Numerical Analysis MATH50003 (2023–24) Problem Sheet 8

Problem 1 Give explicit formulae for $\hat{f}_k$ and $\hat{f}_k^n$ for the following functions:

$$\cos\theta, \cos 4\theta, \sin^4\theta, \frac{3}{3 - e^{i\theta}}, \frac{1}{1 - 2e^{i\theta}}$$

SOLUTION

(1) Just expand in complex exponentials to find that

$$\cos\theta = \frac{\exp(i\theta) + \exp(-i\theta)}{2}$$

that is $\hat{f}_1 = \hat{f}_{-1} = 1/2$, $\hat{f}_k = 0$ otherwise. Therefore for $p \in \mathbb{Z}$ we have

$$\hat{f}_k^1 = \hat{f}_1 + \hat{f}_{-1} = 1$$
$$\hat{f}_{2p}^2 = 0, \hat{f}_{2p+1}^2 = \hat{f}_1 + \hat{f}_{-1} = 1$$
$$\hat{f}_{1+np}^n = \hat{f}_{-1+np}^n = 1/2, \hat{f}_k^n = 0$$

for $n = 3, 4, \ldots$.

(2) Similarly

$$\cos 4\theta = \frac{\exp(4i\theta) + \exp(-4i\theta)}{2}$$

that is $\hat{f}_4 = \hat{f}_{-4} = 1/2$, $\hat{f}_k = 0$ otherwise. Therefore for $p \in \mathbb{Z}$ we have

$$\hat{f}_p^1 = \hat{f}_4 + \hat{f}_{-4} = 1$$
$$\hat{f}_{2p}^2 = \hat{f}_4 + \hat{f}_{-4} = 1, \hat{f}_{2p+1}^2 = 0$$
$$\hat{f}_{3p}^3 = 0, \hat{f}_{3p\pm1}^3 = \hat{f}_{\pm4} = 1/2$$
$$\hat{f}_{4p}^4 = \hat{f}_{-4} + \hat{f}_4 = 1, \hat{f}_{4p\pm1}^4 = 0, \hat{f}_{4p+2}^4 = 0$$
$$\hat{f}_{5p}^5 = 0, \hat{f}_{5p+1}^5 = \hat{f}_{-4} = 1/2, \hat{f}_{5p-1}^5 = \hat{f}_4 = 1/2, \hat{f}_{5p\pm2}^5 = 0$$
$$\hat{f}_{6p}^6 = 0, \hat{f}_{6p\pm1}^6 = 0, \hat{f}_{6p+2}^6 = \hat{f}_{-4} = 1/2, \hat{f}_{6p-2}^6 = \hat{f}_4 = 1/2, \hat{f}_{6p+3}^6 = 0$$
$$\hat{f}_{7p}^7 = 0, \hat{f}_{7p\pm1}^7 = 0, \hat{f}_{7p\pm2}^7 = 0, \hat{f}_{7p\pm3}^7 = \hat{f}_{\mp4} = 1/2$$
$$\hat{f}_{8p}^8 = \hat{f}_{8p\pm1}^8 = \hat{f}_{8p\pm2}^8 = \hat{f}_{8p\pm3}^8 = 0, \hat{f}_{8p+4}^8 = \hat{f}_4 + \hat{f}_{-4} = 1$$
$$\hat{f}_{k+pn}^n = \hat{f}_k \text{ for } -4 \leq k \leq 4, 0 \text{ otherwise.}$$

(3) Here we have:

$$(\sin\theta)^4 = \left(\frac{\exp(i\theta) - \exp(-i\theta)}{2i}\right)^4 = \left(\frac{\exp(2i\theta) - 2 + \exp(-2i\theta)}{-4}\right)^2$$
$$= \frac{\exp(4i\theta) - 4\exp(2i\theta) + 6 - 4\exp(-2i\theta) + \exp(-2i\theta)}{16}$$

that is $\hat{f}_{-4} = \hat{f}_4 = 1/16$, $\hat{f}_{-2} = \hat{f}_2 = -1/4$, $f_0 = 3/8$, $\hat{f}_k = 0$ otherwise. Therefore for $p \in \mathbb{Z}$

we have

$$\hat{f}_p^1 = \hat{f}_{-4} + \hat{f}_{-2} + \hat{f}_0 + \hat{f}_2 + \hat{f}_4 = 0$$

$$\hat{f}_k^2 = 0$$

$$\hat{f}_{3p}^3 = \hat{f}_0 = 3/8, \hat{f}_{3p+1}^3 = \hat{f}_{-2} + \hat{f}_4 = -3/16, \hat{f}_{3p-1}^3 = \hat{f}_2 + \hat{f}_{-4} = -3/16$$

$$\hat{f}_{4p}^4 = \hat{f}_0 + \hat{f}_{-4} + \hat{f}_4 = 1/2, \hat{f}_{4p\pm1}^4 = 0, \hat{f}_{4p+2}^4 = \hat{f}_2 + \hat{f}_{-2} = -1/2$$

$$\hat{f}_{5p}^5 = \hat{f}_0 = 3/8, \hat{f}_{5p+1}^5 = \hat{f}_{-4} = 1/16, \hat{f}_{5p-1}^5 = \hat{f}_4 = 1/16, \hat{f}_{5p+2}^5 = \hat{f}_2 = -1/4, \hat{f}_{5p-2}^5 = \hat{f}_{-2} = -1/4$$

$$\hat{f}_{6p}^6 = \hat{f}_0 = 3/8, \hat{f}_{6p\pm1}^6 = 0, \hat{f}_{6p+2}^6 = \hat{f}_2 + \hat{f}_{-4} = -3/16, \hat{f}_{6p-2}^6 = \hat{f}_{-2} + \hat{f}_4 = -3/16, \hat{f}_{6p+3}^6 = 0$$

$$\hat{f}_{7p}^7 = \hat{f}_0 = 3/8, \hat{f}_{7p\pm1}^7 = 0, \hat{f}_{7p\pm2}^7 = \hat{f}_{\pm2} = -1/4, \hat{f}_{7p\pm3}^7 = \hat{f}_{\mp4} = 1/16$$

$$\hat{f}_{8p}^8 = \hat{f}_0 = 3/8, \hat{f}_{8p\pm1}^8 = 0, \hat{f}_{8p\pm2}^8 = \hat{f}_{\pm2} = -1/4, \hat{f}_{8p\pm3}^8 = 0, \hat{f}_{8p+4}^8 = \hat{f}_4 + \hat{f}_{-4} = 1/8$$

$\hat{f}_{k+pn}^n = \hat{f}_k$ for $-4 \leq k \leq 4$, 0 otherwise.

(4) Under the change of variables $z = e^{i\theta}$ we can use Geoemtric series to determine

$$\frac{3}{3-z} = \frac{1}{1-z/3} = \sum_{k=0}^{\infty} \frac{z^k}{3^k}$$

That is $\hat{f}_k = 1/3^k$ for $k \geq 0$, and $\hat{f}_k = 0$ otherwise. We then have for $0 \leq k \leq n-1$

$$\hat{f}_{k+pn}^n = \sum_{\ell=0}^{\infty} \frac{1}{3^{k+\ell n}} = \frac{1}{3^k}\frac{1}{1-1/3^n} = \frac{3^n}{3^{n+k}-3^k}$$

(5) Now make the change of variables $z = e^{-i\theta}$ to get:

$$\frac{1}{1-2/z} = \frac{1}{-2/z}\frac{1}{1-z/2} = \frac{1}{-2/z}\sum_{k=0}^{\infty}\frac{z^k}{2^k} = -\sum_{k=1}^{\infty}\frac{e^{-ik\theta}}{2^k}$$

That is $\hat{f}_k = -1/2^{-k}$ for $k \leq -1$ and 0 otherwise. We then have for $-n \leq k \leq -1$

$$\hat{f}_{k+pn}^n = -\sum_{\ell=0}^{\infty}\frac{1}{2^{-k+\ell n}} = -\frac{1}{2^{-k}}\frac{1}{1-1/2^n} = -\frac{2^{n+k}}{2^n-1}$$

END

Problem 2 Prove that if the first $\lambda - 1$ derivatives $f(\theta), f'(\theta), \ldots, f^{(\lambda-1)}(\theta)$ are 2π-periodic and $f^{(\lambda)}$ is uniformly bounded that

$$|\hat{f}_k| = O(|k|^{-\lambda}) \qquad \text{as } |k| \to \infty$$

Use this to show for the Taylor case $(0 = \hat{f}_{-1} = \hat{f}_{-2} = \cdots)$ that

$$|f(\theta) - \sum_{k=0}^{n-1}\hat{f}_k e^{ik\theta}| = O(n^{1-\lambda}) \qquad \text{as } n \to \infty$$

SOLUTION A straightforward application of integration by parts yields the result

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}d\theta = \frac{(-i)^\lambda}{2\pi k^\lambda}\int_0^{2\pi} f^{(\lambda)}(\theta)e^{-ik\theta}d\theta$$

given that $f^{(\lambda)}$ is uniformly bounded, the second part follows directly from this result

$$|\sum_{k=n}^{\infty} \hat{f}_k e^{ik\theta}| \leq \sum_{k=n}^{\infty} |\hat{f}_k| \leq C \sum_{k=n}^{\infty} k^{-\lambda}$$

for some constant C.

END

Problem 3(a) If f is a trigonometric polynomial ($\hat{f}_k = 0$ for $|k| > m$) show for $n \geq 2m + 1$ that we can exactly recover f:

$$f(\theta) = \sum_{k=-m}^{m} \hat{f}_k^n e^{ik\theta}$$

SOLUTION This follows from

$$\hat{f}_k^n = \sum_{p=-\infty}^{\infty} \hat{f}_{k+pn} = \hat{f}_k$$

if $-m \leq k \leq m$ as if $p > 0$ we have $k + pn \geq k + 2m + 1 \geq m + 1$ hence $\hat{f}_{k+pn} = 0$ and if $k < 0$ we have $k - pn \leq k - 2m - 1 \leq -m - 1$ hence $\hat{f}_{k+pn} = 0$.

END

Problem 3(b) For the general (non-Taylor) case and $n = 2m + 1$, prove convergence for

$$f_{-m:m}(\theta) := \sum_{k=-m}^{m} \hat{f}_k^n e^{ik\theta}$$

to $f(\theta)$ as $n \to \infty$. What is the rate of convergence if we know that the first $\lambda - 1$ derivatives $f(\theta), f'(\theta), \ldots, f^{(\lambda-1)}(\theta)$ are 2π-periodic and $f^{(\lambda)}$ is uniformly bounded?

SOLUTION

Observe that by aliasing (see corollary in lecture notes) and triangle inequality we have the following

$$|\hat{f}_k^n - \hat{f}_k| \leq \sum_{p=1}^{\infty} (|\hat{f}_{k+pn}| + |\hat{f}_{k-pn}|)$$

Using the result from Problem 2 yields

$$|\hat{f}_k^n - \hat{f}_k| \leq \frac{C}{n^\lambda} \sum_{p=1}^{\infty} \frac{1}{\left(p + \frac{k}{n}\right)^\lambda} + \frac{1}{\left(p - \frac{k}{n}\right)^\lambda}$$

now we pick $|q| < \frac{1}{2}$ (such that the estimate below will hold for both summands above) and construct an integral with convex and monotonocally decreasing integrand such that

$$(p + q)^{-\lambda} < \int_{p-\frac{1}{2}}^{p+\frac{1}{2}} (x + q)^{-\lambda} dx$$

more over summing over the left-hand side from 1 to ∞ yields a bound by the integral:

$$\int_{\frac{1}{2}}^{\infty} (x + q)^{-\lambda} dx = \frac{1}{\lambda}\left(\frac{1}{2} + q\right)^{-\lambda+1}$$

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Finally let $q = \pm\frac{k}{n}$ to achieve the rate of convergence

$$|\hat{f}_k^n - \hat{f}_k| \le \frac{C_\lambda}{n^\lambda}\left(\left(\frac{1}{2} + k/n\right)^{-\lambda+1} + \left(\left(\frac{1}{2} - k/n\right)\right)^{-\lambda+1}\right)$$

where C_λ is a constant depending on λ. Note that it is indeed important to split the n coefficients equally over the negative and positive coefficients as stated in the notes, due to the estatime we used above.

Finally, we have:

$$
\begin{aligned}
|f(\theta) - f_{-m:m}(\theta)| = |\sum_{k=-m}^{m}(\hat{f}_k - \hat{f}_k^n)z^k + \sum_{k=m+1}^{\infty}\hat{f}_k z^k + \sum_{k=-\infty}^{-m-1}\hat{f}_k z^k| \\
\le \sum_{k=-m}^{m}|\hat{f}_k - \hat{f}_k^n| + \sum_{k=m+1}^{\infty}|\hat{f}_k| + \sum_{k=-\infty}^{-m-1}|\hat{f}_k| \\
\le \sum_{k=-m}^{m}\frac{C_\lambda}{n^\lambda}\left(\left(\frac{1}{2} + k/n\right)^{-\lambda+1} + \left(\left(\frac{1}{2} - k/n\right)\right)^{-\lambda+1}\right) + \sum_{k=m+1}^{\infty}|\hat{f}_k| + \sum_{k=-\infty}^{-m-1}|\hat{f}_k| \\
= \frac{C_\lambda}{n^\lambda}2^\lambda + \sum_{k=m+1}^{\infty}|\hat{f}_k| + \sum_{k=-\infty}^{-m-1}|\hat{f}_k| \\
= O(n^{-\lambda}) + O(n^{1-\lambda}) + O(n^{1-\lambda}) \\
= O(n^{1-\lambda})
\end{aligned}
$$

END

Problem 3(c) Show that $f_{-m:m}(\theta)$ interpolates f at $\theta_j = 2\pi j/n$ for $n = 2m+1$.

SOLUTION Note from the aliasing property we have

$$\hat{f}_k^n e^{ik\theta_j} = \hat{f}_k^n e^{2\pi ikj/n} = \hat{f}_{k+n}^n e^{2\pi i(k+n)j/n}$$
$$= \hat{f}_{k+n}^n e^{i(k+n)\theta_j}$$

Thus we have

$$
\begin{aligned}
f_{-m:m}(\theta_j) &= \sum_{k=-m}^{-1}\hat{f}_k^n e^{ik\theta_j} + \sum_{k=0}^{m}\hat{f}_k^n e^{ik\theta_j} \\
&= \sum_{k=-m}^{-1}\hat{f}_{k+n}^n e^{i(k+n)\theta_j} + \sum_{k=0}^{m}\hat{f}_k^n e^{ik\theta_j} \\
&= \sum_{k=n-m}^{n-1}\hat{f}_k^n e^{i(k)\theta_j} + \sum_{k=0}^{m}\hat{f}_k^n e^{ik\theta_j} = f_n(\theta_j) = f(\theta_j)
\end{aligned}
$$

END

Problem 4(a) Show for $0 \le k, \ell \le n-1$

$$\frac{1}{n}\sum_{j=1}^{n}\cos k\theta_j \cos \ell\theta_j = \begin{cases} 1 & k = \ell = 0 \\ 1/2 & k = \ell \\ 0 & \text{otherwise} \end{cases}$$

for $\theta_j = \pi(j - 1/2)/n$. Hint: Be careful as the θ_j differ from before, and only cover half the period, $[0, \pi]$. Using symmetry may help. You may also consider replacing cos with complex exponentials:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

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SOLUTION The case $k = l = 0$ is immediate. Otherwise, we have,

$$\frac{1}{n}\sum_{j=1}^{n}\cos(k\theta_j)\cos(l\theta_j) = \frac{1}{4n}\sum_{j=1}^{n}\left[e^{i(k+l)\theta_j} + e^{-i(k+l)\theta_j} + e^{i(k-l)\theta_j} + e^{-i(k-l)\theta_j}\right].$$

For $\omega = \exp(i\pi/n)$ and any m not a multiple of $2n$ we have

$$\sum_{j=1}^{n}e^{im\theta_j} = \sum_{j=0}^{n-1}e^{im\pi(j+1/2)/n} = e^{im\pi/(2n)}\sum_{j=0}^{n-1}e^{im\pi j/n} = \omega^{m/2}\sum_{j=0}^{n-1}\omega^{mj}$$

$$= \omega^{m/2}\frac{\omega^{nm}-1}{\omega^m-1} = \omega^{m/2}\frac{(-1)^m-1}{\omega^m-1}$$

and hence

$$\sum_{j=1}^{n}[e^{im\theta_j} + e^{-im\theta_j}] = \omega^{m/2}\frac{(-1)^m-1}{\omega^m-1} + \omega^{-m/2}\frac{(-1)^m-1}{\omega^{-m}-1}$$

$$= \omega^{m/2}\frac{(-1)^m-1}{\omega^m-1} + \omega^{m/2}\frac{(-1)^m-1}{1-\omega^m} = 0.$$

Since $0 < k + l \le 2n - 2$ we know $k + l$ is not a multiple of $2n$ hence

$$\sum_{j=1}^{n}\left[e^{i(k+l)\theta_j} + e^{-i(k+l)\theta_j}\right] = 0.$$

Now if $k = l$ we have

$$\sum_{j=1}^{n}e^{i(k-l)\theta_j} = \sum_{j=1}^{n}e^{-i(k-l)\theta_j} = n.$$

Otherwise $k - l \ne 0$ but also $1 - n \le k - l \le n - 1$ hence $k - l$ cannot be a multiple of $2n$. And thus

$$\sum_{j=1}^{n}\left[e^{i(k-l)\theta_j} + e^{i(l-k)\theta_j}\right] = 0.$$

END

Problem 4(b) Consider the Discrete Cosine Transform (DCT)

$$C_n := \begin{bmatrix} \sqrt{1/n} & & & \\ & \sqrt{2/n} & & \\ & & \ddots & \\ & & & \sqrt{2/n} \end{bmatrix}\begin{bmatrix} 1 & \cdots & 1 \\ \cos\theta_1 & \cdots & \cos\theta_n \\ \vdots & \ddots & \vdots \\ \cos(n-1)\theta_1 & \cdots & \cos(n-1)\theta_n \end{bmatrix}$$

for $\theta_j = \pi(j - 1/2)/n$. Prove that C_n is orthogonal: $C_n^\top C_n = C_n C_n^\top = I$.

SOLUTION

The components of C without the diagonal matrix, which we may call $\hat{C}$ are

$$\hat{C}_{ij} = \cos((j-1)\theta_{i-1}),$$

where $\theta_j = \pi(j - 1/2)/n$. Recalling that the elements of matrix multiplication are given by

$$(ab)_{ij} := \sum_{k=1}^{n}a_{ik}b_{kj}$$

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we find that
$$(\hat{C}_n \hat{C}_n^\top)_{ij} = \sum_{k=1}^n \cos((i-1)\theta_{k-1}) \cos((k-1)\theta_{j-1}).$$

By using the previous problem and the terms on the diagonal matrices which ensure that the $1/2$ terms become 1 we know how to compute all of these entries and find that it is the identity.

END

Problem 5 What polynomial interpolates $\cos z$ at $1, \exp(2\pi\mathrm{i}/3)$ and $\exp(-2\pi\mathrm{i}/3)$?

SOLUTION For $\omega = \exp(2\pi\mathrm{i}/3)$, we use the DFT:

$$
\begin{pmatrix} \hat{f}_0^3 \\ \hat{f}_1^3 \\ \hat{f}_2^3 \\ \hat{f}_2^3 \end{pmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \exp(-2\pi\mathrm{i}/3) & \exp(2\pi\mathrm{i}/3) \\ 1 & \exp(2\pi\mathrm{i}/3) & \exp(-2\pi\mathrm{i}/3) \end{bmatrix} \begin{pmatrix} \cos(1) \\ \cos(\exp(2\pi\mathrm{i}/3)) \\ \cos(\exp(-2\pi\mathrm{i}/3)) \end{pmatrix}
$$

$$
= \frac{1}{3} \begin{pmatrix} \cos(1) + \cos(\exp(2\pi\mathrm{i}/3)) + \cos(\exp(-2\pi\mathrm{i}/3)) \\ \cos(1) + \exp(-2\pi\mathrm{i}/3)\cos(\exp(2\pi\mathrm{i}/3)) + \exp(2\pi\mathrm{i}/3)\cos(\exp(-2\pi\mathrm{i}/3)) \\ \cos(1) + \exp(2\pi\mathrm{i}/3)\cos(\exp(2\pi\mathrm{i}/3)) + \exp(-2\pi\mathrm{i}/3)\cos(\exp(-2\pi\mathrm{i}/3)) \end{pmatrix}
$$

That is, the polynomial is

$$
\frac{\cos(1) + \cos(\exp(2\pi\mathrm{i}/3)) + \cos(\exp(-2\pi\mathrm{i}/3))}{3}
$$
$$
+ \frac{\cos(1) + \exp(-2\pi\mathrm{i}/3)\cos(\exp(2\pi\mathrm{i}/3)) + \exp(2\pi\mathrm{i}/3)\cos(\exp(-2\pi\mathrm{i}/3))}{3} z
$$
$$
+ \frac{\cos(1) + \exp(2\pi\mathrm{i}/3)\cos(\exp(2\pi\mathrm{i}/3)) + \exp(-2\pi\mathrm{i}/3)\cos(\exp(-2\pi\mathrm{i}/3))}{3} z^2.
$$

END