# MATH50003 Numerical Analysis

## I.3 Dual Numbers

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Julia Tips:

1. Don’t install via package manger
2. Pull latest Github repos
3. Make a copy of lab to modify to avoid conflicts

# Part I

## Calculus on a Computer

1. Rectangular rules for integration
2. Divided differences for differentiation
3. Dual numbers for differentiation

Divided differences can have large errors.

Is it even possible to algorithmically calculate derivatives to high accuracy?

Yes: if we have access to the code.

**Analysis**
Divided differences

↓

**Algebra**
Dual numbers

Dual numbers are a commutative ring that allow us to differentiate

**Definition 1** (dual numbers)

$$\mathbb{D} := \{a + b\epsilon \quad : \quad a, b \in \mathbb{R}, \quad \epsilon^2 = 0\}$$

Compare with complex numbers:

$$\mathbb{C} := \{a + b\mathrm{i} \quad : \quad a, b \in \mathbb{R}, \quad \mathrm{i}^2 = -1\}$$

# Addition/multiplication

## Follows from simple algebra

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}$$

$$(a + b\epsilon) + (c + d\epsilon) = (a + c) + (b + d)\epsilon$$

$$(a + b\mathrm{i})(c + d\mathrm{i}) = (ac - bd) + (bc + ad)\mathrm{i}$$

$$(a + b\epsilon)(c + d\epsilon) = ac + (bc + ad)\epsilon$$

Complex

Dual

# I.3.1 Differentiating polynomials

## Addition/multiplication ⇒ dual numbers compute derivatives

**Theorem 2** (polynomials on dual numbers)**.** *Suppose $p$ is a polynomial. Then*

$$p(a + b\epsilon) = p(a) + bp'(a)\epsilon$$

**Example 1** (differentiating polynomial)**.** Consider computing $p'(2)$ where

$$p(x) = (x-1)(x-2) + x^2.$$

# I.3.2 Differentiating other functions

## Theorem 1 gives us a rule to extend differentiation via duals

Motivation: consider a Taylor series

$$f(x) = \sum_{k=0}^{\infty} f_k x^k$$

And assume $a$ is in radius of convergence.

What will $f(a + b\epsilon)$ return?

**Definition 2** (dual extension)**.** Suppose a real-valued function $f$ is differentiable at $a$. If

$$f(a+b\epsilon) = f(a) + bf'(a)\epsilon$$

then we say that it is a *dual extension at $a$*.

Examples:

$$\begin{aligned}
\exp(a+b\epsilon) &:= \exp(a) + b\exp(a)\epsilon \\
\sin(a+b\epsilon) &:= \sin(a) + b\cos(a)\epsilon \\
\cos(a+b\epsilon) &:= \cos(a) - b\sin(a)\epsilon \\
\log(a+b\epsilon) &:= \log(a) + \frac{b}{a}\epsilon \\
\sqrt{a+b\epsilon} &:= \sqrt{a} + \frac{b}{2\sqrt{a}}\epsilon \\
|a+b\epsilon| &:= |a| + b\,\mathrm{sign}\,a\,\epsilon
\end{aligned}$$

**Lemma 2** (product and chain rule). *If $f$ is a dual extension at $g(a)$ and $g$ is a dual extension at $a$, then $q(x) := f(g(x))$ is a dual extension at $a$. If $f$ and $g$ are dual extensions at $a$ then $r(x) := f(x)g(x)$ is also dual extensions at $a$. In other words:*

$$\begin{aligned} q(a+b\epsilon) &= q(a) + bq'(a)\epsilon \\ r(a+b\epsilon) &= r(a) + br'(a)\epsilon \end{aligned}$$

**Example 2** (differentiating non-polynomial)**.** Consider differentiating $f(x) = \exp(x^2 + \mathrm{e}^x)$ at the point $a = 1$ by evaluating on the duals:

# How do we implement this on a computer?