Practice Exam 2 Solutions

CS181: Fall 2021

1. $(1|01)^*(0|\epsilon)$. There may well be other clean ways to write this.

2. No. Consider the following simple example. Let M be an NFA consisting of a single state (the start state) which is also an accepting state, and no transitions. The language accepted by M is $\{\epsilon\}$ (only the empty string). The language accepted by the NFA M' is the empty language, $\emptyset$. Clearly, $\emptyset$ is not the compliment of $\{\epsilon\}$.

3. (a) The current state. (b) The head position. (c) The tape contents.

4. Given a TM M and a string $x \in \{0,1\}^*$, we write $M(x) =\perp$ to denote that M does not halt on input x. We define $\text{EVAL}\colon \{0,1\}^* \to \{0,1\}^* \cup \{\perp\}$ as $\text{EVAL}(\langle M \rangle, x) = M(x)$. The universality theorem states that there exists a TM U_{TM} which computes EVAL. That is, for all (M, x), we have $U_{TM}(\langle M \rangle, x) = M(x)$.

5. Use a machine that supposedly computes F to compute HALT.

6. 6.1: A function $F\colon \{0,1\}^* \to \{0,1\}$ is called *a semantic property* (or just *semantic*) if the following holds: for any two TMs M_1 and M_2, if $M_1(x) = M_2(x)$ for all $x \in \{0,1\}^*$, then $F(\langle M_1 \rangle) = F(\langle M_2 \rangle)$. Remark: In words, a function is semantic if its output is the same given encodings of TMs which compute the same language. That is, F depends on the the *function computed by the machine*, not on the encoding of the machine. **Rice's Theorem** states that all non-trivial[1] semantic properties are uncomputable.

   6.2: The second and the fourth functions are semantic.

7. 7.1 Add ϵ-transitions connecting all accept states of M_1 to the start state of M_2.

   7.2: Add a new state s and label it as the start. Connect s to the start states of M_1 and M_2 via ϵ-transitions.

8. Suppose for the sake of contradiction that L is regular. Thus, by the pumping lemma, there exists $p > 0$ such that the statement of the pumping lemma holds. Consider the string $x = 0^p10^p10^{p^2} \in L$. By PL, we can write $x = abc$ where $0 < |b| \leq |ab| \leq p$ and so $b = 0^q$ for some $0 < q \leq p$. Thus, $ab^2c = 0^{p+q}10^p10^{p^2}$. By PL, we have $ab^2c \in L$. However, clearly $(p+q) \cdot p \neq p^2$ and so $ab^2c \notin L$. This is a contradiction and so it follows that L is not regular.

9. Suppose for the sake of contradiction that L is regular. Thus, by the pumping lemma, there exists $p > 0$ such that the statement of the pumping lemma holds. Consider the string $x = 0^p1^p \in L$. By PL, we can write $x = abc$ where $|b| > 0$. Thus, $ac = 0^q1^p$ where $q < p$ and so clearly $ac \notin L$. However, by PL, we have $ac \in L$. This is a contradiction and so it follows that L is not regular.

[1] By non-trivial, we mean that F is not constant. I.e. F is not the all 1's function or the all 0's function.

10. We design a 2-tape TM. We begin with the first tape initialized as $\triangleright, x[0], x[1], \ldots, x[n-1], \emptyset, \emptyset \ldots$ and the second tape initialized as $\triangleright, \emptyset, \emptyset, \ldots$.

    (a) Copy the contents of tape 1 to tape 2. Move head 1 and head 2 left until we see $\triangleright$.

    (b) Move head 2 right 2 steps.

    (c) If head 2 sees a $\emptyset$, then halt and return the contents of tape 1 between $\triangleright$ and head 1.

    (d) Move head 1 right 1 step.

    (e) Repeat from step (b).

11. No, this program does not compute the toddler function. The first line of the program aims to compute $M(\langle M\rangle)$ by running the universal TM, $U_{TM}(\langle M\rangle, \langle M\rangle)$. However, if M does not halt on input $\langle M\rangle$, then U_{TM} does not halt on input $(\langle M\rangle, \langle M\rangle)$ and so line 1 of the infant program may run forever. A simple example of an input that causes this is the following. Simply let M be a TM which runs an infinite loop given *any input.* Observe that infant($\langle M\rangle$) does not output Toddler($\langle M\rangle$) and so the infant program does not compute the toddler function.