Numerical Analysis MATH50003 (2023–24) Problem Sheet 5

Problem 1(a) Suppose $|\epsilon_k| \leq \epsilon$ and $n\epsilon < 1$. Use induction to show that

$$\prod_{k=1}^n (1+\epsilon_k) = 1 + \theta_n$$

for some constant θ_n satisfying

$$|\theta_n| \leq \underbrace{\frac{n\epsilon}{1-n\epsilon}}_{E_{n,\epsilon}}$$

Problem 1(b) Show for an idealised floating point vector $\boldsymbol{x} \in F_{\infty,S}^n$ that

$$x_1 \oplus \cdots \oplus x_n = x_1 + \cdots + x_n + \sigma_n$$

where

$$|\sigma_n| \leq \|\boldsymbol{x}\|_1 E_{n-1,\epsilon_{\rm m}/2},$$

assuming $n\epsilon_{\rm m} < 2$ and where

$$\|\boldsymbol{x}\|_1 := \sum_{k=1}^n |x_k|.$$

Hint: use the previous part to first write

$$x_1 \oplus \cdots \oplus x_n = x_1(1+\theta_{n-1}) + \sum_{j=2}^n x_j(1+\theta_{n-j+1}).$$

Problem 1(c) For $A \in F_{\infty,S}^{n\times n}$ and $\boldsymbol{x} \in F_{\infty,S}^n$ consider the error in approximating matrix multiplication with idealised floating point: for

$$A\boldsymbol{x} = \begin{pmatrix} \bigoplus_{j=1}^n A_{1,j} \otimes x_j \\ \vdots \\ \bigoplus_{j=1}^n A_{1,j} \otimes x_j \end{pmatrix} + \delta$$

show that

$$\|\delta\|_\infty \leq 2\|A\|_\infty \|\boldsymbol{x}\|_\infty E_{n,\epsilon_{\rm m}/2}$$

where $n\epsilon_{\rm m} < 2$ and the matrix norm is $\|A\|_\infty := \max_k \sum_{j=1}^n |a_{kj}|$.

Problem 2 Derive Backward Euler: use the left-sided divided difference approximation

$$u'(x) \approx \frac{u(x) - u(x-h)}{h}$$

to reduce the first order ODE

$$u(a) = c, \qquad u'(x) + \omega(x)u(x) = f(x)$$

to a lower triangular system by discretising on the grid $x_j = a + jh$ for $h = (b-a)/n$. Hint: only impose the ODE on the gridpoints $x_1, \ldots, x_n$ so that the divided difference does not depend on behaviour at x_{-1}.

Problem 3 Reduce a Schrödinger equation to a tridiagonal linear system by discretising on the grid $x_j = a + jh$ for $h = (b-a)/n$:

$$u(a) = c, \qquad u''(x) + V(x)u(x) = f(x), \qquad u(b) = d.$$