18.06 Problem Set 7 Solutions

Total: 100 points

Prob. 16, Sec. 5.2, Pg. 265: F_n is the determinant of the 1, 1, -1 tridiagonal matrix of order n:

$$F_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \qquad F_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 3 \qquad F_4 = \begin{vmatrix} 1 & -1 & & \\ 1 & 1 & -1 & \\ & 1 & 1 & -1 \\ & & 1 & 1 \end{vmatrix} \neq 4.$$

Expand in cofactors to show that $F_n = F_{n-1} + F_{n-2}$. These determinants are *Fibonacci numbers* 1, 2, 3, 5, 8, 13, ... The sequence usually starts 1, 1, 2, 3 (with two 1's) so our F_n is the usual F_{n+1}.

$\boxed{\text{Solution}}$ (see pg. 535, 4 pts.): The 1, 1 cofactor of the n by n matrix is F_{n-1}. The 1, 2 cofactor has a 1 in column 1, with cofactorF_{n-2}. Multiply by $(-1)^{1+2}$ and also (-1) from the 1, 2 entry to find $F_n = F_{n-1} + F_{n-2}$ (so these determinants are Fibonacci numbers).

Prob. 32, Sec. 5.2, Pg. 268: Cofactors of the 1, 3, 1 matrices in Problem 21 give a recursion $S_n = 3S_{n-1} - S_{n-2}$. Amazingly that recursion produces every second Fibonacci number. Here is the challenge.
Show that S_n is the Fibonacci number F_{2n+2} by proving $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using Fibonacci's rule $F_k = F_{k-1} + F_{k-2}$ starting with $k = 2n + 2$.

$\boxed{\text{Solution}}$ (see pg. 535, 12 pts.): To show that $F_{2n+2} = 3F_{2n} - F_{2n-2}$, keep using Fibonacci's rule:

$$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{n-1} + F_{2n} = 2F_{2n} + (F_{2n} - F_{2n-2}) = 3F_{2n} - F_{2n-2}.$$

Prob. 33, Sec. 5.2, Pg. 268: The symmetric Pascal matrices have determinant 1. If I subtract 1 from the n, n entry, why does the determinant become zero? (Use rule 3 or cofactors.)

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = 1 \text{ (known)} \qquad \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & \mathbf{19} \end{bmatrix} = \mathbf{0} \text{ (to explain).}$$

$\boxed{\text{Solution}}$ (see pg. 535, 12 pts.): The difference from 20 to 19 multiplies its cofactor, which is the determinant of the 3 by 3 Pascal matrix, so equal to 1. Thus the det drops by 1.

Prob. 8, Sec. 5.3, Pg. 279: Find the cofactors of A and multiply AC^{T} to find $\det A$:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \text{and } AC^{\mathrm{T}} = \underline{\qquad}.$$

If you change that 4 to 100, why is $\det A$ unchanged?

$\boxed{\text{Solution}}$ (see pg. 536, 4 pts.): Straightforward computation yields C and $\det A = 3$:

$$C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix} \text{ and } AC^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

This is $(\det A)I$ and $\det A = 3$.
The 1, 3 cofactor of A is 0.
Multiplying by 4 or by 100: no change.

Prob. 28, Sec. 5.3, Pg. 281: Spherical coordinates ρ, ϕ, θ satisfy $x = \rho \sin\phi \cos\theta$ and $y = \rho \sin\phi \sin\theta$ and $z = \rho\cos\phi$. Find the 3 by 3 matrix of partial derivatives: $\partial x/\partial\rho$, $\partial x/\partial\phi$, $\partial x/\partial\theta$ in row 1. Simplify its determinant to $J = \rho^2 \sin\phi$. Then dV in spherical coordinates is $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ the volume of an infinitesimal "coordinate box".

$\boxed{\text{Solution}}$ (4 pts.): The rows are formed by the partials of x, y, z with respect to ρ, ϕ, θ:

$$\begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}.$$

Expanding its determinant J along the bottom row, we get

$$\begin{aligned} J &= \cos\phi(\rho^2\cos\phi\sin\phi)(\cos^2\theta + \sin^2\theta) + \rho^2\sin^3\phi(\cos^2\theta + \sin^2\theta) \\ &= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) = \rho^2\sin\phi. \end{aligned}$$

Prob. 40, Sec. 5.3, Pg. 282: Suppose A is a 5 by 5 matrix. Its entries in row 1 multiply determinants (cofactors) in rows 2–5 to give the determinant. Can you guess a "Jacobi formula" for $\det A$ using 2 by 2 determinants from rows 1–2 *times* 3 by 3 determinants from rows 3–5? Test your formula on the -1, 2, -1 tridiagonal matrix that has determinant 6.

$\boxed{\text{Solution}}$ (12 pts.): A good guess for $\det A$ is the sum, over all pairs i, j with $i < j$, of $(-1)^{i+j+1}$ times the 2 by 2 determinant formed from rows 1–2 and columns i, j times the 3 by 3 determinant formed from rows 3–5 and the complementary columns (this formula is more commonly named after Laplace than Jacobi). There are $\binom{5}{2}$ terms. In the given case, only the first two are nonzero:

$$\det A = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} - \begin{vmatrix} 2 & \\ -1 & -1 \end{vmatrix} \begin{vmatrix} -1 & -1 & \\ & 2 & -1 \\ & -1 & 2 \end{vmatrix} = (3)(4) - (-2)(-3) = 6.$$

Prob. 41, Sec. 5.3, Pg. 282: The 2 by 2 matrix $AB = (2 \text{ by } 3)(3 \text{ by } 2)$ has a "Cauchy–Binet formula" for $\det AB$:

$$\det AB = \text{ sum of (2 by 2 determinants in } A) \text{ (2 by 2 determinants in } B).$$

(a) Guess which 2 by 2 determinants to use from A and B.
(b) Test your formula when the rows of A are 1, 2, 3 and $1, 4, 7$ with $B = A^{\mathrm{T}}$.

$\boxed{\text{Solution}}$ (12 pts.): (a) A good guess is the sum, over all pairs i, j with $i < j$, of the product of the 2 by 2 determinants formed from columns i, j of A and rows i, j of B.

(b) First, $AA^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 14 & 30 \\ 30 & 66 \end{bmatrix}$. So $\det AA^{\mathrm{T}} = 924 - 900 = 24$.

On the other hand, $\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} \begin{vmatrix} 2 & 4 \\ 3 & 7 \end{vmatrix} = 4 + 16 + 4 = 24.$

Prob. 19, Sec. 6.1, Pg. 295: A 3 by 3 matrix B is known to have eigenvalues $0, 1, 2$. This is information is enough to find three of these (give the answers where possible):

(a) the rank of B,
(b) the determinant of $B^{\mathrm{T}}B$,
(c) the eigenvalues of $B^{\mathrm{T}}B$,
(d) the eigenvalues of $(B^2 + I)^{-1}$.

$\boxed{\text{Solution}}$ (4 pts.): (a) The rank is at most 2 since B is singular as 0 is an eigenvalue. The rank is not 0 since B is not 0 as B has a nonzero eigenvalue. The rank is not 1 since a rank-1 matrix has only one nonzero eigenvalue as every eigenvector lies in the column space. Thus the rank is 2.

(b) We have $\det B^{\mathrm{T}}B = \det B^{\mathrm{T}} \det B = (\det B)^2 = 0 \cdot 1 \cdot 2 = 0$.

(c) There is not enough information to find the eigenvalues of $B^{\mathrm{T}}B$. For example,

$$\text{if } B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 2 \end{bmatrix}, \text{ then } B^{\mathrm{T}}B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 4 \end{bmatrix}; \quad \text{if } B = \begin{bmatrix} 0 & 1 & \\ & 1 & \\ & & 2 \end{bmatrix}, \text{ then } B^{\mathrm{T}}B = \begin{bmatrix} 0 & & \\ & 2 & \\ & & 4 \end{bmatrix}.$$

However, the eigenvalues of a triangular matrix are its diagonal entries.

(d) If $Ax = \lambda x$, then $x = \lambda A^{-1}x$; also, any polynomial $p(t)$ yields $p(A)x = p(\lambda)x$. Hence the eigenvalues of $(B^2 + I)^{-1}$ are $1/(0^2 + 1)$ and $1/(1^2 + 1)$ and $1/(2^2 + 1)$, or 1 and $1/2$ and $1/5$.

Prob. 29, Sec. 6.1, Pg. 296: (Review) Find the eigenvalues of A, B, and C:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

$\boxed{\text{Solution}}$ (4 pts.): Since the eigenvalues of a triangular matrix are its diagonal entries, the eigenvalues of A are $1, 4, 6$. Since the characteristic polynomial of B is

$$\det(B - \lambda I) = (-\lambda)(2 - \lambda)(-\lambda) - 1(2 - \lambda)3 = (2 - \lambda)(\lambda^2 - 3),$$

the eigenvalues of B are $2, \pm\sqrt{3}$. Since C is 6 times the projection onto $(1, 1, 1)$, the eigenvalues of C are $6, 0, 0$.

Prob. 6, Sec. 6.2, Pg. 308: Describe all matrices S that diagonalize this matrix A (find all eigenvectors):

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix},$$

Then describe all matrices that diagonalize A^{-1}.

$\boxed{\text{Solution}}$ (see pg. 537, 4 pts.): The columns of S are nonzero multiples of $(2, 1)$ and $(0, 1)$: either order. Same for A^{-1}. Indeed, since the eigenvalues of a triangular matrix are its diagonal entries, the eigenvalues of A are 4, 2. Further, $(2, 1)$ and $(0, 1)$ obviously span the nullspaces of

$$\begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}.$$

Prob. 16, Sec. 6.2, Pg. 309: (Recommended) Find Λ and S to diagonalize A_1 in Problem 15:

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix}.$$

What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of the matrix you see the ____.

$\boxed{\text{Solution}}$ (4 pts.): The columns sum to 1; hence, $A_1 - I$ is singular, and so 1 is an eigenvalue. The two eigenvalues sum to 0.6+0.1; so the other one is -0.3. Further, the nullspaces of

$$\begin{bmatrix} -0.4 & 0.9 \\ 0.4 & -0.9 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.9 & 0.9 \\ 0.4 & 0.4 \end{bmatrix}$$

are obviously spanned by $(9, 4)$ and $(-1, 1)$. Therefore,

$$\Lambda = \begin{bmatrix} 1 & \\ & -0.3 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 9 & -1 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda^k \to \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \quad \text{and}$$
$$S\Lambda^k S^{-1} \to \begin{bmatrix} 9 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \frac{1}{9+4} \begin{bmatrix} 1 & 1 \\ -4 & 9 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 9 & 9 \\ 4 & 4 \end{bmatrix}.$$

In the columns of the last matrix you see the steady state vector.

Prob. 37, Sec. 6.2, Pg. 311: The transpose of $A = S\Lambda S^{-1}$ is $A^{\mathrm{T}} = (S^{-1})^{\mathrm{T}} \Lambda S^{\mathrm{T}}$. The eigenvectors in $A^{\mathrm{T}} y = \lambda y$ are the columns of that matrix $(S^{-1})^{\mathrm{T}}$. They are often called ***left eigenvectors***. How do you multiply matrices to find this formula for A?

$$\textbf{Sum of rank-1 matrices} \quad A = S\Lambda S^{-1} = \lambda_1 x_1 y_1^{\mathrm{T}} + \cdots + \lambda_n x_n y_n^{\mathrm{T}}.$$

$\boxed{\text{Solution}}$ (see pg. 539, 12 pts.): Columns of S times rows of ΛS^{-1} will give r rank-1 matrices (r = rank of A).

Challenge problem: in MATLAB (and in GNU Octave), the command `A=toepliz(v)` produces a symmetric matrix in which each descending diagonal (from left to right) is constant and the first row is v. For instance, if $v = [0\ 1\ 0\ 0\ 0\ 1]$, then `toepliz(v)` is the matrix with 1s on both sides of the main diagonal and on the far corners, and 0s elsewhere. More generally, let $v(n)$ be the vector in $\mathbf{R}^n$ with a 1 in the second and last places and 0s elsewhere, and let `A(n)=toepliz(v(n))`.

(a) Experiment with $n = 5, \ldots, 12$ in MATLAB to see the repeating pattern of $\det A(n)$.

(b) Expand $\det A(n)$ in terms of cofactors of the first row and in terms of cofactors of the first column. Use the known determinant C_n of problem 5.2.13 to recover the pattern found in part (a).

$\boxed{\text{Solution}}$ (12 pts.): (a) The output 2, -4, 2, 0, 2, -4, 2, 0 is returned by this line of code:

```
for n = 5:12; v=zeros(1,n); v(2)=1; v(n)=1; det(toeplitz(v)), endfor.
```

(b) Expand $\det A(n)$ along the first row and then down both first columns to get

$$\det A(n) = -C_{n-2} - (-1)^n + (-1)^{n+1} + (-1)^{n+1}(-1)^n C_{n-2} \quad \text{where } C_n = \begin{cases} 0, & n \text{ odd;} \\ (-1)^{n/2}, & n \text{ even.} \end{cases}$$

Thus $\det A(n) = 2(C_n - (-1)^n)$, which recovers the pattern found in part (a).