Numerical Analysis MATH50003 (2023–24) Problem Sheet 7

Problem 1(a) Show for a unitary matrix $Q \in U(n)$ and a vector $\boldsymbol{x} \in \mathbb{C}^n$ that multiplication by Q preserve the 2-norm: $\|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$.

SOLUTION

$$\|Q\boldsymbol{x}\|^2 = (Q\boldsymbol{x})^\star Q\boldsymbol{x} = \boldsymbol{x}^\star Q^\star Q\boldsymbol{x} = \boldsymbol{x}^\star \boldsymbol{x} = \|\boldsymbol{x}\|^2$$

END

Problem 1(b) Show that the eigenvalues λ of a unitary matrix Q are on the unit circle: $|\lambda| = 1$. Hint: recall for any eigenvalue λ that there exists a unit eigenvector $\boldsymbol{v} \in \mathbb{C}^n$ (satisfying $\|\boldsymbol{v}\| = 1$).

SOLUTION Let $\boldsymbol{v}$ be a unit eigenvector corresponding to λ: $Q\boldsymbol{v} = \lambda\boldsymbol{v}$ with $\|\boldsymbol{v}\| = 1$. Then

$$1 = \|\boldsymbol{v}\| = \|Q\boldsymbol{v}\| = \|\lambda\boldsymbol{v}\| = |\lambda|.$$

END

Problem 1(c) Show for an orthogonal matrix $Q \in O(n)$ that $\det Q = \pm 1$. Give an example of $Q \in U(n)$ such that $\det Q \neq \pm 1$. Hint: recall for any real matrices A and B that $\det A = \det A^\top$ and $\det(AB) = \det A \det B$.

SOLUTION

$$(\det Q)^2 = (\det Q^\top)(\det Q) = \det Q^\top Q = \det I = 1.$$

An example would be a 1×1 complex-valued matrix $\exp(\mathrm{i})$.

END

Problem 1(d) A normal matrix commutes with its adjoint. Show that $Q \in U(n)$ is normal.

SOLUTION

$$QQ^\star = I = Q^\star Q$$

END

Problem 1(e) The spectral theorem states that any normal matrix is unitarily diagonalisable: if A is normal then $A = V\Lambda V^\star$ where $V \in U(n)$ and Λ is diagonal. Use this to show that $Q \in U(n)$ is equal to I if and only if all its eigenvalues are 1.

SOLUTION

Note that Q is normal and therefore by the spectral theorem for normal matrices we have

$$Q = V\Lambda V^\star = VV^\star = I$$

since V is unitary.

END

Problem 2 Consider the vectors

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2\mathrm{i} \\ 2 \end{bmatrix}.$$

Use reflections to determine the entries of orthogonal/unitary matrices Q_1, Q_2, Q_3 such that

$$Q_1\boldsymbol{a} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, Q_2\boldsymbol{a} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}, Q_3\boldsymbol{b} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

SOLUTION

For Q_1: we have

$$\boldsymbol{y} = \boldsymbol{a} - \|\boldsymbol{a}\|\boldsymbol{e}_1 = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$$

$$\boldsymbol{w} = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$Q_1 = Q_{\boldsymbol{w}} = I - \frac{2}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} [-1\ 1\ 1] = I - \frac{2}{3} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

For Q_2: we have

$$\boldsymbol{y} = \boldsymbol{a} + \|\boldsymbol{a}\|\boldsymbol{e}_1 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$\boldsymbol{w} = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$Q_1 = Q_{\boldsymbol{w}} = I - \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} [2\ 1\ 1] = I - \frac{1}{3} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -1 & -2 & -2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

For Q_3 we just need to be careful to conjugate:

$$\boldsymbol{y} = \boldsymbol{b} + \|\boldsymbol{b}\|\boldsymbol{e}_1 = \begin{bmatrix} 4 \\ 2\mathrm{i} \\ 2 \end{bmatrix}$$

$$\boldsymbol{w} = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ \mathrm{i} \\ 1 \end{bmatrix}$$

$$Q_3 = Q_{\boldsymbol{w}} = I - \frac{1}{3} \begin{bmatrix} 2 \\ \mathrm{i} \\ 1 \end{bmatrix} [2\ -\mathrm{i}\ 1] = I - \frac{1}{3} \begin{bmatrix} 4 & -2\mathrm{i} & 2 \\ 2\mathrm{i} & 1 & \mathrm{i} \\ 2 & -\mathrm{i} & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -1 & 2\mathrm{i} & -2 \\ -2\mathrm{i} & 2 & -\mathrm{i} \\ -2 & \mathrm{i} & 2 \end{bmatrix}$$

END

Problem 3(a) What simple rotation matrices $Q_1, Q_2 \in SO(2)$ have the property that:

$$Q_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}, Q_2 \begin{bmatrix} \sqrt{5} \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

SOLUTION

The rotation that takes $[x, y]$ to the x-axis is

$$\frac{1}{\sqrt{x^2+y^2}}\begin{bmatrix} x & y \\ -y & x \end{bmatrix}.$$

Hence we get

$$Q_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$
$$Q_2 = \frac{1}{3}\begin{bmatrix} \sqrt{5} & 2 \\ -2 & \sqrt{5} \end{bmatrix}$$

END

Problem 3(b) Find an orthogonal matrix that is a product of two simple rotations but acting on two different subspaces:

$$Q = \underbrace{\begin{bmatrix} \cos\theta_2 & & -\sin\theta_2 \\ & 1 & \\ \sin\theta_2 & & \cos\theta_2 \end{bmatrix}}_{Q_2}\underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & \\ \sin\theta_1 & \cos\theta_1 & \\ & & 1 \end{bmatrix}}_{Q_1}$$

so that, for $\boldsymbol{a}$ defined above,

$$Q\boldsymbol{a} = \begin{bmatrix} \|\boldsymbol{a}\| \\ 0 \\ 0 \end{bmatrix}.$$

Hint: you do not need to determine θ_1, θ_2, instead you can write the entries of Q_1, Q_2 directly using just square-roots.

SOLUTION

We use Q_1 to introduce a 0 in the second entry by rotating the vector $[1, 2]$:

$$Q_1 = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & \\ -2/\sqrt{5} & 1/\sqrt{5} & \\ & & 1 \end{bmatrix}$$

so that

$$Q_1\boldsymbol{a} = \begin{bmatrix} \sqrt{5} \\ 0 \\ 2 \end{bmatrix}.$$

Now we use the matrix that rotates the vector $[\sqrt{5}, 2]$ whose norm is 3 to deduce the entries

$$Q_2 = \begin{bmatrix} \sqrt{5}/3 & & 2/3 \\ & 1 & \\ -2/3 & & \sqrt{5}/3 \end{bmatrix}$$

so that

$$Q_2Q_1 = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/\sqrt{5} & 1/\sqrt{5} & \\ -2/(3\sqrt{5}) & -4/(3\sqrt{5}) & \sqrt{5}/3 \end{bmatrix}$$

END

Problem 4(a) Show that every matrix $A \in \mathbb{R}^{m\times n}$ has a QR factorisation such that the diagonal of R is non-negative. Make sure to include the case of more columns than rows (i.e. $m < n$).

SOLUTION

We first show for $m < n$ that a QR decomposition exists. Writing

$$A = [\boldsymbol{a}_1|\cdots|\boldsymbol{a}_n]$$

and taking the first m columns (so that it is square) we can write $[\boldsymbol{a}_1|\cdots|\boldsymbol{a}_m] = QR_m$. It follows that $R := Q^\star A$ is right-triangular.

We can write:

$$D = \begin{bmatrix} \mathrm{sign}(r_{11}) & & \\ & \ddots & \\ & & \mathrm{sign}(r_{pp}) \end{bmatrix}$$

where $p = \min(m, n)$ and we define $\mathrm{sign}(0) = 1$. Note that $D^\top D = I$. Thus we can write: $A = QR = QDDR$ where (QD) is orthogonal and DR is upper-triangular with positive entries.

END

Problem 4(b) Show that the QR factorisation of a square invertible matrix $A \in \mathbb{R}^{n\times n}$ is unique, provided that the diagonal of R is positive.

SOLUTION

Assume there is a second factorisation also with positive diagonal

$$A = QR = \tilde{Q}\tilde{R}$$

Then we know

$$Q^\top \tilde{Q} = R\tilde{R}^{-1}$$

Note $Q^\top \tilde{Q}$ is orthogonal, and $R\tilde{R}^{-1}$ has positive eigenvalues (the diagonal), hence all m eigenvalues of $Q^\top \tilde{Q}$ are 1. This means that $Q^\top \tilde{Q} = I$ and hence $\tilde{Q} = Q$ and $\tilde{R} = R$. **END**