Note 1: You must **explain all your calculations clearly** to get full credit.

Note 2: This assessment will be graded out of 40 marks. 90% of the grade for this coursework (36/40) will be assigned for a **correct, well-argued calculation**. Another 10% (4/40) will be awarded at the grader's discretion for **clarity and presentation** of your solution.

Note 3: Due by **midday on Monday February 6th 2023**. Please submit via the `Turnitin` dropbox link provided in Blackboard.

Coursework background: Let $N > 1$ be a positive integer. Given an N-node graph viewed as a resistive electric circuit with all edges having unit conductance, the resistance matrix $\mathbf{R}$ is defined as the matrix with components R_{ij} given by the effective resistance between node i (set to unit voltage) and node j (grounded) with all diagonal elements taken to vanish, i.e. $R_{ii} = 0$. Recall that the effective resistance is just the reciprocal of the effective conductance C_{eff} introduced in lectures.

The *total effective resistance* of the graph is defined in terms of this resistance matrix as

$$R^{(\text{total})} = \sum_{i<j} R_{ij}. \tag{1}$$

Let the orthonormal (i.e., unit length and mutually orthogonal) eigenvectors and corresponding eigenvalues of the N-by-N graph Laplacian $\mathbf{K}$ be denoted by

$$\{\mathbf{e}_k | k = 1, \ldots, N\}, \qquad \{\lambda_k | k = 1, \ldots, N\},$$

where we take

$$\mathbf{e}_1 = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix}, \qquad \lambda_1 = 0$$

and where we denote the elements of $\mathbf{e}_k$ for $k = 2, \ldots, N$ as

$$\mathbf{e}_k = \begin{pmatrix} e_{k1} \\ e_{k2} \\ . \\ . \\ e_{kN} \end{pmatrix}, \qquad k = 2, \ldots, N.$$

The following two formulas can be established relating the elements of the resistance matrix and the total effective resistance to the eigenvectors/eigenvalues of $\mathbf{K}$:

$$R_{ij} = \sum_{k=2}^{N} \frac{1}{\lambda_k} (e_{ki} - e_{kj})^2 \tag{2}$$

and

$$R^{(\text{total})} = N \sum_{k=2}^{N} \frac{1}{\lambda_k}. \tag{3}$$

Coursework exercise: Consider the 4-node graph shown in Figure 1:

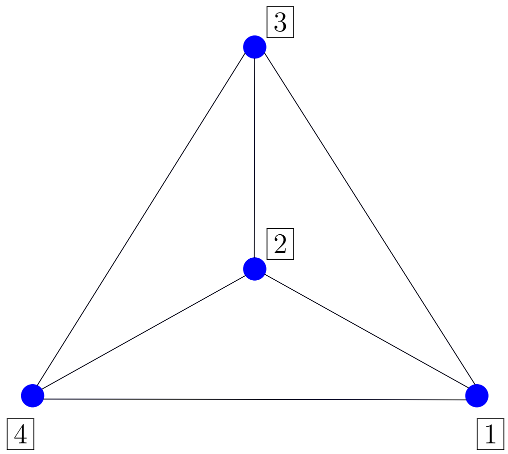

Figure 1: A 4-node graph viewed as an electric circuit.

(a) Using the node labelling given in the figure, find the resistance matrix **R** for this graph.

(b) Hence, using your results from part (a), compute R^{total} for this graph using formula (1).

(c) Find the eigenvalues $\{\lambda_k | k = 1, \ldots, 4\}$ and orthonormal eigenvectors $\{\mathbf{e}_k | k = 1, \ldots, 4\}$ of the graph Laplacian **K**.

(d) Use the results of parts (a) and (c) to verify formula (2).

(e) Use the results of parts (b) and (c) to verify formula (3).

(f) Prove formula (2). *Hint:* consider using the eigenvectors as a basis.

[*Note:* you are not required to prove formula (3)].