MATH50003 Numerical Analysis

III.3 LU and Cholesky factorisations

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Part III

Numerical Linear Algebra

Software
Application
Theory

1. Structured matrices such as banded
2. Differential Equations via finite differences
3. LU and Cholesky factorisation for solving linear systems
4. Polynomial regression for approximating data via least squares
5. Orthogonal matrices such as Householder reflections
6. QR factorisation for solving rectangular least squares problems

LU factorisation:

$$A = LU$$

PLU factorisation:

$$A = P^\top LU$$

Cholesky factorisation:

$$A = LL^\top$$

III.3.1 Outer products

Definition 18 (outer product)**.** Given $\boldsymbol{x} \in \mathbb{F}^m$ and $\boldsymbol{y} \in \mathbb{F}^n$ the *outer product* is:

$$\boldsymbol{x}\boldsymbol{y}^\top := [\boldsymbol{x}y_1|\cdots|\boldsymbol{x}y_n] = \begin{bmatrix} x_1y_1 & \cdots & x_1y_n \\ \vdots & \ddots & \vdots \\ x_my_1 & \cdots & x_my_n \end{bmatrix} \in \mathbb{F}^{m\times n}.$$

Proposition 4 (rank-1)**.** *A matrix $A \in \mathbb{F}^{m\times n}$ has rank 1 if and only if there exists $\boldsymbol{x} \in \mathbb{F}^m$ and $\boldsymbol{y} \in \mathbb{F}^n$ such that*

$$A = \boldsymbol{x}\boldsymbol{y}^\top.$$

III.3.2 LU factorisation $A = LU$

Gaussian elimination w/o pivoting computes an LU factorisation

Example 20 (LU by-hand).

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 1 & 4 & 9 \end{bmatrix}$$

III.3.3 PLU factorisation $A = P^\top LU$

Gaussian elimination w/ pivoting is a PLU factorisation

Permutation matrices:

Theorem 4 (PLU)**.** *A matrix $A \in \mathbb{C}^{n\times n}$ is invertible if and only if it has a PLU decomposition:*

$$A = P^\top LU$$

where the diagonal of L are all equal to 1 and the diagonal of U are all non-zero, and P is a permutation matrix.

III.3.4 Cholesky factorisations $A = LL^\top$

Symmetric positive definite matrices have Cholesky factorisations

Definition 19 (positive definite)**.** A square matrix $A \in \mathbb{R}^{n\times n}$ is *positive definite* if for all $\boldsymbol{x} \in \mathbb{R}^n, x \neq 0$ we have

$$\boldsymbol{x}^\top A\boldsymbol{x} > 0$$

Proposition 5 (conj. pos. def.)**.** *If $A \in \mathbb{R}^{n \times n}$ is positive definite and $V \in \mathbb{R}^{n \times n}$ is non-singular then*

$$V^\top A V$$

is positive definite.

Proposition 6 (diag positivity)**.** *If $A \in \mathbb{R}^{n\times n}$ is positive definite then its diagonal entries are positive: $a_{kk} > 0$.*

Lemma 3 (subslice pos. def.). *If $A \in \mathbb{R}^{n\times n}$ is positive definite and $\boldsymbol{k} = [k_1, \ldots, k_m]^\top \in \{1, \ldots, n\}^m$ is a vector of m integers where any integer appears only once, then $A[\boldsymbol{k}, \boldsymbol{k}] \in \mathbb{R}^{m\times m}$ is also positive definite.*

Theorem 5 (Cholesky and SPD). *A matrix A is symmetric positive definite if and only if it has a Cholesky factorisation*

$$A = LL^\top$$

where L is lower triangular with positive diagonal entries.

Example 21 (Cholesky by hand).

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$