

MATH50003

Numerical Analysis

I.3 Dual Numbers

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Julia Tips:

1. Don't install via package manger
2. Pull latest Github repos
3. Make a copy of lab to modify to avoid conflicts

Part I

Calculus on a Computer

1. Rectangular rules for integration
2. Divided differences for differentiation
3. Dual numbers for differentiation

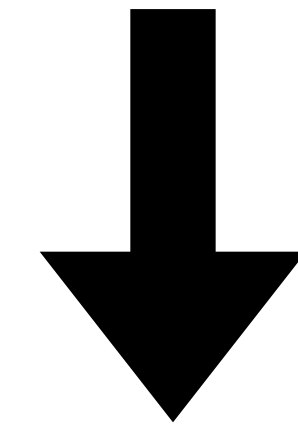
Divided differences can have large errors.

**Is it even possible to
algorithmically calculate
derivatives to high accuracy?**

Yes: if we have access to the code.

Analysis

Divided differences



Algebra

Dual numbers

Dual numbers are a **commutative ring** that allow us to **differentiate**

Definition 1 (dual numbers)

$$\mathbb{D} := \{a + b\epsilon \quad : \quad a, b \in \mathbb{R}, \quad \epsilon^2 = 0\}$$

Compare with complex numbers:

$$\mathbb{C} := \{a + bi \quad : \quad a, b \in \mathbb{R}, \quad i^2 = -1\}$$

Addition/multiplication

Follows from simple algebra

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i$$

Complex

$$(a + b\epsilon) + (c + d\epsilon) = (a + c) + (b + d)\epsilon$$

$$(a + b\epsilon)(c + d\epsilon) = ac + (bc + ad)\epsilon$$

Dual

I.3.1 Differentiating polynomials

Addition/multiplication \Rightarrow dual numbers compute derivatives

Theorem 2 (polynomials on dual numbers). *Suppose p is a polynomial. Then*

$$p(a + b\epsilon) = p(a) + bp'(a)\epsilon$$

Example 1 (differentiating polynomial). Consider computing $p'(2)$ where

$$p(x) = (x - 1)(x - 2) + x^2.$$

I.3.2 Differentiating other functions

Theorem 1 gives us a rule to extend differentiation via duals

Motivation: consider a Taylor series

$$f(x) = \sum_{k=0}^{\infty} f_k x^k$$

And assume a is in radius of convergence.

What will $f(a + b\epsilon)$ return?

Definition 2 (dual extension). Suppose a real-valued function f is differentiable at a . If

$$f(a + b\epsilon) = f(a) + bf'(a)\epsilon$$

then we say that it is a *dual extension at a* .

$$\exp(a + b\epsilon) := \exp(a) + b \exp(a)\epsilon$$

$$\sin(a + b\epsilon) := \sin(a) + b \cos(a)\epsilon$$

$$\cos(a + b\epsilon) := \cos(a) - b \sin(a)\epsilon$$

Examples:

$$\log(a + b\epsilon) := \log(a) + \frac{b}{a}\epsilon$$

$$\sqrt{a + b\epsilon} := \sqrt{a} + \frac{b}{2\sqrt{a}}\epsilon$$

$$|a + b\epsilon| := |a| + b \operatorname{sign} a \epsilon$$

Lemma 2 (product and chain rule). *If f is a dual extension at $g(a)$ and g is a dual extension at a , then $q(x) := f(g(x))$ is a dual extension at a . If f and g are dual extensions at a then $r(x) := f(x)g(x)$ is also dual extensions at a . In other words:*

$$\begin{aligned}q(a + b\epsilon) &= q(a) + bq'(a)\epsilon \\r(a + b\epsilon) &= r(a) + br'(a)\epsilon\end{aligned}$$

Example 2 (differentiating non-polynomial). Consider differentiating $f(x) = \exp(x^2 + e^x)$ at the point $a = 1$ by evaluating on the duals:

**How do we implement this on a
computer?**