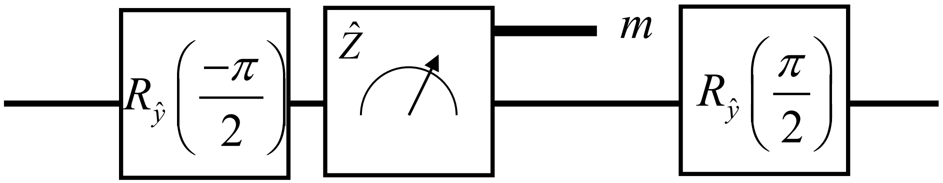
How measurements of X & *Y* are typically done in practice

- Often in the lab, measurements can only be conveniently done for one measurement operator (i.e., one basis, very often *Z*). Other operators can be measured by pre-rotating the qubit, measuring, and post-rotating it back.

- Example: To measure *Y*



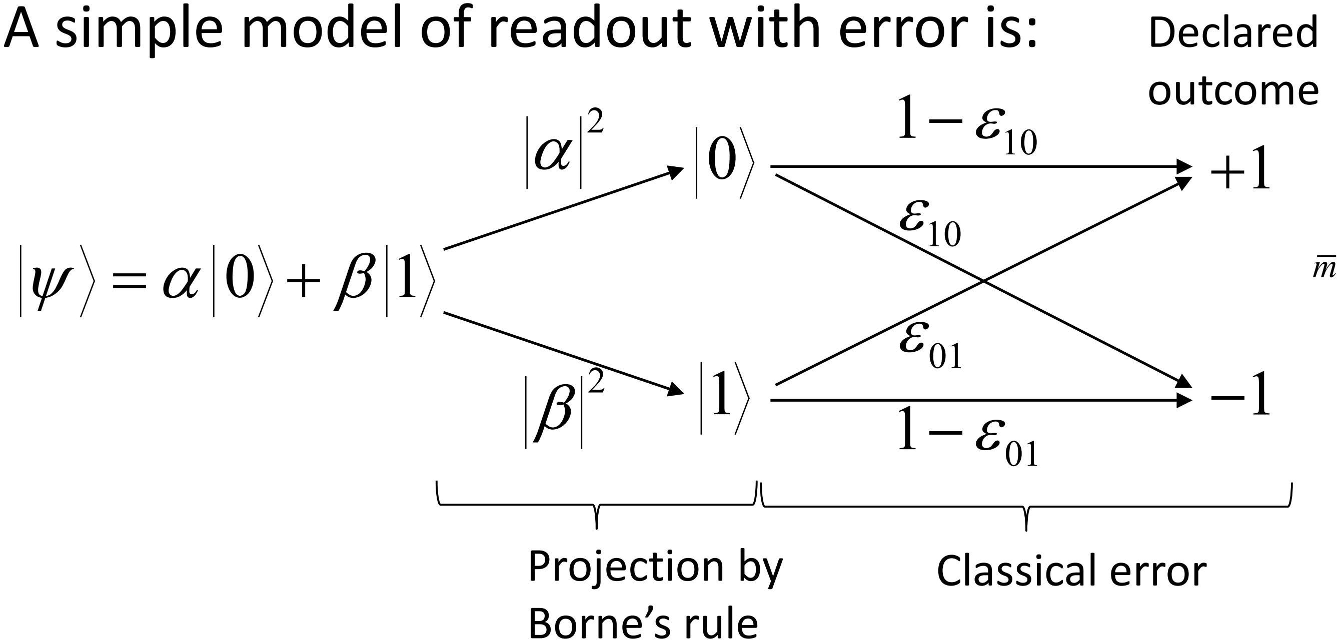
- Example: To measure *X*



Correcting readout error when doing 1-Q state tomography

Errors in the readout can we be corrected to obtain a better estimate of ρ

A simple model of readout with error is:



$$\begin{aligned}\bar{m} &= |\alpha|^2(1-2\varepsilon_{10}) - |\beta|^2(1-2\varepsilon_{01}) \\ &= (\varepsilon_{01}-\varepsilon_{10}) + (1-\varepsilon_{01}-\varepsilon_{10})\left(|\alpha|^2 - |\beta|^2\right)\end{aligned}$$

$$\Rightarrow \bar{m} = \beta_0 + \beta_1 \langle \hat{Z} \rangle$$

Commonly used definitions of readout fidelity:

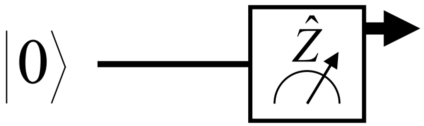
$$F_{\mathrm{ro}} = 1 - (\varepsilon_{01} + \varepsilon_{10})$$ Conservative fidelity

$$F_{\mathrm{a.a.}} = 1 - (\varepsilon_{01} + \varepsilon_{10})/2$$ Average Assignment fidelity

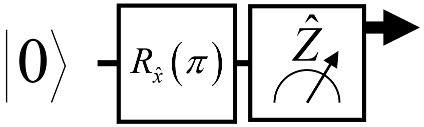
Which one does IBM QX report?

Calibrating the readout

We need to do two calibration experiments:



+1,+1,+1,-1,+1,.....+1, +1 $\longrightarrow$ $\overline{m}_A = \beta_0 + \beta_1$



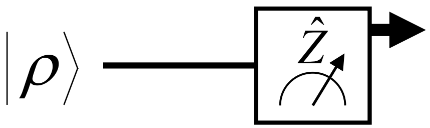
-1,-1,+1,-1,+1,.....-1, -1 $\longrightarrow$ $\overline{m}_B = \beta_0 - \beta_1$

$$\begin{pmatrix} \overline{m}_A \\ \overline{m}_B \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \Rightarrow \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \overline{m}_A \\ \overline{m}_B \end{pmatrix}$$

Here, we assume that initialization into $|0\rangle$ and the Pi pulse are perfect.
This is a fairly good approximation:

- Initialization is usually better than 99%
- Pi pulses have fidelities exceeding 99.9%.

Now that you have the Beta coefficients calibrated, you can use them:



+1,+1,-1,-1,+1,.....+1, +1 $\longrightarrow$ $\overline{m} = \beta_0 + \beta_1 \langle Z \rangle \Rightarrow \langle Z \rangle = \dfrac{\overline{m} - \beta_0}{\beta_1}$

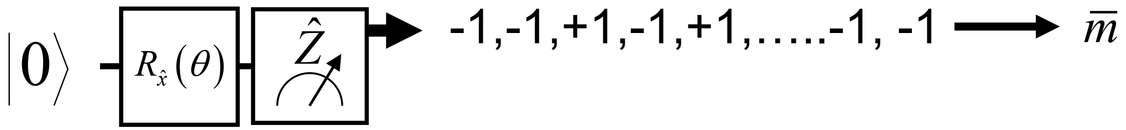
A simple check that everything is nicely calibrated:

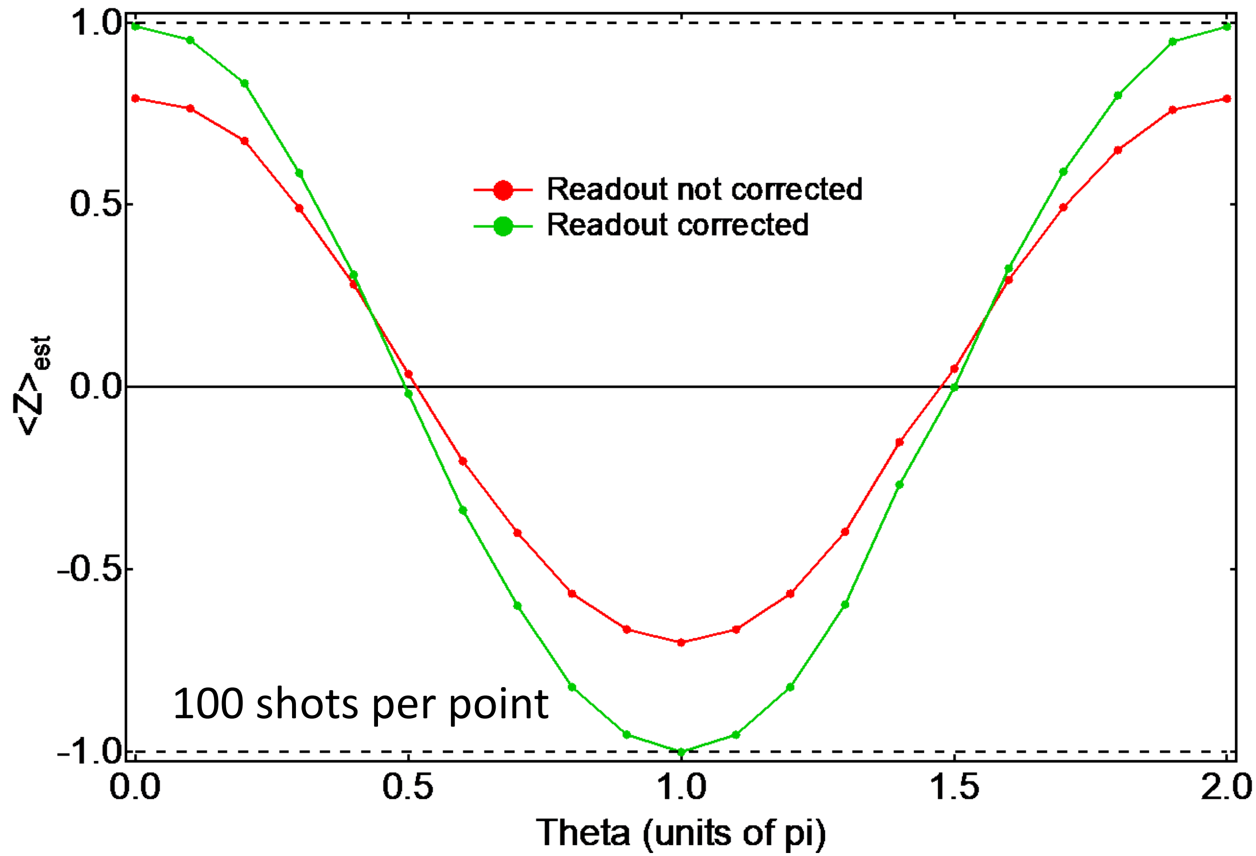
You can check that you recover the full oscillation when doing a Rabi oscillation experiment:

estimate: $\langle Z \rangle_{\text{est}} = \overline{m}$

VS

estimate: $\langle Z \rangle_{\text{est}} = \dfrac{\overline{m} - \beta_0}{\beta_1}$





Two-qubit state tomography

$$
\begin{aligned}
\rho = \frac{1}{4}(II & \\
&+\langle XI\rangle XI+\langle YI\rangle YI+\langle ZI\rangle ZI && \text{Bloch vector of MSQ}\\
&+\langle IX\rangle IX+\langle IY\rangle IY+\langle IZ\rangle IZ && \text{Bloch vector of LSQ}
\end{aligned}
$$

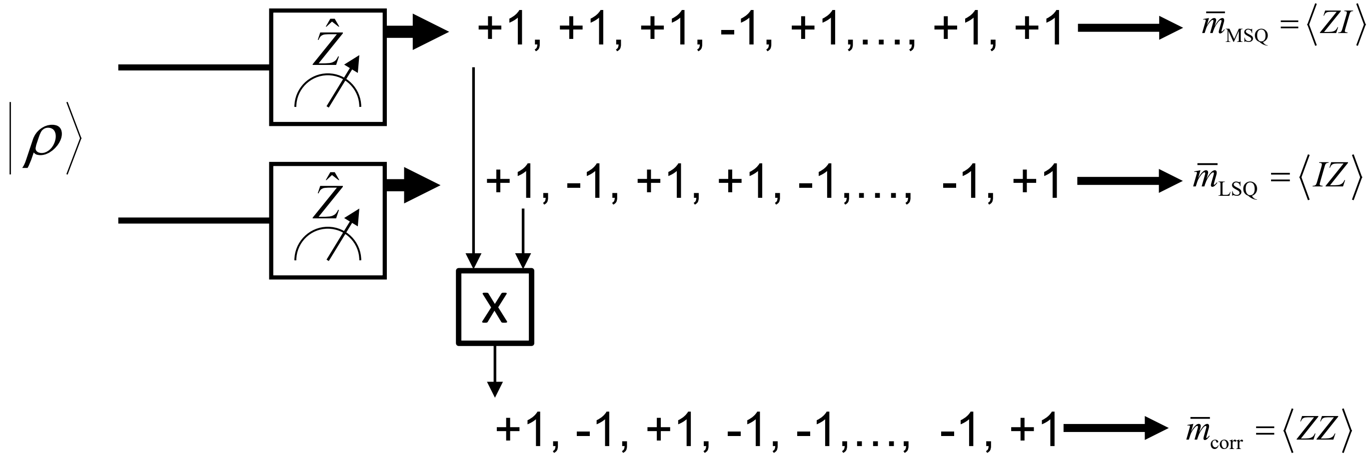
$$
\left.
\begin{aligned}
&+\langle XX\rangle XX+\langle XY\rangle XY+\langle XZ\rangle XZ\\
&+\langle YX\rangle YX+\langle YY\rangle YY+\langle YZ\rangle YZ\\
&+\langle ZX\rangle ZX+\langle ZY\rangle ZY+\langle ZZ\rangle ZZ)
\end{aligned}
\right\}\ \text{Two-qubit correlations}
$$

The fifteen coefficients can be extracted from a total of 9 circuits:

Measurement on MSQ	Measurement on LSQ	Coefficients Extracted
X	X	<XI>, <IX>, <XX>
X	Y	<XI>, <IY>, <XY>
X	Z	<XI>, <IZ>, <XZ>
Y	X	<YI>, <IX>, <YX>
Y	Y	<YI>, <IY>, <YY>
Y	Z	<YI>, <IZ>, <YZ>
Z	X	<ZI>, <IX>, <ZX>
Z	Y	<ZI>, <IY>, <ZY>
Z	Z	<ZI>, <IZ>, <ZZ>

Correcting readout error when doing 2-Q state tomography

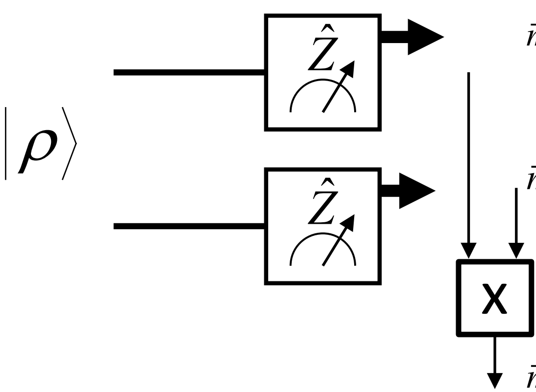
Ideally, as in fundamentals:



In reality, readout is compromised by:

- readout error
- readout crosstalk

As a result of these errors:

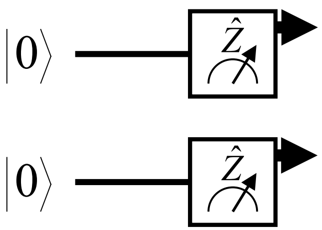


$$\left.\begin{aligned}
\bar{m}_{\mathrm{MSQ}} &= \beta_{0,\mathrm{M}} + \beta_{1,\mathrm{M}} \langle IZ \rangle + \beta_{2,\mathrm{M}} \langle ZI \rangle + \beta_{3,\mathrm{M}} \langle ZZ \rangle \\
\bar{m}_{\mathrm{LSQ}} &= \beta_{0,\mathrm{L}} + \beta_{1,\mathrm{L}} \langle IZ \rangle + \beta_{2,\mathrm{L}} \langle ZI \rangle + \beta_{3,\mathrm{L}} \langle ZZ \rangle \\
\bar{m}_{\mathrm{corr}} &= \beta_{0,\mathrm{corr}} + \beta_{1,\mathrm{corr}} \langle IZ \rangle + \beta_{2,\mathrm{corr}} \langle ZI \rangle + \beta_{3,\mathrm{corr}} \langle ZZ \rangle
\end{aligned}\right\}
\begin{pmatrix} \bar{m}_{\mathrm{MSQ}} \\ \bar{m}_{\mathrm{LSQ}} \\ \bar{m}_{\mathrm{corr}} \end{pmatrix} = \begin{pmatrix} \beta_{0,\mathrm{M}} \\ \beta_{0,\mathrm{L}} \\ \beta_{0,\mathrm{corr}} \end{pmatrix} + \underbrace{\begin{pmatrix} \beta_{1,\mathrm{M}} & \beta_{2,\mathrm{M}} & \beta_{3,\mathrm{M}} \\ \beta_{1,\mathrm{L}} & \beta_{2,\mathrm{L}} & \beta_{3,\mathrm{L}} \\ \beta_{1,\mathrm{corr}} & \beta_{2,\mathrm{corr}} & \beta_{3,\mathrm{corr}} \end{pmatrix}}_{B} \begin{pmatrix} \langle IZ \rangle \\ \langle ZI \rangle \\ \langle ZZ \rangle \end{pmatrix}$$

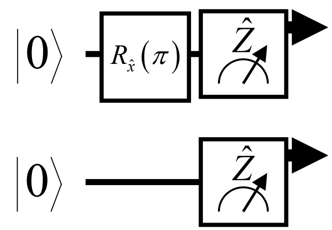
Correcting readout error when doing 2-Q state tomography

Thus, $$\begin{pmatrix} \langle IZ \rangle_{\text{est}} \\ \langle ZI \rangle_{\text{est}} \\ \langle ZZ \rangle_{\text{est}} \end{pmatrix} = B^{-1} \left(\begin{pmatrix} \overline{m}_{\text{MSQ}} \\ \overline{m}_{\text{LSQ}} \\ \overline{m}_{\text{corr}} \end{pmatrix} - \begin{pmatrix} \beta_{0,\text{M}} \\ \beta_{0,\text{L}} \\ \beta_{0,\text{corr}} \end{pmatrix} \right)$$

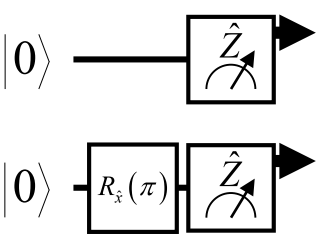
You need four calibration circuits to calibrate the β coefficients:



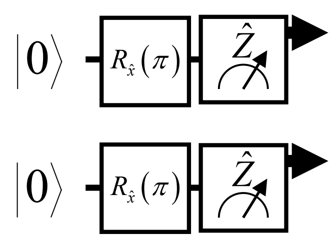
$$\overline{m}_{\text{A,MSQ}} = \beta_{0,\text{M}} + \beta_{1,\text{M}} + \beta_{2,\text{M}} + \beta_{3,\text{M}}$$
$$\overline{m}_{\text{A,corr}} = \beta_{0,\text{corr}} + \beta_{1,\text{corr}} + \beta_{2,\text{corr}} + \beta_{3,\text{corr}}$$
$$\overline{m}_{\text{A,LSQ}} = \beta_{0,\text{L}} + \beta_{1,\text{L}} + \beta_{2,\text{L}} + \beta_{3,\text{L}}$$



$$\overline{m}_{\text{C,MSQ}} = \beta_{0,\text{M}} + \beta_{1,\text{M}} - \beta_{2,\text{M}} - \beta_{3,\text{M}}$$
$$\overline{m}_{\text{C,corr}} = \beta_{0,\text{corr}} + \beta_{1,\text{corr}} - \beta_{2,\text{corr}} - \beta_{3,\text{corr}}$$
$$\overline{m}_{\text{C,LSQ}} = \beta_{0,\text{L}} + \beta_{1,\text{L}} - \beta_{2,\text{L}} - \beta_{3,\text{L}}$$



$$\overline{m}_{\text{B,MSQ}} = \beta_{0,\text{M}} - \beta_{1,\text{M}} + \beta_{2,\text{M}} - \beta_{3,\text{M}}$$
$$\overline{m}_{\text{B,corr}} = \beta_{0,\text{corr}} - \beta_{1,\text{corr}} + \beta_{2,\text{corr}} - \beta_{3,\text{corr}}$$
$$\overline{m}_{\text{B,LSQ}} = \beta_{0,\text{L}} - \beta_{1,\text{L}} + \beta_{2,\text{L}} - \beta_{3,\text{L}}$$



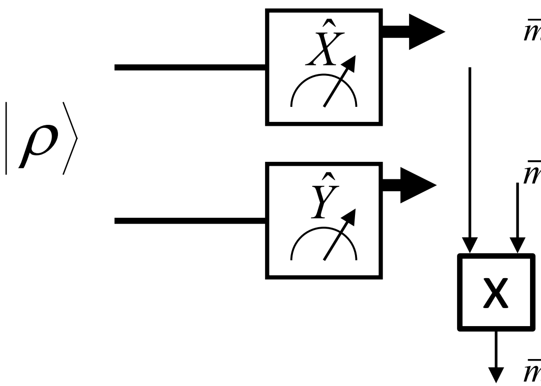
$$\overline{m}_{\text{D,MSQ}} = \beta_{0,\text{M}} - \beta_{1,\text{M}} - \beta_{2,\text{M}} + \beta_{3,\text{M}}$$
$$\overline{m}_{\text{D,corr}} = \beta_{0,\text{corr}} - \beta_{1,\text{corr}} - \beta_{2,\text{corr}} + \beta_{3,\text{corr}}$$
$$\overline{m}_{\text{C,LSQ}} = \beta_{0,\text{L}} - \beta_{1,\text{L}} - \beta_{2,\text{L}} + \beta_{3,\text{L}}$$

$$\begin{pmatrix} \overline{m}_{\text{A,MSQ}} \\ \overline{m}_{\text{B,MSQ}} \\ \overline{m}_{\text{C,MSQ}} \\ \overline{m}_{\text{D,MSQ}} \end{pmatrix} = \underbrace{\begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & -1 & -1 & +1 \end{pmatrix}}_{M_{\text{MSQ}}} \begin{pmatrix} \beta_{0,\text{M}} \\ \beta_{1,\text{M}} \\ \beta_{2,\text{M}} \\ \beta_{3,\text{M}} \end{pmatrix} \Rightarrow \begin{pmatrix} \beta_{0,\text{M}} \\ \beta_{1,\text{M}} \\ \beta_{2,\text{M}} \\ \beta_{3,\text{M}} \end{pmatrix} = M_{\text{MSQ}}^{-1} \begin{pmatrix} \overline{m}_{\text{A,MSQ}} \\ \overline{m}_{\text{B,MSQ}} \\ \overline{m}_{\text{C,MSQ}} \\ \overline{m}_{\text{D,MSQ}} \end{pmatrix}$$

and similarly for LSQ and corr.

Correcting readout error when doing 2-Q state tomography

This approach works similarly when you are measuring in different bases. For example, when you measure *Y* on LSQ and *X* on MSQ:



$$\left.\begin{aligned}
\bar{m}_{\mathrm{MSQ}} &= \beta_{0,\mathrm{M}} + \beta_{1,\mathrm{M}}\langle IY\rangle + \beta_{2,\mathrm{M}}\langle XI\rangle + \beta_{3,\mathrm{M}}\langle XY\rangle \\
\bar{m}_{\mathrm{LSQ}} &= \beta_{0,\mathrm{L}} + \beta_{1,\mathrm{L}}\langle IY\rangle + \beta_{2,\mathrm{L}}\langle XI\rangle + \beta_{3,\mathrm{L}}\langle XY\rangle \\
\bar{m}_{\mathrm{corr}} &= \beta_{0,\mathrm{corr}} + \beta_{1,\mathrm{corr}}\langle IY\rangle + \beta_{2,\mathrm{corr}}\langle XI\rangle + \beta_{3,\mathrm{corr}}\langle XY\rangle
\end{aligned}\right\} \begin{pmatrix} \bar{m}_{\mathrm{MSQ}} \\ \bar{m}_{\mathrm{LSQ}} \\ \bar{m}_{\mathrm{corr}} \end{pmatrix} = \begin{pmatrix} \beta_{0,\mathrm{M}} \\ \beta_{0,\mathrm{L}} \\ \beta_{0,\mathrm{corr}} \end{pmatrix} + \underbrace{\begin{pmatrix} \beta_{1,\mathrm{M}} & \beta_{2,\mathrm{M}} & \beta_{3,\mathrm{M}} \\ \beta_{1,\mathrm{L}} & \beta_{2,\mathrm{L}} & \beta_{3,\mathrm{L}} \\ \beta_{1,\mathrm{corr}} & \beta_{2,\mathrm{corr}} & \beta_{3,\mathrm{corr}} \end{pmatrix}}_{B} \begin{pmatrix} \langle IY\rangle \\ \langle XI\rangle \\ \langle XY\rangle \end{pmatrix}$$

The offset vector and the *B* matrix are the same, regardless of measurement bases (You do not need to recalibrate them).

Thus,

$$\begin{pmatrix} \langle IY\rangle_{\mathrm{est}} \\ \langle XI\rangle_{\mathrm{est}} \\ \langle XY\rangle_{\mathrm{est}} \end{pmatrix} = B^{-1}\left(\begin{pmatrix} \bar{m}_{\mathrm{MSQ}} \\ \bar{m}_{\mathrm{LSQ}} \\ \bar{m}_{\mathrm{corr}} \end{pmatrix} - \begin{pmatrix} \beta_{0,\mathrm{M}} \\ \beta_{0,\mathrm{L}} \\ \beta_{0,\mathrm{corr}} \end{pmatrix}\right).$$