

## Numerical Analysis MATH50003 (2023–24) Problem Sheet 2

**Problem 1** Using dual number arithmetic, compute the following polynomials evaluated at the dual number  $2 + \epsilon$  and use this to deduce their derivative at 2:

$$2x^2 + 3x + 4, (x + 1)(x + 2)(x + 3), (2x + 1)x^3$$

**SOLUTION** (a)

$$2(2 + \epsilon)^2 + 3(2 + \epsilon) + 4 = 2(4 + 4\epsilon) + 6 + 3\epsilon + 4 = 18 + 11\epsilon$$

so the derivative is 11.

(b)

$$(3 + \epsilon)(4 + \epsilon)(5 + \epsilon) = (12 + 7\epsilon)(5 + \epsilon) = 60 + 47\epsilon$$

so the derivative is 47.

(c)

$$(2(2 + \epsilon) + 1)(2 + \epsilon)^3 = (5 + 2\epsilon)(4 + 4\epsilon)(2 + \epsilon) = (20 + 28\epsilon)(2 + \epsilon) = 40 + 76\epsilon$$

so the derivative is 76.

**END**

**Problem 2** What should the following functions applied to dual numbers return for  $x = a + b\epsilon$ :

$$f(x) = x^{100} + 1, g(x) = 1/x, h(x) = \tan x$$

State the domain where these definitions are valid.

**SOLUTION**

$$f(a + b\epsilon) = f(a) + bf'(a)\epsilon = a^{100} + 100ba^{99}\epsilon$$

valid everywhere.

$$g(a + b\epsilon) = \frac{1}{a} - \frac{b}{a^2}\epsilon$$

valid for  $a \neq 0$ .

$$h(a + b\epsilon) = \tan a + b \sec^2 a \epsilon$$

valid for  $a \notin \{k\pi + \pi/2 : k \in \mathbb{Z}\}$ .

**END**

**Problem 3(a)** What is the correct definition of division on dual numbers, i.e.,

$$(a + b\epsilon)/(c + d\epsilon) = s + t\epsilon$$

for what choice of  $s$  and  $t$ ?

**SOLUTION**

As with complex numbers, division is easiest to understand by first multiplying with the conjugate, that is:

$$\frac{a + b\epsilon}{c + d\epsilon} = \frac{(a + b\epsilon)(c - d\epsilon)}{(c + d\epsilon)(c - d\epsilon)}.$$

Expanding the products and dropping terms with  $\epsilon^2$  then leaves us with the definition of division for dual numbers (where the denominator must have non-zero real part):

$$\frac{a}{c} + \frac{bc - ad}{c^2}\epsilon.$$

Thus we have  $s = \frac{a}{c}$  and  $t = \frac{bc - ad}{c^2}$ .

**END**

**Problem 3(b)** A *field* is a commutative ring such that  $0 \neq 1$  and all nonzero elements have a multiplicative inverse, i.e., there exists  $a^{-1}$  such that  $aa^{-1} = 1$ . Can we use the previous part to define  $a^{-1} := 1/a$  to make  $\mathbb{D}$  a field? Why or why not?

**SOLUTION**

Fields require that all nonzero elements have a unique multiplicative inverse. However, this is not the case for dual numbers. To give an explicit counter example, we show that there is no dual number  $z$  which is the inverse of  $0 + \epsilon$ , i.e. a dual number  $z$  such that

$$\frac{(0 + \epsilon)}{(z_r + z_d\epsilon)} = 1 + 0\epsilon.$$

By appropriate multiplication with the conjugate we show that

$$\frac{(0 + \epsilon)(z_r - z_d\epsilon)}{(z_r + z_d\epsilon)(z_r - z_d\epsilon)} = \frac{z_r\epsilon}{z_r^2} = \frac{\epsilon}{z_r}.$$

This proves that no choice of real part  $z_r$  can reach the multiplicative identity  $1 + 0\epsilon$  when starting from the number  $0 + \epsilon$ . More general results for zero real part dual numbers can also be proved.

**END**

**Problem 4** Use dual numbers to compute the derivative of the following functions at  $x = 0.1$ :

$$\exp(\exp x \cos x + \sin x), \prod_{k=1}^3 \left( \frac{x}{k} - 1 \right), \text{ and } f_2^s(x) = 1 + \frac{x-1}{2 + \frac{x-1}{2}}$$

**SOLUTION**

We now compute the derivatives of the three functions by evaluating for  $x = 0.1 + \epsilon$ . For the first function we have:

$$\begin{aligned} & \exp(\exp(0.1 + \epsilon) \cos(0.1 + \epsilon) + \sin(0.1 + \epsilon)) \\ &= \exp((\exp(0.1) + \epsilon \exp(0.1))(\cos(0.1) - \sin(0.1)\epsilon) + \sin(0.1) + \cos(0.1)\epsilon) \\ &= \exp(\exp(0.1) \cos(0.1) + \sin(0.1) + (\exp(0.1)(\cos(0.1) - \sin(0.1)) + \cos(0.1))\epsilon) \\ &= \exp(\exp(0.1) \cos(0.1) + \sin(0.1)) \\ & \quad + \exp(\exp(0.1) \cos(0.1) + \sin(0.1)) \exp(0.1)(\cos(0.1) - \sin(0.1)) + \cos(0.1))\epsilon \end{aligned}$$

therefore the derivative is the dual part

$$\exp(\exp(0.1) \cos(0.1) + \sin(0.1))(\exp(0.1)(\cos(0.1) - \sin(0.1)) + \cos(0.1))$$

For the second function we have:

$$\begin{aligned}
(0.1 + \epsilon - 1) \left( \frac{0.1 + \epsilon}{2} - 1 \right) \left( \frac{0.1 + \epsilon}{3} - 1 \right) &= (-0.9 + \epsilon) (-0.95 + \epsilon/2) (-29/30 + \epsilon/3) \\
&= (171/200 - 1.4\epsilon) (-29/30 + \epsilon/3) \\
&= -1653/2000 + 983\epsilon/600
\end{aligned}$$

Thus the derivative is 983/600.

For the third function we have:

$$\begin{aligned}
1 + \frac{0.1 + \epsilon - 1}{2 + \frac{0.1 + \epsilon - 1}{2}} &= 1 + \frac{-0.9 + \epsilon}{1.55 + \epsilon/2} \\
&= 1 - 18/31 + 2\epsilon/1.55^2
\end{aligned}$$

Thus the derivative is 2/1.55<sup>2</sup>.

**END**