Part I

(i) This is a standard calculus exercise. On use of integration by parts, twice, it can be shown that

$$I_+(n) = \frac{n\pi(1-e(-1)^n)}{1+(n\pi)^2}, \qquad I_-(n) = \frac{n\pi(1-e^{-1}(-1)^n)}{1+(n\pi)^2}.$$

(ii) Writing

$$\sinh y = \sum_{m=1}^{\infty} c_m \sin(m\pi y)$$

and multiplying by $\sin(n\pi y)$ and integrating over the interval $[0,1]$ we find

$$\int_0^1 \sinh y \sin(n\pi y) dy = \int_0^1 \sum_{m=1}^{\infty} c_m \sin(m\pi y) \sin(n\pi y) dy$$

giving

$$\frac{c_n}{2} = \frac{I_+(n) - I_-(n)}{2}, \qquad \text{or} \qquad c_n = I_+(n) - I_-(n). \tag{1}$$

Similarly, writing

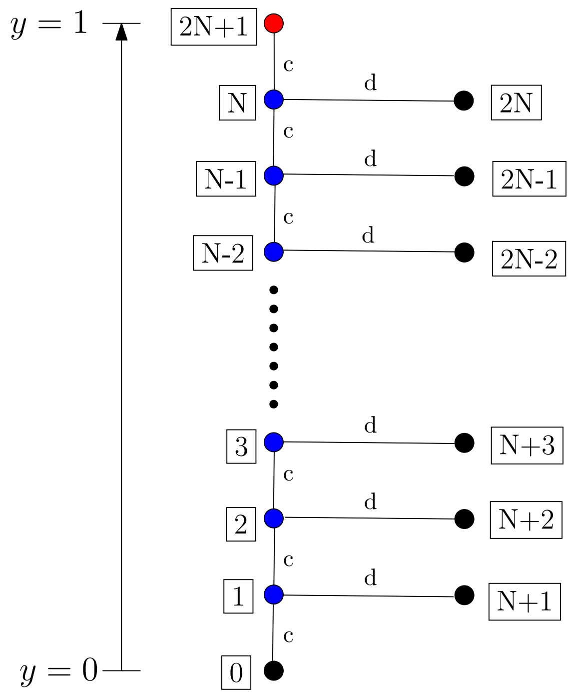
$$\cosh y = \sum_{m=1}^{\infty} d_m \sin(m\pi y)$$

we find

$$\frac{d_n}{2} = \frac{I_+(n) + I_-(n)}{2}, \qquad \text{or} \qquad d_n = I_+(n) + I_-(n). \tag{2}$$

Part II

(a) For the graph



the generalized Laplacian can be found using the usual construction, and using the node ordering given in the question, to be

$$\mathbf{K} = \begin{pmatrix} c\mathbf{K}_N + d\mathbf{I}_N & -d\mathbf{I}_N & -c\mathbf{P} \\ -d\mathbf{I}_N & d\mathbf{I}_N & \mathbf{0} \\ -c\mathbf{P}^T & \mathbf{0} & c\mathbf{I}_2 \end{pmatrix},$$

where the N-by-2 matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) The subsystem to solve for $\hat{\mathbf{x}}$ can therefore be found to be

$$(c\mathbf{K}_N + d\mathbf{I}_N)\hat{\mathbf{x}} = c\mathbf{Pe} = c\begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix}. \tag{3}$$

On division by c, and introduction of $\mu = d/c$, this becomes

$$(\mathbf{K}_N + \mu\mathbf{I}_N)\hat{\mathbf{x}} = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix}.$$

It is known from lectures that the matrix

$$\mathbf{K}_N = \begin{pmatrix} 2 & -1 & 0 & 0 & . & . & 0 & 0 \\ -1 & 2 & -1 & 0 & . & . & 0 & 0 \\ 0 & -1 & 2 & -1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & 2 & -1 \\ 0 & 0 & 0 & 0 & . & . & -1 & 2 \end{pmatrix}, \qquad N \geq 2$$

has orthonormal eigenvectors

$$\mathbf{\Phi}_m = \sqrt{\frac{2}{N+1}}\begin{pmatrix} \sin\left(\frac{m\pi}{N+1}\right) \\ \sin\left(\frac{2m\pi}{N+1}\right) \\ . \\ . \\ \sin\left(\frac{nm\pi}{N+1}\right) \end{pmatrix}, \qquad m = 1, \cdots, N$$

with corresponding eigenvalues

$$\lambda_m = 2 - 2\cos\left(\frac{\pi m}{N+1}\right).$$

Therefore, write

$$\hat{\mathbf{x}} = \sum_{j=1}^{N} a_j \mathbf{\Phi}_j \tag{4}$$

for some coefficients $\{a_j | j = 1, \ldots, N\}$ to be found. On substitution into (3), we find

$$(\mathbf{K}_N + \mu \mathbf{I}_N)\hat{\mathbf{x}} = (\mathbf{K}_N + \mu \mathbf{I}_N) \sum_{j=1}^{N} a_j \mathbf{\Phi}_j = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix}$$

or, using the properties of the eigenvectors,

$$\sum_{j=1}^{N} (\lambda_j + \mu) a_j \mathbf{\Phi}_j = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix}.$$

Now multiply this equation by $\mathbf{\Phi}_m^T$ and use the orthonormality of the eigenvectors:

$$\sum_{j=1}^{N} (\lambda_j + \mu) a_j \mathbf{\Phi}_m^T \mathbf{\Phi}_j = \mathbf{\Phi}_m^T \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix} = \sqrt{\frac{2}{N+1}} \sin\left(\frac{Nm\pi}{N+1}\right). \tag{5}$$

Since $\mathbf{\Phi}_m^T \mathbf{\Phi}_j = \delta_{mj}$ we infer that

$$a_m = \sqrt{\frac{2}{N+1}} \frac{1}{(\mu + \lambda_m)} \sin\left(\frac{Nm\pi}{N+1}\right)$$

so that

$$\hat{\mathbf{x}} = \sum_{m=1}^{N} \sqrt{\frac{2}{N+1}} \frac{1}{(\mu + \lambda_m)} \sin\left(\frac{Nm\pi}{N+1}\right) \mathbf{\Phi}_m. \tag{6}$$

Hence

$$\phi_n = \sum_{m=1}^{N} \frac{2}{N+1} \frac{1}{(\mu + \lambda_m)} \sin\left(\frac{Nm\pi}{N+1}\right) \sin\left(\frac{nm\pi}{N+1}\right)$$

or, on use of the trigonometric identity,

$$\sin\left(\frac{Nm\pi}{N+1}\right) = (-1)^{m+1} \sin\left(\frac{m\pi}{N+1}\right)$$

we can write this as

$$\phi_n = \sum_{m=1}^{N} \frac{2}{N+1} \frac{(-1)^{m+1}}{(\mu + \lambda_m)} \sin\left(\frac{m\pi}{N+1}\right) \sin\left(\frac{nm\pi}{N+1}\right). \tag{7}$$

(c) Using the "weighted mean value property" (which is just current balance at the internal nodes) we arrive at the recurrence relation

$$(2c + d)\phi_n = c\phi_{n+1} + c\phi_{n-1}$$

or

$$\phi_{n+1} - (2 + \mu)\phi_n + \phi_{n-1} = 0.$$

This is a constant-coefficient recurrence relation so therefore try solutions of the form $\phi_n = A\lambda^n$. It has general solution

$$\phi_n = C_1\lambda_+^n + C_2\lambda_-^n,$$

where

$$\lambda_\pm(\mu) = \frac{2 + \mu \pm \sqrt{4\mu + \mu^2}}{2} \tag{8}$$

(these are the two roots of $\lambda^2 - (2 + \mu)\lambda + 1 = 0$) and where we fix the constants C_1 and C_2 using the boundary conditions $\phi_0 = 0$ and "$\phi_{N+1} = 0$" (where the inverted commas are there because we have slightly abused notation because the end node is really the node labelled $\boxed{\text{2N+1}}$ in the diagram). This leads to

$$\phi_n = \frac{\lambda_+(\mu)^n - \lambda_-(\mu)^n}{\lambda_+(\mu)^{N+1} - \lambda_-(\mu)^{N+1}}. \tag{9}$$

4 marks

(d) By the uniqueness theorem, the results in (7) and (9) must correspond. This gives the discrete identity

$$\phi_n = \sum_{m=1}^{N} \frac{2}{N+1} \frac{(-1)^{m+1}}{(\mu + \lambda_m)} \sin\left(\frac{m\pi}{N+1}\right) \sin\left(\frac{nm\pi}{N+1}\right) = \frac{\lambda_+(\mu)^n - \lambda_-(\mu)^n}{\lambda_+(\mu)^{N+1} - \lambda_-(\mu)^{N+1}}. \tag{10}$$

(e) We now take the continuum limit of both sides of this discrete identity.

Left hand side: Suppose we now think of $\phi_n = \Phi(y_n)$ where $y_n = n/(N+1)$ then we can write (7) as

$$\Phi(y_n) = \sum_{m=1}^{N} \frac{2}{N+1} \frac{(-1)^{m+1}}{(\mu+\lambda_m)} \sin\left(\frac{m\pi}{N+1}\right) \sin\left(m\pi y_n\right).$$

With $\mu = 1/(N+1)^2$ and on noting that for any fixed m, as N gets large

$$\lambda_m + \mu = 2 - 2\cos\left(\frac{\pi m}{N+1}\right) + \frac{1}{(N+1)^2} \approx \frac{(\pi m)^2+1}{(N+1)^2}, \qquad \sin\left(\frac{m\pi}{N+1}\right) \approx \frac{m\pi}{N+1}$$

then in the limit $N \to \infty$, we find that, as $N \to \infty$ and renaming $y_n \mapsto y$, (11) has the limit

$$\Phi(y) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{2m\pi}{1+(m\pi)^2} \sin(m\pi y). \tag{11}$$

Right hand side: With $\mu = 1/(N+1)^2$ then as N gets large

$$\lambda_{\pm}(\mu) \approx 1 \pm \frac{1}{N+1} + \dots$$

Hence, since $n = y_n(N+1)$, we can write

$$\begin{aligned}
\phi_n = \Phi(y_n) &= \frac{\lambda_+(\mu)^n - \lambda_-(\mu)^n}{\lambda_+(\mu)^{N+1} - \lambda_-(\mu)^{N+1}} \\
&\approx \frac{(1+1/(N+1)^{(N+1)y_n} - (1-1/(N+1)^{(N+1)y_n}}{(1+1/(N+1))^{N+1} - (1-1/(N+1))^{N+1}} \\
&= \frac{((1+1/(N+1))^{N+1})^{y_n} - ((1-1/(N+1))^{N+1})^{y_n}}{(1+1/(N+1))^{N+1} - (1-1/(N+1))^{N+1}} \to \frac{\sinh y_n}{\sinh 1},
\end{aligned}$$

where we have used the fact that

$$\lim_{N\to\infty} \left(1 \pm \frac{1}{N+1}\right)^{N+1} = e^{\pm 1}.$$

Renaming $y_n \mapsto y$,

$$\Phi(y) = \frac{\sinh y}{\sinh 1}. \tag{12}$$

In summary, the continuous version of the identity is therefore

$$\frac{\sinh y}{\sinh 1} = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{2m\pi}{1+(m\pi)^2} \sin(m\pi y).$$

Note: A check on the analysis in Part II can be made by referring to the answer in Part I where the Fourier sine series of $\sinh y$ is found to be

$$\begin{aligned}
\sinh y &= \sum_{n=1}^{\infty} (I_+(n) - I_-(n)) \sin(n\pi y) \\
&= \sum_{n=1}^{\infty} \left(\frac{n\pi(1 - e(-1)^n)}{1 + (n\pi)^2} - \frac{n\pi(1 - e^{-1}(-1)^n)}{1 + (n\pi)^2} \right) \sin(n\pi y) \\
&= \sum_{n=1}^{\infty} \left(\frac{n\pi(e(-1)^{n+1})}{1 + (n\pi)^2} - \frac{n\pi(e^{-1}(-1)^{n+1})}{1 + (n\pi)^2} \right) \sin(n\pi y) \\
&= \sinh 1 \sum_{n=1}^{\infty} \frac{2n\pi(-1)^{n+1})}{1 + (n\pi)^2} \sin(n\pi y)
\end{aligned}$$

leaving

$$\frac{\sinh y}{\sinh 1} = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{2m\pi}{1 + (m\pi)^2} \sin(m\pi y)$$

which is the same identity as found in Part II.