MATH50003
Numerical Analysis

III.2 Differential Equations via Finite Differences

Dr Sheehan Olver

Part III

Numerical Linear Algebra

Software
Application
Theory

1. Structured matrices such as banded
2. Differential Equations via finite differences
3. LU and Cholesky factorisation for solving linear systems
4. Polynomial regression for approximating data via least squares
5. Orthogonal matrices such as Householder reflections
6. QR factorisation for solving rectangular least squares problems

III.2.1 Indefinite Integration

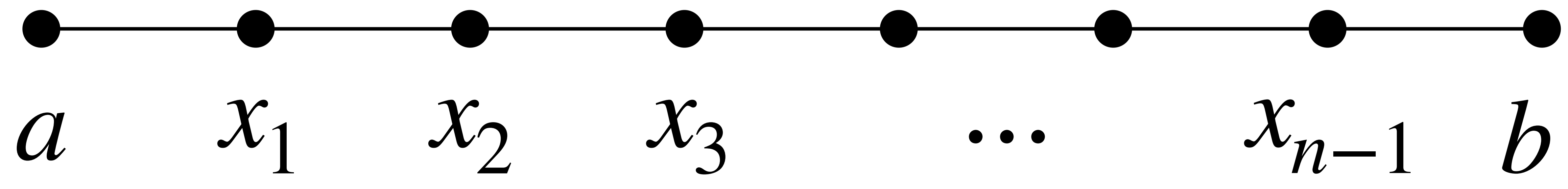
Solve the simplest ODE replacing derivatives w/ divided differences

Indefinite integration can be thought of as an ODE:

$$
\begin{aligned}
u(a) &= c, \\
u'(x) &= f(x)
\end{aligned}
$$

Idea: replace derivatives with divided differences.
Do so in 4 steps.

Step 1: ODE on interval → ODE on grid



$$
\begin{aligned}
u(a) &= c, \\
u'(x) &= f(x)
\end{aligned}
\quad \Longrightarrow \quad
\begin{bmatrix} u(x_0) \\ u'(x_0) \\ u'(x_1) \\ \vdots \\ u'(x_{n-1}) \end{bmatrix}
=
\underbrace{\begin{bmatrix} c \\ f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \end{bmatrix}}_{\boldsymbol{b}}
$$

Step 2: ODE on grid → Divided differences on grid

$$
\begin{bmatrix} u(x_0) \\ u'(x_0) \\ u'(x_1) \\ \vdots \\ u'(x_{n-1}) \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \end{bmatrix}}_{\boldsymbol{b}}
\quad \Rightarrow \quad
\begin{bmatrix} u(x_0) \\ (u(x_1) - u(x_0))/h \\ (u(x_2) - u(x_1))/h \\ \vdots \\ (u(x_n) - u(x_{n-1})/h \end{bmatrix} \approx \boldsymbol{b}
$$

Step 3: Divided differences on grid → Discrete system

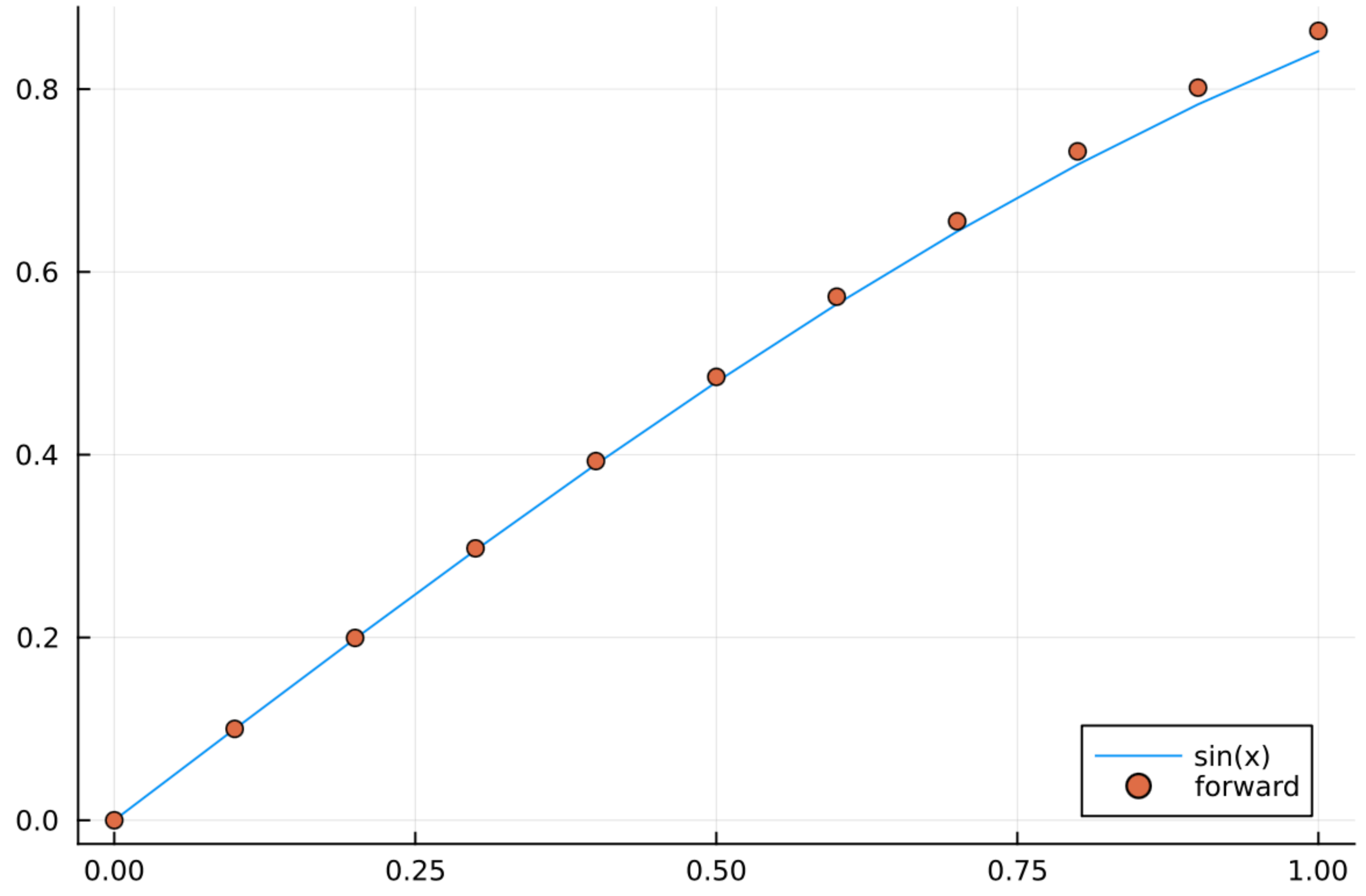
$$\begin{bmatrix} u(x_0) \\ (u(x_1) - u(x_0))/h \\ (u(x_2) - u(x_1))/h \\ \vdots \\ (u(x_n) - u(x_{n-1})/h \end{bmatrix} \approx \boldsymbol{b} \quad \Rightarrow \quad \begin{bmatrix} u_0 \\ (u_1 - u_0)/h \\ (u_2 - u_1)/h \\ \vdots \\ (u_n - u_{n-1})/h \end{bmatrix} = \boldsymbol{b}$$

Step 4: Discrete system → Linear system

$$\begin{bmatrix} u_0 \\ (u_1 - u_0)/h \\ (u_2 - u_1)/h \\ \vdots \\ (u_n - u_{n-1})/h \end{bmatrix} = \boldsymbol{b}$$

$$\underbrace{\begin{bmatrix} 1 & & & \\ -1/h & 1/h & & \\ & \ddots & \ddots & \\ & & -1/h & 1/h \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}}_{\boldsymbol{u}} = \boldsymbol{b}$$

$$u(0) = 0, \qquad u' = \cos x$$



III.2.1 Forward Euler

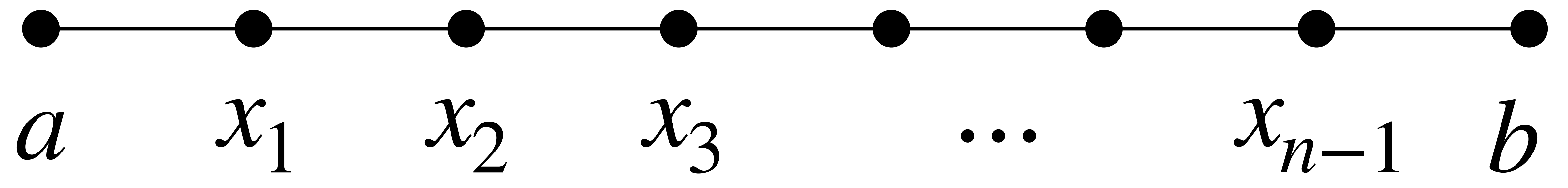
Generalise to first order linear ODEs

Consider general first order linear ODEs:

$$
\begin{aligned}
u(a) &= c \\
u'(x) - \omega(x)u(x) &= f(x)
\end{aligned}
$$

Repeat 4 steps as before.

Step 1: ODE on interval → ODE on grid



$$
\begin{aligned}
u(a) &= c \\
u'(x) - \omega(x)u(x) &= f(x)
\end{aligned}
$$

$$
\begin{bmatrix} u(x_0) \\ u'(x_0) + \omega(x_0)u(x_0) \\ u'(x_1) + \omega(x_1)u(x_1) \\ \vdots \\ u'(x_{n-1}) + \omega(x_{n-1})u(x_{n-1}) \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \end{bmatrix}}_{\boldsymbol{b}}
$$

Step 2: ODE on grid → Divided differences on grid

$$
\begin{bmatrix} u(x_0) \\ u'(x_0) + \omega(x_0)u(x_0) \\ u'(x_1) + \omega(x_1)u(x_1) \\ \vdots \\ u'(x_{n-1}) + \omega(x_{n-1})u(x_{n-1}) \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \end{bmatrix}}_{\boldsymbol{b}}
\quad \Longrightarrow \quad
\begin{bmatrix} u(x_0) \\ (u(x_1) - u(x_0))/h + \omega(x_0)u(x_0) \\ (u(x_2) - u(x_1))/h + \omega(x_1)u(x_1) \\ \vdots \\ (u(x_n) - u(x_{n-1}))/h + \omega(x_{n-1})u(x_{n-1}) \end{bmatrix} \approx \boldsymbol{b}
$$

Step 3: Divided differences on grid → Discrete system

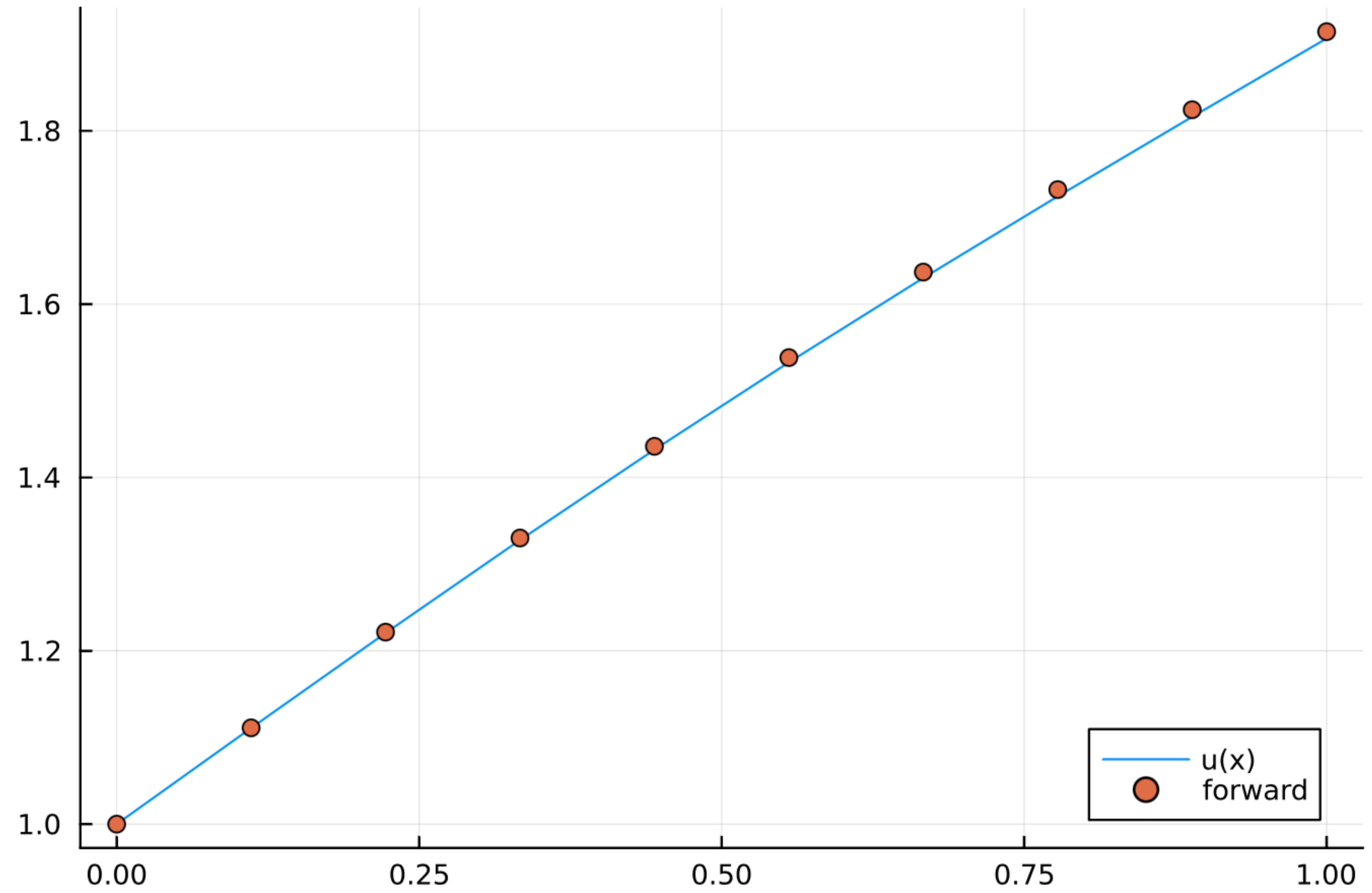
$$\begin{bmatrix} u(x_0) \\ (u(x_1) - u(x_0))/h + \omega(x_0)u(x_0) \\ (u(x_2) - u(x_1))/h + \omega(x_1)u(x_1) \\ \vdots \\ (u(x_n) - u(x_{n-1}))/h + \omega(x_{n-1})u(x_{n-1}) \end{bmatrix} \approx \boldsymbol{b} \quad \Rightarrow \quad \begin{bmatrix} u_0 \\ (u_1 - u_0)/h + \omega(x_0)u_0 \\ (u_2 - u_1)/h + \omega(x_1)u_1 \\ \vdots \\ (u_n - u_{n-1})/h + \omega(x_{n-1})u_{n-1} \end{bmatrix} = \boldsymbol{b}$$

Step 4: Discrete system → Linear system

$$\begin{bmatrix} u_0 \\ (u_1 - u_0)/h + \omega(x_0)u_0 \\ (u_2 - u_1)/h + \omega(x_1)u_1 \\ \vdots \\ (u_n - u_{n-1})/h + \omega(x_{n-1})u_{n-1} \end{bmatrix} = \boldsymbol{b}$$

$$\underbrace{\begin{bmatrix} 1 & & & \\ \omega(x_0) - 1/h & 1/h & & \\ & \ddots & \ddots & \\ & & \omega(x_{n-1}) - 1/h & 1/h \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}}_{\boldsymbol{u}} = \boldsymbol{b}$$

$$u(0) = 1, u' + xu = \mathrm{e}^x$$



III.2.3 Poisson

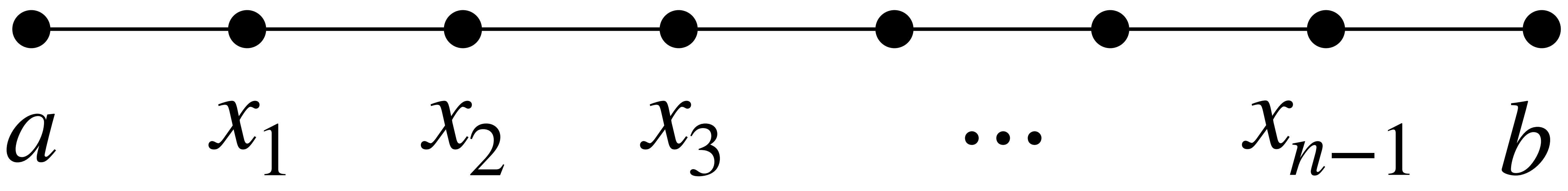
Use second order divided differences

Consider the simplest second order ODE with boundary conditions:

$$
\begin{aligned}
u(0) &= c, \\
u''(x) &= f(x), \\
u(1) &= d
\end{aligned}
$$

Repeat 4 steps as before but with second-order divided differences and one more boundary condition.

Step 1: ODE on interval → ODE on grid



$$
\begin{aligned}
u(0) &= c,\\
u''(x) &= f(x),\\
u(1) &= d
\end{aligned}
$$

$$
\begin{bmatrix} u(x_0) \\ u''(x_1) \\ u''(x_1) \\ \vdots \\ u''(x_{n-1}) \\ u(x_n) \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \\ d \end{bmatrix}}_{\boldsymbol{b}}
$$

Step 2: ODE on grid → Divided differences on grid

$$\begin{bmatrix} u(x_0) \\ u''(x_1) \\ u''(x_1) \\ \vdots \\ u''(x_{n-1}) \\ u(x_n) \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \\ d \end{bmatrix}}_{\boldsymbol{b}} \quad \Longrightarrow \quad \begin{bmatrix} u(x_0) \\ \frac{u(x_0)-2u(x_1)+u(x_2)}{h^2} \\ \frac{u(x_1)-2u(x_2)+u(x_3)}{h^2} \\ \vdots \\ \frac{u(x_{n-2})-2u(x_{n-1})+u(x_n)}{h^2} \\ u(x_n) \end{bmatrix} \approx \boldsymbol{b}$$

Step 3: Divided differences on grid → Discrete system

$$\begin{bmatrix} u(x_0) \\ \frac{u(x_0)-2u(x_1)+u(x_2)}{h^2} \\ \frac{u(x_1)-2u(x_2)+u(x_3)}{h^2} \\ \vdots \\ \frac{u(x_{n-2})-2u(x_{n-1})+u(x_n)}{h^2} \\ u(x_n) \end{bmatrix} \approx \boldsymbol{b}$$

$$\begin{bmatrix} u_0 \\ \frac{u_0-2u_1+u_2}{h^2} \\ \frac{u_1-2u_2+u_3}{h^2} \\ \vdots \\ \frac{u_{n-2}-2u_{n-1}+u_n}{h^2} \\ u_n \end{bmatrix} = \boldsymbol{b}$$

Step 4: Discrete system → Linear system

$$\begin{bmatrix} u_0 \\ \frac{u_0-2u_1+u_2}{h^2} \\ \frac{u_1-2u_2+u_3}{h^2} \\ \vdots \\ \frac{u_{n-2}-2u_{n-1}+u_n}{h^2} \\ u_n \end{bmatrix} = \boldsymbol{b}$$

$$u(0) = \underbrace{1}_{c}$$

$$u''(x) = \underbrace{-4x^2\cos(x^2) - 2\sin(x^2)}_{f(x)}$$

$$u(1) = \underbrace{\cos 1}_{d}$$

$$\underbrace{\begin{bmatrix} 1 & & & & \\ 1/h^2 & -2/h^2 & 1/h & & \\ & \ddots & \ddots & \ddots & \\ & & 1/h^2 & -2/h^2 & 1/h \\ & & & & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}}_{\boldsymbol{u}} = \boldsymbol{b}$$