Practice Exam 3

CS181: Fall 2021

1 Problem

Let G be a CFG with variables V, and terminals Σ. Let L be its language. Which of the following are true?

1. If a string x is in the language L, then exists a way to derive x from the start symbol in at most $2|x|+1$ steps.

   [redacted] ☐

2. If G is in addition in Chomsky normal form, $x \in L$ and x is not the empty string, then there exists a derivation of x takes at most $2|x|-1$ steps.

   [redacted] ☐

3. If G is in addition in Chomsky normal form, $x \in L$ and x is not the empty string, then every derivation of x takes at least $2|x|-1$ steps.

   [redacted] ☐

4. Call G loop-free if there do not exist $\alpha, \beta \in (\Sigma \cup V)^*$ such that $\alpha \Rightarrow_* \beta$ and $\beta \Rightarrow_* \alpha$. If G is in CNF, then G is loop-free.

   [redacted] ☐

2 Problem

In the following, let $F, G : \{0,1\}^* \to \{0,1\}$ denote context-free functions with corresponding languages L, L' respectively. Which of the following are true?

1. The language $\{xy : x \in L, y^{Reverse} \in L\}$ is a context-free language.

   [redacted] ☐

2. The function $H(x) = F(x) + G(x) \mod 2$ is context-free.

   [redacted] ☐

3. The function $H(x) = F(x) \wedge G(x)$ is computable.

   [redacted] ☐

4. The function $H(x) = 1$ if and only if x can be written as $x = x_1 \circ x_2 \circ \ldots \circ x_t$ where for all $1 \leq i \leq t$, $x_i \in L$ or $x_i \in L'$. Then, H is context-free.

   [redacted] ☐

3 Problem

Let $F, G : \{0,1\}^* \to \{0,1\}$ be two functions and suppose that F reduces to G (in the sense defined in class: there exists a computable function $\mathcal{R}$ such that $F(x) = G(\mathcal{R}(x))$ for all x).

True or false: If G is context-free, then F is context-free. Explain your answer in one or two sentences.

[redacted] ☐

4 Problem

Write down the definitions of the following properties of a proof system V for a language L:

1. Effectiveness

   [redacted]

2. Soundness

   [redacted]

3. Completeness

   [redacted]

5 Problem

Suppose we are given a TM M. The following question is about the reduction we saw in class for mapping the computation of M to a QIS.

1. Describe how we mapped a configuration to an integer in class.

   [redacted]

2. Describe the arithmetic formula we designed in class for $LegalStep(m, n)$.

   [redacted]

6 Problem

Give a CFG that generates the following language: $L = \{x; y : |x| \neq |y|, x \neq \varepsilon, y \neq \varepsilon, x, y \in \{0, 1\}^*\}$.

[redacted]

[redacted]

[redacted]

7 Problem

Prove or disprove that the following language is context-free: $L = \{0^m 1 0^n 1 0^{mn} | m \geq 1, n \geq 1\}$.

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

□

8 Problem

Given a TM M, call a state of M useless if the TM doesn't ever move to that state no matter the input. Call a TM M wasteful if M has a useless state.

Define $W : \{0,1\}^* \to \{0,1\}$ as the function that takes as input a TM M and $W(M) = 1$ if M is wasteful and 0 else. (We assume as in class that all binary strings represent TMs.)

Prove that W is uncomputable. You can use any result proved in class or in the homework problems but point out what you are using.

[redacted]

[redacted] □

9 Problem

Let $HZ = \{M \in \{0,1\}^* : \ M \text{ halts on } 0\}$. (Here we view every binary string as corresponding to a TM just as we did in class.)

Prove that there exists an effective, sound, and complete proof system for HZ.

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted] □

10 Problem

Consider the following Python program:

```
def Collatz(n):
    if n==1: return 1
    if n % 2 == 0:
            return Collatz(n/2)
    else:
            return Collatz(3*n+1)
```

Write a QIS for the statement "Collatz(n) halts for every $n > 0$".

[redacted] □