MATH50003 Numerical Analysis

IV.1 Fourier Expansions

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Part IV: Approximation Theory

Introduce more sophisticated mathematical tools for much higher accuracy computations.

Part IV

Approximation Theory

1. Fourier Expansions and approximating Fourier coefficients
2. Discrete Fourier Transforms and interpolation
3. Orthogonal Polynomials and basic properties
4. Classical Orthogonal Polynomials with special structure
5. Gaussian Quadrature for high-accuracy integration

IV.1.1 Basics of Fourier series

Expanding functions in trigonometric polynomials

Definition 31 (Fourier)**.** A function f has a Fourier expansion if

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}_k \mathrm{e}^{\mathrm{i}k\theta}$$

where

$$\hat{f}_k := \langle \mathrm{e}^{\mathrm{i}k\theta}, f \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}k\theta} f(\theta) \mathrm{d}\theta$$

Definition 32 (Fourier-Taylor).

Definition 33 (absolute convergent)**.** We write $\hat{\boldsymbol{f}} \in \ell^1$ if it is absolutely convergent, or in otherwords, the 1-norm of $\hat{\boldsymbol{f}}$ is bounded:

$$\|\hat{\boldsymbol{f}}\|_1 := \sum_{k=-\infty}^{\infty} |\hat{f}_k| < \infty.$$

Theorem 9 (Fourier series equivalence)**.** *If $f, g : \mathbb{T} \to \mathbb{C}$ are continuous and $\hat{f}_k = \hat{g}_k$ for all $k \in \mathbb{Z}$ then $f = g$.*

Proof See Körner 2022 (Theorem 2.4). ■

Theorem 10 (Absolute converging Fourier series)**.** *If* $\hat{\boldsymbol{f}} \in \ell^1$ *then*

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}_k \mathrm{e}^{\mathrm{i}k\theta},$$

which converges uniformly.

Lemma 6 (differentiability and absolutely convergence)**.** *If* $f : \mathbb{R} \to \mathbb{C}$ *and* f' *are periodic and* f'' *is uniformly bounded, then* $\hat{\boldsymbol{f}} \in \ell^1$*.*

IV.1.2 Trapezium rule and discrete Fourier coefficients

Using the Trapezium rule to approximate coefficients has nice structure

Definition 34 (Periodic Trapezium Rule).

Lemma 7 (Discrete orthogonality)**.** *We have:*

$$\sum_{j=0}^{n-1} \mathrm{e}^{\mathrm{i}k\theta_j} = \begin{cases} n & k = \ldots, -2n, -n, 0, n, 2n, \ldots \\ 0 & \textit{otherwise} \end{cases}$$

IV.1.3 Convergence of Approximate Fourier coefficients

Using Trapezium rule leads to a convergent approximation

Definition 35 (Discrete Fourier coefficients)**.** Define the Trapezium rule approximation to the Fourier coefficients by:

$$\hat{f}_k^n := \Sigma_n[\mathrm{e}^{-ik\theta} f(\theta)] = \frac{1}{n}\sum_{j=0}^{n-1} \mathrm{e}^{-ik\theta_j} f(\theta_j)$$

Theorem 11 (discrete Fourier coefficients). *If $\hat{\boldsymbol{f}} \in \ell^1$ (absolutely convergent Fourier coefficients) then*

$$\hat{f}^n_k = \cdots + \hat{f}_{k-2n} + \hat{f}_{k-n} + \hat{f}_k + \hat{f}_{k+n} + \hat{f}_{k+2n} + \cdots$$

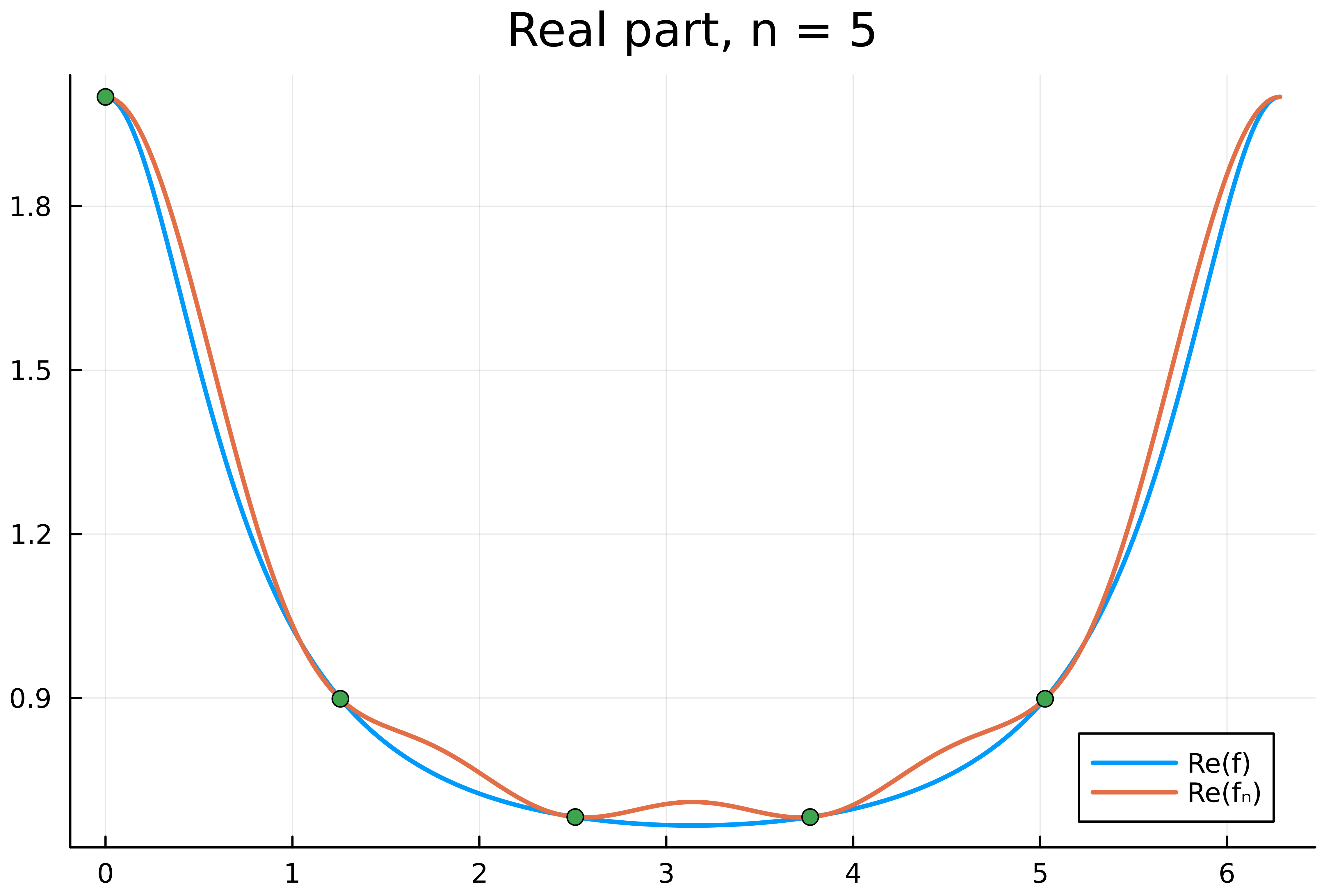
Example 26 (Taylor coefficients via Geometric series).

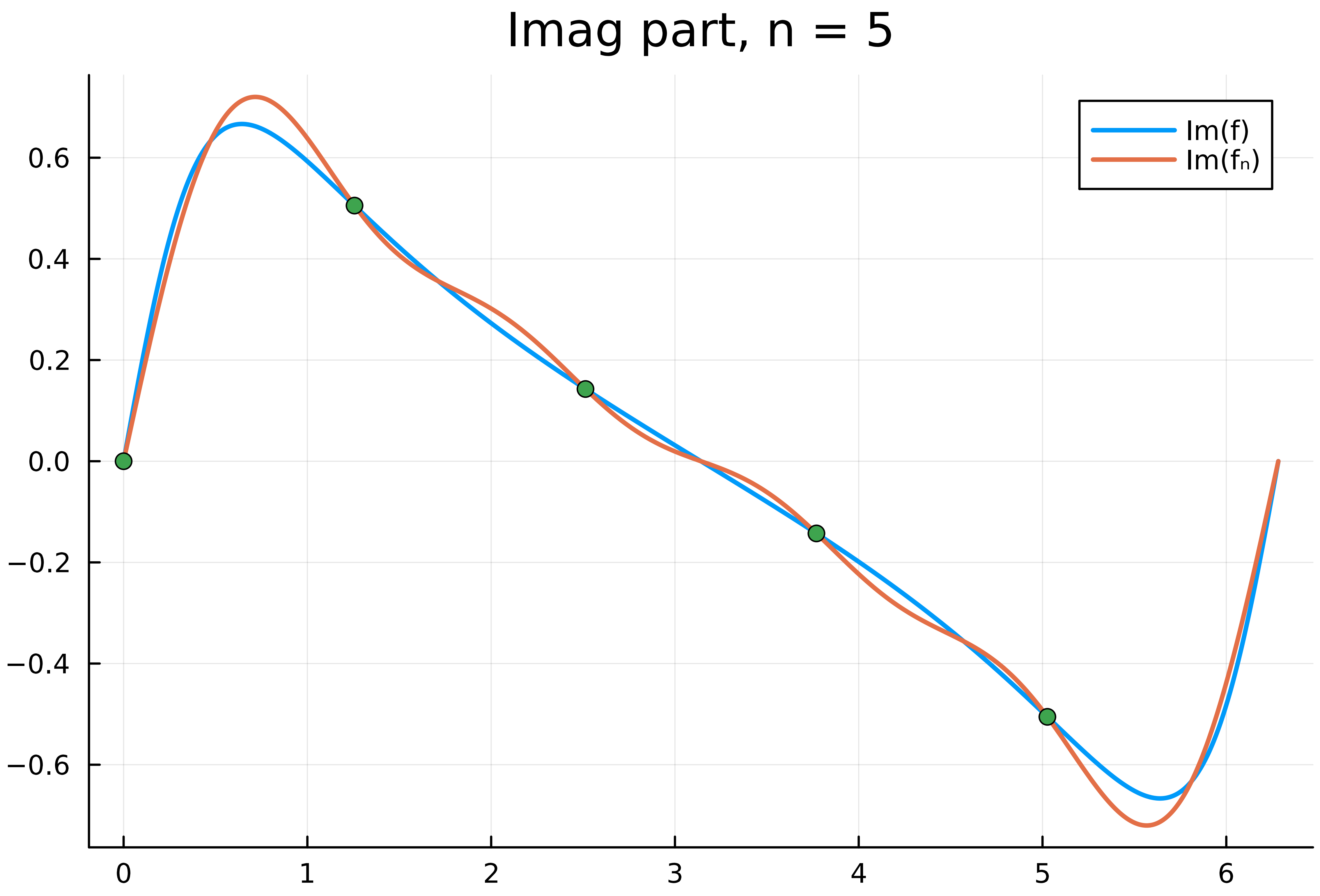
Theorem 12 (Approximate Fourier-Taylor expansions converge)**.** *If* $0 = \hat{f}_{-1} = \hat{f}_{-2} = \cdots$ *and* $\hat{\boldsymbol{f}}$ *is absolutely convergent then*

$$f_n(\theta) = \sum_{k=0}^{n-1} \hat{f}_k^n \mathrm{e}^{\mathrm{i}k\theta}$$

converges uniformly to $f(\theta)$.

$$f(\theta) = \frac{2}{2 - e^{i\theta}}$$





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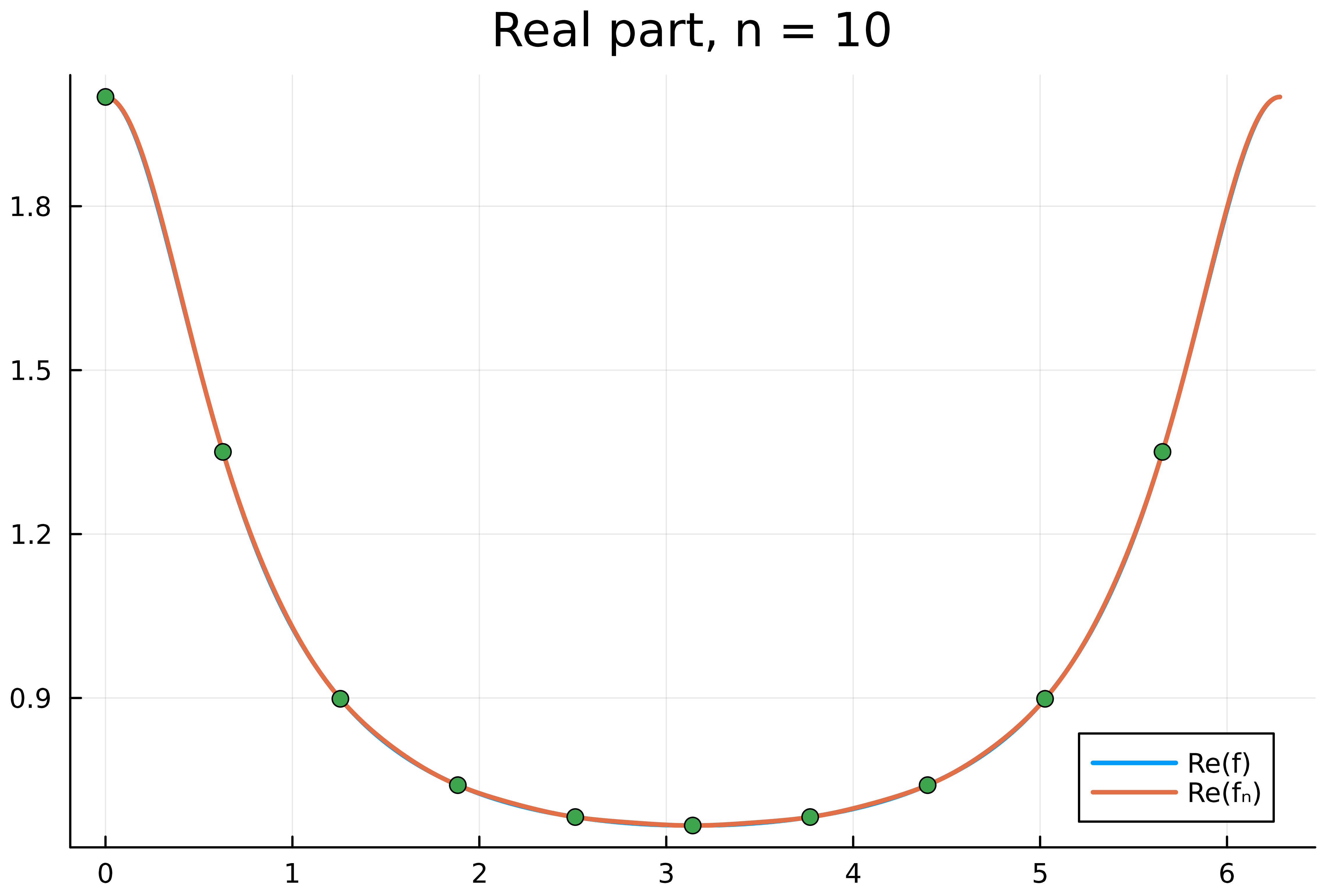
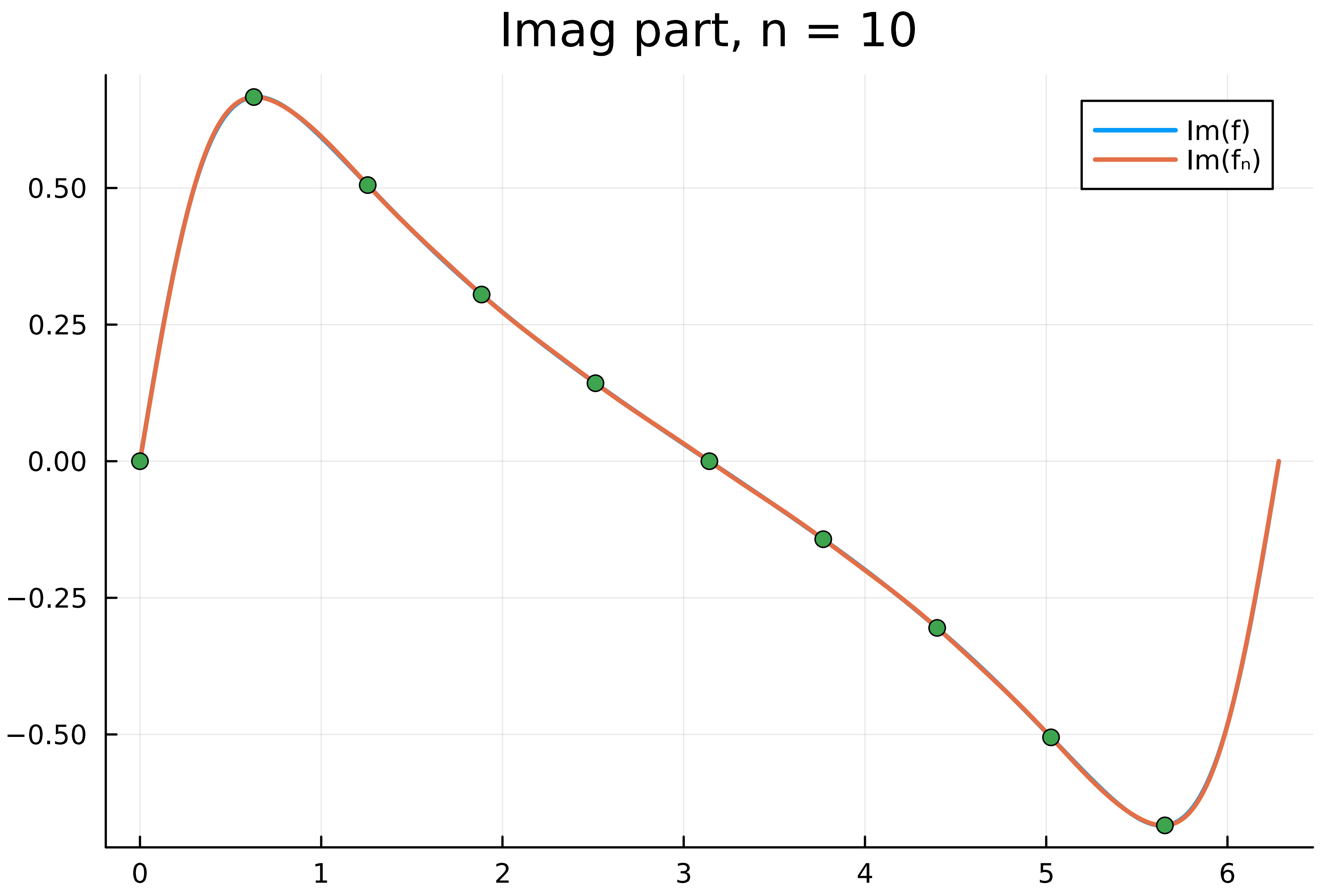
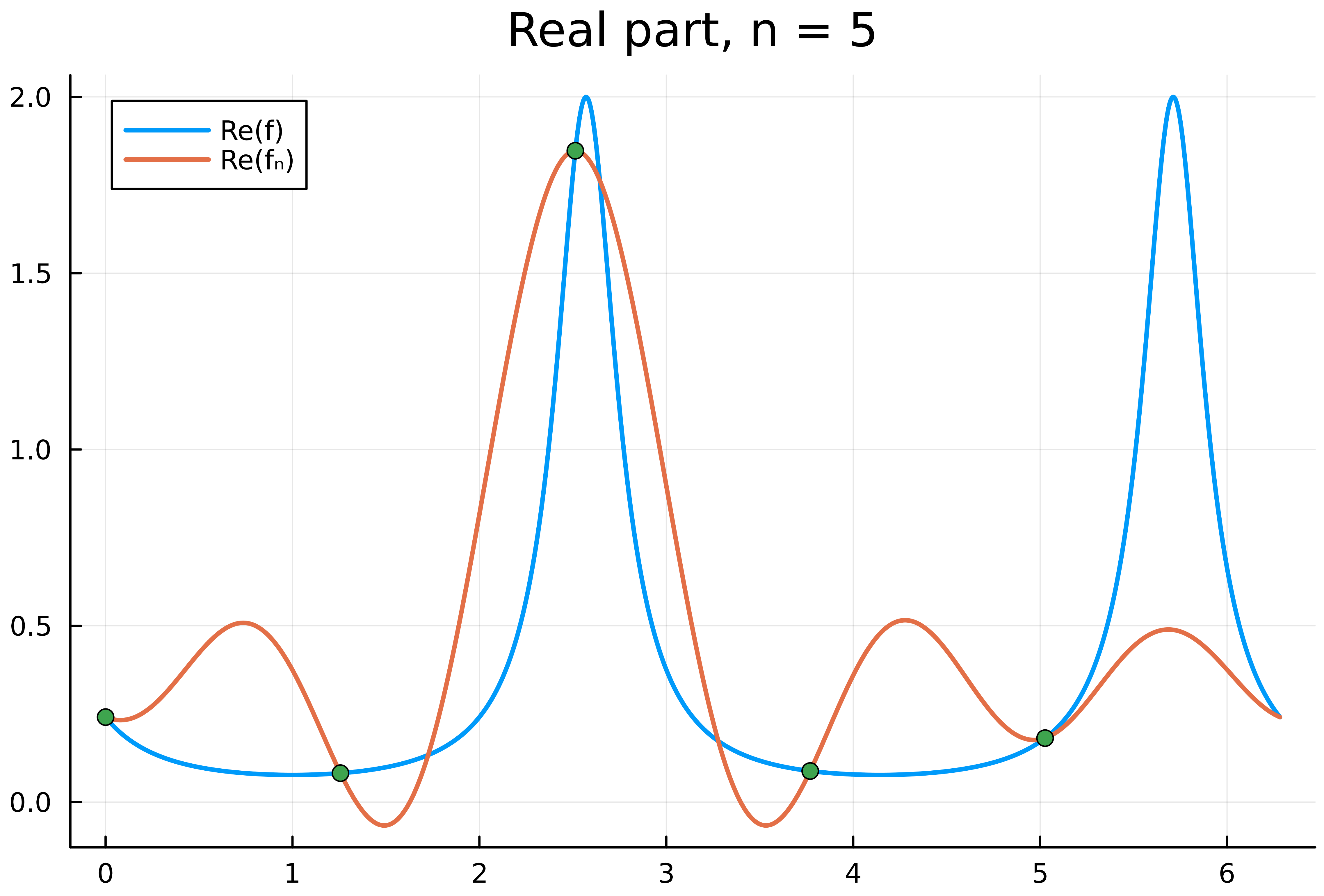
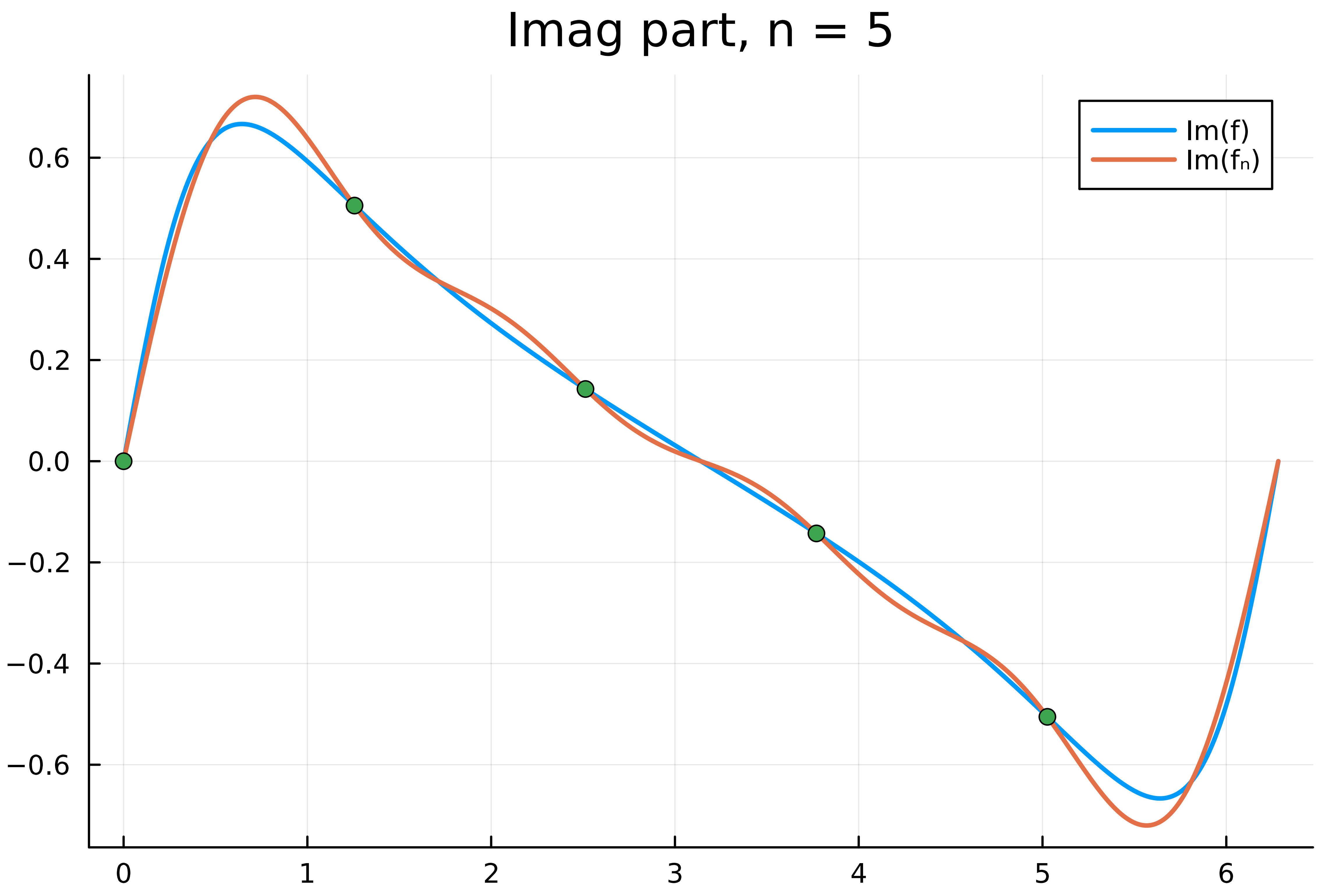




$$f(\theta) = \frac{2}{25\cos(\theta - 1)^2 + 1}$$





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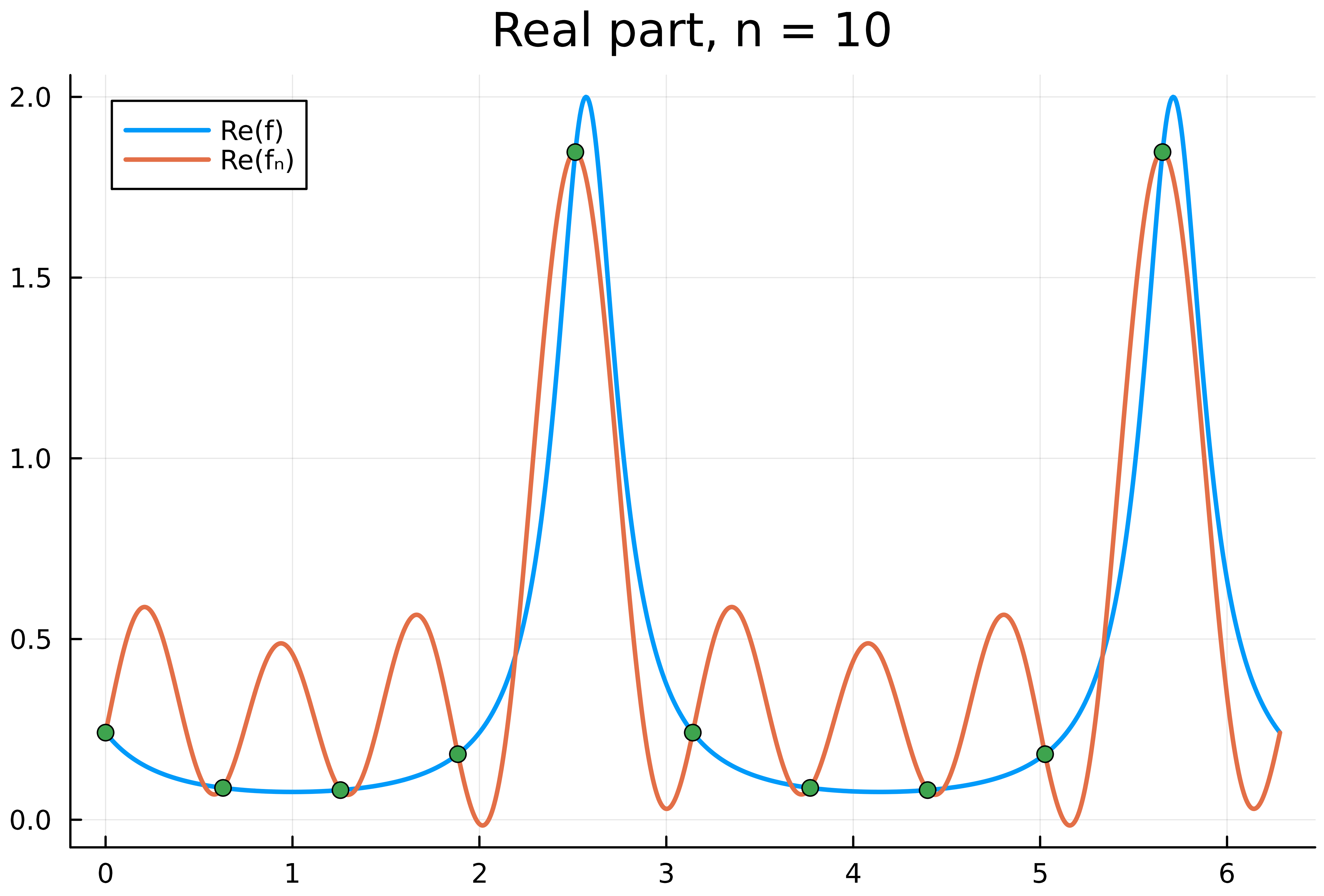

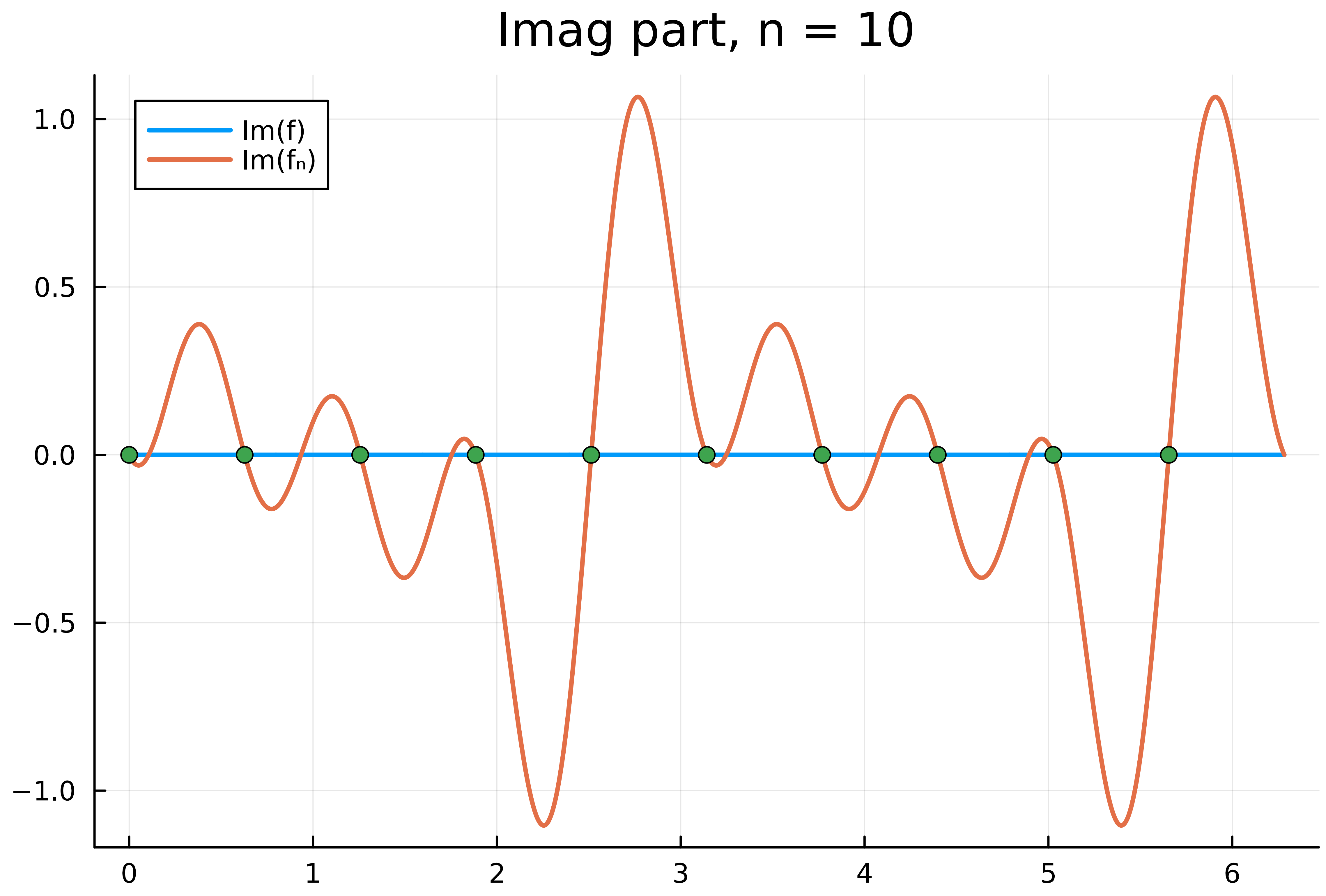

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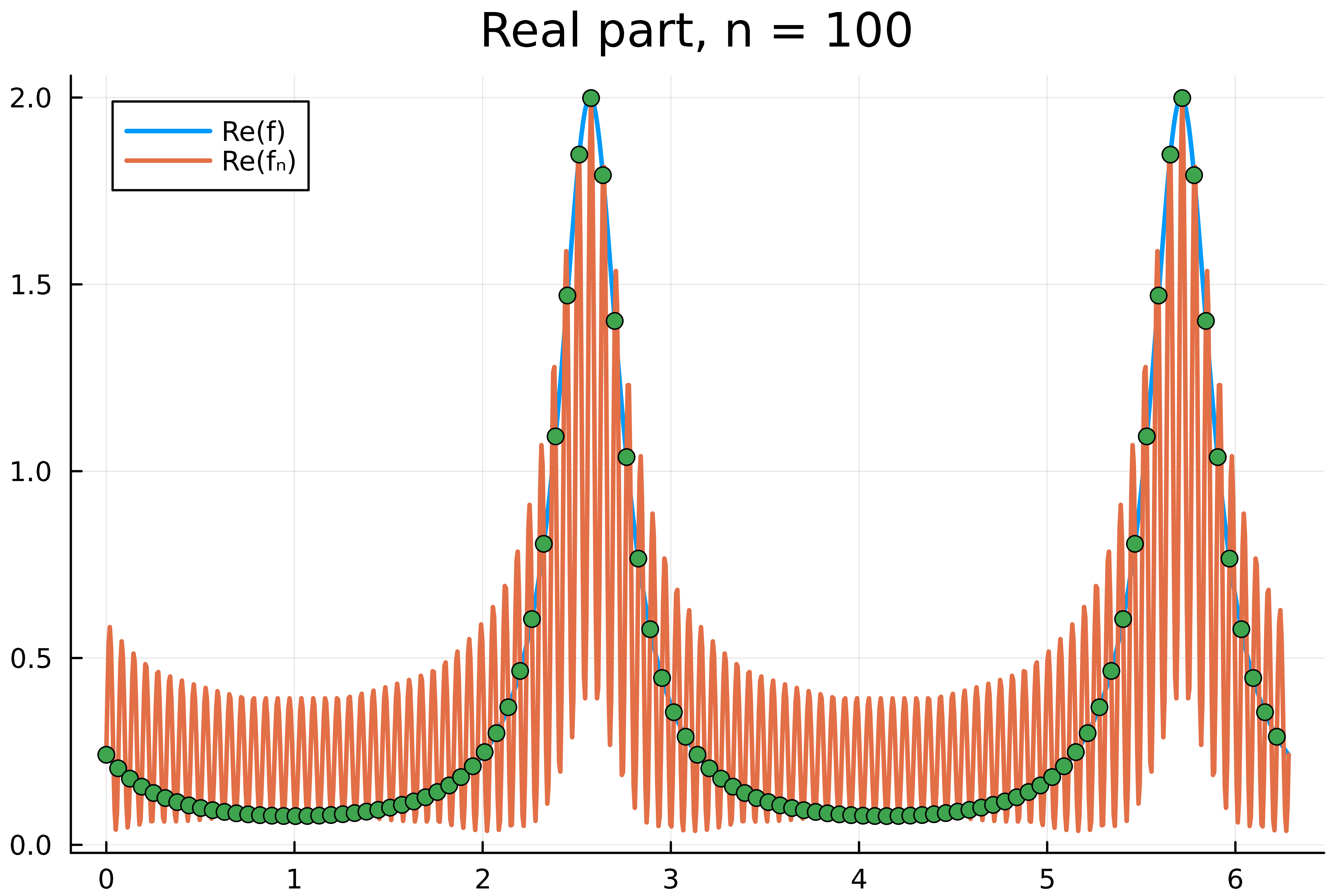
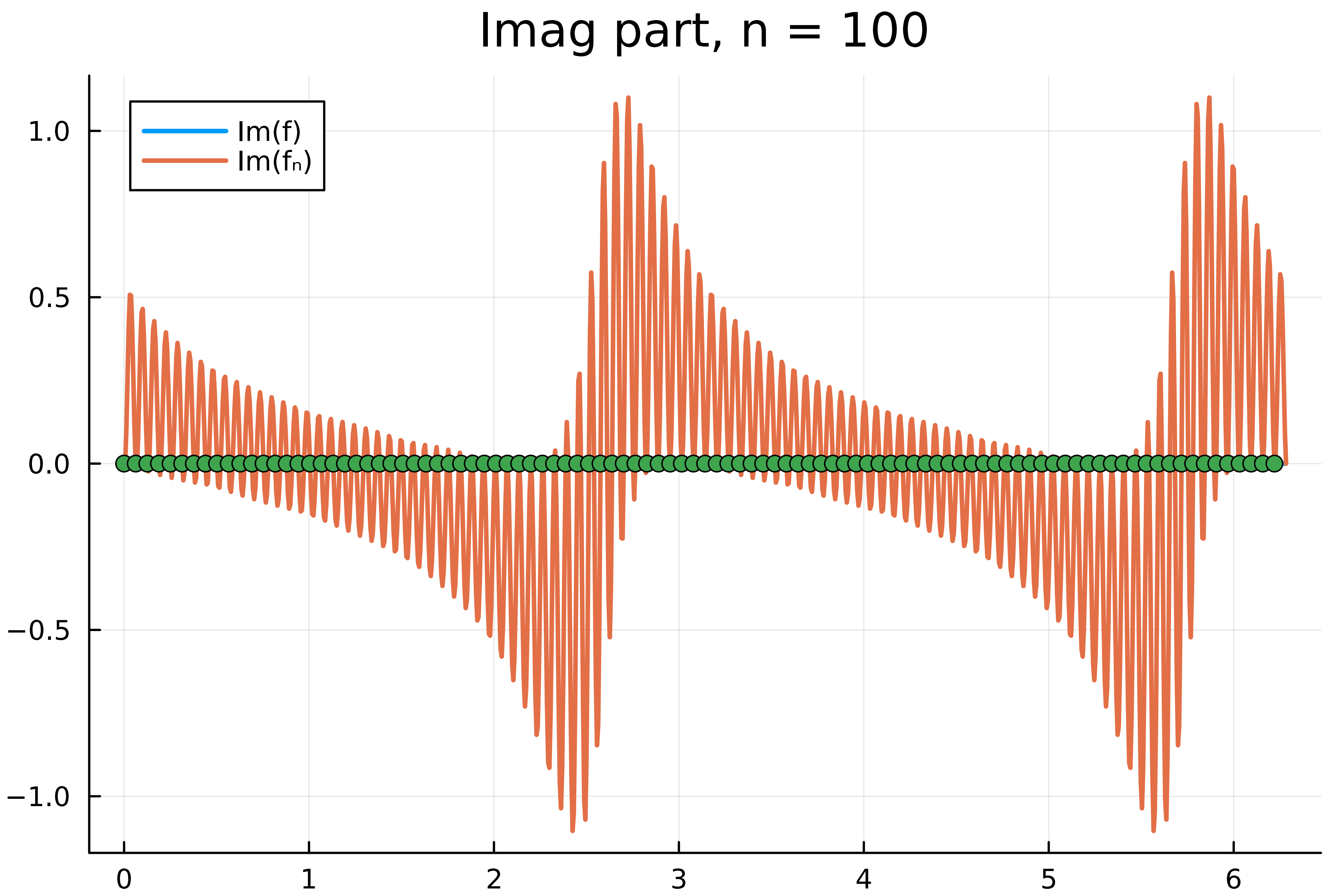
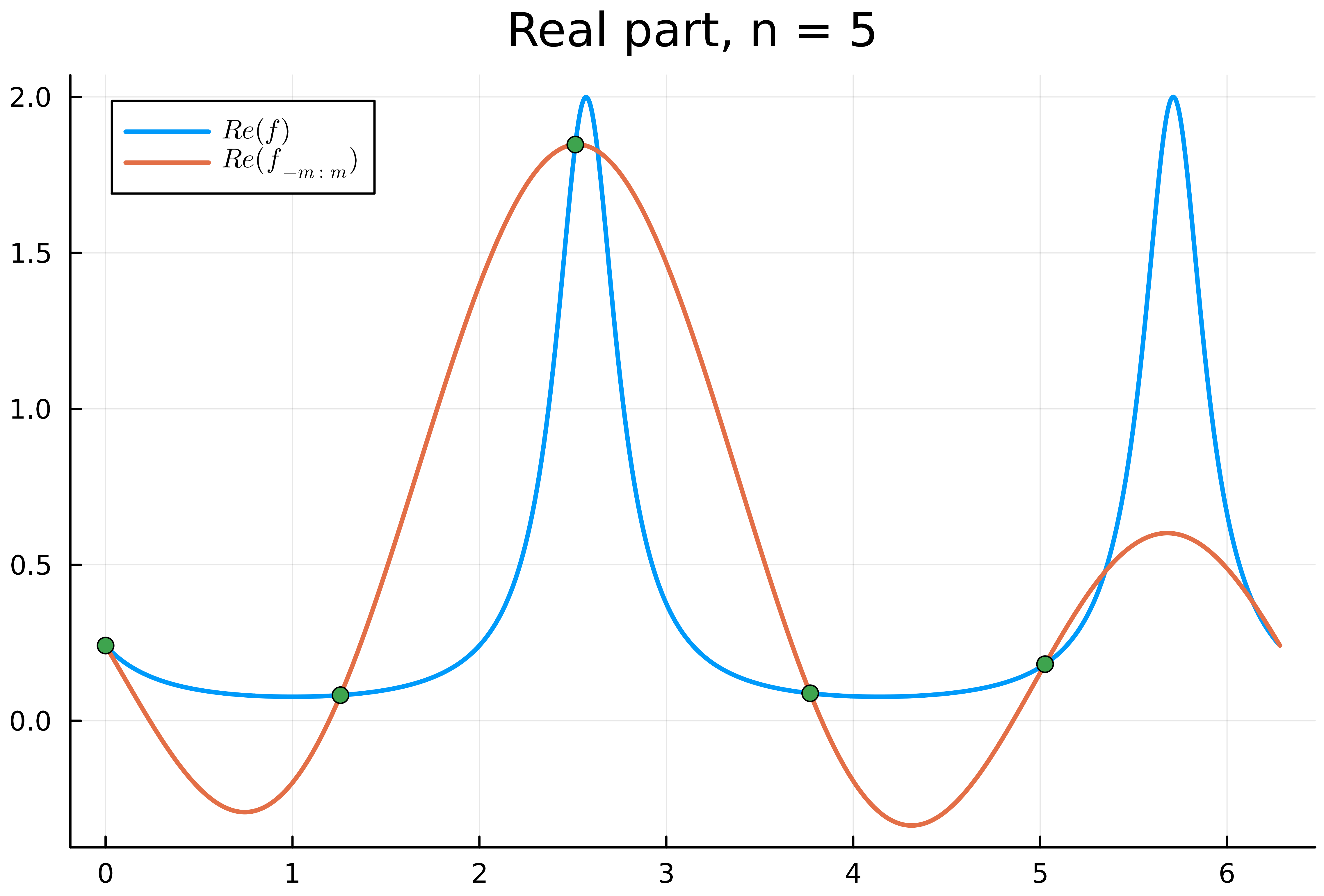


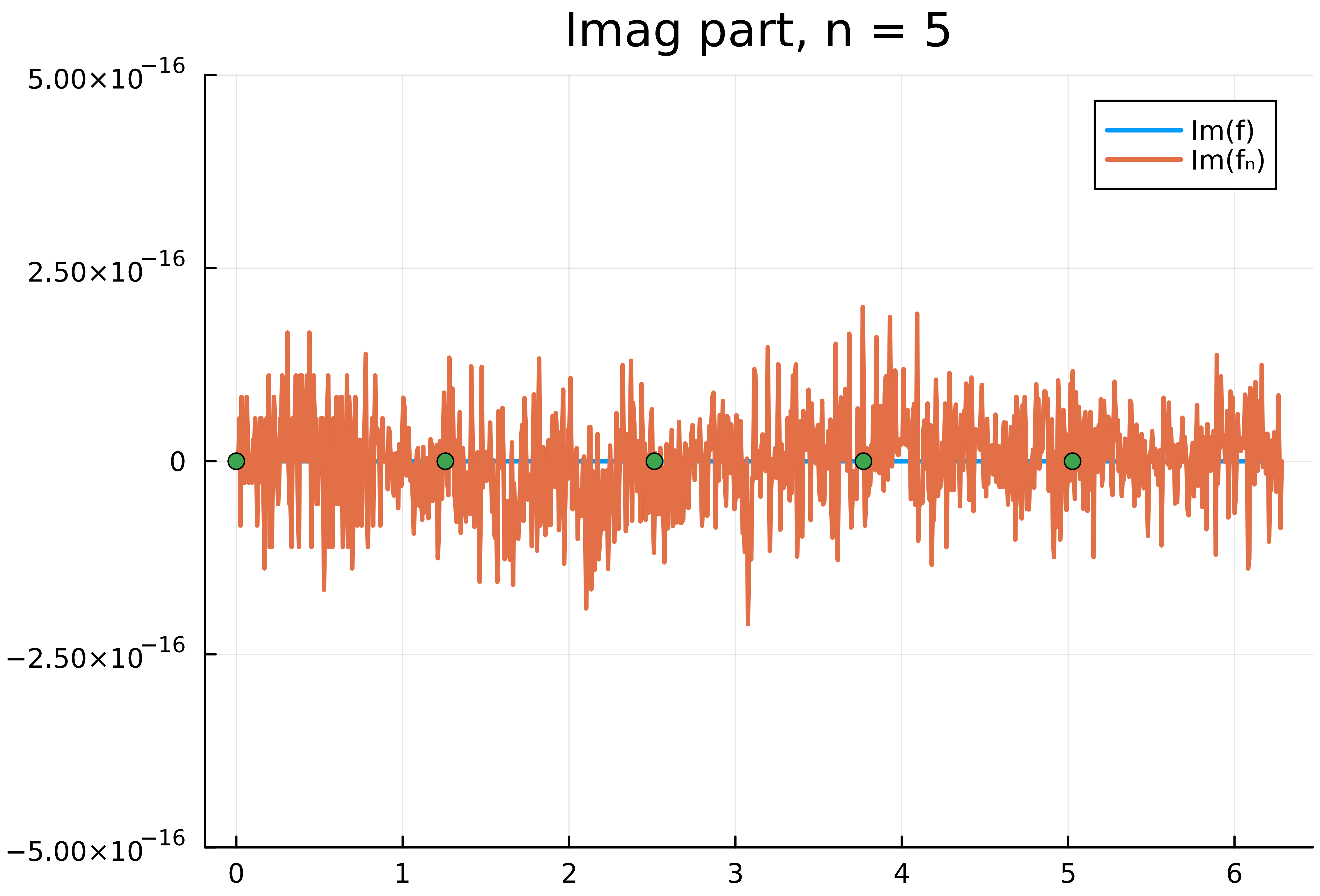


Corollary 2 (aliasing). *For all* $p \in \mathbb{Z}$, $\hat{f}_k^n = \hat{f}_{k+pn}^n$.

Consider $f_{-m:m}(\theta) := \sum_{k=-m}^{m} \hat{f}_k^n \mathrm{e}^{\mathrm{i}k\theta}$ where $n = 2m + 1$.

$$f(\theta) = \frac{2}{25\cos(\theta - 1)^2 + 1}$$





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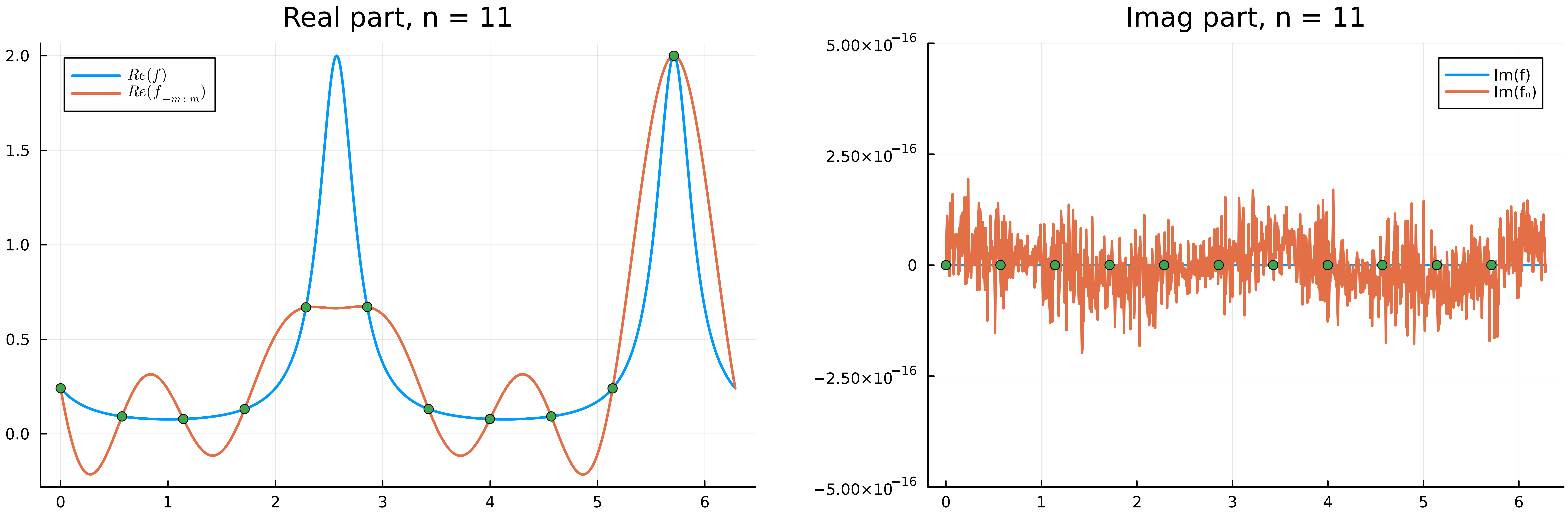

$$f(\theta) = \frac{2}{25\cos(\theta-1)^2+1}$$