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MILP formulation for solving minimum time optimal control problems

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A mixed integer linear programming (MILP) formulation for solving minimum time optimal control problems for linear time-invariant discrete-time systems is presented. The optimization is based on a discretized linear model and minimum final time criteria. Given constraints on both the states and the inputs, the formulation allows for calculation of the minimum time directly. The size of the MILP as a function of model parameters is shown. Finally, some examples from the literature are solved to demonstrate the new method.

1. Introduction

Considerable work has been done on solving the minimum time optimal control problem. Optimization problems that allow a free end time (two-point boundary-value problems) give rise to numerical difficulties because of the presence of switching times (Bryson and Ho 1969).

For a linear discrete time-invariant system, given a discretized linear model with constraints, the problem can be stated as follows.

Controller C1

$$\min t_f \tag{1}$$

subject to

$$x_{k+1} = \Phi x_k + \Gamma u_k \tag{2}$$

$$Wx^* + Xu^* \leqslant \alpha \tag{3}$$

$$Yx_{t_f} + Zu_{t_f} - \beta = 0 \tag{4}$$

$$x_0 \text{ given} \tag{5}$$

where t_f is the final time, $x^* = [x_1^T \quad x_2^T \quad \dots \quad x_{t_f}^T]^T$, $u^* = [u_1^T \quad u_2^T \quad \dots \quad u_{t_f}^T]^T$, x_k and u_k are n_x- and n_y-vectors, respectively, x_0 is the initial condition, and W, X, Y and Z are constant coefficient matrices. The smallest value of t_f for which the final condition (4) is satisfied is the minimum time.

The general constraint (3) may include bounds on the magnitudes of the variables x_k and u_k and their derivatives. Constraint (2) is the linear time-invariant discrete-time model. The terminal condition (4) may require the system to arrive at a set point and be at rest.

Methods based on linear programming (LP) are commonly used to solve for Controller C1. A general review of LP methods can be found in Gutman (1982). The

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idea is that for a given t_f one solves an LP whose objective is to minimize a norm (usually the ∞-norm) of the final condition (4), subject to the constraints (2) and (3). The minimum time is again the smallest t_f for which (4) is zero.

2. Formulation

We are given a discrete linear model of a plant (2) with manipulated variables and/or state constraints (3). The goal is to take the plant from an initial condition (5) to a final condition (4) in the minimum time.

When solving for Controller C1 with LP methods, t_f must be specified and one must iterate over final time. For each t_f one solves an LP whose objective is to minimize a scalar norm of the final condition (4). The minimum time is given as the smallest time for which this norm is zero. In the MILP formulation, final time is expressed in the objective function directly with the use of binary variables, $z[i]$, as follows:

$$t_f = 1 + \sum_{i=1}^{n_{\mathrm{binary}}} z[i]2^{i-1} \tag{6}$$

where n_{binary} is the total number of binary variables. Note that different 0-1 combinations of the $z[i]$ represent each and all of the discrete points from one to H, with $H = 2^{n_{\mathrm{binary}}}$. Also we select $H \leqslant t_f$ so that the terminal condition (4) can be satisfied and Controller C1 has a solution.

We can set up constraint (4) using the binary variables $z[i]$ as follows. We assume that (4) is expressed in terms of the ∞-norm, which is chosen to obtain an MILP rather than an MINLP.

$$\| Yx_{t_f} + Zu_{t_f} - \beta \|_{\infty} = 0 \tag{7}$$

We can set up constraint (7) for each discrete point (from $k = 0$ to t_f), forcing it to go to zero only at final time. Using binary variables, these logical constraints can be expressed as follows:

$$\| Yx_k + Zu_k - \beta \|_{\infty} \leqslant \left(|S_k| - \sum_{i \in S_k} z[i] + \sum_{i \notin S_k} z[i] \right) M \tag{8}$$

where M is an upper bound on (7). $|S_k|$ is the cardinality of the set S_k, and the set S_k is defined by

$$S_k = \left\{ i \,|\, 1 + \sum_{i \in S_k} 2^{i-1} = k \right\} \tag{9}$$

where k goes from one to the horizon H. For example, for $H = 4: S_1 = \phi$, $S_2 = \{1\}$, $S_3 = \{2\}$, $S_4 = \{1, 2\}$.

Constraint (8) is set up so that (7) will be forced to zero only at the final time. At other discrete points, (8) will become superfluous because $\| Yx_k + Zu_k - \alpha_3 \|_{\infty}$ will be bounded by at least M. That is, the right-hand sides of (8) are non-negative, bounding $\| Yx_k + Zu_k - \alpha_3 \|_{\infty}$ always with a larger positive number. That way we avoid imposing the final condition on any other discrete point.

Another advantage of our formulation is its flexibility. For example, in order to track a setpoint change r, constraint (8) becomes

$$\| Cx_k - r \|_{\infty} \leqslant \left(|S_k| - \sum_{i \in S_k} z[i] + \sum_{i \notin S_k} z[i] \right) M \tag{10}$$

where C is the matrix relating the states to the outputs. If in addition to tracking r the plant needs to be at rest at the final time, one would solve for Controller C1 with the additional constraints

$$\|(I-\Phi)x_k-\Gamma u_k\|_\infty \leqslant \left(|S_k| - \sum_{i\in S_k} z[i] + \sum_{i\in S_k} z[i]\right)M \tag{11}$$

where the left-hand side of (11) was obtained by substituting $x_{k+1}=x_k$ into (2).

Finally, one may add the term $\varepsilon\|u_k\|_1$, with $\varepsilon\|u_k\|_1 \ll t_f$, to the objective function (6) so that, whenever more than one control profile can achieve minimum time, the one that requires minimum effort will be chosen.

3. Computational details

The MILP problem to be solved is as follows.

Controller C2

$$\min t_f + \varepsilon\|u_k\|_1 \tag{12}$$

subject to

$$t_f = 1 + \sum_{i=1}^{n_{\text{binary}}} z[i]2^{i-1} \tag{13}$$

$$\|Yx_k + Zu_k - \beta\|_\infty \leqslant \left(|S_k| - \sum_{i\in S_k} z[i] + \sum_{i\notin S_k} z[i]\right)M \tag{14}$$

$$x_{k+1} = \Phi x_k + \Gamma u_k \tag{15}$$

$$x_0 \text{ given} \tag{16}$$

$$Wx^* + Xu^* \leqslant \alpha \tag{17}$$

$$S_k = \left\{ i \,|\, 1 + \sum_{i\in S_k} 2^{i-1} = k \right\} \tag{18}$$

$$z[i] \in (0, 1) \tag{19}$$

For numerical reasons, one must allow for some tolerance on constraint (14). That way one allows for the right-hand side being small instead of zero at the final time. Also, we need to consider the norm of each element of the vector (14) as a separate constraint, increasing the problem size considerably. Table 1 shows the size of the optimization problem as a function of model parameters.

Variables		Equalities		Inequalities	
1	t_f	1	(12)	$\leqslant 2H(n_x+n_u)$	(14)
$\ln H/\ln 2$	$z[i]$	$n_x \cdot H$	(15)	$\leqslant 2H(n_x+n_u)$	(17)
$n_x(H+1)$	x_k	n_x	(16)		
$n_u H$	u_k				
$(n_u+n_x)\cdot H + n_x + \ln H/\ln 2 + 1$		$n_x \cdot (H+1)+1$		$\leqslant 4 \cdot H \cdot (n_x+n_u)$	

Table 1. Size of MILP as a function of model parameters (terminal condition: zero error at rest).

For on-line applications the LP method is most often used. After an initial guess as to the minimum time the search proceeds by increasing (or decreasing) this guess by one until the minimum time is found. This method is very inefficient for start-up or when either large distrubances or set point changes need to be tracked, since the initial guess will usually be far from the actual minimum time (Gutman 1982). For these particular problems the MILP method proposed here is quicker in estimating the minimum time.

More efficient methods for handling the above situation are based on region elimination. We found our method took approximately the same CPU time as interval halving, without considering the time to expand or truncate the LPs between evaluations for the interval halving method. Also, we used the 'Branch & Bound' algorithm available in GAMS/ZOOM (Kendrick and Meeraus 1985), which does not exploit the structure of our problem.

4. Examples

Some minimum time problems will be solved to demonstrate our method. All MILPs were solved using ZOOM with the modelling language GAMS (Kendrick and Meeraus 1985). We solved for final times to take each system from a given initial condition to the origin.

4.1. *Double integrator*

The continuous dynamic model for a double integrator is given by

$$\left.\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ x_0 &= \begin{bmatrix} 10 \\ 10 \end{bmatrix} \end{aligned}\right\} \tag{20}$$

with

$$-1 \leqslant u \leqslant 1 \tag{21}$$

For a discretization time of 1 s the discrete model becomes

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0{\cdot}5 \\ 1 \end{bmatrix} u_k \tag{22}$$

For five binary variables the horizon is 32 s. To understand constraint (8), for example, the set S_k for $k = 18$ is given by $\{2, 5\}$. Constraint (8), for $k = 18$, becomes

$$\| Yx_{18} + Zu_{18} - \beta \|_\infty \leqslant (2 - z[2] - z[5] + z[1] + z[3] + z[4])M \tag{23}$$

If the final time is 18, $z[2] = z[5] = 1$ and $z[1] = z[3] = z[4] = 0$, so that constraint (23) becomes

$$\| Yx_{18} + Zu_{18} - \alpha \|_\infty = 0 \tag{24}$$

The final time for taking (20) to the origin is 25 s, which is the same as that obtained by Gutman (1982).

4.2. *Absorber*

A six-plate absorber controlled by the inlet gas and liquid feedstreams can be described (Lapidus and Luus 1967) by

$$\left.\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma u_k \\ x_0^{\mathrm{T}} &= [-0{\cdot}0306 \quad -0{\cdot}0568 \quad -0{\cdot}0788 \quad -0{\cdot}0977 \quad -0{\cdot}1138 \quad -0{\cdot}1273] \end{aligned}\right\} \qquad (25)$$

Matrices Φ and Γ are given in Appendix E of Carvallo (1989) for a discretization time of 0·5 min. The controls are constrained as follows:

$$\left.\begin{aligned} 0{\cdot}0 \leqslant u[1] \leqslant 1{\cdot}0 \\ -0{\cdot}4167 \leqslant u[2] \leqslant 0{\cdot}972 \end{aligned}\right\} \qquad (26)$$

It took 5·5 min to drive this system to the origin in minimum time. Table 2 shows the resulting minimum times for different input constraints. The first column corresponds to the final time with the manipulated variables bounded by (26). For the next four columns the upper (lower) bounds were increased (decreased) by 0·5, one at a time. In the last column all the bounds were relaxed by 0·5. Note that relaxing the upper bound on $u[1]$ does not affect the final time, whereas relaxing its lower bound has a large effect on it.

$u[1]$	upper	1·0	1·0	1·0	1·0	1·5	1·5
$u[1]$	lower	0·0	0·0	0·0	−0·5	0·0	−0·5
$u[2]$	upper	0·972	0·972	1·472	0·972	0·972	1·472
$u[2]$	lower	−0·4167	−0·9167	−0·4167	−0·4167	−0·4167	−0·9167
t_f	—	5·5	5·0	5·0	3·5	5·5	3·0

Table 2. Absorber final times.

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