# International Comovement of $r^*$: A Case Study of the G7 Countries*

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**Abstract**

The natural rate of interest, $r^*$, is an important input to evaluate monetary policy stance. Considering the cross-country dimension when measuring $r^*$ may improve monetary policy efficacy and reduce errors. In this paper I set up a multi-country state space model to jointly estimate $r^*$ and identify common, regional, and idiosyncratic components of $r^*$ for the G7 countries. I find that common and regional components are the main drivers of the natural rate of interest and the idiosyncratic component plays only a minor role, suggesting the synchronization of the natural rate of interest across countries. Additionally, the natural rate of interest has been declining over several decades and the estimate is lower than the individual estimates found in the literature. Taking into account the international dimension enables policy makers to better understand the dynamics of $r^*$ and helps to reduce the uncertainty associated with forecasting $r^*$.

**JEL: C32, E43, F44**

**Keywords: The natural rate of interest, G7, State space model, Decomposition**

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# 1 Introduction

The natural rate of interest, also known as the equilibrium real interest rate or $r^*$, is the real short-term interest rate at which real GDP equals its potential GDP without transitory shocks (Williams, 2003). Although it is not directly observed, the value of $r^*$ is an important input into monetary policy decisions, as it provides a benchmark for the stance of monetary policy (Taylor, 1993). It is commonly estimated separately country by country, however, including the international dimension[1] may provide new insights. Incorporating a global component would allow policy makers to determine to what extent $r^*$ is influenced by local and global factors.

In this paper, I set up a multi-country state space model to jointly estimate the natural rate of interest for the G7 countries, namely Canada, France, Germany, Italy, Japan, the UK, and the US. In the process of estimating the model, I also decompose country level $r^*$ into common, regional, and idiosyncratic components to identify the dynamics of each component and investigate contributions of each component to $r^*$ for each country.

This paper offers several contributions to the literature. First, it provides the decomposition of the natural rate of interest. In the literature of international business cycles, how the fluctuations of macroeconomic variables comove across countries has been well investigated (i.e. Gregory et al., 1997 and Kose et al., 2003). Recently, the topic extends to the real interest rate (i.e. Del Negro et al., 2019). In this context, this paper adds the decomposition of $r^*$ of the G7 countries to the literature. In addition, this paper also investigates whether only a subset of the G7 countries possess a specific dynamic (i.e. Mitra and Sinclair, 2012).

Second, jointly estimating the natural rate of interest of the G7 countries takes into account the international component. Previous studies focus on estimating the natural rate of interest on an individual country basis (i.e. Holston et al., 2017 and Hamilton et al., 2016) and joint consideration is still scarce. Due to the high integration of macroeconomic variables across countries, estimating the variable by only considering the domestic factors may lead to misspecification and a biased estimation. This paper sheds light on joint estimation in the literature of measuring $r^*$. Furthermore, by exploiting Holston et al. (2017) framework, I also jointly estimate the potential output.

[1] It may be not only essential in a large open economy but also in a small open economy as foreign factors dramatically influence those economies (i.e. Grossman et al., 2019). Also international finance plays a role in influencing $r^*$. For example, a rise in popularity in safe assets as well as the spillover effect influence $r^*$ through the international finance market (i.e. Glick, 2019).

I use quarterly frequency[2] data of output, price, and short term nominal interest rate for the G7 countries ranging from 1979Q4 to 2019Q2. I set up a multi-country state space model which is an extension of Holston et al. (2017) and estimate the parameters using a random walk Metropolis-Hastings algorithm. I then estimate the natural rate of interest using the Kalman filter. At the same time, I decompose the natural rate of interest into common, regional, and idiosyncratic components.

I find that the natural rate of interest has a declining tendency for all of the G7 countries. For example, in the US, the estimated $r^*$ is 3.38% in 1979Q4 and -0.45% in 2019Q2[3]. I also find that the natural rate of interest behaves in a counter-cyclical manner with a large drop during both the energy crisis in the 1980s and the financial crisis in 2007. Furthermore, I find the estimates of the natural rate of interest to be considerably lower than the individual estimates that are found in the literature (i.e. Holston et al., 2017 and Kiley, 2015). In a joint country estimation, Kiley (2019) uncovers a similar finding. These findings indicate that missing international factors may contribute to the overestimation of the variable.

The decomposition exercise reveals that the downward tendency of $r^*$ is mainly captured by the regional component. Further, I find that the contributions of the idiosyncratic components to the country level natural rate of interest are minimal and that the regional and common components are the main drivers of the variable. This suggests the synchronization of $r^*$ across countries. I obtain a similar finding when estimating a three country state space model (Japan, the UK, and the US) with a longer sample size and when estimating the model for the G7 countries with sub-sample periods.

The rest of this paper is organized as follows: Section 2 describes the datasets that are used, Section 3 outlines the methodology (including the state space model and Bayesian estimation), Section 4 presents the results, Section 5 checks robustness, and finally Section 6 concludes.

[2]The quarterly frequency is a long period for measuring interest rate behavior. However, the natural rate of interest is a theoretical "long run" short-term interest rate consistent with potential GDP and stable inflation rate. As $r^*$ is a long run variable, within a quarter, dramatic changes in $r^*$ should be sufficiently captured. Also, since the frequency of GDP is quarterly, the measure of $r^*$ also needs to be quarterly. Understanding the dynamics of $r^*$ along with potential GDP provides a benchmark stance of monetary policy, which complements the high frequency of monetary policy.

[3]Negative $r^*$ can happen if the interest rate at which loan supply (savings) and loan demand (investment) equals is negative. This can be possible if an economy is experiencing a population decline (decline in investment) and a life expectancy rise (a rise in savings). A number of studies have empirically observed a negative $r^*$ (i.e. Brand et al. (2018) in the Euro area and Okazaki and Sudo (2018) in Japan).

## 2 Data

I analyze the following countries: Canada, France, Germany, Italy, Japan, the UK, and the US. Data are of quarterly frequency and cover 1979Q4-2019Q2 based on availability. I use the following three variables for each country: real GDP, core CPI (Core PCE for the US), and 3-month nominal interest rate[4]. Real GDP and core CPI are seasonally adjusted. The data are obtained from the Federal Reserve Economic Data, OECD Data, Datastream, and central banks from those countries.

## 3 Methodology

### 3.1 The Empirical Model

I set up the following state space model, which is an extension of Holston et al. (2017). The model consists of the following measurement equations:

$$\begin{cases} \tilde{y}_{i,t} = a_{y1,i}\tilde{y}_{i,t-1} + a_{y2,i}\tilde{y}_{i,t-2} + \frac{a_{r,i}}{2}(\tilde{r}_{i,t-1} + \tilde{r}_{i,t-2}) + \epsilon_{t,\tilde{y}_i} \\ \pi_{i,t} = b_{\pi,i}\pi_{i,t-1} + (1 - b_{\pi,i})\pi_{i,t-2,4} + b_{y,i}\tilde{y}_{i,t-1} + \epsilon_{t,\pi_i} \end{cases} \tag{1}$$

where $i = \{CA, FR, DE, IT, JP, UK, US\}$, the first equation is the reduced form IS equation and the second equation is the reduced form Phillips curve equation. $\tilde{y}_{i,t} = y_{i,t} - y^*_{i,t}$ represents output gap (log of real GDP minus log of potential GDP), $\tilde{r}_{i,t} = r_{i,t} - r^*_{i,t}$ represents a deviation of real interest rate from the natural rate of interest, $\pi_{i,t}$ represents the inflation rate, and $\pi_{i,t-2,4} = \frac{1}{3}\sum_{j=t-4}^{t-2}\pi_{i,j}$ represents the average of the second to fourth lags of the inflation rates. $\epsilon_{t,\tilde{y}_i}$ and $\epsilon_{t,\pi_i}$ captures the transitory shocks to the measurement equations. Here, the real interest rate is the ex-ante real interest rate: $r_{i,t} = \text{nominal interest rate}_{i,t} - \mathbb{E}_{t-1}[\pi_t]$.[5]

I assume that the natural rate of interest for each country has the following law of motion:

$$\begin{aligned} r^*_{i,t} = \quad & r_t^{*common} + r_t^{*region} + r_{i,t}^{*idiosyncratic} \\ = \quad & g_t^{*common} + z_t^{*common} + g_t^{*region} + z_t^{*region} + g_{i,t}^{*idiosyncratic} + z_{i,t}^{*idiosyncratic} \end{aligned} \tag{2}$$

[4]I use a 3-month maturity interbank rate for Canada, France, Germany, and Italy and use a 3-month maturity government bond yield for the UK and US. I use a 3-month maturity money market rate from 1989Q1 in Japan. Prior to this period in Japan, I use a 10 year maturity government bond yield due to data availability.

[5]where $\mathbb{E}_{t-1}[\pi_t]$ is a four-quarter moving average of past inflation rates as a proxy for inflation expectations based on Holston et al. (2017).

The first equation specifies that the natural rate of interest is a sum of common, regional, and idiosyncratic components. The common component captures the dynamics of $r^*$ for all of the G7 countries. The regional component captures the dynamics of a specific group[6]: North America (Canada and the US) and Europe (France, Germany, Italy, and the UK). I cannot identify the regional component or the idiosyncratic component for Japan given that it is the only Asian country. Thus, $r^{*idiosyncratic}_{JP,t}$ includes all of the dynamics of $r^*_{JP,t}$ that are not captured by the common component.

To obtain the second equation, I assume each common, regional, and idiosyncratic component of the natural rate of interest is a sum of the respective output growth, $g$, and preference shifter, $z$. This specification relies on Holston et al. (2017) based on the theoretical link between $r^*$ and output growth[7].

The following are the state equations:

$$\begin{cases} y^*_{i,t} & = & y^*_{i,t-1} + g^{common}_{t-1} + g^{region}_{t-1} + g^{idiosyncratic}_{i,t-1} & +\epsilon_{t,y^*_i} \\ g^{common}_t & = & g^{common}_{t-1} & +\epsilon_{t,g^{common}} \\ g^{region}_t & = & g^{region}_{t-1} & +\epsilon_{t,g^{region}} \\ g^{idiosyncratic}_{i,t} & = & z^{idiosyncratic}_{i,t-1} & +\epsilon_{t,g^{idiosyncratic}_i} \\ z^{common}_t & = & z^{common}_{t-1} & +\epsilon_{t,z^{common}} \\ z^{region}_t & = & z^{region}_{t-1} & +\epsilon_{t,z^{region}} \\ z^{idiosyncratic}_{i,t} & = & z^{idiosyncratic}_{i,t-1} & +\epsilon_{t,z^{idiosyncratic}_i} \end{cases} \tag{3}$$

The first equation follows a random walk, however, stochastic drift terms enter the equation. These stochastic drift terms each follow a random walk. The rest of the state variables also follow random walks.

The error terms of measurement and state equations follow normal distribution and they are mutually uncorrelated within a country. The error terms are also uncorrelated across countries. Thus, the channels through which shocks transmit to other economies are either regional and common output growth or preference shifter. By the specifications of the model, I decompose the natural rate of interest into common, regional, and idiosyncratic components. The complete

[6]As in the Gravity equation, the closer the countries, the higher the trade flow. Thus proximity is one measure of economic integration. Specifying regional components are found in previous studies (i.e. example Kose et al., 2003 and Mansour, 2003).

[7]How to obtain these results using a simple growth model is shown in Appendix 8.2.

specification of the model can be found in Appendix 8.3.

### 3.2 Estimation

I estimate the parameters using a random walk Metropolis-Hastings algorithm. I take the Bayesian approach over the frequentist approach in order to circumvent the pile-up problem that arises in Maximum Likelihood (Stock and Watson, 1998) and to deal with the relatively large dimension of the state space model and the small sample size. Table 1 summarizes the prior distribution of the estimation.

I set up the prior distributions to be agnostic (only imposing normal and uniform distribution) following Lewis and Vazquez-Grande (2017) and make maximum space for the data to contribute to the parameter estimation. I make 200,000 draws in total but throw away the first 100,000 draws as a burn-in period. After the burn-in period, I take every $10^{th}$ draw. From the 10,000 samples, I take the median, $10^{th}$ %, and $90^{th}$ % of the state variables from the Kalman Filter. In total, there are 76 parameters and 61 state variables to estimate. A detailed explanation of the Bayesian estimation is in Appendix 8.5.

## 4 Results

### 4.1 $r^*$ in the G7 countries

Figure 1 shows the measurement of the natural rate of interest by estimating the model jointly for the G7 countries. The estimates in the figure are the sum of the common, regional (excluding Japan), and idiosyncratic components for each country. The posterior distributions of the parameter estimates are in Appendix 8.1.1. In Figure 1, the black solid lines represent the median series from the Kalman filter. The shaded region represents the $10^{th}$ % to $90^{th}$ % estimation interval. Additionally, the vertical shaded regions represent recession dates indicated by the National Bureau of Economic Research in the US and the Economic Cycle Research Institutes for the other countries. The estimated $r^*$s are clearly time variant with moderate volatility for the G7 countries. Notably, the majority of these countries show a declining tendency of $r^*$, consistent with the literature of the measurement of $r^*$ (i.e. Holston et al., 2017). For example, I find that the estimated $r^*$ for the US is around 4% at the beginning of the sample period and shrinks to around 0% at the end.

The natural rate of interest declines during recessions, indicating that real interest rate needs to be lower to make the economy consistent with the potential GDP. This counter-cyclical component of $r^*$ is also seen in Holston et al. (2017). However, I find that the magnitude is stronger than in the literature. For example, in the US I find that the 1980 recession subsequent to the energy crisis pushes $r^*$ below 0% and the financial crisis in 2007 pushes $r^*$ below -5% . The estimates in the same periods from Holston et al. (2017) are around 2.5% and 1%, respectively. These observations may be due to the differences in single country and joint country estimations. In favor of my findings, Kiley (2019) considers a joint estimation and finds $r^*$ estimates are actually lower than the estimates found in the literature. It is possible that single country estimations miss country interactions and spillover effects that propagate shocks across countries and lower $r^*$.

### 4.2 Decomposition and Synchronization

The estimates of the natural rate of interest for the G7 countries generally show a declining tendency and are counter-cyclical with the business cycles. I now investigate which component(s) plays a role in the decline of the natural rate of interest.

The left panel of Figure 2 exhibits the common component of $r^*$. This component captures variations of $r^*$ that are seen for all of the investigated countries. As opposed to the $r^*$ estimates of the G7 countries, this series is erratic around 0%. The common component does not seem to possess the declining trend and it looks like noise, even though there are some drops during the large recession events (i.e, 1980 and 2007).

The declining tendency in $r^*$ in the G7 countries are captured by the regional components. Both the North American and European components are downward sloping and the North American component seems to possess a slightly steeper decline (around 3% in the beginning and 0% in the end) than the Europe component (around 2% in the beginning and 0% in the end). Additionally, though they share very similar magnitudes of drops during the financial crisis, their dynamics are somewhat different. I find that the North American component and European component have a small discrepancy (the correlation is 51%).

Figure 3 shows the idiosyncratic component of $r^*$. The idiosyncratic components do not have the declining tendency of the country level estimates of the natural rate of interest. Most of the countries have an erratic and relatively constant series, however, estimates in the US and Japan do have declining trends. The idiosyncratic components drop during the recession periods. In general, the magnitudes of the movements of the idiosyncratic components are smaller than the magnitudes of the movements of the regional components.

To understand whether the natural rate of interest is integrated across the investigated countries, I calculate simple correlations of different measures of $r^*$. Tables 2 and 3 show the correlation of country level $r^*$s and idiosyncratic components of country level $r^*$s across the G7 countries.

The correlations of the idiosyncratic components of the natural rate of interest range from -71% to 56% while the correlations of the country level estimates of the natural rate of interest range from 52% to 97%. While the correlations of the idiosyncratic components range from negative to some moderate positive values, the overall correlations of the country level $r^*$s are all above 50%. The idiosyncratic component is a part of the country level $r^*$. This finding indicates that the comovement of country level $r^*$s across the G7 countries are not disrupted by the idiosyncratic components, suggesting a synchronization of $r^*$.

### 4.3 Output Gap

The utilization of Holston et al. (2017) allows the measurement of the output gap. The IS equation in equation (1) implies that there is a negative relationship between output gap and the real interest rate gap (real interest rate minus the natural rate of interest). The negative relationship is shown in Figure 4. The solid curves measured by the left axes represent output gaps and the dotted curves measured by the right axes represent real interest rate gaps. Before the 2000s, both variables show clear trends: output gap increases and real interest rate gap decreases, however, this notion does not apply to Japan and the US. The finding indicates that the economies operated below their potential level after the energy crisis and thus the nominal interest rates needed to be lower than the rate actually was. This phenomena was long lasting. After the 2000s, the slope of the output gap looks almost flat. The deviation of output from its potential and deviation of real interest rate from the natural rate are erratic. However, recently, the countries in the Europe region (excluding the UK) tend to possess a persistently increasing output gap. This can potentially be explained by the monetary policy that is governed by the European Central Bank. This phenomena shifts Italy towards the potential output. However, at the same time, this phenomena also consistently overheats Germany.

## 5 Robustness Analysis

In this section, I conduct two robustness analyses. First, I estimate $r^*$ for three countries (Japan, the UK, and the US) with a longer time series than in the benchmark data. Second, I estimate $r^*$ for the G7 countries excluding two episodes, the energy crisis and the financial crisis.

### 5.1 Three Country Estimation

I estimate a three-country (Japan, the UK, and the US) state space model with a longer time series. The sample period spans from 1962Q1-2019Q4. I use three variables: real GDP, Core CPI (Core PCE for the US), and monetary policy rate. The data are retrieved from Federal Reserve Economic Data, OECD DATA, and central banks from the respective countries. The specification of the decomposition is the following:

$$r^*_{i,t} = \quad r_t^{*common} + r_{i,t}^{*idiosyncratic} \tag{4}$$

The natural rate of interest is decomposed into common and idiosyncratic components. The regional component is excluded in this analysis because the countries are distantly located and it is infeasible to differentiate the regional component with the idiosyncratic component if the regional component is included.

Figure 5 shows the estimates of the natural rate of interest by estimating the model jointly for the 3 countries. Similarly to the benchmark country estimates, the series are steadily declining, ranging from 5-7% in the start period and 0% in the last period.

Figure 6 represents the decomposition of the country estimates of $r^*$. The common component from the three countries show a steady decline in $r^*$. The movement of the idiosyncratic components are erratic except for Japan. Based on these findings, the three countries are driven by the common component. Additionally, $r^*$ in Japan is further pushed downwards by the idiosyncratic component.

Table 4 shows the correlations of $r^*$s and the idiosyncratic components of $r^*$s. As seen in the previous section, the correlations of $r^*$s between countries are high, ranging from 75% to 92%. However, the correlations of the idiosyncratic components across countries range from 13% to 42%. The idiosyncratic components do not seem to have much of an effect on the comovements of country level $r^*$s, indicating the synchronization of $r^*$s across the three countries.

### 5.2 Excluding the Energy Crisis and Financial Crisis

In the benchmark results, the dynamics of the estimated $r^*$ shows relatively large swings during the energy crisis period (the beginning of the sample period) and the financial crisis period (in 2008). The behavior of $r^*$ during these periods can be interpreted as outliers and excluding these periods might dramatically alter the estimated parameters and $r^*$.

In this section, I exclude the following two periods: the energy crisis period which makes the sample period to be 1984Q2-2019Q and the financial crisis period which makes the sample period to be 1979Q4-2007Q4. Aside from changing the sample sizes, the specifications and estimation are the same as the benchmark methodology.

Figure 7 shows the estimated natural rate of interest for the G7 countries excluding the energy crisis and the financial crisis periods. The solid line represents the benchmark median $r^*$, the dashed line represents median $r^*$ excluding the financial crisis period, and the dotted line represents median $r^*$ excluding the energy crisis period. By excluding these periods, the estimated $r^*$ in Italy becomes higher than the benchmark results. Also, when excluding the financial crisis, the estimated $r^*$ in the UK moderately deviates from the benchmark result. However, for most of the countries, $r^*$ is very similar to the benchmark $r^*$. With regard to the energy crisis, the large swings of the benchmark $r^*$ are generally captured by the idiosyncratic component. With regard to the financial crisis, the financial crisis is a really small part of the sample and thus it is possible that the exclusion of the period does not largely influence the dynamics of $r^*$. Overall, the estimated $r^*$ excluding these periods is not dramatically different from the benchmark $r^*$.

Figure 8 plots the common and regional components of $r^*$ excluding the energy crisis and the financial crisis. As seen in the country level estimates, there are minor deviations from the benchmark. However, the overall movements are very similar to the movement of the benchmark results.

With regard to the synchronization of the natural rate of interest across the G7 countries, tables 5, 6, 7, and 8 report the correlations of the natural rate of interest and the idiosyncratic component of the natural rate of interest with different subsample periods.

Looking at the exclusion of the financial crisis period (tables 5 and 6), the range of the correlations of $r^*$ is 6% to 84% and the range of the correlations of the idiosyncratic components of $r^*$ is -69% to 57%. Compared to the benchmark results, the range of correlations do slightly decline. Thus, the synchronization of $r^*$ becomes weaker than the benchmark results, despite that most of the correlations of $r^*$ are still higher than the correlations of the idiosyncratic components. This finding is potentially explained by two factors. The first possibility is the smaller sample size. I drop 30% of the sample size compared to the benchmark and this might reduce the estimation accuracy. Another possibility is that the synchronization accelerated after the financial crisis.

Looking at the exclusion of the energy crisis, (tables 7 and 8), the range of the correlation of $r^*$ is 57% to 94% and the range of the correlations of the idiosyncratic components of $r^*$ is -49% to 46%. This indicates that the idiosyncratic component of $r^*$ does not play a major role in the dynamics of $r^*$ across the G7 countries. Thus, excluding the energy crisis does not distort the finding in Section 4 and supports the synchronization of $r^*$ across the G7 countries.

## 6 Conclusion

In this paper, I estimate a multi-country state space model and jointly estimate the natural rate of interest for the G7 countries. I then decompose the variable into common, regional, and idiosyncratic components. I find that the idiosyncratic component of $r^*$ does not have much of an effect on $r^*$ in the G7 countries. I also find that common and regional components drive the variable. Furthermore, I find that $r^*$ is steadily declining over the sample period and the estimates are lower than the estimates found in literature that does not consider the international component.

These findings are calculated using the G7 countries which are highly advanced economies. These countries have similar characteristics of demographics, productivity, and investment demand, which are typical determinants of $r^*$ in the literature. Thus, the implications of this paper may not apply to developing countries. Looking into the decomposition of $r^*$ in developing countries would be a great topic in future research.

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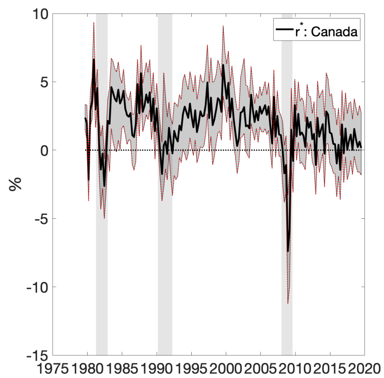
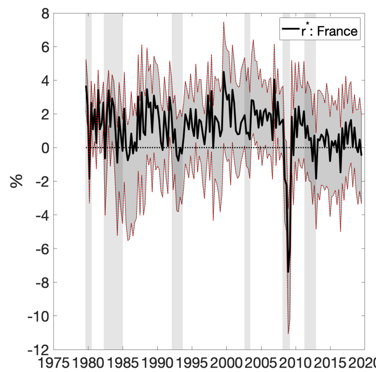
# Figures

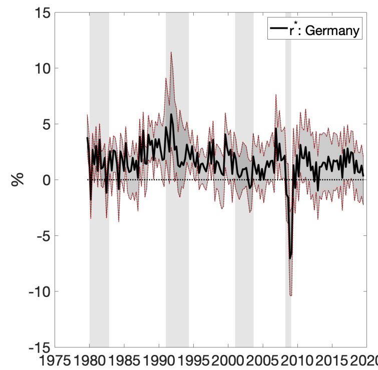
Figure 1: The Natural Rate of Interest




(a) Canada




(b) France

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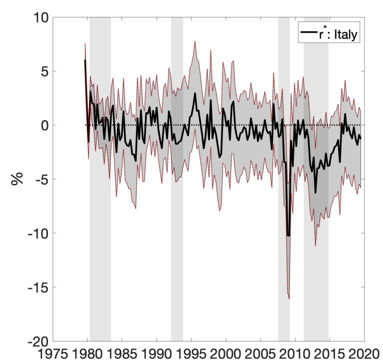


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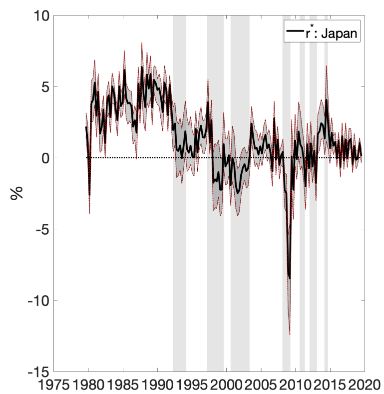


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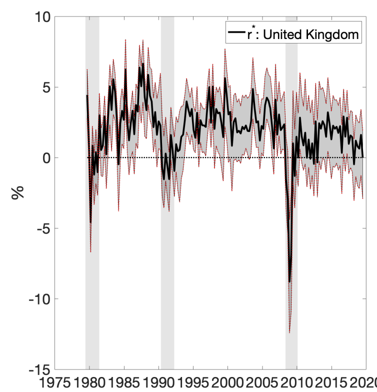
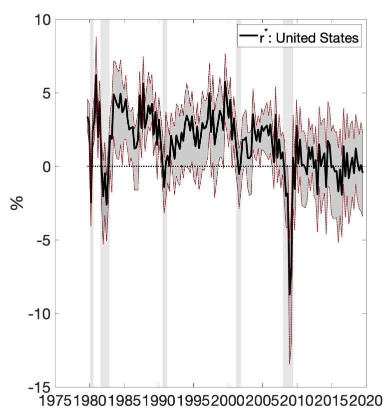
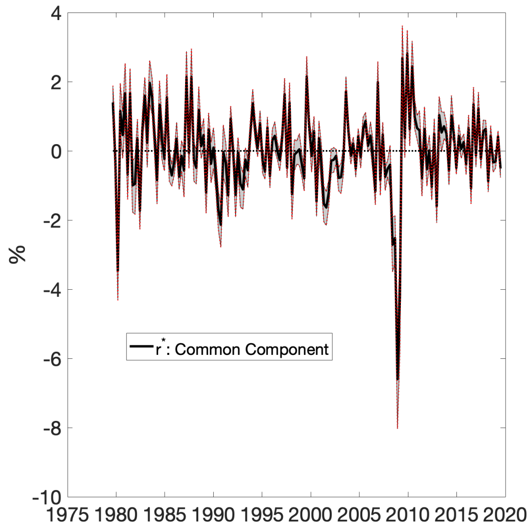



(f) United Kingdom

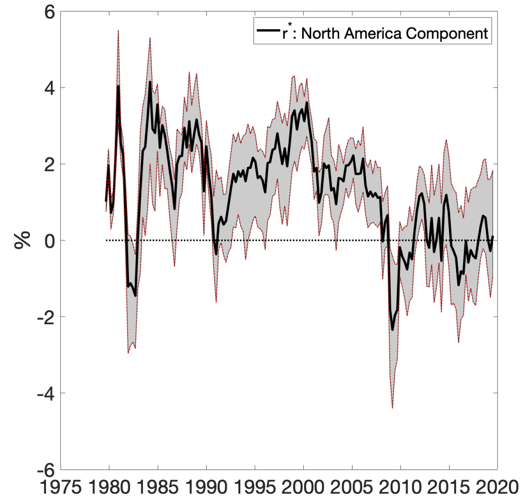



(g) United States

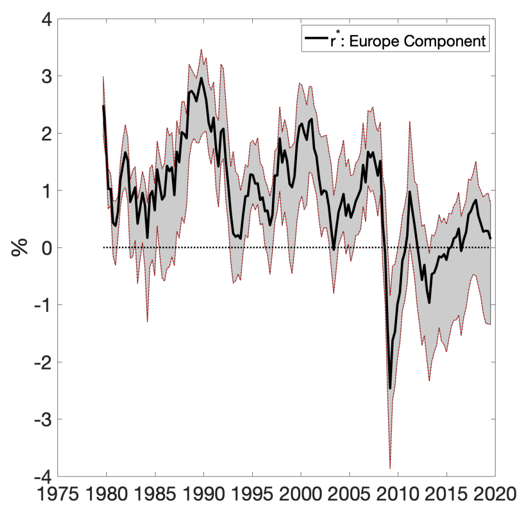
Figure 2: Common Component and Regional Component of $r^*$




(a) Common Component

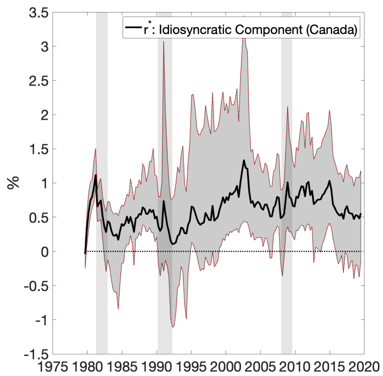



(b) North America Component

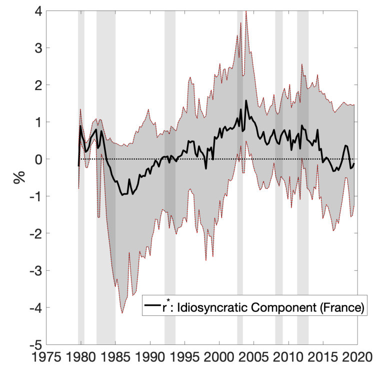



(c) Europe Component

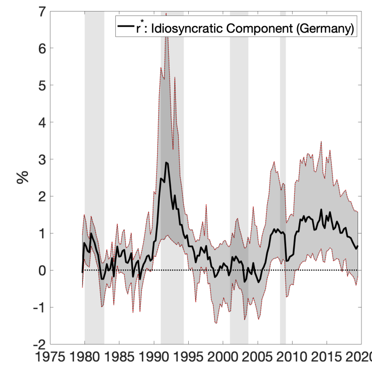
Figure 3: Idiosyncratic Component of $r^*$



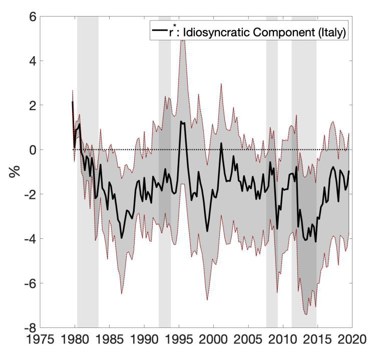

(a) Idiosyncratic (Canada)



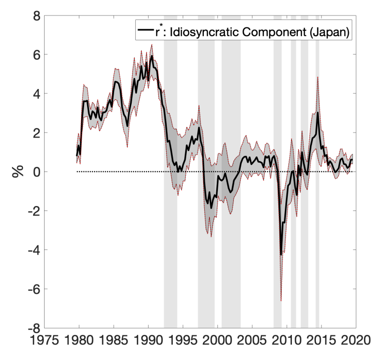

(b) Idiosyncratic (France)



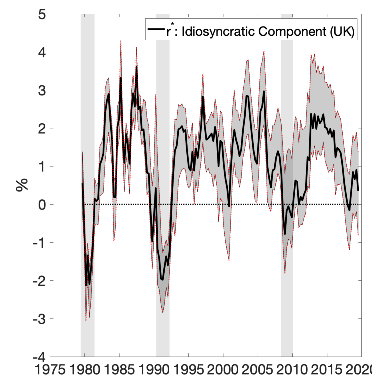

(c) Idiosyncratic (Germany)



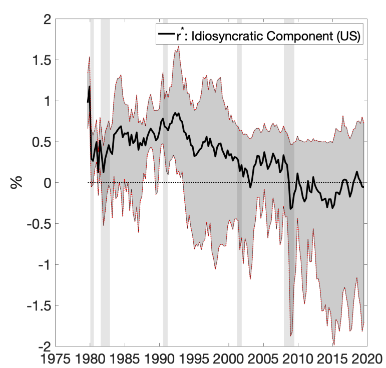

(d) Idiosyncratic (Italy)




(e) Idiosyncratic (Japan)

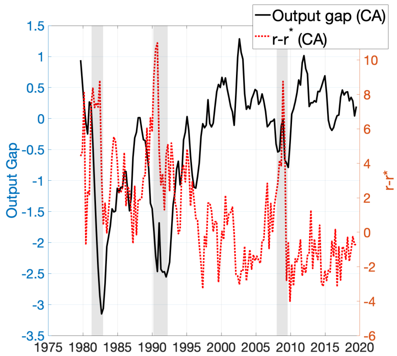



(f) Idiosyncratic (UK)

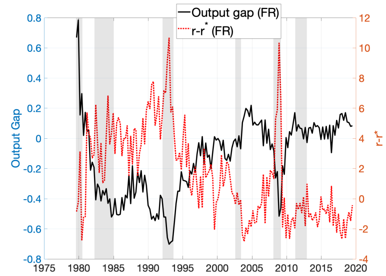



(g) Idiosyncratic (US)

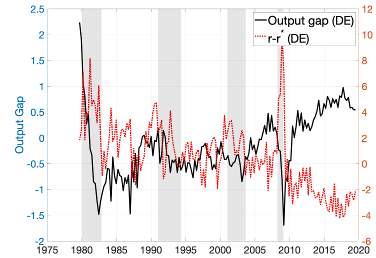
Figure 4: Output Gap and Real Interest Rate Gap



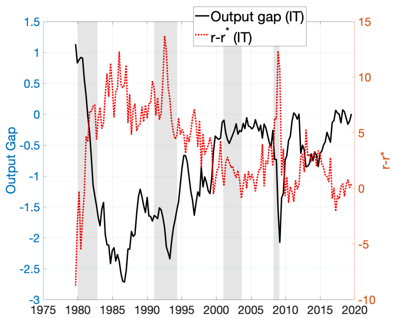

(a) Canada



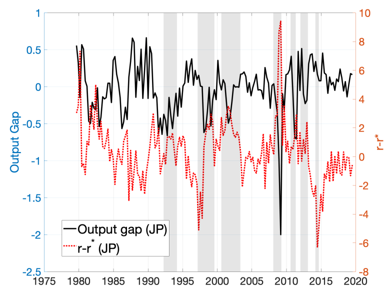

(b) France



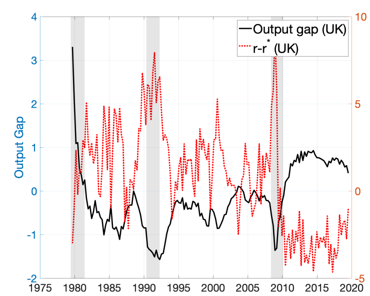

(c) Germany



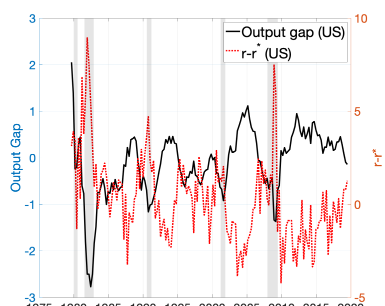

(d) Italy




(e) Japan

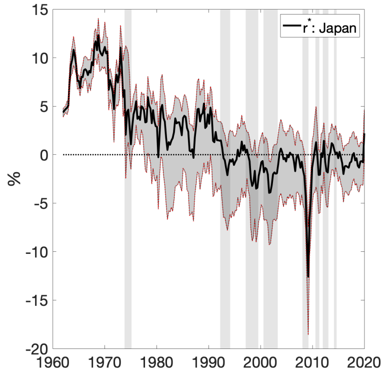



(f) United Kingdom

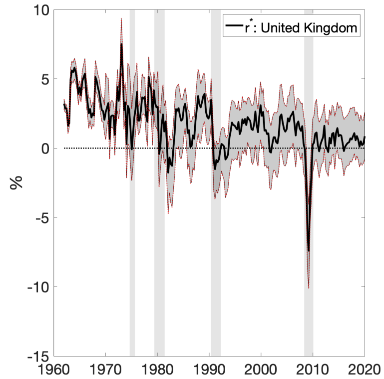



(g) United States

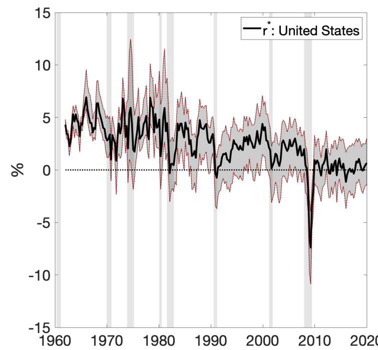
Figure 5: The Natural Rate of Interest (Three Country Estimation)




(a) Japan

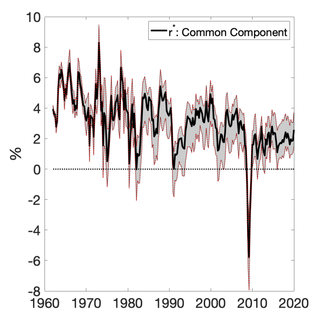



(b) United Kingdom

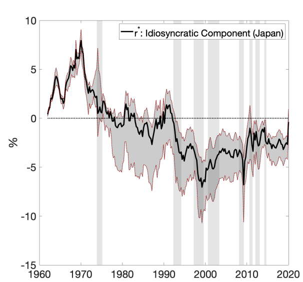



(c) United States

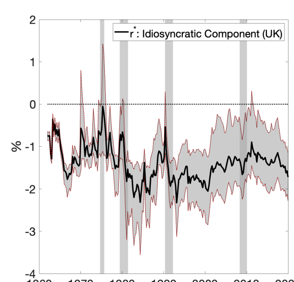
Figure 6: The Natural Rate of Interest Decomposition (Three Country Estimation)



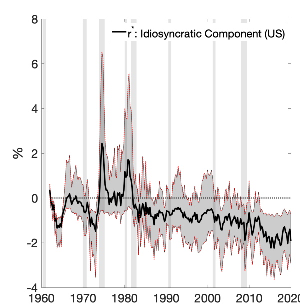

(a) Common Component




(b) Idiosyncratic (Japan)

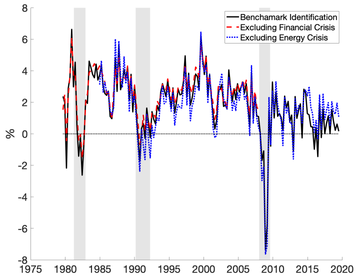



(c) Idiosyncratic (United Kingdom)

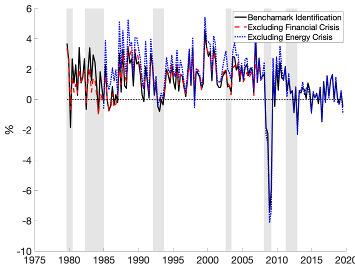



(d) Idiosyncratic (United States)

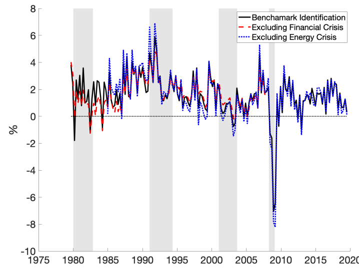
Figure 7: The Natural Rate of Interest (Robustness)



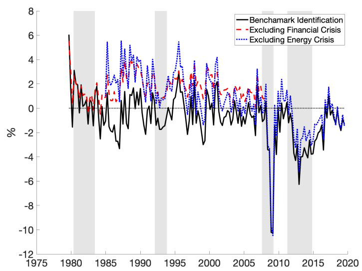

(a) Canada



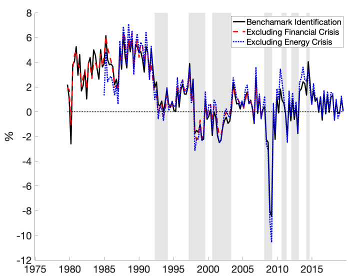

(b) France



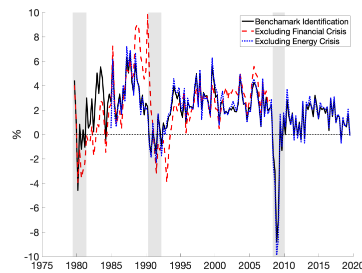

(c) Germany



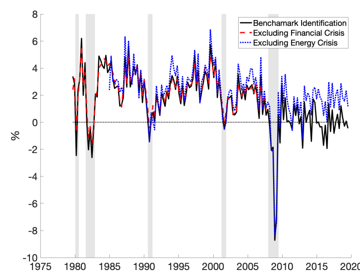

(d) Italy




(e) Japan

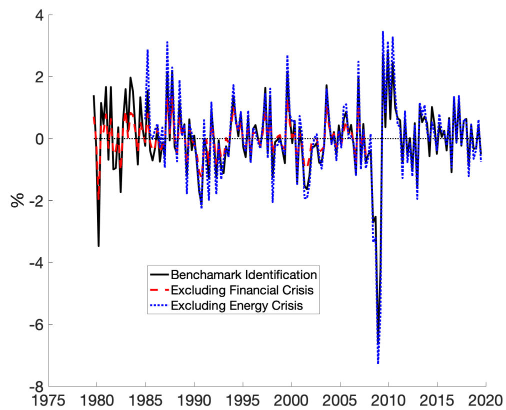



(f) United Kingdom

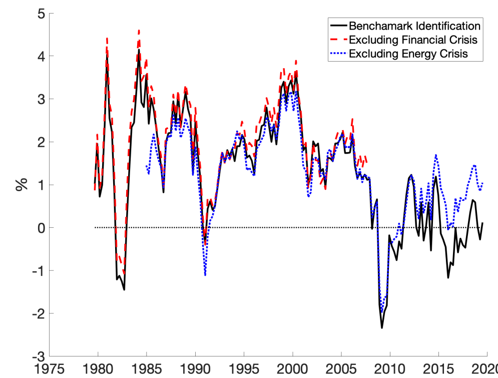



(g) United States

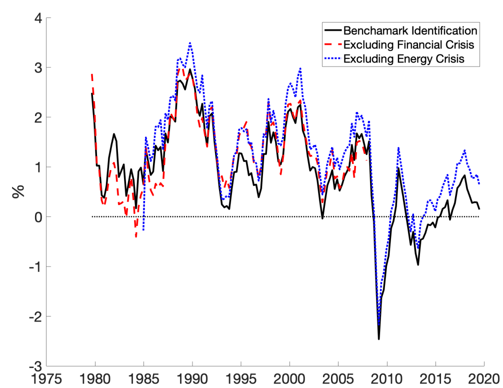
Figure 8: Common Component and Regional Component of $r^*$ (Robustness)




(a) Common Component




(b) North America Component




(c) Europe Component

# 7 Tables

Table 1: Prior Specification

| Parameter | Domain | Density | parameter 1 | parameter 2 |
|---|---|---|---|---|
| $a_{y1,i}$ | $\mathbb{R}$ | Normal | 0 | 2 |
| $a_{y2,i}$ | $\mathbb{R}$ | Normal | 0 | 2 |
| $a_{r,i}$ | $(-\infty, -0.0025]$ | Normal | 0 | 2 |
| $b_{\pi,i}$ | $[0, 1]$ | Uniform | 0 | 1 |
| $b_{y,i}$ | $[0.025, \infty)$ | Normal | 0 | 2 |
| $\sigma_{\tilde{y}_i}$ | $(0, 5]$ | Uniform | 0 | 5 |
| $\sigma_{\pi i}$ | $(0, 5]$ | Uniform | 0 | 5 |
| $\sigma_{y^* i}$ | $(0, 5]$ | Uniform | 0 | 5 |
| $\sigma_{g_i}$ | $(0, 5]$ | Uniform | 0 | 5 |
| $\sigma_{z_i}$ | $(0, 5]$ | Uniform | 0 | 5 |
| $\sigma_{g^{common}}$ | $(0, 5]$ | Uniform | 0 | 5 |
| $\sigma_{z^{common}}$ | $(0, 5]$ | Uniform | 0 | 5 |
| $\sigma_{g^{north-america}}$ | $(0, 5]$ | Uniform | 0 | 5 |
| $\sigma_{z^{north-america}}$ | $(0, 5]$ | Uniform | 0 | 5 |
| $\sigma_{g^{europe}}$ | $(0, 5]$ | Uniform | 0 | 5 |
| $\sigma_{z^{europe}}$ | $(0, 5]$ | Uniform | 0 | 5 |

Note: $i = \{CA, FR, DE, IT, JP, UK, US\}$. Parameter 1 is the mean of the normal distribution and the minimum value of the uniform distribution. Parameter 2 is the standard deviation of the normal distribution and the maximum value of the uniform distribution.

Table 2: Correlation of the Natural Rates in the G7

| | $r^*_{CA}$ | $r^*_{FR}$ | $r^*_{DE}$ | $r^*_{IT}$ | $r^*_{JP}$ | $r^*_{UK}$ | $r^*_{US}$ |
|---|---|---|---|---|---|---|---|
| $r^*_{CA}$ | 1 | | | | | | |
| $r^*_{FR}$ | 0.75 | 1 | | | | | |
| $r^*_{DE}$ | 0.62 | 0.83 | 1 | | | | |
| $r^*_{IT}$ | 0.60 | 0.83 | 0.74 | 1 | | | |
| $r^*_{JP}$ | 0.53 | 0.54 | 0.72 | 0.52 | 1 | | |
| $r^*_{UK}$ | 0.77 | 0.77 | 0.66 | 0.57 | 0.54 | 1 | |
| $r^*_{US}$ | 0.97 | 0.74 | 0.66 | 0.64 | 0.60 | 0.76 | 1 |

The correlations are calculated based on the median estimates of the natural rate of interest.

Table 3: Correlation of Idiosyncratic Component of Natural Rates in the G7

| | $r^{*idio}_{CA}$ | $r^{*idio}_{FR}$ | $r^{*idio}_{DE}$ | $r^{*idio}_{IT}$ | $r^{*idio}_{JP}$ | $r^{*idio}_{UK}$ | $r^{*idio}_{US}$ |
|---|---|---|---|---|---|---|---|
| $r^{*idio}_{CA}$ | 1 | | | | | | |
| $r^{*idio}_{FR}$ | 0.51 | 1 | | | | | |
| $r^{*idio}_{DE}$ | -0.12 | -0.14 | 1 | | | | |
| $r^{*idio}_{IT}$ | -0.05 | 0.24 | -0.08 | 1 | | | |
| $r^{*idio}_{JP}$ | -0.40 | -0.57 | 0.14 | -0.03 | 1 | | |
| $r^{*idio}_{UK}$ | 0.01 | -0.06 | -0.42 | -0.36 | -0.14 | 1 | |
| $r^{*idio}_{US}$ | -0.71 | -0.37 | -0.05 | 0.22 | 0.56 | -0.12 | 1 |

The correlations are calculated based on the median estimates of the natural rate of interest.

Table 4: Correlation of Natural Rates and Idiosyncratic Components of Natural Rates in Japan, UK, and US

| | $r^*_{JP}$ | $r^*_{UK}$ | $r^*_{US}$ | $r^{*idio}_{JP}$ | $r^{*idio}_{UK}$ | $r^{*idio}_{US}$ |
|---|---|---|---|---|---|---|
| $r^*_{JP}$ | 1 | | | | | |
| $r^*_{UK}$ | 0.77 | 1 | | | | |
| $r^*_{US}$ | 0.75 | 0.92 | 1 | | | |
| $r^{*idio}_{JP}$ | 0.92 | 0.48 | 0.48 | 1 | | |
| $r^{*idio}_{UK}$ | 0.38 | 0.36 | 0.17 | 0.42 | 1 | |
| $r^{*idio}_{US}$ | 0.38 | 0.27 | 0.58 | 0.36 | 0.13 | 1 |

The correlations are calculated based on the median estimates of the natural rate of interest.

Table 5: Correlation of $r^*$s in the G7 Excluding Financial Crisis

| | $r^*_{CA}$ | $r^*_{FR}$ | $r^*_{DE}$ | $r^*_{IT}$ | $r^*_{JP}$ | $r^*_{UK}$ | $r^*_{US}$ |
|---|---|---|---|---|---|---|---|
| $r^*_{CA}$ | 1 | | | | | | |
| $r^*_{FR}$ | 0.31 | 1 | | | | | |
| $r^*_{DE}$ | 0.14 | 0.71 | 1 | | | | |
| $r^*_{IT}$ | 0.37 | 0.84 | 0.82 | 1 | | | |
| $r^*_{JP}$ | 0.13 | 0.06 | 0.28 | 0.28 | 1 | | |
| $r^*_{UK}$ | 0.35 | 0.53 | 0.29 | 0.50 | 0.17 | 1 | |
| $r^*_{US}$ | 0.95 | 0.27 | 0.24 | 0.42 | 0.27 | 0.31 | 1 |

The correlations are calculated based on the median estimates of the natural rate of interest.

Table 6: Correlation of $r^{*idio}$s in the G7 Excluding Financial Crisis

| | $r^{*idio}_{CA}$ | $r^{*idio}_{FR}$ | $r^{*idio}_{DE}$ | $r^{*idio}_{IT}$ | $r^{*idio}_{JP}$ | $r^{*idio}_{UK}$ | $r^{*idio}_{US}$ |
|---|---|---|---|---|---|---|---|
| $r^{*idio}_{CA}$ | 1 | | | | | | |
| $r^{*idio}_{FR}$ | 0.50 | 1 | | | | | |
| $r^{*idio}_{DE}$ | -0.38 | -0.26 | 1 | | | | |
| $r^{*idio}_{IT}$ | 0.01 | -0.24 | 0.06 | 1 | | | |
| $r^{*idio}_{JP}$ | -0.28 | -0.43 | 0.17 | 0.16 | 1 | | |
| $r^{*idio}_{UK}$ | 0.21 | -0.00 | 0.47 | -0.24 | 0.06 | 1 | |
| $r^{*idio}_{US}$ | -0.69 | -0.47 | 0.47 | 0.24 | 0.57 | -0.22 | 1 |

The correlations are calculated based on the median estimates of the natural rate of interest.

Table 7: Correlation of $r^{*}$s in the G7 Excluding Energy Crisis

| | $r^{*}_{CA}$ | $r^{*}_{FR}$ | $r^{*}_{DE}$ | $r^{*}_{IT}$ | $r^{*}_{JP}$ | $r^{*}_{UK}$ | $r^{*}_{US}$ |
|---|---|---|---|---|---|---|---|
| $r^{*}_{CA}$ | 1 | | | | | | |
| $r^{*}_{FR}$ | 0.80 | 1 | | | | | |
| $r^{*}_{DE}$ | 0.58 | 0.80 | 1 | | | | |
| $r^{*}_{IT}$ | 0.67 | 0.87 | 0.80 | 1 | | | |
| $r^{*}_{JP}$ | 0.57 | 0.69 | 0.82 | 0.70 | 1 | | |
| $r^{*}_{UK}$ | 0.90 | 0.83 | 0.63 | 0.70 | 0.60 | 1 | |
| $r^{*}_{US}$ | 0.94 | 0.82 | 0.66 | 0.72 | 0.60 | 0.90 | 1 |

The correlations are calculated based on the median estimates of the natural rate of interest.

Table 8: Correlation of $r^{*idio}$s in the G7 Excluding Energy Crisis

| | $r^{*idio}_{CA}$ | $r^{*idio}_{FR}$ | $r^{*idio}_{DE}$ | $r^{*idio}_{IT}$ | $r^{*idio}_{JP}$ | $r^{*idio}_{UK}$ | $r^{*idio}_{US}$ |
|---|---|---|---|---|---|---|---|
| $r^{*idio}_{CA}$ | 1 | | | | | | |
| $r^{*idio}_{FR}$ | 0.05 | 1 | | | | | |
| $r^{*idio}_{DE}$ | -0.27 | -0.21 | 1 | | | | |
| $r^{*idio}_{IT}$ | -0.05 | 0.18 | 0.09 | 1 | | | |
| $r^{*idio}_{JP}$ | 0.08 | -0.05 | 0.40 | 0.37 | 1 | | |
| $r^{*idio}_{UK}$ | 0.02 | -0.01 | -0.47 | -0.20 | -0.21 | 1 | |
| $r^{*idio}_{US}$ | -0.49 | -0.01 | 0.15 | 0.46 | 0.40 | -0.04 | 1 |

The correlations are calculated based on the median estimates of the natural rate of interest.

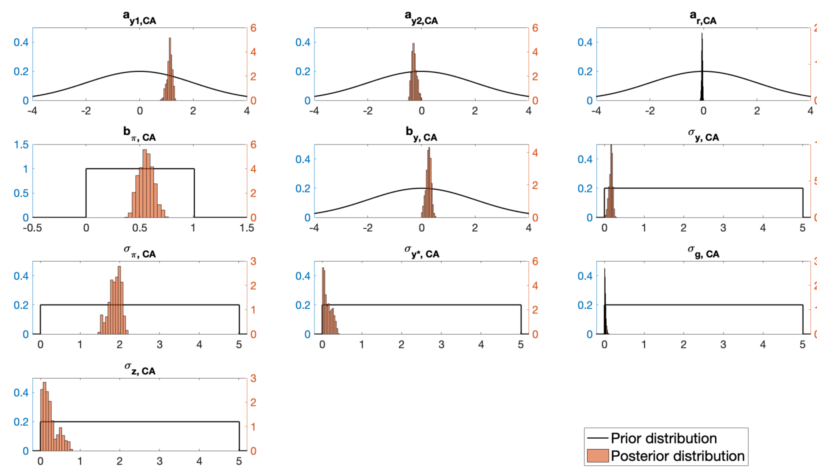
# 8 Appendix

## 8.1 Figures

### 8.1.1 Prior and Posterior Distributions

Figure 9: Prior and Posterior Distribution

(a) Canada




(b) France

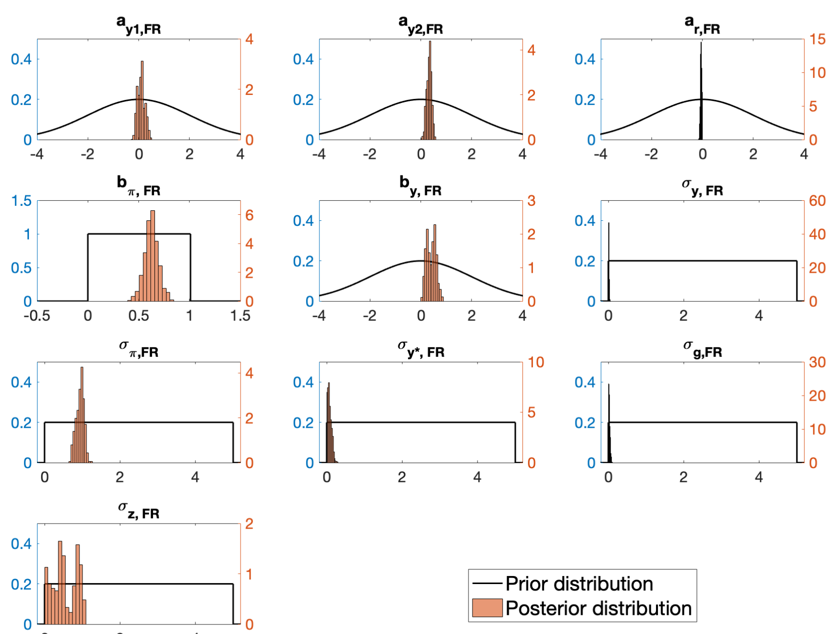
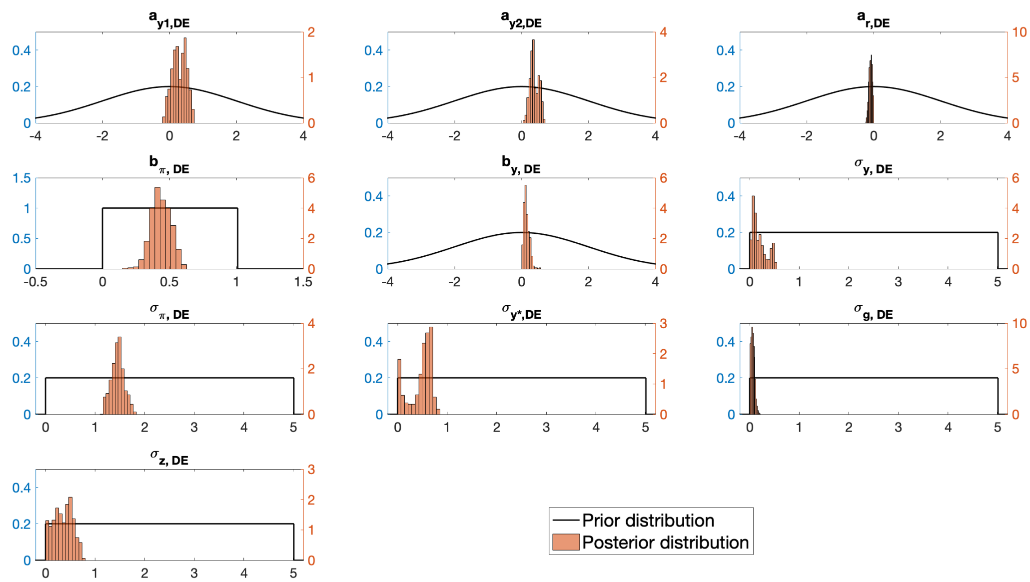


Figure 10: Prior and Posterior Distribution

(a) Germany




(b) Italy

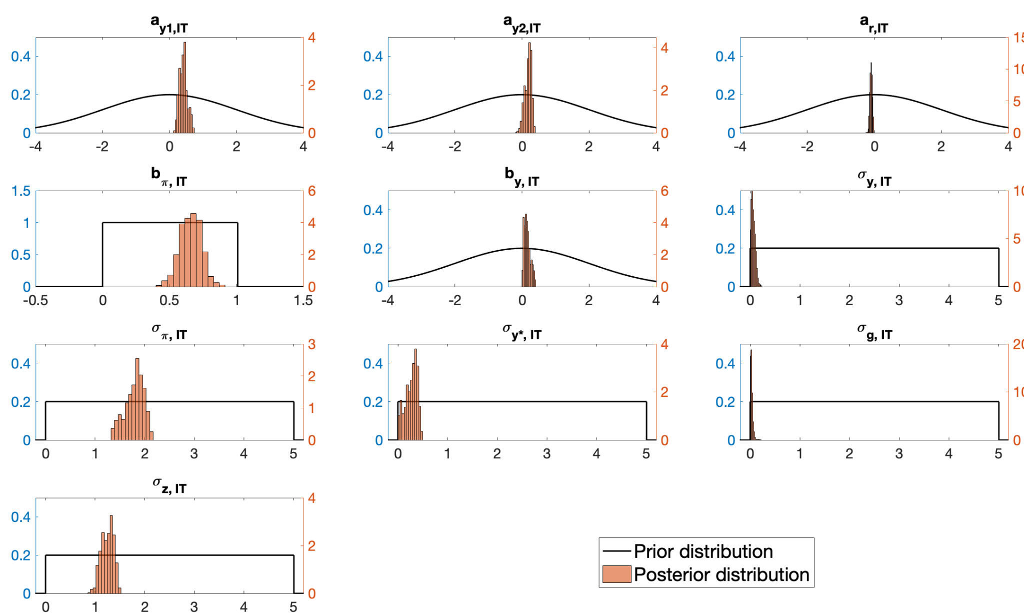
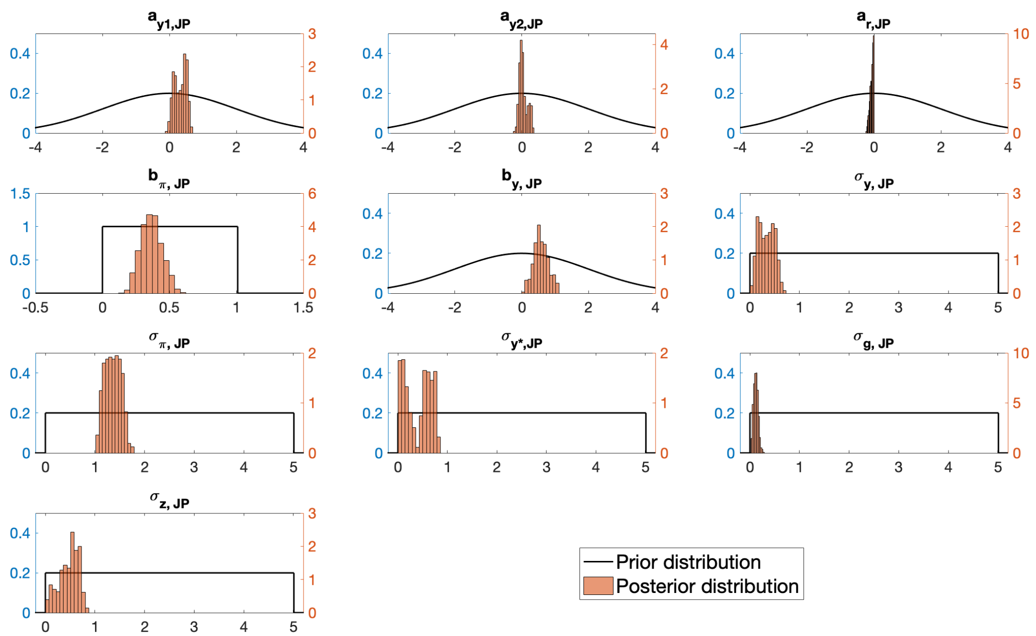


Figure 11: Prior and Posterior Distribution

(a) Japan




(b) United Kingdom

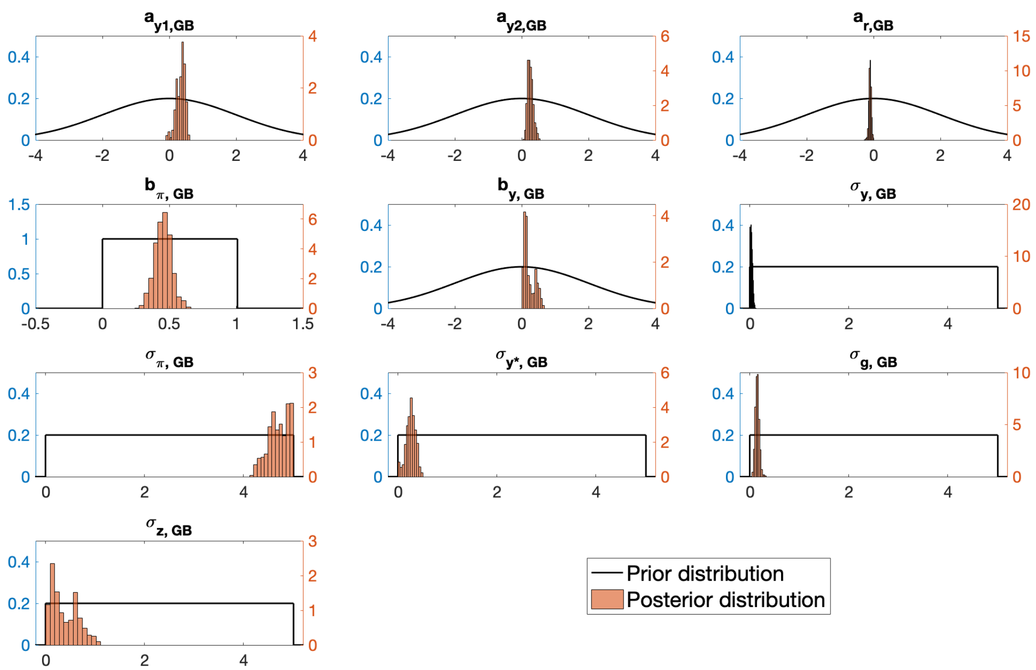
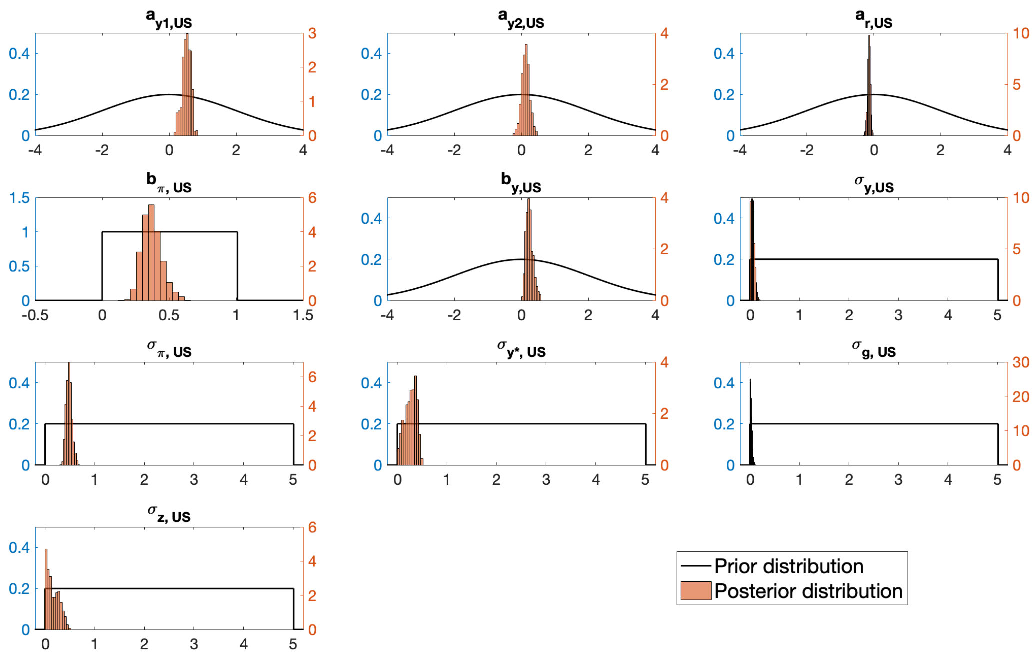
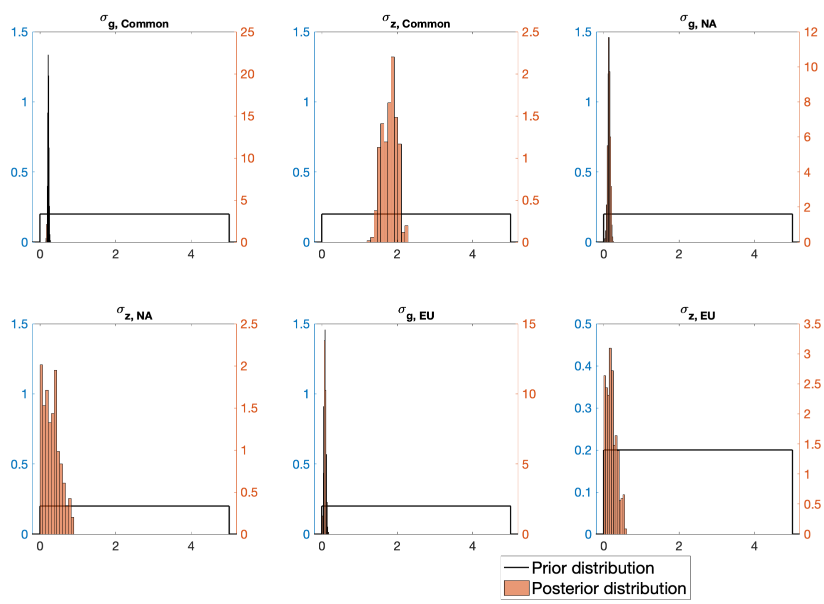


Figure 12: Prior and Posterior Distribution

(a) United States

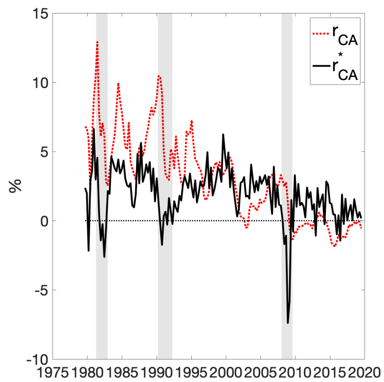



(b) Other Parameters

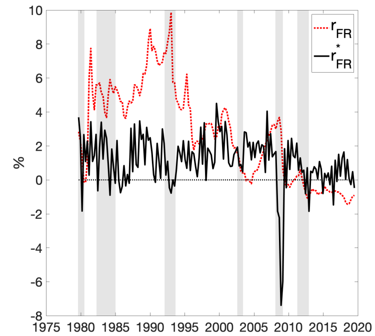


### 8.1.2 Real Interest Rate and the Natural Rate of Interest

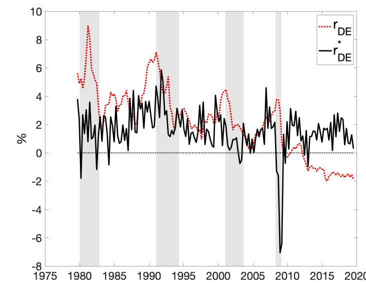
Figure 13: Real Interest Rate and the Natural Rate of Interest



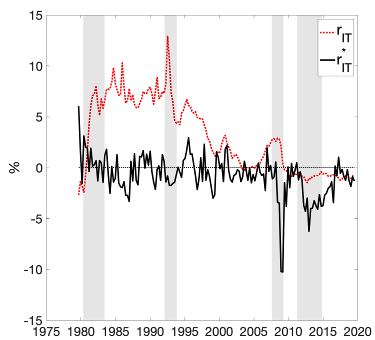

(a) Canada



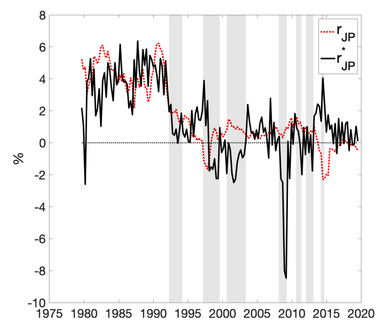

(b) France



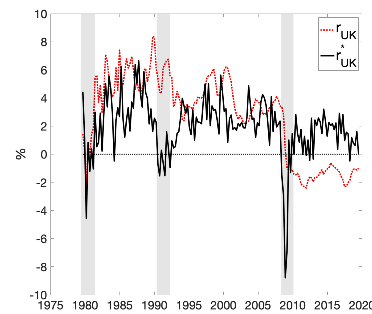

(c) Germany



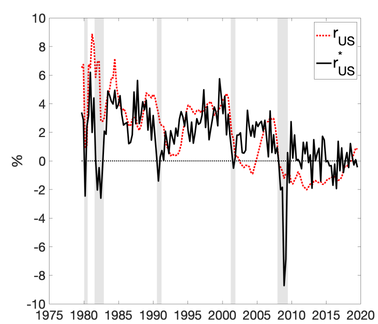

(d) Italy




(e) Japan




(f) United Kingdom




(g) United States

## 8.2 A Simple Model Explaining the Link Between Output Growth and Real Interest Rate

Consider an economy where technology $A_t$ and labor $L_t$ grow at rates $g_A$ and $n$. The production uses labor and capital, and the production function is labor augmenting constant returns to scale. The representative consumer's optimal intertemporal consumption path or Eular equation is:

$$u^{'}(c_t) = \frac{1}{1+\rho}u^{'}(c_{t+1})(F_K(K_t, A_tL_t)+1) \tag{5}$$

where $u^{'}(c_t) = \frac{du(c_t)}{dc_{c_t}}$, $\frac{1}{1+\rho}$ represents time preference with $0 < \frac{1}{1+\rho} < 1$, and $F_K(K_t, A_tL_t) = \frac{\partial F(K_t, A_tL_t)}{\partial K_t} = r_t$.

Here, utility function takes a form of constant relative risk aversion (CRRA):

$$u(c_t) = \frac{c_t^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} \tag{6}$$

Then equation (5) becomes:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\frac{1}{\sigma}} = \frac{1}{1+\rho}(r_t+1) \tag{7}$$

Take logarithm on both sides:

$$\frac{1}{\sigma}\ln\left(\frac{c_{t+1}}{c_t}\right) = \ln(r_t+1) - \ln(1+\rho)$$

Using the first order Taylor approximation:

$$\frac{1}{\sigma}\left(\frac{c_{t+1}-c_t}{c_t}\right) \approx r_t - \rho \tag{8}$$

using the fact that $\ln\left(\frac{x_{t+1}}{x_t}\right) \approx \frac{x_{t+1}-x_t}{x_t}$ and $\ln(x_t+1) \approx x_t$.

Under a balanced growth path and from equation (8), we obtain:

$$r^* = \sigma^{-1}g_A + \rho$$

This is the equation in Holston et al. (2017) that summarizes the one to one relationship of output growth and real interest rate.

## 8.3 A Complete Description of the State Space Model

### 8.3.1 Measurement Equations

## 8.4 Measurement Equations

$$Y_t = AX_t + HM_t + \nu_t$$

or

$$\begin{bmatrix} Y_{CA,t} \\ \vdots \\ Y_{US,t} \end{bmatrix} = \begin{bmatrix} A_{CA} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & A_{US} \end{bmatrix} \begin{bmatrix} x_{CA,t} \\ \vdots \\ x_{US,t} \end{bmatrix} + \begin{bmatrix} H_{CA} & & \mathbf{0} & -\frac{a_{r,CA}}{2}{}_{1\times 4} & \\ & \ddots & & \vdots & \mathrm{h} \\ \mathbf{0} & & H_{US} & -\frac{a_{r,US}}{2}{}_{1\times 4} & \end{bmatrix} M_t + \begin{bmatrix} \nu_{CA,t} \\ \vdots \\ \nu_{US,t} \end{bmatrix}$$

where

$$Y_{i,t} = \begin{bmatrix} y_{i,t} \\ \pi_{i,t} \end{bmatrix}, \quad A_i = \begin{bmatrix} a_{y1,i} & a_{y2,i} & \frac{a_{r,i}}{2} & \frac{a_{r,i}}{2} & 0 & 0 \\ b_{y,i} & 0 & 0 & 0 & b_{\pi,i} & 1-b_{\pi,i} \end{bmatrix}, \quad x_{i,t} = \begin{bmatrix} y_{i,t-1} \\ y_{i,t-1} \\ r_{i,t-1} \\ r_{i,t-2} \\ \pi_{i,t-1} \\ \pi_{i,t-2,4} \end{bmatrix}$$

$$
H_i = \begin{bmatrix} 1 & -a_{y1,i} & -a_{y2,i} & -\frac{a_{r,i}}{2} & -\frac{a_{r,i}}{2} & -\frac{a_{r,i}}{2} & -\frac{a_{r,i}}{2} \\ 0 & -b_{y,i} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} -\frac{a_{r,CA}}{2}{}_{1\times4} & 0_{1\times4} \\ 0_{1\times4} & 0_{1\times4} \\ 0_{1\times4} & -\frac{a_{r,FR}}{2}{}_{1\times4} \\ 0_{1\times4} & 0_{1\times4} \\ 0_{1\times4} & -\frac{a_{r,DE}}{2}{}_{1\times4} \\ 0_{1\times4} & 0_{1\times4} \\ 0_{1\times4} & -\frac{a_{r,IT}}{2}{}_{1\times4} \\ 0_{1\times4} & 0_{1\times4} \\ 0_{1\times4} & 0_{1\times4} \\ 0_{1\times4} & 0_{1\times4} \\ 0_{1\times4} & -\frac{a_{r,UK}}{2}{}_{1\times4} \\ 0_{1\times4} & 0_{1\times4} \\ -\frac{a_{r,US}}{2}{}_{1\times4} & 0_{1\times4} \\ 0_{1\times4} & 0_{1\times4} \end{bmatrix},
$$

$$
Mt = \begin{bmatrix} \xi_{CA,t} \\ \vdots \\ \xi_{US,t} \\ g_{t-1}^{common} \\ g_{t-2}^{common} \\ z_{t-1}^{common} \\ z_{t-2}^{common} \\ g_{t-1}^{north-america} \\ g_{t-2}^{north-america} \\ z_{t-1}^{north-america} \\ z_{t-2}^{north-america} \\ g_{t-1}^{europe} \\ g_{t-2}^{europe} \\ z_{t-1}^{europe} \\ z_{t-2}^{europe} \end{bmatrix}
$$

$$
\text{with} \quad \xi_{i,t} = \begin{bmatrix} y_{i,t}^{*} \\ y_{i,t-1}^{*} \\ y_{i,t-2}^{*} \\ g_{t-1}^{idiosyncratic} \\ g_{t-2}^{idiosyncratic} \\ z_{t-1}^{idiosyncratic} \\ z_{t-2}^{idiosyncratic} \end{bmatrix}
$$

#### 8.4.1 State Equations

$$
M_t = FM_{t-1} + \epsilon_t
$$

or

$$
M_t = \begin{bmatrix} f & & 0 & \omega & z_{CA} \\ & \ddots & & \vdots & \vdots \\ 0 & & f & \omega & z_{US} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \delta \end{bmatrix} M_{t-1} + \begin{bmatrix} \epsilon_{CA,t} \\ \vdots \\ \epsilon_{US,t} \\ e_t \end{bmatrix}
$$

where

$$
f = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \omega = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad z_{CA} = z_{US} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
z_{FR} = z_{DE} = z_{IT} = z_{US} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad z_{JP} = \mathbf{0}_{7\times 8}, \quad \delta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

$$\epsilon_{i,t} = \begin{bmatrix} \epsilon_{t,y_i^*} \\ 0 \\ 0 \\ \epsilon_{t,g_i^{idiosyncratic}} \\ 0 \\ \epsilon_{t,z_i^{idiosyncratic}} \\ 0 \end{bmatrix}, \quad \text{and} \quad e_t = \begin{bmatrix} \epsilon_{t,g^{common}} \\ 0 \\ \epsilon_{t,z^{common}} \\ 0 \\ \epsilon_{t,g^{north-america}} \\ 0 \\ \epsilon_{t,z^{north-america}} \\ 0 \\ \epsilon_{t,g^{europe}} \\ 0 \\ \epsilon_{t,z^{europe}} \\ 0 \end{bmatrix}$$

$$\nu_t \sim \mathcal{N}(\mathbf{0}, R) \qquad \text{where} \qquad R = \begin{bmatrix} R_{CA} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & R_{US} \end{bmatrix} \quad \text{with} \quad R_i = \begin{bmatrix} \sigma_{\tilde{y}_i} & 0 \\ 0 & \sigma_{\pi i} \end{bmatrix}$$

$$\epsilon_t \sim \mathcal{N}(\mathbf{0}, Q) \qquad \text{where} \qquad Q = \begin{bmatrix} q_{CA} & & \mathbf{0} & \Omega & s_{CA} \\ & \ddots & & \vdots & \vdots \\ \mathbf{0} & & q_{US} & \Omega & s_{US} \\ \Omega^T & \dots & \Omega^T & & \\ s_{CA}^T & \dots & s_{US}^T & & \Gamma_{2\times 2} \end{bmatrix}$$

Here

$$q_{CA} = q_{US} = \begin{bmatrix} \sigma^2_{y^*i} + \sigma^2_{g^{common}} + \sigma^2_{g^{north-america}} + \sigma^2_{g_i} & 0 & 0 & \sigma^2_{g_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma^2_{g_i} & 0 & 0 & \sigma^2_{g_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^2_{z_i} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$q_{FR} = q_{DE} = q_{IT} = q_{UK} = \begin{bmatrix} \sigma^2_{y^*i} + \sigma^2_{g^{common}} + \sigma^2_{g^{europe}} + \sigma^2_{g_i} & 0 & 0 & \sigma^2_{g_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma^2_{g_i} & 0 & 0 & \sigma^2_{g_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^2_{z_i} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$q_{JP} = \begin{bmatrix} \sigma^2_{y^*JP} + \sigma^2_{g^{common}} + \sigma^2_{g_i} & 0 & 0 & \sigma^2_{g_{JP}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma^2_{g_{JP}} & 0 & 0 & \sigma^2_{g_{JP}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^2_{z_{JP}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Omega = \begin{bmatrix} \sigma^2_{g^{common}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad s_{CA} = s_{US} = \begin{bmatrix} \sigma^2_{g^{north-america}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$s_{FR} = s_{DE} = s_{IT} = s_{UK} = \begin{bmatrix} 0 & 0 & 0 & 0 & \sigma^2_{g^{europe}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad s_{JP} = \mathbf{0}_{8\times 8}, \quad \text{and}$$

$$\Gamma = \begin{bmatrix} \sigma^2_{g^{common}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2_{z^{common}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^2_{g^{north-america}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{z^{north-america}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{g^{europe}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{z^{europe}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, $\nu_t$ and $\epsilon_t$ follow Gaussian distribution and are mutually uncorrelated.

## 8.5 A Detailed Explanation of the Bayesian Estimation

Given the assumption that the error terms of the state space model follow the Gaussian distribution, the density of the data $f(Y_t, X_t|M_t)$ is:

$$f(Y_t, X_t|M_t) = (2\pi)^{-\frac{1}{2}}|f_{t,t-1}|^{-\frac{1}{2}}exp(-0.5u_{t,t-1}f^{-1}_{t,t-1}u_{t,t-1})$$

where $u_{t,t-1}$ is the predictive error and $f_{t,t-1}$ is the variance of the predictive error of the Kalman filter. The likelihood function of the model is:

$$f(Y, X|\theta) = \prod_{t=1}^{T} f(Y_t, X_t|M_t)$$

where $\theta = \{\theta_{CA}, \ldots, \theta_{US}, \sigma_{g^{common}}, \sigma_{z^{common}}, \sigma_{g^{north-america}}, \sigma_{z^{north-america}}, \sigma_{g^{europe}}, \sigma_{z^{europe}}\}$ with $\theta_i = \{a_{y1,i}, a_{y2,i}, a_{r,i}, b_{\pi,i}, b_{y,i}, \sigma_{\tilde{y}_i}, \sigma_{\pi i}, \sigma_{y^* i}, \sigma_{g_i}, \sigma_{z_i}\}$.

I conducted random walk Metropolis-Hastings approach in the following steps:

**Step 1:** Specify a starting value $\theta_0$ and variance of the shock $\Sigma$.

**Step 2:** Draw a new parameter vector from the random walk equation:

$$\theta_{NEW} = \theta_{OLD} + e \qquad e \sim \mathcal{N}(\mathbf{0}, \Sigma)$$

**Step 3:** Compute the acceptance probability:

$$\alpha = min\left(\frac{f(Y, X|\theta_{NEW})p(\theta_{NEW})}{f(Y, X|\theta_{OLD})p(\theta_{OLD})}, 1\right)$$

where $p(\theta_i)$ is the prior density.

**Step 4:** If $\alpha > a \sim \mathcal{U}(0, 1)$, obtain $\theta_{NEW}$. Otherwise $\theta_{NEW} = \theta_{OLD}$.

Repeat steps 2, 3, and 4 200,000 times. The first 100,000 draws are a burn-in period and after the burn-in period, I draw parameters every 10 draws and have in total 10,000 samples of parameters.