Autumn Take-home Assessment

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Calculus and Applications I, 2022

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Problem 1

The equation

$$y' = 1 + y^2, \quad y(0) = 0, \quad (**)$$

can be solved using separation of variables and integration to find $y = \tan x$.

(a) Find a power series solution of $(**)$ and hence show that

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

[Note: you will need to use the earlier formulas for multiplication of two infinite power series.]

(b) Now get the result above by repeated differentiation of $(**)$ and use of the formula $a_n = \frac{f^{(n)}(0)}{n!}$

Solution.

(a) We assume that a power series solution exists, i.e.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \tag{1}$$

converges for $|x| < R$ for some positive radius of convergence $R > 0$. Then, we can differentiate the power series term by term to find

$$y(x)' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots + (n+1)a_{n+1} x^n + \dots \tag{2}$$

Hence $(**)$ is satisfied if $(2) = 1 + (1)^2$. Therefore, we have

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 1 + \left(\sum_{n=0}^{\infty} a_n x^n \right)^2 \tag{3}$$

by the formulas for multiplication of two infinite power series, we get:

$$\left(\sum_{n=0}^{\infty} a_n x^n \right)^2 = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} a_m a_{n-m} \right) x^n. \tag{4}$$

Substitute (4) into (3), we get:

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 1 + \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} a_m a_{n-m} \right) x^n. \tag{5}$$

Note that $a_0 = 0$ by the initial condition $y(0) = 0$. If the equation (5) is satisfied, the coefficients of different powers of x must match. By inspection

$$
\begin{aligned}
a_0 &= 0 \\
a_1 &= 1 + a_0^2 \Longrightarrow a_1 = 1 \\
2a_2 &= 2a_0 a_1 \Longrightarrow a_2 = 0 \\
3a_3 &= 2a_0 a_2 + a_1^2 \Longrightarrow a_3 = \frac{1}{3} \\
4a_4 &= 2a_0 a_3 + 2a_1 a_2 \Longrightarrow a_4 = 0 \\
5a_5 &= 2a_0 a_4 + 2a_1 a_3 + a_2^2 \Longrightarrow a_5 = \frac{2}{15} \\
&\vdots \\
na_n &= \sum_{m=0}^{n-1} a_m a_{(n-1)-m} \\
(n+1)a_{n+1} &= \sum_{m=0}^{n} a_m a_{n-m} \qquad (6)
\end{aligned}
$$

Therefore, we get a power series solution of $(**)$, which satisfied the recursion formula (6).
$(**)$ can be solved using separation of variable and integration to find $y = \tan x$. The detail is followed:

$$
\begin{aligned}
y' &= 1 + y^2 \\
\frac{dy}{dx} &= 1 + y^2 \\
\frac{1}{1+y^2} dy &= dx \\
\int \frac{1}{1+y^2} dy &= \int dx \\
\tan^{-1} y &= x + C_1 \\
y &= \tan x + C
\end{aligned}
$$

$y(0) = 0$, then $C = 0$. Therefore, $y = \tan x$ is also a solution of $(**)$. Then we have

$$
y(x) = \tan x = \sum_{n=0}^{\infty} a_0 x^n = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots
$$

(b) We have $y' = 1 + y^2 \Longrightarrow f'(0) = 1 \; as \, f(0) = 1$. Then,differentiate $(**)$ at the both sides and repeat:

$$
\begin{aligned}
y'' &= 2yy' \Longrightarrow f''(0) = 0 \\
y^{(3)} &= 2(y')^2 + 2yy'' \Longrightarrow f^{(3)}(0) = 2 \\
y^{(4)} &= 4y'y'' + 2y'y'' + 2yy^{(3)} \Longrightarrow f^{(4)}(0) = 0 \\
y^{(5)} &= 4(y'')^2 + 4y'y^{(3)} + 2(y'')^2 + 2y'y^{(3)} + 2y'y(3) + 2yy^{(4)} \Longrightarrow f^{(5)}(0) = 16 \\
&\vdots
\end{aligned}
$$

Use the formula $a_n = \frac{f^{(n)}(0)}{n!}$

$$
\begin{aligned}
a_0 &= \frac{f(0)}{0!} = 0 \\
a_1 &= \frac{f'(0)}{1!} = 1 \\
a_2 &= \frac{f''(0)}{2!} = 0 \\
a_3 &= \frac{f^{(3)(0)}}{3!} = \frac{1}{3} \\
a_4 &= \frac{f^{(4)}(0)}{4!} = 0 \\
a_5 &= \frac{f^{(5)}(0)}{5!} = \frac{2}{15} \\
&\vdots
\end{aligned}
$$

Then we get

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$