

## Numerical Analysis MATH50003 (2023–24) Problem Sheet 5

**Problem 1(a)** Suppose  $|\epsilon_k| \leq \epsilon$  and  $n\epsilon < 1$ . Use induction to show that

$$\prod_{k=1}^n (1 + \epsilon_k) = 1 + \theta_n$$

for some constant  $\theta_n$  satisfying

$$|\theta_n| \leq \underbrace{\frac{n\epsilon}{1 - n\epsilon}}_{E_{n,\epsilon}}$$

**SOLUTION**

$$\prod_{k=1}^{n+1} (1 + \epsilon_k) = \prod_{k=1}^n (1 + \epsilon_k)(1 + \epsilon_{n+1}) = (1 + \theta_n)(1 + \epsilon_{n+1}) = 1 + \underbrace{\theta_n + \epsilon_{n+1} + \theta_n \epsilon_{n+1}}_{\theta_{n+1}}$$

where

$$\begin{aligned} |\theta_{n+1}| &\leq \frac{n\epsilon}{1 - n\epsilon}(1 + \epsilon) + \epsilon \\ &= \frac{n\epsilon + n\epsilon^2}{1 - (n+1)\epsilon} \frac{1 - (n+1)\epsilon}{1 - n\epsilon} + \frac{\epsilon - (n+1)\epsilon^2}{1 - (n+1)\epsilon} \\ &\leq \frac{(n+1) - n\epsilon}{1 - (n+1)\epsilon} \epsilon \leq \frac{(n+1)\epsilon}{1 - (n+1)\epsilon} \end{aligned}$$

**END**

**Problem 1(b)** Show for an idealised floating point vector  $\mathbf{x} \in F_{\infty,S}^n$  that

$$x_1 \oplus \cdots \oplus x_n = x_1 + \cdots + x_n + \sigma_n$$

where

$$|\sigma_n| \leq \|\mathbf{x}\|_1 E_{n-1,\epsilon_m/2},$$

assuming  $n\epsilon_m < 2$  and where

$$\|\mathbf{x}\|_1 := \sum_{k=1}^n |x_k|.$$

Hint: use the previous part to first write

$$x_1 \oplus \cdots \oplus x_n = x_1(1 + \theta_{n-1}) + \sum_{j=2}^n x_j(1 + \theta_{n-j+1}).$$

**SOLUTION**

Using Problem 2.1 we write:

$$\begin{aligned} (\cdots((x_1 + x_2)(1 + \delta_1) + x_3)(1 + \delta_2) \cdots + x_n)(1 + \delta_{n-1}) &= x_1 \prod_{k=1}^{n-1} (1 + \delta_k) + \sum_{j=2}^n x_j \prod_{k=j-1}^{n-1} (1 + \delta_k) \\ &= x_1(1 + \theta_{n-1}) + \sum_{j=2}^n x_j(1 + \theta_{n-j+1}) \end{aligned}$$

where we have for  $j = 2, \dots, n$

$$|\theta_{n-j+1}| \leq E_{n-j+1,\epsilon_m/2} \leq E_{n-1,\epsilon_m/2}.$$

Thus we have

$$\sum_{j=1}^n x_j(1 + \theta_{n-j+1}) = \sum_{j=1}^n x_j + \underbrace{\sum_{j=1}^n x_j \theta_{n-j+1}}_{\sigma_n}$$

where

$$|\sigma_n| \leq \sum_{j=1}^n |x_j \theta_{n-j+1}| \leq \sup_j |\theta_{n-j+1}| \sum_{j=1}^n |x_j| \leq \|\mathbf{x}\|_1 E_{n-1, \epsilon_m/2}.$$

**END**

**Problem 1(c)** For  $A \in F_{\infty, S}^{n \times n}$  and  $\mathbf{x} \in F_{\infty, S}^n$  consider the error in approximating matrix multiplication with idealised floating point: for

$$A\mathbf{x} = \begin{pmatrix} \bigoplus_{j=1}^n A_{1,j} \otimes x_j \\ \vdots \\ \bigoplus_{j=1}^n A_{n,j} \otimes x_j \end{pmatrix} + \delta$$

show that

$$\|\delta\|_{\infty} \leq 2\|A\|_{\infty} \|\mathbf{x}\|_{\infty} E_{n, \epsilon_m/2}$$

where  $n\epsilon_m < 2$  and the matrix norm is  $\|A\|_{\infty} := \max_k \sum_{j=1}^n |a_{kj}|$ .

**SOLUTION** We have for the  $k$ -th row

$$\bigoplus_{j=1}^n A_{k,j} \otimes x_j = \bigoplus_{j=1}^n A_{k,j} x_j (1 + \delta_j) = \sum_{j=1}^n A_{k,j} x_j (1 + \delta_j) + \sigma_{k,n}$$

where we know  $|\sigma_n| \leq M_k E_{n-1, \epsilon_m/2}$ , where from 1(b) we have

$$M_k = \sum_{j=1}^n |A_{k,j} x_j (1 + \delta_j)| = \sum_{j=1}^n |A_{k,j}| |x_j| (1 + |\delta_j|) \leq 2 \max |x_j| \sum_{j=1}^n |A_{k,j}| \leq 2\|\mathbf{x}\|_{\infty} \|A\|_{\infty}$$

Similarly, we also have

$$\left| \sum_{j=1}^n A_{k,j} x_j \delta_j \right| \leq \|\mathbf{x}\|_{\infty} \|A\|_{\infty} \epsilon_m/2$$

and so the result follows from

$$\epsilon_m/2 + 2E_{n-1, \epsilon_m/2} \leq \frac{\epsilon_m/2 + \epsilon_m(n-1)}{1 - (n-1)\epsilon_m/2} \leq \frac{\epsilon_m n}{1 - n\epsilon_m/2} = 2E_{n, \epsilon_m/2}.$$

**END**

**Problem 2** Derive Backward Euler: use the left-sided divided difference approximation

$$u'(x) \approx \frac{u(x) - u(x-h)}{h}$$

to reduce the first order ODE

$$u'(x) = c, \quad u'(x) + \omega(x)u(x) = f(x)$$

to a lower triangular system by discretising on the grid  $x_j = a + jh$  for  $h = (b-a)/n$ . Hint: only impose the ODE on the gridpoints  $x_1, \dots, x_n$  so that the divided difference does not depend on behaviour at  $x_{-1}$ .

## SOLUTION

We go through all 4 steps (this is to help you understand what to do. In an exam I will still give full credit if you get the right result, even if you don't write down all 4 steps):

(Step 1) Since we need to avoid going off the left in step 2 we start the ODE discretisation at  $x_1$ :

$$\begin{pmatrix} u(x_0) \\ u'(x_1) \\ \vdots \\ u'(x_n) \end{pmatrix} = \underbrace{\begin{pmatrix} c \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}}_{\mathbf{b}}$$

(Step 2) Replace with divided differences:

$$\begin{pmatrix} u(x_0) \\ (u(x_1) - u(x_0))/h \\ \vdots \\ (u(x_n) - u(x_{n-1}))/h \end{pmatrix} \approx \mathbf{b}$$

(Step 3) Replace with discrete system with equality:

$$\begin{pmatrix} u_0 \\ (u_1 - u_0)/h \\ \vdots \\ (u_n - u_{n-1})/h \end{pmatrix} = \mathbf{b}$$

(Step 4) Write as linear system:

$$\begin{bmatrix} 1 & & & & \\ -1/h & 1/h & & & \\ & \ddots & \ddots & & \\ & & -1/h & 1/h & \end{bmatrix} \begin{pmatrix} u_0 \\ \vdots \\ u_n \end{pmatrix} = \mathbf{b}$$

This is the exact same matrix as Forward Euler but with a different right-hand side.

**END**

**Problem 3** Reduce a Schrödinger equation to a tridiagonal linear system by discretising on the grid  $x_j = a + jh$  for  $h = (b - a)/n$ :

$$u(a) = c, \quad u''(x) + V(x)u(x) = f(x), \quad u(b) = d.$$

## SOLUTION

(Step 1)

$$\begin{pmatrix} u(x_0) \\ u''(x_1) + V(x_1)u(x_1) \\ \vdots \\ u''(x_{n-1}) + V(x_{n-1})u(x_{n-1}) \\ u(x_n) \end{pmatrix} = \underbrace{\begin{pmatrix} c \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \\ d \end{pmatrix}}_{\mathbf{b}}$$

(Step 2) Replace with divided differences:

$$\begin{pmatrix} u(x_0) \\ (u(x_0) - 2u(x_1) + u(x_2))/h^2 + V(x_1)u(x_1) \\ \vdots \\ (u(x_{n-2}) - 2u(x_{n-1}) + u(x_n))/h^2 + V(x_{n-1})u(x_{n-1}) \\ u(x_n) \end{pmatrix} \approx \mathbf{b}$$

(Step 3) Replace with discrete system with equality:

$$\begin{pmatrix} u_0 \\ (u_0 - 2u_1 + u_2)/h^2 + V(x_1)u_1 \\ \vdots \\ (u_{n-2} - 2u_{n-1} + u_n)/h^2 + V(x_{n-1})u_{n-1} \\ u_n \end{pmatrix} = \mathbf{b}$$

(Step 4) Write as a tridiagonal linear system:

$$\begin{bmatrix} 1 & & & & & \\ -1/h^2 & V(x_1) - 2/h^2 & 1/h^2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1/h^2 & V(x_{n-1}) - 2/h^2 & 1/h^2 & \\ & & & & 1 & \end{bmatrix} \begin{pmatrix} u_0 \\ \vdots \\ u_n \end{pmatrix} = \mathbf{b}$$

**END**