MATH50003 Numerical Analysis

II.3 Floating Point Arithmetic

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Part II
Representing Numbers

1. Integers via modular arithmetic
2. Reals via floating point
3. Floating point arithmetic and bounding errors
4. Interval arithmetic for rigorous computations

Rounding

How does a computer round a real to a float?

Definition 10 (rounding). $\mathrm{fl}^{\mathrm{up}}_{\sigma,Q,S} : \mathbb{R} \to F_{\sigma,Q,S}$

$$\mathrm{fl}^{\mathrm{down}}_{\sigma,Q,S} : \mathbb{R} \to F_{\sigma,Q,S}$$

$$\mathrm{fl}^{\mathrm{nearest}}_{\sigma,Q,S} : \mathbb{R} \to F_{\sigma,Q,S}$$

Arithmetic

Operations are exact up to rounding

$$x \oplus y := \mathrm{fl}(x + y)$$
$$x \ominus y := \mathrm{fl}(x - y)$$
$$x \otimes y := \mathrm{fl}(x * y)$$
$$x \oslash y := \mathrm{fl}(x / y)$$

Example 16 (decimal is not exact)

II.3.1 Bounding errors

Analysis on rounding errors

Definition 11 (machine epsilon/smallest positive normal number/largest normal number)**.**

$$\epsilon_{\mathrm{m},S} := 2^{-S}.$$

Definition 12 (normalised range)

$$\mathcal{N}_{\sigma,Q,S} := \{x : \min |F^{\mathrm{normal}}_{\sigma,Q,S}| \leq |x| \leq \max F^{\mathrm{normal}}_{\sigma,Q,S}\}$$

Proposition 2 (round bound)**.** *If* $x \in \mathcal{N}$ *then*

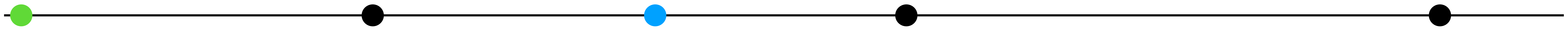
$$\mathrm{fl}^{\mathrm{mode}}(x) = x(1 + \delta_x^{\mathrm{mode}})$$

where the relative error *is bounded by:*

$$|\delta_x^{\mathrm{nearest}}| \leq \frac{\epsilon_{\mathrm{m}}}{2}$$
$$|\delta_x^{\mathrm{up/down}}| < \epsilon_{\mathrm{m}}.$$

(Round Down)

$$x = 2^{q-\sigma}(1.b_1 b_2 \ldots b_S b_{S+1} \ldots)_2$$



$$2^{q-\sigma}$$

$$x_- := \mathrm{fl}^{\mathrm{down}}(x)$$

$$= 2^{q-\sigma}(1.b_1 b_2 \ldots b_S)_2$$

$$x_{\mathrm{h}} := \frac{x_+ + x_-}{2}$$

$$= x_- + 2^{q-\sigma-S-1}$$

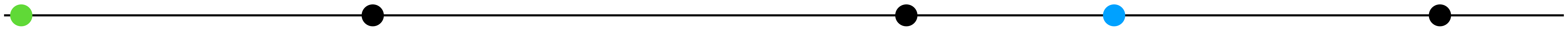
$$= 2^{q-\sigma}(1.b_1 b_2 \ldots b_S 1)_2$$

$$x_+ := \mathrm{fl}^{\mathrm{up}}(x)$$

$$= x_- + 2^{q-\sigma-S}$$

(Round Down)

$$x = 2^{q-\sigma}(1.b_1 b_2 \dots b_S b_{S+1} \dots)_2$$



$2^{q-\sigma}$

$$x_- := \mathrm{fl}^{\mathrm{down}}(x)$$
$$= 2^{q-\sigma}(1.b_1 b_2 \dots b_S)_2$$

$$x_{\mathrm{h}} := \frac{x_+ + x_-}{2}$$
$$= x_- + 2^{q-\sigma-S-1}$$
$$= 2^{q-\sigma}(1.b_1 b_2 \dots b_S 1)_2$$

$$x_+ := \mathrm{fl}^{\mathrm{up}}(x)$$
$$= x_- + 2^{q-\sigma-S}$$

II.3.2 Idealised floating point

A simplifed model for analysis

Definition 13 (idealised floating point)**.** An idealised mathematical model of floating point numbers for which the only subnormal number is zero can be defined as:

$$F_{\infty,S} := \{\pm 2^q \times (1.b_1 b_2 b_3 \ldots b_S)_2 : q \in \mathbb{Z}\} \cup \{0\}$$

Example 17 (bounding a simple computation)

II.3.3 Divided differences floating point error bound

Explain the unexplained error in divided differences

General model of a function implemented in floating point:

$$f(x) = f^{\mathrm{FP}}(x) + \delta_x^f$$

such that

$$|\delta_x^f| \leq c\epsilon_{\mathrm{m}}$$

Theorem 3 (divided difference error bound)**.**

$$\frac{f^{\mathrm{FP}}(x+h) \ominus f^{\mathrm{FP}}(x)}{h} = f'(x) + \delta_{x,h}^{\mathrm{FD}}$$

where

$$|\delta_{x,h}^{\mathrm{FD}}| \leq \frac{|f'(x)|}{2}\epsilon_{\mathrm{m}} + Mh + \frac{4c\epsilon_{\mathrm{m}}}{h}$$

for $M = \sup_{x \leq t \leq x+h} |f''(t)|$.

Corollary 1 (divided differences in practice). *We have*

$$(f^{\mathrm{FP}}(x \oplus h) \ominus f^{\mathrm{FP}}(x)) \oslash h = \frac{f^{\mathrm{FP}}(x+h) \ominus f^{\mathrm{FP}}(x)}{h}$$

whenever $h = 2^{j-n}$ *for* $0 \leq n \leq S$ *and the last binary place of* $x \in F_{\infty,S}$ *is zero, that is* $x = \pm 2^j (1.b_1 \dots b_{S-1}0)_2$.

Heuristic (divided difference with floating-point step)

Now to Lab 4
To see rounding modes.