Numerical Analysis MATH50003 (2023–24) Problem Sheet 1

Problem 1 Assuming f is differentiable, prove the left-point Rectangular rule error formula

$$\int_a^b f(x)\mathrm{d}x = h\sum_{j=0}^{n-1} f(x_j) + \delta$$

where $|\delta| \leq M(b-a)h$ for $M = \sup_{a\leq x\leq b}|f'(x)|$, $h = (b-a)/n$ and $x_j = a + jh$.

SOLUTION

This proof is very similar to the right-point rule, the only difference is we use a different constant in the indefinite integration in the integration-by-parts. First we need to adapt **Lemma 1 (Rect. rule error on one panel)**:

$$\begin{aligned}\int_a^b f(x)\mathrm{d}x &= \int_a^b (x-b)'f(x)\mathrm{d}x = [(x-b)f(x)]_a^b - \int_a^b (x-b)f'(x)\mathrm{d}x \\ &= (b-a)f(a) + \underbrace{\left(-\int_a^b (x-b)f'(x)\mathrm{d}x\right)}_{\varepsilon}.\end{aligned}$$

where

$$|\varepsilon| \leq (b-a)\sup_{a\leq x\leq b}|(x-b)f'(x)| \leq M(b-a)^2$$

Applying this result on $[x_{j-1}, x_j]$ we get

$$\int_{x_{j-1}}^{x_j} f(x)\mathrm{d}x = hf(x_{j-1}) + \delta_j$$

where $|\delta_j| \leq Mh^2$. Splitting the integral into a sum of smaller integrals:

$$\int_a^b f(x)\mathrm{d}x = \sum_{j=1}^n \int_{x_{j-1}}^{x_j} f(x)\mathrm{d}x = h\sum_{j=1}^n f(x_{j-1}) + \underbrace{\sum_{j=1}^n \delta_j}_{\delta}$$

where using the triangular inequality we have

$$|\delta| = \left|\sum_{j=1}^n \delta_j\right| \leq \sum_{j=1}^n |\delta_j| \leq Mnh^2 = M(b-a)h.$$

END

Problem 2(a) Assuming f is twice-differentiable, prove a one-panel Trapezium rule error bound:

$$\int_a^b f(x)\mathrm{d}x = (b-a)\frac{f(a)+f(b)}{2} + \delta$$

where $|\delta| \leq M(b-a)^3$ for $M = \sup_{a\leq x\leq b}|f''(x)|$.

Hint: Recall from the notes

$$\int_a^b \frac{(b-x)f(a)+(x-a)f(b)}{b-a}\mathrm{d}x = (b-a)\frac{f(a)+f(b)}{2}$$

and you may need to use Taylor's theorem. Note that the bound is not sharp and so you may arrive at something sharper like $|\delta| \leq 3(b-a)^3M/4$. The sharpest bound is $|\delta| \leq (b-a)^3M/12$ but that would be a significantly harder challenge to show!

SOLUTION

Recall from the notes:

$$\int_a^b \frac{(b-x)f(a)+(x-a)f(b)}{b-a}\mathrm{d}x = (b-a)\frac{f(a)+f(b)}{2}$$

Thus we can find by integration by parts twice (noting that the integrand vanishes at a and b):

$$\begin{aligned}
\delta &= \int_a^b \left[f(x) - \frac{(b-x)f(a)+(x-a)f(b)}{b-a}\right]\mathrm{d}x \\
&= -\int_a^b (x-b)\left[f'(x) - \frac{f(b)-f(a)}{b-a}\right]\mathrm{d}x \\
&= \frac{(b-a)^2}{2}\left[f'(a) - \frac{f(b)-f(a)}{b-a}\right] + \int_a^b \frac{(x-b)^2}{2} f''(x)\mathrm{d}x
\end{aligned}$$

Applying **Proposition 1** we know

$$\left|f'(a) - \frac{f(b)-f(a)}{b-a}\right| \leq M(b-a)/2$$

Further we have

$$\left|\int_a^b \frac{(x-b)^2}{2} f''(x)\mathrm{d}x\right| \leq \frac{(b-a)^3}{2}M$$

Thus we have the bound

$$|\delta| \leq \frac{(b-a)^2}{2}M(b-a)/2 + \frac{(b-a)^3}{2}M \leq \frac{3(b-a)^3}{4}M \leq (b-a)^3 M.$$

For the sharper 1/12 constant check out the Euler–Maclaurin formula.

END

Problem 2(b) Assuming f is twice-differentiable, prove a bound for the Trapezium rule error:

$$\int_a^b f(x)\mathrm{d}x = h\left[\frac{f(a)}{2} + \sum_{j=1}^{n-1} f(x_j) + \frac{f(b)}{2}\right] + \delta$$

where $|\delta| \leq M(b-a)h^2$ for $M = \sup_{a\leq x\leq b}|f''(x)|$.

SOLUTION

This is very similar to the rectangular rules: applying the preceding result on $[x_{j-1}, x_j]$ we get

$$\int_{x_{j-1}}^{x_j} f(x)\mathrm{d}x = h\frac{f(x_{j-1})+f(x_j)}{2} + \delta_j$$

where $|\delta_j| \leq Mh^3$. Splitting the integral into a sum of smaller integrals:

$$\int_a^b f(x)\mathrm{d}x = \sum_{j=1}^n \int_{x_{j-1}}^{x_j} f(x)\mathrm{d}x = h\left[\frac{f(a)}{2} + \sum_{j=1}^{n-1} f(x_j) + \frac{f(b)}{2}\right] + \underbrace{\sum_{j=1}^n \delta_j}_{\delta}$$

where using the triangular inequality we have

$$|\delta| = \left|\sum_{j=1}^{n} \delta_j\right| \le \sum_{j=1}^{n} |\delta_j| \le Mnh^3 = M(b-a)h^2.$$

END

Problem 3 Assuming f is twice-differentiable, for the left difference approximation

$$f'(x) = \frac{f(x) - f(x-h)}{h} + \delta,$$

show that $|\delta| \le Mh/2$ for $M = \sup_{x-h \le t \le x} |f''(t)|$.

SOLUTION

Almost identical to the right-difference. Use Taylor series to write:

$$f(x-h) = f(x) + f'(x)(-h) + \frac{f''(t)}{2}h^2$$

where $t \in [x-h, x]$, so that

$$f'(x) = \frac{f(x) - f(x-h)}{h} + \underbrace{f''(t)/2h}_{\delta}$$

The bound follows immediately:

$$|\delta| \le |f''(t)/2h| \le Mh/2.$$

END

Problem 4 Assuming f is thrice-differentiable, for the central differences approximation

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \delta,$$

show that $|\delta| \le Mh^2/6$ for $M = \sup_{x-h \le t \le x+h} |f'''(t)|$.

SOLUTION

By Taylor's theorem, the approximation around $x + h$ is

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(t_1)}{6}h^3,$$

for some $t_1 \in (x, x+h)$ and similarly $f(x-h) = f(x) + f'(x)(-h) + \frac{f''(x)}{2}h^2 - \frac{f'''(t_2)}{6}h^3$, for some $t_2 \in (x-h, x)$.

Subtracting the second expression from the first we obtain $f(x+h) - f(x-h) = f'(x)(2h) + \frac{f'''(t_1)+f'''(t_2)}{6}h^3$. Hence,

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \underbrace{\frac{f'''(t_1) + f'''(t_2)}{12}h^2}_{\delta}.$$

Thus, the error can be bounded by $|\delta| \le \frac{M}{6}h^2$.

END

Problem 5 Assuming f is thrice-differentiable, for the second-order derivative approximation

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + \delta$$

show that $|\delta| \leq Mh/3$ for $M = \sup_{x-h \leq t \leq x+h} |f'''(t)|$.

SOLUTION Using the same two formulas as in the previous problem we have $f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(t_1)}{6}h^3$, for some $t_1 \in (x, x+h)$ and $f(x-h) = f(x) + f'(x)(-h) + \frac{f''(x)}{2}h^2 - \frac{f'''(t_2)}{6}h^3$, for some $t_2 \in (x-h, x)$.

Summing the two we obtain $f(x+h) + f(x-h) = 2f(x) + f''(x)h^2 + \frac{f'''(t_1)}{6}h^3 - \frac{f'''(t_2)}{6}h^3$.

Thus, $f''(x) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2} + \frac{f'''(t_2)-f'''(t_1)}{6}h$.

Hence, the error is

$$|\delta| = \left| f''(x) - \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \right| = \left| \frac{f'''(t_2) - f'''(t_1)}{6}h \right| \leq \frac{Mh}{3}.$$

END