MATH50003
Numerical Analysis

IV.2 Discrete Fourier Transform

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Part IV

Approximation Theory

1. Fourier Expansions and approximating Fourier coefficients
2. Discrete Fourier Transforms and interpolation
3. Orthogonal Polynomials and basic properties
4. Classical Orthogonal Polynomials with special structure
5. Gaussian Quadrature for high-accuracy integration

IV.2.1 The discrete Fourier transform

Map from values to approximate Fourier coefficients

Definition 36 (DFT)**.** The *Discrete Fourier Transform (DFT)* is defined as:

$$Q_n := \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \mathrm{e}^{-\mathrm{i}\theta_1} & \mathrm{e}^{-\mathrm{i}\theta_2} & \cdots & \mathrm{e}^{-\mathrm{i}\theta_{n-1}} \\ 1 & \mathrm{e}^{-\mathrm{i}2\theta_1} & \mathrm{e}^{-\mathrm{i}2\theta_2} & \cdots & \mathrm{e}^{-\mathrm{i}2\theta_{n-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \mathrm{e}^{-\mathrm{i}(n-1)\theta_1} & \mathrm{e}^{-\mathrm{i}(n-1)\theta_2} & \cdots & \mathrm{e}^{-\mathrm{i}(n-1)\theta_{n-1}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \cdots & \omega^{-(n-1)^2} \end{bmatrix}$$

for the n-th root of unity $\omega = \mathrm{e}^{2\pi\mathrm{i}/n}$.

Proposition 1 (DFT is Unitary) $Q_n \in U(n)$, that is, $Q_n^\star Q_n = Q_n Q_n^\star = I$.

Example 27 (Computing Sum)**.**

IV.2.2 Interpolation

Approximate Fourier series interpolates at sample points

Corollary 3 (Interpolation).

$$f_n(\theta) := \sum_{k=0}^{n-1} \hat{f}_k^n \mathrm{e}^{\mathrm{i}k\theta}$$

interpolates f *at* θ_j*:*

$$f_n(\theta_j) = f(\theta_j)$$

Example 28 (DFT versus Lagrange).