# C&EE 110:
# Introduction to Statistics and Probability

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# 1 | Set Theory and Random Variables

## 1.1 Probabilistic Sets

A random event, E, has more than 1 possible outcome in the sample space S. S is the collection of all possible event outcomes. We know that E $\subset$ S.

**Ex)** Number in dice roll

$$S = \{1, 2, 3, 4, 5, 6\}$$
$$E_{odd} = \{1, 3, 5\}$$
$$E_{>3} = \{4, 5, 6\}$$

### Operations

We can apply several operations to our sets.

1. Union, denoted $E_1 \cup E_2$
2. Intersection, denoted $E_1 \cap E_2$ or $E_1 E_2$

Consider $E_{odd}$ and $E_{>3}$ above.

$$E_{odd} \cup E_{>3} = \{1, 3, 4, 5, 6\}$$
$$E_{odd} \cap E_{>3} = \{5\}$$

These operations are commutative, associate, and distributive. Intersection has precedence over union.

### Special Events

- S is the event the spans the entire sample space
- $\varnothing$ is the null event, it has no outcomes
- if $E_1$ and $E_2$ are mutually exclusive, $E_1 E_2 = \varnothing$
- if $E_1$ and $E_2$ are collectively exhaustive, $E_1 \cup E_2 = S$
- $\overline{E_1} = S - E_1$, the complement[1] of $E_1$

[1] Demorgan's Laws hold

### Frequentist Probability (Natural Variation)

The probability of occurrence of E is the relative frequency of observations of E ina large number of repeated experiments. Put more formally below,

$$P(E) = \lim_{N\to\infty} \frac{n}{N}, \text{ where n = occurences of E in N observations in S}$$

### Bayesian Probability (Incomplete Knowledge)

The probability of an event E represents analysts' degree of belief that E will occur.

| Frequentist Probability | Bayesian Probability |
|---|---|
| probability of expecting a ground shaking intensity of 1g in next 100 years | probability of finding water on new planet |
| max wind speed in a year | probability that a buidling will collapse under ground shaking intensity of 1g |
| live load on a building | election results |
| *based on previous observations | *not based on previous observations |
| *cannot be reduced through more measurement | *can be reduced if more observations/measurements applied |

## 1.2 Axioms

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. $P(A \cup B) = P(A) + P(B)$, s.t. $AB = \varnothing$

* these axioms are consistent with Frequentist probability

We can derive several rules from these axioms.

1. $P(\overline{E}) = 1 - P(E)$
2. $P(\varnothing) = 0$
3. $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1E_2)$
   - if $E_1$ and $E_2$ are mutually exclusive, then we double count their intersection when using the 3rd axiom; subtracting it leads to the correct value
   - what if we have $> 2$ events? Inclusion/Exclusion rule
   - $P(E_1 \cup E_2 \cup ... \cup E_n) =$

$$\sum_{i=1}^{n} P(E_i) - \sum_{i=1}^{n}\sum_{j=1}^{i-1} P(E_iE_j) + \sum_{i=1}^{n}\sum_{j=1}^{i-1}\sum_{k=1}^{j-1} P(E_iE_jE_k) + ... + (-1)^{n-1}P(E_1E_2...E_n)$$

## Conditional Probability

We may want to determine the probability of an event given another event is guaranteed to occur. This is denoted $P(E_1|E_2)$, which is read as $E_1$ given $E_2$. It essentially redefines the sample space to be $E_2$.

$$P(E_1|E_2) = \begin{cases} \frac{P(E_1E_2)}{P(E_2)} & P(E_2) > 0 \\ 0 & P(E_2) = 0 \end{cases} \tag{1.1}$$

From this equation, it follows that

$$P(E_1E_2) = P(E_1|E_2)P(E_2)$$

This holds in general for n events.

$$P(E_1E_2E_3) = P(E_1|E_2E_3)P(E_2E_3) = P(E_1|E_2E_3)P(E_2|E_3)P(E_3)$$

**Ex)** Applying conditions to operations

$$P(E_1 \cup E_2|E_3) = P(E_1|E_3) + P(E_2|E_3) - P(E_1E_2|E_3)$$

$$P(E_1E_2|E_3) = P(E_1|E_2|E_3)P(E_2|E_3), \text{ which follows from (1.1)}$$

## Independence

Two events are indpendent iff $P(E_1|E_2) = P(E_1)$

We have mutual independence if $P(E_1E_2...E_n) = P(E_1)P(E_2)...P(E_n)$.

## Theorem of Total Probability

Consider an event A and a set of of mutually exclusive and collectively exhaustive events $E_1, E_2, ..., E_3$.

$$P(A) = \sum_{i=1}^{n} P(A|E_i)P(E_i) \tag{1.2}$$

## Bayes' Rule

Consider an event A and a set of of mutually exclusive and collectively exhaustive events $E_1, E_2, ..., E_3$ in S.

$$P(AE_j) = P(E_j|A)P(A) = P(A|E_j)P(E_j)$$
$$P(E_j|A) = \frac{P(A|E_j)P(E_j)}{P(A)}$$
$$P(E_j|A) = \frac{P(A|E_j)P(E_j)}{\sum_{i=1}^{n} P(A|E_i)P(E_i)}$$