MATH50003 Numerical Analysis

II.4 Interval Arithmetic

Dr Sheehan Olver

Part II
Representing Numbers

1. Integers via modular arithmetic
2. Reals via floating point
3. Floating point arithmetic and bounding errors
4. Interval arithmetic for rigorous computations

II.4 Interval Arithmetic

Use set operations with rounding to prove rigorous bounds

For sets $X, Y \subseteq \mathbb{R}$ consider the set operations

$$\begin{aligned} X + Y &:= \{x + y : x \in X, y \in Y\}, \\ XY &:= \{xy : x \in X, y \in Y\}, \\ X/Y &:= \{x/y : x \in X, y \in Y\} \end{aligned}$$

We will use floating point arithmetic to define operations so that

$$\begin{aligned} X + Y &\subseteq X \oplus Y, \\ XY &\subseteq X \otimes Y, \\ X/Y &\subseteq X \oslash Y \end{aligned}$$

Proposition 3 (interval bounds)**.** *For intervals $X = [a, b]$ and $Y = [c, d]$ satisfying $0 < a \leq b$ and $0 < c \leq d$, and $n > 0$, we have:*

$$\begin{aligned} X + Y &= [a + c, b + d] \\ X/n &= [a/n, b/n] \\ XY &= [ac, bd] \end{aligned}$$

Definition 14 (floating point interval arithmetic)**.** For intervals $A = [a, b]$ and $B = [c, d]$ satisfying $0 < a \leq b$ and $0 < c \leq d$, and $n > 0$, define:

$$\begin{aligned}
[a,b] \oplus [c,d] &:= [\mathrm{fl}^{\mathrm{down}}(a+c), \mathrm{fl}^{\mathrm{up}}(b+d)] \\
[a,b] \ominus [c,d] &:= [\mathrm{fl}^{\mathrm{down}}(a-d), \mathrm{fl}^{\mathrm{up}}(b-c)] \\
[a,b] \oslash n &:= [\mathrm{fl}^{\mathrm{down}}(a/n), \mathrm{fl}^{\mathrm{up}}(b/n)] \\
[a,b] \otimes [c,d] &:= [\mathrm{fl}^{\mathrm{down}}(ac), \mathrm{fl}^{\mathrm{up}}(bd)]
\end{aligned}$$

Example 18 (small sum).

Example 19 (exponential with intervals).

$$\exp(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + \underbrace{\exp(t)\frac{x^{n+1}}{(n+1)!}}_{\delta_{x,n}}$$

$$\exp(X) \subseteq \left(\bigoplus_{k=0}^{n} X \owedge k \oslash k! \right) \oplus \left[\mathrm{fl}^{\mathrm{down}}\left(-\frac{3}{(n+1)!} \right), \mathrm{fl}^{\mathrm{up}}\left(\frac{3}{(n+1)!} \right) \right]$$

Let’s implement Interval arithmetic in Lab 4.