

Coursework II

CID number: Trust me, you don't want to know it.

MATH40007: Introduction to Applied Mathematics , 2023

**Imperial College
London**

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Problem 1

Part I: For any integer $n \geq 0$ define

$$I_+(n) \equiv \int_0^1 e^y \sin(n\pi y) dy, \quad I_-(n) \equiv \int_0^1 e^{-y} \sin(n\pi y) dy.$$

(i) Calculate these two integrals explicitly.

(ii) Use the result of part (i) to find the Fourier sine series of both $\sinh y$ and $\cosh y$ over the interval $[0, 1]$ (you should use ideas from the "Calculus and Applications" course).

Part II: Consider the electric circuit shown in the Figure where the vertical edges have conductance c and the horizontal edges have conductance d . Node $2N+1$ is set to unit voltage, while nodes 0 and $N+1$ to $2N$ are grounded (set to zero voltage). Kirchhoff's current law holds at nodes 1 to N . Let $\hat{\mathbf{x}}$ denote the voltages at nodes 1 to N . The nodes should be ordered as follows: $1, 2, \dots, 2N-1, 2N, 0, 2N+1$.

(a) Show that the conductance-weighted Laplacian matrix is

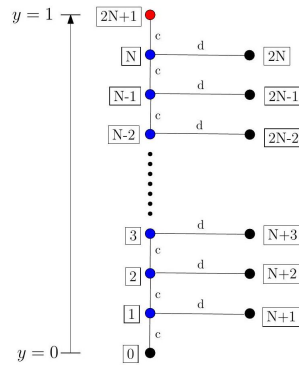
$$\mathbf{K} = \begin{pmatrix} c\mathbf{K}_N + d\mathbf{I}_N & -d\mathbf{I}_N & -c\mathbf{P} \\ -d\mathbf{I}_N & d\mathbf{I}_N & \mathbf{0} \\ -c\mathbf{P}^T & \mathbf{0} & c\mathbf{I}_2 \end{pmatrix},$$

where \mathbf{I}_j denotes the j -by- j identity matrix and \mathbf{K}_N is the N -by- N matrix familiar from lectures. You should find the N -by- 2 matrix \mathbf{P} .

(b) Let $\{\Phi_j \mid j = 1, \dots, N\}$ and $\{\lambda_j \mid j = 1, \dots, N\}$ denote the orthonormal eigenvectors and corresponding eigenvalues of \mathbf{K}_N . By writing

$$\hat{\mathbf{x}} = \sum_{j=1}^N a_j(\mu) \Phi_j, \quad \mu = \frac{d}{c}$$

find the coefficients $\{a_j(\mu) \mid j = 1, \dots, N\}$.



(c) Show that the n -th element of $\hat{\mathbf{x}}$ can also be written as

$$\frac{\lambda_+(\mu)^n - \lambda_-(\mu)^n}{\lambda_+(\mu)^{N+1} - \lambda_-(\mu)^{N+1}}, \quad n = 1, \dots, N,$$

for suitable choices of the parameters $\lambda_{\pm}(\mu)$.

(d) The uniqueness theorem for harmonic potentials discussed in lectures has an analogous version when the conductances are not all equal. Use this fact to establish a discrete identity involving your answers to parts (b) and (c).

(e) Now pick μ to be given by

$$\mu = \frac{1}{(N+1)^2}$$

and introduce the new variable

$$y = \frac{n}{(N+1)}$$

Find the limit of both left- and right-hand sides of the discrete identity you found in part (d) as $N \rightarrow \infty$ with y taken to be fixed.

Solution.

Part I: (i)

$$\begin{aligned} I_+(n) &= \int_0^1 e^y \sin(n\pi y) dy \\ &= \int_0^1 \sin(n\pi y) d(e^y) \\ &= \sin(n\pi y) e^y \Big|_0^1 - n\pi \int_0^1 e^y \cos(n\pi y) dy \\ &= -n\pi \int_0^1 e^y \cos(n\pi y) dy \\ &= -n\pi \int_0^1 \cos(n\pi y) d(e^y) \\ &= -n\pi \cos(n\pi y) e^y \Big|_0^1 - n^2 \pi^2 \int_0^1 e^y \sin(n\pi y) dy \\ &= -n\pi (\cos(n\pi) e - 1) - n^2 \pi^2 I_+(n) \\ (n^2 \pi^2 + 1) I_+(n) &= n\pi (1 - (-1)^n e) \\ I_+(n) &= \frac{n\pi (1 - (-1)^n e)}{n^2 \pi^2 + 1} \\ I_-(n) &= \int_0^1 e^{-y} \sin(n\pi y) dy \\ &= - \int_0^1 \sin(n\pi y) d(e^{-y}) \\ &= -e^{-y} \sin(n\pi y) \Big|_0^1 + n\pi \int_0^1 e^{-y} \cos(n\pi y) dy \\ &= -n\pi \int_0^1 \cos(n\pi y) d(e^{-y}) \end{aligned}$$

$$\begin{aligned}
&= -n\pi \cos(n\pi y) e^{-y} \Big|_0^1 - n^2 \pi^2 \int_0^1 e^{-y} \sin(n\pi y) dy \\
&= -n\pi (\cos(n\pi) e^{-1} - 1) - n^2 \pi^2 I_-(n) \\
I_-(n) + n^2 \pi^2 I_-(n) &= n\pi (1 - (-1)^n e^{-1}) \\
I_-(n) &= \frac{n\pi (1 - (-1)^n e^{-1})}{n^2 \pi^2 + 1}
\end{aligned}$$

As conclusion, we have

$$\begin{aligned}
I_+(n) &= \frac{n\pi (1 - (-1)^n e)}{n^2 \pi^2 + 1} \\
I_-(n) &= \frac{n\pi (1 - (-1)^n e^{-1})}{n^2 \pi^2 + 1}
\end{aligned}$$

- (ii) As $\sinh y$ is an odd function, we have $a_n = 0$ for all n . Therefore, at the interval $[0, 1]$, we have

$$\begin{aligned}
b_n &= 2 \int_0^1 \frac{e^y - e^{-y}}{2} \sin(n\pi y) dy \\
&= \int_0^1 e^y \sin(n\pi y) dy - \int_0^1 e^{-y} \sin(n\pi y) dy \\
&= I_+(n) - I_-(n)
\end{aligned}$$

Hence, the Fourier sine series of $\sinh y$ is

$$\sinh y = \sum_{n=1}^{\infty} b_n \sin(n\pi y) = \sum_{n=1}^{\infty} (I_+(n) - I_-(n)) \sin(n\pi y) = \sum_{n=1}^{\infty} \frac{n\pi (-1)^n (e^{-1} - e)}{n^2 \pi^2 + 1} \sin(n\pi y)$$

For $\cosh y$, it is an even function. We can do the odd extension of $\cosh y$ to get the Fourier sine series of $\cosh y$. Hence,

$$\begin{aligned}
b_n &= 2 \int_0^1 \frac{e^y + e^{-y}}{2} \sin(n\pi y) dy \\
&= \int_0^1 e^y \sin(n\pi y) dy + \int_0^1 e^{-y} \sin(n\pi y) dy \\
&= I_+(n) + I_-(n)
\end{aligned}$$

Hence, the Fourier sine series of $\cosh y$ is

$$\cosh y = \sum_{n=1}^{\infty} b_n \sin(n\pi y) = \sum_{n=1}^{\infty} (I_+(n) + I_-(n)) \sin(n\pi y) = \sum_{n=1}^{\infty} \frac{2n\pi (-1)^n (e^{-1} + e)}{n^2 \pi^2 + 1} \sin(n\pi y)$$

As conclusion, we have

$$\sinh y = \sum_{n=1}^{\infty} \frac{n\pi (-1)^n (e^{-1} - e)}{n^2 \pi^2 + 1} \sin(n\pi y)$$

$$\cosh y = \sum_{n=1}^n \frac{2n\pi(-1)^n(e^{-1} + e)}{n^2\pi + 1} \sin(n\pi y)$$

Part II: (a) By the order given, The conductance-weighted Laplacian matrix of the graph is given by

$$\mathbf{K} = \begin{bmatrix} 2c+d & -c & 0 & \cdots & 0 & 0 & -d & \cdots & 0 & 0 & -c & 0 \\ -c & 2c+d & -c & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -c & 2c+d & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2c+d & -c & 0 & \cdots & -d & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -c & 2c+d & 0 & \cdots & 0 & -d & 0 & -c \\ -d & 0 & 0 & \cdots & 0 & 0 & d & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -d & 0 & 0 & \cdots & d & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -d & 0 & \cdots & 0 & d & 0 & 0 \\ -c & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & c & 0 \\ 0 & 0 & 0 & \cdots & 0 & -c & 0 & \cdots & 0 & 0 & 0 & c \end{bmatrix}$$

which is equal to

$$\mathbf{K} = \begin{bmatrix} c\mathbf{K}_N + d\mathbf{I}_N & -d\mathbf{I}_N & -c\mathbf{P} \\ -d\mathbf{I}_N & d\mathbf{I}_N & \mathbf{0} \\ -c\mathbf{P}^T & \mathbf{0} & c\mathbf{I}_2 \end{bmatrix},$$

where \mathbf{I}_j denotes the j -by- j identity matrix and \mathbf{K}_N is the N -by- N matrix familiar from lectures and the N -by- N matrix \mathbf{P} is given by

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) For this electric circuit, we have:

$$\mathbf{K}\mathbf{X} = \mathbf{f} \quad (1)$$

$$\begin{bmatrix} c\mathbf{K}_N + d\mathbf{I}_N & -d\mathbf{I}_N & -c\mathbf{P} \\ -d\mathbf{I}_N & d\mathbf{I}_N & \mathbf{0} \\ -c\mathbf{P}^T & \mathbf{0} & c\mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \mathbf{0} \\ \hat{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{C}_{\text{eff}} \\ \hat{\mathbf{f}} \end{bmatrix} \quad (2)$$

where $\hat{\mathbf{x}}$ is the vector of the voltages at the nodes 1 to N . Since the nodes at $N+1$ to $2N$ are grounded, the voltages there are all 0 and $\hat{\mathbf{e}}$ is the vector of the voltages at the voltage source $2N+1$ and 0, \mathbf{C}_{eff} is the vector of the effective conductance, and $\hat{\mathbf{f}}$ is the vector of the applied voltages. As KCL holds at nodes

1 to N , the flux of nodes 1 to N are all zero. In details, we have

$$\hat{\mathbf{e}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \hat{\mathbf{f}} = \begin{bmatrix} -f_0 \\ f_0 \end{bmatrix}$$

The linear system (2) is equivalent to

$$c\mathbf{K}_N\hat{\mathbf{x}} + d\mathbf{I}_N\hat{\mathbf{x}} - d\mathbf{I}_N\mathbf{0} - c\mathbf{P}\hat{\mathbf{e}} = \mathbf{0} \quad (3)$$

$$d\mathbf{I}_N\hat{\mathbf{x}} - d\mathbf{I}_N\mathbf{0} = \mathbf{C}_{\text{eff}} \quad (4)$$

$$-c\mathbf{P}^T\hat{\mathbf{e}} = \hat{\mathbf{f}} \quad (5)$$

Let's consider equation (3), it implies that

$$c\mathbf{K}_N\hat{\mathbf{x}} + d\mathbf{I}_N\hat{\mathbf{x}} = c\mathbf{P}\hat{\mathbf{e}} \quad (6)$$

$$c\mathbf{K}_N\hat{\mathbf{x}} + d\hat{\mathbf{x}} = c\mathbf{P}\hat{\mathbf{e}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ c \end{bmatrix} \quad (7)$$

Let us solve equation (7) using the eigenvectors of \mathbf{K}_N , which we learnt in the lecture.

$$\mathbf{K}_N\Phi_j = \lambda\Phi_j, \quad j = 1, 2, \dots, N \quad (8)$$

where

$$\Phi_j = \sqrt{\frac{2}{N+1}} \begin{pmatrix} \sin\left(\frac{j\pi}{N+1}\right) \\ \sin\left(\frac{2j\pi}{N+1}\right) \\ \vdots \\ \sin\left(\frac{Nj\pi}{N+1}\right) \end{pmatrix}, \quad j = 1, \dots, N$$

which has corresponding eigenvalue

$$\lambda_j = 2 - 2\cos\left(\frac{\pi j}{N+1}\right), \quad j = 1, \dots, N.$$

This orthonormal set of vectors can be used as a basis of the solution space. As

$$\hat{\mathbf{x}} = \sum_{j=1}^N a_j(\mu)\Phi_j, \quad \mu = \frac{d}{c}$$

for some set of coefficients $\{a_j(\mu) \mid j = 1, \dots, N\}$ to be determined. The equation (7) now tells us that

$$c\mathbf{K}_N\hat{\mathbf{x}} + d\hat{\mathbf{x}} = c\mathbf{K}_N\left(\sum_{j=1}^N a_j(\mu)\Phi_j\right) \quad (9)$$

$$= c \sum_{j=1}^N a_j(\mu) \lambda_j \Phi_j + d \sum_{j=1}^N a_j(\mu) \Phi_j \quad (10)$$

$$= \sum_{j=1}^N (ca_j(\mu) \lambda_j + da_j(\mu)) \Phi_j = \sum_{j=1}^N a_j(\mu) (c\lambda_j + d) \Phi_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ c \end{bmatrix} \quad (11)$$

The orthonormality of the eigenvectors can be exploited to find the coefficients $a_j(\mu)$. To see this, note that on multiplying (11) by Φ_j^T , it follows that

$$\sum_{j=1}^N a_j(\mu) (c\lambda_j + d) \Phi_m^T \Phi_j = \Phi_m^T \begin{bmatrix} 0 \\ 0 \\ \vdots \\ c \end{bmatrix} = c \sqrt{\frac{2}{N+1}} \sin\left(\frac{Nm\pi}{N+1}\right).$$

By the orthonormality of the eigenvectors, we have

$$\Phi_m^T \Phi_j = \delta_{mj}.$$

where δ_{mj} is the Kronecker delta. Therefore, we have

$$\begin{aligned} a_m(\mu) (c\lambda_m + d) &= c \sqrt{\frac{2}{N+1}} \sin\left(\frac{Nm\pi}{N+1}\right) \\ a_m(\mu) &= \frac{c \sqrt{\frac{2}{N+1}} \sin\left(\frac{Nm\pi}{N+1}\right)}{c\lambda_m + d} \end{aligned}$$

As $\mu = \frac{d}{c}$, we have

$$a_m(\mu) = \sqrt{\frac{2}{N+1}} \frac{\sin\left(\frac{Nm\pi}{N+1}\right)}{\lambda_m + \mu} = \sqrt{\frac{2}{N+1}} \frac{\sin\left(\frac{Nm\pi}{N+1}\right)}{(2 - 2\cos\left(\frac{Nm\pi}{N+1}\right)) + \mu}$$

Therefore, we have the coefficients $\{a_j(\mu) | j = 1, \dots, N\}$ as

$$a_j(\mu) = \sqrt{\frac{2}{N+1}} \frac{\sin\left(\frac{Nj\pi}{N+1}\right)}{(2 - 2\cos\left(\frac{Nj\pi}{N+1}\right)) + \mu}$$

- (c) As KCL holds at nodes 1 to N and node $2N+1$ is set to unit voltage and node 0 is grounded, we have

$$x_0 = 1, \quad x_{2N+1} = 1$$

For $n = 1, 2, \dots, N$

$$\begin{aligned} c(x_{n+1} - x_n) &= dx_n + c(x_{n-1} - x_n) \\ x_n &= \frac{c}{2c+d} (x_{n+1} + x_{n-1}) \end{aligned}$$

$$= \frac{x_{n+1} + x_{n-1}}{\mu + 2}$$

where $\mu = \frac{d}{c}$. Therefore, we have

$$(2 + \mu)x_n = x_{n-1} + x_{n+1}, \quad n = 1, 2, \dots, N$$

We get such a recursion relation and it is linear, then we can solve it like this. We can transfer the relation into a characteristic equation,

$$(2 + \mu)\lambda^n = \lambda^{n-1} + \lambda^{n+1} \quad (12)$$

$$\lambda^{n+1} - (2 + \mu)\lambda^n + \lambda^{n-1} = 0 \quad (13)$$

$$\lambda^{n-1}(\lambda^2 - (2 + \mu)\lambda + 1) = 0 \quad (14)$$

As $n = 1, 2, \dots, N$, $\lambda^{n-1} \neq 0$, then it must have $\lambda^2 - (2 + \mu)\lambda + 1 = 0$, then we can get the solution of λ ,

$$\lambda = \frac{1}{2}(\mu + 2 \pm \sqrt{4\mu + \mu^2})$$

Therefore, we have

$$x_n = \frac{(\frac{1}{2}(\mu + 2 + \sqrt{4\mu + \mu^2}))^n - (\frac{1}{2}(\mu + 2 - \sqrt{4\mu + \mu^2}))^n}{(\frac{1}{2}(\mu + 2 + \sqrt{4\mu + \mu^2}))^{N+1} - (\frac{1}{2}(\mu + 2 - \sqrt{4\mu + \mu^2}))^{N+1}}$$

We set

$$\begin{aligned} \lambda_+(\mu) &= \frac{1}{2}(\mu + 2 + \sqrt{4\mu + \mu^2}) \\ \lambda_-(\mu) &= \frac{1}{2}(\mu + 2 - \sqrt{4\mu + \mu^2}), \end{aligned}$$

Then, we have

$$x_n = \frac{\lambda_+(\mu)^n - \lambda_-(\mu)^n}{\lambda_+(\mu)^{N+1} - \lambda_-(\mu)^{N+1}} \quad n = 1, 2, \dots, N$$

- (d) By the uniqueness theorem of harmonic potentials, the results from (b) and (c) must be equal. Therefore, we have

$$\frac{\lambda_+(\mu)^n - \lambda_-(\mu)^n}{\lambda_+(\mu)^{N+1} - \lambda_-(\mu)^{N+1}} = \sum_{j=1}^N \frac{2}{N+1} \frac{\sin(\frac{Nj\pi}{N+1})}{(2 - 2\cos(\frac{j\pi}{N+1})) + \mu} \sin(\frac{nj\pi}{N+1}) \quad (15)$$

$$\frac{\lambda_+(\mu)^n - \lambda_-(\mu)^n}{\lambda_+(\mu)^{N+1} - \lambda_-(\mu)^{N+1}} = \frac{2}{N+1} \sum_{j=1}^N \frac{\sin(\frac{Nj\pi}{N+1})}{(2 - 2\cos(\frac{j\pi}{N+1})) + \mu} \sin(\frac{nj\pi}{N+1}) \quad (16)$$

which is the discrete identity.

(e) We take the limit $N \rightarrow \infty$ at the both sides of identity (16) and use

$$\mu = \frac{1}{(N+1)^2}, \quad y = \frac{n}{N+1}.$$

then we get,

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{(\frac{1}{2}(\mu + 2 + \sqrt{4\mu + \mu^2}))^n - (\frac{1}{2}(\mu + 2 - \sqrt{4\mu + \mu^2}))^n}{(\frac{1}{2}(\mu + 2 + \sqrt{4\mu + \mu^2}))^{N+1} - (\frac{1}{2}(\mu + 2 - \sqrt{4\mu + \mu^2}))^{N+1}} \\ = \lim_{N \rightarrow \infty} \sum_{j=1}^{\infty} \frac{2}{N+1} \frac{\sin(\frac{Nj\pi}{N+1})}{(2 - 2\cos(\frac{j\pi}{N+1})) + \frac{1}{(N+1)^2}} \sin(j\pi y) \end{aligned}$$

As $N \rightarrow \infty$, the $\frac{j\pi}{N+1}$ is very small and we use the Taylor series,

$$\begin{aligned} 2 - 2\cos(\frac{j\pi}{N+1}) &= 2(1 - \cos(\frac{j\pi}{N+1})) = 2(1 - (1 - \frac{1}{2!} \frac{j^2\pi^2}{(N+1)^2} + \dots)) = \frac{j^2\pi^2}{(N+1)^2} + \dots \\ \sin(\frac{j\pi}{N+1}) &= \frac{\pi j}{N+1} + \dots \end{aligned}$$

Let us see the limitation again. From the Calculus and Applications course, we know that,

$$\begin{aligned} \lim_{N \rightarrow \infty} \left(\frac{1}{2} \left(\left(\frac{1}{(N+1)^2} \right) + 2 + \sqrt{4 \frac{1}{(N+1)^2} + \frac{1}{(N+1)^2}} \right) \right)^n - \\ \left(\frac{1}{2} \left(\left(\frac{1}{(N+1)^2} \right) + 2 - \sqrt{4 \frac{1}{(N+1)^2} + \frac{1}{(N+1)^2}} \right) \right)^n = 0 \\ \lim_{N \rightarrow \infty} \left(\frac{1}{2} \left(\left(\frac{1}{(N+1)^2} \right) + 2 + \sqrt{4 \frac{1}{(N+1)^2} + \frac{1}{(N+1)^2}} \right) \right)^{N+1} - \\ \left(\frac{1}{2} \left(\left(\frac{1}{(N+1)^2} \right) + 2 - \sqrt{4 \frac{1}{(N+1)^2} + \frac{1}{(N+1)^2}} \right) \right)^{N+1} = e - \frac{1}{e} \end{aligned}$$

For the identity, use the Taylor expansion of $2 - 2\cos(\frac{j\pi}{N+1})$,

$$\begin{aligned} \frac{0}{e - e^{-1}} &= \lim_{N \rightarrow \infty} \sum_{j=1}^{\infty} \frac{2(N+1) \sin(\frac{Nj\pi}{N+1})}{j^2\pi^2 + 1} \sin(j\pi y) \\ 0 &= \lim_{N \rightarrow \infty} \sum_{j=1}^{\infty} \frac{2(N+1)(e - e^{-1}) \sin(\frac{j\pi(N+1) - j\pi}{N+1})}{j^2\pi^2 + 1} \sin(j\pi y) \\ 0 &= \lim_{N \rightarrow \infty} \sum_{j=1}^{\infty} \frac{2(N+1)(e - e^{-1})(\sin(j\pi) \cos(\frac{j\pi}{N+1}) - \cos(j\pi) \sin(\frac{j\pi}{N+1}))}{j^2\pi^2 + 1} \sin(j\pi y) \\ 0 &= \lim_{N \rightarrow \infty} \sum_{j=1}^{\infty} \frac{2(N+1)(e - e^{-1})(-(-1)^j \sin(\frac{j\pi}{N+1}))}{j^2\pi^2 + 1} \sin(j\pi y) \end{aligned}$$

We use the Taylor expansion of $\sin(\frac{j\pi}{N+1})$,

$$\begin{aligned} 0 &= \lim_{N \rightarrow \infty} \sum_{j=1}^{\infty} \frac{2(N+1)(e - e^{-1})(-1)^j \frac{\pi j}{N+1}}{j^2 \pi^2 + 1} \sin(j\pi y) \\ 0 &= 2 \sum_{j=1}^{\infty} \frac{j\pi(e^{-1} - e)(-1)^j}{j^2 \pi^2 + 1} \sin(j\pi y) \\ 0 &= \sum_{j=1}^{\infty} \frac{j\pi(e^{-1} - e)(-1)^j}{j^2 \pi^2 + 1} \sin(j\pi y) \end{aligned}$$

We can observe that

$$\sum_{j=1}^{\infty} \frac{j\pi(e^{-1} - e)(-1)^j}{j^2 \pi^2 + 1} \sin(j\pi y)$$

is the Fourier sine series of $\sinh y$ from Part I.

Since

$$\begin{aligned} y &= \frac{n}{N+1} \rightarrow 0 \quad \text{when } N \rightarrow \infty \\ \sinh y &= 0 \quad \text{when } y = 0 \end{aligned}$$

We have

$$\sum_{j=1}^{\infty} \frac{j\pi(e^{-1} - e)(-1)^j}{j^2 \pi^2 + 1} \sin(j\pi y) = \sinh 0 = 0$$

Therefore, the value of $\sinh y$ when $y = 0$ is coincide with what we calculate in (e). It is clear that the Fourier sine series of $\sinh y$ is zero when y is fixed as $\frac{n}{1+N}$.