**Numerical Analysis MATH50003 (2023–24) Problem Sheet 4**

**Problem 1** Suppose $x = 1.25$ and consider 16-bit floating point arithmetic ($F_{16}$). What is the error in approximating $x$ by the nearest float point number $\mathrm{fl}(x)$? What is the error in approximating $2x$, $x/2$, $x + 2$ and $x - 2$ by $2 \otimes x$, $x \oslash 2$, $x \oplus 2$ and $x \ominus 2$?

**SOLUTION** None of these computations have errors since they are all exactly representable as floating point numbers. **END**

**Problem 2** Show that $1/5 = 2^{-3}(1.1001100110011\ldots)_2$. What are the exact bits for $1 \oslash 5$, $1 \oslash 5 \oplus 1$ computed using half-precision arithmetic ($F_{16} := F_{15,5,10}$) (using default rounding)?

**SOLUTION**

For the first part we use Geometric series:

$$\begin{aligned} 2^{-3}(1.1001100110011\ldots)_2 &= 2^{-3}\left(\sum_{k=0}^{\infty} \frac{1}{2^{4k}} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^{4k}}\right) \\ &= \frac{3}{2^4}\frac{1}{1 - 1/2^4} = \frac{3}{2^4 - 1} = \frac{1}{5} \end{aligned}$$

Write $-3 = 12 - 15$ hence we have $q = 12 = (01100)_2$. Since $1/5$ is below the midpoint (the midpoint would have been the first magenta bit was 1 and all other bits are 0) we round down and hence have the bits:

0 01100 1001100110

Adding 1 we get:

$$1 + 2^{-3} * (1.1001100110)_2 = (1.0011001100 11)_2 \approx (1.0011001101)_2$$

Here we write the exponent as $0 = 15 - 15$ where $q = 15 = (01111)_2$. Thus we have the bits:

0 01111 0011001101

**END**

**Problem 3** Prove the following bounds on the *absolute error* of a floating point calculation in idealised floating-point arithmetic $F_{\infty,S}$ (i.e., you may assume all operations involve normal floating point numbers):

$$\begin{aligned} (\mathrm{fl}(1.1) \otimes \mathrm{fl}(1.2)) \oplus \mathrm{fl}(1.3) &= 2.62 + \varepsilon_1 \\ (\mathrm{fl}(1.1) \ominus 1) \oslash \mathrm{fl}(0.1) &= 1 + \varepsilon_2 \end{aligned}$$

such that $|\varepsilon_1| \leq 11\epsilon_{\mathrm{m}}$ and $|\varepsilon_2| \leq 40\epsilon_{\mathrm{m}}$, where $\epsilon_{\mathrm{m}}$ is machine epsilon.

**SOLUTION**

The first problem is very similar to what we saw in lecture. Write

$$(\mathrm{fl}(1.1) \otimes \mathrm{fl}(1.2)) \oplus \mathrm{fl}(1.3) = (1.1(1 + \delta_1)1.2(1 + \delta_2)(1 + \delta_3) + 1.3(1 + \delta_4))(1 + \delta_5)$$

where we have $|\delta_1|, \ldots, |\delta_5| \leq \epsilon_{\mathrm{m}}/2$. We first write

$$1.1(1 + \delta_1)1.2(1 + \delta_2)(1 + \delta_3) = 1.32(1 + \varepsilon_1)$$

where, using the bounds:

$$|\delta_1\delta_2|, |\delta_1\delta_3|, |\delta_2\delta_3| \leq \epsilon_{\mathrm{m}}/4, |\delta_1\delta_2\delta_3| \leq \epsilon_{\mathrm{m}}/8$$

we find that

$$|\varepsilon_1| \leq |\delta_1| + |\delta_2| + |\delta_3| + |\delta_1\delta_2| + |\delta_1\delta_3| + |\delta_2\delta_3| + |\delta_1\delta_2\delta_3| \leq (3/2 + 3/4 + 1/8) \leq 5/2\epsilon_{\rm m}$$

Then we have

$$1.32(1+\varepsilon_1) + 1.3(1+\delta_4) = 2.62 + \underbrace{1.32\varepsilon_1 + 1.3\delta_4}_{\varepsilon_2}$$

where

$$|\varepsilon_2| \leq (15/4 + 3/4)\epsilon_{\rm m} \leq 5\epsilon_{\rm m}.$$

Finally,

$$(2.62 + \varepsilon_2)(1+\delta_5) = 2.62 + \underbrace{\varepsilon_2 + 2.62\delta_5 + \varepsilon_2\delta_5}_{\varepsilon_3}$$

where, using $|\varepsilon_2\delta_5| \leq 3\epsilon_{\rm m}$ we get,

$$|\varepsilon_3| \leq (5 + 3/2 + 3)\epsilon_{\rm m} \leq 10\epsilon_{\rm m}.$$

For the second part, we do:

$$(\mathrm{fl}(1.1) \ominus 1) \oslash \mathrm{fl}(0.1) = \frac{(1.1(1+\delta_1) - 1)(1+\delta_2)}{0.1(1+\delta_3)}(1+\delta_4)$$

where we have $|\delta_1|, \ldots, |\delta_4| \leq \epsilon_{\rm m}/2$. Write

$$\frac{1}{1+\delta_3} = 1 + \varepsilon_1$$

where, using that $|\delta_3| \leq \epsilon_{\rm m}/2 \leq 1/2$, we have

$$|\varepsilon_1| \leq \left|\frac{\delta_3}{1+\delta_3}\right| \leq \frac{\epsilon_{\rm m}}{2}\frac{1}{1-1/2} \leq \epsilon_{\rm m}.$$

Further write

$$(1+\varepsilon_1)(1+\delta_4) = 1 + \varepsilon_2$$

where

$$|\varepsilon_2| \leq |\varepsilon_1| + |\delta_4| + |\varepsilon_1||\delta_4| \leq (1 + 1/2 + 1/2)\epsilon_{\rm m} = 2\epsilon_{\rm m}.$$

We also write

$$\frac{(1.1(1+\delta_1) - 1)(1+\delta_2)}{0.1} = 1 + \underbrace{11\delta_1 + \delta_2 + 11\delta_1\delta_2}_{\varepsilon_3}$$

where

$$|\varepsilon_3| \leq (11/2 + 1/2 + 11/4) \leq 9\epsilon_{\rm m}$$

Then we get

$$(\mathrm{fl}(1.1) \ominus 1) \oslash \mathrm{fl}(0.1) = (1+\varepsilon_3)(1+\varepsilon_2) = 1 + \underbrace{\varepsilon_3 + \varepsilon_2 + \varepsilon_2\varepsilon_3}_{\varepsilon_4}$$

and the error is bounded by:

$$|\varepsilon_4| \leq (9 + 2 + 18)\epsilon_{\rm m} \leq 29\epsilon_{\rm m}.$$

**END**

**Problem 4** Let $x \in [0,1] \cap F_{\infty,S}$. Assume that $f^{\rm FP} : F_{\infty,S} \to F_{\infty,S}$ satisfies $f^{\rm FP}(x) = f(x) + \delta_x$ where $|\delta_x| \leq c\epsilon_{\rm m}$ for all $x \in [0,1]$. Show that

$$\frac{f^{\rm FP}(x+h) \ominus f^{\rm FP}(x-h)}{2h} = f'(x) + \varepsilon$$

where absolute error is bounded by

$$|\varepsilon| \leq \frac{|f'(x)|}{2}\epsilon_{\rm m} + \frac{M}{3}h^2 + \frac{2c\epsilon_{\rm m}}{h},$$

where we assume that $h = 2^{-n}$ for $n \leq S$.

**SOLUTION**

In floating point we have

$$\begin{aligned}
\frac{f^{\rm FP}(x+h) \ominus f^{\rm FP}(x-h)}{2h} &= \frac{f(x+h) + \delta_{x+h} - f(x-h) - \delta_{x-h}}{2h}(1+\delta_1) \\
&= \frac{f(x+h) - f(x-h)}{2h}(1+\delta_1) + \frac{\delta_{x+h} - \delta_{x-h}}{2h}(1+\delta_1)
\end{aligned}$$

Applying Taylor's theorem we get

$$(f^{\rm FP}(x+h) \ominus f^{\rm FP}(x-h))/(2h) = f'(x) + \underbrace{f'(x)\delta_1 + \delta^{\rm T}_{x,h}(1+\delta_1) + \frac{\delta_{x+h} - \delta_{x-h}}{2h}(1+\delta_1)}_{\delta^{\rm CD}_{x,h}}$$

where

$$|\delta^{\rm CD}_{x,h}| \leq \frac{|f'(x)|}{2}\epsilon_{\rm m} + \frac{M}{3}h^2 + \frac{2c\epsilon_{\rm m}}{h}$$

**END**

**Problem 5** For intervals $X = [a,b]$ and $Y = [c,d]$ satisfying $0 < a < b$ and $0 < c < d$, and $n > 0$ prove that

$$\begin{aligned}
X/n &= [a/n, b/n] \\
XY &= [ac, bd]
\end{aligned}$$

Generalise (without proof) these formulæ to the case $n < 0$ and to where there are no restrictions on positivity of $a, b, c, d$. You may use the min or max functions.

**SOLUTION**

For $X/n$: if $x \in X$ then $a/n \leq x/n \leq b/n$ means $x \in [a/n, b/n]$. Similarly, if $z \in [a/n, b/n]$ then $a \leq nz \leq b$ hence $nz \in X$ and therefore $z \in X/n$.

For $XY$: if $x \in X$ and $y \in Y$ then $ac \leq xy \leq bd$ means $xy \in [ac, bd]$. Note $ac, bd \in XY$. To employ convexity we take logarithms. In particular if $z \in [ac, bd]$ then $\log a + \log c \leq \log z \leq \log b + \log d$. Hence write

$$\log z = (1-t)(\log a + \log c) + t(\log b + \log d) = \underbrace{(1-t)\log a + t\log b}_{\log x} + \underbrace{(1-t)\log c + t\log d}_{\log y}$$

i.e. we have $z = xy$ where

$$\begin{aligned}
x &= \exp((1-t)\log a + t\log b) = a^{1-t}b^t \in X \\
y &= \exp((1-t)\log c + t\log d) = c^{1-t}d^t \in Y.
\end{aligned}$$

The generalisation to negative cases proceeds by being a bit careful with the signs. Eg if $n < 0$ we need to swap the order hence we get:

$$A/n = \begin{cases} [a/n, b/n] & n > 0 \\ [b/n, a/n] & n < 0 \end{cases}$$

For multiplication we just use min and max in a naive fashion:

$$AB = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)].$$

**END**

**Problem 6(a)** Compute the following floating point interval arithmetic expression assuming half-precision $F_{16}$ arithmetic:

$$[1, 1] \ominus ([1, 1] \oslash 6)$$

Hint: it might help to write $1 = (0.1111\ldots)_2$ when doing subtraction.

**SOLUTION** Note that

$$\frac{1}{6} = \frac{1}{2}\frac{1}{3} = 2^{-3}(1.010101\ldots)_2$$

Thus

$$[1, 1] \oslash 6 = 2^{-3}[(1.0101010101)_2, (1.0101010110)_2]$$

And hence

$$\begin{aligned} [1, 1] \ominus ([1, 1] \oslash 6) &= [1, 1] \ominus [(0.0010101010101)_2, (0.0010101010110)_2] \\ &= [\mathrm{fl}^{\mathrm{down}}(0.110101010101{\color{magenta}0111111\ldots})_2, \mathrm{fl}^{\mathrm{up}}(0.11010101010{\color{magenta}011111\ldots})_2] \\ &= 2^{-1}[(1.1010101010)_2, (1.1010101011)_2] = [0.8330078125, 0.83349609375] \end{aligned}$$

**END**

**Problem 6(b)** Writing

$$\sin\ x = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \delta_{x,2n+1}$$

Prove the bound $|\delta_{x,2n+1}| \leq 1/(2n+3)!$, assuming $x \in [0, 1]$.

**SOLUTION**

We have from Taylor's theorem up to order $x^{2n+2}$:

$$\sin\ x = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \underbrace{\frac{\sin^{2n+3}(t) x^{2n+3}}{(2n+3)!}}_{\delta_{x,2n+1}}.$$

The bound follows since all derivatives of sin are bounded by 1 and we have assumed $|x| \leq 1$.

**END**

**Problem 6(c)** Combine the previous parts to prove that:

$$\sin 1 \in [(0.11010011000)_2, (0.11010111101)_2] = [0.82421875, 0.84228515625]$$

You may use without proof that $1/120 = 2^{-7}(1.000100010001\ldots)_2$.

**SOLUTION** Using $n = 1$ we have

$$\sum_{k=0}^{1} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^2}{3!} \in x \ominus ((x \otimes x) \oslash 6).$$

Noting that in floating point $1 \otimes 1 = 1$ (ie it is exact) we compute

$$\begin{aligned}
\sin 1 &\in [1,1] \ominus [1,1] \oslash 6 \oplus [\mathrm{fl}^{\mathrm{down}}(-1/120), \mathrm{fl}^{\mathrm{up}}(1/120)] \\
&= [(0.11010101010)_2, (0.11010101011)_2] \oplus [-(0.0000001000100010)_2, (0.00000010001000101)_2] \\
&= [\mathrm{fl}^{\mathrm{down}}(0.11010011000{\color{magenta}11101111\ldots})_2, \mathrm{fl}^{\mathrm{up}}(0.11010111100{\color{magenta}000101})_2] \\
&= [(0.11010011000)_2, (0.11010111101)_2] = [0.82421875, 0.84228515625]
\end{aligned}$$

**END**