Numerical Analysis MATH50003 (2023–24) Problem Sheet 9

Problem 1 Construct the monic and orthonormal polynomials up to degree 3 for the weights $\sqrt{1-x^2}$ and $1-x$ on $[-1,1]$. What are the top 3×3 entries of the corresponding Jacobi matrices? Hint: for the first weight, find a recursive formula for $\int_{-1}^1 x^k\sqrt{1-x^2}\mathrm{d}x$ using a change-of-variables.

SOLUTION

Weight 1 ($\sqrt{1-x^2}$)

Following the hint, we first calculate $\int_{-1}^1 x^k\sqrt{1-x^2}\mathrm{d}x$. By symmetry, it's zero when k is odd and double the integral on $[0,1]$ when k is even.

$$\underbrace{\int_0^1 x^k\sqrt{1-x^2}\mathrm{d}x}_{I_k} =_{x=\sin t} \underbrace{\int_0^{\pi/2}\sin^k(t)\cos^2(t)\mathrm{d}t}_{I_k} = \underbrace{\int_0^{\pi/2}\sin^k t\mathrm{d}t}_{J_k} - \underbrace{\int_0^{\pi/2}\sin^{k+2} t\mathrm{d}t}_{J_{k+2}}.$$

Meanwhile,

$$J_k = -\int_0^{\pi/2}\sin^{k-1} t\mathrm{d}(\cos t) =_{\text{integral by part}} (k-1)I_{k-2}.$$

Putting the above 2 equations together, we have $I_k = (k-1)I_{k-2} - (k+1)I_k$, so $I_k = \frac{k-1}{k+2}I_{k-2}$. Since $I_0 = \pi/4$ we have $I_2 = \pi/16$ and $I_4 = \pi/32$ hence

$$\begin{aligned}\int_{-1}^1\sqrt{1-x^2}\mathrm{d}x &= \frac{\pi}{2}, \int_{-1}^1 x\sqrt{1-x^2}\mathrm{d}x = 0, \int_{-1}^1 x^2\sqrt{1-x^2}\mathrm{d}x = \frac{\pi}{8},\\ \int_{-1}^1 x^3\sqrt{1-x^2}\mathrm{d}x &= 0, \int_{-1}^1 x^4\sqrt{1-x^2}\mathrm{d}x = \frac{\pi}{16}.\end{aligned}$$

Let $p_0(x) = 1$, then $\|p_0\|^2 = 2I_0 = \pi/2$. We know from the 3-term recurrence that

$$xp_0(x) = a_0p_0(x) + p_1(x)$$

where

$$a_0 = \frac{\langle p_0, xp_0\rangle}{\|p_0\|^2} = 0.$$

Thus $p_1(x) = x$ and $\|p_1\|^2 = 2I_2 = \pi/8$. From

$$xp_1(x) = c_0p_0(x) + a_1p_1(x) + p_2(x)$$

we have

$$\begin{aligned}c_0 &= \frac{\langle p_0, xp_1\rangle}{\|p_0\|^2} = 2I_2/2I_0 = 1/4\\ a_1 &= \frac{\langle p_1, xp_1\rangle}{\|p_1\|^2} = 0\\ p_2(x) &= xp_1(x) - c_0 - a_1p_1(x) = x^2 - 1/4\\ \|p_2\|^2 &= 2I_4 - I_2 + 1/8I_0 = \pi/32\end{aligned}$$

Finally, from

$$xp_2(x) = c_1p_1(x) + a_2p_2(x) + p_3(x)$$

we have

$$\begin{aligned}
c_1 &= \frac{\langle p_1, xp_2\rangle}{\|p_1\|^2} = (2I_4 - 1/2I_2)/(\pi/8) = 1/4 \\
a_2 &= \frac{\langle p_2, xp_2\rangle}{\|p_2\|^2} = 0 \\
p_3(x) &= xp_2(x) - c_1p_1(x) - a_2p_2(x) = x^3 - x/2
\end{aligned}$$

We need one more constant: from

$$xp_3(x)p_2(x) = (x^4 - x^2/2)(x^2 - 1/4) = x^6 - 3x^4/4 - x^2/8$$

we find (since $I_6 = 5I_4/8 = 5\pi/256$)

$$c_2 = \frac{\langle p_2, xp_3\rangle}{\|p_2\|^2} = 2\frac{I_6 - 3I_4/4 + I_2/8}{\pi/32} = 1/4$$

We see from this that

$$x[p_0(x), p_1(x), \ldots] = [p_0(x), p_1(x), \ldots]\begin{bmatrix} 0 & 1/4 & & & \\ 1 & 0 & 1/4 & & \\ & 1 & 0 & 1/4 & \\ & & 1 & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

To make this symmetric we choose

$$\begin{aligned}
k_0 &= \|p_0\|^{-1} = \sqrt{2/\pi} \\
k_1 &= k_0\sqrt{4} = 2\sqrt{2/\pi} \\
k_2 &= k_1\sqrt{4} = 4\sqrt{2/\pi} \\
k_3 &= k_2\sqrt{4} = 8\sqrt{2/\pi}
\end{aligned}$$

Giving us (also computable from norms of $p_n(x)$):

$$\begin{aligned}
q_0(x) &= \sqrt{2/\pi} \\
q_1(x) &= 2\sqrt{2/\pi x} \\
q_2(x) &= 4\sqrt{2/\pi}(x^2 - 1/4) \\
q_3(x) &= 8\sqrt{2/\pi}(x^3 - x/2)
\end{aligned}$$

with Jacobi matrix

$$J = \begin{bmatrix} 0 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ & 1/2 & 0 & 1/2 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

Weight 2 $(1 - x)$

Here we have polynomials so computing the moments are easier. We have $p_0(x) = 1$ hence

$$\|p_0\|^2 = \int_{-1}^{1}(1 - x)\mathrm{dx} = 2$$

Hence

$$
\begin{aligned}
a_0 &= \frac{\langle xp_0, p_0\rangle}{\|p_0\|^2} = \frac{\int_{-1}^{1} x(1-x)\mathrm{dx}}{2} = -\frac{1}{3} \Rightarrow \\
p_1(x) &= xp_0(x) - a_0 p_0(x) = x + 1/3 \Rightarrow \\
\|p_1\|^2 &= \int_{-1}^{1} (-x^3 + x^2/3 + 5x/9 + 1/9)\mathrm{dx} = \frac{4}{9} \Rightarrow \\
c_0 &= \frac{\langle xp_1, p_0\rangle}{\|p_0\|^2} = \frac{2}{9}, \\
a_1 &= \frac{\langle xp_1, p_1\rangle}{\|p_1\|^2} = -\frac{1}{15} \Rightarrow \\
p_2(x) &= xp_1(x) - a_1 p_1(x) - c_0 p_0(x) = x^2 + \frac{2x}{5} - \frac{1}{5} \Rightarrow \\
\|p_2\|^2 &= \frac{8}{75} \\
c_1 &= \frac{\langle xp_2, p_1\rangle}{\|p_1\|^2} = \frac{6}{25}, \\
a_2 &= \frac{\langle xp_2, p_2\rangle}{\|p_2\|^2} = -\frac{1}{35} \Rightarrow \\
p_3(x) &= xp_2(x) - a_2 p_2(x) - c_1 p_1(x) = x^3 + \frac{3x^2}{7} - \frac{3x}{7} - \frac{3}{35} \\
c_2 &= \frac{\langle xp_3, p_2\rangle}{\|p_2\|^2} = \frac{12}{49}.
\end{aligned}
$$

Thus the multiplication matrix is

$$
x[p_0(x), p_1(x), \ldots] = [p_0(x), p_1(x), \ldots] \begin{bmatrix} -1/3 & 2/9 & & & \\ 1 & -1/15 & 6/25 & & \\ & 1 & -1/35 & 12/49 & \\ & & 1 & \ddots & \ddots \\ & & & \ddots & \ddots \end{bmatrix}
$$

To make this symmetric we choose

$$
\begin{aligned}
k_0 &= \|p_0\|^{-1} = 1/\sqrt{2} \\
k_1 &= k_0\sqrt{9/2} = 3/2 \\
k_2 &= k_1\sqrt{25/6} = 15/(2\sqrt{6}) \\
k_3 &= k_2\sqrt{49/12} = 35/(4\sqrt{2})
\end{aligned}
$$

That is

$$
\begin{aligned}
q_0(x) &= 1/\sqrt{2} \\
q_1(x) &= \frac{3}{2}(x + 1/3) \\
q_2(x) &= \frac{15}{2\sqrt{6}}(x^2 + \frac{2x}{5} - \frac{1}{5}) \\
q_3(x) &= \frac{35}{4\sqrt{2}}(x^3 + \frac{3x^2}{7} - \frac{3x}{7} - \frac{3}{35}).
\end{aligned}
$$

with Jacobi matrix

$$J = \begin{bmatrix} -1/3 & \sqrt{2}/3 & & \\ \sqrt{2}/3 & -1/15 & \sqrt{6}/5 & \\ & \sqrt{6}/5 & -1/35 & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

END

Problem 2 Prove Theorem 13: a precisely degree n polynomial

$$p(x) = k_n x^n + O(x^{n-1})$$

satisfies

$$\langle p, f_m \rangle = 0$$

for all polynomials f_m of degree $m < n$ of degree less than n if and only if $p(x) = c\pi_n$ for some constant c, where π_n are monic orthogonal polynomials.

SOLUTION As $\{\pi_0, \ldots, \pi_n\}$ are a basis of all polynomials of degree n, we can write

$$r(x) = \sum_{k=0}^{m} a_k \pi_k(x)$$

Thus if $p(x) = c\pi_n(x)$, by linearity of inner products we have

$$\langle p, r \rangle = \langle c\pi_n, \sum_{k=0}^{m} a_k \pi_k \rangle = \sum_{k=0}^{m} c a_k \langle \pi_n, \pi_k \rangle = 0.$$

Now suppose

$$p(x) = cx^n + O(x^{n-1})$$

and consider $p(x) - c\pi_n(x)$ which is of degree $n-1$. It satisfies for $k \leq n-1$

$$\langle \pi_k, p - c\pi_n \rangle = \langle \pi_k, p \rangle - c\langle \pi_k, \pi_n \rangle = 0.$$

Thus $p - c\pi_n$ is zero, i.e., $p(x) = c\pi_n(x)$.

END

Problem 3 If $w(-x) = w(x)$ for a weight supported on $[-b, b]$ show that $a_n = 0$. Hint: first show that the (monic) polynomials $p_{2n}(x)$ are even and $p_{2n+1}(x)$ are odd.

SOLUTION

An integral is zero if its integrand is odd. Moreover an even function times an odd function is odd and an odd function times an odd fuction is even. Note that $p_0(x)$ and $w(x)$ are even and x is odd.

We see that a_0 is zero:

$$\langle p_0, x p_0(x) \rangle = \int_{-b}^{b} x w(x) \mathrm{d}x = 0$$

since $xw(x)$ is odd, which shows that

$$p_1(x) = x p_0(x)$$

is odd. We now proceed by induction. Assume that p_{2n} is even and p_{2n-1} is odd. We have:

$$\langle p_{2n}, x p_{2n}(x) \rangle = \int_{-b}^{b} x w(x) p_{2n}(x)^2 \mathrm{d}x = 0$$

since $xw(x)p_{2n}(x)^2$ is odd, therefore $a_{2n} = 0$. Thus from

$$p_{2n+1}(x) = (xp_{2n}(x) - c_{2n-1}p_{2n-1}(x))/b_{2n}$$

we see that p_{2n+1} is odd. Then

$$\langle p_{2n+1}, xp_{2n+1}(x)\rangle = \int_{-b}^{b} xw(x)p_{2n+1}(x)^2 \mathrm{d}x = 0$$

since $xw(x)p_{2n+1}(x)^2$ is odd, therefore $a_{2n+1} = 0$. and hence

$$p_{2n+2}(x) = (xp_{2n+1}(x) - c_{2n}p_{2n}(x))/b_{2n+1}$$

is even.

END

Problem 4(a) Prove that

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

SOLUTION

We need to verify

$$p_n(x) := \frac{\sin(n+1)\theta}{\sin\theta}$$

are

1. graded polynomials
2. orthogonal w.r.t. $\sqrt{1-x^2}$ on $[-1,1]$, and
3. have the leading coefficient 2^n.

Then uniqueness will guarantee that $p_n(x) = U_n(x)$.

(2) follows under a change of variables

$$\begin{aligned}\int_{-1}^{1} p_n(x)p_m(x)\sqrt{1-x^2}\mathrm{d}x &= \int_0^{\pi} p_n(\cos\theta)p_m(\cos\theta)\sin^2\theta \mathrm{d}\theta \\ &= \int_0^{\pi} \sin(n+1)\theta\sin(m+1)\theta \mathrm{d}\theta = \frac{\pi}{2}\delta_{mn}\end{aligned}$$

where the last step can be shown by substituting $\sin k\theta = (\exp(ik\theta) - \exp(-ik\theta)/(2i)$.

To see that they are graded, first note that

$$p_0(x) = \sin\theta/\sin\theta = 1, p_1(x) = \frac{\sin 2\theta}{\sin\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta} = 2x.$$

Now for $n = 1, 2, \ldots$ use the fact that

$$xp_n(x) = \frac{\cos\theta\sin(n+1)\theta}{\sin\theta} = \frac{\sin(n+2)\theta + \sin n\theta}{2\sin\theta}$$

In other words $2xp_n(x) = p_{n+1}(x) + p_{n-1}(x)$, i.e. $p_{n+1}(x) = 2xp_n(x) + p_{n-1}(x)$. By induction it follows that

$$p_n(x) = 2^n x^n + O(x^{n-1})$$

which also proves (3).

END

Problem 4(b) Show that

$$xU_0(x) = U_1(x)/2$$
$$xU_n(x) = \frac{U_{n-1}(x)}{2} + \frac{U_{n+1}(x)}{2}.$$

SOLUTION

The first result is trivial. For the other parts, from the solution to the previous part we know $2xU_n(x) = U_{n+1}(x) + U_{n-1}(x)$ and this result is a reordering.

END

Problem 5 Use the fact that orthogonal polynomials are uniquely determined by their leading order coefficient and orthogonality to lower dimensional polynomials to show that:

$$T_n'(x) = nU_{n-1}(x).$$

SOLUTION

We need to verify that $T_n'(x)$

1. graded polynomials
2. orthogonal w.r.t. $\sqrt{1-x^2}$ on $[-1,1]$, and
3. have the leading coefficient $n2^n$.

(1) and (3) are clear:

$$T_n'(x) = n2^{n-1}x^{n-1} + O(x^{n-2}).$$

(2) For f_m degree $m < n-1$ we have

$$\int_{-1}^{1} T_n'(x) f_m(x)\sqrt{1-x^2}\mathrm{d}x = -\int_{-1}^{1} T_n(x) \underbrace{(f_m'(x)(1-x^2) - xf_m)}_{\text{degree } m+1<n}(1-x^2)^{-1/2}\mathrm{d}x = 0.$$

END

Problem 6(a) Consider Hermite polynomials orthogonal with respect to the weight $\exp(-x^2)$ on $\mathbb{R}$ with the normalisation

$$H_n(x) = 2^n x^n + O(x^{n-1}).$$

Prove the Rodrigues formula

$$H_n(x) = (-1)^n \exp(x^2) \frac{\mathrm{d}^n}{\mathrm{d}x^n} \exp(-x^2).$$

Hint: use integration-by-parts.

SOLUTION Define

$$p_n(x) := (-1)^n \exp(x^2) \frac{\mathrm{d}^n}{\mathrm{d}x^n} \exp(-x^2)$$

We need to verify that p_n

1. are graded polynomials
2. are orthogonal to all lower degree polynomials on $\mathbb{R}$, and
3. have the right leading coefficient 2^n.

Comparing the Rodrigues formula for n and $n-1$, we find that

$$(-1)^n \exp(-x^2) p_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^{n-1}\exp(-x^2)p_{n-1}(x)\right)$$

which reduces to $p_n(x) = 2xp_{n-1}(x) - p'_{n-1}(x)$.

(1) and (3) then follows from induction since $p_0(x) = 1$.

(2) follows by integration by parts. If r_m is any degree $m < n$ polynomial we have:

$$\begin{aligned}\int_{-\infty}^{\infty} p_n(x) r_m(x)\exp(-x^2)\mathrm{d}x &= \int_{-\infty}^{\infty}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp(-x^2)r(x)\mathrm{d}x = -\int_{-\infty}^{\infty}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\exp(-x^2)r'(x)\mathrm{d}x \\ &= \cdots \text{integration by parts } n \text{ times} \ldots = (-1)^n\int_{-\infty}^{\infty}\exp(-x^2)r_m^{(n)}(x) = 0\end{aligned}$$

Thus $p_n(x) = H_n(x)$ by uniqueness.

END

Problem 6(b) What are $k_n^{(1)}$ and $k_n^{(2)}$ such that

$$H_n(x) = 2^n x^n + k_n^{(1)} x^{n-1} + k_n^{(2)} x^{n-2} + O(x^{n-3})$$

SOLUTION

From the previous part we know:

$$\begin{aligned} H_{n+1}(x) &= 2xH_n(x) - H_n'(x) = 2x(2^n x^n + k_n^{(1)}x^{n-1} + k_n^{(2)}x^{n-2} + O(x^{n-3})) - (n2^n x^{n-1} + O(x^{n-2}) \\ &= 2^{n+1}x^{n+1} + 2k_n^{(1)}x^n + (2k_n^{(2)} - n2^n)x^{n-1} + O(x^{n-2}) \end{aligned}$$

hence

$$\begin{aligned} k_{n+1}^{(1)} &= 2k_n^{(1)}, \\ k_{n+1}^{(2)} &= 2k_n^{(2)} - n2^n \end{aligned}$$

Since $k_0^{(1)} = 0$, we have $k_n^{(1)} = 0$ (which also follows by symmetry in the weight). For the second recurrence, lets see the pattern for the first few:

$$\begin{aligned} k_0^{(2)} &= k_1^{(2)} = 0 \\ k_2^{(2)} &= -2 \\ k_3^{(2)} &= 2 \times (-2) - 2 \times 2^2 = -3 \times 2^2 = -12 \\ k_4^{(2)} &= 2 \times (-3 \times 2^2) - 3 \times 2^3 = -6 \times 2^3 = -48 \\ k_5^{(2)} &= 2 \times (-6 \times 2^3) - 4 \times 2^4 = -10 \times 2^4 = -160 \end{aligned}$$

From this the pattern is clear:

$$k_n^{(2)} = -(\sum_{k=1}^{n-1} k)2^{n-1} = -n(n-1)2^{n-2}.$$

This can be confirmed by induction:

$$k_{n+1}^{(2)} = 2k_n^{(2)} - n2^n = -n(n-1)2^{n-1} - n2^n = -n(n+1)2^{n-1}.$$

END

Problem 6(c) Deduce the 3-term recurrence relationship for $H_n(x)$.

SOLUTION

Our goal is to find a_n, b_n and c_n such that

$$xH_n(x) = c_{n-1}H_{n-1}(x) + a_nH_n(x) + b_nH_{n+1}(x).$$

Matching terms we have $b_n = 1/2$ and $a_n = 0$ so that

$$\begin{aligned} c_{n-1}H_{n-1}(x) &= xH_n(x) - H_{n+1}(x)/2 = 2^n x^{n+1} + k_n^{(2)}x^{n-1} - 2^n x^{n+1} - k_{n+1}^{(2)}/2x^{n-1} + O(x^{n-2}) \\ &= (k_n^{(2)} - k_{n+1}^{(2)}/2)x^{n-1} + O(x^{n-2}) \\ &= (-n(n-1)2^{n-2} + n(n+1)2^{n-2})x^{n-1} + O(x^{n-2}) \\ &= n2^{n-1}x^{n-1} + O(x^{n-2}). \end{aligned}$$

Therefore we choose

$$c_{n-1} = \frac{n2^{n-1}}{2^{n-1}} = n.$$

END

Problem 6(d) Prove that $H_n'(x) = 2nH_{n-1}(x)$. Hint: show orthogonality of H_n' to all lower degree polynomials, and that the normalisation constants match.

SOLUTION

We have for f_m degree $m < n-1$, using integration by parts

$$\langle H_n', f_m \rangle = \int_{-\infty}^{\infty} H_n'(x) f_m(x) \mathrm{e}^{-x^2} \mathrm{d}x = \int_{-\infty}^{\infty} H_n(x) \underbrace{(f_m'(x) - 2xf_m)}_{\text{degree } m+1<n} \mathrm{e}^{-x^2} \mathrm{d}x = 0.$$

Further,

$$H_n'(x) = n2^n x^{n-1} + O(x^{n-1}) = 2n(2^{n-1}x^{n-1} + O(x^{n-1}))$$

hence the normalisation constants match.

END