Background: Given an N-node graph viewed as a resistive electric circuit with all edges having unit conductance, the resistance matrix $\mathbf{R}$ is defined as the matrix with components R_{ij} given by the effective resistance between node i (set to unit voltage) and node j (grounded) with all diagonal elements taken to vanish, i.e. $R_{ii} = 0$.

The *total effective resistance* of the graph is defined in terms of this resistance matrix as

$$R^{(\text{total})} = \sum_{i<j} R_{ij}. \tag{1}$$

Let the unit-length eigenvectors and corresponding eigenvalues of the N-by-N graph Laplacian $\mathbf{K}$ be denoted by

$$\{\mathbf{e}_k | k = 1, \ldots, N\}, \qquad \{\lambda_k | k = 1, \ldots, N\}$$

where we take

$$\mathbf{e}_1 = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix}, \qquad \lambda_1 = 0$$

and where we denote the elements of $\mathbf{e}_k$ for $k = 2, \ldots, N$ as

$$\mathbf{e}_k = \begin{pmatrix} e_{k1} \\ e_{k2} \\ . \\ . \\ e_{kN} \end{pmatrix}, \qquad k = 2, \ldots, N.$$

The following two formulas can be established relating the resistance matrix and total effective resistance to the Laplace spectrum:

$$R_{ij} = \sum_{k=2}^{N} \frac{1}{\lambda_k} (e_{ki} - e_{kj})^2 \tag{2}$$

and

$$R^{(\text{total})} = N \sum_{k=2}^{N} \frac{1}{\lambda_k}. \tag{3}$$

Suggested solutions:

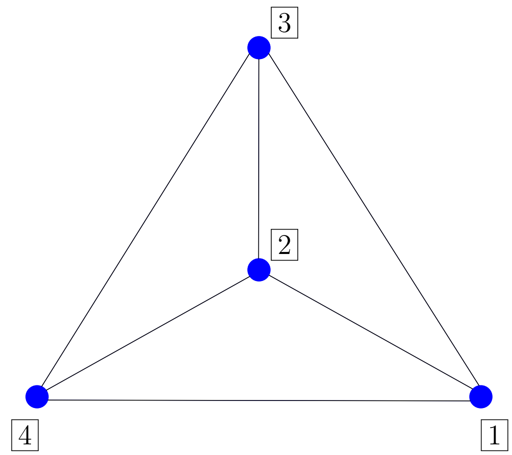

Figure 1: Example 4-node graph

(a): There are "4 choose 2", or

$$\binom{4}{2} = \frac{4!}{2!2!} = 6 \tag{4}$$

choices of i and j. However, since the graph is complete, *any* choice of two nodes to set to 0/1 voltage will be identical to any other choice (this is clear after suitable rearrangement of the graph, say). It is therefore enough to find the effective conductance when, say, node $\boxed{2}$ is grounded and node $\boxed{3}$ is set to unit voltage. Once this is found, the required effective resistance is its reciprocal.

The required effective conductance can be found by the "method of equivalent circuits". The sequence of diagrams in Figure 2 shows how to do this. The key point is that, by the symmetry of the graph and the electrified nodes, we expect the two nodes $\boxed{1}$ and $\boxed{4}$ where KCL holds to be at the same voltage; this means we can combine these nodes (while preserving all edges; although the edge between $\boxed{1}$ and $\boxed{4}$ can be eliminated because it will not carry a current). Using this fact, and constructing a sequence of "equivalent circuits" where the rules for conductors in series and in parallel are used, namely,

$$\frac{c_a c_b}{c_a + c_b} \text{ (in series)}, \qquad c_a + c_b, \text{ (in parallel)}, \tag{5}$$

the required effective resistance is $1/2$. See figure where the reduction is explained

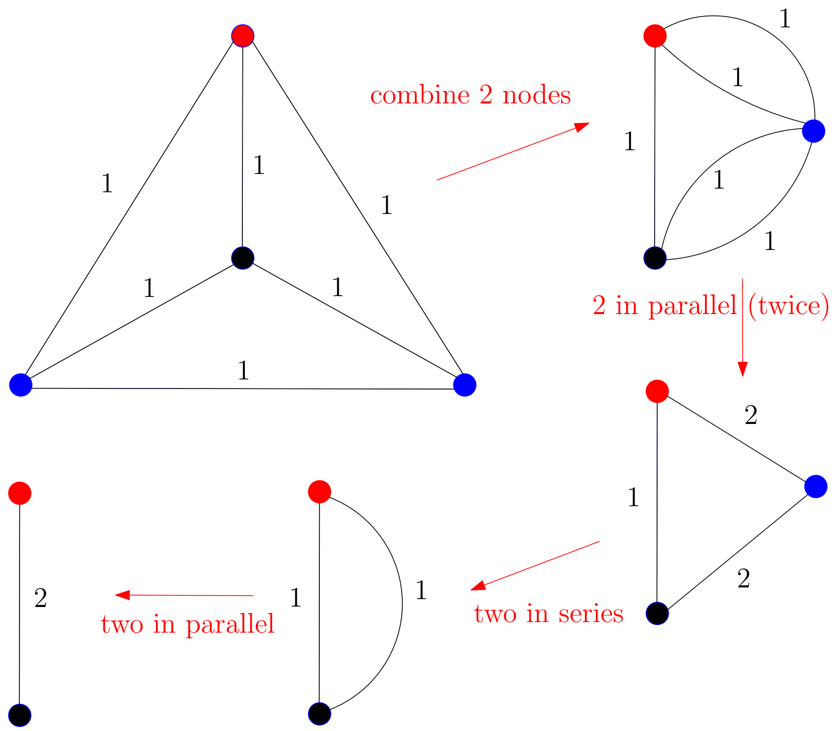

Figure 2: Effective conductance when one of the vertices on the triangle being at unit voltage with the central node grounded. Notes $\boxed{1}$ and $\boxed{4}$ are expected to be at the same voltage, so they can be combined while preserving edges. The effective conductance is computed to be 2 so the effective resistance is 1/2.

pictorially. Thus, the resistance matrix is

$$\mathbf{R} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}. \tag{6}$$

Alternative approaches: Another approach is to spot that the 6 possible node pair choices split into two "types": 3 choices are where a central node is grounded and one of the 3 satellite nodes is set to unit voltage. Then these effective conductances are all equal to 2, as just computed by the method of equivalent circuits above. Another 3 choices are when two of the nodes at the vertices of the exterior triangle are set to 0/1 voltage. Although this is also equivalent to the circuit solved above, it can be solved alternatively by linear algebra as follows.

Suppose node $\boxed{3}$ is set to unit voltage and node $\boxed{4}$ is grounded. A calculation using the Laplacian matrix and the notion of *Schur complements* goes as follows:

$$\mathbf{K}\mathbf{x} = f_0 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \tag{7}$$

where f_0 is to be determined and

$$\mathbf{K} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

We can write this is sub-block form

$$\begin{pmatrix} \mathbf{P} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{P} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \mathbf{e} \end{pmatrix} = f_0 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \tag{8}$$

where

$$\mathbf{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}. \tag{9}$$

Hence, on solving

$$\mathbf{P}\hat{\mathbf{x}} = -\mathbf{Q}^T\mathbf{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{10}$$

we find

$$\hat{\mathbf{x}} = -\mathbf{P}^{-1}\mathbf{Q}^T\mathbf{e} = \frac{1}{8}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}. \tag{11}$$

Therefore,

$$f_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \mathbf{Q}\hat{\mathbf{x}} + \mathbf{P}\mathbf{e} = (\mathbf{P} - \mathbf{Q}\mathbf{P}^{-1}\mathbf{Q}^T)\mathbf{e} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \tag{12}$$

from which we infer $f_0 = 2$ so that the required effective resistance is $1/f_0 = 1/2$ (note that the matrix $\mathbf{P} - \mathbf{Q}\mathbf{P}^{-1}\mathbf{Q}^T$ is known as the Schur complement of $\mathbf{P}$ in $\mathbf{K}$).

Note: Yet another approach follows an example done in lecture notes where a column and a row are eliminated (corresponding to the grounded node, say) leaving the 3-by-3 linear system that can be solved by hand. With the 4th column and row eliminated, the reduced system becomes

$$\mathbf{K} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} C_{\text{eff}} \\ 0 \\ 0 \end{pmatrix}$$

which can be solved by hand easily (even more easily if one notices *a priori* that by symmetry $x_2 = x_3$):

$$x_2 = x_3 = \frac{1}{2}, \qquad C_{\text{eff}} = 2.$$

10 marks

(b) It follows that the total effective resistance is

$$R^{(\text{total})} = 6 \times \frac{1}{2} = 3.$$

2 marks

(c) Notice that the Laplacian of this complete graph is

$$\mathbf{K} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

and an exercise from Problem Sheet 1 shows that this can be written as

$$\mathbf{K} = 4\mathbf{I} - \mathbf{J},$$

where

$$\mathbf{J} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

From its simple form – it is clearly a rank-1 matrix (see solutions of Problem Sheet 1) – eigenvectors of $\mathbf{J}$ are easy to spot by inspection. The thing to ensure, however, *is that they are of unit length and are mutually orthogonal* (otherwise the results of the "Background" section do not apply as stated). Use of procedures such as Gram-Schmidt is acceptable, if needed. The following choices satisfy these conditions:

$$\mathbf{x}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix},$$

where $\mathbf{x}_1$ has eigenvalue 4 and the other 3 have zero eigenvalues. These eigenvectors are also eigenvectors of $\mathbf{K}$ but with eigenvalues $0, 4, 4, 4$.

Note 1: Since the eigenspace corresponding to eigenvalue 4 has dimension 3 then there are lots of *non-orthogonal*, but still linearly independent, eigenvectors! It is perfectly fine to use Gram-Schmidt orthogonalization here.

Note 2: $\mathbf{K}$ also happens to be a circulant matrix, and the above orthogonal eigenvectors can be derived using that property also. Another exercise of Problem Sheet 1 gives an indication of this approach, which is also acceptable.

8 marks

(d) We know now that all the effective resistances equal $1/2$. The formula to verify is

$$R_{ij} = \sum_{k=2}^{N} \frac{1}{\lambda_k}(e_{ki} - e_{kj})^2.$$

We have to check all off choices $i < j$. Notice that the first eigenvector never contributes to the sum.

$\underline{i = 1, j = 2:}$

$$R_{12} = \sum_{k=2}^{N} \frac{1}{\lambda_k}(e_{k1} - e_{k2})^2 = \frac{1}{4}\left(0 + \left(\frac{1}{\sqrt{2}}\right)^2 + 1 + \left(\frac{1}{\sqrt{2}}\right)^2\right) = \frac{1}{2}$$

$\underline{i = 1, j = 3:}$

$$R_{13} = \sum_{k=2}^{N} \frac{1}{\lambda_k}(e_{k1} - e_{k3})^2 = \frac{1}{4}\left(0 + \left(\frac{2}{\sqrt{2}}\right)^2 + 0 + 0\right) = \frac{1}{2}$$

$\underline{i = 1, j = 4:}$

$$R_{14} = \sum_{k=2}^{N} \frac{1}{\lambda_k}(e_{k1} - e_{k4})^2 = \frac{1}{4}\left(0 + \left(\frac{1}{\sqrt{2}}\right)^2 + 1 + \left(\frac{1}{\sqrt{2}}\right)^2\right) = \frac{1}{2}$$

$\underline{i = 2, j = 3:}$

$$R_{23} = \sum_{k=2}^{N} \frac{1}{\lambda_k}(e_{k2} - e_{k3})^2 = \frac{1}{4}\left(0 + \left(\frac{1}{\sqrt{2}}\right)^2 + 1 + \left(\frac{1}{\sqrt{2}}\right)^2\right) = \frac{1}{2}$$

$\underline{i = 2, j = 4:}$

$$R_{24} = \sum_{k=2}^{N} \frac{1}{\lambda_k}(e_{k2} - e_{k4})^2 = \frac{1}{4}\left(0 + 0 + 0 + \left(\frac{2}{\sqrt{2}}\right)^2\right) = \frac{1}{2}$$

$\underline{i = 3, j = 4}$:

$$R_{34} = \sum_{k=2}^{N} \frac{1}{\lambda_k}(e_{k3} - e_{k4})^2 = \frac{1}{4}\left(0 + \left(\frac{1}{\sqrt{2}}\right)^2 + 1 + \left(\frac{1}{\sqrt{2}}\right)^2\right) = \frac{1}{2}$$

4 marks

(e) The 3 eigenvalues needed to verify this formula are $\lambda_2 = \lambda_3 = \lambda_4 = 4$ and

$$R^{(\text{total})} = N \sum_{n=2}^{N} \frac{1}{\lambda_n} = 4\left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) = 3.$$

2 marks

(f) Without loss of generality we can reorder node positions in the vectors so that node $\boxed{1}$ is set to unit voltage and node $\boxed{2}$ is grounded, i.e., we can take $i = 1, j = 2$. Then we need to solve

$$\mathbf{Kx} = \mathbf{f} = f_0 \begin{pmatrix} 1 \\ -1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \tag{13}$$

for

$$R_{\text{eff}}^{(12)} = \frac{1}{f_0},$$

where $\mathbf{x}$ is the vector of node potentials having the form

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ \phi_3 \\ \phi_4 \\ . \\ . \\ \phi_N \end{pmatrix}.$$

Write the solution

$$\mathbf{x} = \sum_{k=1}^{N} a_k \mathbf{e}_k \tag{14}$$

for some set of coefficients $\{a_k | k = 1, \ldots, N\}$ to be determined. On substitution of (14) into (13) we find

$$\mathbf{Kx} = \mathbf{K} \sum_{k=1}^{N} a_k \mathbf{e}_k = \sum_{k=2}^{N} a_k \lambda_k \mathbf{e}_k = \mathbf{f}. \tag{15}$$

Multiplication by $\mathbf{e}_m^T$ gives

$$\sum_{k=2}^{N} a_k \lambda_k \mathbf{e}_m^T \mathbf{e}_k = \mathbf{e}_m^T \mathbf{f} = f_0(e_{m1} - e_{m2}). \tag{16}$$

For $m = 2, \ldots, N$, and on use of the orthogonality of the eigenvectors, we therefore have

$$a_m = \frac{f_0(e_{m1} - e_{m2})}{\lambda_m}, \qquad m = 2, \ldots, N. \tag{17}$$

Note that the coefficient a_1 is not determined by these considerations. However, we can write

$$\mathbf{x} = a_1 \mathbf{e}_1 + \sum_{k=2}^{N} \frac{f_0(e_{k1} - e_{k2})}{\lambda_k} \mathbf{e}_k. \tag{18}$$

But node 2 is grounded implying, by inspection of the second component of (18),

$$0 = \frac{a_1}{\sqrt{N}} + \sum_{k=2}^{N} \frac{f_0(e_{k1} - e_{k2})}{\lambda_k} e_{k2} \tag{19}$$

which provides the required expression for a_1. Now we can use the fact that node 1 is at unit voltage to deduce, by inspection of the first component of (18), that

$$1 = \frac{a_1}{\sqrt{N}} + \sum_{k=2}^{N} \frac{f_0(e_{k1} - e_{k2})}{\lambda_k} e_{k1} = -\sum_{k=2}^{N} \frac{f_0(e_{k1} - e_{k2})}{\lambda_k} e_{k2} + \sum_{k=2}^{N} \frac{f_0(e_{k1} - e_{k2})}{\lambda_k} e_{k1}, \tag{20}$$

where we have used (19). We infer

$$1 = \sum_{k=2}^{N} \frac{f_0(e_{k1} - e_{k2})}{\lambda_k} (e_{k1} - e_{k2}) \tag{21}$$

implying

$$R_{\text{eff}}^{(12)} = \frac{1}{f_0} = \sum_{k=2}^{N} \frac{(e_{k1} - e_{k2})^2}{\lambda_k}, \tag{22}$$

where we have used the fact that the effective resistance is the inverse of the effective conductance. Finally, the required more general result follows on setting $1 \mapsto i$ and $2 \mapsto j$ since the initial reordering of node labels does not affect the result.

10 marks