

## Numerical Analysis MATH50003 (2023–24) Problem Sheet 8

**Problem 1** Give explicit formulae for  $\hat{f}_k$  and  $\hat{f}_k^n$  for the following functions:

$$\cos \theta, \cos 4\theta, \sin^4 \theta, \frac{3}{3 - e^{i\theta}}, \frac{1}{1 - 2e^{i\theta}}$$

### SOLUTION

(1) Just expand in complex exponentials to find that

$$\cos \theta = \frac{\exp(i\theta) + \exp(-i\theta)}{2}$$

that is  $\hat{f}_1 = \hat{f}_{-1} = 1/2$ ,  $\hat{f}_k = 0$  otherwise. Therefore for  $p \in \mathbb{Z}$  we have

$$\begin{aligned}\hat{f}_k^1 &= \hat{f}_1 + \hat{f}_{-1} = 1 \\ \hat{f}_{2p}^2 &= 0, \hat{f}_{2p+1}^2 = \hat{f}_1 + \hat{f}_{-1} = 1 \\ \hat{f}_{1+np}^n &= \hat{f}_{-1+np}^n = 1/2, \hat{f}_k^n = 0\end{aligned}$$

for  $n = 3, 4, \dots$

(2) Similarly

$$\cos 4\theta = \frac{\exp(4i\theta) + \exp(-4i\theta)}{2}$$

that is  $\hat{f}_4 = \hat{f}_{-4} = 1/2$ ,  $\hat{f}_k = 0$  otherwise. Therefore for  $p \in \mathbb{Z}$  we have

$$\begin{aligned}\hat{f}_p^1 &= \hat{f}_4 + \hat{f}_{-4} = 1 \\ \hat{f}_{2p}^2 &= \hat{f}_4 + \hat{f}_{-4} = 1, \hat{f}_{2p+1}^2 = 0 \\ \hat{f}_{3p}^3 &= 0, \hat{f}_{3p\pm 1}^3 = \hat{f}_{\pm 4} = 1/2 \\ \hat{f}_{4p}^4 &= \hat{f}_{-4} + \hat{f}_4 = 1, \hat{f}_{4p\pm 1}^4 = 0, \hat{f}_{4p+2}^4 = 0 \\ \hat{f}_{5p}^5 &= 0, \hat{f}_{5p+1}^5 = \hat{f}_{-4} = 1/2, \hat{f}_{5p-1}^5 = \hat{f}_4 = 1/2, \hat{f}_{5p\pm 2}^5 = 0 \\ \hat{f}_{6p}^6 &= 0, \hat{f}_{6p\pm 1}^6 = 0, \hat{f}_{6p+2}^6 = \hat{f}_{-4} = 1/2, \hat{f}_{6p-2}^6 = \hat{f}_4 = 1/2, \hat{f}_{6p\pm 3}^6 = 0 \\ \hat{f}_{7p}^7 &= 0, \hat{f}_{7p\pm 1}^7 = 0, \hat{f}_{7p\pm 2}^7 = 0, \hat{f}_{7p\pm 3}^7 = \hat{f}_{\mp 4} = 1/2 \\ \hat{f}_{8p}^8 &= \hat{f}_{8p\pm 1}^8 = \hat{f}_{8p\pm 2}^8 = \hat{f}_{8p\pm 3}^8 = 0, \hat{f}_{8p+4}^8 = \hat{f}_4 + \hat{f}_{-4} = 1 \\ \hat{f}_{k+pn}^n &= \hat{f}_k \text{ for } -4 \leq k \leq 4, 0 \text{ otherwise.}\end{aligned}$$

(3) Here we have:

$$\begin{aligned}(\sin \theta)^4 &= \left( \frac{\exp(i\theta) - \exp(-i\theta)}{2i} \right)^4 = \left( \frac{\exp(2i\theta) - 2 + \exp(-2i\theta)}{-4} \right)^2 \\ &= \frac{\exp(4i\theta) - 4\exp(2i\theta) + 6 - 4\exp(-2i\theta) + \exp(-4i\theta)}{16}\end{aligned}$$

that is  $\hat{f}_{-4} = \hat{f}_4 = 1/16$ ,  $\hat{f}_{-2} = \hat{f}_2 = -1/4$ ,  $\hat{f}_0 = 3/8$ ,  $\hat{f}_k = 0$  otherwise. Therefore for  $p \in \mathbb{Z}$

we have

$$\begin{aligned}
\hat{f}_p^1 &= \hat{f}_{-4} + \hat{f}_{-2} + \hat{f}_0 + \hat{f}_2 + \hat{f}_4 = 0 \\
\hat{f}_k^2 &= 0 \\
\hat{f}_{3p}^3 &= \hat{f}_0 = 3/8, \hat{f}_{3p+1}^3 = \hat{f}_{-2} + \hat{f}_4 = -3/16, \hat{f}_{3p-1}^3 = \hat{f}_2 + \hat{f}_{-4} = -3/16 \\
\hat{f}_{4p}^4 &= \hat{f}_0 + \hat{f}_{-4} + \hat{f}_4 = 1/2, \hat{f}_{4p+1}^4 = 0, \hat{f}_{4p+2}^4 = \hat{f}_2 + \hat{f}_{-2} = -1/2 \\
\hat{f}_{5p}^5 &= \hat{f}_0 = 3/8, \hat{f}_{5p+1}^5 = \hat{f}_{-4} = 1/16, \hat{f}_{5p-1}^5 = \hat{f}_4 = 1/16, \hat{f}_{5p+2}^5 = \hat{f}_2 = -1/4, \hat{f}_{5p-2}^5 = \hat{f}_{-2} = -1/4 \\
\hat{f}_{6p}^6 &= \hat{f}_0 = 3/8, \hat{f}_{6p+1}^6 = 0, \hat{f}_{6p+2}^6 = \hat{f}_2 + \hat{f}_{-4} = -3/16, \hat{f}_{6p-2}^6 = \hat{f}_{-2} + \hat{f}_4 = -3/16, \hat{f}_{6p+3}^6 = 0 \\
\hat{f}_{7p}^7 &= \hat{f}_0 = 3/8, \hat{f}_{7p+1}^7 = 0, \hat{f}_{7p+2}^7 = \hat{f}_{\pm 2} = -1/4, \hat{f}_{7p+3}^7 = \hat{f}_{\mp 4} = 1/16 \\
\hat{f}_{8p}^8 &= \hat{f}_0 = 3/8, \hat{f}_{8p+1}^8 = 0, \hat{f}_{8p+2}^8 = \hat{f}_{\pm 2} = -1/4, \hat{f}_{8p+3}^8 = 0, \hat{f}_{8p+4}^8 = \hat{f}_4 + \hat{f}_{-4} = 1/8 \\
\hat{f}_{k+pn}^n &= \hat{f}_k \text{ for } -4 \leq k \leq 4, 0 \text{ otherwise.}
\end{aligned}$$

(4) Under the change of variables  $z = e^{i\theta}$  we can use Geometric series to determine

$$\frac{3}{3-z} = \frac{1}{1-z/3} = \sum_{k=0}^{\infty} \frac{z^k}{3^k}$$

That is  $\hat{f}_k = 1/3^k$  for  $k \geq 0$ , and  $\hat{f}_k = 0$  otherwise. We then have for  $0 \leq k \leq n-1$

$$\hat{f}_{k+pn}^n = \sum_{\ell=0}^{\infty} \frac{1}{3^{k+\ell n}} = \frac{1}{3^k} \frac{1}{1-1/3^n} = \frac{3^n}{3^{n+k} - 3^k}$$

(5) Now make the change of variables  $z = e^{-i\theta}$  to get:

$$\frac{1}{1-2/z} = \frac{1}{-2/z} \frac{1}{1-z/2} = \frac{1}{-2/z} \sum_{k=0}^{\infty} \frac{z^k}{2^k} = - \sum_{k=1}^{\infty} \frac{e^{-ik\theta}}{2^k}$$

That is  $\hat{f}_k = -1/2^{-k}$  for  $k \leq -1$  and 0 otherwise. We then have for  $-n \leq k \leq -1$

$$\hat{f}_{k+pn}^n = - \sum_{\ell=0}^{\infty} \frac{1}{2^{-k+\ell n}} = - \frac{1}{2^{-k}} \frac{1}{1-1/2^n} = - \frac{2^{n+k}}{2^n - 1}$$

**END**

**Problem 2** Prove that if the first  $\lambda - 1$  derivatives  $f(\theta), f'(\theta), \dots, f^{(\lambda-1)}(\theta)$  are  $2\pi$ -periodic and  $f^{(\lambda)}$  is uniformly bounded that

$$|\hat{f}_k| = O(|k|^{-\lambda}) \quad \text{as } |k| \rightarrow \infty$$

Use this to show for the Taylor case ( $0 = \hat{f}_{-1} = \hat{f}_{-2} = \dots$ ) that

$$|f(\theta) - \sum_{k=0}^{n-1} \hat{f}_k e^{ik\theta}| = O(n^{1-\lambda}) \quad \text{as } n \rightarrow \infty$$

**SOLUTION** A straightforward application of integration by parts yields the result

$$\hat{f}_k = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-ik\theta} d\theta = \frac{(-i)^\lambda}{2\pi k^\lambda} \int_0^{2\pi} f^{(\lambda)}(\theta) e^{-ik\theta} d\theta$$

given that  $f^{(\lambda)}$  is uniformly bounded, the second part follows directly from this result

$$\left| \sum_{k=n}^{\infty} \hat{f}_k e^{ik\theta} \right| \leq \sum_{k=n}^{\infty} |\hat{f}_k| \leq C \sum_{k=n}^{\infty} k^{-\lambda}$$

for some constant  $C$ . The result then follows by the dominant convergence test:

$$\sum_{k=n}^{\infty} k^{-\lambda} \leq \int_{n-1}^{\infty} k^{-\lambda} dk = O(n^{1-\lambda}).$$

**END**

**Problem 3(a)** If  $f$  is a trigonometric polynomial ( $\hat{f}_k = 0$  for  $|k| > m$ ) show for  $n \geq 2m + 1$  that we can exactly recover  $f$ :

$$f(\theta) = \sum_{k=-m}^m \hat{f}_k^n e^{ik\theta}$$

**SOLUTION** This follows from

$$\hat{f}_k^n = \sum_{p=-\infty}^{\infty} \hat{f}_{k+pn} = \hat{f}_k$$

if  $-m \leq k \leq m$  as if  $p > 0$  we have  $k + pn \geq k + 2m + 1 \geq m + 1$  hence  $\hat{f}_{k+pn} = 0$  and if  $k < 0$  we have  $k - pn \leq k - 2m - 1 \leq -m - 1$  hence  $\hat{f}_{k+pn} = 0$ .

**END**

**Problem 3(b)** For the general (non-Taylor) case and  $n = 2m + 1$ , prove convergence for

$$f_{-m:m}(\theta) := \sum_{k=-m}^m \hat{f}_k^n e^{ik\theta}$$

to  $f(\theta)$  as  $n \rightarrow \infty$ . What is the rate of convergence if we know that the first  $\lambda - 1$  derivatives  $f(\theta), f'(\theta), \dots, f^{(\lambda-1)}(\theta)$  are  $2\pi$ -periodic and  $f^{(\lambda)}$  is uniformly bounded?

**SOLUTION**

Observe that by aliasing (see corollary in lecture notes) and triangle inequality we have the following

$$|\hat{f}_k^n - \hat{f}_k| \leq \sum_{p=1}^{\infty} (|\hat{f}_{k+pn}| + |\hat{f}_{k-pn}|)$$

Using the result from Problem 2 yields

$$|\hat{f}_k^n - \hat{f}_k| \leq \frac{C}{n^\lambda} \sum_{p=1}^{\infty} \frac{1}{\left(p + \frac{k}{n}\right)^\lambda} + \frac{1}{\left(p - \frac{k}{n}\right)^\lambda}$$

now we pick  $|q| < \frac{1}{2}$  (such that the estimate below will hold for both summands above) and construct an integral with convex and monotonically decreasing integrand such that

$$(p + q)^{-\lambda} < \int_{p-\frac{1}{2}}^{p+\frac{1}{2}} (x + q)^{-\lambda} dx$$

more over summing over the left-hand side from 1 to  $\infty$  yields a bound by the integral:

$$\int_{\frac{1}{2}}^{\infty} (x + q)^{-\lambda} dx = \frac{1}{\lambda} \left(\frac{1}{2} + q\right)^{-\lambda+1}$$

Finally let  $q = \pm \frac{k}{n}$  to achieve the rate of convergence

$$|\hat{f}_k^n - \hat{f}_k| \leq \frac{C_\lambda}{n^\lambda} \left( \left( \frac{1}{2} + k/n \right)^{-\lambda+1} + \left( \left( \frac{1}{2} - k/n \right) \right)^{-\lambda+1} \right)$$

where  $C_\lambda$  is a constant depending on  $\lambda$ . Note that it is indeed important to split the  $n$  coefficients equally over the negative and positive coefficients as stated in the notes, due to the estimate we used above.

Finally, we have:

$$\begin{aligned} |f(\theta) - f_{-m:m}(\theta)| &= \left| \sum_{k=-m}^m (\hat{f}_k - \hat{f}_k^n) z^k + \sum_{k=m+1}^{\infty} \hat{f}_k z^k + \sum_{k=-\infty}^{-m-1} \hat{f}_k z^k \right| \\ &\leq \sum_{k=-m}^m |\hat{f}_k - \hat{f}_k^n| + \sum_{k=m+1}^{\infty} |\hat{f}_k| + \sum_{k=-\infty}^{-m-1} |\hat{f}_k| \\ &\leq \sum_{k=-m}^m \frac{C_\lambda}{n^\lambda} \left( \left( \frac{1}{2} + k/n \right)^{-\lambda+1} + \left( \left( \frac{1}{2} - k/n \right) \right)^{-\lambda+1} \right) + \sum_{k=m+1}^{\infty} |\hat{f}_k| + \sum_{k=-\infty}^{-m-1} |\hat{f}_k| \\ &= \frac{C_\lambda}{n^\lambda} 2^\lambda + \sum_{k=m+1}^{\infty} |\hat{f}_k| + \sum_{k=-\infty}^{-m-1} |\hat{f}_k| \\ &= O(n^{-\lambda}) + O(n^{1-\lambda}) + O(n^{1-\lambda}) \\ &= O(n^{1-\lambda}) \end{aligned}$$

**END**

**Problem 3(c)** Show that  $f_{-m:m}(\theta)$  interpolates  $f$  at  $\theta_j = 2\pi j/n$  for  $n = 2m + 1$ .

**SOLUTION** Note from the aliasing property we have

$$\begin{aligned} \hat{f}_k^n e^{ik\theta_j} &= \hat{f}_k^n e^{2\pi i k j / n} = \hat{f}_{k+n}^n e^{2\pi i (k+n) j / n} \\ &= \hat{f}_{k+n}^n e^{i(k+n)\theta_j} \end{aligned}$$

Thus we have

$$\begin{aligned} f_{-m:m}(\theta_j) &= \sum_{k=-m}^{-1} \hat{f}_k^n e^{ik\theta_j} + \sum_{k=0}^m \hat{f}_k^n e^{ik\theta_j} \\ &= \sum_{k=-m}^{-1} \hat{f}_{k+n}^n e^{i(k+n)\theta_j} + \sum_{k=0}^m \hat{f}_k^n e^{ik\theta_j} \\ &= \sum_{k=n-m}^{n-1} \hat{f}_k^n e^{i(k)\theta_j} + \sum_{k=0}^m \hat{f}_k^n e^{ik\theta_j} = f_n(\theta_j) = f(\theta_j) \end{aligned}$$

**END**

**Problem 4(a)** Show for  $0 \leq k, \ell \leq n-1$

$$\frac{1}{n} \sum_{j=1}^n \cos k\theta_j \cos \ell\theta_j = \begin{cases} 1 & k = \ell = 0 \\ 1/2 & k = \ell \\ 0 & \text{otherwise} \end{cases}$$

for  $\theta_j = \pi(j - 1/2)/n$ . Hint: Be careful as the  $\theta_j$  differ from before, and only cover half the period,  $[0, \pi]$ . Using symmetry may help. You may also consider replacing  $\cos$  with complex exponentials:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

**SOLUTION** The case  $k = l = 0$  is immediate. Otherwise, we have,

$$\frac{1}{n} \sum_{j=1}^n \cos(k\theta_j) \cos(l\theta_j) = \frac{1}{4n} \sum_{j=1}^n \left[ e^{i(k+l)\theta_j} + e^{-i(k+l)\theta_j} + e^{i(k-l)\theta_j} + e^{-i(k-l)\theta_j} \right].$$

For  $\omega = \exp(i\pi/n)$  and any  $m$  not a multiple of  $2n$  we have

$$\begin{aligned} \sum_{j=1}^n e^{im\theta_j} &= \sum_{j=0}^{n-1} e^{im\pi(j+1/2)/n} = e^{im\pi/(2n)} \sum_{j=0}^{n-1} e^{im\pi j/n} = \omega^{m/2} \sum_{j=0}^{n-1} \omega^{mj} \\ &= \omega^{m/2} \frac{\omega^{nm} - 1}{\omega^m - 1} = \omega^{m/2} \frac{(-1)^m - 1}{\omega^m - 1} \end{aligned}$$

and hence

$$\begin{aligned} \sum_{j=1}^n [e^{im\theta_j} + e^{-im\theta_j}] &= \omega^{m/2} \frac{(-1)^m - 1}{\omega^m - 1} + \omega^{-m/2} \frac{(-1)^m - 1}{\omega^{-m} - 1} \\ &= \omega^{m/2} \frac{(-1)^m - 1}{\omega^m - 1} + \omega^{m/2} \frac{(-1)^m - 1}{1 - \omega^m} = 0. \end{aligned}$$

Since  $0 < k + l \leq 2n - 2$  we know  $k + l$  is not a multiple of  $2n$  hence

$$\sum_{j=1}^n [e^{i(k+l)\theta_j} + e^{-i(k+l)\theta_j}] = 0.$$

Now if  $k = l$  we have

$$\sum_{j=1}^n e^{i(k-l)\theta_j} = \sum_{j=1}^n e^{-i(k-l)\theta_j} = n.$$

Otherwise  $k - l \neq 0$  but also  $1 - n \leq k - l \leq n - 1$  hence  $k - l$  cannot be a multiple of  $2n$ . And thus

$$\sum_{j=1}^n [e^{i(k-l)\theta_j} + e^{i(l-k)\theta_j}] = 0.$$

**END**

**Problem 4(b)** Consider the Discrete Cosine Transform (DCT)

$$C_n := \begin{bmatrix} \sqrt{1/n} & & & \\ & \sqrt{2/n} & & \\ & & \ddots & \\ & & & \sqrt{2/n} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \cos \theta_1 & \cdots & \cos \theta_n \\ \vdots & \ddots & \vdots \\ \cos(n-1)\theta_1 & \cdots & \cos(n-1)\theta_n \end{bmatrix}$$

for  $\theta_j = \pi(j - 1/2)/n$ . Prove that  $C_n$  is orthogonal:  $C_n^\top C_n = C_n C_n^\top = I$ .

**SOLUTION**

The components of  $C_n$  are

$$\mathbf{e}_k^\top C_n \mathbf{e}_j = \frac{1}{\sqrt{n}} \begin{cases} 1 & k = 1 \\ \sqrt{2} & k \neq 1 \end{cases} \cos((k-1)\theta_{j-1}),$$

where  $\theta_j = \pi(j - 1/2)/n$ . We find using the previous part:

$$\begin{aligned} \mathbf{e}_k^\top C_n C_n^\top \mathbf{e}_\ell &= \left( \begin{cases} 1 & k = \ell = 1 \\ \sqrt{2} & k, \ell = 1, k \neq \ell \\ 2 & k, \ell \neq 1 \end{cases} \right) \frac{1}{n} \sum_{j=1}^n \cos((k-1)\theta_{j-1}) \cos((\ell-1)\theta_{j-1}) \\ &= \left( \begin{cases} 1 & k = \ell = 1 \\ \sqrt{2} & k, \ell = 1, k \neq \ell \\ 2 & k, \ell \neq 1 \end{cases} \right) \left( \begin{cases} 1 & k = \ell = 1 \\ 1/2 & k = \ell \\ 0 & \text{otherwise} \end{cases} \right) = \delta_{k\ell}. \end{aligned}$$

**END**

**Problem 5** What polynomial interpolates  $\cos z$  at  $1, \exp(2\pi i/3)$  and  $\exp(-2\pi i/3)$ ?

**SOLUTION** For  $\omega = \exp(2\pi i/3)$ , we use the DFT:

$$\begin{aligned} \begin{pmatrix} \hat{f}_0^3 \\ \hat{f}_1^3 \\ \hat{f}_2^3 \end{pmatrix} &= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \exp(-2\pi i/3) & \exp(2\pi i/3) \\ 1 & \exp(2\pi i/3) & \exp(-2\pi i/3) \end{bmatrix} \begin{pmatrix} \cos(1) \\ \cos(\exp(2\pi i/3)) \\ \cos(\exp(-2\pi i/3)) \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} \cos(1) + \cos(\exp(2\pi i/3)) + \cos(\exp(-2\pi i/3)) \\ \cos(1) + \exp(-2\pi i/3) \cos(\exp(2\pi i/3)) + \exp(2\pi i/3) \cos(\exp(-2\pi i/3)) \\ \cos(1) + \exp(2\pi i/3) \cos(\exp(2\pi i/3)) + \exp(-2\pi i/3) \cos(\exp(-2\pi i/3)) \end{pmatrix} \end{aligned}$$

That is, the polynomial is

$$\begin{aligned} &\frac{\cos(1) + \cos(\exp(2\pi i/3)) + \cos(\exp(-2\pi i/3))}{3} \\ &+ \frac{\cos(1) + \exp(-2\pi i/3) \cos(\exp(2\pi i/3)) + \exp(2\pi i/3) \cos(\exp(-2\pi i/3))}{3} z \\ &+ \frac{\cos(1) + \exp(2\pi i/3) \cos(\exp(2\pi i/3)) + \exp(-2\pi i/3) \cos(\exp(-2\pi i/3))}{3} z^2. \end{aligned}$$

**END**