Coursework I

CID number: This number does not exist!

MATH40002: Analysis I

Imperial College London

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Problem 1

Prove that if

$$f$$

and

$$g$$

are continuous functions such that

$$f(q) = g(q)$$

for all

$$q \in \mathbb{Q}$$

, then

$$f = g$$

.

Solution.

Proof. We prove this by using the sequential continuity. It states that if

$$f$$

is continuous at

$$x$$

, then for any sequence

$$\lim_{n \to \infty} x_n = x$$

, we have

$$\lim_{n \to \infty} f(x_n) = f(x)$$

.

Let

$$x$$

be any real number. Since

$$\mathbb{Q}$$

is dense in

$$\mathbb{R}$$

, we can always find a sequence of rational numbers

$$\{q_n\} \subset \mathbb{Q}$$

, which satisfies,

$$\lim_{n \to \infty} q_n = x$$

for any real number

$$x$$

.

As

$$f(q) = g(q)$$

for all

$$q \in \mathbb{Q}$$

, we have

$$f\left(q_n\right) = g\left(q_n\right)$$

for any

$$n \in \mathbb{N}$$

. It follows that

$$\lim_{n \to \infty} f\left(q_n\right) = \lim_{n \to \infty} g\left(q_n\right)$$

Since

$$f$$

and

$$g$$

are continuous functions, then they are continuous at any

$$x \in \mathbb{R}$$

by the definition of continuous function. Then, by the sequential continuity we stated before and the equation (1), we have

$$\lim_{n \to \infty} f\left(q_n\right) = f(x).$$

$$\lim_{n \to \infty} g\left(q_n\right) = g(x).$$

By the equations (2), (3), (4), we immediately get this:

$$f(x) = g(x) \forall x \in \mathbb{R}$$

It means that

$$f = g$$

. $\square$

Problem 2

Prove that the finite union of bounded sets is bounded.

Solution.

Proof. Suppose we have a collection of finite bounded sets, which is

$$\{S_1, S_2, S_3, \dots, S_n\}$$

. Each bounded set has its own upper bound, which is

$$\{M_1, M_2, M_3, \dots, M_n\}$$

respectively and has its own lower bound, which is

$$\{m_1, m_2, m_3, \dots, m_n\}$$

respectively. We claim that the finite union of these sets

$$U_{i=1}^{n} S_i$$

is upper bounded by

$$\max\{M_1, M_2, M_3, \dots, M_n\}$$

and lower bounded by

$$\min\{m_1, m_2, m_3 \dots, m_n\}$$

.

Since any element

$$x \in U_{i=1}^{n} S_i$$

must be in at least one of the bounded sets

$$\{S_1, S_2, S_3, \dots, S_n\}$$

, then for any

$$x \in U_{i=1}^{n} S_i$$

, we have

$$x \leq \max\{M_1, M_2, M_3, \dots, M_n\}$$

and

$$x \geq \min\{m_1, m_2, m_3 \dots, m_n\}$$

. Therefore, the finite union of bounded sets is also bounded. □