FYS4150 - Computational Physics - Project 2

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Abstract

This report looks at different algorithms to solve the one-dimensional Poisson's equation with Dirichlet boundary condition. The algorithms are compared by speed and floating point error in the solution. The results show how different ways of writing code effect speed and floating point error, and discusses how the machines architecture and functionally are the reason for the performance difference.

1 Theory

Side 24

$$j\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$$

$$\Psi\left(\vec{x},t\right) = \psi\left(\vec{x}\right)\varphi\left(t\right)$$

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V\psi = E\psi$$

Side 133

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi}\right] + V\psi = E\psi$$

$$\psi\left(r,\theta,\phi\right) = R\left(r\right)Y\left(\theta,\phi\right) + C$$

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{2mr^2}{\hbar^2}\left(V-E\right) + \frac{1}{Y}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right) = \frac{2mr^2C}{RY\hbar^2}\left(V-E\right)$$

2 Eigenvalue algorithms

2.1 Diagonalize by matrix similarity

If we have a eigenvalue problem for a matrix $\mathbb{A}$ with eigenvector $\mathbf{x}$ and eigenvalue λ

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}\,,$$

we can transform it by a basis matrix $\mathbf{S}$ to a similar matrix $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ with the same eigenvalue λ but with a different eigenvector $\mathbf{S}^{-1}\mathbf{x}$;

$$\lambda\mathbf{S}^{-1}\mathbf{x} = \mathbf{S}^{-1}\mathbf{A}\mathbf{x} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{S}^{-1}\mathbf{x} = \mathbf{B}\mathbf{S}^{-1}\mathbf{x}\,.$$

If we are able to diagonalize out matrix $\mathbf{A}$ the eigenvalue problem becomes trivial to solve. However to directly diagonalize matrix $\mathbf{A}$ of size $n \times n$ requires to solve a n-th polynomial problem, which is a very hard task. So we choose a easier way by iteratively zero out more and more non-diagonal elements of matrix $\mathbf{A}$, which will eventually give us the diagonal matrix

$$\mathbf{D} = \prod_i \mathbf{S}_i^{-1}\mathbf{A}\prod_i \mathbf{S}_i\,.$$

To make the problem simple to solve we define a basis matrix for transformation as such

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{ii} & \cdots & s_{ij} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{ji} & \cdots & s_{jj} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}, \tag{1}$$

where the strategy is for this basis matrix $\mathbf{S}$ will make $b_{ij} = b_{ji} = 0$ when we do a matrix similarity transformation of $\mathbf{A}$ to $\mathbf{B}$. The matrix elements of $\mathbf{S}^{-1}$ is then given by

$$s^{-}_{k\ell} = \begin{cases} \frac{s_{jj}}{s_{ii}s_{jj}-s_{ij}s_{ji}} & \text{for } k = \ell = i \\ -\frac{s_{ij}}{s_{ii}s_{jj}-s_{ij}s_{ji}} & \text{for } k = i \text{ and } \ell = j \\ -\frac{s_{ji}}{s_{ii}s_{jj}-s_{ij}s_{ji}} & \text{for } k = j \text{ and } \ell = i \\ \frac{s_{ii}}{s_{ii}s_{jj}-s_{ij}s_{ji}} & \text{for } k = \ell = j \\ 1 & \text{for } k = \ell \in \mathbb{N}_1^n / \left\{i, j\right\} \\ 0 & \text{elsewhere.} \end{cases}$$

To see how this matrix similarity transformation behaves we must do the cumbersome matrix multiplication of the transformation;

$$\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{ii}^{-} & \cdots & s_{ij}^{-} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{ji}^{-} & \cdots & s_{jj}^{-} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{ji} & \cdots & a_{jj} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \mathbf{S}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{ii}^{-}a_{i1} + s_{ij}^{-}a_{j1} & s_{ii}^{-}a_{i2} + s_{ij}^{-}a_{j2} & \cdots & s_{ii}^{-}a_{ii} + s_{ij}^{-}a_{ji} & \cdots & s_{ii}^{-}a_{ij} + s_{ij}^{-}a_{jj} & \cdots & s_{ii}^{-}a_{in} + s_{ij}^{-}a_{jn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{ji}^{-}a_{i1} + s_{jj}^{-}a_{j1} & s_{ji}^{-}a_{i2} + s_{jj}^{-}a_{j2} & \cdots & s_{ji}^{-}a_{ii} + s_{jj}^{-}a_{ji} & \cdots & s_{ji}^{-}a_{ij} + s_{jj}^{-}a_{jj} & \cdots & s_{ji}^{-}a_{in} + s_{jj}^{-}a_{jn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \mathbf{S}$$

$$= \begin{bmatrix} a_{11}^{-} & a_{12}^{-} & \cdots & a_{1i}^{-} & \cdots & a_{1j}^{-} & \cdots & a_{1n}^{-} \\ a_{21}^{-} & a_{22}^{-} & \cdots & a_{2i}^{-} & \cdots & a_{2j}^{-} & \cdots & a_{2n}^{-} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1}^{-} & a_{i2}^{-} & \cdots & a_{ii}^{-} & \cdots & a_{ij}^{-} & \cdots & a_{in}^{-} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j1}^{-} & a_{j2}^{-} & \cdots & a_{ji}^{-} & \cdots & a_{jj}^{-} & \cdots & a_{jn}^{-} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1}^{-} & a_{n2}^{-} & \cdots & a_{ni}^{-} & \cdots & a_{nj}^{-} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{ii} & \cdots & s_{ij} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{ji} & \cdots & s_{jj} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^{-} & a_{12}^{-} & \cdots & a_{1i}^{-}s_{ii} + a_{1j}^{-}s_{ji} & \cdots & a_{1i}^{-}s_{ij} + a_{1j}^{-}s_{jj} & \cdots & a_{1n}^{-} \\ a_{21}^{-} & a_{22}^{-} & \cdots & a_{2i}^{-}s_{ii} + a_{2j}^{-}s_{ji} & \cdots & a_{2i}^{-}s_{ij} + a_{2j}^{-}s_{jj} & \cdots & a_{2n}^{-} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1}^{-} & a_{i2}^{-} & \cdots & a_{ii}^{-}s_{ii} + a_{ij}^{-}s_{ji} & \cdots & a_{ii}^{-}s_{ij} + a_{ij}^{-}s_{jj} & \cdots & a_{in}^{-} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j1}^{-} & a_{j2}^{-} & \cdots & a_{ji}^{-}s_{ii} + a_{jj}^{-}s_{ji} & \cdots & a_{ji}^{-}s_{ij} + a_{jj}^{-}s_{jj} & \cdots & a_{jn}^{-} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1}^{-} & a_{n2}^{-} & \cdots & a_{ni}^{-}s_{ii} + a_{nj}^{-}s_{ji} & \cdots & a_{ni}^{-}s_{ij} + a_{nj}^{-}s_{jj} & \cdots & a_{nn}^{-} \end{bmatrix},$$

which gives us the matrix elements of the similarity matrix $\mathbf{B}$

$$b_{k\ell} = \begin{cases} \bar{s_{ii}} a_{ii} s_{ii} + \bar{s_{ij}} a_{ji} s_{ii} + \bar{s_{ii}} a_{ij} s_{ji} + \bar{s_{ij}} a_{jj} s_{ji} & \text{for } k = \ell = i \\ \bar{s_{ii}} a_{ii} s_{ij} + \bar{s_{ij}} a_{ji} s_{ij} + \bar{s_{ii}} a_{ij} s_{jj} + \bar{s_{ij}} a_{jj} s_{jj} & \text{for } k = i \text{ and } \ell = j \\ \bar{s_{ji}} a_{ii} s_{ii} + \bar{s_{jj}} a_{ji} s_{ii} + \bar{s_{ji}} a_{ij} s_{ji} + \bar{s_{jj}} a_{jj} s_{ji} & \text{for } k = j \text{ and } \ell = i \\ \bar{s_{ji}} a_{ii} s_{ij} + \bar{s_{jj}} a_{ji} s_{ij} + \bar{s_{ji}} a_{ij} s_{jj} + \bar{s_{jj}} a_{jj} s_{jj} & \text{for } k = \ell = j \\ \bar{s_{ki}} a_{i\ell} + \bar{s_{kj}} a_{j\ell} & \text{for } k \in \{i, j\} \text{ and } \ell \in \mathbb{N}_1^n / \{i, j\} \\ a_{ki} s_{i\ell} + a_{kj} s_{j\ell} & \text{for } \ell \in \{i, j\} \text{ and } k \in \mathbb{N}_1^n / \{i, j\} \\ a_{k\ell} & \text{elsewhere.} \end{cases} \tag{2}$$

What we want our matrix similarity transformation to do is to set $b_{ij} = b_{ji} = 0$, but we also need to ensure that the basis matrix $\mathbf{S}$ for the transformation is invertible. Hence we have the following three equation we want to solve

$$s_{ii} s_{jj} - s_{ij} s_{ji} = 1 \tag{3}$$

$$a_{ji} {s_{ij}}^2 - \left(a_{ii} - a_{jj}\right) s_{jj} s_{ij} - a_{ij} {s_{jj}}^2 = 0 \tag{4}$$

$$a_{ij} {s_{ji}}^2 - \left(a_{jj} - a_{ii}\right) s_{ii} s_{ji} - a_{ji} {s_{ii}}^2 = 0\,. \tag{5}$$

If the matrix $\mathbf{A}$ is indeed diagonalizable then the above three equations above solvable. We further observer that we have four unknowns in the three equations, which means that we can choose one of them as we want, I suggest $s_{ii} = 1$. I will not show the solution here, but it involves in principle in two second order equations $ax^2 + bx + c = 0$ which has a well known analytical solution $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Be aware of that the coefficient a may be zero which would make it a first order equations to solve instead. If the coefficient $b = 0$ in addition to $a = 0$, then the matrix $\mathbf{A}$ is not diagonalizable.

There are on last thing we need to consider regarding the diagonalizing algorithm, and that is that the matrix elements b_{ix}, b_{jx}, b_{xi} and b_{xj} for all $x \in \mathbb{N}_1^n / \{i, j\}$ may change from zero to a value if we are not careful. Which may undermine the work we try to do when we diagonalize the matrix. What we see in (2) is if we lock k to i and let ℓ go through all possibilities $\ell \in \mathbb{N}_1^n / \{k\}$ exactly once, we ensure that the values we have zeroed out for $k = i$ does not change when we do successive transformation with the same $k = i$;

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \sim \begin{bmatrix} b_{11}^{(1)} & b_{12}^{(1)} & \cdots & 0 \\ b_{21}^{(1)} & b_{22}^{(1)} & \cdots & b_{2n}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2}^{(1)} & \cdots & b_{nn}^{(1)} \end{bmatrix} \sim \begin{bmatrix} b_{11}^{(n-1)} & 0 & \cdots & 0 \\ 0 & b_{22}^{(n-1)} & \cdots & b_{2n}^{(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2}^{(n-1)} & \cdots & b_{nn}^{(n-1)} \end{bmatrix} \to \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Note that we have now transformed such that $a_{ki} = a_{ii}\delta_{ik} = a_{ki}$, where δ_{ik} is the Kronecker delta function. Now we change indices to $i' \neq i$ and let $j \in \mathbb{N}_1^n / \{i, i'\}$, because we are done with the row and column i. What we now realize in (2) is that the elements of matrix $\mathbb{B}$ in the row or column i is given by $b_{i\ell} = a_{ii'} s_{i'\ell} + a_{ij} s_{j\ell} = 0$ because $k \notin \{i', j\}$ and $i' \neq i \neq j$ which means that $a_{ii'} = 0$ and $a_{ij} = 0$, and $b_{ki} = \bar{s_{ki'}} a_{i'i} + \bar{s_{kj}} a_{ji} = 0$ because $\ell \notin \{i', j\}$ and $i' \neq i \neq j$ which means that $a_{i'i} = 0$ and $a_{ji} = 0$. In other words the matrix elements in row or column i that was zeroed out does not change when we continue to zero out matrix elements on row and column i', which means that we can continue the iteration until the matrix $\mathbf{A}$ is completely diagonalized;

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \sim \begin{bmatrix} b_{11}^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & b_{22}^{(1)} & b_{23}^{(1)} & \cdots & 0 \\ 0 & b_{32}^{(1)} & b_{33}^{(1)} & \cdots & b_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & b_{n3}^{(1)} & \cdots & b_{nn}^{(1)} \end{bmatrix} \sim \begin{bmatrix} b_{11}^{(n-2)} & 0 & 0 & \cdots & 0 \\ 0 & b_{22}^{(n-2)} & 0 & \cdots & 0 \\ 0 & 0 & b_{33}^{(n-2)} & \cdots & b_{3n}^{(n-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & b_{n3}^{(n-2)} & \cdots & b_{nn}^{(n-2)} \end{bmatrix}$$

$$\sim \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

2.2 Jacobi's eigenvalue algorithm

The Jacobi's eigenvalue algorithm is a special case of diagonalize by matrix similarity algorithm discussed in section 2.1, where we use a two dimensional rotation matrix in n dimensional space

$$\mathbf{R}_{i,j,\theta} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{ii} = \cos\theta & \cdots & r_{ij} = -\sin\theta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{ji} = \sin\theta & \cdots & r_{jj} = \cos\theta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}, \tag{6}$$

which satisfies the basis matrix $\mathbf{S}$ in (1). The rotation matrix can easily be shown to be a orthogonal matrix by realizing $\mathbf{R}_{-\theta} = \mathbf{R}_{\theta}{}^{-1} = \mathbf{R}_{\theta}{}^{\mathrm{T}}$, because when we rotate an angle θ, we need to rotate an angle $-\theta$ to get back (or you could realize this by a more cumbersome way by actually do the matrix inversion).

By doing the matrix similarity transformation $\mathbf{B} = \mathbf{R}_{i,j,-\theta}\mathbf{A}\mathbf{R}_{i,j,\theta}$, we get the following similar matrix elements according to (2)

$$b_{k\ell} = \begin{cases} a_{ii}\cos^2\theta + a_{jj}\sin^2\theta + \left(a_{ij} + a_{ji}\right)\sin\theta\cos\theta & \text{for } k = \ell = i \\ a_{ij}\cos^2\theta - a_{ji}\sin^2\theta + \left(a_{jj} - a_{ii}\right)\sin\theta\cos\theta & \text{for } k = i \text{ and } \ell = j \\ a_{ji}\cos^2\theta - a_{ij}\sin^2\theta + \left(a_{jj} - a_{ii}\right)\sin\theta\cos\theta & \text{for } k = j \text{ and } \ell = i \\ a_{jj}\cos^2\theta + a_{ii}\sin^2\theta - \left(a_{ij} + a_{ji}\right)\sin\theta\cos\theta & \text{for } k = \ell = j \\ a_{i\ell}\cos\theta + a_{j\ell}\sin\theta & \text{for } k = i \text{ and } \ell \in \mathbb{N}_1^n / \left\{i, j\right\} \\ a_{j\ell}\cos\theta - a_{i\ell}\sin\theta & \text{for } k = j \text{ and } \ell \in \mathbb{N}_1^n / \left\{i, j\right\} \\ a_{ki}\cos\theta + a_{kj}\sin\theta & \text{for } \ell = i \text{ and } k \in \mathbb{N}_1^n / \left\{i, j\right\} \\ a_{kj}\cos\theta - a_{ki}\sin\theta & \text{for } \ell = j \text{ and } k \in \mathbb{N}_1^n / \left\{i, j\right\} \\ a_{k\ell} & \text{elsewhere.} \end{cases}$$

The strategy is to set b_{ij} and b_{ji} to zero, which enables us by iteration to zero out every element in the matrix except the diagonal elements (as described in section 2.1). Unfortunately we have only one variable, namely θ, which makes it impossible to diagonalize a general matrix $\mathbf{A}$ by similarity. However when $\mathbf{A}$ is a symmetric matrix, $a_{ij} = a_{ji}$, which gives us only one equation to solve to set both b_{ij} and b_{ji} to zero. For a symmetric matrix $\mathbf{A}$ we have the following elements of the similarity matrix $\mathbf{B}$

$$b_{k\ell} = \begin{cases} a_{ii}\cos^2\theta + a_{jj}\sin^2\theta + 2a_{ij}\sin\theta\cos\theta & \text{for } k = \ell = i \\ a_{ij}\left(\cos^2\theta - \sin^2\theta\right) + \left(a_{jj} - a_{ii}\right)\sin\theta\cos\theta & \text{for } k = i \text{ and } \ell = j, \text{ or } k = j \text{ and } \ell = i \\ a_{jj}\cos^2\theta + a_{ii}\sin^2\theta - 2a_{ij}\sin\theta\cos\theta & \text{for } k = \ell = j \\ a_{i\ell}\cos\theta + a_{j\ell}\sin\theta & \text{for } k = i \text{ and } \ell \in \mathbb{N}_1^n / \left\{i, j\right\} \\ a_{j\ell}\cos\theta - a_{i\ell}\sin\theta & \text{for } k = j \text{ and } \ell \in \mathbb{N}_1^n / \left\{i, j\right\} \\ a_{ki}\cos\theta + a_{kj}\sin\theta & \text{for } \ell = i \text{ and } k \in \mathbb{N}_1^n / \left\{i, j\right\} \\ a_{kj}\cos\theta - a_{ki}\sin\theta & \text{for } \ell = j \text{ and } k \in \mathbb{N}_1^n / \left\{i, j\right\} \\ a_{k\ell} & \text{elsewhere.} \end{cases} \tag{7}$$

Now we are able to solve the following equation

$$b_{ij} = b_{ji} = a_{ij}\left(\cos^2\theta - \sin^2\theta\right) + \left(a_{jj} - a_{ii}\right)\sin\theta\cos\theta = 0\,,$$

and by using the trigonometric relation $\cos^2\theta + \sin^2\theta = 1$ we can rewrite to

$$\cos^4\theta - \cos^2\theta + \frac{{a_{ij}}^2}{\left(a_{ii} - a_{jj}\right)^2 + 4{a_{ij}}^2} = 1\,.$$

By solving this as a second order equation with regard to $\cos^2\theta$ and using the trigonometric relation $\cos^2\theta + \sin^2\theta = 1$ again, we get

$$\theta = \arccos\sqrt{\frac{1}{2}\left(1 + \sqrt{\frac{\left(a_{ii} - a_{jj}\right)^2}{\left(a_{ii} - a_{jj}\right)^2 + 4{a_{ij}}^2}}\right)} = \arcsin\sqrt{\frac{1}{2}\left(1 - \sqrt{\frac{\left(a_{ii} - a_{jj}\right)^2}{\left(a_{ii} - a_{jj}\right)^2 + 4{a_{ij}}^2}}\right)}. \tag{8}$$

We can diagonalize the matrix $\mathbf{A}$ by doing matrix similarity transformation iteratively as described in section 2.1. However we can optimize Jacobi's eigenvalue algorithm by using the Frobenius norm.

2.3 QR algorithm and tridiagonal matrices

We want to find the eigenvalues of a tridiagonal symmetric matrix $\mathbf{A}_0$ of $n \times n$ which do QR decomposition on

$$\mathbf{A}_0 = \mathbf{S}_0\mathbf{U}_0\,,$$

where S_0 is the orthogonal matrix $\mathbf{Q}$ and $\mathbf{U}_0$ is the upper triangle matrix $\mathbf{R}$ in the QR decomposition. We can show that the similar matrix $\mathbf{A}_1$ to $\mathbf{A}_0$ is linked to the QR decomposition as follows

$$\mathbf{A}_1 = \mathbf{S}_0^{\mathrm{T}}\mathbf{A}_0\mathbf{S}_0 = \mathbf{S}_0^{\mathrm{T}}\mathbf{S}_0\mathbf{U}_0\mathbf{S}_0 = \mathbf{U}_0\mathbf{S}_0\,,$$

where I have used the orthogonal property of $\mathbf{S}_0$ such that $\mathbf{S}_0^{\mathrm{T}}\mathbf{S}_0 = \mathbf{I}$ ($\mathbf{I}$ is the identity martrix). Now imagine that we use QR decomposition on $\mathbf{A}_1$ and find the similarity matrix $\mathbf{A}_2$ in the same manner. We can now show by induction that this process for step $i+1$ by using the QR decomposition on $\mathbf{A}_i$;

$$\mathbf{A}_i = \mathbf{S}_i\mathbf{U}_i\,,$$

and we now find the similarity matrix $\mathbf{A}_{i+1}$ to $\mathbf{A}_i$ as follows

$$\mathbf{A}_{i+1} = \mathbf{S}_i^{\mathrm{T}}\mathbf{A}_i\mathbf{S}_i = \mathbf{S}_i^{\mathrm{T}}\mathbf{S}_i\mathbf{U}_i\mathbf{S}_i = \mathbf{U}_i\mathbf{S}_i\,.$$

We assume that $\mathbf{A}_i$ is a symmetric tridiagonal matrix with the elements

$$a_{i,k,\ell} = \begin{cases} a_{i,k,k+|\,\ell-k\,|} & \text{when } \ell \in \mathbb{N}_{k-1}^{k+1} \\ 0 & \text{otherwise,} \end{cases}$$

which I will show by induction. To find the upper matrix we use successively rotation matrices $\mathbf{R}_{i,j}$ to eliminate the elements in the lower triangle part of the tridiagonal matrix $\mathbf{A}_i$

$$\mathbf{U}_i = \prod_{j=1}^{n-1} \mathbf{R}_{i,j}^{\mathrm{T}}\mathbf{A}_i\,,$$

where the rotation matrix $\mathbf{R}_{i,j}$ has the elements

$$r_{i,j,k,\ell} = \begin{cases} \cos\theta_{i,j} & \text{when } j = k = \ell \\ -\sin\theta_{i,j} & \text{when } j = k = \ell - 1 \\ \sin\theta_{i,j} & \text{when } j = k - 1 = \ell \\ \cos\theta_{i,j} & \text{when } j = k - 1 = \ell - 1 \\ 1 & \text{when } k = \ell \in \mathbb{N}_1^n/\mathbb{N}_j^{j+1} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the rotation matrix is orthogonal which means that the elements of $\mathbf{R}_{i,j}^{\mathrm{T}}$ is given by $r_{i,j,k,\ell}^{\mathrm{T}} = r_{i,j,\ell,k}$. I define the matrix

$$\mathbf{A}_{i,j+1} = \begin{cases} \mathbf{A}_i & \text{when } j = 0 \\ \mathbf{R}_{i,j}^{\mathrm{T}}\mathbf{A}_{i,j} & \text{when } j \in \mathbb{N}_1^{n-1}\,, \end{cases}$$

which means that we are going to show that $\mathbf{U}_i = \mathbf{A}_{i,n}$, where the matrix $\mathbf{A}_{i,j}$ has the elements $a_{i,j,k\ell}$. I start by doing the matrix multiplication $\mathbf{A}_{i,2} = \mathbf{R}_{i,1}^{\mathrm{T}}\mathbf{A}_{i,1}$;

$$a_{i,2,k,\ell} = \sum_{m=1}^{n} r^{\mathrm{T}}_{i,1,k,m} a_{i,1,m,\ell} = \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,1,k,\ell+m} a_{i,1,\ell+m,\ell} = \begin{cases} \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,1,k,\ell+m} a_{i,1,\ell+m,\ell} & \text{when } (k, \ell+m) \in {}^2\mathbb{N}_1^2 \\ a_{i,1,k,\ell} & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,1,k,\ell+m} a_{i,1,\ell+m,\ell} & \text{when } (k, \ell+m) \in {}^2\mathbb{N}_1^2 \text{ and } \ell \in \mathbb{N}_1^3 \\ a_{i,1,k,\ell} & \text{otherwise.} \end{cases}$$

The notation ${}^2\mathbb{N}_1^2 = \mathbb{N}_1^2 \times \mathbb{N}_1^2$, which means all the permutations of the natural number from 1 to 2; $(1,1)$, $(1,2)$, $(2,1)$ and $(2,2)$. We want our transformation to zero out the lower triangle element $(2,1)$

$$a_{i,2,2,1} = \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,1,2,m+1} a_{i,1,m+1,1} = \sum_{m=0}^{1} r^{\mathrm{T}}_{i,1,2,m+1} a_{i,1,m+1,1} = 0\,,$$

which means that

$$a_{i,2,k,\ell} = \begin{cases} \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,1,k,\ell+m} a_{i,1,\ell+m,\ell} & \text{when } (k, \ell+m) \in {}^2\mathbb{N}_1^2, (k,\ell) \neq (2,1) \text{ and } \ell \in \mathbb{N}_1^3 \\ 0 & \text{when } (k,\ell) = (2,1) \\ a_{i,1,k,\ell} & \text{otherwise.} \end{cases}$$

Note that all the lower off diagonal elements $a_{i,2,k,\ell}$ is the same as $a_{i,1,k,\ell}$, expect $a_{i,2,2,1} = 0$. Note also that $a_{i,2,1,3}$ is the only upper triangle element that isn't necessarily zero that was zero in matrix $\mathbf{A}_{i,1}$. If we now continue the matrix multiplication to step j, $\mathbf{A}_{i,j+1} = \mathbf{R}^{\mathrm{T}}_{i,j} \mathbf{A}_{i,j}$;

$$a_{i,j+1,k,\ell} = \sum_{m=1}^{n} r^{\mathrm{T}}_{i,j,k,m} a_{i,j,m,\ell} = \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,j,k,\ell+m} a_{i,j,\ell+m,\ell} = \begin{cases} \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,j,k,\ell+m} a_{i,j,\ell+m,\ell} & \text{when } (k, \ell+m) \in {}^2\mathbb{N}_j^{j+1} \\ a_{i,j,k,\ell} & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,j,k,\ell+m} a_{i,j,\ell+m,\ell} & \text{when } (k, \ell+m) \in {}^2\mathbb{N}_j^{j+1} \text{ and } \ell \in \mathbb{N}_{j-1}^{j+2} \\ a_{i,j,k,\ell} & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,j,k,\ell+m} a_{i,j,\ell+m,\ell} & \text{when } (k, \ell+m) \in {}^2\mathbb{N}_j^{j+1} \text{ and } \ell \in \mathbb{N}_{j}^{j+2} \\ a_{i,j,k,\ell} & \text{otherwise,} \end{cases}$$

where I have used that $a_{i,j,j,j-1} = 0$ due to zero out of the previous rotation. We want now want our transformation to zero out the lower triangle element $(j+1, j)$

$$a_{i,j+1,j+1,j} = \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,j,j+1,j+m} a_{i,j,j+m,j} = \sum_{m=0}^{1} r^{\mathrm{T}}_{i,j,j+1,j+m} a_{i,j,j+m,j} = 0$$

where I have used the fact that $r^{\mathrm{T}}_{i,j,j+1,j-1} = 0$. This means that

$$a_{i,j+1,k,\ell} = \begin{cases} \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,j,k,\ell+m} a_{i,j,\ell+m,\ell} & \text{when } (k, \ell+m) \in {}^2\mathbb{N}_j^{j+1}, (k,\ell) \neq (j+1, j) \text{ and } \ell \in \mathbb{N}_j^{j+2} \\ 0 & \text{when } (k,\ell) = (j+1, j) \\ a_{i,j,k,\ell} & \text{otherwise.} \end{cases}$$

I have now shown that all the lower off diagonal elements $a_{i,j+1,k,\ell}$ is the same as $a_{i,j,k\ell}$, expect $a_{i,j+1,j+1,j} = 0$. Then using the assumption that $\mathbf{A}_i$ is a tridiagonal matrix, induction gives us that $\mathbf{A}_{i,n} = \mathbf{U}_i$ is a upper triangle matrix. Furthermore $a_{i,j+1,j,j+2}$ is the only upper triangle element that isn't zero that was zero in the matrix $\mathbf{A}_{i,j}$, hence induction give us the following upper triangle matrix elements

$$u_{i,k,\ell} = \begin{cases} a_{i,n,k,\ell} & \text{when } \ell \in \mathbb{N}_k^{k+2} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbf{A}_i = \mathbf{S}_i \mathbf{U}_i$ means that

$$\mathbf{S}_i^{\mathrm{T}} = \prod_{j=1}^{n-1} \mathbf{R}_{i,j}^{\mathrm{T}},$$

and due to the similarity transformation $\mathbf{A}_{i+1} = \mathbf{S}_i^{\mathrm{T}} \mathbf{A}_i \mathbf{S}_i = \prod_{j=1}^{n-1} \mathbf{R}_{i,j}^{\mathrm{T}} \mathbf{A}_i \prod_{i=1}^{n-1} \mathbf{R}_{i,n-j}$ we have that

$$\mathbf{S}_i = \prod_{j=1}^{n-1} \mathbf{R}_{i,n-j}.$$

To find the similarity matrix $\mathbf{A}_{i+1}$ to $\mathbf{A}_i$ we use successively rotation matrices $\mathbf{R}_{i,j}$ by defining the matrix

$$\mathbf{U}_{i,j} = \begin{cases} \mathbf{U}_i & \text{when } j = n \\ \mathbf{U}_{i,j+1} \mathbf{R}_{i,j} & \text{when } j \in \mathbb{N}_1^{n-1}, \end{cases}$$

where $\mathbf{U}_{i,1} = \mathbf{A}_{i+1}$. Doing the matrix multiplication $\mathbf{U}_{i,n-1} = \mathbf{U}_{i,n} \mathbf{R}_{i,n-1}$;

$$u_{i,n-1,k,\ell} = \sum_{m=1}^{n} u_{i,n,k,m} r_{i,n-1,m,\ell} = \sum_{m=0}^{2} u_{i,n,k,k+m} r_{i,n-1,k+m,\ell} = \begin{cases} \sum_{m=0}^{2} u_{i,n,k,k+m} r_{i,n-1,k+m,\ell} & \text{when } (k+m,\ell) \in {}^2\mathbb{N}_{n-1}^n \\ u_{i,n,k,\ell} & \text{otherwise} \end{cases}$$
$$= \begin{cases} \sum_{m=0}^{2} u_{i,n,k,k+m} r_{i,n-1,k+m,\ell} & \text{when } (k+m,\ell) \in {}^2\mathbb{N}_{n-1}^n \text{ and } k \in \mathbb{N}_{n-3}^n \\ u_{i,n,k,\ell} & \text{otherwise.} \end{cases}$$

Note that all lower off diagonal elements $u_{i,n-1,k,\ell}$ is the same as $u_{i,n,k,\ell}$, except $u_{i,n-1,n,n-1}$ which now may be different than zero. If we now continue the matrix multiplication to step $n - j + 1$, $\mathbf{U}_{i,j-1} = \mathbf{U}_{i,j} \mathbf{R}_{i,j-1}$;

$$u_{i,j-1,k,\ell} = \sum_{m=1}^{n} u_{i,j,k,m} r_{i,j-1,m,\ell} = \sum_{m=0}^{2} u_{i,j,k,k+m} r_{i,j-1,k+m,\ell} = \begin{cases} \sum_{m=0}^{2} u_{i,j,k,k+m} r_{i,j-1,k+m,\ell} & \text{when } (k+m,\ell) \in {}^2\mathbb{N}_{j-1}^j \\ u_{i,j+1,k,\ell} & \text{otherwise} \end{cases}$$
$$= \begin{cases} \sum_{m=0}^{2} u_{i,j,k,k+m} r_{i,j-1,k+m,\ell} & \text{when } (k+m,\ell) \in {}^2\mathbb{N}_{j-1}^j \text{ and } k \in \mathbb{N}_{j-3}^j \\ u_{i,j,k,\ell} & \text{otherwise,} \end{cases}$$

and we see that the lower off diagonal elements $u_{i,j-1,k,\ell}$ is the same as $u_{i,j,k,\ell}$, except $u_{i,j-1,j,j-1}$ which now may be different than zero. Hence by induction I shown that the only non-zero lower off diagonal elements in $\mathbf{A}_{i+1} = \mathbf{U}_{i,1}$ is the elements $u_{i,1,k,k-1}$;

$$a_{i+1,k,\ell} = \begin{cases} u_{i,1,k,\ell} & \text{when } \ell \in \mathbb{N}_{k-1}^{n} \\ 0 & \text{otherwise.} \end{cases}$$

But we can actually show that $\mathbf{A}_{i+1}$ is tridiagonal matrix by using the assumption that $\mathbf{A}_i$ is a symmetric matrix. This we can do by showing that when $\mathbf{A}_i$ is a symmetric then $\mathbf{A}_{i+1}$ is also symmetric by doing the matrix multiplication of the similarity transformation $\mathbf{A}_{i+1}^{\mathrm{T}} = \left(\mathbf{S}_i^{\mathrm{T}} \mathbf{A}_i \mathbf{S}_i\right)^{\mathrm{T}}$;

$$a_{i+1,k,\ell}^{\mathrm{T}} = \left(\sum_{m=1}^{n} \sum_{o=1}^{n} s_{i,k,m}^{\mathrm{T}} a_{i,m,o} s_{i,o,\ell} \right)^{\mathrm{T}} = \sum_{m=1}^{n} \sum_{o=1}^{n} s_{i,\ell,m}^{\mathrm{T}} a_{i,m,o} s_{i,o,k} = a_{i+1,\ell,k} \, .$$

Since we have shown that $\mathbf{A}_{i+1}$ is symmetric be due to that $\mathbf{A}_i$ is symmetric, and that $\mathbf{A}_{i+1}$ has only non-zero elements in the lower off diagonal $a_{i,k,k-1}$, means that $a_{i,k,k+1}$ is the only higher off diagonal elements that are non-zero as well;

$$a_{i+1,k,\ell} = \begin{cases} a_{i+1,k,k+|\,\ell-k\,|} & \text{when } \ell \in \mathbb{N}_{k-1}^{k+1} \\ 0 & \text{otherwise.} \end{cases}$$

And we have indeed shown by induction that $\mathbf{A}_i$ is symmetric tridiagonal matrix because the initial matrix $\mathbf{A}_0$ is symmetric tridiagonal matrix. Now if the off diagonal elements decreases for each QR transformation, then the similarity matrices $\mathbf{A}_i$ will converge to a diagonal matrix, and hence we have found the eigenvalues.

Now lets take a look at how we can calculate the elements in the symmetric tridiagonal matrix $\mathbf{A}_{i+1}$ we are primarily interested in calculating the elements $u_{i,j-1,j,j}$ and $u_{i,j-1,j,j-1}$ in $\mathbf{U}_{i,j-1}$, because this elements does not change in next matrix $\mathbf{U}_{i,j-2}$. So let us see what we need to calculate these elements;

$$u_{i,j-1,j,j} = \sum_{m=0}^{2} u_{i,j,j,j+m} r_{i,j-1,j+m,j} = u_{i,j,j,j} r_{i,j-1,j,j}$$
$$u_{i,j-1,j,j-1} = \sum_{m=0}^{2} u_{i,j,j,j+m} r_{i,j-1,j+m,j-1} = u_{i,j,j,j} r_{i,j-1,j,j-1}$$

So we need $u_{i,j,j,j}$ to calculate the elements $u_{i,j-1,j,j}$ and $u_{i,j-1,j,j-1}$, which means that we need to calculate $u_{i,j-1,j-1,j-1}$ as well;

$$u_{i,j-1,j-1,j-1} = \sum_{m=0}^{2} u_{i,j,j-1,j+m-1} r_{i,j-1,j+m-1,j-1} = u_{i,j,j-1,j-1} r_{i,j-1,j-1,j-1} + u_{i,j,j-1,j} r_{i,j-1,j,j-1} \, .$$

The matrix elements $u_{i,j,j-1,j-1}$ and $u_{i,j,j-1,j}$ are the same for the previous $\mathbf{U}_{i,k}$ with $k \in \mathbb{N}_{j+1}^{n-1}$, in fact these elements haven't changed since the transformation to $\mathbf{A}_{i,j}$, which means that

$$u_{i,j-1,j-1,j-1} = a_{i,j,j-1,j-1} r_{i,j-1,j-1,j-1} + a_{i,j,j-1,j} r_{i,j-1,j,j-1} \,.$$

Hence for the elements in interest we have

$$u_{i,j-1,j,j} = \begin{cases} a_{i,n,n,n} r_{i,n-1,n,n} & \text{when } j = n \\ \left(a_{i,j+1,j,j} r_{i,j,j,j} + a_{i,j+1,j,j+1} r_{i,j,j+1,j} \right) r_{i,j-1,j,j} & \text{otherwise} \end{cases}$$

$$u_{i,j-1,j,j-1} = \begin{cases} a_{i,n,n,n} r_{i,n-1,n,n-1} & \text{when } j = n \\ \left(a_{i,j+1,j,j} r_{i,j,j,j} + a_{i,j+1,j,j+1} r_{i,j,j+1,j} \right) r_{i,j-1,j,j-1} & \text{otherwise.} \end{cases}$$

Next step is to calculate the elements $a_{i,j+1,j,j}$ and $a_{i,j+1,j,j+1}$;

$$\begin{aligned} a_{i,j+1,j,j} &= \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,j,j,j+m} a_{i,j,j+m,j} = r^{\mathrm{T}}_{i,j,j,j} a_{i,j,j,j} + r^{\mathrm{T}}_{i,j,j,j+1} a_{i,j,j+1,j} \\ &= r^{\mathrm{T}}_{i,j,j,j} a_{i,j,j,j} + r^{\mathrm{T}}_{i,j,j,j+1} a_{i,j+1,j} \\ a_{i,j+1,j,j+1} &= \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,j,j,j+m+1} a_{i,j,j+m+1,j+1} = r^{\mathrm{T}}_{i,j,j,j} a_{i,j,j,j+1} + r^{\mathrm{T}}_{i,j,j,j+1} a_{i,j,j+1,j+1} \\ &= r^{\mathrm{T}}_{i,j,j,j} a_{i,j,j,j+1} + r^{\mathrm{T}}_{i,j,j,j+1} a_{i,j+1,j+1} \end{aligned}$$

where we recognize that the elements $a_{i,j,j+1,j}$ and $a_{i,j,j+1,j+1}$ haven't been changed yet by the previous transformations $\mathbf{A}_{i,k}$ with $k \in \mathbb{N}_1^{j-1}$, and are therefore the same as the original elements $a_{i,j+1,j}$ and $a_{i,j+1,j+1}$ in $\mathbf{A}_0$. So we need both $a_{i,j,j,j}$ and $a_{i,j,j,j+1}$ to calculate $a_{i,j+1,j,j}$ and $a_{i,j+1,j,j+1}$, and therefore we want to calculate $a_{i,j+1,j+1,j+1}$ and $a_{i,j+1,j+1,j+2}$ to prepare for next rotation;

$$\begin{aligned} a_{i,j+1,j+1,j+1} &= \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,j,j+1,j+m+1} a_{i,j,j+m+1,j+1} = r^{\mathrm{T}}_{i,j,j+1,j} a_{i,j,j,j+1} + r^{\mathrm{T}}_{i,j,j+1,j+1} a_{i,j,j+1,j+1} \\ &= r^{\mathrm{T}}_{i,j,j+1,j} a_{i,j,j,j+1} + r^{\mathrm{T}}_{i,j,j+1,j+1} a_{i,j+1,j+1} \\ a_{i,j+1,j+1,j+2} &= \sum_{m=-1}^{1} r^{\mathrm{T}}_{i,j,j+1,j+m+2} a_{i,j,j+m+2,j+2} = r^{\mathrm{T}}_{i,j,j+1,j+1} a_{i,j,j+1,j+2} = r^{\mathrm{T}}_{i,j,j+1,j+1} a_{i,j+1,j+2} \,, \end{aligned}$$

where we recognize again that $a_{i,j,j+1,j+1}$ and $a_{i,j,j+1,j+2}$ haven't been changed yet by the previous transformations $\mathbf{A}_{i,k}$ with $k \in \mathbb{N}_1^{j-1}$, and are therefore the same as the original elements $a_{i,j+1,j+1}$ and $a_{i,j+1,j+2}$. Lastly I will show to calculate the rotation matrix $\mathbf{R}_{i,j}$ from the already established relation

$$a_{i,j+1,j+1,j} = \sum_{m=0}^{1} r^{\mathrm{T}}_{i,j,j+1,j+m} a_{i,j,j+m,j} = a_{i,j,j+1,j} \cos\theta_{i,j} - a_{i,j,j,j} \sin\theta_{i,j} = 0 \,,$$

which gives

$$\begin{aligned} \cos\theta_{i,j} &= \frac{a_{i,j,j,j}}{\sqrt{{a_{i,j,j,j}}^2 + {a_{i,j,j+1,j}}^2}} \\ \sin\theta_{i,j} &= \frac{a_{i,j,j+1,j}}{\sqrt{{a_{i,j,j,j}}^2 + {a_{i,j,j+1,j}}^2}} \,. \end{aligned}$$

3 Numerical implementation

3.1 Jacobi's eigenvalue algorithm

We can optimize the calculation of diagonal elements in (7) by rewriting them by using the trigonometrical relation $\cos^2\theta + \sin^2\theta = 1$;

$$\begin{aligned} b_{ii} &= a_{ii}\cos^2\theta + a_{jj}\sin^2\theta + 2a_{ij}\sin\theta\cos\theta = a_{ii}\left(1-\sin^2\theta\right) + a_{jj}\sin^2\theta + 2a_{ij}\sin\theta\cos\theta \\ &= a_{ii} + \left[\left(a_{jj}-a_{ii}\right)\sin^2\theta + 2a_{ij}\sin\theta\cos\theta\right] \end{aligned} \tag{9}$$

$$\begin{aligned} b_{jj} &= a_{jj}\cos^2\theta + a_{ii}\sin^2\theta - 2a_{ij}\sin\theta\cos\theta = a_{jj}\left(1-\sin^2\theta\right) + a_{ii}\sin^2\theta - 2a_{ij}\sin\theta\cos\theta \\ &= a_{jj} - \left[\left(a_{jj}-a_{ii}\right)\sin^2\theta + 2a_{ij}\sin\theta\cos\theta\right] \end{aligned} \tag{10}$$

Note that it's better to use $\sin^2\theta$ instead of $\cos^2\theta$, because when we look at

$$\begin{aligned} b_{ii} &= a_{jj} + \left[\left(a_{ii}-a_{jj}\right)\cos^2\theta + 2a_{ij}\cos\theta\sin\theta\right] \\ b_{jj} &= a_{ii} - \left[\left(a_{ii}-a_{jj}\right)\cos^2\theta + 2a_{ij}\cos\theta\sin\theta\right], \end{aligned}$$

we see that a_{jj} corresponds to b_{ii} and a_{ii} corresponds to b_{jj}, which means that we need to backup either a_{ii} or a_{jj} before we do the calculation, so we don't overwrite the value before we have used in the calculation. However for the $\sin^2\theta$ expressions we have that a_{ii} corresponds to b_{ii} and a_{jj} corresponds to b_{jj}, which means that we don't need to backup a_{ii} or a_{jj}, and we can use the program operator += directly on the diagonal elements.

4 Results

4.1 Comparing different solvers

Number	tqli (lib.cpp)	QR	Jacobi (lecture)	Jacobi	Jacobi (FD)	Jacobi (RD)	Jacobi (FC)	Jacobi (RC)	Jacobi (FR)	Jacobi (RR)
10	2.7e-05	7.6e-05	0.000114	0.000107	8.5e-05	7.3e-05	0.000332	9.7e-05	0.000322	0.000114
100	0.008963	0.00919	0.375987	0.354782	0.047262	0.043355	0.471775	0.123497	0.475447	0.107761

Table 4.1.1: *Time used in seconds to solve the different eigenvalue and eigenvector solvers. The Jacobi solvers and QR algorithm terminates when off-diagonal elements are less than* 10^{-6}. *Used the case of single electron and* $\rho_{\max} = 10$.

Number	QR	Jacobi (lecture)	Jacobi	Jacobi (FD)	Jacobi (RD)	Jacobi (FC)	Jacobi (RC)	Jacobi (FR)	Jacobi (RR)
10	-6.94463	-6.94463	-6.94463	-6.94463	-6.94462	-6.94445	-6.94462	-6.94445	-6.9446
100	-5.35675	-5.35046	-5.35104	-5.34988	-5.35022	-5.3323	-5.35485	-5.3323	-5.34152

Table 4.1.2: *Order of relative error to result of eigenvalues to the tqli in lib.cpp. The Jacobi solvers and QR algorithm terminates when off-diagonal elements are less than* 10^{-6}. *Used the case of single electron and* $\rho_{\max} = 10$.

Number	QR	Jacobi (lecture)	Jacobi	Jacobi (FD)	Jacobi (RD)	Jacobi (FC)	Jacobi (RC)	Jacobi (FR)	Jacobi (RR)
10	857	129	128	324	269	1144	402	1144	384
100	115618	12830	12613	65886	61013	568584	165170	568584	147280

Table 4.1.3: *Number of rotations. The Jacobi solvers and QR algorithm terminates when off-diagonal elements are less than* 10^{-6}. *Used the case of single electron and* $\rho_{\max} = 10$.

4.2 Single electron harmonic oscillator

$n_{\text{step}} \backslash \rho_{\max}$	1	2	4	10	100	1000
4	0.697139	-0.220224	-0.768751	0.365163	2.51358	4.5149
10	0.821045	0.00785614	-1.36144	-0.801571	1.82361	3.83
20	0.843252	0.0435069	-2.1136	-1.06262	1.24455	3.26818
50	0.850155	0.0543602	-2.30935	-1.86932	0.341044	2.49633
100	0.851202	0.0559976	-2.18102	-2.46733	-0.881219	1.89877

Table 4.2.1: *Order of relative error of the three lower eigenvalues with* $\ell = 0$ *for a single electron trapped in a harmonic oscillator well. The relative error is compared to the exact eigenvalues* $\lambda_0 = 3$, $\lambda_1 = 7$ *and* $\lambda_2 = 11$. *Solved with my QR algorithm with off-diagonal elements are less than* 10^{-6}.

$n_{\text{step}} \backslash \rho_{\max}$	1	2	4	10	100	1000
4	0.697139	-0.220224	-0.768751	0.365163	2.51358	4.5149
10	0.821045	0.00785616	-1.36144	-0.801571	1.82361	3.83
20	0.843252	0.043507	-2.11361	-1.06262	1.24455	3.26818
50	0.850155	0.0543607	-2.30944	-1.86933	0.341044	2.49633
100	0.851202	0.0559969	-2.18109	-2.46733	-0.881219	1.89877

Table 4.2.2: *Order of relative error of the three lower eigenvalues with* $\ell = 0$ *for a single electron trapped in a harmonic oscillator well. The relative error is compared to the exact eigenvalues* $\lambda_0 = 3$, $\lambda_1 = 7$ *and* $\lambda_2 = 11$. *Solved with my tqli in lib.cpp*

4.3 Two electrons harmonic oscillator with repulsive Coulomb interaction

$n_{\text{step}} \backslash \rho_{\max}$	10	20	40	60	80	100	1000	10000
10	0.290567	-0.513702	-2.33983	-3.26877	-2.3104	-1.85744	0.843536	2.89531
100	0.293728	-0.507142	-2.13443	-2.16045	-2.16411	-2.16886	-1.85758	0.925858
1000	0.293765	-0.507065	-2.13251	-2.15583	-2.15587	-2.15592	-2.1691	-1.869

Table 4.3.1: *Order of relative error of the ground state with* $\ell = 0$ *and* $\omega_r = 0.01$ *for two electrons trapped in a harmonic oscillator well with repulsive Coulomb interaction. The relative error is compared to the exact eigenvalues* $\lambda_0 = 0.105041$. *Solved with my QR algorithm with off-diagonal elements less than* 10^{-6}.

$n_{\text{step}} \backslash \rho_{\max}$	10	20	40	60	80	100	1000	10000
10	-1.30549	-1.24813	-0.0891626	0.437514	0.74154	0.959596	3.00158	5.00201
100	-1.07571	-1.08172	-1.10691	-1.15374	-1.23304	-1.37533	1.04043	3.07579
1000	-1.07375	-1.07381	-1.07405	-1.07446	-1.07502	-1.07575	-1.38447	1.04884

Table 4.3.2: *Order of relative error of the ground state with* $\ell = 0$ *and* $\omega_r = 0.5$ *for two electrons trapped in a harmonic oscillator well with repulsive Coulomb interaction. The relative error is compared to the exact eigenvalues* $\lambda_0 = 2.05658$. *Solved with my QR algorithm with off-diagonal elements less than* 10^{-6}.

$n_{\text{step}} \backslash \rho_{\max}$	10	20	40	60	80	100	1000	10000
10	-1.57751	-0.640193	0.442244	0.862346	1.13519	1.33955	3.35808	5.35827
100	-0.922724	-0.932575	-0.975505	-1.0641	-1.26443	-2.23825	1.41663	3.43226
1000	-0.919542	-0.919639	-0.92003	-0.920683	-0.9216	-0.922782	-2.43711	1.42468

Table 4.3.3: *Order of relative error of the ground state with* $\ell = 0$ *and* $\omega_r = 1$ *for two electrons trapped in a harmonic oscillator well with repulsive Coulomb interaction. The relative error is compared to the exact eigenvalues* $\lambda_0 = 3.62193$. *Solved with my QR algorithm with off-diagonal elements less than* 10^{-6}.

$n_{\text{step}} \backslash \rho_{\max}$	10	20	40	60	80	100	1000	10000
10	-0.153129	0.690821	1.34894	1.71137	1.96484	2.16032	4.16326	6.16329
100	-0.649211	-0.684726	-0.906906	-0.927603	-0.381454	-0.0261586	2.23493	4.2374
1000	-0.638443	-0.638769	-0.640077	-0.642271	-0.645372	-0.649412	-0.0134	2.24275

Table 4.3.4: *Order of relative error of the ground state with* $\ell = 0$ *and* $\omega_r = 5$ *for two electrons trapped in a harmonic oscillator well with repulsive Coulomb interaction. The relative error is compared to the exact eigenvalues* $\lambda_0 = 14.1863$. *Solved with my QR algorithm with off-diagonal elements less than* 10^{-6}.

5 Conclusion

6 Attachments

The files produced in working with this project can be found at
https://github.com/Eimund/UiO/tree/master/FYS4150/Project%202

The source files developed are

7 Resources

1. QT Creator 5.3.1 with C11
2. Armadillo
3. Eclipse Standard/SDK - Version: Luna Release (4.4.0) with PyDev for Python
4. Ubuntu 14.04.1 LTS
5. ThinkPad W540 P/N: 20BG0042MN with 32 GB RAM

References

[1] Morten Hjorth-Jensen, *FYS4130 - Project 2 - Schrödinger's equation for two electrons in a three-dimensional harmonic oscillator well*, University of Oslo, 2014

[2] Morten Hjorth-Jensen, *Computational Physics - Lecture Notes Fall 2014*, University of Oslo, 2014

[3] M. Taut, *Two electrons in an external oscillator potential: Particular analytic solutions of a Coulomb correlation problem*, Physical Review A Volume 48 Number 5, 1993

[4] http://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors

[5] http://en.wikipedia.org/wiki/Jacobi_eigenvalue_algorithm

[6] http://en.wikipedia.org/wiki/Matrix_similarity

[7] http://en.wikipedia.org/wiki/Rotation_matrix

[8] http://en.wikipedia.org/wiki/Orthogonal_matrix

[9] http://en.wikipedia.org/wiki/Matrix_norm#Frobenius_norm

[10] http://en.wikipedia.org/wiki/Planck_constant

[11] http://en.wikipedia.org/wiki/Electron

[12] http://en.wikipedia.org/wiki/QR_algorithm