Exam 3. December 6, 2021

CS181: Fall 2021

Guidelines:

- The exam is closed book and closed notes. Do not open the exam until instructed to do so.
- Write your solutions clearly and when asked to do so, provide complete proofs. Unless explicitly asked not to use a specific result, you may use results and theorems from class without proofs or details as long as you state what you are using.
- I recommend taking a quick look at all the questions first and then deciding what order to tackle to them in. Even if you don't solve the problems fully, attempts that show some understanding of the questions and relevant topics will get partial credit.
- You can use extra sheets for scratch work, but you can only use the white space (it should be more than enough) on the exam sheets for your final solutions. The exams will be scanned into gradescope so for each problem, only the corresponding white space will be used for grading.
- Most importantly, make sure you adhere to the policies for academic honesty set out on the course webpage. The policies will be enforced strictly and any cheating reported with the score automatically becoming zero.
- Write clearly and legibly. All the best!

Problem	Points	Maximum
1		5
2		3
3		3
4		3
5		3
6		3
7		3
8		3
Total		26

Name	
UID	

1 Problem

The answers to the following should fit in the white space below the question.

1. Which of the following statements are true? No need for explanations. [2 points]

   (a) Define the function $F : \{0,1\}^* \to \{0,1\}$ as $F(M) = 1$ if and only if there are no smaller-size Turing machines equivalent to it. The function F is semantic.

   [redacted] $\square$

   (b) The language $L \subseteq \{0,1\}^*$ defined below is computable:

   $$L = \{\langle M\rangle : \langle M\rangle \text{ has length at most 100 and encodes a TM that halts on zero}\}.$$

   [redacted] $\square$

   (c) If we have a reduction from a function F to HALTONZERO then F is uncomputable.

   [redacted] $\square$

   (d) If we have reductions from a function F to G and from a function G to H, then we also have a reduction from F to H.

   [redacted] $\square$

2. Write a QIS that is equivalent to the *Goldbach Conjecture*, i.e., the statement "Every even number greater than two is the sum of two primes." [1.5 point]

   (You can assume that you have access to a formula $Prime(p)$ that returns true if and only if p is a prime as in the homework.)

   [redacted]

   [redacted]

   [redacted] □

3. Call a TM M *egotist* if on any input it just prints its own encoding three times (i.e., ignores inputs and just talks about itself ...). That is for any input w, $M(w) = \langle M \rangle \langle M \rangle \langle M \rangle$. Show that there exists an egotist Turing machine. [1.5 points]

   [redacted] □

   [redacted]

   [redacted]

   [redacted]

   [redacted]

   [redacted] □

2 Problem

Call a TM M unbounded if there are an infinite number of inputs x such that $M(x) = 1$.

Consider the function $F : \{0,1\}^* \to \{0,1\}$ defined as $F(M) = 1$ if and only if M is an unbounded TM.

Prove that F is uncomputable. **Do not** use Rice's theorem for this problem but give a reduction from any of the uncomputable problems we discussed in class. [3 points]

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3 Problem

Call a function $F : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$ that takes two Turing machines as input *doubly-semantic* if $F(A,B) = F(A',B')$ whenever TM A is equivalent to A' and TM B is equivalent to B'. Call a doubly-semantic function non-trivial if it is not the constant *zero* or constant *one* function.

Prove that every non-trivial doubly-semantic function is uncomputable. [3 points]

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4 Problem

Show that the language $L = \{x : x \text{ contains more 1's than 0's}\}$ is context-free. [3 points]

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[redacted]

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5 Problem

For a string $x \in \{0,1\}^*$, let $Reverse(x)$ denote the string backwards (e.g., $Reverse(000110) = 011000$) and let $Complement(x)$ denote the string with all bits flipped (e.g., $Complement(000110) = 111001$).

Show that the language $L = \{u \circ Reverse(Complement(u)) : u \in \{0,1\}^*\}$ is context-free. [3 points]

As examples, $001011, 10110010, 111000$ are in L as defined above (you must have u followed by the string obtained by reversing the complement of u), whereas $00101, 11100$ are not.

[redacted]

[redacted]

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6 Problem

Show that the language $L = \{10^n 10^{2^n} : n > 0\}$ is not context-free. For instance $10100, 10010000, 1000100000000$ are in L.

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7 Problem

Recall that we call a Turing machine M *not-empty* if and only if there is some input string w such that $M(w) = 1$.

Consider the language $L = \{x \in \{0,1\}^* : x \text{ is the encoding of a Turing machine that is not-empty}\}$. Prove that there is a effective, sound, complete verifier for L. [3 points]

For full-credit you have to describe the verifier (in any high-level programming language or pseudo-code) and write a sentence or two explaining why your verifier satisfies the three properties for L.

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8 Problem

Consider a TM $F : \{0,1\}^* \to \{0,1\}^*$ that always halts which takes an encoding of a TM M as input and produces another encoding TM as some output (i.e., F takes one program as input and then returns another program as output after doing some processing).

Prove that for any such TM manipulator as above, there exists a *fixed point*, that is, there is a TM M such that the TM $F(M)$ (the TM whose encoding is given by the output of F on M) is equivalent to M. [3 points]

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[redacted] $\square$