MATH50003
Numerical Analysis

III.6 QR Factorisation

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Part III

Numerical Linear Algebra

Software
Application
Theory

1. Structured matrices such as banded
2. Differential Equations via finite differences
3. LU and Cholesky factorisation for solving linear systems
4. Polynomial regression for approximating data via least squares
5. Orthogonal matrices such as Householder reflections
6. QR factorisation for solving rectangular least squares problems

Definition 29 (QR factorisation)**.** The *QR factorisation* is

$$A = QR = \underbrace{\begin{bmatrix} \boldsymbol{q}_1 | \cdots | \boldsymbol{q}_m \end{bmatrix}}_{Q \in U(m)} \underbrace{\begin{bmatrix} \times & \cdots & \times \\ & \ddots & \vdots \\ & & \times \\ & & 0 \\ & & \vdots \\ & & 0 \end{bmatrix}}_{R \in \mathbb{C}^{m \times n}}$$

Definition 30 (Reduced QR factorisation)**.** The *reduced QR factorisation*

$$A = \hat{Q}\hat{R} = \underbrace{\begin{bmatrix} \boldsymbol{q}_1 | \cdots | \boldsymbol{q}_n \end{bmatrix}}_{\hat{Q} \in \mathbb{C}^{m \times n}} \underbrace{\begin{bmatrix} \times & \cdots & \times \\ & \ddots & \vdots \\ & & \times \end{bmatrix}}_{\hat{R} \in \mathbb{C}^{n \times n}}$$

QR gives reduced QR

Embedded in a QR factorisation is the reduced QR

III.6.1 Reduced QR and Gram–Schmidt

Gram–Schmidt is a way of computing the reduced QR

Define

$$\begin{aligned}
\boldsymbol{v}_j &:= \boldsymbol{a}_j - \sum_{k=1}^{j-1} \underbrace{\boldsymbol{q}_k^\star \boldsymbol{a}_j}_{r_{kj}} \boldsymbol{q}_k \\
r_{jj} &:= \|\boldsymbol{v}_j\| \\
\boldsymbol{q}_j &:= \frac{\boldsymbol{v}_j}{r_{jj}}
\end{aligned}$$

Theorem (Gram–Schmidt and reduced QR) Define $\boldsymbol{q}_j$ and r_{kj} as above (with $r_{kj} = 0$ if $k > j$). Then a reduced QR factorisation is given by:

$$A = \underbrace{\left[\boldsymbol{q}_1 | \cdots | \boldsymbol{q}_n\right]}_{\hat{Q} \in \mathbb{C}^{m \times n}} \underbrace{\begin{bmatrix} r_{11} & \cdots & r_{1n} \\ & \ddots & \vdots \\ & & r_{nn} \end{bmatrix}}_{\hat{R} \in \mathbb{C}^{n \times n}}$$

III.6.2 Householder reflections and QR

Householder is a more stable way to compute QR

Theorem 7 (QR)**.** *Every matrix $A \in \mathbb{C}^{m\times n}$ has a QR factorisation:*

$$A = QR$$

where $Q \in U(m)$ and $R \in \mathbb{C}^{m\times n}$ is right triangular.

III.6.3 QR and least squares

Use QR to solve least squares problems

Theorem 8 (least squares via QR)**.** *Suppose $A \in \mathbb{C}^{m \times n}$ with $m \geq n$ has full rank. Given a QR factorisation $A = QR$ then*

$$\boldsymbol{x} = \hat{R}^{-1}\hat{Q}^\star \boldsymbol{b}$$

minimises $\|A\boldsymbol{x} - \boldsymbol{b}\|$.