RIEMANN-ROCH THEOREM AND APPLICATIONS

Bachelor's Thesis

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Eidesstattliche Erklärung

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Saarbrücken, March, 2020 Eileen Oberringer

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Contents

Introduction

This thesis centres on the famous Riemann-Roch Theorem in mathematics, especially algebraic geometry, named after Bernhard Riemann and Gustav Roch. During his study of compact complex manifolds of dimension one (today called *compact Riemann surfaces*), Riemann searched for a way to determine the number of linearly independent meromorphic functions on such manifolds. This is accomplished by the Theorem of Riemann-Roch.

Riemann's original proof was very technical, so we are going to treat the theorem in a geometric way by using the language of *divisors* on *nonsingular projective curves*. In this context, the theorem concerns the dimension of vectorspaces $H^0(X, D)$ of rational functions on a curve X with prescribed zeros and allowed poles.

The Riemann-Roch Theorem is a very deep result, so it requires a lot of previous knowledge. Therefore, in the first chapter, we gather the essential definitions and results, needed to prove the theorem. In particular, we will define the notion of *divisors* D and their *degree* $\deg(D)$. *Divisors* are the fundamental objects appearing in the theorem, since they specify the prescriptions concerning the zeros and poles of the rational functions in $H^0(X, D)$.

After defining the general notion of *divisors* and after a small introduction into the notion of *differentials*, we will also define the *canonical divisor*. This is the most important kind of *divisor* on a curve, since it is an intrinsic object. It is used to characterise the *genus*, which is the most significant birational invariant of a curve.

The first partial step towards the Riemann-Roch Theorem is a lower bound for the dimensions of the spaces $H^0(X, D)$, given as

$$\dim(H^0(X, D)) \geq \deg(D) + 1 - g.$$

Here, g denotes the genus of the given curve. This inequality will be treated in detail in the second chapter along with some helpful properties it implies.

In the second to last chapter, we come to the Riemann-Roch Theorem itself. With the help of Gustav Roch, Riemann found the missing term, which makes the aforementioned inequality into an equality and obtained the precise formula

$$\dim(H^0(X, D)) = \deg(D) + 1 - g + \dim(H^0(X, K_X - D)),$$

depending on the *canonical divisor* K_X of X. We will state and prove this formula. Our proof follows that of Max Noether and Alexander von Brill, which utilises an inductive argument while distinguishing different cases.

In the last chapter, we will give a small introduction to some applications of the Riemann-Roch Theorem. In particular, we introduce Clifford's Theorem as well as Hurwitz' Formula.

A few remarks on notation

In the entire thesis, an *affine* respectively a *projective curve* X corresponds to an affine respectively projective variety of dimension 1 over $\mathbb{C}$.

The *affine* respectively *homogeneous coordinate ring* of X is denoted by $A(X) := \mathbb{C}[X_1, ..., X_n]/I(X)$ respectively by $S(X) := \mathbb{C}[X_0, ..., X_n]/I(X)$. The corresponding *field of rational functions* is denoted by $K(X)$. Furthermore, for each point $P \in X$, $\mathcal{O}_{X,P} \subset K(X)$ denotes the *local ring of* X *at* P with maximal ideal $\mathfrak{m}_P$.

An *affine* respectively a *projective plane curve* C corresponds to a curve $C := V(F) \subset \mathbb{A}^2$ for an irreducible polynomial $F \in \mathbb{C}[X_1, X_2]$ respectively to a curve $C := V(F) \subset \mathbb{P}^2$ for an irreducible, homogeneous polynomial $F \in \mathbb{C}[X_0, X_1, X_2]$. We call F the *defining polynomial* of the plane curve.

If X is a curve, possibly with singularities, then, via *Noether normalisation*, we can always find a birational morphism $\rho\colon X' \to X$, where X' is a nonsingular curve. We call such a curve X' *nonsingular model* of X.

Chapter 1

Background

1.1 Preliminary Results

To be able to understand the main definitions and results of this thesis, we first have to do some preparation. This section gives a collection of results, establishing a sufficient background for understanding this thesis.

For the beginning, we consider the local ring $\mathcal{O}_{X,P}$ at a nonsingular point P on a curve X and we claim that this is a discrete valuation ring. This is an important result, since many further definitions and results will build on it. To prove the claim, we want to use the following proposition.

Proposition 1.1. *Let R be a Noetherian, local integral domain, let $\mathfrak{m} \subset R$ be its maximal ideal and $k := R/\mathfrak{m}$ be its residue field. Assume that* $\dim(R) = 1$. *Then the following are equivalent:*

(i) *R is a discrete valuation ring.*

(ii) *R is integrally closed.*

(iii) *$\mathfrak{m}$ is a principal ideal.*

(iv) $\dim_k (\mathfrak{m}/\mathfrak{m}^2) = 1$.

(v) *Every non-zero ideal of R is a power of $\mathfrak{m}$.*

(vi) *There exists an element $x \in R$, such that every ideal $I \subset R$ is of the form $I = (x^n)$ for some $n \in \mathbb{N}$.*

Proof. See [AM94, Proposition 9.2]. □

Now, we are able to prove the aforementioned claim by showing that one and a fortiori all of the equivalent statements from the proposition above hold.

Proposition 1.2. *Let X be a curve. For each nonsingular point $P \in X$, the local ring $\mathcal{O}_{X,P}$ is a discrete valuation ring.*

Proof. Let $P \in X$ be a nonsingular point. Let $\mathcal{O}_{X,P}$ be the local ring at P and let $\mathfrak{m}_P$ be its maximal ideal. We know that the coordinate ring $S(X)$ is a Noetherian integral domain of dimension 1. Hence, $\mathcal{O}_{X,P}$ is a Noetherian local integral domain, which is regular because P is a nonsingular point.

Now $\mathcal{O}_{X,P} \simeq S(X)_{\mathfrak{m}'_P}$, where

$$\mathfrak{m}'_P := \langle f \in S(X) \ : \ f(P) = 0 \rangle.$$

Since $S(X)$ is integral, $\mathfrak{m}'_P$ is a prime ideal and since $S(X)$ is Noetherian, $\mathfrak{m}'_P = \langle f_1, ..., f_n \rangle$ for some irreducible $f_i \in \mathcal{O}_{X,P}$, for $1 \le i \le n$, $n \in \mathbb{N}$.

Claim: $n = 1$, i.e. $\mathfrak{m}'_P$ is principal.

Proof of claim. Assume $n \ge 2$. Then we have an ascending chain of prime ideals $(0) \subsetneq (f_1) \subsetneq (\mathfrak{m}'_P)$ of length 2, contradicting $\dim S(X) = 1$. Hence, the maximal ideal $\mathfrak{m}'_P S(X)_{\mathfrak{m}'_P}$ of $S(X)_{\mathfrak{m}'_P} \simeq \mathcal{O}_{X,P}$ is principal, which means that $(0) \subseteq \mathfrak{m}'_P S(X)_{\mathfrak{m}'_P}$ is the only chain of prime ideals in $\mathcal{O}_{X,P}$, such that $\mathcal{O}_{X,P}$ is of dimension 1. □

Now, $\mathcal{O}_{X,P}$ is a regular local Noetherian integral domain of dimension 1 and therefore

$$1 = \dim(\mathcal{O}_{X,P}) = \dim\left(\mathfrak{m}_P/\mathfrak{m}_P^2\right).$$

By Proposition 1.1, the local ring $\mathcal{O}_{X,P}$ is a discrete valuation ring, which was to be shown. □

As we have seen above, for all nonsingular points $P \in X$, the maximal ideal $\mathfrak{m}_P$ of $\mathcal{O}_{X,P}$ is principal. If $t \in \mathcal{O}_{X,P}$ generates $\mathfrak{m}_P$, we call t a **local parameter** at P.

Next, we give some basic definitions for projective plane curves, in order to state Bézout's Theorem, which we need in further results. Roughly speaking, Bézout's Theorem gives us the number of intersection points of two projective plane curves with special assumptions, as we will see below.

At first, we want to define the notion of the *multiplicity* of a point on an affine plane curve.

Definition 1.3 (Multiplicity of a point)**.** Let $C \subset \mathbb{A}^2$ be an affine plane curve with defining polynomial $F \subset \mathbb{C}[X_1, X_2]$ and let $P := (0,0) \in C$. Write

$$F = F_0 + F_1 + ... + F_d,$$

where $d := \deg(F)$ and where for all $0 \le i \le d$, F_i denotes a homogeneous polynomial of degree i. We define the **multiplicity** $m_P(F)$ of P on C to be the smallest integer r, such that $F_r \neq 0$. If we have a point $(0,0) \neq Q \in C$, then we first make a linear change of coordinates and the same definition works for Q.

Bézout's Theorem makes a statement about projective plane curves, so we also have to define the notion of the multiplicity in the projective case. The next definition shows how to get an affine plane curve from a projective plane curve and vice versa, so we can make use of the previous definition and can transfer the notion of the multiplicity in the affine case into the projective case.

Definition 1.4. Let $C \subset \mathbb{P}^2$ be a projective plane curve with defining polynomial $F \in \mathbb{C}[X_0, X_1, X_2]$. We call

$$F_D := F(1, X_1, X_2) \in \mathbb{C}[X_1, X_2]$$

the **dehomogenisation** of F and call the curve $C_D \subset \mathbb{A}^2$ with defining polynomial F_D the **corresponding affine** curve. Conversely let $C \subset \mathbb{A}^2$ be an affine plane curve with defining polynomial $F \in \mathbb{C}[X_1, X_2]$. Given an extra variable X_0, we call

$$F_H := X_0^{\deg(F)} F\left(\frac{X_1}{X_0}, \frac{X_2}{X_0}\right) \in \mathbb{C}[X_0, X_1, X_2]$$

the **homogenisation** of F and call the curve $C_H \subset \mathbb{P}^2$ with defining polynomial F_H the **corresponding projective** curve.

At this point, we are able to define the notion of the multiplicity of a point for projective plane curves.

Definition 1.5. Let $C \subset \mathbb{P}^2$ be a projective plane curve with defining polynomial $F \in \mathbb{C}[X_0, X_1, X_2]$ and let $(x_0, x_1, x_2) =: P \in C$ be a point. We define the **multiplicity** of $P \in C$ to be

$$m_P(F) := m_{P'}(F_D),$$

where $P' := (x_1, x_2)$.

The following definition introduces a special kind of multiplicity. Roughly speaking, it generalises the notion of counting the intersection points of two projective plane curves. Since we mentioned above that Bézout's Theorem just characterises the number of intersection points of two projective plane curves, the definition will be crucial to introduce the theorem.

Definition 1.6 (Intersection multiplicity)**.** Let $C_1 \subset \mathbb{P}^2$ and $C_2 \subset \mathbb{P}^2$ be two projective plane curves with defining polynomials $F_1 \in \mathbb{C}[X_0, X_1, X_2]$ respectively $F_2 \in \mathbb{C}[X_0, X_1, X_2]$. Let $F_{1,D} \in \mathbb{C}[X_1, X_2]$ and $F_{2,D} \in \mathbb{C}[X_1, X_2]$ be the dehomogenisations of F_1 respectively F_2 and let $P \in \mathbb{P}^2$ be a point. We define

$$I(P; C_1 \cap C_2) := \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{P}^2,P}/(F_{1,D}, F_{2,D}))$$

to be the **intersection multiplicity** of C_1 and C_2.

At this point, we have everything that we need to state the theorem of Bézout. We only provide a reference for the proof.

Theorem 1.7 (Bézout's Theorem)**.** *Let $C_1 \subset \mathbb{P}^2$ and $C_2 \subset \mathbb{P}^2$ be two projective plane curves. Assume that C_1 and C_2 have no common component. Then*

$$\sum_{P \in \mathbb{P}^2} I(P; C_1 \cap C_2) = \deg(C_1)\deg(C_2).$$

Proof. See [Ful69, Chapter 5.3, Bézout's Theorem]. □

The next definition formalises the intuitive understanding of a *tangent* at a point and, associated with it, introduces the notion of an *ordinary multiple point*, which is just characterised by the number of its *tangents*.

Definition 1.8. Let $C \subset \mathbb{P}^2$ be a projective plane curve with defining polynomial $F \in \mathbb{C}[X_0, X_1, X_2]$. A line L is said to be **tangent** to C at $P \in \mathbb{P}^2$, if

$$I(P; C \cap L) > m_P(F).$$

If C has $m_P(F)$ distinct tangent lines at P, we call the point $P \in C$ an **ordinary multiple point** of C.

Ordinary multiple points of a curve are interesting because many results, which we need to prove the Riemann-Roch Theorem, hold for curves whose singular points are all ordinary multiple points. Furthermore, to a projective plane curve, we can always find a birational plane curve with only ordinary multiple points. So, as one may suspect, we want to use this result.

In the next couple of definitions, we want to make a digression into *cycle*-theory.

Definition 1.9. A **zero-cycle** Z on $\mathbb{P}^2$ is a finite sum

$$Z := \sum_{P \in \mathbb{P}^2} \eta_P \cdot P,$$

where $\eta_P \in \mathbb{Z}$. We define the **degree** of Z to be

$$\deg(Z) := \sum_{P \in \mathbb{P}^2} \eta_P.$$

If each $\eta_P \geq 0$, we say that Z is **positive**.

Remark 1.10. The notion of a cycle corresponds to the dimension of the objects, we consider. In the example of a *zero*-cycle, we study points $P \in C$, i.e. objects of dimension *zero*. We can define the notion of a cycle in a much more general way.

Definition 1.11 (k-cycle)**.** Let X be a projective curve and let k be a natural number. A ***k*-cycle** Z on X is an element of the free abelian group generated by the closed irreducible subvarieties of X of dimension k, i.e. a finite sum

$$Z := \sum_{Y \subset X} \eta_Y \cdot Y,$$

where $\eta_Y \in \mathbb{Z}$ and Y is a subvariety of X of dimension k.

Remark 1.12. In the next chapter, we will define the notion of a *divisor* on a curve, which is the most important object in this thesis. The *divisor* is a zero-cycle on a projective curve, as we will see later.

The next definition builds on the definition of the intersection multiplicity of two projective plane curves and introduces a zero-cycle with precisely these numbers as coefficients of the points.

Definition 1.13. Let $C_1 \subset \mathbb{P}^2$ and $C_2 \subset \mathbb{P}^2$ be two projective plane curves with defining polynomials $F_1 \in \mathbb{C}[X_0, X_1, X_2]$ respectively $F_2 \in \mathbb{C}[X_0, X_1, X_2]$. Assume that C_1 and C_2 have no common component. We define

$$C_1 \cdot C_2 := \sum_{P \in \mathbb{P}^2} I(P; C_1 \cap C_2) \cdot P$$

to be the **intersection cycle** of C_1 and C_2.

Remark 1.14. By Bézout's Theorem, $C_1 \cdot C_2$ is a positive zero-cycle of degree $\deg(C_1)\deg(C_2)$.

In the following important theorem of Max Noether, all the work that we have done so far flows together. Intuitively, the theorem answers the following question:

Consider three projective plane curves C_1, C_2 and C_3 with defining polynomials F_1, F_2 and F_3 in $\mathbb{C}[X_0, X_1, X_2]$. Assume that for the intersection cycles $C_3 \cdot C_1$ and $C_2 \cdot C_1$ it holds that

$$C_3 \cdot C_1 \geq C_2 \cdot C_1,$$

i.e. C_3 intersects C_1 in a "bigger" cycle, or "more often" than C_2 does. When does a curve C_4 exist, such that

$$C_4 \cdot C_1 = C_3 \cdot C_1 - C_2 \cdot C_1,$$

where $\deg(C_4) = \deg(C_3) - \deg(C_2)$? This means that we can distinguish the different behaviours of the curves when intersecting them with C_1. This is equivalent to finding polynomials $A, B \in \mathbb{C}[X_0, X_1, X_2]$, such that we can write F_3 as a sum $F_3 = AF_1 + BF_2$ because then we have

$$C_3 \cdot C_1 = V(A)C_1 \cdot C_1 + V(B)C_2 \cdot C_1 = V(B)C_2 \cdot C_1 = V(B) \cdot C_1 + C_2 \cdot C_1.$$

The following definition introduces a local version of the considerations above.

Definition 1.15 (Noether's Conditions)**.** Let $P \in \mathbb{P}^2$ be a point and let $C_1 \subset \mathbb{P}^2$ and $C_2 \subset \mathbb{P}^2$ be two projective plane curves with defining polynomials F_1 respectively F_2 in $\mathbb{C}[X_0, X_1, X_2]$. Assume that C_1 and C_2 have no common component through P. Let $C_3 \subset \mathbb{P}^2$ be another projective plane curve with defining polynomial $F_3 \in \mathbb{C}[X_0, X_1, X_2]$. We say that **Noether's Conditions** are satisfied with respect to C_3 at the point $P \in \mathbb{P}^2$, if there exist $a, b \in \mathcal{O}_{\mathbb{P}^2,P}$ with

$$F_{3,D} = aF_{1,D} + bF_{2,D}.$$

Now, we are able to introduce the theorem of Max Noether, which we mentioned before. It establishes that solving the problem globally is equivalent to solving it locally. In particular, it answers the aforementioned question.

Theorem 1.16 (Noether's Fundamental Theorem)**.** *Let C_1, C_2 be two projective plane curves with defining polynomials F_1 respectively F_2 in $\mathbb{C}[X_0, X_1, X_2]$. Assume that C_1 and C_2 have no common component. Let $C_3 \subset \mathbb{P}^2$ be another projective plane curve with defining polynomial $F_3 \in \mathbb{C}[X_0, X_1, X_2]$. Then there exist polynomials $A \in \mathbb{C}[X_0, X_1, X_2]$ and $B \in \mathbb{C}[X_0, X_1, X_2]$ of degree* $\deg(F_3) - \deg(F_1)$ *respectively* $\deg(F_3) - \deg(F_2)$*, with*

$$F_3 = AF_1 + BF_2,$$

if and only if Noether's Conditions are satisfied at each point $P \in C_1 \cap C_2$.

Since Noether's Fundamental Theorem is very powerful, we first have to introduce some supplementary results to prove the theorem.

Lemma 1.17. *Let $Y \subset \mathbb{A}^n$ be an algebraic set and let $P_1, \dots, P_N$ be points not contained in Y. Then there exist polynomials $F_1, \dots, F_N \in \mathbb{C}[X_1, \dots, X_n]$, such that $F_i(P_j) = \delta_{i,j}$.*

Proof. Let $1 \leq i \leq N$. We have a proper inclusion

$$Y_1 := Y \cup \bigcup_{\substack{i=1 \\ i \neq j}}^{N} \{P_i\} \subsetneq Y \cup \bigcup_{i=1}^{N} \{P_i\} =: Y_2,$$

which corresponds to a proper inclusion $I(Y_1) \supsetneq I(Y_2)$. This implies that there exists a polynomial $F_j' \in I(Y_1)$, such that $F_j' \notin I(Y_2)$, i.e. $F_j'(P_i) = 0$ for $i \neq j$ and $F_j'(P_j) \neq 0$. Now, the polynomial

$$F_j := \frac{F_j'}{F_j'(P_j)}$$

does the job. □

Proposition 1.18. *Let I be an ideal in $\mathbb{C}[X_1, \dots, X_n]$. Suppose that the set $V(I) := \{P_1, \dots, P_N\}$ is finite. Then there is a natural isomorphism*

$$\mathbb{C}[X_1, \dots, X_n]/I \simeq \prod_{i=1}^{N} \mathcal{O}_{\mathbb{A}^n, P_i}/I\mathcal{O}_{\mathbb{A}^n, P_i}.$$

Proof. First, for each $1 \leq i \leq N$, let $I_i := I(\{P_i\})$ and let $R_i := \mathcal{O}_{\mathbb{A}^n, P_i}/I\mathcal{O}_{\mathbb{A}^n, P_i}$. Furthermore, let $R := \mathbb{C}[X_1, \dots, X_n]/I$. Note that for each $1 \leq i \leq N$, the ideal I_i contains I. The natural homomorphisms

$$\varphi_i \colon R \to R_i, \;\; F + I \mapsto F_{P_i} + I\mathcal{O}_{\mathbb{A}^n, P_i}.$$

induce the homomorphism

$$\varphi\colon R \to \prod_{i=1}^{N} R_i,\; F + I \mapsto (\varphi_1(F+I), \dots, \varphi_N(F+I)).$$

We want to show that φ is even an isomorphism.

Using Hilbert's Nullstellensatz, we get

$$\sqrt{I} = I(V(I)) = I(\{P_1, \dots, P_N\}) = \bigcap_{i=1}^{N} I_i$$

and since $\bigcap_{i=1}^{N} I_i = \sqrt{I}$ is finitely generated, there exists $k \in \mathbb{N}$ with $(\bigcap_{i=1}^{N} I_i)^k \subset I$. Since for all $1 \le i \le N$ the I_i are maximal, for any $i \neq j$, we have that I_i and I_j are comaximal. Inductively, I_j and $\bigcap_{i\neq j} I_i$ are comaximal, so we obtain

$$\bigcap_{i=1}^{N} I_i^k = (I_1^k \cdots I_N^k) = (I_1 \cdots I_N)^k = \left(\bigcap_{i=1}^{N} I_i\right)^k \subset I. \tag{1.1}$$

By Lemma 1.17, we may for all $1 \le i \le N$ choose polynomials F_i in $\mathbb{C}[X_1, \dots, X_n]$, such that $F_i(P_j) = 0$ for $i \neq j$ and $F_i(P_j) = 1$ for $i = j$.

Define $E_i := 1 - (1 - F_i^k)^k$. Then $E_i = F_i^k D_i$ for some D_i by the binomial theorem. This implies $E_i \in I_j^k$ for $i \neq j$, so in particular, by (1.1), we get

$$E_i - E_i^2 = E_i(1 - F_i^k)^k \in I. \tag{1.2}$$

Let now $e_i := E_i + I \in R$ be the image of E_i in R. Then by (1.2), we have $e_i e_j = e_i^2 = e_i$ for $i = j$ and $e_i e_j = 0$ for $i \neq j$. Furthermore

$$1 - \sum_{i=1}^{N} E_i = (1 - E_j) - \sum_{i\neq j}^{N} E_i \in I,$$

which implies $\sum_{i=1}^{N} e_i = 1$.

Claim: Fix $1 \le i \le N$. Assume there exists $G \in \mathbb{C}[X_1, \dots, X_n]$, such that $G(P_i) \neq 0$. Then, there exists an element $r \in R$, such that $rg = e_i$, where $g := G + I \in R$.

Proof of claim. Since $G(P_i) \neq 0$, we may assume that $G(P_i) = 1$. Let $H := 1 - G$. Then, by definition, $H \in I_i$. Furthermore,

$$(1 - H)(E_i + HE_i + \dots + H^{k-1}E_i) = E_i - H^k E_i,$$

implying $H^k E_i \in I$. Thus, $g(e_i + he_i + \dots + h^{k-1}e_i) = e_i$ for $h := H + I$. So the element

$$r := (e_i + he_i + \dots + h^{k-1}e_i) \in R$$

does the job. This proves the claim. □

Now, by using the claim, we want to show that φ is an isomorphism. First we show that φ is injective. To this end, let $F \in \mathbb{C}[X_1, \dots, X_n]$ and let $f := F + I \in R$ be its image in R. Suppose $\varphi(f) = 0$. Then $\varphi_i(f) = 0$ for each $1 \leq i \leq N$, i.e. for each $1 \leq i \leq N$, there exists a $G_i \in \mathbb{C}[X_1, \dots, X_n]$ with $G_i(P_i) \neq 0$ and $G_i F \in I$.

By the claim, for fixed $1 \leq i \leq N$, there exists an $r_i \in R$ with $r_i g_i = e_i$. But then,

$$f = \left(\sum_{i=1}^{n} e_i \right) f = \sum_{i=1}^{N} r_i g_i f = 0,$$

since $\sum_{i=1}^{n} e_i = 1$ and each $r_i g_i f = 0$ by assumption. This shows injectivity.

Finally, we want to prove that φ is surjective. To this end, let

$$f' := \left(\frac{a_1}{s_1} + I\mathcal{O}_{\mathbb{A}^n, P_1}, \dots, \frac{a_N}{s_N} + I\mathcal{O}_{\mathbb{A}^n, P_N} \right) \in \prod_{i=1}^{N} R_i,$$

where $s_i(P_i) \neq 0$ by definition of φ_i. Thus, we can use the claim on each s_i, so for each $1 \leq i \leq N$, there exists r_i with $r_i s_i = e_i$. But then for each $1 \leq i \leq N$, we have $\frac{a_i}{s_i} = \frac{r_i a_i}{e_i} = a_i r_i \in R_i$, since e_i is a unit. This implies for each $1 \leq i \leq N$ that

$$\varphi_i \left(\sum_{j=1}^{N} r_j a_j e_j \right) = \varphi_i(r_i a_i) = \frac{a_i}{s_i}.$$

It follows that $\varphi(\sum_{j=1}^{N} r_j a_j e_j) = f'$. This shows surjectivity.

In summary, we get that $\varphi \colon R \to \prod_{i=1}^{N} R_i$ is an isomorphism, which was to be shown. $\square$

Proof of Noether's Fundamental Theorem. "$\Rightarrow$": Let $F_3 = AF_1 + BF_2$. By dehomogenising each polynomial, we get

$$F_{3,D} = A_D F_{1,D} + B_D F_{2,D}$$

at any $P \in C_1 \cap C_2$, which are exactly Noether's Conditions. So this direction is clear.

"$\Leftarrow$": Assume that Noether's Conditions are satisfied for each point P in $C_1 \cap C_2$. Since C_1 and C_2 have no common component, we may assume by Bézout's Theorem that $C_1 \cap C_2$ is finite.

We may assume that after a projective change of coordinates, no point $P \in C_1 \cap C_2$ lies on the line $X_0 = 0$ at infinity, i.e. $V(F_1, F_2, X_0) = \varnothing$. Thus, for each point $(x_0, x_1, x_2) =: P \in C_1 \cap C_2$, we have that $x_0 \neq 0$, say $P = (1, x_1, x_2)$. Thus, we can consider the dehomogenisations of F_1, F_2 and F_3.

By Noether's Conditions, it holds that $F_{3,D} \in (F_{1,D}, F_{2,D})$, i.e. the class of $F_{3,D}$ in $\mathcal{O}_{\mathbb{P}^2, P}/(F_{1,D}, F_{2,D})$ is zero for each point $P := (1, x_1, x_2) \in C_1 \cap C_2$.

Claim: The class $[F_{3,D}]$ in $\mathbb{C}[X_1, X_2]/(F_{1,D}, F_{2,D})$ is zero.

Proof of claim. To prove the claim, we want to use the Proposition 1.18. Let $I := (F_{1,D}, F_{2,D}) \subset \mathbb{C}[X_1, X_2]$. Then $V(I)$ is a finite set, as we have seen above, say $V(I) := \{P_1, \dots, P_N\}$. The assumptions of Proposition 1.18 are satisfied, so we obtain an isomorphism

$$\mathbb{C}[X_1, X_2]/I \simeq \prod_{i=1}^{N} \mathcal{O}_{\mathbb{A}^2,P_i}/I\mathcal{O}_{\mathbb{A}^2,P_i} \simeq \prod_{i=1}^{N} \mathbb{C}[X_1, X_2]_{P_i}/I\mathbb{C}[X_1, X_2]_{P_i},$$

which sends the class $[F]$ to the tuple $([F_{P_1}], \dots, [F_{P_n}])$. But this indeed implies the claim, because the preimage of $([0], \dots, [0])$ under this isomorphism has to be $[0]$. □

Thus, the class $[F_{3,D}]$ in $\mathbb{C}[X_1, X_2]/(F_{1,D}, F_{2,D})$ is zero, i.e. there exist a, b in $\mathbb{C}[X_1, X_2]$ with

$$F_{3,D} = aF_{1,D} + bF_{2,D}.$$

Again by homogenising $F_{3,D}$ with respect to the variable X_0, we get the polynomial

$$F_3 = (F_{3,D})_H = X_0^{\deg(F_3)} F_3 \left(\frac{X_1}{X_0}, \frac{X_2}{X_0} \right) := A'F_1 + B'F_2,$$

for polynomials A', B' in $\mathbb{C}[X_0, X_1, X_2]$. Finally, if we have $A' = \sum_{i \in I} A'_i$ and $B' = \sum_{i \in I} B'_i$, where A'_i, B'_i are homogeneous of degree $i \in \mathbb{N}$, then

$$F_3 = A'_s F_1 + B'_t F_2,$$

where $s = \deg(F_3) - \deg(F_1)$ and $t = \deg(F_3) - \deg(F_2)$. This finishes the proof. □

Remark 1.19. Noether's Fundamental Theorem is also called "$AF + BG$ Theorem", which refers to the fact that the theorem implies that we can write a curve as an equation of two other curves.

Next, we provide a helpful condition for a point to satisfy Noether's Conditions. We will use it to prove further results in the next chapters.

Proposition 1.20. *Let $C_1 \subset \mathbb{P}^2$ be a projective plane curve and let $P \in C_1$ be an ordinary multiple point of multiplicity r on C_1. Consider the birational morphism $\rho\colon C'_1 \to C_1$ from the nonsingular model C'_1 onto C_1 and for $1 \leq i \leq r$, let $\rho^{-1}(P) = \{P_1, \dots, P_r\}$. Let $C_2 \subset \mathbb{P}^2$ and $C_3 \subset \mathbb{P}^2$ be further projective plane curves with defining polynomials F_2 respectively F_3 in $\mathbb{C}[X_0, X_1, X_2]$. Then Noether's Conditions are satisfied at $P \in C_1$ with respect to C_3, if $\nu_{P_i}(F_3) \geq \nu_{P_i}(F_2) + r - 1$ for all $1 \leq i \leq r$.*

Proof. See [Ful69, Chapter 7.5, Proposition 3]. □

1.2 Divisors

In this section, we define the notion of a *divisor* on a nonsingular projective curve X, which is the central object for Riemann-Roch's Theorem. Remember that this is just a instantiation of a zero-cycle. First we give the general definition of a *divisor*.

Definition 1.21 (Divisor)**.** Let X be a nonsingular projective curve. A **divisor** D on X is a formal finite sum

$$D := \sum_{P \in X} \eta_P \cdot P,$$

where each η_P assignes an integer to a specific point P. The **degree** of D is defined as

$$\deg(D) := \sum_{P \in X} \eta_P.$$

We call D **effective** or **positive**, if $\eta_P \geq 0$ for all $P \in X$ and say that $\sum_{P \in X} \eta_P \cdot P \geq \sum_{P \in X} \mu_P \cdot P$, if each $\eta_P \geq \mu_P$. We define the **support** of D to be $\text{Supp}(D) := \{P \in X : \eta_P \neq 0\}$.

Remark 1.22. The set of all divisors on a projective curve X, denoted by

$$\text{Div}(X) := \left\{ D = \sum_{P \in X} \eta_P \cdot P : \text{ all but finitely many } \eta_P = 0 \right\},$$

defines an abelian group. It is the **free abelian group** generated by the points of X.

As mentioned in the previous chapter, we can see immediately that a divisor is the same as a zero-cycle on a nonsingular projective curve.

Next, we want to define a special kind of divisor, which is defined through the field of rational functions $K(X)$ of a nonsingular projective curve X. This defines the most important kind of divisor for introducing the *Riemann-Roch spaces* $H^0(X, D)$. It is defined locally and the definition makes use of the discrete valuation rings $\mathcal{O}_{X,P}$. Therefore, recall that we saw in Proposition 1.2 that $\mathcal{O}_{X,P}$ is indeed a discrete valuation ring.

Definition 1.23. Let X be a nonsingular projective curve and let $f \in K(X)^*$. We define the **divisor** of f to be

$$\text{div}(f) := \sum_{P \in X} \nu_P(f) \cdot P,$$

where $\nu_P(f)$ denotes the discrete valuation associated to $\mathcal{O}_{X,P}$. We call $\text{div}(f)$ a **principal divisor**. Let $H \subset \mathbb{P}^n$ be a hypersurface with defining polynomial $F \in \mathbb{C}[X_0, \ldots, X_n]$. We define

$$X \cdot H := \sum_{P \in X \cap H} I(P; X \cap H) \cdot P$$

to be the **intersection divisor** of X and H.

Remark 1.24. By Bézout's Theorem, it holds that $\deg(X \cdot H) = \deg(X) \deg(H)$.

Before we come to an illustration of principal divisors on curves, we define what the divisor of a projective plane curve is.

Definition 1.25. Let C be a projective plane curve. Let $\rho \colon C' \to C$ be a birational morphism to the nonsingular model C'. Let C_1 be another projective plane curve, with defining polynomial $F_1 \in \mathbb{C}[X_0, X_1, X_2]$. Let $P \in C'$ and let $Q := \rho(P)$. Take any $F_2 \in \mathbb{C}[X_0, X_1, X_2]$ of the same degree as F_1 and such that $F_2(Q) \neq 0$. Consider $\frac{F_1}{F_2} \in K(\mathbb{P}^2)$ and its image $f_1/f_2 \in K(C_1) \simeq K(C_1')$. Define

$$\mu_P(F_2) := \nu_P\left(\frac{f_1}{f_2}\right).$$

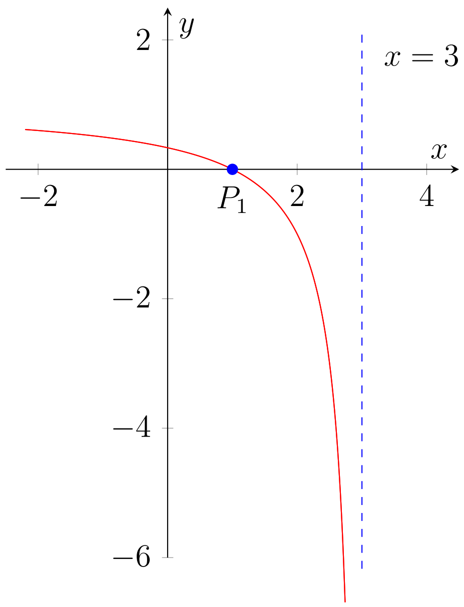
Note that $\nu_P\left(\frac{f_1}{f_2}\right) = \nu_P(f_1)$, since $f_2(P) \neq 0$. Define the **divisor** of F_1 to be

$$\operatorname{div}(F_1) := \sum_{P \in C'} \mu_P(F_1) \cdot P.$$

Remark 1.26. We have $\operatorname{div}(F_1) := \sum_{P \in C_1'} \mu_P(F_1) \cdot P = \sum_{Q \in C} I(Q; C \cap C_1)$ and by Bézout's Theorem, $\operatorname{div}(F_1)$ is a divisor of degree $\deg(C) \deg(C_1)$.

We can also define principal divisors on affine curves in the same way as we did for projective curves. This is indeed easier to visualise, as the following example shows.

Example 1.27. Consider the rational function $f(x) = \frac{x-1}{x-3}$ on $\mathbb{A}^1_{\mathbb{R}}$. We want to calculate the divisor of f.



As we can see, $f(x) = \frac{x-1}{x-3}$ has a zero at $x = 1$ and a pole at $x = 3$, both with multiplicity 1. Consider the discrete valuation rings $\mathcal{O}_{\mathbb{A}^1_{\mathbb{R}},1}$ and $\mathcal{O}_{\mathbb{A}^1_{\mathbb{R}},3}$

and notice that the discrete valuation of f at $P \in \{1, 3\}$ is given by the multiplicity of the point P. Thus, we can calculate the divisor of f precisely as

$$\operatorname{div}(f) = 1 \cdot (x = 1) - 1 \cdot (x = 3).$$

Remark 1.28. More generally, we can split each principal divisor into two parts. Let X be a nonsingular projective curve and let $f \in K(X)^*$ be a rational function. We define $(f)_0 := \sum_{\nu_P(f)>0} \nu_P(f) \cdot P$ to be the **divisor of zeros**, which counts the zeros of f and $(f)_\infty := -\sum_{\nu_P(f)<0} \nu_P(f) \cdot P$ to be the **divisor of poles**, which counts the poles of f. In summary for each rational function $f \in K(X)^*$, we get a representation

$$\operatorname{div}(f) = (f)_0 - (f)_\infty.$$

Proposition 1.29. *Let X be a nonsingular projective curve and let $f \in K(X)^*$ be a rational function. Then $\deg(\operatorname{div}(f)) = 0$. In particular, the number of poles and zeros of f are the same, if they are counted with multiplicity.*

Proof. Let $C \subset \mathbb{P}^2$ be the plane model of X. Assume that C has degree n. Let $K(C) \simeq K(X)$ be its field of rational functions. Let $f = \frac{f_1}{f_2} \in K(X)^*$. We identify f with its image in $K(C)^*$. Then, $\frac{f_1}{f_2}$ is the image of an element $\frac{F_1}{F_2} \in K(\mathbb{P}^2)$, where $F_1, F_2 \in \mathbb{C}[X_0, X_1, X_2]$ are of degree m. Now,

$$\begin{aligned}
\operatorname{div}(f) = \sum_{P \in X} \nu_P\left(\frac{f_1}{f_2}\right) \cdot P &= \sum_{P \in X} \nu_P(f_1) \cdot P - \sum_{P \in X} \nu_P(f_2) \cdot P \\
&= \sum_{P \in X} \mu_P(F_1) \cdot P - \sum_{P \in X} \mu_P(F_2) \cdot P \\
&= \operatorname{div}(F_1) - \operatorname{div}(F_2).
\end{aligned}$$

By Remark 1.26, $\operatorname{div}(F_1)$ and $\operatorname{div}(F_2)$ both have degree mn, so $\operatorname{div}(f)$ has degree zero. □

Through principal divisors, we are in the position to define an equivalence relation on the set of divisors on a curve X.

Definition 1.30. Let X be a nonsingular projective curve. We define a relation $\sim$ on $\operatorname{Div}(X)$ as follows. For $D_1, D_2 \in \operatorname{Div}(X)$ let

$$D_1 \sim D_2 \iff \exists f \in K(X)^* : D_2 - D_1 = \operatorname{div}(f).$$

We say that D_1 and D_2 are **linearly equivalent**.

Remark 1.31. The relation above defines an equivalence relation.

Definition 1.32. Let X be a nonsingular curve. Let $D \in \operatorname{Div}(X)$. We define the **linear system associated to** D as

$$|D| := \{D' \in \operatorname{Div}(X) : D' \geq 0 \text{ and } D \sim D'\}.$$

From the definition, we immediately obtain the following properties for linearly equivalent divisors.

Proposition 1.33. *Let X be a nonsingular projective curve. Let $D_1, {D_1}', D_2, {D_2}'$ be divisors on X. Then:*

(i) $D_1 \sim 0$ *if and only if* $D = \operatorname{div}(f)$ *for some rational function* $f \in K(X)^*$.

(ii) *If* $D_1 \sim D_2$, *then* $\deg(D_1) = \deg(D_2)$.

(iii) *If* $D_1 \sim D_2$ *and* ${D_1}' \sim {D_2}'$, *then* $D_1 + {D_1}' \sim D_2 + {D_2}'$.

(iv) $D_1 \sim D_2$ *if and only if there exist* $F_1, F_2 \in \mathbb{C}[X_0, X_1, X_2]$ *of the same degree, such that* $D_1 + \operatorname{div}(F_1) = D_2 + \operatorname{div}(F_2)$.

The next definition is crucial to introduce the Residue Theorem, which will be an important ingredient for the proof of the Riemann-Roch Theorem. Therefore, recall that for each projective curve, there exists a birational nonsingular projective model. In the sequel, we want to define an important divisor on such a nonsingular model, called the *divisor of multiple points.*

Definition 1.34 (Divisor of multiple points)**.** Let $C_1 \subset \mathbb{P}^2$ be a projective plane curve with defining polynomial $F_1 \in \mathbb{C}[X_0, X_1, X_2]$. Assume that C_1 has only ordinary multiple points. Consider the birational morphism $\rho\colon C_1' \to C_1$ from the nonsingular model C_1' onto C_1. For each $Q \in C_1'$ let $r_Q := m_{\rho(Q)}(F_1)$. Define the divisor

$$E := \sum_{Q \in C_1'} (r_Q - 1) \cdot Q$$

to be the **divisor of multiple points** on C_1'. It is an effective divisor of degree

$$\deg(E) = \sum_{P \in C_1} m_P(F_1)(m_P(F_1) - 1).$$

Let $C_2 \subset \mathbb{P}^2$ be another projective plane curve with defining polynomial $F_2 \in \mathbb{C}[X_0, X_1, X_2]$. If $\operatorname{div}(F_2) \geq E$, then we call C_2 an **adjoint** of C_1.

We come to the just mentioned Residue Theorem.

Theorem 1.35 (Residue Theorem)**.** *Let $C_1 \subset \mathbb{P}^2$ be a projective plane curve with defining polynomial $F_1 \in \mathbb{C}[X_0, X_1, X_2]$. Assume that C_1 has only ordinary multiple points. Consider the birational morphism $\rho\colon C_1' \to C_1$ from the nonsingular model C_1' onto C_1 and let E be the divisor of multiple points on C_1'. Let $D_1, D_2 \in \operatorname{Div}(C_1')$ be linearly equivalent and effective. Let $C_2 \subset \mathbb{P}^2$ be an adjoint to C_1 of degree m with defining polynomial $F_2 \in \mathbb{C}[X_0, X_1, X_2]$, such that*

$$\operatorname{div}(F_2) = D_1 + E + D_3$$

for some effective divisor $D_3 \in \operatorname{Div}(C_1')$. Then there exists an adjoint $C_3 \subset \mathbb{P}^2$ of C_1 with defining polynomial $F_3 \in \mathbb{C}[X_0, X_1, X_2]$ of degree m such that

$$\operatorname{div}(F_3) = D_2 + E + D_3.$$

Proof. Since D_1 and D_2 are linearly equivalent, by Proposition 1.33 (iv), there exist $G_1, G_2 \in \mathbb{C}[X_0, X_1, X_2]$ with

$$D_1 + \mathrm{div}(G_1) = D_2 + \mathrm{div}(G_2).$$

Let $X_1, X_2 \subset \mathbb{C}[X_0, X_1, X_2]$ be the projective plane curves with G_1, G_2, respectively, as defining polynomials of degree $n = \deg(X_1) = \deg(X_2)$. Then

$$\begin{aligned}\mathrm{div}(F_2 G_1) &= \mathrm{div}(F_2) + \mathrm{div}(G_1)\\ &= D_1 + E + D_3 + D_2 + \mathrm{div}(G_2) - D_1\\ &= \mathrm{div}(G_2) + D_2 + E + D_3 \geq \mathrm{div}(G_2) + E,\end{aligned}$$

because D_3 and D_2 are effective. By applying Proposition 1.20 to C_1, X_2 and $C_2 X_1$, we get that Noether's Conditions are satisfied for each point $P \in C_1$. Now, applying Noether's Fundamental Theorem, there are F_4 and F_3 in $\mathbb{C}[X_0, X_1, X_2]$, defining two plane curves X_4, C_3 with $\deg(C_3) = m$, such that

$$F_2 G_1 = F_4 F_1 + F_3 G_2,$$

Recall that when forming the divisor $\mathrm{div}(F)$ of a polynomial, we consider the discrete valuations ν_P inside the rings $\mathcal{O}_{C_1',P}$. Consequently, if F is a multiple of the defining polynomial F_1 of C_1, then $\mathrm{div}(F)$ vanishes. We obtain

$$\mathrm{div}(F_3) = \mathrm{div}(F_2 G_1) - \mathrm{div}(G_2) = D_2 + E + D_3,$$

which finishes the proof. □

1.3 Canonical Divisors

At this point, we want to define the *canonical divisor*. This is, as the name may suggest, the most important divisor on a curve. To define it, we first have to do some preparation, by introducing the notion of *differentials*.

1.3.1 The Space Of Differentials

In this subsection, we give a sufficient background for understanding *differentials*. First, we have to define the notion of a *derivation*.

Definition 1.36. Let $\mathbb{C} \subset R$ be a ring and let M be an R-module. Furthermore, let $D\colon R \to M$ be a $\mathbb{C}$-linear map, satisfying $D(xy) = yD(x) + xD(y)$ for all $x, y \in R$. We call the map D a **derivation** of R into M over $\mathbb{C}$.

Each derivation of a ring can be uniquely extended to a deriation of a field, as the following proposition shows.

Proposition 1.37. *Let R be an integral domain containing $\mathbb{C}$. Let $K = Q(R)$ be the quotient field of R and let V be a vector space over K. Then any derivation $D\colon R \to V$ extends uniquely to a derivation $D'\colon K \to V$.*

Proof. Let $z \in K$ with $z = xy^{-1}$, where $x \in R, y \in \mathbb{R} - \{0\}$. Then

$$z = xy^{-1} \iff x = zy \Rightarrow D(x) = yD(z) + zD(y).$$

We define the derivation $D' \colon K \to V$ via

$$D'(z) := (D(x) - zD(y))y^{-1}.$$

Claim: The derivation D' is well-defined.

Proof of claim. Let $z = x_1{y_1}^{-1} = x_2{y_2}^{-1} \in K$, with $x_1 \neq x_2,\ y_1 \neq y_2$. We want to show that $D'(x_1{y_1}^{-1}) = D'(x_2{y_2}^{-1})$. We have

$$\begin{aligned} & y_2D(x_1) - zy_2D(y_1) = y_1D(x_2) - zy_1D(y_2) \\ \iff \quad & y_2D(x_1) - x_2D(y_1) = y_1D(x_2) - x_1D(y_2) \\ \iff \quad & y_2D(x_1) + x_1D(y_2) = y_1D(x_2) + x_2D(y_1) \\ \iff \quad & D(x_1y_2) = D(x_2y_1), \end{aligned}$$

which holds, since $x_1y_2 = x_2y_1$. □

Claim: This notion is unique.

Proof of claim. Assume, we have two derivations $D_1' \colon K \to V,\ D_2' \colon K \to V$ extending D. Let $z = xy^{-1} \in K$ be as above. We have $D_1'(x) = D(x) = D_2'(x)$ and thus

$$yD_1'(z) + zD(y) = yD_2'(z) + zD(y),$$

implying

$$yD_1'(z) = yD_2'(z).$$

Thus, $y(D_1'(z) - D_2'(z)) = 0$ and since R is an integral domain, it follows $D_1'(z) - D_2'(z) = 0$. Because $z \in K$ was arbitrary, we have $D_1' = D_2'$. This shows uniqueness. □

Thus, D extends uniquely to the well-defined derivation $D' \colon K \to V$, which completes the proof. □

At this point, we are able to define the *module of differentials* of R over $\mathbb{C}$. Since the differentials should behave like differentials in calculus, we define them in the following way.

Definition 1.38 (Module of differentials)**.** Let R be a $\mathbb{C}$-algebra and for each $x \in R$ let dx be a symbol. Define F to be the free module $F := \bigoplus_{x \in R} R dx$ and define N to be the submodule of F generated by the relations

(i) $d(x+y) - dx - dy,\ x, y \in R$

(ii) $d(\lambda x) - \lambda dx,\ x \in R, \lambda \in \mathbb{C}$

(iii) $d(xy) - xdy - ydx$

We define $\Omega_{\mathbb{C}}(R) := F/N$ to be the quotient module of F and N and call $\Omega_{\mathbb{C}}(R)$ the **module of differentials** of R over $\mathbb{C}$. We identify the class $[dx]$ with the symbol dx. Let $d\colon R \to \Omega_{\mathbb{C}}(R)$ be the map that sends x to the class dx. This defines a derivation.

Lemma 1.39. *Let R be a $\mathbb{C}$-algebra, let M be an R-module and let $D\colon R \to M$ be a derivation. Then there is a unique homomorphism*

$$\varphi\colon \Omega_{\mathbb{C}}(R) \to M.$$

Proof. We use the notation from Definition 1.38. First we define a map $\varphi'\colon F \to M$, where $F := \bigoplus_{x\in R} R dx$ as above. Take $\sum_{x\in R} r_x dx \in F$ and send it to $\sum_{x\in R} r_x D(x)$. Then $\varphi'(N) = 0$ by definition of φ and N, such that φ' induces a map $\varphi\colon \Omega_{\mathbb{C}}(R) \to M$, which shows the claim. □

Thus, if $R = \mathbb{C}[x_1, \ldots, x_n]$, then $\Omega_{\mathbb{C}}(R)$ is generated by $dx_1, \ldots, dx_n$. If $k = \mathbb{C}(x_1, \ldots, x_n)$, then $\Omega_{\mathbb{C}}(k)$ is a finite-dimensional vector space over k, generated by $dx_1, \ldots, dx_n$.

More concretely, for an algebraic function field k in one variable, $\Omega_{\mathbb{C}}(k)$ even forms a one-dimensional vector space, as the following proposition shows.

Proposition 1.40. *Let k be an algebraic function field in one variable over $\mathbb{C}$. The above defined module $\Omega_{\mathbb{C}}(k)$ forms a one-dimensional vector space over k. If $f \in k$ is non-constant, then df forms a basis for $\Omega_{\mathbb{C}}(k)$.*

Proof. By [Har77, Chapter 1, Theorem 4.4], there exists an affine plane curve $C \subset \mathbb{A}^2$ with function field $K(C) = k$. Let $F \in \mathbb{C}[X_1, X_2]$ be its defining polynomial and define $x_1 := X_1 + (F)$ and $x_2 := X_2 + (F)$. Let $R := \mathbb{C}[X_1, X_2]/(F)$. Then $R = \mathbb{C}[x_1, x_2]$ and $K(C) = Q(R) = \mathbb{C}(x_1, x_2)$. Since F is irreducible, we may assume that $\frac{\partial F}{\partial X_1} \neq 0$, because otherwise F would be a polynomial in one variable over $\mathbb{C}$ and hence would be reducible. We have already seen above that dx_1 and dx_2 generate $\Omega_{\mathbb{C}}(K(C))$ over $\mathbb{C}$. We also have

$$0 = d(F(x_1, x_2)) = \frac{\partial F}{\partial X_1}(x_1, x_2)dx_1 + \frac{\partial F}{\partial X_2}(x_1, x_2)dx_2$$

and therefore $dx_2 = udx_1$, where

$$u := -\frac{\frac{\partial F}{\partial X_1}(x_1, x_2)}{\frac{\partial F}{\partial X_2}(x_1, x_2)} = \frac{dx_2}{dx_1}.$$

Hence, each dx_2 can be expressed by dx_1, implying that dx_1 generates $\Omega_{\mathbb{C}}(K(C))$. This implies $\dim_{\mathbb{C}}(\Omega_{\mathbb{C}}(K(C)) \leq 1$. We want to show that

$$\dim_{\mathbb{C}}(\Omega_{\mathbb{C}}(K(C))) = 1.$$

To this end, we need to show that $\Omega_{\mathbb{C}}(K(C))$ is non-zero. To do so, we want to use Proposition 1.37 and Lemma 1.39 by showing that there exists a non-zero derivation $D\colon R \to V$ for some vector space V over $\mathbb{C}$.

Indeed, define $V := K(C)$. Let $G \in \mathbb{C}[X_1, X_2]$ be a polynomial and consider the class $[G] \in R$. We define

$$D([G]) := \frac{\partial G}{\partial X_1}(x_1, x_2) + u\frac{\partial G}{\partial X_2}(x_1, x_2)$$

with u defined as above. We need to show that D is indeed well-defined.

Let $G_1, G_2 \in \mathbb{C}[X_1, X_2]$ with $[G_1] = [G_2]$, such that $G_1 + (F) = G_2 + (F)$, i.e. there exists a polynomial $A \in \mathbb{C}[X_1, X_2]$, such that $G_2 - G_1 = AF$. We want to show that $D([G_1]) = D([G_2])$. We have

$$D([G_1]) - D([G_2] = D([G_1] - [G_2]) = D([G_1 - G_2]) = D([AF]) = 0.$$

Thus, $D([G_1]) = D([G_2])$, so $D\colon R \to V$ is a well-defined derivation. Furthermore, D is non-zero because

$$D(x_1) = \frac{\partial x_1}{X_1}(x_1, x_2) + u\frac{\partial x_1}{X_2}(x_1, x_2) = 1 - 0 = 1.$$

In conclusion, it follows that $\Omega_{\mathbb{C}}(K(C)) \neq \{0\}$. Thus, $\dim(\Omega_{\mathbb{C}}(K(C))) = 1$, which completes the proof. $\square$

We just saw that for any $f, t \in k$, where $t \notin \mathbb{C}$, there exists a unique element $v \in k$ with $df = vdt$. We call this element v the **derivative** of f with respect to t and usually write $v = \frac{df}{dt}$ or v' for short.

The next proposition shows that for each element f in a discrete valuation ring, the derivative of f also belongs there. We need this result to make sure that we can define divisors of differentials. This is important because the *canonical divisor* is defined in this way.

Proposition 1.41. *Let k be an algebraic function field in one variable over $\mathbb{C}$. Let R be a discrete valuation ring of k and let t be a generator for the maximal ideal of R. Let $f \in R$. Then $\frac{df}{dt} \in R$.*

Proof. As in the proof of Proposition 1.40, there exists an affine plane curve $C \subset \mathbb{A}^2$ with function field $K(C) = k$. Let $F \in \mathbb{C}[X_1, X_2]$ be its defining polynomial. Let $x_1 := X_1 + (F)$ and $x_2 := X_2 + (F)$. We may assume that $R = \mathcal{O}_{C,P}$ and $P := (0, 0)$ is a nonsingular point on C. Let $f \in K(C)$ and fix $t \in \mathcal{O}_{C,P} \subset K(C)$. We want to show that $f' = \frac{df}{dt} \in \mathcal{O}_{C,P}$, i.e. we want to show that $\nu_P(f') \geq 0$.

To this end, let $x_1' := \frac{dx_1}{dt}$, $x_2' := \frac{dx_2}{dt}$ and choose $N \in \mathbb{N}$ large enough, such that $\nu_P(x_1'), \nu_P(x_2') \geq -N$ in R.

First we show that for $f \in \mathbb{C}[X_1, X_2]/(F) = \mathbb{C}[x_1, x_2]$, it holds that $\nu_P(f') \geq -N$. Indeed,

$$f' = \frac{\partial f}{\partial X_1}(x_1, x_2)x_1' + \frac{\partial f}{\partial X_2}(x_1, x_2)x_2'$$

and each summand is greater or equal to $-N$, which implies $\nu_P(f') \geq -N$.

Now, let $f \in R$. Then there exist $g, h \in \mathbb{C}[x_1, x_2]$, $h(P) \neq 0$ with $f = gh^{-1}$. Then $f' = h^{-2}(hg' - gh')$ by the product rule, which implies

$$-N \leq \nu_P(f') = \nu_P(h^{-2}) + \nu_P(hg' - gh').$$

Finally, for each $i < N$ and $g \in R$, we can write

$$f = \sum_{i<N} \lambda_i t^i + t^N g,$$

where $\lambda_i \in \mathbb{C}$. But then

$$f' = \sum_{i<N} (i\lambda_i t^{i-1}) + gNt^{N-1} + t^N g'.$$

Since $t \in R$ and $\nu_P(g) \geq 0$, we also have that the first two summands are in R. It remains to show that $t^N g'$ is in R. But since above, we showed that $\nu_P(g') \geq -N$, we have that

$$\nu_P(t^N g') = \nu_P(t^N) + \nu_P(g') = N\nu_P(t) + \nu_P(g') \geq N - N = 0.$$

Thus, $f' \in R$, which was to be shown. □

Definition 1.42 (Space of rational differential forms)**.** Let X be a nonsingular projective curve and let $K(X)$ be its field of rational functions. We call $\Omega := \Omega_{\mathbb{C}}(K(X))$ the space of **rational differential forms** on X.

1.3.2 The Canonical Divisor

In this subsection, we will finally define the last and most important kind of divisor, which we need to formulate the Riemann-Roch Theorem. This will be accomplished by extending our definition of divisors of rational function to rational differential forms.

Definition 1.43. Let X be a nonsingular projective curve and let $K(X)$ be its field of rational functions. Let $0 \neq \omega \in \Omega$ with $\omega = gdf$ and let $P \in X$. Consider $\mathcal{O}_{X,P}$ and choose a local parameter $t \in \mathcal{O}_{X,P} \subset K(X)$, such that $\omega = hdt$ for a rational function $h \in K(X)^*$, $dt \in \Omega$. We define $\nu_P(\omega) := \nu_P(h)$ to be the **discrete valuation** of ω at P and define the **divisor** of ω on Ω to be

$$\operatorname{div}(\omega) := \sum_{P \in X} \nu_P(\omega) \cdot P.$$

We call $K_X := \operatorname{div}(\omega)$ a **canonical divisor** on X, **associated to** ω.

Remark 1.44. The notion above is indeed well-defined.

(i) Every differential $0 \neq \omega \in \Omega$ can be written as $\omega = hdt$ for $h \in K(X)^*$ because Ω is a one-dimensional vector space with basis dt.

(ii) Let $s \in \mathcal{O}_{X,P}$ be another local parameter. Then we can write s as $s = (\alpha + f)t$, where $0 \neq \alpha \in \mathbb{C}$ and $f \in (t)$, because X is nonsingular and so $\dim_{\mathbb{C}} (\mathfrak{m}_P/\mathfrak{m}_P^2) = 1$. But then, we have that

$$ds = d((\alpha + f)t)) = (\alpha + f)dt + fdt = (\alpha + f + tf')dt$$

with $f + tf' \in \mathfrak{m}_P$. Hence, we have that

$$\frac{ds}{dt} = \alpha + f + tf'$$

is a unit in $\mathcal{O}_{X,P}$ and the discrete valuation of ω at P is independent of the choice of the local parameter.

(iii) Let $0 \neq \omega' \in \Omega$ be another differential. Then we can write $\omega' = f\omega$ for $f \in K(X)^*$. But then we have

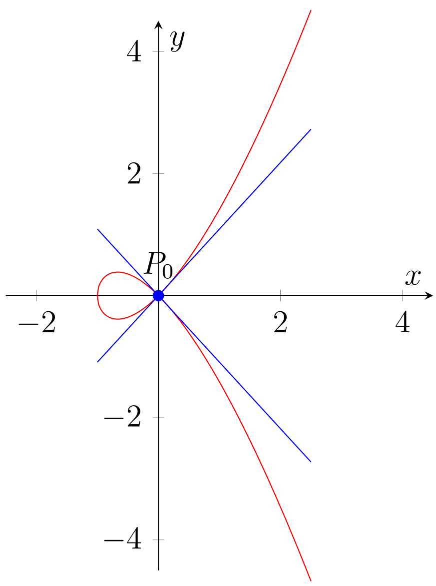
$$\operatorname{div}(\omega') = \operatorname{div}(f\omega) = \operatorname{div}(f) + \operatorname{div}(\omega),$$

so that $\operatorname{div}(\omega') \sim \operatorname{div}(\omega)$. Let now $\operatorname{div}(\omega) =: K_X \sim K'_X := \operatorname{div}(\omega')$. Then there is $f \in K(X)^*$, such that $K'_X = \operatorname{div}(f) + K_X$. But then, we have $K'_X = \operatorname{div}(f\omega)$. Hence the canonical divisors form an equivalence class under linear equivalence. In particular, all canonical divisors have the same degree.

In summary, we get that the canonical divisor is well-defined.

Remark 1.45. Now, besides the fact that we need for the local rings $\mathcal{O}_{X,P}$ to be discrete valuation rings, another reason why we define divisors on nonsingular curves points out. For the canonical divisor, it is important that we have a unique tangent line at each point. If we work on singular curves, this is not guaranteed, as the following example illustrates.

Example 1.46. Consider $Y = V(x^3 - y^2) \subset \mathbb{A}^2$.



At the point P_0 we have two possible tangent lines and hence we cannot define the canonical divisor on Y.

1.4 The Vector Spaces $H^0(X, D)$

At this point, we have everything together to define the vector spaces of rational functions, on which the Riemann-Roch Theorem centres.

Definition 1.47 (Riemann-Roch space)**.** Let X be a nonsingular projective curve and let $D \in \mathrm{Div}(X)$ be a divisor. We define

$$H^0(X, D) := \{f \in K(X)^* : \mathrm{div}(f) + D \geq 0\} \cup \{0\}$$

to be the space of rational functions with **poles up to order** D, or the **Riemann-Roch space** of D.

The spaces defined above are of finite dimension, as the following proposition shows.

Proposition 1.48. *Let X be a nonsingular projective curve. For a given divisor $D \in \mathrm{Div}(X)$, the space $H^0(X, D)$ forms a finite-dimensional vector space over $\mathbb{C}$, i.e. the following hold:*

(i) *Let $D_1, D_2 \in \mathrm{Div}(X)$ with $D_1 \leq D_2$. Then $H^0(X, D_1) \subset H^0(X, D_2)$ and $\dim_{\mathbb{C}}(H^0(X, D_2)/H^0(X, D_1)) \leq \deg(D_2 - D_1)$.*

(ii) *Let $D \in \mathrm{Div}(X)$. Then $H^0(X, 0) = \mathbb{C}$ and if $\deg(D) < 0$, then $H^0(X, D) = 0$.*

(iii) *Let $D \in \mathrm{Div}(X)$. Then $H^0(X, D)$ is finite-dimensional. If $\deg(D) \geq 0$, then $\dim(H^0(X, D)) \leq \deg(D) + 1$.*

(iv) *Let $D_1 \sim D_2 \in \mathrm{Div}(X)$. Then $\dim(H^0(X, D_1)) = \dim(H^0(X, D_2))$.*

Proof. Let $D \in \mathrm{Div}(X)$. Clearly, $H^0(X, D)$ forms a vector space over $\mathbb{C}$. Furthermore:

(i) Since $D_1 \leq D_2$, we may write $D_2 = D_1 + P_1 + P_2 + \cdots + P_s$ with $P_i \in X,\ 1 \leq i \leq s$. Then we have that

$$H^0(X, D_1) \subset H^0(X, D_1 + P_1) \subset \cdots \subset H^0(X, D_1 + P_1 + P_2 + \cdots + P_s),$$

so it is enough to show that $\dim_{\mathbb{C}}(H^0(X, D + P)/H^0(X, D)) \leq 1$ for $P \in X$ and $D \in \mathrm{Div}(X)$. To this end, let $t \in \mathcal{O}_{X,P}$ be a local parameter and let $r := \eta_P \in \mathbb{Z}$ be the coefficient of P in D_2. We define a map

$$\varphi\colon H^0(X, D + P) \to \mathbb{C},\ f \mapsto (t^{r+1} f)(P).$$

The map is well defined since $\nu_P(f) \geq -r - 1$ by definition of $H^0(X, D + P)$. Furthermore, the map φ is linear with kernel $\ker(\varphi) = H^0(X, D)$. Thus, we get an injective linear map

$$\tilde{\varphi}\colon H^0(X, D + P)/H^0(X, D) \to \mathbb{C}$$

and hence, $\dim_{\mathbb{C}}(H^0(X, D + P)/H^0(X, D)) \leq 1$.

(ii) This follows from Proposition 1.29.

(iii) Let $\deg(D_1) = n \geq 0$ and let $P \in X$. Put $D_2 := D_1 - (n+1) \cdot P$. Then

$$\deg(D_2) = \deg(D_1) - \deg((n+1) \cdot P) = n - n - 1 = -1 < 0,$$

thus by (ii), $H^0(X, D_2) = 0$. It follows from (i) that

$$\dim_{\mathbb{C}}(H^0(X, D_1)/H^0(X, D_2)) \leq \deg(D_2 - D_1) = n + 1 - n = 1 \leq n + 1.$$

Thus, $\dim(H^0(X, D_1)) \leq n + 1$.

(iv) Let $D_1 \sim D_2$, i.e. there exists a rational function $f \in K(X)^*$ with $D_2 = D_1 + \operatorname{div}(f)$. We define

$$\rho\colon H^0(X, D_1) \to H^0(X, D_2),\ g \mapsto gf.$$

This clearly defines an isomorphism, so that $H^0(X, D_1) \simeq H^0(X, D_2)$ and hence $\dim(H^0(X, D_1)) = \dim(H^0(X, D_2))$.

In summary, we have that $H^0(X, D)$ is a finite-dimensional vector space. □

Definition 1.49. Let X be a nonsingular projective curve and let $D \in \operatorname{Div}(X)$. We define

$$h^0(X, D) := \dim(H^0(X, D))$$

to be the **dimension** of the vector space $H^0(X, D)$.

Now, at least we know that the dimension of the vector spaces defined above is finite. The point of both Riemann's Theorem and the Riemann-Roch Theorem is to calculate this dimension. This will be done in detail in the following sections.

As a byproduct of the proof of Proposition 1.48, we obtain the following results.

Corollary 1.50. *Let X be a nonsingular projective curve and let D, D' be two divisors on X such that $D \leq D'$. Then*

$$h^0(X, D') \leq h^0(X, D) + \deg(D' - D).$$

Proof. Since $D \leq D'$ and by Proposition 1.48, we have

$$h^0(X, D') - h^0(X, D) = \dim_{\mathbb{C}}\left(H^0(X, D')/H^0(X, D)\right) \leq \deg(D' - D),$$

which shows the claim. □

Proposition 1.51. *Let X be a nonsingular projective curve and let $D \in \operatorname{Div}(X)$ with $h^0(X, D) > 0$. Let $0 \neq f \in H^0(X, D)$. Then, $f \notin H^0(X, D - P)$ for all but finitely many $P \in X$. In particular, $h^0(X, D - P) = h^0(X, D) - 1$ for all but finitely many $P \in X$.*

Proof. Let $0 \neq f \in H^0(X,D)$. Thus, $0 \leq \operatorname{div}(f) + D := \sum_{i=1}^{k} \eta_i \cdot P_i$. Choose $P \in X$ with $P \notin \{P_1, \ldots, P_k\}$. Then,

$$\operatorname{div}(f) + D - P = \left(\sum_{i=1}^{n} \eta_i \cdot P_i\right) - P \ngeq 0.$$

This proves the claim. □

Considering the above proposition, it is natural to ask if there are divisors D for which the formula $h^0(X, D - P) = h^0(X, D) - 1$ holds for every point $P \in X$. This leads to the definition of basepoint free linear systems.

Definition 1.52. Let X be a nonsingular projective curve. Let D be a divisor on X and let $|D|$ be the linear system associated to D. A point $P \in X$ is called **basepoint** of $|D|$, if $D' - P \geq 0$ for all $D' \in |D|$. If $|D|$ does not possess any base points, then $|D|$ is called **basepoint free**.

Remark 1.53. Proposition 1.51 implies that a divisor D on X is basepoint free if and only if $h^0(X, D - P) = h^0(X, D) - 1$ for all $P \in X$.

Proposition 1.54. *Let X be a nonsingular projective curve and let D be a divisor on X. Then, $h^0(X,D) > 0$ if and only if D is linearly equivalent to an effective divisor.*

Proof. Easy. □

Sometimes, we just want to consider Riemann-Roch spaces on a subset of points of the curve. We will introduce this notion in the next definition.

Definition 1.55. Let X be a nonsingular projective curve and let $T \subset X$ be a subset of X. Let $D = \sum_{P \in X} \eta_P \cdot P \in \operatorname{Div}(X)$. We define

$$\deg^T(D) := \sum_{P \in T} \eta_P$$

to be the **degree** of the divisor D, **restricted to** $T \subset X$. Furthermore, we define

$$H^0(X,D)^T := \{f \in K(X)^* : (\operatorname{div}(f) + D)|_T \geq 0\} \cup \{0\}$$

to be the Riemann-Roch space of D, **restricted to** the subset T.

Lemma 1.56. *Let X be a nonsingular projective curve and let $D_1, D_2 \in \operatorname{Div}(X)$ with $D_1 \leq D_2$. Let $T \subset X$. Then $H^0(X, D_1)^T \subset H^0(X, D_2)^T$ and if $T \subset X$ is finite, we have*

$$\dim_{\mathbb{C}}(H^0(X,D_2)^T / H^0(X,D_1)^T) = \deg^T(D_2 - D_1).$$

Proof. The first part is clear. For the second part, we proceed in the same way as in the proof of Proposition 1.48 (i). □

As mentioned before, the main goal is to calculate the dimension $h^0(X, D)$ for some divisor D on a projective curve X. The next proposition gives the first bound, with which we come a step closer to Riemann's Theorem.

Proposition 1.57. *Let X be a nonsingular projective curve and let $f \in K(X)^*$ be non-constant. Let $Z := (f)_0$ and let $n := [K(X) : \mathbb{C}(f)]$, i.e. a $\mathbb{C}(f)$-basis of $K(X)$ has n elements. Then the following hold:*

(i) *Z is an effective divisor of degree $\leq n$.*

(ii) *There exists a constant τ such that $h^0(X, rZ) \geq rn - \tau$ for each $r \in \mathbb{C}$. Furthermore, $\deg(Z) = n$.*

Proof.

(i) Write $Z = \sum_{P \in X} \eta_P \cdot P$ and let $m = \deg(Z)$. We want to show that $m \leq n$.

Let $T := \{P \in X : \eta_P > 0\}$ and choose $f_1, \dots, f_m$ in $H^0(X, 0)^T$, such that

$$[f_1], \dots, [f_m] \in H^0(X, 0)/H^0(X, -Z)$$

form a basis for the vector space $H^0(X, 0)/H^0(X, -Z)$. By Lemma 1.56, $H^0(X, 0)/H^0(X, -Z)$ has dimension $\deg^T(0 + Z) = \deg^T(Z) = m$. We want to show that $\{f_1, \dots, f_m\}$ is linearly independent over $\mathbb{C}(f)$. This implies that $m \leq n$.

Indeed, assume that $\{f_1, \dots, f_m\}$ is linearly dependent over $\mathbb{C}(f)$. Then there exists $\varphi_i := \frac{p_i}{q_i} \in \mathbb{C}(f)$, such that $\sum_{i=1} \varphi_i f_i = 0$. By clearing each denominator of φ_i, we obtain

$$\sum_{i=1}^{m} \prod_{\substack{j=1 \\ j \neq i}}^{m} p_i q_i f_i = 0. \tag{1.3}$$

Now, $\prod_{j=1, j \neq i}^{m} p_i q_i \in \mathbb{C}[f]$ for all $1 \leq i \leq m$, so we may write

$$\prod_{\substack{j=1 \\ j \neq i}}^{m} p_i q_j = \sum_{k=0}^{n_i} \lambda_k^{(i)} f^k = \lambda_0^{(i)} + f \sum_{k=1}^{n_i} \lambda_k^{(i)} f^{k-1}.$$

Now, by letting $\lambda_i := \lambda_0^{(i)}, \mathrm{h}_i := \sum_{k=1}^{n_i} \lambda_k^{(i)} f^{k-1}$ and $g_i := \lambda_i + f h_i$, we may assume (1.3) to be of the form

$$\sum_{i=1}^{m} g_i f_i = \sum_{i=1}^{m} (\lambda_i + f h_i) f_i = 0,$$

where not all $\lambda_i = 0$. But then

$$\sum_{i=1}^{m} \lambda_i f_i = -f \sum_{i=1}^{m} h_i f_i \in H^0(X, -Z)^T$$

and hence $\sum_{i=1}^{m} \lambda_i [f_i] = 0$, but not all λ_i are zero, which contradicts the fact that $[f_1], \dots, [f_m]$ form a basis for $H^0(X, 0)/H^0(X, -Z)$. Hence, $\{f_1, \dots, f_m\}$ is linearly independent over $\mathbb{C}(f)$ and therefore $m \leq n$.

(ii) Let $\{g_1, \ldots, g_n\}$ be a $\mathbb{C}(f)$ basis for $K(X)$. Since the field extension is finite, each g_i, where $1 \le i \le n$, satisfies an equation of the form

$$g_i^{n_i} + a_{i_1} g_i^{n_i - 1} + \cdots + a_{i_{n_i - 1}} g_i + a_{i_{n_i}} = 0, \tag{1.4}$$

with $a_{ij} \in \mathbb{C}[f^{-1}],\ n_i \in \mathbb{N}$.

If $P \notin T$, then f^{-1} has a zero along P and hence $\nu_P(a_{ij}) \ge 0$. If $\nu_P(g_i) < 0$ and $P \notin T$, then for each $0 \le j \le r$, we have $\nu_P(g_i^{n_i}) < \nu_P(a_{ij} g_i^{n_i - j})$, contradicting (1.4).

Now, we can write

$$\operatorname{div}(g_i) = \sum_{P \in T} \nu_P(g_i) \cdot P + \sum_{P \notin T} \nu_P(g_i) \cdot P.$$

By the above considerations, $\sum_{P \notin T} \nu_P(g_i) \cdot P$ is effective. By definition of T, for $t_i \ge \max_{P \in T}\{|\nu_P(g_i)|\}$, we have $\operatorname{div}(g_i) + t_i Z \ge 0$. If we choose $t := \max\{t_1, \ldots, t_n\}$, then $\operatorname{div}(g_i) + tZ \ge 0$ for all $1 \le i \le n$. But then $g_i f^{-j} \in H^0(X, (r+t)Z)$ for $1 \le i \le n,\ 0 \le j \le r$. Since the g_i are linearly independent over $\mathbb{C}(f) \subset K(X)$ and $\{1, f^{-1}, \ldots, f^{-r}\}$ is linearly independent over $\mathbb{C}$, the elements $g_i f^{-j}$ are linearly independent over $\mathbb{C}$ for $1 \le i \le n,\ 0 \le j \le r$. Hence,

$$h^0(X, (r+t)Z) \ge n(r+1)$$

and

$$\begin{aligned} h^0(X, (r+t)Z) &= h^0(X, rZ) + \dim_{\mathbb{C}}(H^0(X, (r+t)Z)/H^0(X, rZ)) \\ &\le h^0(X, rZ) + tm. \end{aligned}$$

If we let $\tau := -n + tm$, we obtain

$$h^0(X, rZ) \ge n(r+1) - tm = rn - \tau$$

for each $r \in \mathbb{C}$.

Now,

$$rn - \tau \le h^0(X, rZ) \le rm + 1.$$

Thus, for large r, we get the missing inequality, i.e. $n \le m$. This finishes the proof.

□

Chapter 2

Riemann's Theorem

In this chapter, we are going to give a lower bound for the dimension $h^0(X, D)$ of the vector space $H^0(X, D)$ for a divisor D on a nonsingular projective curve X. This forms the so called Riemann's Theorem by Bernhard Riemann and is the first step towards the development of the Riemann-Roch Theorem.

Theorem 2.1 (Riemann's Theorem)**.** *Let X be a nonsingular projective curve. Then there is an integer m, such that*

$$h^0(X, D) \geq \deg(D) + 1 - m$$

for each divisor $D \in \operatorname{div}(X)$.

The claim follows directly from the next proposition.

Proposition 2.2. *Let X be a nonsingular projective curve. Let*

$$s(D) := \deg(D) + 1 - h^0(X, D)$$

for all divisors $D \in \operatorname{Div}(X)$. Then, for arbitrary divisors $D_1, D_2 \in \operatorname{Div}(X)$, it holds:

(i) *$s(0) = 0$, i.e. $m \geq 0$.*

(ii) *If $D_1 \sim D_2$, then $s(D_1) = s(D_2)$.*

(iii) *If $D_1 \leq D_2$, then $s(D_1) \leq s(D_2)$.*

(iv) *If $f \in K(X)^*$ is a rational function and $Z := (f)_0$ denotes its divisor of zeros, then $s(rZ) = \tau + 1$ for all large $r > 0$, where τ is defined as in Proposition 1.57.*

(v) *For any $D_1 \in \operatorname{Div}(X)$, there exists another divisor $D_2 \in \operatorname{Div}(X)$, linearly equivalent to D_1, such that for all large $r \geq 0$, we have that $D_2 \leq rZ$, where Z is as in (iv).*

In particular, for any $D \in \operatorname{Div}(X)$, there exists $m \in \mathbb{Z}$ with $m \geq s(D)$.

Proof. The first three statements follow directly from Proposition 1.48.

(iv) Let $f \in K(X)^*$, $f \notin \mathbb{C}$. Let $Z = (f)_0$ be the divisor of zeros of f and let τ be the smallest integer as of Proposition 1.57, such that $h^0(X, rZ) \geq rn - \tau$ for each $r \in \mathbb{C}$. Then we have that $s(rZ) \leq \tau + 1$ for all $r > 0$ and that $rZ \leq (r+1)Z$. By using (iii), we get that $s(rZ) \leq s((r+1)Z)$, which implies $s(rZ) = \tau + 1$ for all large $r > 0$.

(v) Let $m := \tau + 1$. Let $Z := \sum_{P \in X} \eta_P \cdot P$ and let $D_1 := \sum_{P \in X} \mu_P \cdot P$. Furthermore, let $g := f^{-1}$, f as above and let

$$S := \{P \in X : \mu_P > 0 \text{ and } \nu_P(g) > 0\}.$$

Now, we want to show that there exists a divisor D_2, which is linearly equivalent to D_1 and such that for all large $r \geq 0$, we have that $D_2 \leq rZ$. So, we want to show that there exists a rational function $h \in K(X)^*$ with $D_2 - D_1 = \operatorname{div}(h)$. To this end, let

$$h := \prod_{P \in S} (g - g(P))^{\mu_P}.$$

Now, if $\nu_P(g) \geq 0$, then we get $\mu_P - \nu_P(h) \leq 0 \leq r\eta_P$ for each $P \in S$. Conversely, if $\nu_P(g) < 0$, then $\eta_P > 0$. Hence, for large r, we get $\mu_P - \nu_P(h) \leq r\eta_P$. This implies $D_2 \leq rZ$, so h does the job.

In summary, we showed that for each $D \in \operatorname{Div}(X)$, there exists an integer m with

$$m \geq s(D) = \deg(D) + 1 - h^0(X, D).$$

This completes the proof. □

The smallest non-negative integer m we get from Riemann's Theorem, we call the **genus** of a curve X and denote it by g. For a canonical divisior K_X on X, it holds that $h^0(X, K_X) = g$, which we will prove in the next chapter. It indeed makes sense to speak of "the" genus of a curve, because we have seen in Remark 1.44 that the canonical divisors of a curve are all linearly equivalent to each other.

The following corollaries give us some cases where we even get equality in Riemann's Theorem.

Corollary 2.3. *Let X be a nonsingular projective curve. Let D_1, D_2, D_3 be three divisors on X with $h^0(X, D_1) = \deg(D_1) + 1 - g$ and $D_2 \sim D_3 \geq D_1$. Then*

$$h^0(X, D_2) = \deg(D_2) + 1 - g.$$

Proof. The proof is analogous to that of Proposition 2.2. □

Corollary 2.4. *Let X be a nonsingular projective curve. There exists an integer N, such that for all divisors $D \in \operatorname{Div}(X)$ with $\deg(D) > N$,*

$$h^0(X, D) = \deg(D) + 1 - g.$$

Proof. Let $D_1 \in \mathrm{Div}(X)$ with

$$h^0(X, D_1) = \deg(D_1) + 1 - g.$$

Let $N := \deg(D_1) + g$ and let $D \in \mathrm{Div}(X)$ with $\deg(D) > N$. Then we have

$$\begin{aligned}\deg(D - D_1) + 1 - g &= \deg(D) - \deg(D_1) + 1 - g \\ &\geq \deg(D_1) + g - \deg(D_1) - g + 1 = 1 > 0.\end{aligned}$$

So by Riemann's Theorem, $h^0(X, D - D_1) > 0$.

So there exists $f \in h^0(X, D - D_1)$, i.e. $\mathrm{div}(f) + D - D_1 \geq 0$, which implies that $D \sim D + \mathrm{div}(f) \geq D_1$ and hence, by Corollary 2.3, we have

$$h^0(X, D) = \deg(D) + 1 - g.$$

Thus, N does the job. □

For large divisors D on X, we can calculate the dimension $h^0(X, D)$. Our goal is to calculate it for all divisors D on a curve X, which we will get with the help of Roch and his and Riemann's common Theorem in the next chapter. But first, we want to consider some helpful propositions, which allow us to calculate the genus of some curves with special assumptions. This indeed helps us to see the usefulness of Riemann's Theorem.

Theorem 2.5. *Let $V(m; r_1P_1, \ldots, r_nP_n)$ be the set of all projective plane curves $C \subset \mathbb{P}^2$ of degree m with defining polynomial $F \in \mathbb{C}[X_0, X_1, X_2]$, such that $m_{P_i}(F) \geq r_i$ for all $1 \leq i \leq n$. Then, $V(m; r_1P_1, \ldots, r_nP_n)$ is a linear subvariety of $\mathbb{P}^{(m(m+3))/2}$ of dimension greater or equal than*

$$\frac{m(m+3)}{2} - \sum_{i=1}^{n} \frac{r_i(r_i+1)}{2}$$

and if $d \geq (\sum_{i=1}^{n} r_i) - 1$, then

$$\dim(V(m; r_1P_1, \ldots, r_nP_n)) = \frac{m(m+3)}{2} - \sum_{i=1}^{n} \frac{r_i(r_i+1)}{2}.$$

Proof. See [Ful69, Chapter 5.2, Theorem 1] □

Proposition 2.6. *Let $C \subset \mathbb{P}^2$ be a projective plane curve with defining polynomial $F \in \mathbb{C}[X_0, X_1, X_2]$. Assume that C has only ordinary multiple points and let $n := \deg(C)$. Define $s_P := m_P(F)$ for $P \in C$. Then the genus g of C is given by*

$$g = \frac{(n-1)(n-2)}{2} - \sum_{P \in C} \frac{s_P(s_P - 1)}{2}.$$

Proof. Let $\rho\colon C' \to C$ be the birational morphism from the nonsingular model C' onto C. It would be helpful for the proof, if we were able to calculate $h^0(C', D)$ explicitly. To this end, we want to use Corollary 2.4, which means that we need to find some large $N \in \mathbb{N}$. Then for all $D \in \operatorname{Div}(C')$ with degree larger than N, we can calculate $h^0(C', D)$. By the Residue Theorem, we find all effective divisors on C' which are linearly equivalent to such divisors D on C'. With that, we can calculate g as follows.

After a projective change of coordinates, we may assume by Bézout's Theorem that the line $X_0 = 0$ intersects C in n distinct points $P_1, \dots, P_n \in \mathbb{P}^2$. Let $E := \sum_{Q \in C'} (r_Q - 1) \cdot Q$, where $r_Q := m_{\rho(Q)}(F) = s_P$ for $\rho(Q) =: P \in C$, be the divisor of multiple points on C'. Furthermore, let

$$E_m := m \sum_{i=1}^{n} P_i - E,$$

which is a divisor of degree $mn - \sum_{P \in C} s_P(s_P - 1)$.

Let V_m be the vector space of all defining polynomials $G \subset \mathbb{C}[X_0, X_1, X_2]$ of adjoints of C of degree m. Now, the adjoints of C are just given by polynomials $G \subset \mathbb{C}[X_0, X_1, X_2]$, which satisfy

$$\operatorname{div}(G) \geq E = \sum_{Q \in C'} (r_Q - 1) \cdot Q,$$

i.e. for each $P \in C$, it holds that $m_P(G) \geq s_P - 1$. By applying Theorem 2.5 with the space $V_m = V(m; E)$, we get that

$$\dim(V_m) \geq \frac{m(m+3)}{2} - \sum_{P \in C} \frac{s_P(s_P - 1)}{2}.$$

Since here V_m is the vector space of defining polynomials and not the projective space of curves, the dimension increases by 1, i.e

$$\begin{aligned} \dim(V_m) &\geq \frac{m(m+3)}{2} + 1 - \sum_{P \in C} \frac{s_P(s_P - 1)}{2} \\ &= \frac{(m+1)(m+2)}{2} - \sum_{P \in C} \frac{s_P(s_P - 1)}{2} \end{aligned}$$

and if m is large, we even get equality.

Now, let $\varphi\colon V_m \to H^0(C', E_m)$, $G \mapsto \frac{G}{X_0}^m$, where $\frac{G}{X_0^m}$ is identified with its image in $K(C')$. This map is clearly well-defined and linear, with $\varphi(G) = 0$ if and only if F divides G.

Claim: φ is surjective.

Proof of claim. Let $f \in H^0(C', E_m)$ be arbitrary. As in the proof of Proposition 1.29, we identify f with an element $\frac{F_1}{F_2} \in K(\mathbb{P}^2)$ in $K(C')$, where

$F_1, F_2 \in \mathbb{C}[X_0, X_1, X_2]$ and $\deg(F_1) = \deg(F_2)$, $F_2 \neq 0$. Then

$$\begin{aligned}\operatorname{div}(F_1 {X_0}^m) = \operatorname{div}(f F_2 X_0^m) &= \operatorname{div}(f) + \operatorname{div}(F_2) + \operatorname{div}(X_0^m)\\ &\geq -E_m + \operatorname{div}(F_2) + \operatorname{div}(X_0^m)\\ &= -m\sum_{i=1}^n P_i + E + \operatorname{div}(F_2) + m\sum_{i=1}^n P_i\\ &= \operatorname{div}(F_2) + E.\end{aligned}$$

Applying Proposition 1.20 to $F_1 X_0^m$, there exist $A, B \in \mathbb{C}[X_0, X_1, X_2]$ with

$$F_1 {X_0}^m = AF_2 + BF$$

and therefore $f = \frac{F_1}{F_2} = \frac{A}{X_0^m}$ in $K(C) \simeq K(C')$, so $\varphi(A) = f$. Since the element $f \in H^0(C', E_m)$ was arbitrary, we obtain the surjectivity of φ. □

Let

$K_{m-n} := \{G \in \mathbb{C}[X_0, X_1, X_2] : G$ defines a projective curve of degree $m-n\}$

and let $\psi\colon K_{m-n} \to V_m$, $G \mapsto FG$. This map is clearly injective and the sequence of vector spaces

$$0 \longrightarrow K_{m-n} \xrightarrow{\psi} V_m \xrightarrow{\varphi} H^0(C', E_m) \longrightarrow 0 \tag{2.1}$$

is exact. Thus

$$\begin{aligned}h^0(C', E_m) &= \dim(V_m) - \dim(K_{m-n})\\ &= \deg(E_m) + 1 - \left(\frac{(n-1)(n-2)}{2} - \sum_{P\in C}\frac{s_P(s_P-1)}{2}\right)\end{aligned}$$

for large $m \in \mathbb{N}$. We finish the proof by applying Corollary 2.4. Indeed, since E_m becomes larger as m becomes larger, the assumptions of Corollary 2.4 hold and we get

$$h^0(C', E_m) = \deg(E_m) + 1 - g,$$

forcing

$$g = \frac{(n-1)(n-2)}{2} - \sum_{P\in C}\frac{s_P(s_P-1)}{2}.$$

This completes the proof. □

Now, we are able to calculate the genus of some special kind of curves. As mentioned above, Proposition 2.6 is crucial for the proof of the Riemann-Roch Theorem. In the proof of Proposition 2.6, we also showed the following important result.

Corollary 2.7. *Let $C \subset \mathbb{P}^2$ be a plane curve of degree n with defining polynomial $F \in \mathbb{C}[X_0, X_1, X_2]$ and let $\rho\colon C' \to C$ be the birational morphism from the nonsingular model C' onto C. Let E_m be defined as in the proof above. Then any $g \in H^0(C', E_m)$ can be written as the image of an element $\frac{G}{X_0^m}$ in $K(C')$, where G is an adjoint of C of degree m. Furthermore,*

$$\deg(E_{n-3}) = 2g - 2 \text{ and } h^0(C', E_{n-3}) \geq g.$$

Proof. The first part was shown in the proof of Proposition 2.6.

For the second part, remark that from (2.1), we obtain $V_m = H^0(C', E_m)$ for $m < n$. Furthermore, observe that that

$$\deg(E_{n-3}) = (n-3)n - \sum_{P \in C} s_P(s_P - 1),$$

where $s_P := m_P(F)$ for $P \in C$. By Proposition 2.6, we have

$$g = \frac{(n-1)(n-2)}{2} - \sum_{P \in C} \frac{s_P(s_P - 1)}{2}.$$

Thus,

$$\begin{aligned} 2g - 2 &= (n-1)(n-2) - 2 \sum_{P \in C} \frac{s_P(s_P - 1)}{2} - 2 \\ &= (n^2 - 3n) - \sum_{P \in C} s_P(s_P - 1) = \deg(E_{n-3}), \end{aligned}$$

which was to be shown. □

Using both, the just proved result and the following proposition, we may calculate the degree of the canonical divisor.

Proposition 2.8. *Let $C_1 \subset \mathbb{P}^2$ be a projective plane curve of degree $n \geq 3$ with only ordinary multiple points and let $F \in \mathbb{C}[X_0, X_1, X_2]$ be its defining polynomial. Let $\rho\colon C_1' \to C_1$ be the birational morphism from the nonsingular model C_1' onto C_1 and let E be the divisor of multiple points on C_1'. Let $C_2 \subset \mathbb{P}^2$ be another projective plane curve of degree $n - 3$ with defining polynomial $F_2 \in \mathbb{C}[X_0, X_1, X_2]$. Then, $\operatorname{div}(F_2) - E$ is a canonical divisor. In particular: If $n = 3$, then $\operatorname{div}(F_2) = 0$.*

Proof. As in the previous proof, assume that the line $L_0 := V(X_0)$ intersects C_1 in n distinct points $P_1, \ldots, P_n \in \mathbb{P}^2$. Assume that $(0, 0, 1)$ does not lie on C and that no tangent to C at a multiple point passes through $(0, 0, 1)$.

Let x_1 be the image of $\frac{X_1}{X_0}$ and let x_2 be the image of $\frac{X_2}{X_0}$ in $K(C_1)$. Furthermore, let

$$f_{x_1} := \frac{\partial F}{\partial X_1}(1, x_1, x_2) \text{ and } f_{x_2} := \frac{\partial F}{\partial X_2}(1, x_1, x_2)$$

and let $E_m := m \sum_{i=1}^{n} P_i - E$ be as defined in Proposition 2.6.

Take $\omega \in \Omega_{\mathbb{C}}(K(C_1'))$, such that $\omega = dx_2$. We want to show that

$$\operatorname{div}(\omega) = E_{n-3} + \operatorname{div}(f_{x_1}). \tag{2.2}$$

Since

$$\begin{aligned}\operatorname{div}(f_{x_1}) &= \operatorname{div}\left(\frac{\partial F}{\partial X_1}(1, x_1, x_2)\right) \\ &= \operatorname{div}\left(\frac{\frac{\partial F}{\partial X_1}}{X_0^{n-1}}\right) = \operatorname{div}\left(\frac{\partial F}{\partial X_1}\right) - \operatorname{div}(X_0^{n-1}),\end{aligned}$$

showing (2.2) is equivalent to showing that

$$\begin{aligned}&\operatorname{div}(dx_2) - \operatorname{div}\left(\frac{\partial F}{\partial X_1}\right) \\ &= E_{n-3} - \operatorname{div}(X_0^{n-1}) \\ &= (n-3)\sum_{i=1}^{n} P_i - E - (n-1)\sum_{i=1}^{n} P_i = -2\sum_{i=1}^{n} P_i - E.\end{aligned} \tag{2.3}$$

First, note that

$$dx_2 = -\frac{f_{x_1}}{f_{x_2}} dx_1 = -\frac{\frac{\partial F}{\partial X_1}}{\frac{\partial F}{\partial X_2}} dx_1.$$

Thus, for each $P \in C_1'$, we have

$$\nu_P(dx_2) - \nu_P\left(\frac{\partial F}{\partial X_1}\right) = \nu_P(dx_1) - \nu_P\left(\frac{\partial F}{\partial X_2}\right).$$

Let $P_i \in C_1 \cap L_0$ and let $Q := \rho^{-1}(P_i)$. Then $x_1^{-1} = \frac{X_0}{X_1}$ is a local parameter in $\mathcal{O}_{C_1',Q}$ and $dx_1 = -x_1^2 dx_1^{-1}$, which implies that $\nu_Q(dx_1) = -2$. Since $\frac{\partial F}{\partial X_2}(P_i) \neq 0$, both sides of equation (2.3) have coefficient -2 at Q.

Let $P := (1, a, b) \in C_1$ and let $Q := \rho^{-1}(P)$. Since $dx_2 = d(x_2 - a)$ and by using the fact that derivatives do not change by translations, we may assume that $P = (1, 0, 0)$.

Let $L_1 := V(X_1)$. We distinguish between two cases.

Case 1: The line L_1 is a tangent to C_1 at P. Then, by hypothesis, P is not a multiple point. This implies that x_2 is a local parameter of $\mathcal{O}_{C_1',Q}$ and $(\partial F/\partial X_1)(P) \neq 0$. Thus,

$$\nu_Q(dx_2) = \nu_Q\left(\frac{\partial F}{\partial X_1}\right) = 0$$

and we are done.

Case 2: The line L_1 is not a tangent to C at P. Then x_1 is a local parameter in $\mathcal{O}_{C_1',Q}$. Hence, $\nu_Q(dx_1) = 0$ and $\nu_Q(f_{x_2}) = r_Q^{-1}$ and we are done.

This finishes the proof. □

Proposition 2.9. *Let X be a nonsingular projective curve. Let K_X be the canonical divisor on X. Then* $\deg(K_X) = 2g - 2$ *and* $h^0(X, K_X) \geq g$.

Proof. We may assume that $K_X = E_{n-3}$, defined as above, since we already have seen in Proposition 2.8 that E_{n-3} is a canonical divisor on X. Then we can use Corollary 2.7 and conclude. □

Chapter 3

The Riemann-Roch Theorem

This chapter forms the heart of this thesis. The goal is to introduce and to prove the Riemann-Roch Theorem, which presents an explicit formula for calculating $h^0(X, D)$. Our proof follows the classical proof of Max Noether and Alexander von Brill.

This proof's main ingredient is the so-called Noether's Reduction Lemma, which allows us to reduce the statement of the Riemann-Roch Theorem to simpler cases.

We have seen in Corollary 2.3 that for divisors of large degree we can already calculate $h^0(X, D)$. Thus, we already know one special case, where the Riemann-Roch Theorem holds. In the sequel, we want to use this result and extend it to the general case, which then gives us the Riemann-Roch Theorem.

Lemma 3.1 (Noether's Reduction Lemma)**.** *Let X be a nonsingular projective curve and let K_X be the canonical divisor on X. Let $D \in \mathrm{Div}(X)$ be a divisor and let $P \in X$ be a point. Assume that*

$$h^0(X, D) > 0 \textit{ and } h^0(X, K_X - (D + P)) < h^0(X, K_X - D).$$

Then

$$h^0(X, D + P) = h^0(X, D).$$

Proof. The goal is to show that for an element $f \in H^0(X, D + P)$, we already have $f \in H^0(X, D)$. To do so, we want to use the Residue Theorem, so we first have to do some preparation in finding divisors for which the assumptions hold. To this end, we use the fact that for each projective curve, we can find a birational projective plane curve with only ordinary multiple points and reduce to such a curve. Then we can use multiple results, proved in the last chapter.

Let $C \subset \mathbb{P}^2$ be a birational projective plane curve to X with only ordinary multiple points and with defining polynomial $F \in \mathbb{C}[X_0, X_1, X_2]$. Let C be of degree n and let P a nonsingular point on C.

Let L_0 be the line defined by $X_0 = 0$. Since C consists of only ordinary multiple points, we have

$$H := L_0 \cdot C = \sum_{i=1}^{n} P_i,$$

for distinct points P_i, where $P_i \neq P$ for all $1 \leq i \leq n$. Furthermore, let

$$E = \sum_{Q \in X} (r_Q - 1) r_Q \cdot Q$$

be the divisor of multiple points of X and let

$$E_{n-3} := (n-3)H - E$$

be the divisor, defined as in the proof of Proposition 2.6. As of the proof of Proposition 2.9, this is a canonical divisor and since all canonical divisors are linearly equivalent, we may assume that $K_X = E_{n-3}$. Furthermore, by Proposition 1.54, we may assume that $D \geq 0$. Thus,

$$H^0(X, K_X - D) \subset H^0(X, E_{n-3}).$$

Let $g \in H^0(X, K_X - D)$ with $g \notin H^0(X, K_X - (D + P))$. By Corollary 2.7, we can write g as the image of some element $\frac{G}{{X_0}^{n-3}}$ for an adjoint G in $\mathbb{C}[X_0, X_1, X_2]$ of C, which is of degree $n-3$. Thus, by definition of an adjoint, we can write

$$\operatorname{div}(G) \geq E \iff \operatorname{div}(G) = E + D'$$

for some divisor D' on X. For a divisor $A \geq 0$ on X with $A \ngeq P$, define $D' =: D + A$. Then

$$\operatorname{div}(G) = E + D + A.$$

Now, take a line Z through the point P, which is defined by an equation $z = 0$. Let $Z \cdot C = P + B$, where B is a divisor given by $n-1$ nonsingular points, which are all distinct from P, be the intersection divisor of Z and C. Then

$$\operatorname{div}(zG) = \operatorname{div}(z) + \operatorname{div}(G) = \operatorname{div}(z) + D + E + A = (D + P) + E + (A + B).$$

Let $f \in H^0(X, D + P)$. Then $\operatorname{div}(f) + D + P \geq 0$. Let D' be a linearly equivalent divisor to D with $D' - D = \operatorname{div}(f)$. We want to show that we already have $f \in H^0(X, D)$, i.e. we want to show that D' is effective. This is guaranteed by the Residue Theorem. Indeed, since D and D' are linearly equivalent, $D + P \sim D' + P$ and since both $D + P$ and $D' + P$ are effective, we can apply the Residue Theorem. Thus, there exists a curve C_1 of degree $n-2$ with defining polynomial $F_1 \in \mathbb{C}[X_0, X_1, X_2]$ such that

$$\operatorname{div}(F_1) = (D' + P) + E + (A + B).$$

Finally, B consists of $n-1$ distinct points, lying on one line and C_1 is a curve of degree $n-2$. This implies by Bézout's Theorem that Z has to be a component of C, i.e. $F_1(P)=0$. But since $E+A+B$ does not contain the point P, we have $D'+P \geq P$, i.e. $D' \geq 0$, which implies $f \in H^0(X,D)$. This finishes the proof. $\square$

Now, we can introduce and prove the Riemann-Roch Theorem.

Theorem 3.2 (Riemann-Roch Theorem). *Let X be a nonsingular projective curve and let K_X be the canonical divisor on X. Then*

$$h^0(X,D) = \deg(D) + 1 - g + h^0(X, K_X - D)$$

for each divisor D on X.

Proof. Consider the equation

$$h^0(X,D) = \deg(D) + 1 - g + h^0(X, K_X - D)$$

for an arbitrary divisor D on X. The "new" object is $h^0(X, K_X - D)$, for which we have to distinguish between two cases:

Case 1: $h^0(X, K_X - D) = 0$. In this case, we can use induction on $h^0(X,D)$.

1.1: If $h^0(X,D) = 0$, then applying Riemann's Theorem to D and to $K_X - D$ gives

$$0 = h^0(X,D) \geq \deg(D) + 1 - g$$

and

$$\begin{aligned} 0 = h^0(X, K_X - D) &\geq \deg(K_X - D) + 1 - g \\ &= \deg(K_X) - \deg(D) + 1 - g = 2g - 2 - \deg(D) + 1 - g \\ &= g - \deg(D) - 1 = -(\deg(D) + 1 - g) \iff 0 \leq \deg(D) + 1 - g, \end{aligned}$$

where we used Proposition 2.9. By combining, we get

$$h^0(X,D) = \deg(D) + 1 - g + h^0(X, K_X - D),$$

which was to be shown.

1.2: If $h^0(X,D) = 1$, we may assume that $D \geq 0$, so by Proposition 2.9, we obtain

$$g \leq h^0(X, K_X).$$

Furthermore, $K_X - D \leq K_X$, so by Corollary 1.50, we obtain

$$h^0(X, K_X) \leq h^0(X, K_X - D) + \deg(D) = \deg(D).$$

But this implies that $g \leq \deg(D)$, so by Riemann's Theorem, we conclude.

1.3: If $h^0(X, D) > 1$, by Proposition 1.51, we may choose a point $P \in X$, such that

$$h^0(X, D - P) = h^0(X, D) - 1.$$

Then we can use Noether's Reduction Lemma and obtain

$$h^0(X, K_X - (D - P)) = h^0(X, K_X - D) = 0.$$

By induction hypothesis, we have that

$$h^0(X, D - P) = \deg(D - P) + 1 - g + h^0(X, K_X - (D - P)),$$

which implies

$$h^0(X, D) = \deg(D) + 1 - g + h^0(X, K_X - D),$$

which was to be shown.

Case 2: $h^0(X, K_X - D) > 0$.

2.1: If $h^0(X, D) = 0$, then we also have that

$$h^0(X, K_X - (K_X - D)) = h^0(X, D) = 0.$$

Thus, we can proceed as in Case 1 on the divisor $K_X - D$ and obtain

$$\begin{aligned} h^0(X, K_X - D) &= \deg(K_X - D) + 1 - g + h^0(X, K_X - (K_X - D)) \\ &= g - 1 - \deg(D) + h^0(X, D). \end{aligned}$$

and get the result.

2.2: Let $h^0(X, D) > 0$. We prove this by induction on $h^0(X, K_X - D)$. Since $h^0(X, D) > 0$, we can again assume that $D \geq 0$. Choose P such that $h^0(X, K_X - (D + P)) = h^0(X, K_X - D) - 1$. This is guaranteed by Proposition 1.51. Then we can use Noether's Reduction Lemma and obtain

$$h^0(X, D + P) = h^0(X, D). \tag{3.1}$$

By induction hypothesis, we have

$$\begin{aligned} h^0(X, K_X - (D + P)) &= \deg(K_X - D - P) + 1 - g + h^0(X, D + P) \\ &= \deg(K_X) - \deg(D + P) + 1 - g + h^0(X, D + P) \\ &= g - 1 - \deg(D + P) + h^0(X, D + P) \end{aligned}$$

and by (3.1), we obtain

$$\begin{aligned} h^0(X, D) &= h^0(X, D + P) \\ &= \deg(D + P) + 1 - g + h^0(X, K_X - (D + P)) \\ &= \deg(D) + 1 - g + h^0(X, K_X - D). \end{aligned}$$

This completes the proof. $\square$

The next few corollaries are some direct consequences of the Riemann-Roch Theorem, which illustrate the usefulness of the theorem.

At first, we recall a well-known result, first stated when we introduced the Riemann-Roch spaces. We claimed that $h^0(X, K_X) = g$ but we were not able to prove it then. At this point, we are.

Corollary 3.3. *Let X be a nonsingular projective curve and let K_X be the canonical divisor on X. Then $h^0(X, K_X) = g$.*

Proof. By the Riemann-Roch Theorem, we have that

$$\begin{aligned} & h^0(X, 0) = \deg(0) + 1 - g + h^0(X, K_X) \\ \iff & 1 = 1 - g + h^0(X, K_X) \\ \iff & h^0(X, K_X) = g. \end{aligned}$$

This shows the claim. $\square$

We also have already seen the following result in the previous chapter. But the proof was somehow difficult. The Riemann-Roch Theorem gives us a much simpler proof.

Corollary 3.4. *Let X be a nonsingular projective curve and let K_X be the canonical divisor on X. Then $\deg(K_X) = 2g - 2$.*

Proof. By the Riemann-Roch Theorem, we have that

$$\begin{aligned} & h^0(X, K_X) = \deg(K_X) + 1 - g + h^0(X, 0) \\ \iff & g = \deg(K_X) + 1 - g + 1 \\ \iff & \deg(K_X) = 2g - 2, \end{aligned}$$

which proves the claim. $\square$

Using the previous two corollaries, we even can introduce two more.

Corollary 3.5. *Let X be a nonsingular projective curve. Let D be a divisor on X with $\deg(D) \geq 2g - 1$. Then $h^0(X, D) = \deg(D) + 1 - g$.*

Proof. We want to show that $h^0(X, K_X - D) = 0$. Indeed,

$$\deg(K_X - D) = \deg(K_X) - \deg(D).$$

Thus, by assumption,

$$\deg(K_X - D) = \deg(K_X) - \deg(D) \leq 2g - 2 - (2g - 1) = -1.$$

Now, Proposition 1.48 implies that $h^0(X, K_X - D) = 0$, which is what we wanted to prove. $\square$

Remark 3.6. Recall Corollary 2.3, where we stated that for all divisors bigger than some "large" N, Riemann's inequality becomes an equality. Now, Corollary 3.5 gives us an explicit bound for this N.

We also derive a sufficient condition for a complete linear system to be basepoint free and prove that the canonical linear system $|K_X|$ is basepoint free.

Corollary 3.7. *Let X be a nonsingular projective curve. Let D be a divisor on X with $\deg(D) \geq 2g$. Then $h^0(X, D-P) = h^0(X, D) - 1$ for all $P \in X$. In particular, $|D|$ is basepoint free.*

Proof. By assumption, we know that $\deg(D) \geq 2g$ and therefore

$$\deg(D-P) \geq 2g-1.$$

Thus, by using Corollary 3.5, we obtain

$$h^0(X, D-P) = \deg(D-P) + 1 - g = \deg(D) - g$$

and

$$h^0(X, D) = \deg(D) + 1 - g.$$

By combining, it follows that

$$h^0(X, D) - 1 = (\deg(D) + 1 - g) - 1 = \deg(D) - g = h^0(X, D-P).$$

This finishes the proof. □

Proposition 3.8. *Let X be a nonsingular projective curve. Let K_X be its canonical divisor. Then $|K_X|$ is basepoint free.*

Proof. Applying the Riemann-Roch Theorem to $K_X - P$, we obtain

$$\begin{aligned} h^0(X, K_X - P) &= \deg(K_X - P) + 1 - g + h^0(X, P) \\ &= \deg(K_X) - \deg(P) + 1 - g + h^0(X, P) = \mathrm{h}^0(X, K_X) - 1. \end{aligned}$$

By Remark 1.53, K_X is basepoint free. □

All together, we can assert that the Riemann-Roch Theorem indeed is a very helpful tool. As the next chapter will show, we can also deduce some interesting applications from it.

Chapter 4

Applications of the Riemann-Roch Theorem

In this final chapter, we want to consider some applications of the Riemann-Roch Theorem.

4.1 Clifford's Theorem

We begin with Clifford's Theorem, named after William K. Clifford. Roughly speaking, this theorem gives another, more concrete, upper bound for the dimension $h^0(X, D)$ of the Riemann-Roch space $H^0(X, D)$, which is only dependent on the degree of the divisor.

Theorem 4.1 (Clifford's Theorem)**.** *Let X be a nonsingular projective curve and let K_X be a canonical divisor on X. Let D be a divisor on X with $h^0(X, D) > 0$ and $h^0(X, K_X - D) > 0$. Then*

$$h^0(X, D) \leq \frac{1}{2}\deg(D) + 1.$$

Proof. First, we may assume that D is effective and that $K_X = D + D'$ for $D' \in \mathrm{Div}(X)$ and $D' \geq 0$. Furthermore, assume that $h^0(X, D-P) < h^0(X, D)$ for all $P \in X$. Otherwise, the proof works the same and we even get a better inequality.

Let $f \in H^0(X, D)$ with $f \notin H^0(X, D-P)$ for each $P \in \mathrm{Supp}(D')$. Consider the linear map

$$\varphi\colon H^0(X, D')/H^0(X, 0) \to H^0(X, K_X)/H^0(X, D), \ [h] \mapsto [hf],$$

which is well-defined, since $H^0(X, 0) \simeq \mathbb{C}$.

Claim: The map φ is injective.

Proof of claim. Assume that $\varphi([h]) = 0$. Then $hf \in H^0(X, D)$, i.e.

$$0 \leq \mathrm{div}(hf) + D = \mathrm{div}(h) + \mathrm{div}(f) + D.$$

We need to show that $h \in H^0(X,0) \simeq \mathbb{C}$. Assume that $h \notin \mathbb{C}$. Then $h \in H^0(X,D')$ implies that there exists a point $P \in \operatorname{Supp}(D')$ such that $\nu_P(h) < 0$. Now,

$$0 \leq \operatorname{div}(f) + \operatorname{div}(h) + D = \operatorname{div}(f) + D - P + \operatorname{div}(h) + P$$

and this can only be true if $f \in H^0(X, D-P)$, contradicting the assumption of f. Hence, the claim follows. □

Thus,

$$\begin{aligned}\dim_{\mathbb{C}}(H^0(X,D')/H^0(X,0)) = h^0(X,D') - 1 &\leq g - h^0(X,D)\\ &= \dim_{\mathbb{C}}(H^0(X,K_X)/H^0(X,D)).\end{aligned}$$

Applying the Riemann-Roch Theorem to $D' = K_X - D$, we obtain

$$h^0(X, K_X - D) - 1 = \deg(K_X - D) + 1 - g + h^0(X,D) - 1 \leq g - h^0(X,D).$$

In particular,

$$\begin{aligned}&\deg(K_X) - \deg(D) - g + h^0(X,D) \leq g - h^0(X,D)\\ \Longleftrightarrow\ &2g - 2 - g - g + 2h^0(X,D) \leq \deg(D)\\ \Longleftrightarrow\ &h^0(X,D) \leq \frac{1}{2}\deg(D) + 1.\end{aligned}$$

This completes the proof. □

4.2 Hurwitz' Formula

In this section, we consider Hurwitz' Formula, named after Adolf Hurwitz. First, we introduce some basic definitions, which are needed to state the formula. We start by defining the notion of the *degree of a morphism*.

Definition 4.2. Let X_1, X_2 be two nonsingular projective curves and let $\varphi\colon X_1 \to X_2$ be a finite morphism of curves. Then φ induces a finite extension $K(X_2) \subset K(X_1)$ of function fields. We define the **degree** of φ to be

$$\deg(\varphi) := [K(X_1) : K(X_2)].$$

Remark 4.3. Note that the extension $K(X_2) \subset K(X_1)$ is separable, since the characteristic of $\mathbb{C}$ is zero.

The following definition introduces the so-called *ramification divisor*. As we will see, this is an essential ingredient in Hurwitz' Formula.

Definition 4.4 (Ramification)**.** Let X_1 and X_2 be two nonsingular projective. Let $\varphi\colon X_1 \to X_2$ be a finite morphism curves. Let $P \in X_1$ be a point and let $Q := \varphi(P)$. Let $f \in K(X_2)$. We may identify f as an element of $K(X_1)$ via the map $f \mapsto f \circ \varphi$. We define $\nu_P(f) := \nu_P(f \circ \varphi)$, where ν_P is the discrete valuation associated to $\mathcal{O}_{X_1,P}$. We have the natural ring homomorphism

$$\varphi^{\#}\colon \mathcal{O}_{X_2,Q} \to \mathcal{O}_{X_1,P},$$
$$h \mapsto h \circ \varphi.$$

If $t \in \mathcal{O}_{X_2,Q}$ is a local parameter at Q, we define $\nu_P(t) := \nu_P(t \circ \varphi)$. We call $e_P := \nu_P(t)$ the **ramification index** of P. If $e_P > 1$, we say that φ is **ramified** at P and that Q is a **branch point** of φ. If $e_P = 1$, we say that φ is **unramified** at P. For $dt \in \Omega_{\mathbb{C}}(K(X_2))$, we define $\nu_P(dt) := \nu_P(d(t \circ \varphi))$. Note that $\nu_P(dt) = e_P - 1$. We call

$$R := \sum_{P \in X_1} \nu_P(dt) \cdot P = \sum_{P \in X_1} (e_P - 1) \cdot P$$

the **ramification divisor** of φ and call a point $P \in \operatorname{Supp}(R)$ a **ramification point**.

The next definition shows, how we can *pull back* divisors on X_2 to divisors on X_1. We will do it pointwise and then extend linearly.

Definition 4.5. Let X_1 and X_2 be two nonsingular projective curves. Let $\varphi\colon X_1 \to X_2$ be a finite morphism. Then, for any point $Q \in X_2$, we define

$$\varphi^*\colon \operatorname{Div}(X_2) \to \operatorname{Div}(X_1) \text{ by } \varphi^*(Q) := \sum_{P \in \varphi^{-1}(Q)} e_p \cdot P.$$

By extending φ^* linearly, we obtain a homomorphism, which we call the **pullback of divisors** on X_2 to X_1.

The next proposition is crucial for the proof of Hurwitz' Formula.

Proposition 4.6. *Let X_1 and X_2 be two nonsingular projective curves. Let $\varphi\colon X_1 \to X_2$ be a finite morphism. Let K_{X_1} be the canonical divisor on X_1 and let K_{X_2} be the canonical divisor on X_2. Then*

$$K_{X_1} \sim \varphi^* K_{X_2} + R.$$

Proof. Let $\omega \in \Omega_{\mathbb{C}}(K(X_2))$. We define the pullback $\varphi^*(\omega) \in \Omega_{\mathbb{C}}(K(X_1))$ as follows: For $P \in X_1$, let $Q := \varphi(P)$. Choose a local parameter $t \in \mathcal{O}_{X_2,Q}$ such that $\omega = f dt$ for $f \in K(X_2)$. Now, define

$$\varphi^*(\omega) := (f \circ \varphi) d(t \circ \varphi).$$

Note that $\operatorname{div}(\varphi^*(\omega)) \sim K_{X_1}$. To obtain the result, we compare the coefficients of $\operatorname{div}(\varphi^*(\omega))$ with those of $\varphi^*(K_{X_2}) + R$.

By Proposition 3.8, $|K_{X_2}|$ is basepoint free, so we may choose ω such that the support of K_{X_2} does not contain any branch points. In particular, if $K_{X_2} = \sum_{Q\in X_2} \mu_Q \cdot Q$, then

$$\varphi^*(K_{X_2}) = \sum_{Q\in X_2} \sum_{P\in\varphi^{-1}(Q)} \mu_Q \cdot P \tag{4.1}$$

Let P be any point on X_1 and let $Q := \varphi(P)$. We have to distinguish between three cases.

Case 1: $P \in \operatorname{Supp}(R)$. Then Q is a branch point, so by assumption on K_{X_2}, $Q \notin \operatorname{Supp}(K_{X_2})$. In particular, if $\omega = f dt$, then $\nu_Q(f) = 0$, i.e. f is a unit in $\mathcal{O}_{X_2,Q}$. Consequently, $f \circ \varphi$ is a unit in $\mathcal{O}_{X_1,P}$, so $\nu_P(f) = 0$. Since Q is a branch point, $\nu_P(t) = e_P > 1$ and consequently $\nu_P(dt) = e_P - 1 > 0$. Now

$$\begin{aligned} \nu_P(\varphi^*(\omega)) &= \nu_P(f\circ\varphi) + \nu_P(d(t\circ\varphi)) \\ &= \nu_P(f) + \nu_P(dt) \\ &= e_P - 1. \end{aligned}$$

Consequently, for each ramification point $P \in \operatorname{Supp}(R)$, $\operatorname{div}(\varphi^*(\omega))$ has the same coefficient $e_P - 1$ as R.

Case 2: $P \notin \operatorname{Supp}(R)$ and $Q \notin \operatorname{Supp}(K_{X_2})$. Then $\nu_P(dt) = e_P - 1$. Furthermore, since $Q \notin \operatorname{Supp}(K_{X_2})$, $\nu_P(f) = 0$ by the same argument as in case 1. We obtain

$$\nu_P(\varphi^*(\omega)) = \nu_P(f) + \nu_P(dt) = 0.$$

Case 3: $P \notin \operatorname{Supp}(R)$ and $Q \in \operatorname{Supp}(K_{X_2})$. Then $\nu_P(dt) = e_P - 1 = 0$. Let $\omega = f dt$. Then $\nu_Q(f)\mu_Q \neq 0$ by assumption. On the one hand, if $\mu_Q > 0$, then f has a zero of order μ_Q at Q. Consequently, the rational function $f \circ \varphi \in K(X_1)$ has a zero of order μ_Q at each point in $\varphi^{-1}(Q)$. On the other hand, if $\mu_Q < 0$, then f has a pole of order $-\mu_Q$ at Q. Consequently, $f\circ\varphi$ has a pole of order $-\mu_Q$ at each point in $\varphi^{-1}(Q)$. Thus, we obtain that

$$\nu_P(f) = \nu_P(f\circ\varphi) = \nu_Q(f) = \mu_Q.$$

Hence,

$$\nu_P(\varphi^*(\omega)) = \nu_P(f) + \nu_P(dt) = \mu_Q$$

for each point $P \in \varphi^{-1}(Q)$. Comparing with (4.1), we see that for each point $P \in \varphi^{-1}(Q)$, $\operatorname{div}(\varphi^*(\omega))$ has the same coefficient μ_Q as $\varphi^*(K_{X_2})$.

Putting everything together, we see that $\operatorname{div}(\varphi^*(\omega))$ has coefficient

- zero for each point $P \notin \operatorname{Supp}(R)$ and $Q \notin \operatorname{Supp}(K_{X_2})$,
- $e_P - 1$ for each point $P \in \operatorname{Supp}(R)$,
- μ_Q for each point $P \in \varphi^{-1}(Q),\ Q \in X_2$.

This yields $\operatorname{div}(\varphi^*(\omega)) = \varphi^*(K_{X_2}) + R$ and the claim follows. □

We just need one more result to finally state and prove Hurwitz' Formula. Since the proof does not utilise any of the results of this thesis, we only provide a reference.

Proposition 4.7. *Let X_1 and X_2 be two nonsingular projective curves. Let $\varphi\colon X_1 \to X_2$ be a finite morphism. Then for any divisor $D \in \operatorname{Div}(X_2)$, we have that*

$$\deg(\varphi^* D) = \deg(\varphi)\deg(D).$$

Proof. See [Har77, Chapter 2, Proposition 6.9]. □

Theorem 4.8 (Hurwitz' Formula)**.** *Let X_1 and X_2 be two nonsingular projective curves of genus g_{X_1}, g_{X_2} respectively. Furthermore, let $\varphi\colon X_1 \to X_2$ be a finite morphism of degree $n \in \mathbb{N}$. Then*

$$2g_{X_1} - 2 = n(2g_{X_2} - 2) + \deg(R).$$

Proof. We want to use Proposition 4.6 and Proposition 4.7. Since we have $K_{X_1} \sim \varphi^* K_{X_2} + R$, it follows that

$$\begin{aligned} 2g_{X_2} - 2 = \deg(K_{X_1}) &= \deg(\varphi^* K_{X_2} + R) \\ &= n\deg(K_{X_2}) + \deg(R) \\ &= n(2g_{X_2} - 2) + \deg(R). \end{aligned}$$

This finishes the proof. □

As a direct implication of Hurwitz' Formula, we obtain the following.

Corollary 4.9. *Let X_1 and X_2 be two nonsingular projective curves. Let $\varphi\colon X_1 \to X_2$ be a finite morphism. Then the degree of the ramification divisor R is an even number.*

Proof. This follows directly from Hurwitz' Formula, since

$$2 \mid n(2g_{X_2} - 2) + \deg(R).$$

Thus, in particular, 2 has to divide $\deg(R)$. □

Finally, we want to use Hurwitz' Formula to state one more corollary, which we then use to give a short proof of Lüroth's Theorem.

Corollary 4.10. *Let $\varphi\colon X_1 \to X_2$ be a finite morphism of nonsingular projective curves. Let g_{X_1} be the genus of X_1 and g_{X_2} be the genus of X_2. Then $g_{X_1} \geq g_{X_2}$.*

Proof. For $n = 1$ and $\deg(R) = 0$, Hurwitz' Formula yields $g_{X_1} = g_{X_2}$. Since $\deg(R) \geq 0$ and $n \geq 1$, the claim follows. □

As the final result of this thesis, we want to state and prove Lüroth's Theorem, named after Jacob Lüroth. It states that every field that lies between $\mathbb{C}$ and a pure transcendental extension $\mathbb{C}(x)$ of $\mathbb{C}$ must be generated as a pure transcendental extension of $\mathbb{C}$ by a single element of $\mathbb{C}(x)$.

Theorem 4.11 (Lüroth's Theorem)**.** *Let L be field, such that $\mathbb{C} \subset L \subset \mathbb{C}(x)$, where x is transcendental over $\mathbb{C}$. Then $L = \mathbb{C}(y)$ for some $y \in \mathbb{C}(x)$.*

Proof. We may assume that $L \neq \mathbb{C}$. Then L has transcendence degree one over $\mathbb{C}$. Consequently, L is a function field of a nonsingular projective curve X and the inclusion $L \subset \mathbb{C}(x)$ corresponds to a finite morphism $\varphi\colon \mathbb{P}^1 \to X$. By Corollary 4.10, we have $g_X \leq g_{\mathbb{P}^1} = 0$, forcing $g_X = 0$.

Hence, by [Har77, Chapter 4, Example 1.3.5.], X is birational to $\mathbb{P}^1$, so $L \simeq \mathbb{C}(y)$ for some $y \in \mathbb{C}(x)$ and we are done. □

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