C&EE 110:
Introduction to Statistics and Probability

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1 | Sets and Probablity Theory

1.1 Probabilistic Sets

A random event, E, has more than 1 possible outcome in the sample space S. S is the collection of all possible event outcomes. We know that E $\subset$ S.

Ex) Number in dice roll

$$S = \{1, 2, 3, 4, 5, 6\}$$
$$E_{odd} = \{1, 3, 5\}$$
$$E_{>3} = \{4, 5, 6\}$$

Operations

We can apply several operations to our sets.

1. Union, denoted $E_1 \cup E_2$
2. Intersection, denoted $E_1 \cap E_2$ or $E_1 E_2$

Consider E_{odd} and $E_{>3}$ above.

$$E_{odd} \cup E_{>3} = \{1, 3, 4, 5, 6\}$$
$$E_{odd} \cap E_{>3} = \{5\}$$

These operations are commutative, associate, and distributive. Intersection has precedence over union.

Special Events

- S is the event the spans the entire sample space
- $\varnothing$ is the null event, it has no outcomes
- if E_1 and E_2 are mutually exclusive, $E_1 E_2 = \varnothing$
- if E_1 and E_2 are collectively exhaustive, $E_1 \cup E_2 = S$
- $\overline{E_1} = S - E_1$, the complement[1] of E_1

[1]Demorgan's Laws hold

Frequentist Probability (Natural Variation)

The probability of occurrence of E is the relative frequency of observations of E ina large number of repeated experiments. Put more formally below,

$$P(E) = \lim_{N\to\infty} \frac{n}{N}, \text{ where n = occurences of E in N observations in S}$$

Bayesian Probability (Incomplete Knowledge)

The probability of an event E represents analysts' degree of belief that E will occur.

Frequentist Probability	Bayesian Probability
probability of expecting a ground shaking intensity of 1g in next 100 years	probability of finding water on new planet
max wind speed in a year	probability that a buidling will collapse under ground shaking intensity of 1g
live load on a building	election results
*based on previous observations	*not based on previous observations
*cannot be reduced through more measurement	*can be reduced if more observations/measurements applied

1.2 Axioms

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. $P(A \cup B) = P(A) + P(B)$, s.t. $AB = \varnothing$

* these axioms are consistent with Frequentist probability

We can derive several rules from these axioms.

1. $P(\overline{E}) = 1 - P(E)$
2. $P(\varnothing) = 0$
3. $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1E_2)$
    - if E_1 and E_2 are mutually exclusive, then we double count their intersection when using the 3rd axiom; subtracting it leads to the correct value
    - what if we have > 2 events? Inclusion/Exclusion rule
    - $P(E_1 \cup E_2 \cup ... \cup E_n) =$

$$\sum_{i=1}^{n} P(E_i) - \sum_{i=1}^{n}\sum_{j=1}^{i-1} P(E_iE_j) + \sum_{i=1}^{n}\sum_{j=1}^{i-1}\sum_{k=1}^{j-1} P(E_iE_jE_k) + ... + (-1)^{n-1}P(E_1E_2...E_n)$$

Conditional Probability

We may want to determine the probability of an event given another event is guaranteed to occur. This is denoted $P(E_1|E_2)$, which is read as E_1 given E_2. It essentially redefines the sample space to be E_2.

$$P(E_1|E_2) = \begin{cases} \frac{P(E_1E_2)}{P(E_2)} & P(E_2) > 0 \\ 0 & P(E_2) = 0 \end{cases} \tag{1.1}$$

From this equation, it follows that

$$P(E_1E_2) = P(E_1|E_2)P(E_2)$$

This holds in general for n events.

$$P(E_1E_2E_3) = P(E_1|E_2E_3)P(E_2E_3) = P(E_1|E_2E_3)P(E_2|E_3)P(E_3)$$

Ex) Applying conditions to operations

$$P(E_1 \cup E_2|E_3) = P(E_1|E_3) + P(E_2|E_3) - P(E_1E_2|E_3)$$

$$P(E_1E_2|E_3) = P(E_1|E_2|E_3)P(E_2|E_3), \text{ which follows from 1.1}$$

Independence

Two events are indpendent iff $P(E_1|E_2) = P(E_1)$

We have mutual independence if $P(E_1E_2 \dots E_n) = P(E_1)P(E_2) \dots P(E_n)$.

Theorem of Total Probability

Consider an event A and a set of of mutually exclusive and collectively exhaustive events $E_1, E_2, \dots, E_3$.

$$P(A) = \sum_{i=1}^{n} P(A|E_i)P(E_i) \tag{1.2}$$

Bayes' Rule

Consider an event A and a set of of mutually exclusive and collectively exhaustive events $E_1, E_2, \dots, E_3$ in S.

$$P(AE_j) = P(E_j|A)P(A) = P(A|E_j)P(E_j)$$
$$P(E_j|A) = \frac{P(A|E_j)P(E_j)}{P(A)}$$
$$P(E_j|A) = \frac{P(A|E_j)P(E_j)}{\sum_{i=1}^{n} P(A|E_i)P(E_i)}$$

where equation 1.2 is used to subtitute P(A)

2 Random Variables

A **random variable** is a variable whose specific value cannot be predicted with certaintiy before an experiement. They take on a numerical value for each possible event in the sample space.

Ex) Random variables are easy to define. For example,

- X = magnitutde of a future earthquake
- Y = yield stress of a material
- Z = peak wind pressure during a given year

For a random variable X, its outcomes are denoted $x_1, x_2, \ldots, x_n$. For an outcome x_i, we denote the probability of that outcome as $P(X = x_i)$.

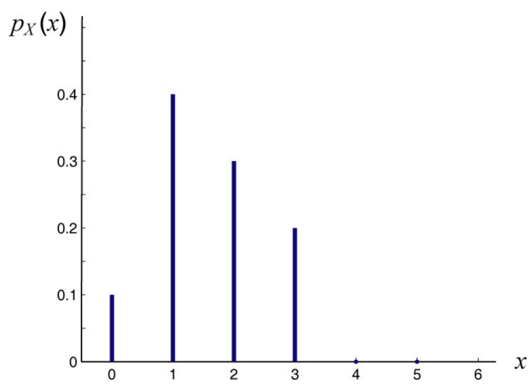
2.1 Discrete Random Variables

A random variable is called **discrete** if the number of outcomes is countable. For example, for X = the number of cars on a bridge at a certain time, X is discrete.

Distributions of discrete random variables can be quantified in 2 ways.

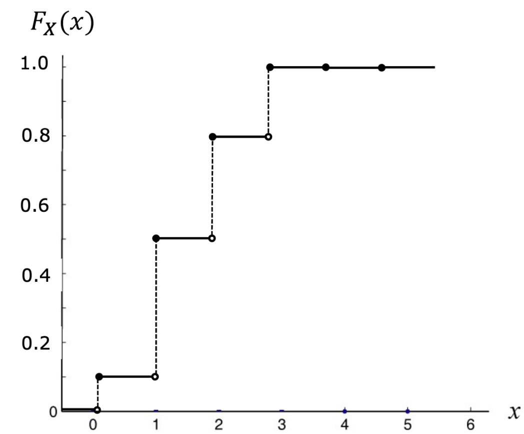
1. Probability Mass Function

$$p_X(x_i) = P(X = x_i)$$



2. Cumulative Distribution Function

$$F_X(x_i) = P(X \leq x_i)$$



Intuitively, adding up $p_X(x_i)$ for all i is equal to $F_x(a)$.

$$F_X(a) = \sum_{\text{all } x_i \leq a} p_X(x_i)$$

Rules of Discrete Random Variables

- $0 \leq p_X(x_i) \leq 1$
- $\sum_{\text{all } x_i} p_X(x_i) = 1$
- $F_X(-\infty) = 0$
- $F_X(+\infty) = 1$
- $F_X(b) \geq F_X(a)$ if $b \geq a$

All of these rules are fairly intuitive. For example, the probability of any event must be between 0 and 1. Additionally, the sum of all events in a sample space must be 1.

2.2 Continuous Random Variables

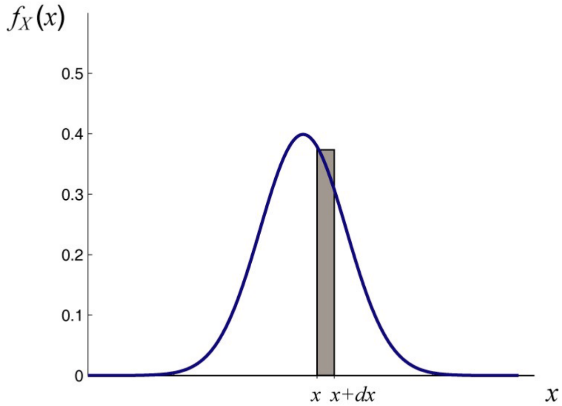
Random variables are said to be **continuous** if they can take on any real value. As a result, there are ∞ possible values for a random variable X. It follows that

$$P(X = x_i) = \frac{1}{\infty} = 0, \text{ for all } i$$

We can describe the distribution of continuous random variables in 2 ways.

1. Probability Density Function

$$f_X(x_i)dx = P(x_i < X < x_i + dx)$$

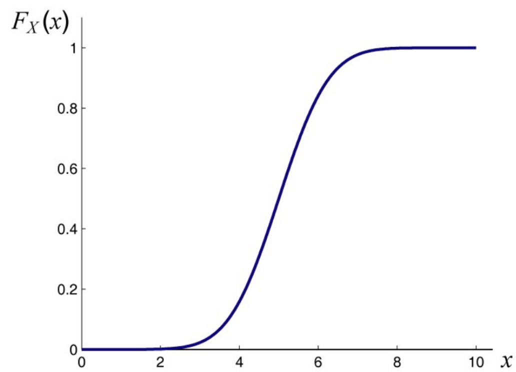


We know that occurences in different intervals are mutually exclusive, so it follows that

$$P(a < X \leq b) = \int_a^b f_X(x)dx$$

2. Cumulative Distribuion Function

$$F_X(x_i) = P(X \leq x_i)$$



Additionally,

$$F_X(x_i) = P(X \leq x_i) = \int_{-\infty}^{+\infty} f_X(u)du$$

and it follows from the Fundamental Theorem of Calculus that

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Rules of Continuous Random Variables

- $f_X(x) \geq 0$
- $\int_{-\infty}^{+\infty} f_X(x)dx = 1 = S$
- $F_X(-\infty) = 0$
- $F_X(\infty) = 1$
- $F_X(b) \geq F_X(a)$ if $b \geq a$