Numerical Analysis MATH50003 (2023–24) Problem Sheet 2

Problem 1 Show that dual numbers $\mathbb{D}$ are a *commutative ring*, that is, for all $a, b, c \in \mathbb{D}$ the following are satisfied:

1. *additive associativity*: $(a + b) + c = a + (b + c)$
2. *additive commutativity*: $a + b = b + a$
3. *additive identity*: There exists $0 \in \mathbb{D}$ such that $a + 0 = a$.
4. *additive inverse*: There exists $-a$ such that $(-a) + a = 0$.
5. *multiplicative associativity*: $(ab)c = a(bc)$
6. *multiplictive commutativity*: $ab = ba$
7. *multiplictive identity*: There exists $1 \in \mathbb{D}$ such that $1a = a$.
8. *distributive*: $a(b + c) = ab + ac$

Problem 2 What should the following functions applied to dual numbers return for $x = a+b\epsilon$:

$$f(x) = x^{10} + 1, g(x) = 1/x, h(x) = \tan x$$

State the domain where these definitions are valid.

Problem 3(a) What is the correct definition of division on dual numbers, i.e.,

$$(a + b\epsilon)/(c + d\epsilon) = s + t\epsilon$$

for what choice of s and t?

Problem 3(b) A *field* is a commutative ring such that $0 \neq 1$ and all nonzero elements have a multiplicative inverse, i.e., there exists a^{-1} such that $aa^{-1} = 1$. Can we use Problem 4(b) to define $a^{-1} := 1/a$ to make $\mathbb{D}$ a field? Why or why not?

Problem 4 Use dual numbers to compute the derivative of the following functions at $x = 0.1$:

$$\exp(\exp x \cos x + \sin x), \prod_{k=1}^{3} \left(\frac{x}{k} - 1\right), \text{ and } f_2^{\mathrm{s}}(x) = 1 + \frac{x - 1}{2 + \frac{x-1}{2}}$$