MATH50003 Numerical Analysis

III.5 Orthogonal and Unitary Matrices

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Part III

Numerical Linear Algebra

Software
Application
Theory

1. Structured matrices such as banded
2. Differential Equations via finite differences
3. LU and Cholesky factorisation for solving linear systems
4. Polynomial regression for approximating data via least squares
5. Orthogonal matrices such as Householder reflections
6. QR factorisation for solving rectangular least squares problems

Definition 23 (orthogonal/unitary matrix)**.** A square real matrix is *orthogonal* if its inverse is its transpose:

$$O(n) = \{Q \in \mathbb{R}^{n\times n} : Q^\top Q = I\}$$

A square complex matrix is *unitary* if its inverse is its adjoint:

$$U(n) = \{Q \in \mathbb{C}^{n\times n} : Q^\star Q = I\}.$$

Here the adjoint is the same as the conjugate-transpose: $Q^\star := \bar{Q}^\top$.

Properties of orthogonal/unitary matrices

III.5.1 Rotations

Rotations in $\mathbb{R}^2$ correspond to 2×2 orthogonal matrices

Definition 24 (Special Orthogonal and Rotations). *Special Orthogonal Matrices* are

$$SO(n) := \{Q \in O(n) | \det Q = 1\}$$

And (simple) *rotations* are $SO(2)$.

Definition 25 (two-arg arctan).

$$\operatorname{atan}(b, a) := \begin{cases} \operatorname{atan}\frac{b}{a} & a > 0 \\ \operatorname{atan}\frac{b}{a} + \pi & a < 0 \text{ and } b > 0 \\ \operatorname{atan}\frac{b}{a} - \pi & a < 0 \text{ and } b < 0 \\ \pi/2 & a = 0 \text{ and } b > 0 \\ -\pi/2 & a = 0 \text{ and } b < 0 \end{cases}$$

Proposition 8 (simple rotation)**.** *A* 2×2 rotation matrix *through angle* θ *is*

$$Q_\theta := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

We have $Q \in SO(2)$ *if and only if* $Q = Q_\theta$ *for some* $\theta \in \mathbb{R}$.

Proposition 9 (rotation of a vector)**.** *The matrix*

$$Q = \frac{1}{\sqrt{a^2 + b^2}} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

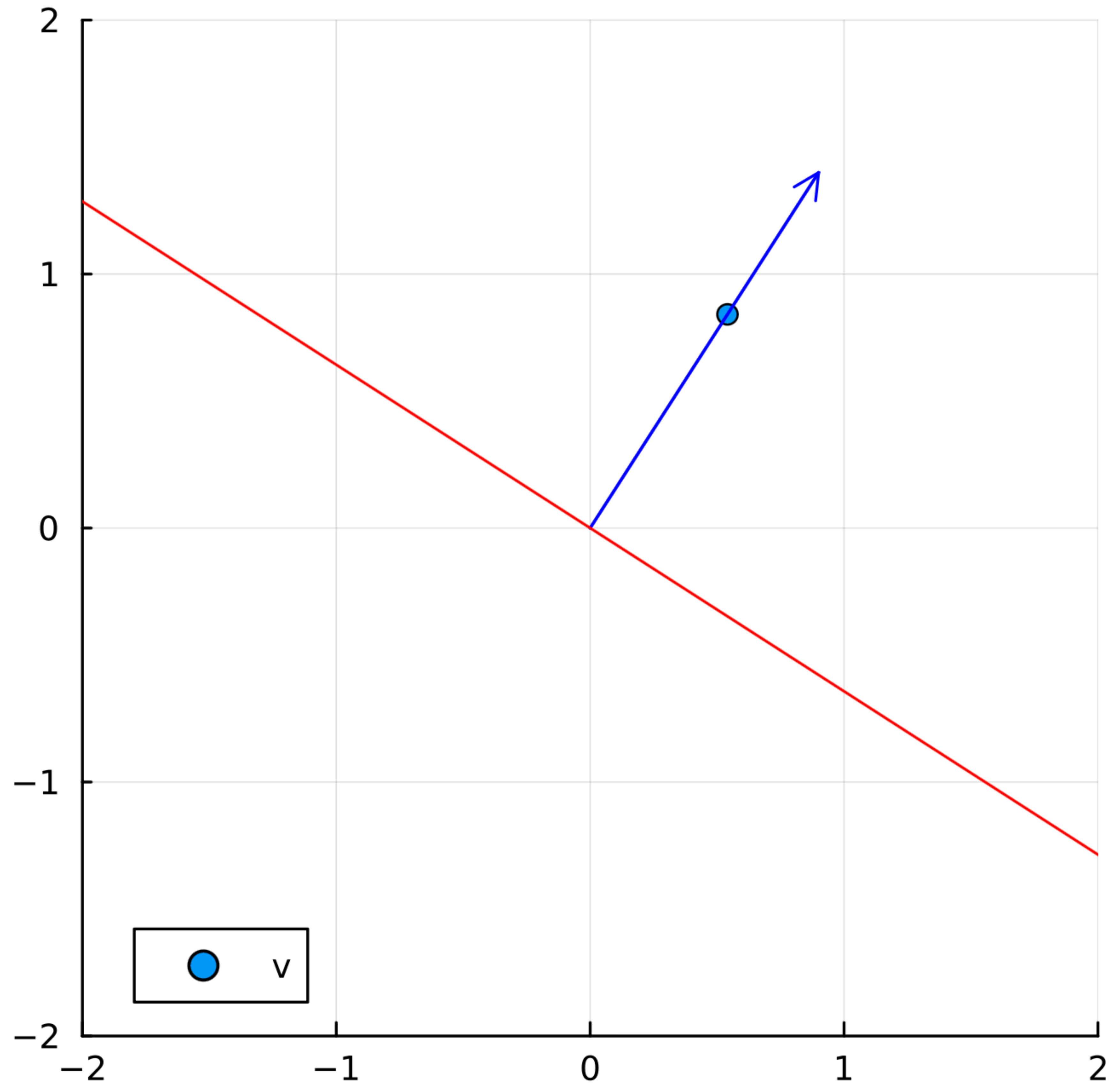
is a rotation matrix ($Q \in SO(2)$) satisfying

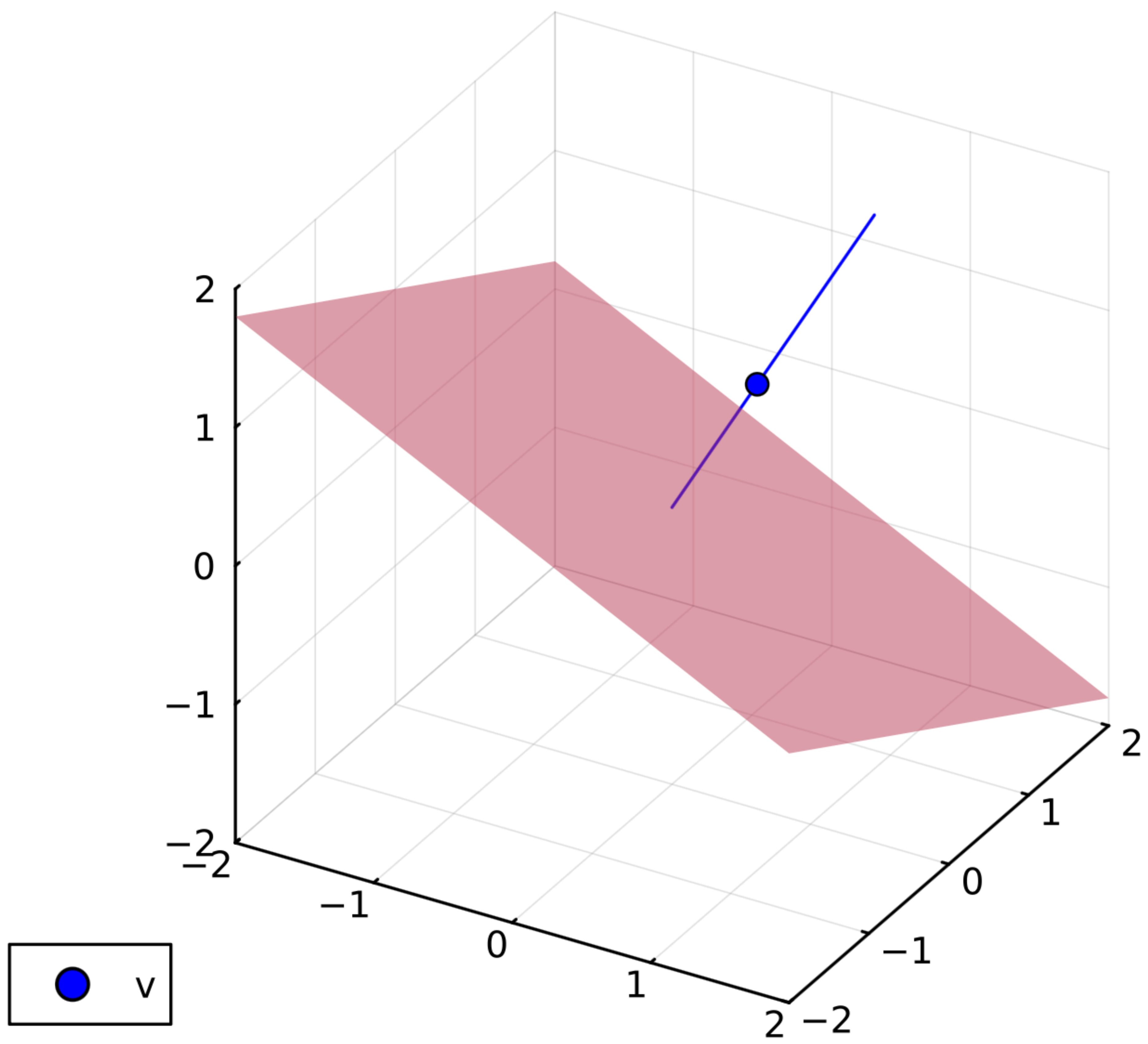
$$Q \begin{bmatrix} a \\ b \end{bmatrix} = \sqrt{a^2 + b^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Example 23 (rotating a vector).

III.5.2 Reflections

Every unit vector corresponds to a reflection, which is unitary





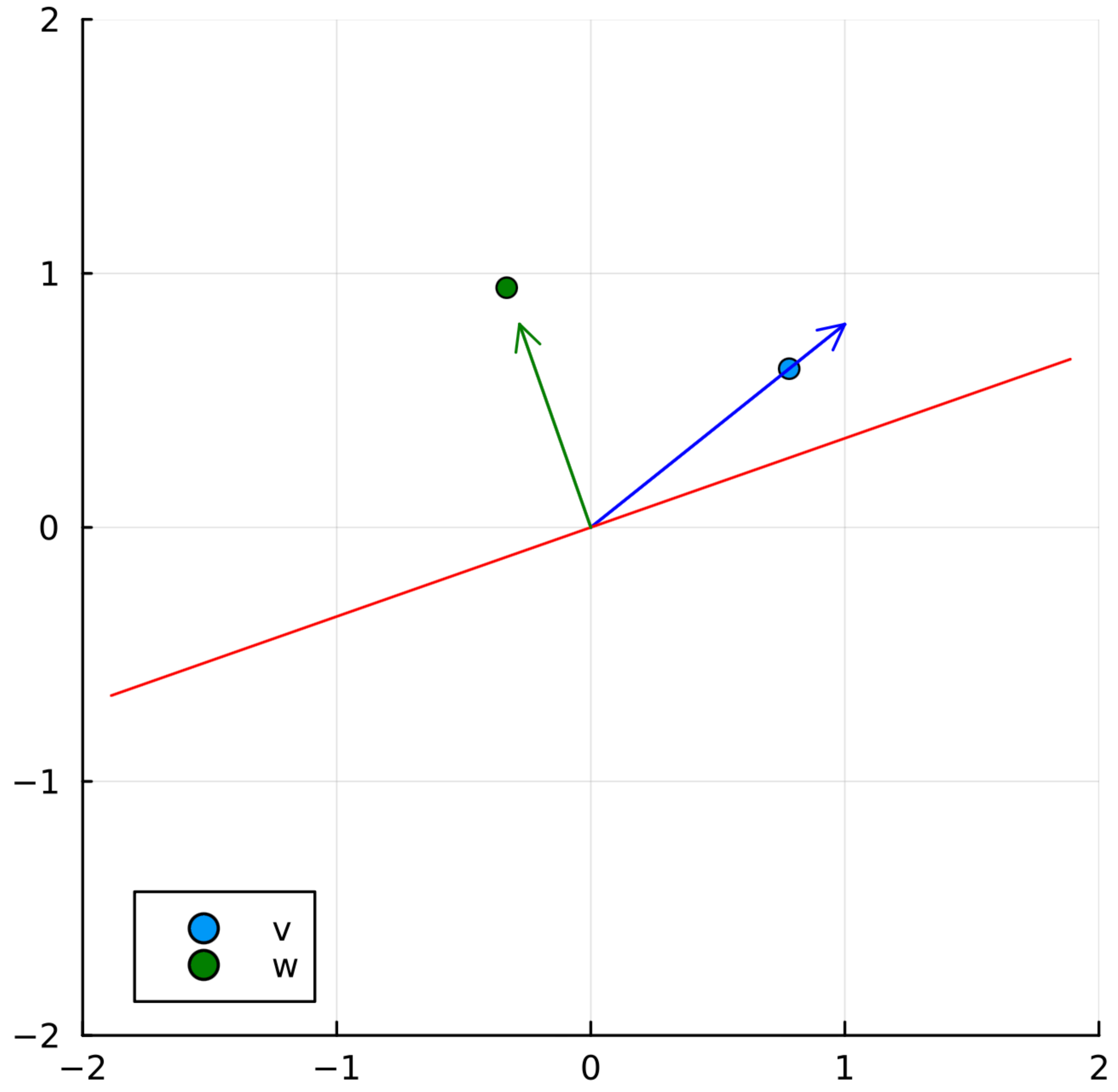
Definition 26 (reflection matrix)**.** Given a unit vector $\boldsymbol{v} \in \mathbb{C}^n$ (satisfying $\|\boldsymbol{v}\| = 1$), define the corresponding *reflection matrix* as:

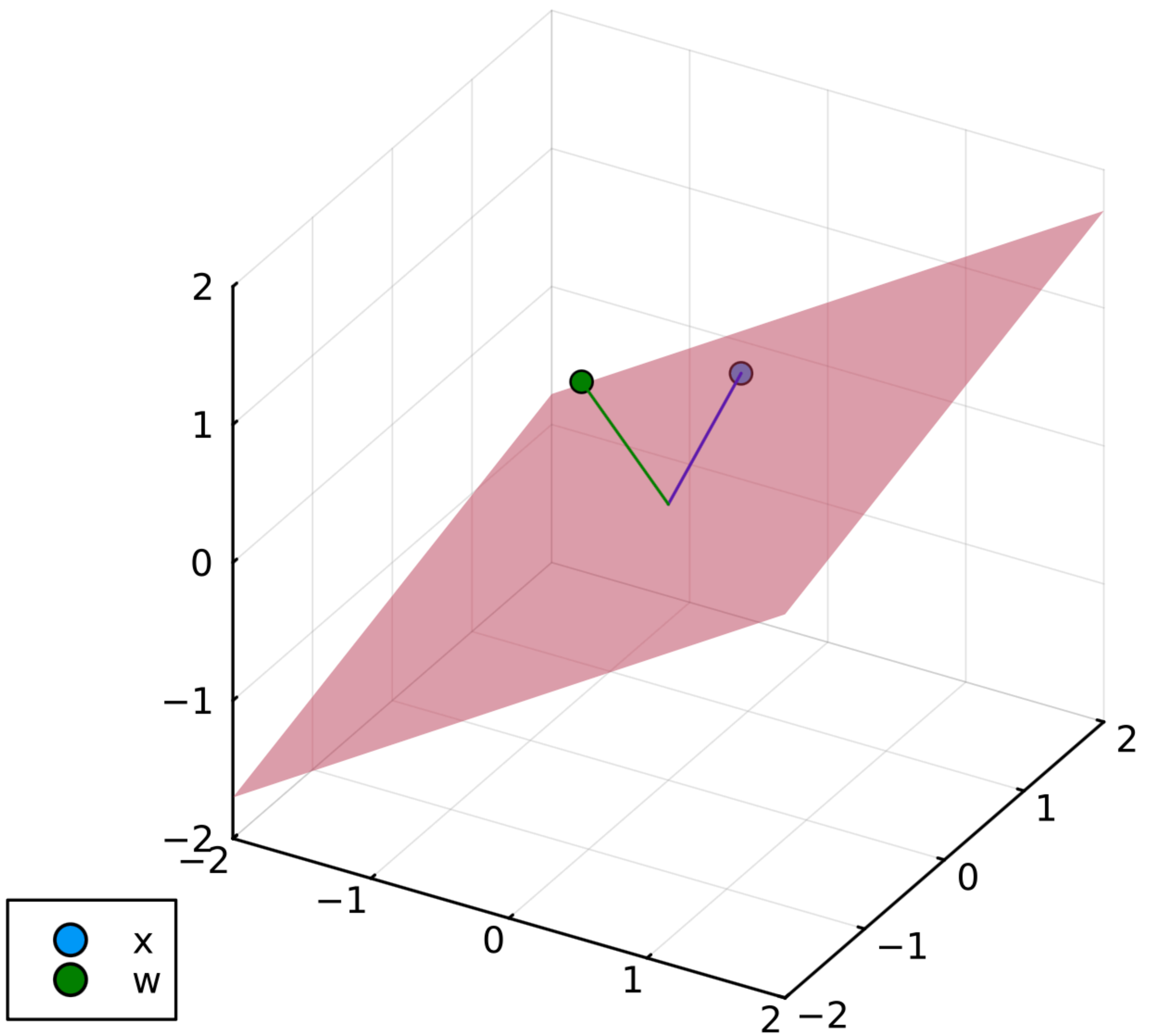
$$Q_{\boldsymbol{v}} := I - 2\boldsymbol{v}\boldsymbol{v}^\star$$

Example 24 (reflection through 2-vector).

Householder reflections

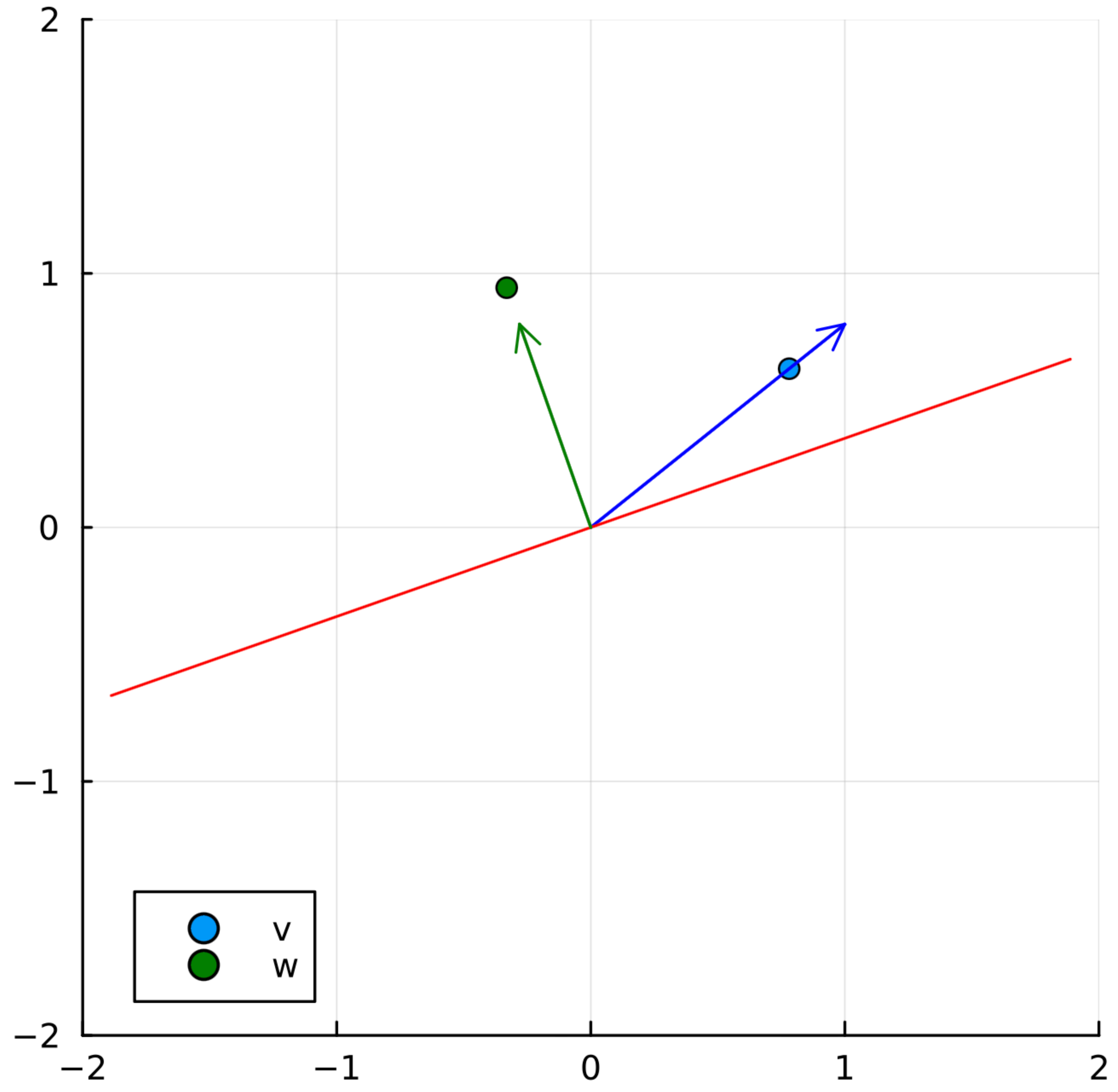
Reflect to the x-axis

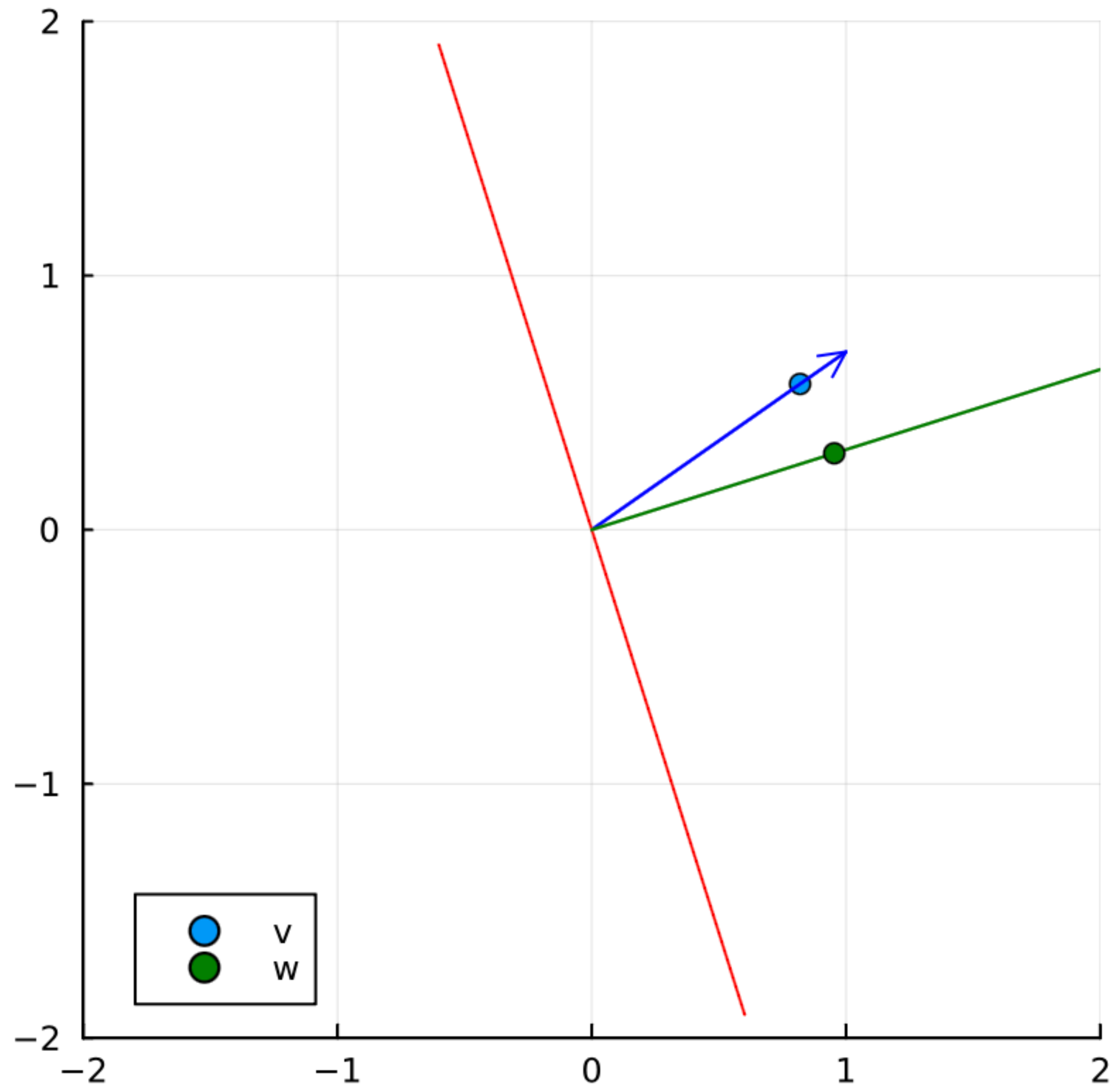




Householder reflections

Reflect to the x-axis (2 ways)





Definition 27 (Householder reflection, real case)**.** For a given vector $\boldsymbol{x} \in \mathbb{R}^n$, define the Householder reflection

$$Q_{\boldsymbol{x}}^{\pm,\mathrm{H}} := Q_{\boldsymbol{w}}$$

for $\boldsymbol{y} = \mp\|\boldsymbol{x}\|\boldsymbol{e}_1 + \boldsymbol{x}$ and $\boldsymbol{w} = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}$. The default choice in sign is:

$$Q_{\boldsymbol{x}}^{\mathrm{H}} := Q_{\boldsymbol{x}}^{-\mathrm{sign}(x_1),\mathrm{H}}.$$

Lemma 4 (Householder reflection maps to axis)**.** *For* $\boldsymbol{x} \in \mathbb{R}^n$,

$$Q_{\boldsymbol{x}}^{\pm,\mathrm{H}}\boldsymbol{x} = \pm\|\boldsymbol{x}\|\boldsymbol{e}_1$$

Definition 28 (Householder reflection, complex case)**.** For a given vector $\boldsymbol{x} \in \mathbb{C}^n$, define the Householder reflection as

$$Q_{\boldsymbol{x}}^{\mathrm{H}} := Q_{\boldsymbol{w}}$$

for $\boldsymbol{y} = \operatorname{csign}(x_1)\|\boldsymbol{x}\|\boldsymbol{e}_1 + \boldsymbol{x}$ and $\boldsymbol{w} = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}$, for $\operatorname{csign}(z) = \mathrm{e}^{\mathrm{i}\arg z}$.

Lemma 5 (Householder reflection maps to axis, complex case)**.** *For* $\boldsymbol{x} \in \mathbb{C}^n$,

$$Q_{\boldsymbol{x}}^{\mathrm{H}}\boldsymbol{x} = -\operatorname{csign}(x_1)\|\boldsymbol{x}\|\boldsymbol{e}_1$$