Numerical Analysis MATH50003 (2023–24) Problem Sheet 7

Problem 1(a) Show for a unitary matrix $Q \in U(n)$ and a vector $\boldsymbol{x} \in \mathbb{C}^n$ that multiplication by Q preserve the 2-norm: $\|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$.

Problem 1(b) Show that the eigenvalues λ of a unitary matrix Q are on the unit circle: $|\lambda| = 1$. Hint: recall for any eigenvalue λ that there exists a unit eigenvector $\boldsymbol{v} \in \mathbb{C}^n$ (satisfying $\|\boldsymbol{v}\| = 1$).

Problem 1(c) Show for an orthogonal matrix $Q \in O(n)$ that $\det Q = \pm 1$. Give an example of $Q \in U(n)$ such that $\det Q \neq \pm 1$. Hint: recall for any real matrices A and B that $\det A = \det A^\top$ and $\det(AB) = \det A \det B$.

Problem 1(d) A normal matrix commutes with its adjoint. Show that $Q \in U(n)$ is normal.

Problem 1(e) The spectral theorem states that any normal matrix is unitarily diagonalisable: if A is normal then $A = V\Lambda V^\star$ where $V \in U(n)$ and Λ is diagonal. Use this to show that $Q \in U(n)$ is equal to I if and only if all its eigenvalues are 1.

Problem 2 Consider the vectors

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2\mathrm{i} \\ 2 \end{bmatrix}.$$

Use reflections to determine the entries of orthogonal/unitary matrices Q_1, Q_2, Q_3 such that

$$Q_1\boldsymbol{a} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, Q_2\boldsymbol{a} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}, Q_3\boldsymbol{b} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

Problem 3(a) What simple rotation matrices $Q_1, Q_2 \in SO(2)$ have the property that:

$$Q_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}, Q_2 \begin{bmatrix} \sqrt{5} \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Problem 3(b) Find an orthogonal matrix that is a product of two simple rotations but acting on two different subspaces:

$$Q = \underbrace{\begin{bmatrix} \cos\theta_2 & & -\sin\theta_2 \\ & 1 & \\ \sin\theta_2 & & \cos\theta_2 \end{bmatrix}}_{Q_2} \underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & \\ \sin\theta_1 & \cos\theta_1 & \\ & & 1 \end{bmatrix}}_{Q_1}$$

so that, for $\boldsymbol{a}$ defined above,

$$Q\boldsymbol{a} = \begin{bmatrix} \|\boldsymbol{a}\| \\ 0 \\ 0 \end{bmatrix}.$$

Hint: you do not need to determine θ_1, θ_2, instead you can write the entries of Q_1, Q_2 directly using just square-roots.

Problem 4(a) Show that every matrix $A \in \mathbb{R}^{m\times n}$ has a QR factorisation such that the diagonal of R is non-negative. Make sure to include the case of more columns than rows (i.e. $m < n$).

Problem 4(b) Show that the QR factorisation of a square invertible matrix $A \in \mathbb{R}^{n\times n}$ is unique, provided that the diagonal of R is positive.