Numerical Analysis MATH50003 (2023–24) Problem Sheet 9

Problem 1 Construct the monic and orthonormal polynomials up to degree 3 for the weights $\sqrt{1-x^2}$ and $1-x$ on $[-1,1]$. What are the top 3×3 entries of the corresponding Jacobi matrices? Hint: for the first weight, find a recursive formula for $\int_{-1}^{1} x^k \sqrt{1-x^2} \mathrm{d}x$ using a change-of-variables.

Problem 2 Prove Theorem 13: a precisely degree n polynomial

$$p(x) = k_n x^n + O(x^{n-1})$$

satisfies

$$\langle p, f_m \rangle = 0$$

for all polynomials f_m of degree $m < n$ of degree less than n if and only if $p(x) = c\pi_n$ for some constant c, where π_n are monic orthogonal polynomials.

Problem 3 If $w(-x) = w(x)$ for a weight supported on $[-b, b]$ show that $a_n = 0$. Hint: first show that the (monic) polynomials $p_{2n}(x)$ are even and $p_{2n+1}(x)$ are odd.

Problem 4(a) Prove that

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

Problem 4(b) Show that

$$\begin{aligned} xU_0(x) &= U_1(x)/2 \\ xU_n(x) &= \frac{U_{n-1}(x)}{2} + \frac{U_{n+1}(x)}{2}. \end{aligned}$$

Problem 5 Use the fact that orthogonal polynomials are uniquely determined by their leading order coefficient and orthogonality to lower dimensional polynomials to show that:

$$T_n'(x) = nU_{n-1}(x).$$

Problem 6(a) Consider Hermite polynomials orthogonal with respect to the weight $\exp(-x^2)$ on $\mathbb{R}$ with the normalisation

$$H_n(x) = 2^n x^n + O(x^{n-1}).$$

Prove the Rodrigues formula

$$H_n(x) = (-1)^n \exp(x^2) \frac{\mathrm{d}^n}{\mathrm{d}x^n} \exp(-x^2).$$

Hint: use integration-by-parts.

Problem 6(b) What are $k_n^{(1)}$ and $k_n^{(2)}$ such that

$$H_n(x) = 2^n x^n + k_n^{(1)} x^{n-1} + k_n^{(2)} x^{n-2} + O(x^{n-3})$$

Problem 6(c) Deduce the 3-term recurrence relationship for $H_n(x)$.

Problem 6(d) Prove that $H_n'(x) = 2nH_{n-1}(x)$. Hint: show orthogonality of H_n' to all lower degree polynomials, and that the normalisation constants match.