Numerical Analysis MATH50003 (2023–24) Problem Sheet 7

Problem 1(a) Show for a unitary matrix $Q \in U(n)$ and a vector $\boldsymbol{x} \in \mathbb{C}^n$ that multiplication by Q preserve the 2-norm: $\|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$.

SOLUTION

$$\|Q\boldsymbol{x}\|^2 = (Q\boldsymbol{x})^\star Q\boldsymbol{x} = \boldsymbol{x}^\star Q^\star Q\boldsymbol{x} = \boldsymbol{x}^\star\boldsymbol{x} = \|\boldsymbol{x}\|^2$$

END

Problem 1(b) Show that the eigenvalues λ of a unitary matrix Q are on the unit circle: $|\lambda| = 1$. Hint: recall for any eigenvalue λ that there exists a unit eigenvector $\boldsymbol{v} \in \mathbb{C}^n$ (satisfying $\|\boldsymbol{v}\| = 1$).

SOLUTION Let $\boldsymbol{v}$ be a unit eigenvector corresponding to λ: $Q\boldsymbol{v} = \lambda\boldsymbol{v}$ with $\|\boldsymbol{v}\| = 1$. Then

$$1 = \|\boldsymbol{v}\| = \|Q\boldsymbol{v}\| = \|\lambda\boldsymbol{v}\| = |\lambda|.$$

END

Problem 1(c) Show for an orthogonal matrix $Q \in O(n)$ that $\det Q = \pm 1$. Give an example of $Q \in U(n)$ such that $\det Q \neq \pm 1$. Hint: recall for any real matrices A and B that $\det A = \det A^\top$ and $\det(AB) = \det A \det B$.

SOLUTION

$$(\det Q)^2 = (\det Q^\top)(\det Q) = \det Q^\top Q = \det I = 1.$$

An example would be a 1×1 complex-valued matrix $\exp(\mathrm{i})$.

END

Problem 1(d) A normal matrix commutes with its adjoint. Show that $Q \in U(n)$ is normal.

SOLUTION

$$QQ^\star = I = Q^\star Q$$

END

Problem 1(e) The spectral theorem states that any normal matrix is unitarily diagonalisable: if A is normal then $A = V\Lambda V^\star$ where $V \in U(n)$ and Λ is diagonal. Use this to show that $Q \in U(n)$ is equal to I if and only if all its eigenvalues are 1.

SOLUTION

Note that Q is normal and therefore by the spectral theorem for normal matrices we have

$$Q = V\Lambda V^\star = VV^\star = I$$

since V is unitary.

END

Problem 2 Consider the vectors

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2\mathrm{i} \\ 2 \end{bmatrix}.$$

Use reflections to determine the entries of orthogonal/unitary matrices Q_1, Q_2, Q_3 such that

$$Q_1\boldsymbol{a} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, Q_2\boldsymbol{a} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}, Q_3\boldsymbol{b} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

SOLUTION

For Q_1: we have

$$\boldsymbol{y} = \boldsymbol{a} - \|\boldsymbol{a}\|\boldsymbol{e}_1 = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$$

$$\boldsymbol{w} = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$Q_1 = Q_{\boldsymbol{w}} = I - \frac{2}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} [-1 \ 1 \ 1] = I - \frac{2}{3} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

For Q_2: we have

$$\boldsymbol{y} = \boldsymbol{a} + \|\boldsymbol{a}\|\boldsymbol{e}_1 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$\boldsymbol{w} = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$Q_1 = Q_{\boldsymbol{w}} = I - \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} [2 \ 1 \ 1] = I - \frac{1}{3} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -1 & -2 & -2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

For Q_3 we just need to be careful to conjugate:

$$\boldsymbol{y} = \boldsymbol{b} + \|\boldsymbol{b}\|\boldsymbol{e}_1 = \begin{bmatrix} 4 \\ 2\mathrm{i} \\ 2 \end{bmatrix}$$

$$\boldsymbol{w} = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ \mathrm{i} \\ 1 \end{bmatrix}$$

$$Q_3 = Q_{\boldsymbol{w}} = I - \frac{1}{3} \begin{bmatrix} 2 \\ \mathrm{i} \\ 1 \end{bmatrix} [2 \ -\mathrm{i} \ 1] = I - \frac{1}{3} \begin{bmatrix} 4 & -2\mathrm{i} & 2 \\ 2\mathrm{i} & 1 & \mathrm{i} \\ 2 & -\mathrm{i} & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -1 & 2\mathrm{i} & -2 \\ -2\mathrm{i} & 2 & -\mathrm{i} \\ -2 & \mathrm{i} & 2 \end{bmatrix}$$

END

Problem 3(a) What simple rotation matrices $Q_1, Q_2 \in SO(2)$ have the property that:

$$Q_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}, Q_2 \begin{bmatrix} \sqrt{5} \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

SOLUTION

The rotation that takes $[x, y]$ to the x-axis is

$$\frac{1}{\sqrt{x^2+y^2}} \begin{bmatrix} x & y \\ -y & x \end{bmatrix}.$$

Hence we get

$$Q_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$
$$Q_2 = \frac{1}{3} \begin{bmatrix} \sqrt{5} & 2 \\ -2 & \sqrt{5} \end{bmatrix}$$

END

Problem 3(b) Find an orthogonal matrix that is a product of two simple rotations but acting on two different subspaces:

$$Q = \underbrace{\begin{bmatrix} \cos\theta_2 & & -\sin\theta_2 \\ & 1 & \\ \sin\theta_2 & & \cos\theta_2 \end{bmatrix}}_{Q_2} \underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & \\ \sin\theta_1 & \cos\theta_1 & \\ & & 1 \end{bmatrix}}_{Q_1}$$

so that, for $\boldsymbol{a}$ defined above,

$$Q\boldsymbol{a} = \begin{bmatrix} \|\boldsymbol{a}\| \\ 0 \\ 0 \end{bmatrix}.$$

Hint: you do not need to determine θ_1, θ_2, instead you can write the entries of Q_1, Q_2 directly using just square-roots.

SOLUTION

We use Q_1 to introduce a 0 in the second entry by rotating the vector $[1, 2]$:

$$Q_1 = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & \\ -2/\sqrt{5} & 1/\sqrt{5} & \\ & & 1 \end{bmatrix}$$

so that

$$Q_1\boldsymbol{a} = \begin{bmatrix} \sqrt{5} \\ 0 \\ 2 \end{bmatrix}.$$

Now we use the matrix that rotates the vector $[\sqrt{5}, 2]$ whose norm is 3 to deduce the entries

$$Q_2 = \begin{bmatrix} \sqrt{5}/3 & & 2/3 \\ & 1 & \\ -2/3 & & \sqrt{5}/3 \end{bmatrix}$$

so that

$$Q_2 Q_1 = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/\sqrt{5} & 1/\sqrt{5} & \\ -2/(3\sqrt{5}) & -4/(3\sqrt{5}) & \sqrt{5}/3 \end{bmatrix}$$

END

Problem 4(a) Show that every matrix $A \in \mathbb{R}^{m \times n}$ has a QR factorisation such that the diagonal of R is non-negative. Make sure to include the case of more columns than rows (i.e. $m < n$).

SOLUTION

We first show for $m < n$ that a QR decomposition exists. Writing

$$A = [\boldsymbol{a}_1 | \cdots | \boldsymbol{a}_n]$$

and taking the first m columns (so that it is square) we can write $[\boldsymbol{a}_1 | \cdots | \boldsymbol{a}_m] = QR_m$. It follows that $R := Q^\star A$ is right-triangular.

We can write:

$$D = \begin{bmatrix} \operatorname{sign}(r_{11}) & & \\ & \ddots & \\ & & \operatorname{sign}(r_{pp}) \end{bmatrix}$$

where $p = \min(m, n)$ and we define $\operatorname{sign}(0) = 1$. Note that $D^\top D = I$. Thus we can write: $A = QR = QDDR$ where (QD) is orthogonal and DR is upper-triangular with positive entries.

END

Problem 4(b) Show that the QR factorisation of a square invertible matrix $A \in \mathbb{R}^{n \times n}$ is unique, provided that the diagonal of R is positive.

SOLUTION

Assume there is a second factorisation also with positive diagonal

$$A = QR = \tilde{Q}\tilde{R}$$

Then we know

$$Q^\top \tilde{Q} = R\tilde{R}^{-1}$$

Note $Q^\top \tilde{Q}$ is a product of orthogonal matrices so is also orthogonal. It's eigenvalues are the same as $R\tilde{R}^{-1}$, which is upper triangular. The eigenvalues of an upper triangular matrix are the diagonal entries, which in this case are all positive. Since all eigenvalues of an orthogonal matrix are on the unit circle (see Q1(b) above) we know all m eigenvalues of $Q^\top \tilde{Q}$ are 1. By Q1(e) above, this means that $Q^\top \tilde{Q} = I$. Hence

$$\tilde{Q} = (Q^\top)^{-1} = Q$$

and

$$\tilde{R} = (\tilde{Q})^{-1} A = R.$$

END