

## Numerical Analysis MATH50003 (2023–24) Revision Sheet

**Problem 1(a)** State which real number is represented by an IEEE 16-bit floating point number (with  $\sigma = 15$ ,  $Q = 5$ , and  $S = 10$ ) with bits

$$1\ 01000\ 00000000001$$

**SOLUTION** The sign bit is 1 so the answer is negative. The exponent bits correspond to

$$q = 2^3 = 8$$

The significand is

$$(1.00000000001)_2 = 1 + 2^{-10}$$

So this represents

$$-2^{8-\sigma}(1 + 2^{-10}) = -2^{-7}(1 + 2^{-10})$$

**END**

**Problem 1(b)** How are the following real numbers rounded to the nearest  $F_{16}$ ?

$$1/2, 1/2 + 2^{-12}, 3 + 2^{-9} + 2^{-10}, 3 + 2^{-10} + 2^{-11}.$$

**SOLUTION**  $1/2$  is already a float. We have

$$1/2 + 2^{-12} = (0.1000000000001)_2 = 2^{-1}(1.000000000001)_2$$

This is exactly at the midpoint so is rounded down so the last bit is 0, that is, it is rounded to  $1/2$ . Next we have

$$3 + 2^{-9} + 2^{-10} = (11.00000000011)_2 = 2(1.100000000011)_2.$$

This time we are exactly at the midpoint but we round up so the last bit is 0 giving us

$$2(1.1000000001)_2 = 3 + 2^{-8}.$$

Finally,

$$3 + 2^{-10} + 2^{-11} = 2(1.1000000000011)_2$$

This we round up since we are above the midpoint giving us

$$2(1.1000000001)_2 = 3 + 2^{-9}.$$

**END**

**Problem 2(a)** Consider a Lower triangular matrix with floating point entries:

$$L = \begin{bmatrix} \ell_{11} & & & \\ \ell_{21} & \ell_{22} & & \\ \vdots & \ddots & \ddots & \\ \ell_{n1} & \cdots & \ell_{n,n-1} & \ell_{nn} \end{bmatrix} \in F_{\sigma,Q,S}^{n \times n}$$

and a vector  $\mathbf{x} \in F_{\sigma,Q,S}^n$ , where  $F_{\sigma,Q,S}$  is a set of floating-point numbers. Denoting matrix-vector multiplication implemented using floating point arithmetic as

$$\mathbf{b} := \text{lowermul}(L, \mathbf{x})$$

express the entries  $b_k := \mathbf{e}_k^\top \mathbf{b}$  in terms of  $\ell_{kj}$  and  $x_k := \mathbf{e}_k^\top \mathbf{x}$ , using rounded floating-point operations  $\oplus$  and  $\otimes$ .

**SOLUTION**

$$b_k = \bigoplus_{j=1}^k (\ell_{kj} \otimes x_j)$$

**END**

**Problem 2(b)** Assuming all operations involve normal floating numbers, show that your approximation has the form

$$L\mathbf{x} = \text{lowermul}(L, \mathbf{x}) + \boldsymbol{\epsilon}$$

where, for  $\epsilon_m$  denoting machine epsilon and  $E_{n,\epsilon} := \frac{n\epsilon}{1-n\epsilon}$  and assuming  $n\epsilon_m < 2$ ,

$$\|\boldsymbol{\epsilon}\|_1 \leq 2E_{n,\epsilon_m/2} \|L\|_1 \|\mathbf{x}\|_1.$$

Here we use the matrix norm  $\|A\|_1 := \max_j \sum_{k=1}^n |a_{kj}|$  and the vector norm  $\|\mathbf{x}\|_1 := \sum_{k=1}^n |x_k|$ . You may use the fact that

$$x_1 \oplus \cdots \oplus x_n = x_1 + \cdots + x_n + \sigma_n$$

where

$$|\sigma_n| \leq \|\mathbf{x}\|_1 E_{n-1,\epsilon_m/2}.$$

**SOLUTION**

We have

$$b_k = \left( \bigoplus_{j=1}^k \ell_{kj} \otimes x_j \right) = \left( \bigoplus_{j=1}^k \ell_{kj} x_j (1 + \delta_j) \right) = \left( \sum_{j=1}^k \ell_{kj} x_j (1 + \delta_j) \right) + \sigma_k$$

where

$$|\sigma_k| \leq M_k E_{k-1,\epsilon_m/2}$$

for

$$M_k := \sum_{j=1}^k |\ell_{kj}| |x_j| (1 + \delta_j) \leq 2 \sum_{j=1}^k |\ell_{kj}| |x_j|.$$

Thus

$$b_k = \mathbf{e}_k^\top L\mathbf{x} + \underbrace{\sum_{j=1}^k \ell_{kj} x_j \delta_j}_{\varepsilon_k} + \sigma_k.$$

where

$$|\varepsilon_k| \leq \sum_{j=1}^k |\ell_{kj}| |x_j| (|\delta_j| + 2E_{k-1,\epsilon_m/2}) \leq 2E_{k,\epsilon_m/2} \sum_{j=1}^k |\ell_{kj}| |x_j|$$

where we use

$$\begin{aligned} (|\delta_j| + 2E_{k-1,\epsilon_m/2}) &\leq \frac{\epsilon_m}{2} + 2 \frac{(k-1)\epsilon_m/2}{1 - (k-1)\epsilon_m/2} \\ &= \frac{\epsilon_m/2 - (k-1)\epsilon_m^2/4 + 2(k-1)\epsilon_m/2}{1 - (k-1)\epsilon_m/2} \\ &\leq \frac{2k\epsilon_m/2}{1 - k\epsilon_m/2} = 2E_{k,\epsilon_m/2}. \end{aligned}$$

We then have using  $E_{k,\epsilon_m/2} \leq E_{n,\epsilon_m/2}$ ,

$$\begin{aligned}\|\epsilon\|_1 &= \sum_{k=1}^n |\epsilon_k| \leq 2E_{n,\epsilon_m/2} \sum_{k=1}^n \sum_{j=1}^k |\ell_{kj}| |x_j| \\ &= 2E_{n,\epsilon_m/2} \sum_{j=1}^n |x_j| \sum_{k=1}^{n-j+1} |\ell_{kj}| \leq 2E_{n,\epsilon_m/2} \sum_{j=1}^n |x_j| \|L\|_1 \\ &= 2E_{n,\epsilon_m/2} \|L\|_1 \|\mathbf{x}\|_1.\end{aligned}$$

**END**

**Problem 3** What is the dual extension of square-roots? I.e. what should  $\sqrt{a+b\epsilon}$  equal assuming  $a > 0$ ?

**SOLUTION**

$$\sqrt{a+b\epsilon} = \sqrt{a} + \frac{b}{2\sqrt{a}}\epsilon$$

**END**

**Problem 4** Use the Cholesky factorisation to determine whether the following matrix is symmetric positive definite:

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

**SOLUTION**

Here  $\alpha_1 = 2$  and  $\mathbf{v} = [2, 1]$  giving us

$$\begin{aligned}A_2 &= \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 3/2 \end{bmatrix}\end{aligned}$$

Thus  $\alpha_2 = 1$  and  $\mathbf{v} = [1]$  giving us

$$A_3 = [3/2 - 1] = [1/2]$$

As  $\alpha_3 = 1/2 > 0$  we know a Cholesky decomposition exists hence  $A$  is SPD. In particular we have computed  $A = LL^\top$  where

$$L = \begin{bmatrix} \sqrt{2} & & \\ \sqrt{2} & 1 & \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} \end{bmatrix}$$

**END**

**Problem 5** Use reflections to determine the entries of an orthogonal matrix  $Q$  such that

$$Q \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}.$$

**SOLUTION**

$$\begin{aligned}
\mathbf{x} &:= [2, 1, 2], \|\mathbf{x}\| = 3 \\
\mathbf{y} &:= \|\mathbf{x}\|\mathbf{e}_1 + \mathbf{x} = [5, 1, 2], \|\mathbf{y}\| = \sqrt{30} \\
\mathbf{w} &:= \mathbf{y}/\|\mathbf{y}\| = [5, 1, 2]/\sqrt{30} \\
Q &:= I - 2\mathbf{w}\mathbf{w}^\top = I - \frac{1}{15} \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} [512] = I - \frac{1}{15} \begin{bmatrix} 25 & 5 & 10 \\ 5 & 1 & 2 \\ 10 & 2 & 4 \end{bmatrix} \\
&= \frac{1}{15} \begin{bmatrix} -10 & -5 & -10 \\ -5 & 14 & -2 \\ -10 & -2 & 11 \end{bmatrix}
\end{aligned}$$

**END****Problem 6** For the function  $f(\theta) = \sin 3\theta$ , state explicit formulae for its Fourier coefficients

$$\hat{f}_k := \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-ik\theta} d\theta$$

and their discrete approximation:

$$\hat{f}_k^n := \frac{1}{n} \sum_{j=0}^{n-1} f(\theta_j) e^{-ik\theta_j}.$$

for *all* integers  $k$ ,  $n = 1, 2, \dots$ , where  $\theta_j = 2\pi j/n$ .**SOLUTION**

We have

$$f(\theta) = \sin 3\theta = \frac{\exp(3i\theta)}{2} - \frac{\exp(-3i\theta)}{2i}$$

hence  $\hat{f}_3 = 1/(2i)\hat{f}_{-3} = -1/(2i)$  and  $\hat{f}_k = 0$  otherwise. Thus we have:

$$\begin{aligned}
\hat{f}_k^1 &= \sum_{k=-\infty}^{\infty} \hat{f}_k = \hat{f}_{-3} + \hat{f}_3 = 0, \\
\hat{f}_{2k}^2 &= 0, \hat{f}_{2k+1}^2 = \hat{f}_{-3} + \hat{f}_3 = 0, \\
\hat{f}_{3k}^3 &= \hat{f}_{-3} + \hat{f}_3 = 0, \hat{f}_{3k+1}^3 = \hat{f}_{3k-1}^3 = 0, \\
\hat{f}_{4k}^4 &= \hat{f}_{4k+2}^4 = 0, \hat{f}_{4k+1}^4 = \hat{f}_{-3} = -1/(2i), \hat{f}_{4k+3}^4 = \hat{f}_3 = 1/(2i) \\
\hat{f}_{5k}^5 &= \hat{f}_{5k+1}^5 = \hat{f}_{5k+4}^5, \hat{f}_{5k+2}^5 = \hat{f}_{-3} = -1/(2i), \hat{f}_{5k+3}^5 = \hat{f}_3 = 1/(2i), \\
\hat{f}_{6k}^6 &= \hat{f}_{6k+1}^6 = \hat{f}_{6k+2}^6 = \hat{f}_{6k+4}^6 = \hat{f}_{6k+5}^6, \hat{f}_{6k+3}^6 = \hat{f}_{-3} + \hat{f}_3 = 0
\end{aligned}$$

For  $n > 6$  we have

$$\hat{f}_{-3+nk}^n = \hat{f}_{-3} = -\frac{1}{2i}, \hat{f}_{3+nk}^n = \hat{f}_3 = \frac{1}{2i}$$

and all other  $\hat{f}_k^n = 0$ .**END****Problem 7** Consider orthogonal polynomials

$$H_n(x) = 2^n x^n + O(x^{n-1})$$

as  $x \rightarrow \infty$  and  $n = 0, 1, 2, \dots$ , orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)w(x)dx, \quad w(x) = \exp(-x^2)$$

Construct  $H_0(x)$ ,  $H_1(x)$ ,  $H_2(x)$  and hence show that  $H_3(x) = 8x^3 - 12x$ . You may use without proof the formulae

$$\int_{-\infty}^{\infty} w(x)dx = \sqrt{\pi}, \int_{-\infty}^{\infty} x^2 w(x)dx = \sqrt{\pi}/2, \int_{-\infty}^{\infty} x^4 w(x)dx = 3\sqrt{\pi}/4.$$

### SOLUTION

Because  $w(x) = w(-x)$  we know that  $a_k$  is zero. We further know that  $H_0(x) = 1$  with  $\|H_0\|^2 = \sqrt{\pi}$  and  $H_1(x) = 2x$  with

$$\|H_1\|^2 = 4 \int_{-\infty}^{\infty} x^2 w(x)dx = 2\sqrt{\pi}.$$

We have

$$xH_1(x) = c_0 H_0(x) + H_2(x)/2$$

where

$$c_0 = \frac{\langle xH_1(x), H_0(x) \rangle}{\|H_0\|^2} = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

Hence  $H_2(x) = 2xH_1(x) - H_0(x) = 4x^2 - 2$ , which satisfies

$$\|H_2\|^2 = 16 \int_{-\infty}^{\infty} x^4 w(x)dx - 16 \int_{-\infty}^{\infty} x^2 w(x)dx + 4 \int_{-\infty}^{\infty} w(x)dx = (12 - 8 + 4)\sqrt{\pi} = 8\sqrt{\pi}.$$

We further have

$$\langle xH_2(x), H_1(x) \rangle = \int_{-\infty}^{\infty} (8x^4 - 4x^2)w(x)dx = (6 - 2)\sqrt{\pi} = 4\sqrt{\pi}$$

Finally we have

$$xH_2(x) = c_1 H_1(x) + H_3(x)/2$$

where

$$c_1 = \frac{\langle xH_2(x), H_1(x) \rangle}{\|H_1\|^2} = \frac{4\sqrt{\pi}}{2\sqrt{\pi}} = 2$$

Hence

$$H_3(x) = 2xH_2(x) - 4H_1(x) = 8x^3 - 12x.$$

**END**

**Problem 8(a)** Derive the 3-point Gauss quadrature formula

$$\int_{-\infty}^{\infty} f(x) \exp(-x^2)dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

with analytic expressions for  $x_j$  and  $w_j$ .

### SOLUTION

We know  $x_k$  are the roots of  $H_3(x) = 8x^3 - 12x$  hence we have  $x_2 = 0$  and the other roots satisfy

$$2x^2 - 3 = 0,$$

i.e.,  $x_1 = -\sqrt{3/2}$  and  $x_2 = \sqrt{3/2}$ . To deduce the weights the easiest approach is to use Lagrange interpolation. An alternative is to orthonormalise. Note the Jacobi matrix satisfies

$$x[H_0|H_1|H_2|H_3|\dots] = [H_0|H_1|H_2|H_3|\dots] \underbrace{\begin{bmatrix} 0 & 1 & & \\ 1/2 & 0 & 2 & \\ & 1/2 & 0 & \ddots \\ & & 1/2 & \ddots \\ & & & \ddots \end{bmatrix}}_X$$

To find  $q_k = d_k H_k$ , orthonormalised versions of Hermite, we need to choose  $d_k$  to symmetrise  $X$ , that is for  $D = \text{diag}(d_0, d_1, \dots)$  we have

$$x[q_0|q_1|\dots] = x[H_0|H_1|\dots]D = [H_0|H_1|\dots]XD = [q_0|q_1|\dots]D^{-1}XD$$

where

$$D^{-1}XD = \begin{bmatrix} 0 & d_1/d_0 & & \\ d_0/(2d_1) & 0 & 2d_2/d_1 & \\ & d_1/(2d_2) & 0 & \ddots \\ & & d_2/(2d_3) & \ddots \\ & & & \ddots \end{bmatrix}$$

Note  $d_0 = 1/\sqrt{\int_{-\infty}^{\infty} \exp(-x^2)dx} = 1/\pi^{1/4}$  then we have

$$\begin{aligned} d_0^2 &= 2d_1^2 \Rightarrow d_1 = 1/(\sqrt{2}\pi^{1/4}) \\ d_1^2 &= 4d_2^2 \Rightarrow d_2 = 1/(2\sqrt{2}\pi^{1/4}) \end{aligned}$$

We thus have

$$\begin{aligned} w_1 &= \frac{1}{q_0(-\sqrt{3/2})^2 + q_1(-\sqrt{3/2})^2 + q_2(-\sqrt{3/2})^2} = \frac{1}{d_0^2 + 4d_1^2(3/2) + d_2^2(6-2)^2} = \frac{\sqrt{\pi}}{6} \\ w_2 &= \frac{1}{q_0(0)^2 + q_1(0)^2 + q_2(0)^2} = \frac{1}{d_0^2 + d_2^2(2)^2} = \frac{2\sqrt{\pi}}{3} \\ w_3 &= w_1 = \frac{\sqrt{\pi}}{6}. \end{aligned}$$

**END**

**Problem 8(b)** Compute the 2-point and 3-point Gaussian quadrature rules associated with  $w(x) = 1$  on  $[-1, 1]$ .

**SOLUTION**

For the weights  $w(x) = 1$ , the orthogonal polynomials of degree  $\leq 3$  are the Legendre polynomials,

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= \frac{1}{2}(3x^2 - 1), \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \end{aligned}$$

which can be found from, e.g, the Rodriguez formula or by direct construction. We can normalise each to get  $q_j(x) = P_j(x)/\|P_j\|$ , with  $\|P_j\|^2 = \int_{-1}^1 P_j^2 dx$ . This gives,

$$\begin{aligned} q_0(x) &= \frac{1}{\sqrt{2}}, \\ q_1(x) &= \sqrt{\frac{3}{2}}x, \\ q_2(x) &= \sqrt{\frac{5}{8}}(3x^2 - 1), \\ q_3(x) &= \sqrt{\frac{7}{8}}(5x^3 - 3x). \end{aligned}$$

For the first part we use the roots of  $P_2(x)$  which are  $\mathbf{x} = \{\pm \frac{1}{\sqrt{3}}\}$ . The weights are,

$$w_j = \frac{1}{\alpha_j^2} = \frac{1}{q_0(x_j)^2 + q_1(x_j)^2} = \frac{1}{\frac{1}{2} + \frac{3}{2}x_j^2},$$

where  $\alpha_j$  is the same as in III.6 Lemma 2, so that,

$$w_1 = w_2 = 1,$$

and the Gaussian Quadrature rule is,

$$\Sigma_2^w[f] = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

For the second part, we use the roots of  $P_3(x)$  which are  $\mathbf{x} = \{0, \pm\sqrt{\frac{3}{5}}\}$ . The weights are then,

$$w_j = \frac{1}{\alpha_j^2} = \frac{1}{q_0(x_j)^2 + q_1(x_j)^2 + q_2(x_j)^2} = \frac{1}{\frac{9}{8} - \frac{9}{4}x_j^2 + \frac{45}{8}x_j^4}$$

Giving us,

$$\begin{aligned} w_1 = w_3 &= \frac{1}{\frac{9}{8} - \frac{9}{4}\frac{3}{5} + \frac{45}{8}\frac{9}{25}} = \frac{5}{9} \\ w_2 &= \frac{8}{9} \end{aligned}$$

Then the Gaussian Quadrature rule is,

$$\Sigma_3^w[f] = \frac{1}{9} \left[ 5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right) \right]$$

**END**

**Problem 9** Solve Problem 4(b) from PS8 using **Lemma 12 (discrete orthogonality)** with  $w(x) = 1/\sqrt{1-x^2}$  on  $[-1, 1]$ . That is, use the connection of  $T_n(x)$  with  $\cos n\theta$  to show that the Discrete Cosine Transform

$$C_n := \begin{bmatrix} \sqrt{1/n} & & & \\ & \sqrt{2/n} & & \\ & & \ddots & \\ & & & \sqrt{2/n} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \cos \theta_1 & \cdots & \cos \theta_n \\ \vdots & \ddots & \vdots \\ \cos(n-1)\theta_1 & \cdots & \cos(n-1)\theta_n \end{bmatrix}$$

for  $\theta_j = \pi(j - 1/2)/n$  is an orthogonal matrix.

**SOLUTION**

By the Lemma 3 (Discrete Orthogonality), we have,

$$\begin{aligned}\Sigma_n^w[q_l q_m] &= \frac{\pi}{n} \sum_{j=1}^n q_l(x_j) q_m(x_j) = \delta_{lm}, \\ \sum_{j=1}^n q_l(x_j) q_m(x_j) &= \frac{n}{\pi} \delta_{lm},\end{aligned}$$

By the previous question, for the weight  $w(x) = \frac{1}{\sqrt{1-x^2}}$  we have  $q_0(x_j) = \frac{1}{\sqrt{\pi}}$ ,  $q_k(x_j) = \sqrt{\frac{2}{\pi}} \cos(k\theta_j)$ . For  $l = m = 0$  then we have,

$$\begin{aligned}\frac{1}{\pi} \sum_{j=1}^n \cos(l\theta_j) \cos(m\theta_j) &= \sum_{j=1}^n q_l(x_j) q_m(x_j) = \frac{n}{\pi} \delta_{lm} = \frac{n}{\pi} \\ &\Rightarrow \frac{1}{n} \sum_{j=1}^n \cos(l\theta_j) \cos(m\theta_j) = 1\end{aligned}$$

Now, for  $l = m \neq 0$ , we have,

$$\begin{aligned}\frac{2}{\pi} \sum_{j=1}^n \cos(l\theta_j) \cos(m\theta_j) &= \sum_{j=1}^n q_l(x_j) q_m(x_j) = \frac{n}{\pi} \delta_{lm} = \frac{n}{\pi} \\ &\Rightarrow \frac{1}{n} \sum_{j=1}^n \cos(l\theta_j) \cos(m\theta_j) = \frac{1}{2}\end{aligned}$$

Finally, for  $l \neq m$ , we have,

$$C_{lm} \sum_{j=1}^n \cos(l\theta_j) \cos(m\theta_j) = \sum_{j=1}^n q_l(x_j) q_m(x_j) = \frac{n}{\pi} \delta_{lm} = 0,$$

for some constant  $C_{lm} \neq 0$  which is  $\frac{1}{\pi}$  if  $l = 0$  or  $m = 0$  and  $\frac{2}{\pi}$  otherwise (it doesn't matter what it is so long as it is not 0). Therefore, for  $l \neq m$  we have,

$$\frac{1}{n} \sum_{j=1}^n \cos(l\theta_j) \cos(m\theta_j) = 0$$

**END**