18.06 Quiz 2 April 7, 2010 Professor Strang

Your PRINTED name is: —————————————————————— **1.**

Your recitation number or instructor is ——————————————— **2.**

3.

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1. (33 points)

(a) Find the matrix P that projects every vector b in R^3 onto the line in the direction of $a = (2, 1, 3)$.

Solution The general formula for the orthogonal projection onto the column space of a matrix $\mathbf{A}$ is

$$P = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T.$$

Here,

$$\mathbf{A} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \qquad \text{so that} \qquad \mathbf{P} = \frac{1}{14}\begin{bmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{bmatrix}$$

Remarks:

- Since we're projecting onto a one-dimensional space, $\mathbf{A}^T\mathbf{A}$ is just a number and we can write things like $P = (\mathbf{A}\mathbf{A}^T)/(\mathbf{A}^T\mathbf{A})$. *This won't work in general.*

- You don't have to know the formula to do this. The ith column of $\mathbf{P}$ is, pretty much by definition, the projection of e_i ($e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$) onto the line in the direction of a. And this is something you should know how to do without a formula.

RUBRIC: There was some leniency for computational errors, but otherwise there weren't many opportunities for partial credit.

(b) What are the column space and nullspace of P? Describe them geometrically and also *give a basis for each space.*

Solution The column space is the line in R^3 in the direction of $a = (2, 1, 3)$. One basis for it is

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

and there's not really much choice in giving this basis (you can rescale by a non-zero constant).

The nullspace is the plane in R^3 that is perpendicular to $a = (2, 1, 3)$ (i.e., $2x + y + z = 0$.)
One basis for it is

$$\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

though there are a lot of different looking choices for it (any two vectors that are perpendicular to a and not in the same line will work).

RUBRIC: 6 points for giving a correct basis, and 4 points for giving the complete geometric description. Note that it is not correct to say e.g., $N(\mathbf{P}) = R^2$. It is correct to say that $N(\mathbf{P})$ is a (2-dimensional) plane in R^3, but this is not a complete geometric description unless you say (geometrically) *which* plane it is: the one perpendicular to a/to the line through a.

(c) What are *all* the eigenvectors of P and their corresponding eigenvalues? (You can use the geometry of projections, not a messy calculation.) The diagonal entries of P add up to _____ .

Solution The diagonal entries of P add up to $1 =$ the sum of the eigenvalues

Since $\mathbf{P}$ is a projection, it's only possible eigenvalues are $\lambda = 0$ (with multiplicity equal to the dimension of the nullspace, here 2) and $\lambda = 1$ (with multiplicity equal to the dimension of the column space, here 1). So, a complete list of eigenvectors and eigenvalues is:

- $\lambda = 0$ with multiplicity 2. The eigenvectors for $\lambda = 0$ are precisely the vectors in the null space. That is, all linear combinations of $\begin{bmatrix} 3 & 0 & -2 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & 2 & 0 \end{bmatrix}^T$.
- $\lambda = 1$ with multiplicity 1. The eigenvectors for $\lambda = 1$ are precisely the vectors in the column space. That is, all multiples of $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}^T$.

RUBRIC: 2 points for the sum of eigenvalues, 4 points for a full list (with multiplicities) of eigenvalues, and 4 points for a complete description of all eigenvectors. In light of the emphasized "*all*," you'd lose 1 point if you gave two eigenvectors for $\lambda = 0$ and didn't say that all (at least non-zero) linear combinations were also eigenvectors for $\lambda = 0$.

2. (34 points)

(a) $p = A\widehat{x}$ is the vector in $C(A)$ nearest to a given vector b. If A has independent columns, what equation determines $\widehat{x}$? What are all the vectors perpendicular to the error $e = b - A\widehat{x}$? What goes wrong if the columns of A are dependent?

Solution $\widehat{x}$ is determined by the equation $\widehat{x} = (A^T A)^{-1} A^T b$ (since A has independent columns, $A^T A$ is invertible whether or not A is square). The vectors perpendicular to an arbitrary error vector are the elements of the column space of A. If the columns of A are dependent, $A^T A$ is no longer invertible, and there is no unique nearest vector (i.e. there are multiple solutions).

RUBRIC: 4 points for the determining equation (1 point off for actually inverting $A^T A$ or saying that it was invertible), 3 points for identifying the column space, and three points for identifying the multiple solutions (1 point off if you just say that $A^T A$ is not invertible). Note that you cannot write $A^{-1}B$ as $\frac{B}{A}$: this only works for numbers because multiplication and division are commutative, which is not true for matrices.

(b) Suppose $A = QR$ where Q has orthonormal columns and R is upper triangular invertible. Find $\widehat{x}$ and p in terms of Q and R and b (not A).

Solution Since $Q^T Q = I$ and R is invertible, we obtain

$$\widehat{x} = (A^T A)^{-1} A^T b = ((QR)^T (QR))^{-1} (QR)^T b$$
$$= (R^T Q^T QR)^{-1} R^T Q^T b = R^{-1}(R^T)^{-1} R^T Q^T b = R^{-1} Q^T b$$
$$p = (QR)\widehat{x} = QQ^T b$$

Note that QQ^T is not the identity matrix in general.

RUBIC: 6 points for finding $\widehat{x}$, 4 points for p. One point off from each if the equations are not simplified, more points off for bad form, having variables other than Q, R and b, etc.

(c) If q_1 and q_2 are any orthonormal vectors in R^5, give a formula for the projection p of any vector b onto the plane spanned by q_1 and q_2 (write p as a combination of q_1 and q_2).

Solution $p = (q_1^T b)q_1 + (q_2^T b)q^2$.

RUBRIC: little partial credit. If you identified the difference between b and p instead, you may have gotten some points.

3. (33 points) This problem is about the n by n matrix A_n that has zeros on its main diagonal and all other entries equal to -1. In MATLAB $A_n = \text{eye}(n) - \text{ones}(n)$.

(a) Find the determinant of A_n. Here is a suggested approach:

Start by adding all rows (except the last) to the last row, and then factoring out a constant. (You could check $n = 3$ to have a start on part b.)

Solution Following the hint, add all of the rows to the last row (which does not change the determinant). Thus the matrix becomes

$$
\begin{bmatrix}
0 & -1 & -1 & \cdots & -1 \\
-1 & 0 & -1 & \cdots & -1 \\
-1 & -1 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & & \vdots \\
-(n-1) & -(n-1) & -(n-1) & \cdots & -(n-1)
\end{bmatrix} .
$$

Next, pull out the factor of $-(n-1)$ from the last row. As the determinant is linear in each row separately, we get

$$
\begin{vmatrix}
0 & -1 & -1 & \cdots & -1 \\
-1 & 0 & -1 & \cdots & -1 \\
-1 & -1 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & & \vdots \\
-(n-1) & -(n-1) & -(n-1) & \cdots & -(n-1)
\end{vmatrix}
= (1-n)
\begin{vmatrix}
0 & -1 & -1 & \cdots & -1 \\
-1 & 0 & -1 & \cdots & -1 \\
-1 & -1 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & & \vdots \\
1 & 1 & 1 & \cdots & 1
\end{vmatrix} .
$$

Next, add the last row back to each of the other rows (which again keeps the determinant the same). So now we want to find

$$
(1-n)
\begin{vmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
1 & 1 & 1 & \cdots & 1
\end{vmatrix} .
$$

This matrix is lower triangular. So its determinant is the product of the entries on its diagonal. Thus the above quantity is $(1-n)$.

Alternately, one can find the determinant of the matrix by finding all its eigenvalues. As $A_n = I - \texttt{ones}(n)$, we know that $N(A_n - I) = N(-\texttt{ones}(n))$. The latter nullspace has dimension $n - 1$. Thus 1 is an eigenvalue of multiplicity $n - 1$, and the corresponding eigenvectors are all the nonzero vectors whose entries add up to 0.

In addition, all of the rows of A_n add up to $1 - n$. So $1 - n$ is an eigenvalue with eigenvector $(1, 1, \ldots, 1)$. Thus we have found all of the eigenvectors and eigenvalues. The determinant is the product of the eigenvalues, so it is $1^{n-1} \cdot (1 - n)$ or $1 - n$.

RUBRIC: 2 points for following the hint, 2 points for pulling out the factor of $(1 - n)$ correctly, 2 points for adding the last row to the other rows, 2 points for the correct answer.

(b) For any invertible matrix A, the $(1, 1)$ entry of A^{-1} is the ratio of _____ .

So the $(1, 1)$ entry of A_4^{-1} is _____ .

Solution Cramer's rule gives $A^{-1} = \frac{1}{|A|} C^{\mathrm{T}}$ where C is the cofactor matrix, whose (i, j) entry is $(-1)^{i+j}|M_{ij}|$ where M_{ij} is the submatrix obtained by deleting row i and column j of the (arbitrary) invertible matrix A. Thus the entry with $i = j = 1$ is $|M_{11}|/|A|$.

In the case where $A = A_n$, the submatrix M_{11} is A_{n-1}; so the desired formula is $|A_{n-1}|/|A_n|$. Now, $|A_n| = 1 - n$ by part (a). So $|A_4| = -3$ and $|A_3| = -2$. Thus the $(1, 1)$ entry of A_4^{-1} is $2/3$.

RUBRIC: 5 points for the correct ratio, 5 points for the correct application to the current problem. If the wrong ratio was given, then no credit was given for applying it.

(c) Find *two orthogonal eigenvectors* with $A_3\, x = x$. (So $\lambda = 1$ is a double eigenvalue.)

Solution In solution 2 of part (a) above, we saw that the eigenvectors are all the nonzero vectors whose entries add up to 0. Two obvious such vectors are $(1, -1, 0)$ and $(0, 1, -1)$, but there are many more linearly independent pairs.

However, $(1, -1, 0)$ and $(0, 1, -1)$ are not orthogonal! So we must find another pair. We can use the Gram–Schmidt process to get orthogonal vectors, or we can just try to guess two orthogonal vectors whose entries add up to 1. For example, $(1, -1, 0)$ and $(1, 1, -2)$ work. (Note that the vectors are not required to have unit length.)

RUBRIC: up to 5 points for a correct method, 2 points for finding linearly independent vectors, 3 points for orthogonality.

(d) What is the third eigenvalue of A_3 and a corresponding eigenvector?

Solution In solution 2 of part (a) above, we saw that the third eigenvalue is -2 and a corresponding eigenvector is $(1, 1, 1)$.

Another way to proceed is to notice that the trace of A_3 is 0. However, the trace is the sum of the eigenvalues, and two of them are 1. So the third must be -2. Alternatively, in part (a), we saw that $|A_3| = -2$. However, the determinant is the product of the eigenvalues, and two of them are 1. So the third must be -2.

A third way to proceed is to find the characteristic polynomial of A_3, which is $\lambda^3 - 3\lambda + 2$. Since 1 is a double root, we can find the third root by dividing twice by $\lambda - 1$.

RUBRIC: 5 points for the eigenvalue, 5 points for a corresponding eigenvector.

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