MATH50003 Numerical Analysis

III.4 Polynomial Interpolation and Regression

Dr Sheehan Olver

Part III

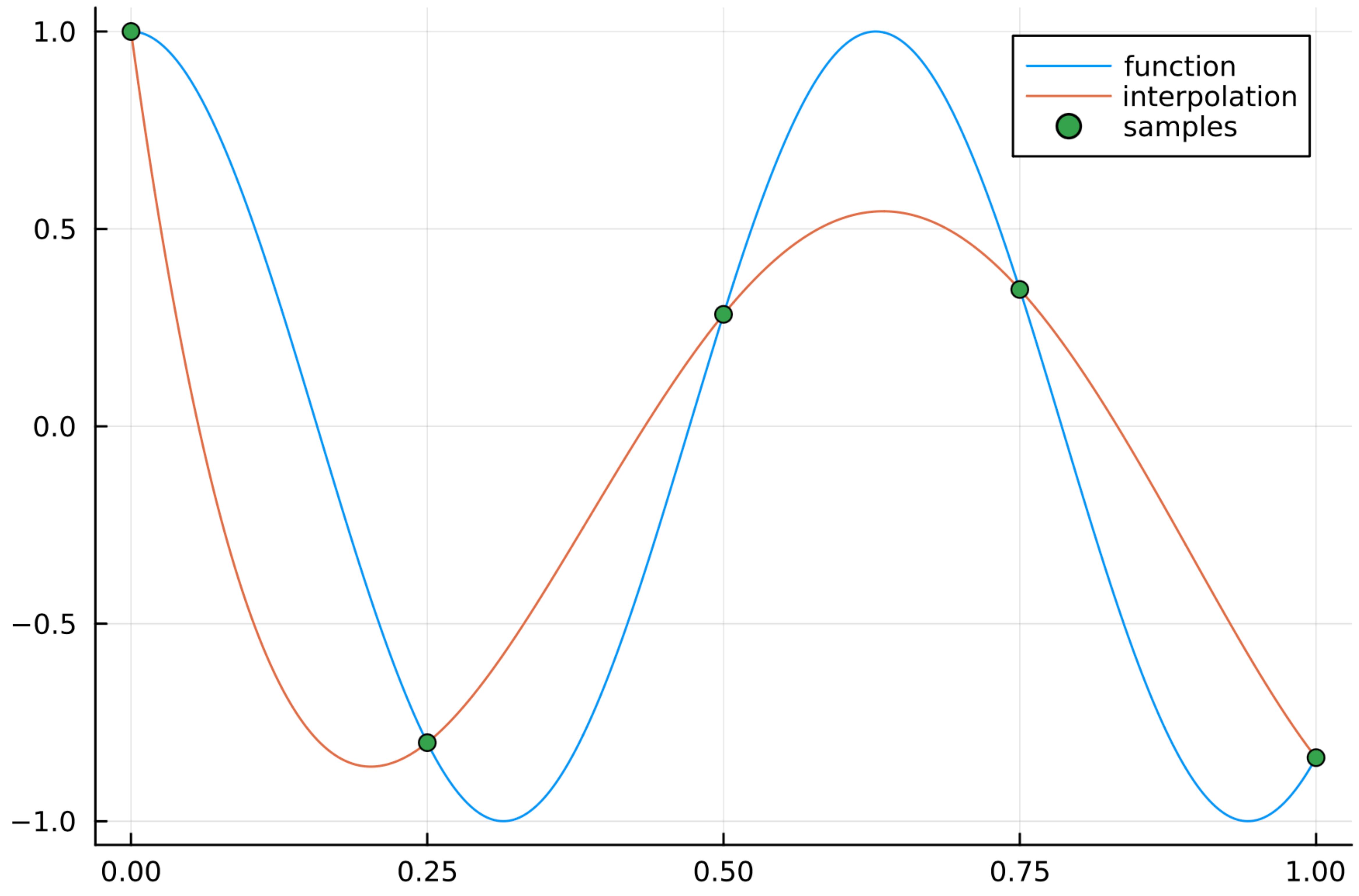
Numerical Linear Algebra

Software
Application
Theory

1. Structured matrices such as banded
2. Differential Equations via finite differences
3. LU and Cholesky factorisation for solving linear systems
4. Polynomial regression for approximating data via least squares
5. Orthogonal matrices such as Householder reflections
6. QR factorisation for solving rectangular least squares problems

III.4.1 Polynomial interpolation

Find a polynomial equal to data at a grid



Definition 20 (interpolatory polynomial)**.** Given *distinct* points $\boldsymbol{x} = [x_1, \ldots, x_n]^\top \in \mathbb{F}^n$ and *data* $\boldsymbol{f} = [f_1, \ldots, f_n]^\top \in \mathbb{F}^n$, a degree $n-1$ *interpolatory polynomial* $p(x)$ satisfies

$$p(x_j) = f_j$$

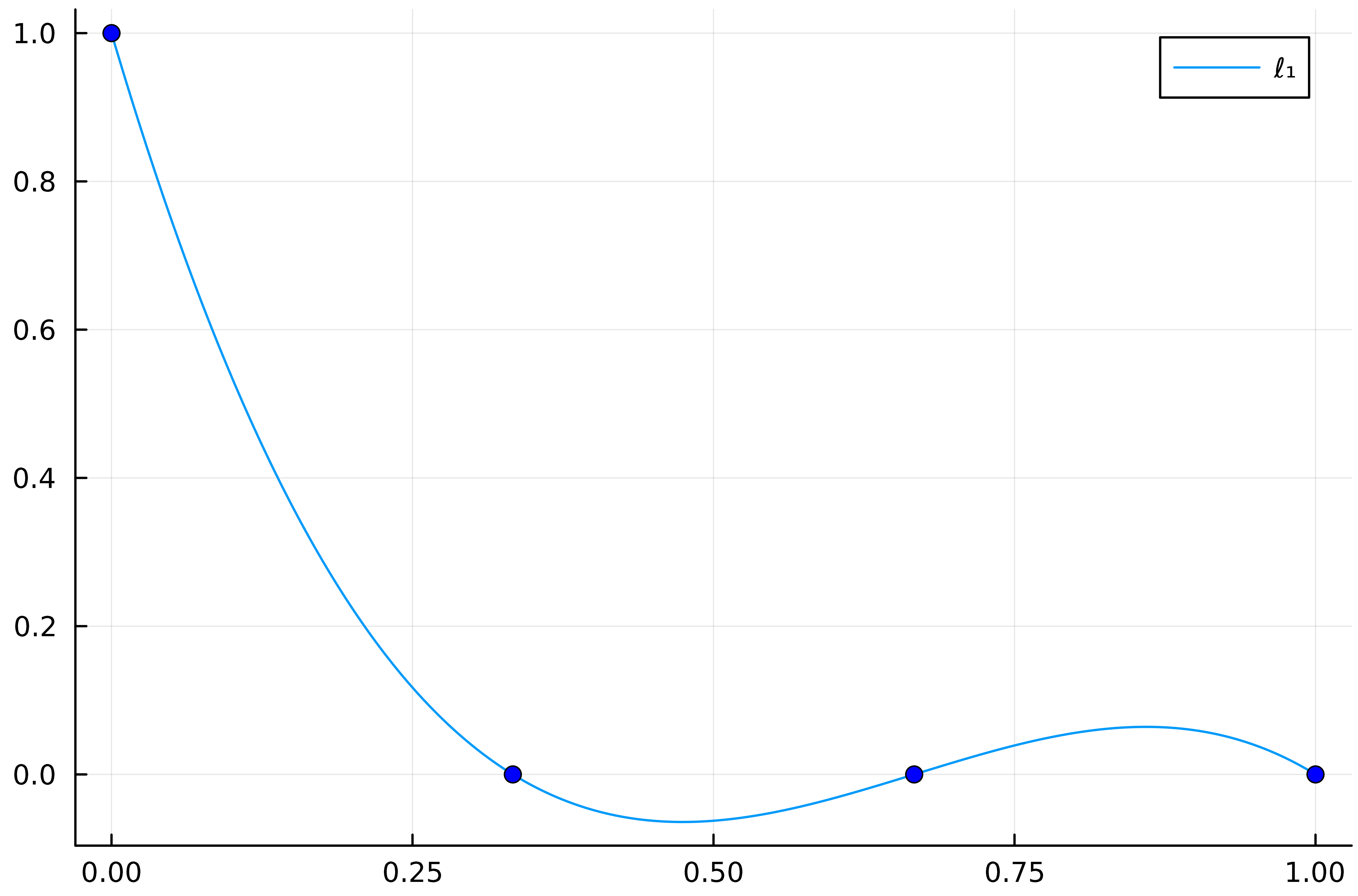
Definition 21 (Vandermonde)**.** The *Vandermonde matrix* associated with $\boldsymbol{x} \in \mathbb{F}^m$ is the matrix

$$V_{\boldsymbol{x},n} := \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & \cdots & x_m^{n-1} \end{bmatrix} \in \mathbb{F}^{m\times n}.$$

Proposition 7 (interpolatory polynomial uniqueness)**.** *Interpolatory polynomials are unique and therefore square Vandermonde matrices are invertible.*

Definition 22 (Lagrange basis polynomial)**.** The *Lagrange basis polynomial* is defined as

$$\ell_k(x) := \prod_{j \neq k} \frac{x - x_j}{x_k - x_j} = \frac{(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}$$



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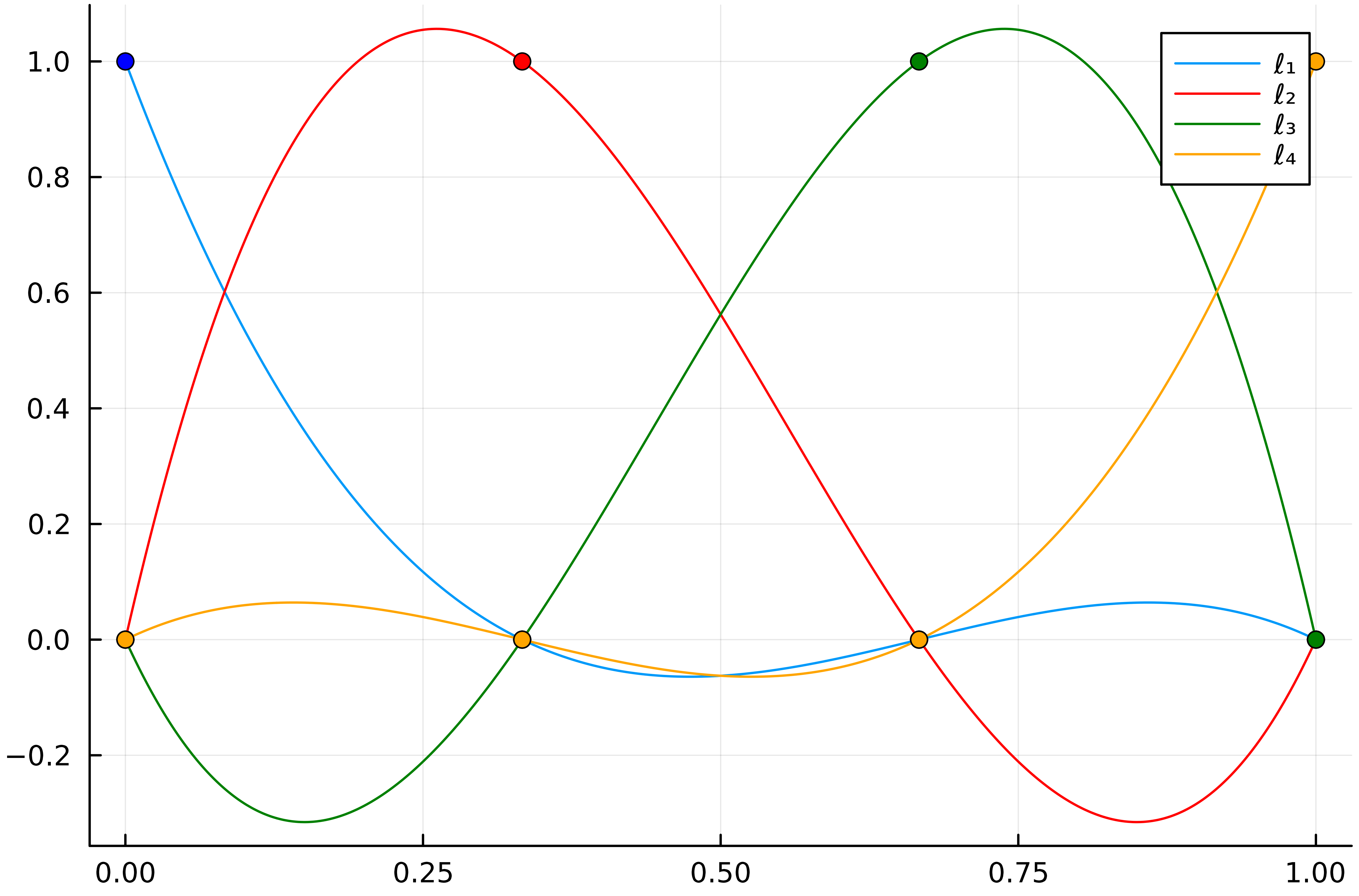


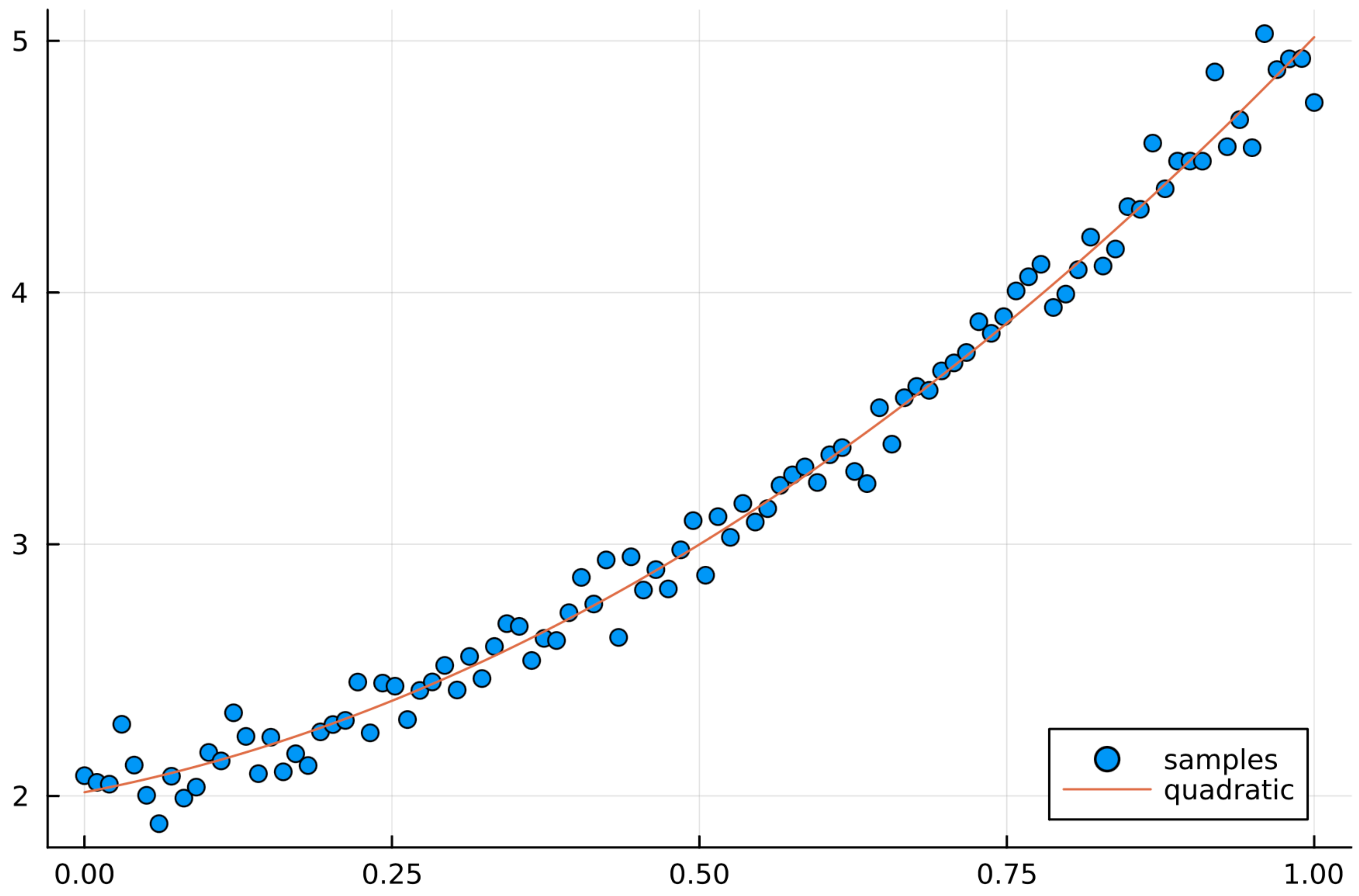
Theorem 6 (Lagrange interpolation). *The unique interpolation polynomial is:*

$$p(x) = f_1 \ell_1(x) + \cdots + f_n \ell_n(x)$$

Example 22 (interpolating an exponential)**.**

III.4.2 Polynomial regression

How to fit a polynomial to lots of data?



Find a polynomial such that:

$$\begin{bmatrix} p(x_1) \\ \vdots \\ p(x_m) \end{bmatrix} \approx \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}}_{\boldsymbol{f}}.$$