Note 1: You must **explain all your calculations clearly** to get full credit.

Note 2: This assessment will be graded out of 40 marks. 90% of the grade for this coursework (36/40) will be assigned for a **correct, well-argued calculation**. Another 10% (4/40) will be awarded at the grader's discretion for **clarity and presentation** of your solution.

Note 3: Due by **midday on Monday March 13th 2023**. Please submit via the `Turnitin` dropbox link provided in Blackboard.

Part I: For any integer $n \geq 0$ define

$$I_+(n) \equiv \int_0^1 e^y \sin(n\pi y)dy, \qquad I_-(n) \equiv \int_0^1 e^{-y} \sin(n\pi y)dy.$$

(i) Calculate these two integrals explicitly.

(ii) Use the result of part (i) to find the Fourier sine series of both $\sinh y$ and $\cosh y$ over the interval $[0, 1]$ (you should use ideas from the "Calculus and Applications" course).

Part II: Consider the electric circuit shown in the Figure where the vertical edges have conductance c and the horizontal edges have conductance d. Node $\boxed{\text{2N+1}}$ is set to unit voltage, while nodes $\boxed{0}$ and $\boxed{\text{N+1}}$ to $\boxed{\text{2N}}$ are grounded (set to zero voltage). Kirchhoff's current law holds at nodes $\boxed{1}$ to $\boxed{\text{N}}$. Let $\hat{\mathbf{x}}$ denote the voltages at nodes $\boxed{1}$ to $\boxed{\text{N}}$. The nodes should be ordered as follows: $\boxed{1}$, $\boxed{2}$, $\ldots$, $\boxed{\text{2N-1}}$, $\boxed{\text{2N}}$, $\boxed{0}$, $\boxed{\text{2N+1}}$.

(a) Show that the conductance-weighted Laplacian matrix is

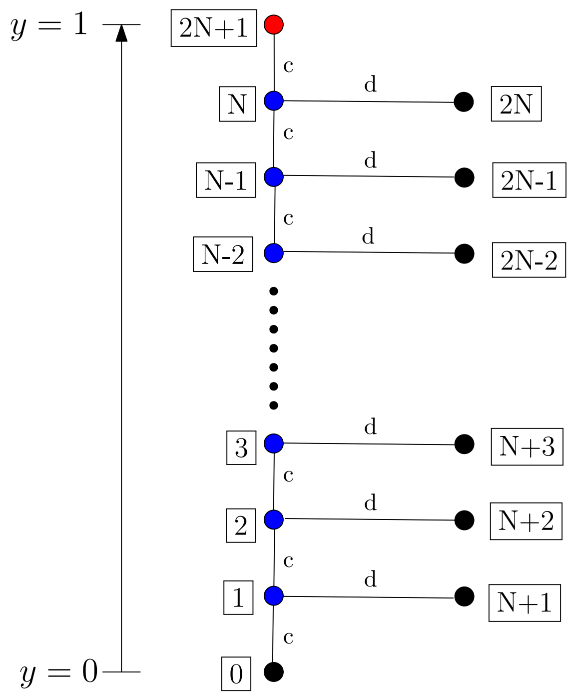
$$\mathbf{K} = \begin{pmatrix} c\mathbf{K}_N + d\mathbf{I}_N & -d\mathbf{I}_N & -c\mathbf{P} \\ -d\mathbf{I}_N & d\mathbf{I}_N & \mathbf{0} \\ -c\mathbf{P}^T & \mathbf{0} & c\mathbf{I}_2 \end{pmatrix},$$

where $\mathbf{I}_j$ denotes the j-by-j identity matrix and $\mathbf{K}_N$ is the N-by-N matrix familiar from lectures. You should find the N-by-2 matrix $\mathbf{P}$.

(b) Let $\{\mathbf{\Phi_j} | j = 1, \ldots, N\}$ and $\{\lambda_j | j = 1, \ldots, N\}$ denote the orthonormal eigenvectors and corresponding eigenvalues of $\mathbf{K}_N$. By writing

$$\hat{\mathbf{x}} = \sum_{j=1}^{N} a_j(\mu)\mathbf{\Phi}_j, \qquad \mu = \frac{d}{c},$$

find the coefficients $\{a_j(\mu) | j = 1, \ldots, N\}$.



(c) Show that the n-th element of $\hat{\mathbf{x}}$ can also be written as

$$\frac{\lambda_+(\mu)^n - \lambda_-(\mu)^n}{\lambda_+(\mu)^{N+1} - \lambda_-(\mu)^{N+1}}, \qquad n = 1, \ldots, N,$$

for suitable choices of the parameters $\lambda_\pm(\mu)$.

(d) The uniqueness theorem for harmonic potentials discussed in lectures has an analogous version when the conductances are not all equal. Use this fact to establish a discrete identity involving your answers to parts (b) and (c).

(e) Now pick μ to be given by

$$\mu = \frac{1}{(N+1)^2}$$

and introduce the new variable

$$y = \frac{n}{(N+1)}.$$

Find the limit of **both** left- and right-hand sides of the discrete identity you found in part (d) as $N \to \infty$ with y taken to be fixed.

THE END