

## Numerical Analysis MATH50003 (2023–24) Problem Sheet 10

**Problem 1** What are the upper  $3 \times 3$  sub-block of the multiplication matrix  $X$  / Jacobi matrix  $J$  for the monic and orthonormal polynomials with respect to the following weights on  $[-1, 1]$ :

$$1 - x, \sqrt{1 - x^2}, 1 - x^2$$

### SOLUTION

**Monic** We know that for monic ( $b_n = 1$ ) orthogonal polynomials we can write the upper  $3 \times 3$  block in the form

$$X_3 = \begin{bmatrix} a_0 & c_0 & 0 \\ 1 & a_1 & c_1 \\ 0 & 1 & a_2 \end{bmatrix}$$

1.

$$w(x) = 1 - x$$

Take  $\pi_0(x) = 1$  (monic) and note

$$\|\pi_0\|^2 = \int_{-1}^1 (1 - x) dx = 2$$

From

$$x\pi_0(x) = a_0\pi_0(x) + \pi_1(x)$$

we deduce

$$a_0 = \langle x\pi_0, \pi_0 \rangle / \|\pi_0\|^2 = \frac{\int_{-1}^1 (1 - x)x dx}{2} = -\frac{1}{3}$$

i.e.

$$\pi_1(x) = (x - a_0)\pi_0(x) = x + 1/3.$$

and note that

$$\|\pi_1\|^2 = \int_{-1}^1 (1 - x)(x + 1/3)^2 dx = 4/9.$$

From

$$x\pi_1(x) = c_0\pi_0(x) + a_1\pi_1(x) + \pi_2(x)$$

we deduce

$$c_0 = \langle x\pi_1, \pi_0 \rangle / \|\pi_0\|^2 = \frac{\int_{-1}^1 (1 - x)x(x + 1/3) dx}{2} = \frac{2}{9}$$

and

$$a_1 = \langle x\pi_1, \pi_1 \rangle / \|\pi_1\|^2 = \frac{9}{4} \int_{-1}^1 (1 - x)x(x + 1/3)^2 dx = -\frac{1}{15}$$

Thus

$$\pi_2(x) = (x - a_1)\pi_1(x) - c_0\pi_0(x) = (x + 1/15)(x + 1/3) - 2/9 = x^2 + 2x/5 - 1/5.$$

And once again as before:

$$c_1 = \frac{\langle \pi_1, x\pi_2 \rangle}{\|\pi_1\|^2} = \frac{\int_{-1}^1 (x + \frac{1}{3})(x^2 + \frac{2}{5}x - \frac{1}{5})x(1 - x) dx}{\int_{-1}^1 (x + \frac{1}{3})^2(1 - x) dx} = \frac{6}{25}$$

and

$$a_2 = \frac{\langle \pi_2, x\pi_2 \rangle}{\|\pi_2\|^2} = \frac{\int_{-1}^1 (x^2 + \frac{2}{5}x - \frac{1}{5})^2 x(1-x) dx}{\int_{-1}^1 (x^2 + \frac{2}{5}x - \frac{1}{5})^2 (1-x) dx} = -\frac{1}{35}$$

Thus we have

$$X_3 = \begin{bmatrix} -1/3 & 2/9 & \\ 1 & -1/15 & 6/25 \\ & 1 & -1/35 \end{bmatrix}$$

2.

$$w(x) = \sqrt{1-x^2}$$

Take  $\pi_0(x) = k_0 = 1$  (monic) so that

$$\|\pi_0\|^2 = \int_{-1}^1 \sqrt{1-x^2} = \frac{\pi}{2}.$$

From PS9 we know that  $a_k = 0$ . Thus from the recurrence we have

$$x\pi_0(x) = \pi_1(x)$$

and hence

$$\pi_1(x) = x\pi_0(x) = x.$$

Likewise for

$$x\pi_1(x) = c_0\pi_0(x) + \pi_2(x)$$

we have

$$c_0 = \frac{\langle \pi_0, x\pi_1 \rangle}{\|\pi_0\|^2} = \frac{\int_{-1}^1 x^2 \sqrt{1-x^2} dx}{\pi/2} = \frac{\pi/8}{\pi/2} = \frac{1}{4}$$

i.e.

$$\pi_2(x) = x\pi_1(x) - c_0 = x^2 - \frac{1}{4}.$$

Finally:

$$x\pi_2(x) = c_1\pi_1(x) + \pi_3(x)$$

and thus

$$c_1 = \frac{\langle \pi_1, x\pi_2 \rangle}{\|\pi_1\|^2} = \frac{\int_{-1}^1 (x^2 - \frac{1}{4})x^2 \sqrt{1-x^2} dx}{\int_{-1}^1 x^2 \sqrt{1-x^2} dx} = \frac{\pi/32}{\pi/8} = \frac{1}{4}$$

Thus we have

$$X_3 = \begin{bmatrix} 0 & 1/4 & \\ 1 & 0 & 1/4 \\ & 1 & 0 \end{bmatrix}$$

3.

$$w(x) = 1 - x^2$$

Take  $\pi_0(x) = k_0 = 1$  (monic). Again due to  $w(x) = w(-x)$  from recurrence we have

$$x\pi_0(x) = \pi_1(x)$$

Then from

$$x\pi_1(x) = c_0\pi_0(x) + \pi_2(x)$$

we find

$$c_0 = \frac{\langle \pi_0, x\pi_1 \rangle}{\|\pi_0\|^2} \frac{\int_{-1}^1 x^2(1-x^2)dx}{4/15} = \frac{4/15}{4/3} = \frac{1}{5}$$

Finally,

$$x\pi_2(x) = c_1\pi_1(x) + \pi_3(x)$$

and thus

$$c_1 = \frac{\langle \pi_1, x\pi_2 \rangle}{\|\pi_1\|^2} = \frac{\int_{-1}^1 (x^2 - \frac{1}{5})x^2(1-x^2)dx}{\int_{-1}^1 x^2(1-x^2)dx} = \frac{32/525}{4/15} = \frac{8}{35}$$

Thus we have

$$X_3 = \begin{bmatrix} 0 & 1/5 & \\ 1 & 0 & 8/35 \\ & 1 & 0 \end{bmatrix}$$

**Orthonormal** The hard way to solve this problem is to compute  $\|\pi_n\|$  for each case. Instead, we use a trick for computing the orthonormal variants: III.3 Corollary 6 tells us that if we find constants  $\alpha_n$  and define

$$q_n(x) := \alpha_n \pi_n(x)$$

so that  $\|q_0\| = 1$  and the resulting Jacobi matrix is symmetric then  $q_n$  must be orthonormal. Note that the three-term recurrence for  $q_n$  satisfies

$$\begin{aligned} xq_0 &= x\alpha_0\pi_0 = \alpha_0 a_0 \pi_0 + \alpha_0 \pi_1 = a_0 q_0 + \frac{\alpha_0}{\alpha_1} q_1 \\ xq_m &= x\alpha_n \pi_n = \alpha_n c_{n-1} \pi_{n-1} + a_n \alpha_n \pi_n + \alpha_n \pi_{n+1} = \frac{\alpha_n c_{n-1}}{\alpha_{n-1}} q_{n-1} + a_n q_n + \frac{\alpha_n}{\alpha_{n+1}} q_{n+1} \end{aligned}$$

This is easier to see using linear algebra:

$$\begin{aligned} x[q_0|q_1|\dots] &= x[\pi_0|\pi_1|\dots] \begin{bmatrix} \alpha_0 & & \\ & \alpha_1 & \\ & & \ddots \end{bmatrix} = [\pi_0|\pi_1|\dots] X \begin{bmatrix} \alpha_0 & & \\ & \alpha_1 & \\ & & \ddots \end{bmatrix} \\ &= [q_0|q_1|\dots] \begin{bmatrix} \alpha_0^{-1} & & \\ & \alpha_1^{-1} & \\ & & \ddots \end{bmatrix} X \begin{bmatrix} \alpha_0 & & \\ & \alpha_1 & \\ & & \ddots \end{bmatrix} \\ &= [q_0|q_1|\dots] \underbrace{\begin{bmatrix} a_0 & c_0\alpha_1/\alpha_0 & & \\ \alpha_0/\alpha_1 & a_1 & c_1\alpha_2/\alpha_1 & \\ & \alpha_1/\alpha_2 & a_2 & \ddots \\ & & \ddots & \ddots \end{bmatrix}}_{\tilde{X}} \end{aligned}$$

Thus to make this symmetric we need  $\tilde{c}_n := c_n \alpha_{n+1} / \alpha_n = \alpha_n / \alpha_{n+1} =: \tilde{b}_n$ , i.e.,  $\alpha_{n+1} = \alpha_n / \sqrt{\tilde{c}_n}$ , in other words,

$$\alpha_n = \frac{\alpha_0}{\prod_{k=0}^{n-1} \sqrt{\tilde{c}_k}}.$$

Moreover, we see with this choice that  $\tilde{c}_n = \sqrt{\tilde{b}_n} = \sqrt{\tilde{c}_n}$ .

1.

$$w(x) = 1 - x$$

. We know  $q_0(x) = \alpha_0 = 1/\|\pi_0\| = 1/\sqrt{2}$ . Then  $\alpha_1 = 1/\sqrt{2c_0} = 3/2$  (hence  $q_1(x) = \alpha_1\pi_1(x) = 3x/2 + 1/2$ ),

which tells us

$$\tilde{c}_0 = c_0\alpha_1/\alpha_0 = \sqrt{2}/3 = \tilde{b}_0(= \sqrt{c_0}).$$

Then  $\alpha_2 = \alpha_1/\sqrt{c_1} = 15/(2\sqrt{6})$  which tells us  $\tilde{c}_1 = c_1\alpha_2/\alpha_1 = \sqrt{6}/5 = \tilde{b}_1(= \sqrt{c_1})$ . In other words we have,

$$\tilde{X}_3 = \begin{bmatrix} -1/3 & \sqrt{2}/3 & \\ \sqrt{2}/3 & -1/15 & \sqrt{6}/5 \\ & \sqrt{6}/5 & -1/35 \end{bmatrix}$$

2.

$$w(x) = \sqrt{1 - x^2}$$

We can just jump ahead since we know the answer is just with  $\sqrt{c_n}$  in place of  $b_n$  and  $c_n$ :

$$\tilde{X}_3 = \begin{bmatrix} 0 & 1/2 & \\ 1/2 & 0 & 1/2 \\ & 1/2 & 0 \end{bmatrix}$$

3.

$$w(x) = 1 - x^2$$

:

$$\tilde{X}_3 = \begin{bmatrix} 0 & 1/\sqrt{5} & \\ 1/\sqrt{5} & 0 & \sqrt{8/35} \\ & \sqrt{8/35} & 0 \end{bmatrix}$$

**END**

**Problem 2** Compute the roots of the Legendre polynomial  $P_3(x)$ , orthogonal with respect to  $w(x) = 1$  on  $[-1, 1]$ , by computing the eigenvalues of a  $3 \times 3$  truncation of the Jacobi matrix.

**SOLUTION**

We have,  $P_0(x) = 1$ . Though recall that in order to use Lemma (zeros), the Jacobi matrix must be symmetric and hence the polynomials orthonormal. So Take  $Q_0(x) = 1/\|P_0(x)\| = \frac{1}{\sqrt{2}}$ . Then we have, by the three term recurrence relationship,

$$xQ_0(x) = a_0Q_0(x) + b_0Q_1(x),$$

and taking the inner product of both sides with  $Q_0(x)$  we get,

$$a_0 = \langle xQ_0(x), Q_0(x) \rangle = \int_{-1}^1 x/2 dx = 0.$$

Next recall that  $P_1(x) = x$  and so  $Q_1(x) = x/\|P_1(x)\| = \sqrt{\frac{3}{2}}x$ . We then have, taking the inner product of the first equation above with  $Q_1(x)$ ,

$$b_0 = \langle xQ_0(x), Q_1(x) \rangle = \int_{-1}^1 \frac{\sqrt{3}}{2} x^2 dx = \frac{1}{\sqrt{3}},$$

and also  $b_0 = c_0$  by the Corollary 8 (Jacobi matrix). We have,

$$a_1 = \langle xQ_1(x), Q_1(x) \rangle = \int_{-1}^1 \frac{3}{2}x^3 dx = 0.$$

Recall that  $P_2(x) = \frac{1}{2}(3x^2 - 1)$ , so that  $Q_2(x) = P_2(x)/\|P_2(x)\| = \sqrt{\frac{5}{8}}(3x^2 - 1)$ , and that,

$$xQ_1(x) = c_0Q_0(x) + a_1Q_1(x) + b_1Q_2(x).$$

Taking inner the inner product of both sides with  $Q_2(x)$ , we see that,

$$c_1 = b_1 = \langle xQ_1(x), Q_2(x) \rangle = \int_{-1}^1 \sqrt{\frac{5}{8}} \cdot \sqrt{\frac{3}{2}}(3x^2 - 1) \cdot x \cdot x dx = \frac{2}{\sqrt{15}}.$$

Finally,

$$a_2 = \langle Q_2(x), xQ_2(x) \rangle = \frac{5}{8} \int_{-1}^1 (3x^2 - 1)^2 x dx = 0.$$

This gives us the truncated Jacobi matrix,

$$X_3 = \begin{bmatrix} a_0 & b_0 & 0 \\ b_0 & a_1 & b_1 \\ 0 & b_1 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{15}} \\ 0 & \frac{2}{\sqrt{15}} & 0 \end{bmatrix},$$

whose eigenvalues are the zeros of  $Q_3(x)$ , and hence the zeros of  $P_3(x)$  since they are the same up to a constant. To work out the eigenvalues, we have,

$$\begin{aligned} |X_3 - \lambda I| &= \begin{vmatrix} -\lambda & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\lambda & \frac{2}{\sqrt{15}} \\ 0 & \frac{2}{\sqrt{15}} & -\lambda \end{vmatrix} = 0 \\ \Leftrightarrow -\lambda(\lambda^2 - \frac{4}{15}) - \frac{1}{\sqrt{3}} \cdot \frac{-\lambda}{\sqrt{3}} &= 0 \\ \Leftrightarrow -\lambda^3 + \frac{3}{5}\lambda &= 0, \end{aligned}$$

which has solutions  $\lambda = 0, \pm\sqrt{\frac{3}{5}}$

**END**

**Problem 3** Compute the interpolatory quadrature rule for  $w(x) = \sqrt{1-x^2}$  with the points  $[-1, 1/2, 1]$ .

**SOLUTION**

For the points  $\mathbf{x} = \{-1, 1/2, 1\}$  we have the Lagrange polynomials:

$$\ell_1(x) = \left( \frac{x - 1/2}{-1 - 1/2} \right) \cdot \left( \frac{x - 1}{-1 - 1} \right) = \frac{1}{3} \left( x^2 - \frac{3}{2}x + \frac{1}{2} \right),$$

and

$$\ell_2(x) = -\frac{4}{3}x^2 + \frac{4}{3}, \ell_3(x) = x^2 + \frac{1}{2}x - \frac{1}{2},$$

similarly. We can then compute the weights,

$$w_j = \int_{-1}^1 \ell_j(x) w(x) dx,$$

using,

$$\int_{-1}^1 x^k \sqrt{1-x^2} dx = \begin{cases} \frac{\pi}{2} & k=0 \\ 0 & k=1 \\ \frac{\pi}{8} & k=2 \end{cases}$$

to find,

$$w_j = \begin{cases} \frac{\pi}{8} & j=1 \\ \frac{\pi}{2} & j=2 \\ -\frac{\pi}{8} & j=3, \end{cases}$$

so that the interpolatory quadrature rule is:

$$\Sigma_3^{w,\mathbf{x}}(f) = \frac{\pi}{2} \left( \frac{1}{4}f(-1) + f(1/2) - \frac{1}{4}f(1) \right)$$

**END**

**Problem 4** Compute the 2-point interpolatory quadrature rule associated with roots of orthogonal polynomials for the weights  $\sqrt{1-x^2}$ , 1, and  $1-x$  on  $[-1, 1]$  by integrating the Lagrange bases.

**SOLUTION** For  $w(x) = \sqrt{1-x^2}$  the orthogonal polynomial of degree 2 is  $U_2(x) = 4x^2 - 1$ , with roots  $\mathbf{x} = \{x = \pm \frac{1}{2}\}$ . The Lagrange polynomials corresponding to these roots are,

$$\begin{aligned} \ell_1(x) &= \frac{x - 1/2}{-1/2 - 1/2} = \frac{1}{2} - x, \\ \ell_2(x) &= \frac{x + 1/2}{1/2 + 1/2} = x + \frac{1}{2} \end{aligned}$$

We again work out the weights

$$w_j = \int_{-1}^1 \ell_j(x) w(x) dx,$$

to find,

$$w_1 = w_2 = \frac{\pi}{4},$$

and thus the interpolatory quadrature rule is,

$$\Sigma_2^{w,\mathbf{x}}(f) = \frac{\pi}{4} (f(-1/2) + f(1/2)).$$

For  $w(x) = 1$ , the orthogonal polynomial of degree 2 is, using Legendre Rodriguez formula:

$$P_2(x) = \frac{1}{(-2)^2 2!} \frac{d^2}{dx^2} (1-x^2)^2 = -\frac{1}{2} + \frac{3}{2}x^2.$$

This has roots  $\mathbf{x} = \{\pm \frac{1}{\sqrt{3}}\}$ . We then have,

$$\begin{aligned} \ell_1(x) &= -\frac{\sqrt{3}}{2}x + \frac{1}{2} \\ \ell_2(x) &= \frac{3}{2}x + \frac{1}{2}, \end{aligned}$$

from which we can compute the weights,

$$w_1 = w_2 = 1,$$

which give the quadrature rule:

$$\Sigma_2^{w, \mathbf{x}}(f) = \left[ f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \right]$$

Finally, with  $w(x) = 1 - x$  we use the solution to PS9, which states that

$$p_2(x) = x^2 + 2x/5 - 1/5$$

which has roots,  $\mathbf{x} = \left\{-\frac{1}{5} \pm \frac{\sqrt{6}}{5}\right\}$ . The Lagrange polynomials are then,

$$\begin{aligned} \ell_1(x) &= \frac{x - (-\frac{1}{5} + \frac{\sqrt{6}}{5})}{-\frac{1}{5} - \frac{\sqrt{6}}{5} - (-\frac{1}{5} + \frac{\sqrt{6}}{5})} \\ &= \frac{x - (-\frac{1}{5} + \frac{\sqrt{6}}{5})}{-\frac{2\sqrt{6}}{5}} \\ &= -\frac{5}{2\sqrt{6}}x - \frac{1}{2\sqrt{6}} + \frac{1}{2} \\ \ell_2(x) &= \frac{x - (-\frac{1}{5} - \frac{\sqrt{6}}{5})}{\frac{2\sqrt{6}}{5}} \\ &= \frac{5}{2\sqrt{6}}x + \frac{1}{2\sqrt{6}} + \frac{1}{2} \end{aligned}$$

From which we can compute the weights,

$$\begin{aligned} w_1 &= 1 + \frac{\sqrt{6}}{9}, \\ w_2 &= 1 - \frac{\sqrt{6}}{9}, \end{aligned}$$

giving the quadrature rule,

$$\Sigma_2^{w, \mathbf{x}}(f) = \left[ \left(1 + \frac{\sqrt{6}}{9}\right) f\left(-\frac{1}{5} - \frac{\sqrt{6}}{5}\right) + \left(1 - \frac{\sqrt{6}}{9}\right) f\left(-\frac{1}{5} + \frac{\sqrt{6}}{5}\right) \right]$$

**END**

**Problem 5(a)** For the matrix

$$J_n = \begin{bmatrix} 0 & 1/\sqrt{2} & & & \\ 1/\sqrt{2} & 0 & 1/2 & & \\ & 1/2 & 0 & \ddots & \\ & & \ddots & \ddots & 1/2 \\ & & & 1/2 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

use the relationship with the Jacobi matrix associated with  $T_n(x)$  to prove that, for  $x_j = \cos \theta_j$ , and  $\theta_j = (n - j + 1/2)\pi/n$ ,

$$J_n = Q_n \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix} Q_n^\top$$

where

$$\mathbf{e}_1^\top Q_n \mathbf{e}_j = \frac{1}{\sqrt{n}}, \quad \mathbf{e}_k^\top Q_n \mathbf{e}_j = \sqrt{\frac{2}{n}} \cos(k-1)\theta_j.$$

You may use without proof the sums-of-squares formula

$$1 + 2 \sum_{k=1}^{n-1} \cos^2 k\theta_j = n.$$

### SOLUTION

Recall the three term recurrence for the Chebyshev Polynomials  $T_n$ ,

$$\begin{aligned} xT_0(x) &= T_1(x), \\ xT_n(x) &= \frac{T_{n-1}(x)}{2} + \frac{T_{n+1}(x)}{2}, \end{aligned}$$

and hence it has the multiplication matrix

$$x[T_0|T_1|\cdots] = [T_0|T_1|\cdots] \underbrace{\begin{bmatrix} 0 & 1/2 & & \\ 1 & 0 & 1/2 & \\ & 1/2 & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix}}_X$$

To find the Jacobi matrix we need to symmetrise this, that is, we write

$$[q_0(x)|q_1(x)|\cdots] = [T_0(x)|T_1(x)|\cdots] \underbrace{\begin{bmatrix} \beta_0 & & & \\ & \beta_1 & & \\ & & \beta_2 & \\ & & & \ddots \end{bmatrix}}_K$$

so that

$$x[q_0(x)|q_1(x)|\cdots] = [q_0(x)|q_1(x)|\cdots] \underbrace{K^{-1}XK}_J$$

where

$$K^{-1}XK = \begin{bmatrix} 0 & \beta_1/(2\beta_0) & & \\ \beta_0/\beta_1 & 0 & \beta_2/(2\beta_1) & \\ & \beta_1/(2\beta_2) & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

First recall that the change-of-variables  $x = \cos \theta$  tells us

$$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \pi$$

hence  $q_0(x) = \beta_0 = 1/\sqrt{\pi}$ . From this we find that

$$\frac{\beta_0}{\beta_1} = \frac{\beta_1}{2\beta_0} \Rightarrow \beta_1 = \sqrt{2/\pi}.$$



Other equations give us:

$$\frac{\beta_n}{2\beta_{n+1}} = \frac{\beta_{n+1}}{2\beta_n} \Rightarrow \beta_{n+1} = \beta_n = \sqrt{2/\pi}.$$

Hence since  $\beta_1/(2\beta_0) = 1/\sqrt{2}$  and  $\beta_{n+1}/(2\beta_n) = 1/2$  we have

$$J = \begin{bmatrix} 0 & 1/\sqrt{2} & & & \\ 1/\sqrt{2} & 0 & 1/2 & & \\ & 1/2 & 0 & 1/2 & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

The roots of  $q_n(x)$  are the roots of  $T_n(x) = \cos n \arccos x$ , i.e.,  $x_j = \cos \theta_j$  for  $\theta_j = (n - j + 1/2)\pi/n$ . Thus we know that we can diagonalise  $J_n$  as

$$J_n = Q_n \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix} Q_n^\top$$

where

$$Q_n = \begin{bmatrix} q_0(x_1) & \cdots & q_0(x_n) \\ \vdots & \cdots & \vdots \\ q_{n-1}(x_1) & \cdots & q_{n-1}(x_n) \end{bmatrix} \begin{bmatrix} \alpha_1^{-1} & & \\ & \ddots & \\ & & \alpha_n^{-1} \end{bmatrix}$$

where

$$\alpha_j = \sqrt{q_0(x_j)^2 + \cdots + q_{n-1}(x_j)^2} = \frac{1}{\sqrt{\pi}} \sqrt{1 + 2 \sum_{k=1}^{n-1} \cos k\theta_j} = \sqrt{\frac{n}{\pi}}.$$

Thus we have

$$\begin{aligned} \mathbf{e}_1^\top Q_n \mathbf{e}_j &= \frac{q_0(x_j)}{\alpha_j} = \frac{1}{\sqrt{n}} \\ \mathbf{e}_k^\top Q_n \mathbf{e}_j &= \frac{q_{k-1}(x_j)}{\alpha_j} = \sqrt{\frac{2}{n}} \cos(k-1)\theta_j. \end{aligned}$$

**END**

**Problem 5(b)** Show for  $w(x) = 1/\sqrt{1-x^2}$  that the Gaussian quadrature rule is

$$Q_n^w[f] = \frac{\pi}{n} \sum_{j=1}^n f(x_j)$$

where  $x_j = \cos \theta_j$  for  $\theta_j = (j - 1/2)\pi/n$ .

**SOLUTION** This follows immediately from the previous parts as  $x_j$  are the eigenvalues of  $J_n$  and the weights in Gauss quadrature have the form

$$\frac{1}{\alpha_j^2} = \frac{\pi}{n}.$$

**END**

**Problem 5(c)** Give an explicit formula for the polynomial that interpolates  $\exp x$  at the points  $x_1, \dots, x_n$  as defined above, in terms of Chebyshev polynomials with the coefficients defined in terms of a sum involving only exponentials, cosines and  $\theta_j = (n - j + 1/2)\pi/n$ .

**SOLUTION**

From Theorem 18 we know the interpolatory polynomial is

$$f_n(x) = \sum_{k=0}^{n-1} c_k^n q_k(x)$$

where  $q_0(x) = 1/\sqrt{\pi}$  and  $q_n(x) = \sqrt{2/\pi} T_n(x)$  and

$$c_k^n = \Sigma_n^w[\exp(x)q_k] = \frac{\pi}{n} \sum_{j=1}^n \exp(\cos \theta_j) \cos(k\theta_j)$$

for  $\theta_j = (n - j + 1/2)\pi/n$ .

**END**