Review of Probability Distributions

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Pairing and Ordering of Objects

Unique pairing of objects

- given n objects, how many possible ways of selecting unique pairs, without caring about ordering ?
  - let's consider a matrix $n \times n$
  - every element in the matrix, except the leading diagonal, is a paring
  - since the two parts on each side of the diagonal are identical (order does not count), we have

$$n_{pairs} = (n^2 - n)/2 = n(n-1)/2$$

Unique ordering of objects

- given n objects, how many possible ways of ordering them ?
  - we have n options to select the first element
  - $n-1$ for the second, $n-2$ for the third, ...
  - therefore it is

$$n(n-1)(n-2)\dots 2 \cdot 1 = n!$$

Combinations and Permutations

- in the english language the word *"combination"* is used loosely, without specifying if the order of the object is relevant
- examples:
  - when buying an ice cream, we select a *combination* of mint, chocolate and stracciatella. We do not care about the order of the three flavours on the cone
  - the *combination* of my bike locker is 4-3-6-9. In this case, the order of the numbers really matters!
- when we select k elements from a set of n objects
  - if the order of selection is NOT important, we have a combination
  - but if the order matters, we have a permutation

➜ a permutation is and ordered combination

Permutations - ORDER MATTERS

- there are two types of permutations

Repetition IS allowed

- given n objects, how many sequences of r elements ($r \leq n$) can be built ?

  Example: given n letters, how many words of r characters can be built with those letters ?
  - each object (character) has n different possibilities, therefore it is

$$n^r$$

Repetition is NOT allowed

- given n objects, we select r elements ($r \leq n$) from the set
- how many unique selections are possible ?
  - there are n ways to select the first, $n-1$ for the second, and $n-r+1$ for the r-th
  - we get:

$$n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!} = {}^{n}P_r$$

- this is called permutations, ${}^{n}P_r$. Note that ${}^{n}P_n = n!$

Combinations - ORDER IS NOT IMPORTANT

- there are two types of combinations

Repetition is NOT allowed

- we now select r objects, has in the previous case, but we are not concerned about the order
- the number of ways of selecting r object from a set of n without regard to the order of selection is called combinations, ${}^{n}C_r$

$$ {}^{n}C_r = \frac{{}^{n}P_r}{n!} = \binom{n}{r} = \frac{n!}{r!\,(n-r)!} $$

- this is the binomial coefficient, also called *n choose r*

Repetition IS allowed

- finally, the number of ways of choosing r objects from a set of n with replacement and without caring about the order is

$$ \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!} $$

- this is sometimes called *n multichoose r*

Application: the Birthday Paradox

The Problem

- in a large room, full of people, how many of them do you have to ask before there is a 50% chance that any of two, ore more, share a common birthday ?

- assuming $n = 365$ birthday/year and equally probable, we consider r people and we combine them so that they do not share a common birthday

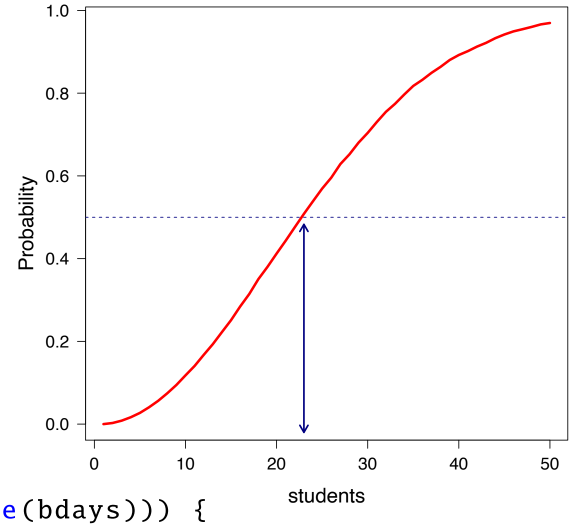
$$ A = n\,(n-1)\;\ldots\;(n-r+1) = \frac{n!}{(n-r)!} $$

- the way of assigning n birthday to r people is $B = n^r$
- the probability of no common birthday is A/B
- therefore the probability of at least one birthday is

$$ P(\text{birthday} \geq 1) = 1 - \frac{A}{B} = 1 - \frac{n!}{(n-r)!}\,\frac{1}{n^r} $$

Computation of the birthday problem

```
First element with prob>0.5:  23
```



R code

```
n_people_tot <- 50
pbday <- rep(0, n_people_tot)
for (k in 2:n_people_tot) {
  n_tests = 1E5; cb <- 0
  for (i in 1:n_tests) {
    bdays <- sample(1:365, k ,
                    replace=TRUE)
        if (length(bdays) > length(unique(bdays))) {
            cb = cb + 1
        }
    }
    pbday[k] <- cb/n_tests
    message(paste("k:", k, "pb(",k,"):",pbday[k]))
}
pfunc <- function(f, b) function(a) f(a,b)
p50_index <- Position(pfunc(`>`, 0.5), pbday)

message(paste("First␣element␣with␣prob>0.5:", p50_index))
```


Probability Distributions

- two basic types: discrete distributions and continuous distributions
- discrete distribution : finite or countable set of possible outcomes of the random variable
- continuous distribution : a random variable can have outcomes in an interval of the real line
- probability densities are a way to specify probability distributions
- the cumulative distribution function (CDF) is defined by

$$F(x) = P(X \leq x)$$

- for discrete distributions:

$$F(x_j) = P(X \leq x_j) = \sum_j p_j$$

- while for continuous distributions:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du$$

Probability Distributions

- with distribution functions, we compute the probability for intervals, $(c, d]$ as

$$P(c < X \le d) = P(X \le d) - P(X \le c) = F(d) - F(c)$$

- the expectation, or expected value reflects the location of a distribution

$$E[X] = \sum_i x_i p(x_i) \qquad E[X] = \int_{-\infty}^{+\infty} x\, f(x)\, dx$$

- the variance reflects the dispersion of the distribution:

$$var(X) = E[X - E[X]]^2 = E[X^2] - (E[X])^2$$

- properties:

$$\begin{array}{ll} E[a + bX] = a + bE[X] & var(a + bX) = b^2 var(X) \\ E[X + Y] = E[X] + E[Y] & var(X + Y) = var(X) + var(Y) + 2cov(X, Y) \end{array}$$

- with the covariance of the two variables

$$cov(X, Y) = E[(X - E[X])(Y - E[Y]] = E[XY] - E[X]E[Y]$$

Moments of a distribution

- they are analogous to the center-of-mass and to the momentum of inertia

Algebraic Moments

- the moment of order k about the origin is

$$\mu'_k \equiv E[x^k] = \int x^k f(x) dx \quad \text{and} \quad \sum_j x_j^k p_j$$

Central Moments

- the moment of order k about the mean are

$$\mu'_k \equiv E[(x - \mu)^k] = \int (x - \mu)^k f(x) dx \quad \text{and} \quad \sum_j (x_j - \mu)^k p_j$$

$$\begin{array}{ll} \mu'_0 = 1 & \mu_0 = 1 \\ \mu'_1 = \mu & \mu_1 = 0 \\ \mu'_2 = \mu + \sigma^2 & \mu_2 = \sigma^2 \end{array}$$

- the higher order moments become interesting only for studying the behavior of $f(x)$ for large $|x - \mu|$
- for a symmetric distribution, all odd central moments vanish ➜ non zero values are a possible measure of the skewness of a distribution

Probability Distributions in R

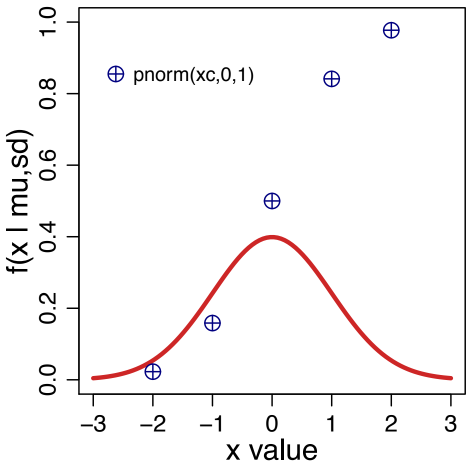
- all standard distributions available
- naming convention: a core name is associated with each distribution, and a prefix is appended to indicate the four basic associated functions:
  - `d` for the probability density function (pdf)
  - `p` for the cumulative density function (cdf)
  - `q` for the quantile function
  - `r` for the sampling from the distribution
- note that `pcore_name()` and `qcore_name()` are one the inverse of one another

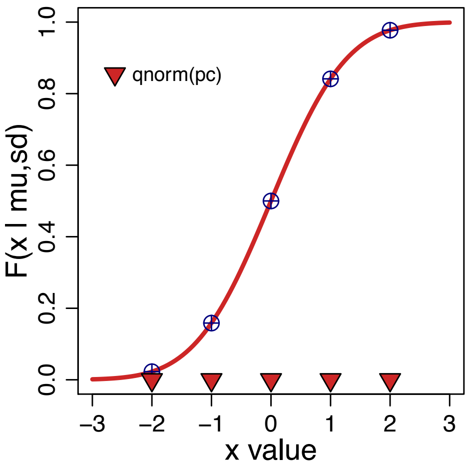
```
xc <- seq(-2,2,1)
pc <- pnorm(xc,0,1)
qc <- qnorm(pc)
```

xc: -2 -1 0 1 2

pc: 0.023 0.159 0.5 0.841 0.978

qc: -2 -1 0 1 2



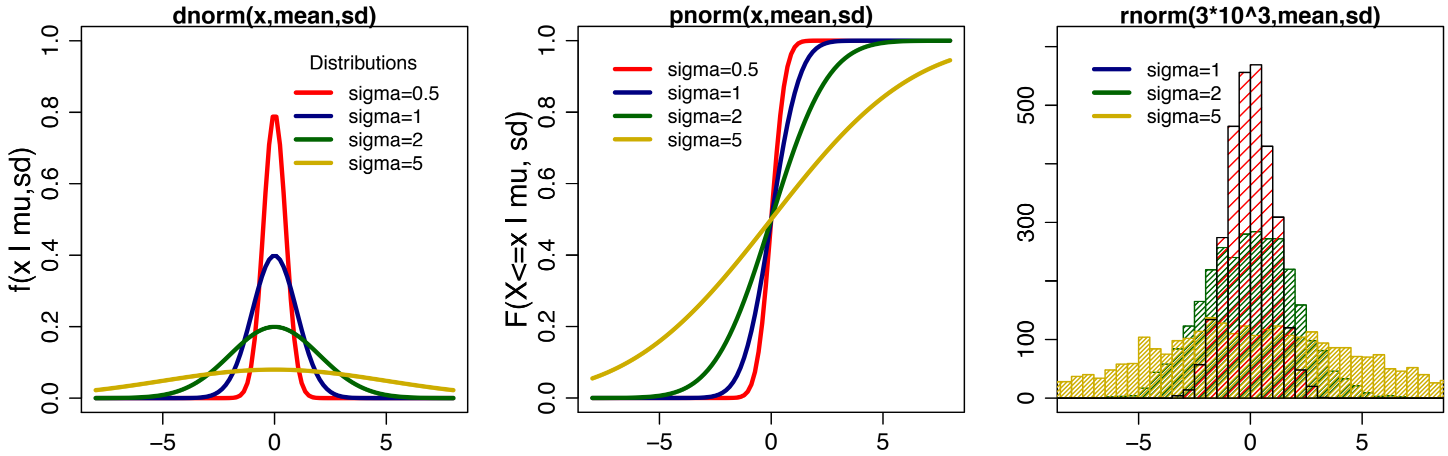


Standard Probability Distributions in R

Distribution	Core name	Parameters	Default values
Beta	`beta`	`shape1, shape2`	
Binomial	`binom`	`size, prob`	
Cauchy	`cauchy`	`location, scale`	0, 1
Chi-square	`chisq`	`df`	
Exponential	`exp`	`1/mean`	1
Fisher	`f`	`df1, df2`	
Gamma	`gamma`	`shape, 1/scale`	`NA`, 1
Geometric	`geom`	`prob`	
Hypergeometric	`hyper`	`m, n, k`	
Log-Normal	`lnorm`	`mean, sd`	0,1
Logistic	`logis`	`location, scale`	0,1
Normal	`norm`	`mean, sd`	0,1
Poisson	`pois`	`lambda`	
Student	`t`	`df`	
Uniform	`unif`	`min, max`	0,1
Weibull	`weibull`	`shape`	

Probability Distributions in R: normal distribution

- `dnorm(x, mean = 0, sd = 1)` gives a density of a normal distribution .i.e. the pdf
- `pnorm(q, mean = 0, sd = 1)` returns the distribution function, i.e. the cdf
- `rnorm(n, mean = 0, sd = 1)` generates random numbers from a normal distribution function
- `qnorm(p, mean = 0, sd = 1` is the quantile function



Standard Discrete Distributions

Bernoulli process

- it is a process with only two possible outcomes: success with probability p and failure with probability $1-p$ (also called q, since $q = 1-p$)
- if we call the two outcomes, 0 and 1, we can define $x \in [0,1]$, and

$$P(X=1) = p$$

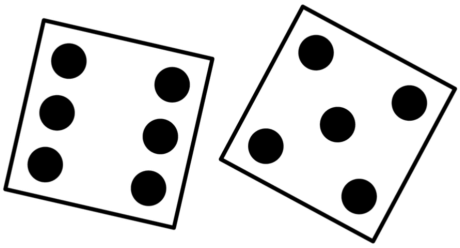
$$P(X=0) = 1-p = q$$

- the expected value and variance are

$$E[x] = p \quad \text{and} \quad Var(x) = p(1-p)$$

Examples

- the toss of a coin
- the draw of a die



Binomial distribution

- the sum of n independent Bernoulli trials, follows a Binomial distribution

$$Bn(x \,|\, p, n) = \binom{n}{x} p^x (1-p)^{n-x}$$

- it gives the probability of $\mathtt{x}$ successes in $\mathtt{n}$ independent Bernoulli trials
- the expected value and variance are

$$E[x] = np \quad \text{and} \quad Var(x) = np(1-p)$$

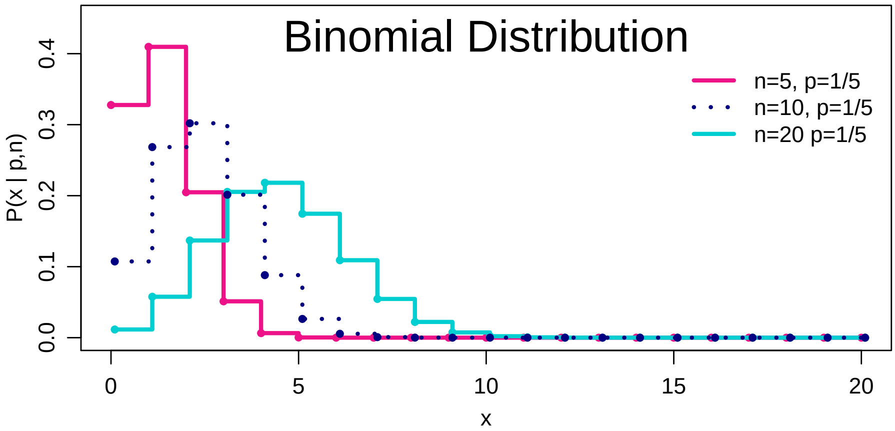
$$\sum_{j=0}^{n} \binom{n}{j} p^j \, (1-p)^{n-j} = (p+1-p)^n = 1$$

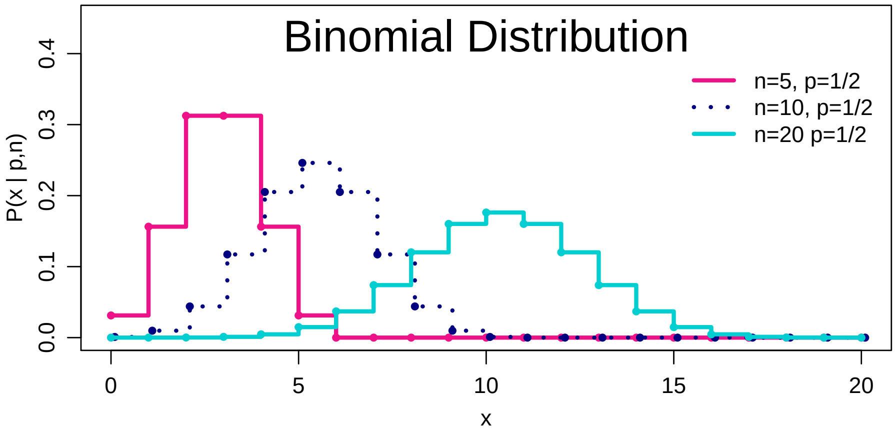
Examples

- multiple toss of a coin, or coins
- draw of dice
- drawing x red balls from an urn with n red and white balls (the fraction of red balls is p). Draws are done with replacement (➜ p remains constant)

Binomial distribution examples

- the distribution is symmetric when $p = 1/2$, and otherwise skew
- the distribution gets increasingly symmetric for higher values of n
- when n becomes large, it takes and approximate Gaussian shape





Binomial distribution - exercise

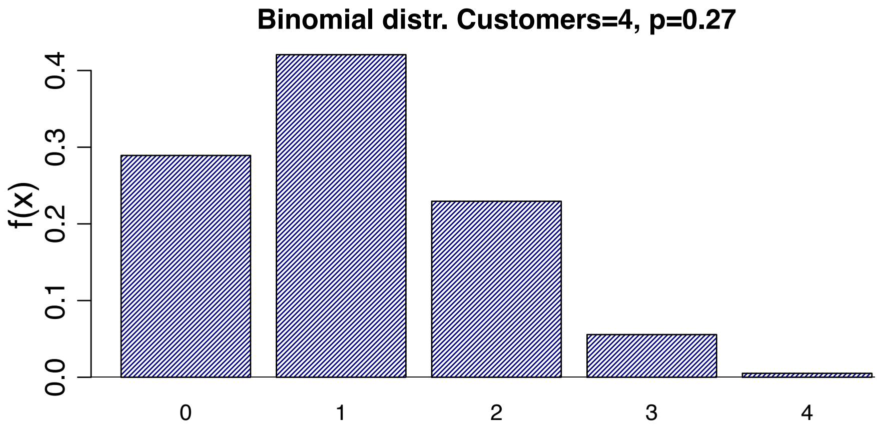
Problem

- in a restaurant 8 entrees of fish, 12 of beef and 10 of poultry are served
- what is the probability that 2 of the 4 next customers order fish entrees ?

Solution

```
cust <- 4
p <- 4/15
x <- 0:4
ap <- dbinom(x,cust,p)
barplot(ap, names=x, col='navy', xlab='x', ylab='f(x)', density=40,
        main = sprintf("Binomial distr. Customers=%d, p=%.2f",cust,p),
  cex.lab=1.5, cex.axis=1.25, cex.main=1.25, cex.sub=1.5)
cat(paste(c("P(2|np) = ", ap[3],'\n')))
```

```
P(2|np) = 0.229451851851852
```



Example: histogramming events

- we are interested in just the events contained in one bin of the histogram
  - A : we get the event of that particular bin (success)
  - $\overline{A}$: correspond to the events in any other bin (failure)
- the probability of having $x_\circ$ out of n events in the bin follows a Binomial distribution:

$$E[x] = np \quad \text{and} \quad Var(x) = np(1-p)$$

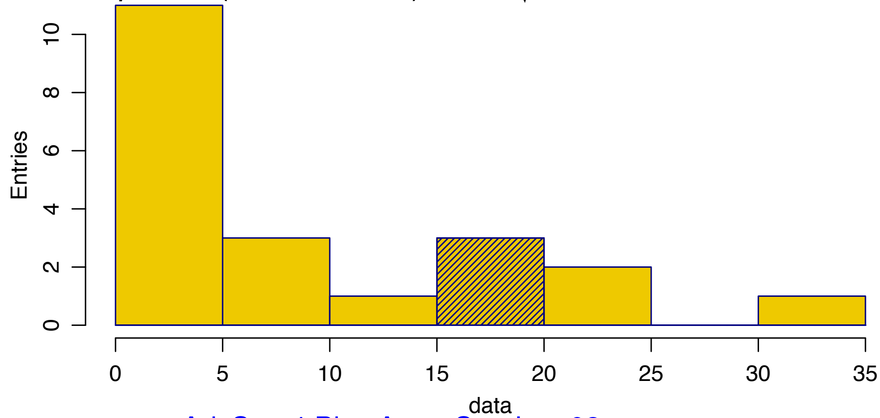
- p can be estimated as the ratio $p = x_\circ / n$:

$$E[x] = np = n\frac{x_\circ}{n} = x_\circ \quad \text{and} \quad Var(x) = x_\circ(1 - \frac{x_\circ}{n})$$

➜ the error on the number of the events is not $\sqrt{x_\circ}$, but a smaller quantity, $\sqrt{x_\circ(1-p)}$. Only in the limit $p \to 0$ (Poisson limit), $\sigma = \sqrt{x_\circ}$

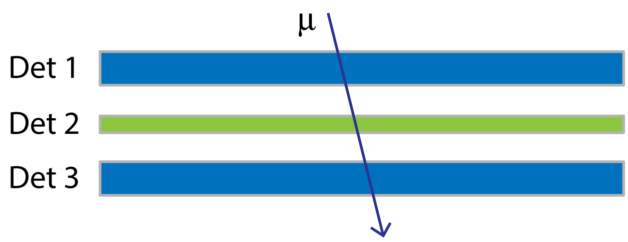
$\sqrt{3} = 1.73$

$\sqrt{3(1-3/21)} = 1.60$



Example: detection efficiency

- we want to compute the efficiency of a detector and evaluate the uncertainty on the measurement
- a muon-like signal has been registered by `Det1` and `Det3`
- what is the detection efficiency of our `Det2` ?
- detection is a Bernoulli process:



$$\epsilon_2 = \frac{N_{det2}}{N_{det1\&det3}} \quad \textit{with} \quad N_{det2} \subset N_{det1\&det3}$$

- since we are interested in a relative number of success in a trial,

$$E\left[\frac{r}{n}\right] = \frac{1}{n}E[r] = p \quad \text{and} \quad Var\left(\frac{r}{n}\right) = \frac{1}{n^2}V(r) = \frac{p(1-p)}{n} = \frac{pq}{n}$$

- in our case, p is the ratio of events detected with `Det2` with respect to those seen by both `Det1` and `Det3`
- therefore:

$$\sigma(\epsilon_2) = \sqrt{\frac{\epsilon_2(1-\epsilon_2)}{N_{det1\&det3}}}$$

The drunk-man and the home keys problem

The background information

- a man comes back home pretty drunk
- he has 8 keys and tries them randomly to unlock his door apartment
- after each trial he loses memory
- we watch him and bet on the attempt on which he will succeed
- $n_{try} = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$
- on which number would you bet ?

The problem

- E_j : the door gets unlocked in attempt j, with $j = 1, 2, \ldots$
- we know that: $P(E_j \big| \overline{\bigcup_{j<i} E_j}) = 1/8$

$$f(1) = P(E_1) = p = 1/8$$

$$f(2) = P(E_2 \cdot \overline{E}_1) = P(E_2 \big| \overline{E}_1) \cdot P(\overline{E}_1) = p \cdot (1-p)$$

$$f(3) = P(E_3 \cdot \overline{E}_2 \cdot \overline{E}_1) = P(E_3 \big| \cdot \overline{E}_2 \cdot \overline{E}_1) \cdot P(\overline{E}_2 \big| \cdot \overline{E}_1) \cdot P(\cdot \overline{E}_1) = p \cdot (1-p)^2$$

$$f(x) = p \cdot (1-p)^{x-1}$$

Geometric distribution

- our probabilities follow a geometric distribution with $p = 1/8$

$f(1) = p = 1/8 = 0.125$ ✓ our best bet!

$f(2) = p(1-p) = 1/8(7/8) = 0.109$

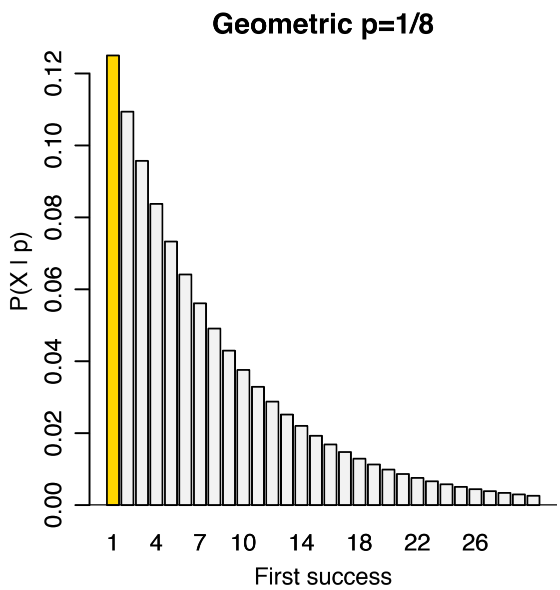
$f(3) = p(1-p)^2 = 0.096$

$f(4) = p(1-p)^3 = 0.084$

$\ldots$

- the geometric distribution gives the number of trials to get the first success

$$Geo(x|p) = p\,(1-p)^x$$



- the expected value and variance are

$$E[x] = \frac{1}{p} \quad \text{and} \quad Var(x) = \frac{1-p}{p^2}$$

- useful relations:

$$P(x \le r) = 1 - (1-p)^r = q^r \quad \text{and} \quad P(x > r) = q^r$$

Geometric distribution examples (1)

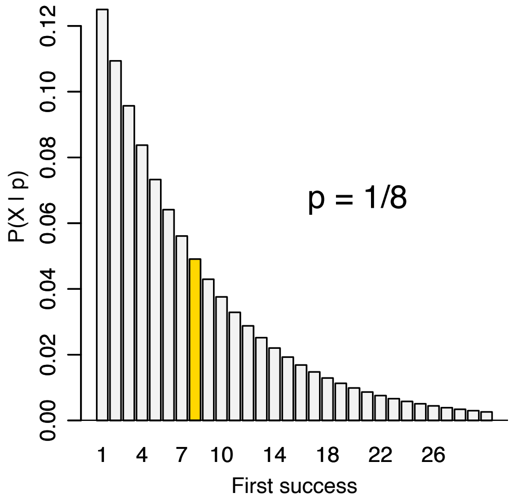
Drunk-man

- the first trial is the most probable
- but

$$E[X] = 1/p = 8$$

and

$$\sigma = \sqrt{(1-p)/p^2} = 7.5$$

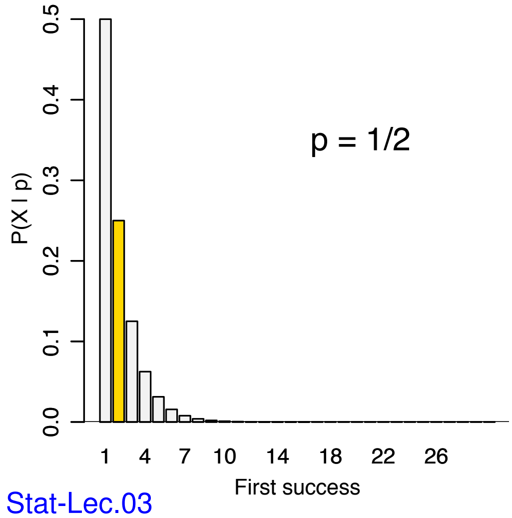


Coin tossing

- if we apply it to the tossing of one coin, we get
- $p_{max} = p = 1/2$
- and $E[X] = 1/p = 2$

  and

  $\sigma = \sqrt{(1-p)/p^2} = 1.4$



Geometric distribution examples (2)

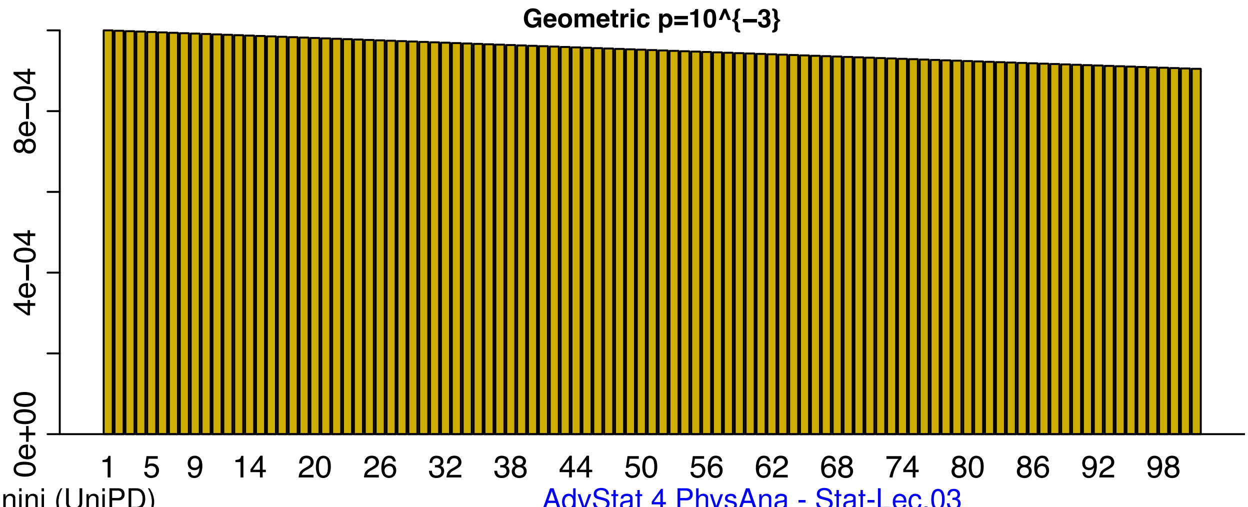
Rare Events

- let's decrease the probability of the event

$$E[X] = 1/p = 10^3$$

$$Var(X) = \frac{\sqrt{1-p}}{p} \xrightarrow[p\to 0]{} \frac{1}{p}$$

- rare moments might happen at any moment
  (even if they have a negligible probability to happen at any moment)



Multinomial distribution

- it is a generalization of the binomial distribution to the case with more than 2 possible outcomes
- labeling the disjoint outcomes $A_1, A_2, \ldots, A_r$, we define $P(A_j) = p_j$, with $1 \leq j \leq r$
- in n independent trials, x_j denotes the number of times that A_j occurs
- assuming, by construction, $n = x_1 + x_2 + \ldots + x_r$, we have

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_r = x_r | p_1, p_2, \ldots p_r, n) = \frac{n!}{x_1! x_2! \ldots x_r!} p_1^{x_1} p_2^{x_2} \ldots p_r^{x_r}$$

Properties

- the expectation for class A_j is $E[x_j] = np_j$
- the variance for class A_j is $Var(x_j) = np_j(1 - p_j)$
- the covariance for classes A_i, A_j is $cov(x_i, x_j) = -n\, p_i\, p_j$
- when n becomes large, the distribution tends to a multinormal distribution

Multinomial distribution - exercise

Problem

- in a certain town, at 20:00, 30% of the TV audience watches the news, 25% a TV show, and the rest other programs
- What is the probability that, selecting 7 random viewers, exactly 3 watch the news and at least 2 what the TV show ?

Solution

- the probabilities are $p_1 = 3/10$, $p_2 = 1/4$, $p_3 = 9/20$
- the sum of the trials $i + j + k = 7$
  - we write

$$P\left(i, j, k \middle| n = 7\right) = \frac{7!}{i!\ j!\ k!} \left(\frac{3}{10}\right)^i \left(\frac{1}{4}\right)^j \left(\frac{9}{20}\right)^k$$

  - and we compute

$$\begin{aligned} P(i = 3, j \geq 2 | n = 7) &= P(3, 2, 2|7) + P(3, 3, 1|7) + P(3, 4, 0|7) \\ &= \tfrac{7!}{3!\ 2!\ 2!} \left(\tfrac{3}{10}\right)^3 \left(\tfrac{1}{4}\right)^2 \left(\tfrac{9}{20}\right)^2 + \tfrac{7!}{3!\ 3!} \left(\tfrac{3}{10}\right)^3 \left(\tfrac{1}{4}\right)^3 \left(\tfrac{9}{20}\right) \\ &+ \tfrac{7!}{3!\ 4!} \left(\tfrac{3}{10}\right)^3 \left(\tfrac{1}{4}\right)^4 \simeq 0.103 \end{aligned}$$

Multinomial distribution marginalization

- let suppose we have a multinomial distribution $P(X_1, X_2, \ldots X_r)$ and we want to find the marginal probability $P(X_1)$

$$
\begin{aligned}
P(X_1) &= \sum_{x_2+x_3+\ldots+x_r=n-x_1} \frac{n!}{x_1!x_2!\ldots x_r!} p_1^{x_1} p_2^{x_2} \ldots p_r^{x_r} \\
&= \frac{n!}{x_1!(n-x_1)!} p_1^{x_1} \sum_{x_2+x_3+\ldots+x_r=n-x_1} \frac{(n-x_1)!}{x_2!\ldots x_r!} p_2^{x_2} \ldots p_r^{x_r} \\
&= \frac{n!}{x_1!(n-x_1)!} p_1^{x_1} (p_2+\ldots+p_r)^{n-x_1} \\
&= \frac{n!}{x_1!(n-x_1)!} p_1^{x_1} (1-p_1)^{n-x_1}
\end{aligned}
$$

- where the multinomial expansion has been used, and also the fact that $p_1 + p_2 + \ldots + p_r = 1$
- the obtained distribution coincides with the binomial distribution

Poisson process

- let's consider an event that might happen at a given time, with the following conditions:
  - the probability of 1 count in Δt is proportional to Δt itself, with Δt a 'small' value
  - calling r, the intensity of the process,

$$p = P('1 \text{ count in } \Delta t') = r\Delta t$$

- moreover:
  - $P(\geq 2 \text{ counts}) \lll P(1 \text{ count})$
  - what happens in one interval does not depend on other intervals ➜ it has a memory-less property

Examples

- accidents occurring at an intersection
- γ-s emitted from a radioactive substance
- customers entering a post office
- earthquakes in Italy

Poisson distribution

- the Poisson distribution can be derived by the Binomial distribution, in the limit where the rate of success, p, is very small
- we divide a finite time interval, T, in n small intervals:

$$T = n \, \Delta T$$

- and we consider the possible occurrence of an event, an independent Bernoulli trial, in each small interval Δt

$$p = r \, \Delta T = r \frac{T}{n}$$

- if the number of of trials is large, the total number of successes, np, is however considerable: $np = rT = \lambda$
- mathematically, in the limit $p \to 0$, $n \to \infty$ and $np = \lambda$ remaining constant, we get

$$\mathrm{Bn}\left(r \middle| n \, p\right) \to Poi\left(r \middle| \lambda\right)$$

- λ depends only on the intensity of the process, r, and on the finite time of observation

$$\mathrm{Poi}(r|\lambda) = \frac{\lambda^r}{r!} \exp(-\lambda)$$

Poisson distribution

- Given the Poisson distribution function:

$$\mathrm{Poi}\left(r \middle| \lambda\right) = \frac{\lambda^r}{r!} \exp(-\lambda)$$

- the expected value and variance are

$$E[x] = \lambda \quad \text{and} \quad Var(x) = \lambda$$

- Asymptotically, for growing λ values, the Poisson distribution becomes identical to the normal distribution

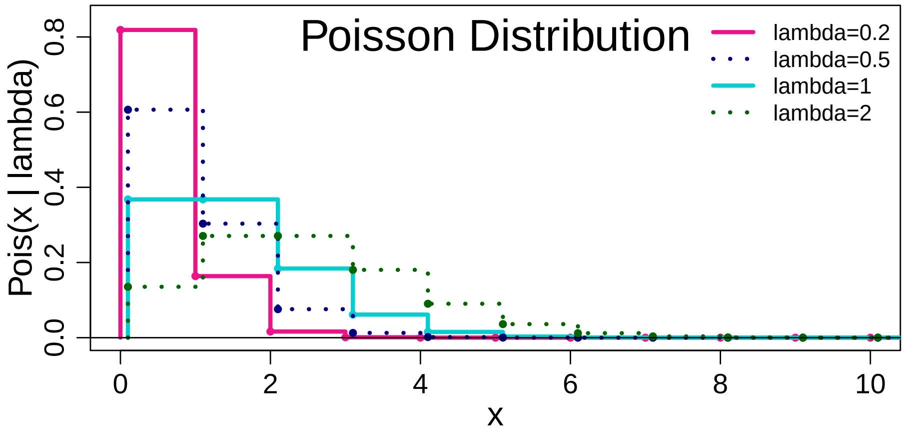
  the similarity is rather close already at $\lambda = 20$
- an interesting property is:

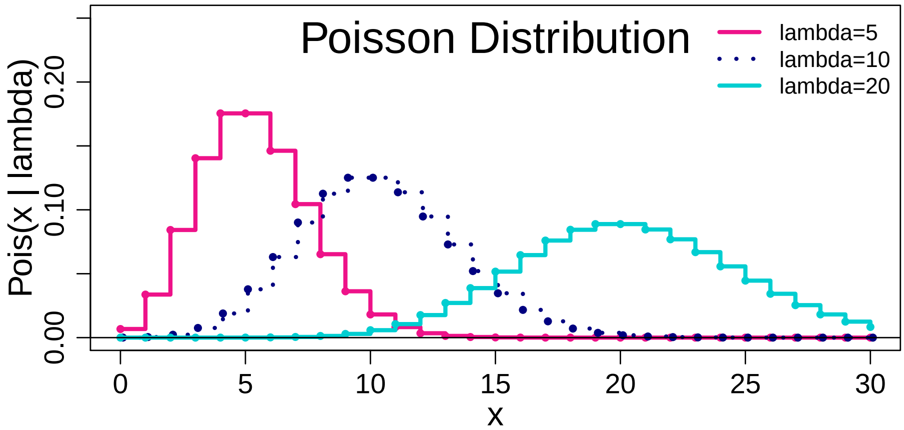
$$\mathrm{Poi}\left(r \middle| \lambda\right) = Poi\left(r - 1 \middle| \lambda\right) \frac{\lambda}{r}$$

- it is possible to demonstrate that the sum of any independent Poisson variables is itself a Poisson variable with mean value equal to the sum of the individual means

Poisson distribution examples

- the distribution is very asymmetric for small λ and it has a tail to the right of the mean
- the distribution gets increasingly symmetric for higher values of λ
- already for $\lambda = 20$ is very similar to the normal distribution (but it has only integer values)

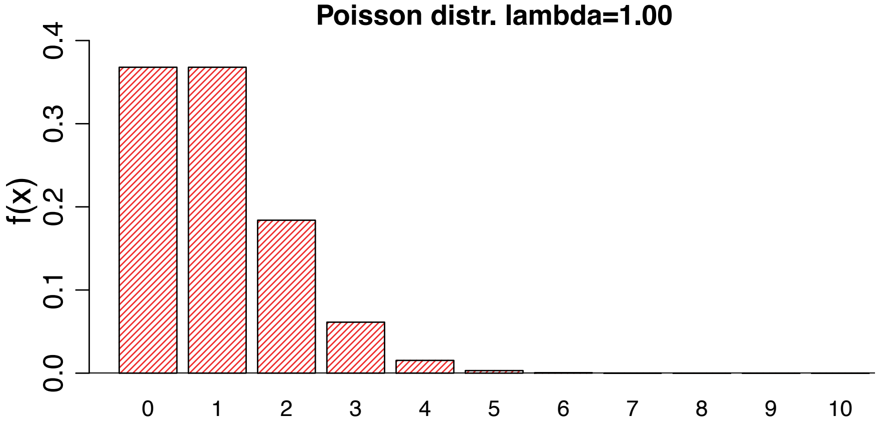




Poisson distribution - exercise 1

Problem

- the average number of received wrong phone calls per week is 7
- what is the probability to get, tomorrow, A) two wrong calls ? B) at least one wrong call ?



Solution

- assuming we get a large number of calls, the number of wrong calls follows, to a good approximation, a Poisson distribution
- we assume $\lambda = 1$

```
lambda <- 1
x <- 0:10
ap <- dpois(x,lambda)
barplot(ap, names=x, col='firebrick2', xlab='x', ylab='f(x)', density=30,
        main = sprintf("Poisson␣distr.␣lambda=%.2f",lambda),
        ylim=c(0,0.415),
        cex.lab=1.5, cex.axis=1.25, cex.main=1.25, cex.sub=1.5)
cat(paste(c("P(2|lambda)␣=␣", ap[3],'\n')))
cat(paste(c("P(>=1|lambda)␣=␣", 1 - ap[1],'\n')))
```

```
P(2|lambda) = 0.184
P(>=1|lambda) = 0.632
```

Poisson distribution - exercise 2

Problem

- a radioactive substance emits on average 3.9 α/s per gram
- compute the probability that, in the next second, the number of emitted alpha particles is

  A) at most 6

  B) at least 2

  C) at least 3 and at most 6

Solution

- every gram of element has n atoms
- From the information we have, $E[X] = np = \lambda = 3.9$

$$P(x|\lambda) = \frac{\lambda^x}{x!} \exp(-\lambda)$$

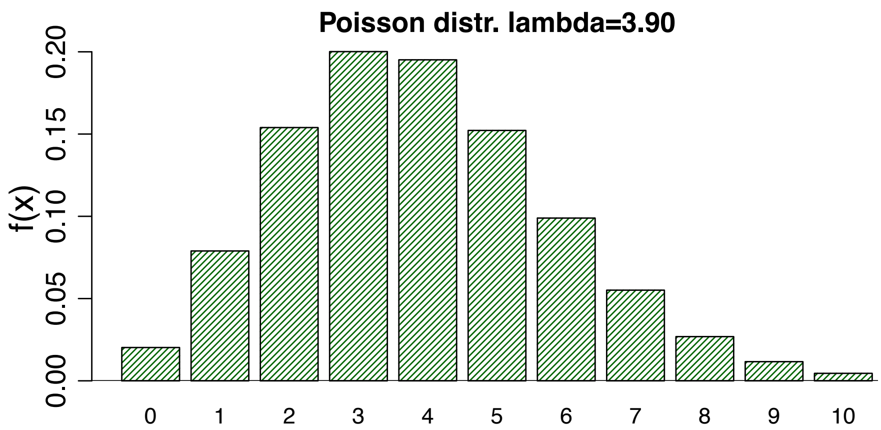
$$\text{A)} \quad P(x \le 6) = \sum_{x=0}^{6} \frac{3.9^x}{x!} \exp(-3.9)$$

$$\text{B)} \quad P(x \ge 2) = 1 - P(x \le 1) = 1 - \sum_{x=0}^{1} \frac{3.9^x}{x!} \exp(-3.9)$$

$$\text{C)} \quad P(3 \le x \le 6) = \sum_{x=3}^{6} \frac{3.9^x}{x!} \exp(-3.9)$$

Poisson distribution - exercise 2

```
P(<=6)  = 0.899483035093612
P(>=2)  = 0.900814633915558
P(2<X<=6)  = 0.646357932463829
```




```
lambda <- 3.9
x <- 0:10
ap <- dpois(x,lambda)

barplot(ap, names=x, col='darkgreen', xlab='x', ylab='f(x)', density=30,
        main = sprintf("Poisson distr. lambda=%.2f",lambda),
        ylim=c(0,0.21),
        cex.lab=1.5, cex.axis=1.25, cex.main=1.25, cex.sub=1.5)
abline(0,0)

P_6 = sum(ap[x<=6])
P_2 = 1 - sum(ap[x<=1])

cat(paste(c("P(<=6) = ", P_6,'\n')))
cat(paste(c("P(>=2) = ", P_2,'\n')))

pp <- ppois(x, lambda)
P_36 = pp[x==6] - pp[x==2]
cat(paste(c("P(2<X<=6) = ", P_36,'\n')))
```

Pascal or Negative Binomial distribution

- the probability of obtaining the r-th success in n trials, is given by the Negative Binomial, or Pascal, distribution
- since in $n-1$ trials we had $r-1$ successes, the probability is given by the Binomial distribution:

$$\mathrm{Bn}\left(r\middle|n,p\right) = \binom{n-1}{r-1} p^{r-1}(1-p)^{n-1-r+1} = \binom{n-1}{r-1} p^{r-1}(1-p)^{n-r}$$

- but we got the r-th success at the n-th trial, therefore

$$\mathrm{Bneg}\left(r\middle|n,p\right) = \binom{n-1}{r-1} p^{r}(1-p)^{n-r}$$

- the expected value and variance are

$$E[x] = \frac{r}{p} \quad \text{and} \quad Var(x) = \frac{r(1-p)}{p^2}$$

Pascal distribution - exercise

Problem

- Ann and Maggie are playing cards until one of them wins 5 games
- suppose all games are independent and the probability that Ann wins is 58%

  A) what is the probability that they complete in 7 games

  B) if the series ends in 7 games, what is the probability that Ann wins ?

Solution to A

- `X`: number of games played until Ann wins 5 games
- `Y`: number of games played until Maggie wins 5 games
- both `X` and `Y` follow a Pascal distribution

$$P(X=7, r=5) = \binom{6}{4} 0.58^5 0.42^2 = 0.174$$

$$P(Y=7, r=5) = \binom{6}{4} 0.42^5 0.58^2 = 0.066$$

- we get $P(X=7, r=5) + P(Y=7, r=5) = 0.24$

Pascal distribution - exercise

Solution to B

- A: Ann wins
- B: the series ends in 7 games

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(X=7)}{P(X=7)+P(Y=7)} = \frac{0.17}{0.24} = 0.71$$

Solution with R

```
dnbinom(x, size, prob, mu)
```

```
The negative binomial distribution with 'size' = n and 'prob' = p
...
for x = 0, 1, 2, ..., n > 0 and 0 < p <= 1.

This represents the number of failures which occur in a sequence
of Bernoulli trials before a target number of successes is
reached.  The mean is mu = n(1-p)/p and variance n(1-p)/p^2.

P_Ann    <- dnbinom(2,5,0.58) # 0.173672
P_Maggie <- dnbinom(2,5,0.42) # 0.0659468
```