

## Numerical Analysis MATH50003 (2023–24) Problem Sheet 9

**Problem 1** Construct the monic and orthonormal polynomials up to degree 3 for the weights  $\sqrt{1-x^2}$  and  $1-x$  on  $[-1, 1]$ . What are the corresponding Jacobi matrices? Hint: for the first weight, first compute  $\int_{-1}^1 x^k \sqrt{1-x^2} dx$  for  $0 \leq k \leq 2$  using a change-of-variables.

**Problem 2** Prove Theorem 13: a precisely degree  $n$  polynomial

$$p(x) = k_n x^n + O(x^{n-1})$$

satisfies

$$\langle p, f_m \rangle = 0$$

for all polynomials  $f_m$  of degree  $m < n$  of degree less than  $n$  if and only if  $p(x) = c\pi_n$  for some constant  $c$ , where  $\pi_n$  are monic orthogonal polynomials.

**Problem 3** If  $w(-x) = w(x)$  for a weight supported on  $[-b, b]$  show that  $a_n = 0$ . Hint: first show that the (monic) polynomials  $p_{2n}(x)$  are even and  $p_{2n+1}(x)$  are odd.

**Problem 4(a)** Prove that

$$U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}.$$

**Problem 4(b)** Show that

$$\begin{aligned} xU_0(x) &= U_1(x)/2 \\ xU_n(x) &= \frac{U_{n-1}(x)}{2} + \frac{U_{n+1}(x)}{2}. \end{aligned}$$

**Problem 5** Use the fact that orthogonal polynomials are uniquely determined by their leading order coefficient and orthogonality to lower dimensional polynomials to show that:

$$T'_n(x) = nU_{n-1}(x).$$

**Problem 6(a)** Consider Hermite polynomials orthogonal with respect to the weight  $\exp(-x^2)$  on  $\mathbb{R}$  with the normalisation

$$H_n(x) = 2^n x^n + O(x^{n-1}).$$

Prove the Rodrigues formula

$$H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2).$$

Hint: use integration-by-parts.

**Problem 6(b)** What are  $k_n^{(1)}$  and  $k_n^{(2)}$  such that

$$H_n(x) = 2^n x^n + k_n^{(1)} x^{n-1} + k_n^{(2)} x^{n-2} + O(x^{n-3})$$

**Problem 6(c)** Deduce the 3-term recurrence relationship for  $H_n(x)$ .

**Problem 6(d)** Prove that  $H'_n(x) = 2nH_{n-1}(x)$ . Hint: show orthogonality of  $H'_n$  to all lower degree polynomials, and that the normalisation constants match.