Numerical Analysis MATH50003 (2023–24) Problem Sheet 2

Problem 1 Using dual number arithmetic, compute the following polynomials evaluated at the dual number $2+\epsilon$ and use this to deduce their derivative at 2:

$$2x^2+3x+4, (x+1)(x+2)(x+3), (2x+1)x^3$$

SOLUTION (a)

$$2(2+\epsilon)^2+3(2+\epsilon)+4 = 2(4+4\epsilon)+6+3\epsilon+4 = 18+11\epsilon$$

so the derivative is 11.

(b)

$$(3+\epsilon)(4+\epsilon)(5+\epsilon) = (12+7\epsilon)(5+\epsilon) = 60+47\epsilon$$

so the derivative is 47.

(c)

$$(2(2+\epsilon)+1)(2+\epsilon)^3 = (5+2\epsilon)(4+4\epsilon)(2+\epsilon) = (20+28\epsilon)(2+\epsilon) = 40+76\epsilon$$

so the derivative is 76.

END

Problem 2 What should the following functions applied to dual numbers return for $x = a+b\epsilon$:

$$f(x) = x^{100}+1, g(x) = 1/x, h(x) = \tan x$$

State the domain where these definitions are valid.

SOLUTION

$$f(a+b\epsilon) = f(a)+bf'(a)\epsilon = a^{100}+1+100ba^{99}\epsilon$$

valid everywhere.

$$g(a+b\epsilon) = \frac{1}{a} - \frac{b}{a^2}\epsilon$$

valid for $a \neq 0$.

$$h(a+b\epsilon) = \tan a + b\sec^2 a\epsilon$$

valid for $a \notin \{k\pi + \pi/2 : k \in \mathbb{Z}\}$.

END

Problem 3(a) What is the correct definition of division on dual numbers, i.e.,

$$(a+b\epsilon)/(c+d\epsilon) = s+t\epsilon$$

for what choice of s and t?

SOLUTION

As with complex numbers, division is easiest to understand by first multiplying with the conjugate, that is:

$$\frac{a+b\epsilon}{c+d\epsilon} = \frac{(a+b\epsilon)(c-d\epsilon)}{(c+d\epsilon)(c-d\epsilon)}.$$

Expanding the products and dropping terms with ϵ^2 then leaves us with the definition of division for dual numbers (where the denominator must have non-zero real part):

$$\frac{a}{c} + \frac{bc - ad}{c^2}\epsilon.$$

Thus we have $s = \frac{a}{c}$ and $t = \frac{bc-ad}{c^2}$.

END

Problem 3(b) A *field* is a commutative ring such that $0 \neq 1$ and all nonzero elements have a multiplicative inverse, i.e., there exists a^{-1} such that $aa^{-1} = 1$. Can we use the previous part to define $a^{-1} := 1/a$ to make $\mathbb{D}$ a field? Why or why not?

SOLUTION

Fields require that all nonzero elements have a unique multiplicative inverse. However, this is not the case for dual numbers. To give an explicit counter example, we show that there is no dual number z which is the inverse of $0 + \epsilon$, i.e. a dual number z such that

$$\frac{(0 + \epsilon)}{(z_r + z_d\epsilon)} = 1 + 0\epsilon.$$

By appropriate multiplication with the conjugate we show that

$$\frac{(0 + \epsilon)(z_r - z_d\epsilon)}{(z_r + z_d\epsilon)(z_r - z_d\epsilon)} = \frac{z_r\epsilon}{z_r^2} = \frac{\epsilon}{z_r}.$$

This proves that no choice of real part z_r can reach the multiplicative identity $1 + 0\epsilon$ when starting from the number $0 + \epsilon$. More general results for zero real part dual numbers can also be proved.

END

Problem 4 Use dual numbers to compute the derivative of the following functions at $x = 0.1$:

$$\exp(\exp x \cos x + \sin x), \prod_{k=1}^{3} \left(\frac{x}{k} - 1\right), \text{ and } f_2^{\mathrm{s}}(x) = 1 + \frac{x - 1}{2 + \frac{x-1}{2}}$$

SOLUTION

We now compute the derivatives of the three functions by evaluating for $x = 0.1 + \epsilon$. For the first function we have:

$$\begin{aligned}
&\exp(\exp(0.1 + \epsilon)\cos(0.1 + \epsilon) + \sin(0.1 + \epsilon)) \\
&= \exp((\exp(0.1) + \epsilon\exp(0.1))(\cos(0.1) - \sin(0.1)\epsilon) + \sin(0.1) + \cos(0.1)\epsilon) \\
&= \exp(\exp(0.1)\cos(0.1) + \sin(0.1) + (\exp(0.1)(\cos(0.1) - \sin(0.1)) + \cos(0.1))\epsilon) \\
&= \exp(\exp(0.1)\cos(0.1) + \sin(0.1)) \\
&\qquad + \exp(\exp(0.1)\cos(0.1) + \sin(0.1))\exp(0.1)(\cos(0.1) - \sin(0.1)) + \cos(0.1))\epsilon
\end{aligned}$$

therefore the derivative is the dual part

$$\exp(\exp(0.1)\cos(0.1) + \sin(0.1))(\exp(0.1)(\cos(0.1) - \sin(0.1)) + \cos(0.1))$$

For the second function we have:

$$\begin{aligned}
(0.1 + \epsilon - 1)\left(\frac{0.1 + \epsilon}{2} - 1\right)\left(\frac{0.1 + \epsilon}{3} - 1\right) &= (-0.9 + \epsilon)(-0.95 + \epsilon/2)(-29/30 + \epsilon/3) \\
&= (171/200 - 1.4\epsilon)(-29/30 + \epsilon/3) \\
&= -1653/2000 + 983\epsilon/600
\end{aligned}$$

Thus the derivative is 983/600.

For the third function we have:

$$\begin{aligned} 1 + \frac{0.1 + \epsilon - 1}{2 + \frac{0.1+\epsilon-1}{2}} &= 1 + \frac{-0.9 + \epsilon}{1.55 + \epsilon/2} \\ &= 1 - 18/31 + 2\epsilon/1.55^2 \end{aligned}$$

Thus the derivative is $2/1.55^2$.

END