

Practice Exam 3

CS181: Fall 2021

1 Problem

Let G be a CFG with variables V , and terminals Σ . Let L be its language. Which of the following are true?

1. If a string x is in the language L , then exists a way to derive x from the start symbol in at most $2|x| + 1$ steps.

2. If G is in addition in Chomsky normal form, $x \in L$ and x is not the empty string, then there exists a derivation of x takes at most $2|x| - 1$ steps.

3. If G is in addition in Chomsky normal form, $x \in L$ and x is not the empty string, then every derivation of x takes at least $2|x| - 1$ steps.

4. Call G loop-free if there do not exist $\alpha, \beta \in (\Sigma \cup V)^*$ such that $\alpha \Rightarrow_* \beta$ and $\beta \Rightarrow_* \alpha$. If G is in CNF, then G is loop-free.

2 Problem

In the following, let $F, G : \{0, 1\}^* \rightarrow \{0, 1\}$ denote context-free functions with corresponding languages L, L' respectively. Which of the following are true?

1. The language $\{xy : x \in L, y^{Reverse} \in L\}$ is a context-free language.

2. The function $H(x) = F(x) + G(x) \bmod 2$ is context-free.

3. The function $H(x) = F(x) \wedge G(x)$ is computable.

4. The function $H(x) = 1$ if and only if x can be written as $x = x_1 \circ x_2 \circ \dots \circ x_t$ where for all $1 \leq i \leq t$, $x_i \in L$ or $x_i \in L'$. Then, H is context-free.

3 Problem

Let $F, G : \{0, 1\}^* \rightarrow \{0, 1\}$ be two functions and suppose that F reduces to G (in the sense defined in class: there exists a computable function \mathcal{R} such that $F(x) = G(\mathcal{R}(x))$ for all x).

True or false: If G is context-free, then F is context-free. Explain your answer in one or two sentences.

4 Problem

Write down the definitions of the following properties of a proof system V for a language L :

- ## 1. Effectiveness





- ## 2. Soundness




- ### 3. Completeness

5 Problem

Suppose we are given a TM M . The following question is about the reduction we saw in class for mapping the computation of M to a QIS.

1. Describe how we mapped a configuration to an integer in class.

2. Describe the arithmetic formula we designed in class for $LegalStep(m, n)$.

6 Problem

Give a CFG that generates the following language: $L = \{x; y : |x| \neq |y|, x \neq \varepsilon, y \neq \varepsilon, x, y \in \{0, 1\}^*\}$.

Gender	Percentage
Male	~85%
Female	~15%

7 Problem

Prove or disprove that the following language is context-free: $L = \{0^m 10^n 10^{mn} \mid m \geq 1, n \geq 1\}$.



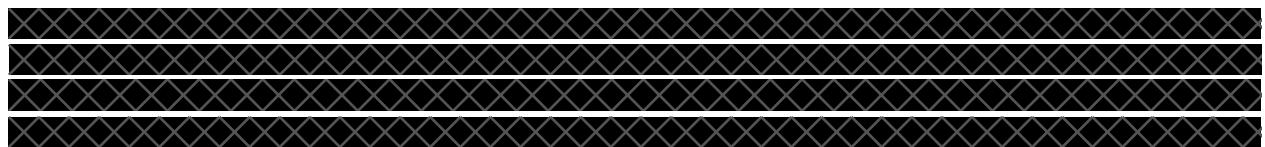
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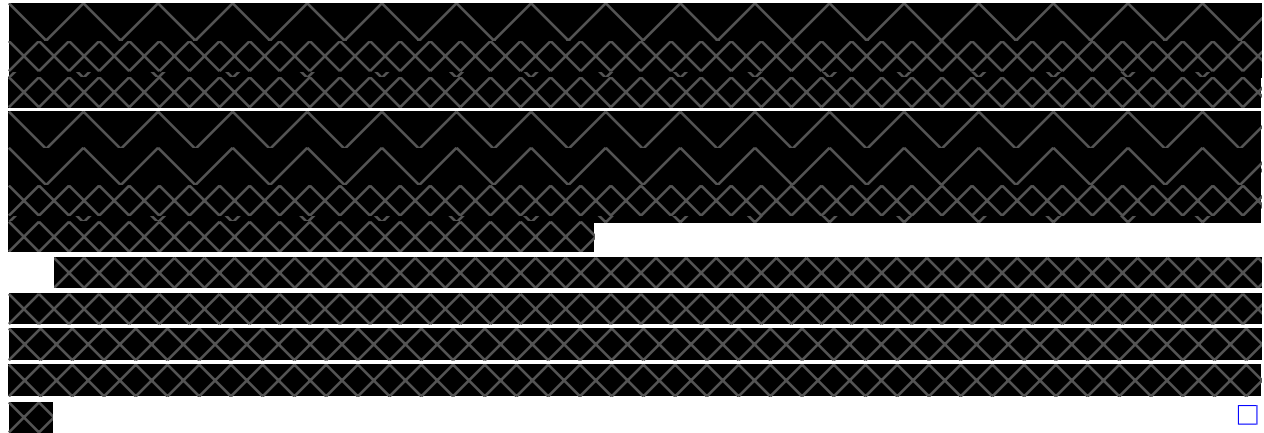
8 Problem

Given a TM M , call a state of M useless if the TM doesn't ever move to that state no matter the input. Call a TM M wasteful if M has a useless state.

Define $W : \{0, 1\}^* \rightarrow \{0, 1\}$ as the function that takes as input a TM M and $W(M) = 1$ if M is wasteful and 0 else. (We assume as in class that all binary strings represent TMs.)

Prove that W is uncomputable. You can use any result proved in class or in the homework problems but point out what you are using.





9 Problem

Let $HZ = \{M \in \{0,1\}^* : M \text{ halts on } 0\}$. (Here we view every binary string as corresponding to a TM just as we did in class.)

Prove that there exists an effective, sound, and complete proof system for HZ .



10 Problem

Consider the following Python program:

```
def Collatz(n):  
    if n==1: return 1  
    if n % 2 == 0:  
        return Collatz(n/2)  
    else:  
        return Collatz(3*n+1)
```

Write a QIS for the statement “Collatz(n) halts for every $n > 0$ ”.

