Review of Probability Distributions - II

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The Exponential distribution

- Let $\{N(t), t \geq 0\}$ be a Poisson process
- $N(t)$ represents the number of events occurred at or prior to time t
- If T_1 is the time arrival of the 1^{st} event.
- T_j represents the elapsed time between the events T_j and T_{j-1}
- the ordered set $\{T_1, T_2, \ldots, T_n\}$ is a sequence of inter-arrival times of the Poisson process
- setting

$$P\left(N(t) = n\right) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

- we can evaluate the probability distribution function of the random variables T_j:

$$P\left(T_1 > t\right) = P\left(N(t) = 0\right) = e^{-\lambda t}$$

$$P\left(T_1 \leq t\right) = 1 - P\left(T_1 > t\right) = 1 - e^{-\lambda t}$$

- a Poisson process is stationary and possesses independent increments: at any time t the process probabilistically starts all over again

➜ the inter-arrival time of any two consecutive events has the same distribution as T_1

The Exponential distribution

- the cumulative distribution is therefore

$$F(t) = \begin{cases} 1 - e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- and the probability density function is

$$f(t) = \frac{dF(t)}{dt} = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- the expected value and variance are

$$E[x] = \frac{1}{\lambda} \quad \text{and} \quad Var(x) = \frac{1}{\lambda^2}$$

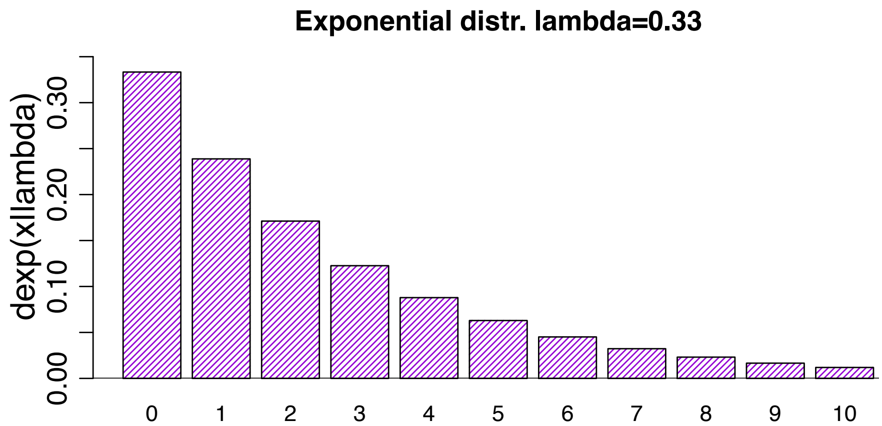
Examples

- the inter-arrival time between two customers in a shop
- the duration of my next telephone call
- the time between two accidents at an intersection
- time until the next baby is born in a hospital
- the time to failure of the next chip in a large group of such devices when all of them are initially fault free

Exponential distribution - exercise

Problem

- suppose that every three months, an earthquake of some entity happens in Italy
- what is the probability that the next earthquake happens after three but before seven months ?



Solution

- X: the time, in months, until the next earthquake
- X is exponential with $\lambda = 1/3$

```
P(3<X<7) = 0.270907473307037
```

$$P(3 < X < 7) = F(7) - F(3) = (1 - e^{7/3}) - (1 - e^{3/3})$$

```
lambda <- 1/3; x <- 0:10; ap <- dexp(x,lambda)

barplot(ap, names=x, col='darkviolet', xlab='x', ylab='dexp(x|lambda)',
        density=30,
        main = sprintf("Exponential␣distr.␣lambda=%.2f",lambda),
        ylim=c(0,0.375),
        cex.lab=1.5, cex.axis=1.25, cex.main=1.25, cex.sub=1.5)

 cat(paste(c("P(3<X<7)␣=␣", pexp(7,lambda) - pexp(3,lambda),'\n')))
```

The memory-less feature of the Exp distr

- a non-negative random variable X is memory-less if

$$P(X > s + t \mid X > t) = P(X > s) \qquad \forall s, t \geq 0$$

- since

$$P(X > s + t, X > t) = P(X > s + t \mid X > t) P(X > t)$$

$$\frac{P(X > s + t, X > t)}{P(X > t)} = P(X > s)$$

- and

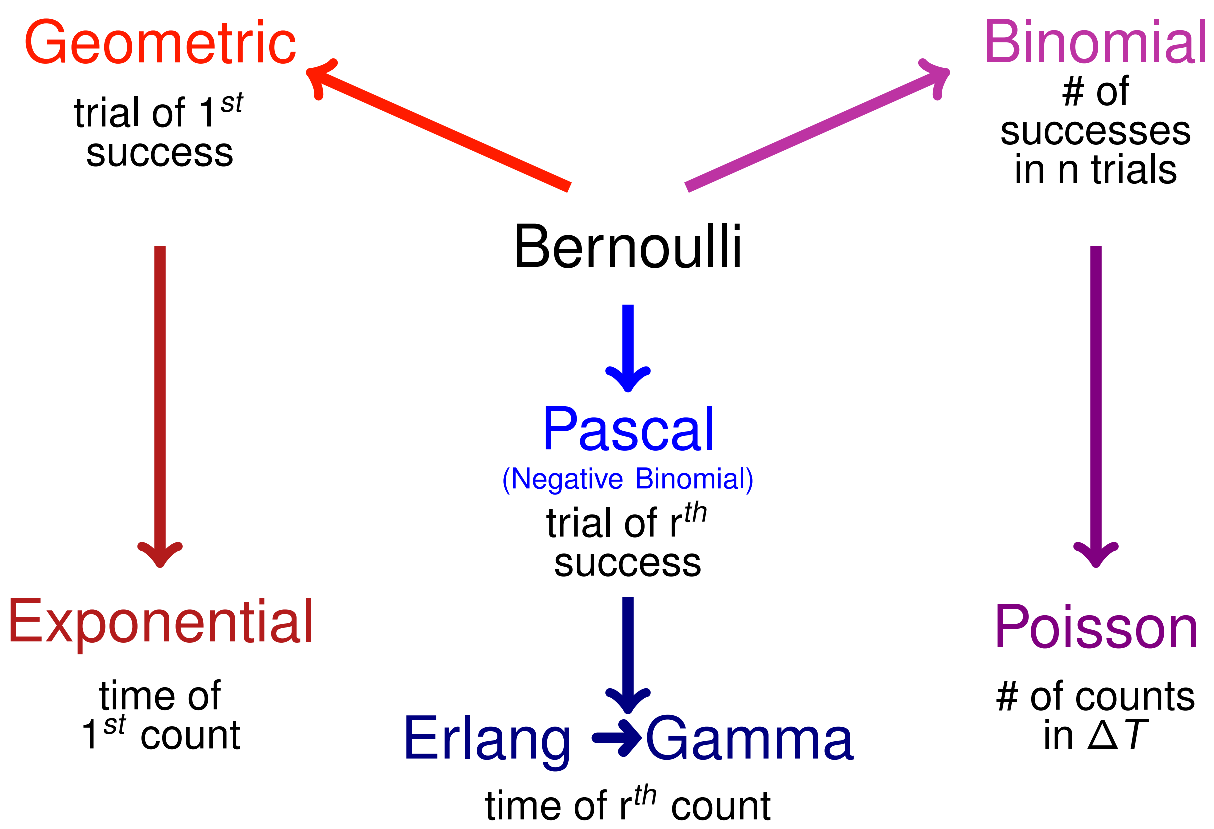
$$P(X > s + t) = P(X > s) \cdot P(X > t)$$

- since
$P(X > s + t) = 1 - (1 - \exp(-\lambda(s + t))) = \exp(-\lambda(s + t)) = \exp(-\lambda s) \exp(-\lambda t)$

- and

$P(X > s) = 1 - (1 - \exp(-\lambda s)) = \exp(-\lambda s))$
$P(X > t) = 1 - (1 - \exp(-\lambda t)) = \exp(-\lambda t)$

Summary of discrete probability distributions



Hypergeometric distribution

- suppose we have a box containing B black stones and $N-B$ white stones and we draw them, randomly, without replacement
- if the number of drawn items, n, does not exceed the number of black or white balls, i.e. $n \leq min(B, N-B)$
- and if X identifies the number of black stones extracted, its probability distribution follows the Hypergeometric distribution

$$P(x \mid N, B, n) = \frac{\binom{B}{x}\binom{N-B}{n-x}}{\binom{N}{n}} \quad \text{with } x = \{0, 1, 2, \ldots, n\}$$

- the expected value and variance are

$$E[x] = \frac{nB}{N} \quad \text{and} \quad Var(x) = \frac{nB(N-B)}{N^2}\left(1 - \frac{n-1}{N-1}\right)$$

- note that if sampling is done with replacement, X follows a binomial distribution with parameters n and B/N

$$E[x] = n\frac{B}{N} \quad \text{and} \quad Var(x) = n\frac{B}{N}\left(1 - \frac{B}{N}\right)$$

Standard Continuous Distributions

The Uniform Distribution

- a random variable $X \sim \mathcal{U}(a, b)$ follows a uniform distribution if the pdf is given by the following:

$$f(X) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & otherwise \end{cases}$$

- the cumulative density function is

$$F(X) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x > b \end{cases}$$

- and the expected value and variance are

$$E[X] = \frac{a+b}{2} \quad \text{and} \quad Var(x) = \frac{(b-a)^2}{12}$$

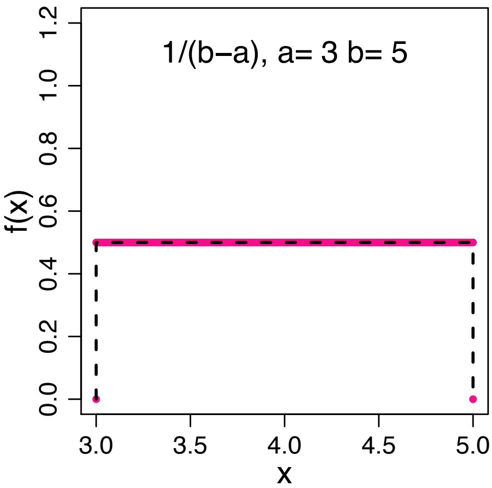
The Uniform distribution in R

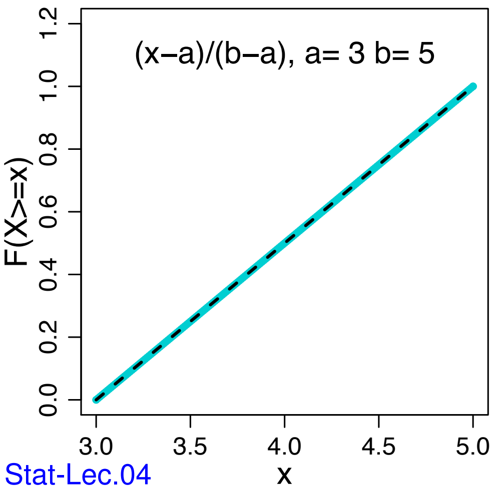
- we have four pre-defined functions:
  - `dunif(x, min=0, max=1)` returns the probability density function
  - `punif(q, min=0, max=1)` gives the cumulative distribution function
  - `qunif(p, min=0, max=1)` is the quantile returning function
  - `runif(n, min=0, max=1)` generate a vector with random values from a uniform distribution
- if not specified, the default interval is $(0, 1)$

```
x <- seq(3, 5, 0.01)
a <- min(x); b <- max(x)
xp <- c(a, x, b)

yp1 <- c(0, dunif(x, a, b), 0)
plot(xp, yp1)

yp2 <- c(0, punif(x, a, b), 1)
plot(xp, yp2)
```





Example: sum of two Uniform distributions

- let's suppose two random variables, x_1 and x_2 follow a uniform distribution, $x_j \sim \mathcal{U}(0,1)$
- let's compute the $y = x_1 + x_2$ distribution function

$$f(y) = \begin{cases} y & 0 \le y \le 1 \\ 2-y & 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

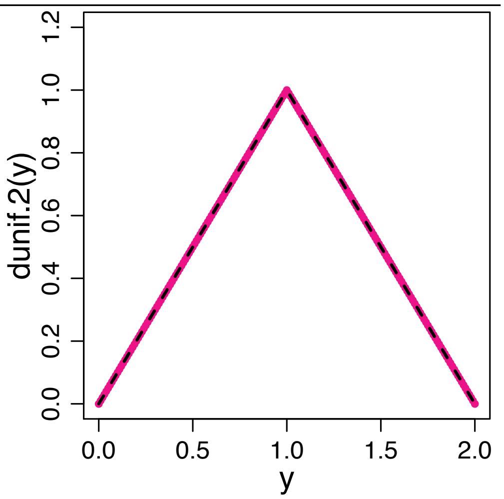
- integrating $f(y)$ in the domain we get

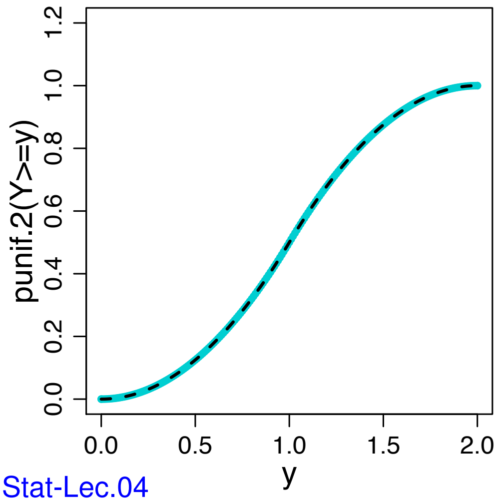
$$F(y) = \begin{cases} 0 & y < 0 \\ y^2/2 & 0 \le y \le 1 \\ -y^2/2 + 2y - 1 & 1 \le y \le 2 \\ 1 & y > 2 \end{cases}$$

- the expected value and variance are

$$E[X] = \int_0^1 y f(y) dy = 1 \;, \quad E[X^2] = \frac{7}{6}$$

$$Var(x) = \int_0^1 (y-1)^2 f(y) dy = \frac{1}{6}$$





Example: sampling from a user's pdf

- all cumulative distributions are monotone increasing functions in the interval $[0,1]$
- if the analytical form of $F(X)$ is known, it is also invertible:

$$F^{-1}(y) = \inf\{x : F(x) \ge y\} \quad u \in [0,1]$$

- there is a 1:1 correspondence between CDFs, since they have the same image

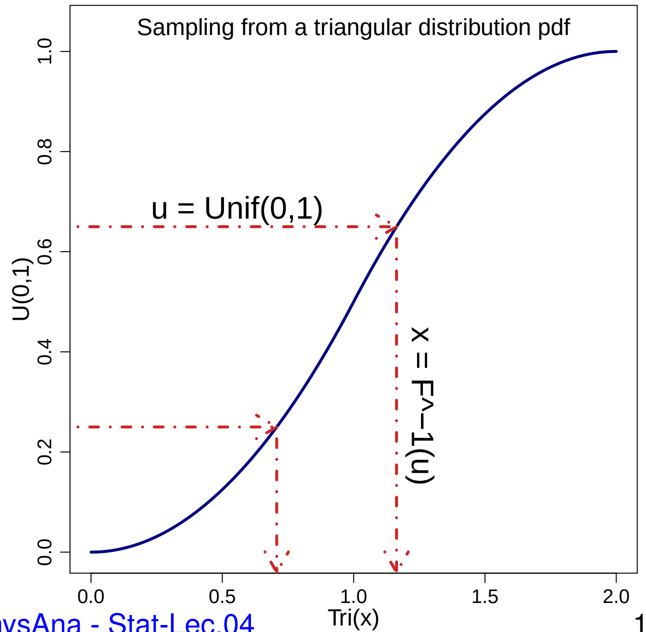
- given X and Y with CDFs $F(X)$ and $G(Y)$
- we ask for the same probability, and search for x_i and y_i such that

$$F(x_i) \equiv P(X \le x_i) = G(y_i) \equiv P(Y \le y_i)$$

- assuming

$$\begin{aligned} G(y) &= \mathcal{U}(0,1) = u \\ &\rightarrow F(x_i) = u \\ &\rightarrow x_i = F^{-1}(u) \end{aligned}$$

- this is called the inverse transform sampling method

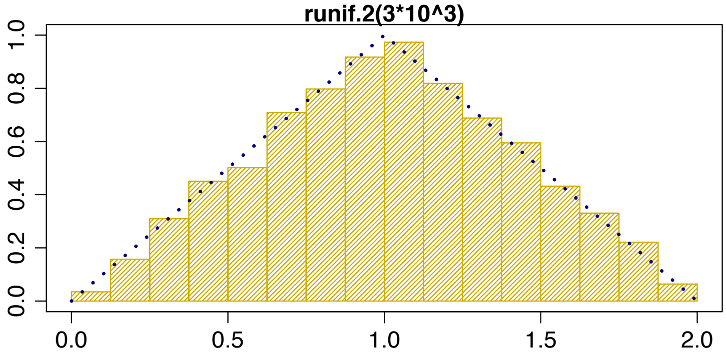


The Inverse Transform on the Triangular distribution

Algorithm

1) extract a sample from $\mathcal{U}(0,1)$
2) compute $F^{-1}(u) = x$
3) release x as sampled from our $F(x)$

- we define the new `rinf.2()` and `qinf.2()` functions




```r
dunif.2 <- function(x) {
  duni2 <- ifelse(x <= 1,
                  x,
                  2 - x)
  return (duni2)
}

runif.2 <- function(n) {
  us <- runif(n)
  runi2 <- ifelse(us <= 0.5,
                  sqrt(2 * us),
              2 - sqrt(2*(1 - us)))
  return (runi2)
}
```

```r
punif.2 <- function(x) {
  puni2 <- ifelse(x <= 1,
                  0.5 * x^2,
             (4*x - x^2 - 2)/2)
  return (puni2)
}

qunif.2 <- function(p) {
  quni2 <- ifelse(p<=0.5,
                  sqrt(2*p),
              2 - sqrt(2*(1 - p)))
  return (quni2)
}
```


Integrating a pdf with R

- we have computed the mean and variance values, analytically

$$E[X] = \int_0^1 y\, f(y)\, dy \quad \text{and} \quad E[X^2] = \int_0^1 y^2\, f(y)\, dy$$

- and now we ask R to do it for us

```
Evaluate the mean value and variance, by integration
The mean value of the distribution is: 1
and the variance: 0.166666666666667
```

```r
# Evaluate integral of pdf
# using a anonymous function
E.X.integral <- integrate(function(x) {x * dunif.2(x)},
                          lower=0, upper=2)

E.X <- E.X.integral$value

E.X2.integral <- integrate(function(x) {x^2 * dunif.2(x)},
                           lower=0, upper=2)

E.X2 <- E.X2.integral$value

Var.X <- E.X2 - E.X^2

cat(paste("The mean value of the distribution is: ", E.X, '\n'))
cat(paste("and the variance: ", Var.X, '\n'))
```

The `integrate()` R function

- an adaptive quadrature of functions of one variable over a finite or infinite interval

```
integrate(f, lower, upper, ...)
```

- `f()` is an R function taking a numeric first argument and returning a numeric vector of the same length.

```
x.integral <- integrate(function(x) {x*dunif.2(x)}, lower=0, upper=2)

> class(x.integral)
[1] "integrate"
> summary(x.integral)
             Length Class  Mode
value        1      -none- numeric
abs.error    1      -none- numeric
subdivisions 1      -none- numeric
message      1      -none- character
call         4      -none- call
> names(x.integral)
[1] "value"        "abs.error"    "subdivisions" "message"      "call"
```

```
> x.integral$value
[1] 1
> x.integral$abs.error
[1] 1.110223e-14
> x.integral$subdivisions
[1] 2
```

```
> x.integral$message
[1] "OK"
> x.integral$call
integrate(f = function(x) {
    x * dunif.2(x)
}, lower = 0, upper = 2)
```


Inequalities

- we will discuss three important inequalities:
- Markov's inequality
- Jensen's inequality
- Chebyshev's inequality

- they are very useful when we do not have enough information about the distribution of random variables
- but we can calculate their expected values and/or variances
- using the, bounds on probabilities can be derived

Markov's Inequality

- X is a non-negative random variable with $E[x] = \mu$
- for any $k > 0$

$$P(X \geq k) \leq \frac{\mu}{k}$$

Proof

- let's do it for a discrete random variable X, with pdf $p(x)$ over a set A
- let $B \subset A$, defined as $B = \{x \in A : x \geq k\}$

$$E[X] = \sum_{x \in A} x\, p(x) \geq \sum_{x \in B} x\, p(x)$$

$$\geq k \sum_{x \in B} p(x) = k\; P(X \geq k)$$

- in a similar way it can be demonstrated for continuous variables

Markov's Inequality application

Exercise

- a post office handles, on average, 10^4 letters per day

(a) what is the probability that, tomorrow, it will handle at least $1.5\ 10^4$ letters ?

(b) and less than $1.5\ 10^4$ letters ?

Solution

- the average value of handled letters is $E[X] = 10^4$
- from Markov's inequality

$$P(X \geq 1.5\ 10^4) \leq \frac{E[X]}{1.5\ 10^4} = \frac{2}{3}$$

- the second question is answered using the normalization of the probability

$$P(X < 1.5\ 10^4) = 1 - P(X \geq 1.5\ 10^4) = \frac{1}{3}$$

Jensen's Inequality

- the variance of a random variable is always a positive value

$$Var(X) = E[X^2] - (E[X])^2 \geq 0$$

- therefore the most basic moment inequality is

$$E[X^2] \geq (E[X])^2$$

Jensen's inequality

- let X be a random variable with finite mean $\mu = E[X]$
- let $g(x) : \mathbb{R} \longmapsto \mathbb{R}$, a convex function (i.e. $d^2g/dx^2 > 0$)

$$g\,(E[X]) \leq E\,[g\,(X)]$$

Example

- X, positive random variable with $E[X] = \mu$, finite
- we consider $g(x) = x^{-1}$, $x > 0$
- g is convex, since $g'' = 2 \cdot x^{-3} > 0$, $\forall x > 0$
- from Jensen's inequality:

$$E\left[\frac{1}{x}\right] \geq \frac{1}{E[X]} \iff E\left[\frac{1}{x}\right] \cdot \frac{1}{E[X]} \geq 0$$

Chebyshev's Inequality

- X is a non-negative random variable with $E[x] = \mu$ and $Var(x) = \sigma^2$
- for any $k > 0$

$$P\left(\left|X - \mu\right| \geq k\right) \leq \frac{\sigma^2}{k^2}$$

Proof

- we know that

$$(X - \mu)^2 \geq 0$$

- therefore, applying Markov's inequality

$$P\left(\,(X - \mu)^2 \geq k^2\right) \leq \frac{E\left[(X - \mu)^2\right]}{k^2} = \frac{\sigma^2}{k^2}$$

but since $(X - \mu)^2 \geq k^2 \;\rightarrow \left|X - \mu\right| \geq k$, Chebyshev's inequality follows

Chebyshev's Inequality implications

- if $k = r\sigma$

$$P\left(|X-\mu| \geq r\sigma\right) \leq \frac{\sigma^2}{r^2\sigma^2} = \frac{1}{r^2}$$

- meaning that the probability that X deviates from its expected value at least r standard deviations is less than $1/r^2$

- as an example
  $P(|X-\mu| \geq 2\sigma) \leq 1/4 = 25\%$
  $P(|X-\mu| \geq 4\sigma) \leq 1/16 = 6.25\%$
  $P(|X-\mu| \geq 10\sigma) \leq 1/100 = 1\%$

- since

$$1 - P\left(|X-\mu| < r\sigma\right) = P\left(|X-\mu| \geq r\sigma\right) \leq \frac{1}{r^2}$$

- it follows that

$$P\left(|X-\mu| < r\sigma\right) \geq 1 - \frac{1}{r^2}$$

Chebyshev's Inequality application

Exercise

- the same a post office handles, on average, 10^4 letters per day, with a variance of 2000 letters

(a) what is the probability it will handle between 8000 and 12000 letters, tomorrow ?

Solution

- we know that $E[X] = 10^4$ and $\sigma^2 = Var(X) = 2 \cdot 10^3$
- we need to evaluate

$$\begin{aligned} P(8\cdot 10^3 < X < 12\cdot 10^3) &= P(|X-10^4| < 2\cdot 10^3) \\ &= 1 - P(|X-10^4| \geq 2\cdot 10^3) \end{aligned}$$

- since $k\sigma = 2000 \ \rightarrow k = 2000/\sigma = 2000/\sqrt{2000}$
- therefore

$$P(|X-10^4| \geq 2\cdot 10^3) = 1 - P(8\cdot 10^3 < X < 12\cdot 10^3) \geq \frac{2\cdot 10^3}{(2\cdot 10^3)^2} = 5\cdot 10^{-4}$$

- and

$$P(8\cdot 10^3 < X < 12\cdot 10^3) \geq 1 - 5\cdot 10^{-4} = 0.9995$$

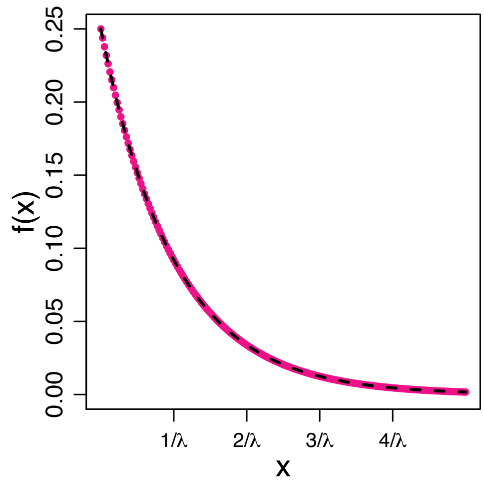
Exponential Random Variables

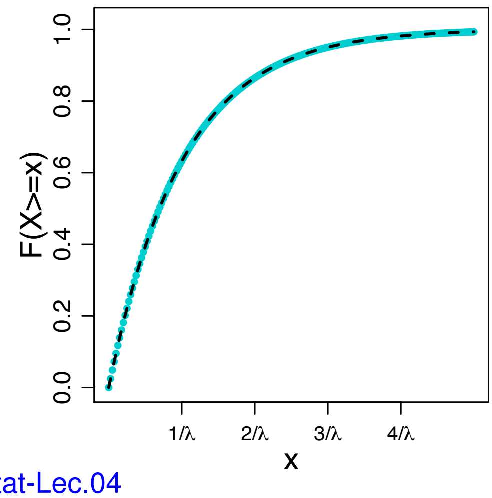
- consider a Poisson process $\{N(t) : t \geq 0\}$ where $N(t)$ represents the number of events that happened at or before time t:
  - T_1 is the time of the first event
  - T_2 is the time between the first and the second event
  - T_j is the time between events $j-1$ and j
- this sequence, also called inter-arrival times follows an exponential distribution, $X \sim \mathrm{Exp}(\lambda)$

$$f(x) = \lambda\, \mathrm{e}^{-\lambda x} \quad \text{with } \lambda > 0$$

- the expected value and variance are

$$E[X] = \frac{1}{\lambda} \qquad Var(x) = \frac{1}{\lambda^2}$$





$\mathrm{Exp}(\lambda)$ property: lack of memory

- important feature of the exponential distribution is the memory-less property
- a positive random variable X is called memory-less if $\forall s,t \geq 0$,

$$P\left(X > s + t \,\middle|\, X > s\right) = P\left(X > t\right)$$

--> suppose you are in front of an elevator
--> and you have already waited for three minutes $(s = 3)$
--> the probability to wait for another two minutes $(t = 2)$ is the same as you just arrived in front of the same elevator
$\rightrightarrows$ but this is only true for an exponential distribution

Proof

- our requirements is

$$\begin{aligned}
P\left(X > s + t \,\middle|\, X > s\right) &= \frac{P\left(X > s + t\right)}{P\left(X > s\right)} \\
&= \frac{\mathrm{e}^{-(s+t)/\lambda}}{\mathrm{e}^{-s/\lambda}} \\
&= \mathrm{e}^{-t/\lambda} = P\left(X > t\right)
\end{aligned}$$

- therefore our hypothesis follows: X is memory-less

Analogy between $\mathrm{Exp}(\lambda)$ and $\mathrm{Geo}(p)$

- a Bernoulli trial is performed successively and independently
  - the number of trials until the first success occurs follows $\mathrm{Geo}(p)$
  - but also the number of trials between two consecutive successes follows $\mathrm{Geo}(p)$
- let's now consider a Poisson process
  - the time it will take until the first event occurs is $\mathrm{Exp}(\lambda)$
  - the time between two consecutive events is also $\mathrm{Exp}(\lambda)$

➜ moreover $\mathrm{Exp}(\lambda)$ is the only memory-less continuous distribution, and $\mathrm{Geo}(p)$ is the only memory-less discrete distribution

The Erlang distribution

- let's consider again a Poisson process $\{N(t) : t \geq 0\}$ where $N(t)$ represents the number of events that happened at or before time t:
  - being T_j the time between events $j-1$ and j, the sequence $\{T_1, T_2, \ldots\}$ distributes as $\mathrm{Exp}(\lambda)$
- let now X be the time of the n-th event
- X follows a so-called Gamma distribution with parameters n and λ

$$f(x) = \frac{x^{n-1}\, \lambda^n\, e^{-\lambda x}}{(n-1)!}$$

➜ Exponential is the time to wait for the first event to occur
➜ Gamma is the time to wait for the n-th event to occur

- an Erlang distribution with parameters $(1, \lambda)$ is an exponential distribution:

$$\mathrm{Gamma}(1, \lambda) \sim \mathrm{Exp}(\lambda)$$

From Erlang to Gamma distributions

- we want to extend the parameters of the Erlang distribution from (n, λ) to (r, λ), where r is a real and positive number
- the factorial $(n-1)!$ can be extended using the Gamma function, $\Gamma : (0, \infty) \longmapsto \mathbb{R}$:

$$\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx$$

- the function has the same property of the factorials:

  $n! = n \cdot (n-1)!$

  $\Gamma(r+1) = r \cdot \Gamma(x)$ with $\quad r > 1$

- if r is integer, we get back the factorials:

$$\Gamma(1) = \int_0^\infty \mathrm{e}^{-x}\, dx = 1$$

- and

$$\begin{aligned} \Gamma(2) &= (2-1)\,\Gamma(2-1) = 1 = 1! \\ \Gamma(3) &= (3-1)\,\Gamma(3-1) = 2 \cdot 1 = 2! \\ &\cdots \\ \Gamma(n+1) &= n! \end{aligned}$$

The Gamma distribution

- a random variable X follows a gamma distribution, $X \sim \mathrm{Gamma}(\alpha, \lambda)$, if the pdf has the form

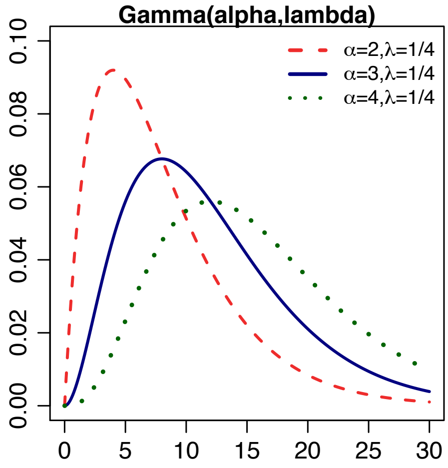
$$f(x) = \frac{1}{\Gamma(\alpha)}\, x^{\alpha-1} \lambda^{\alpha}\, e^{-\lambda x} \quad \text{with } x \geq 0$$

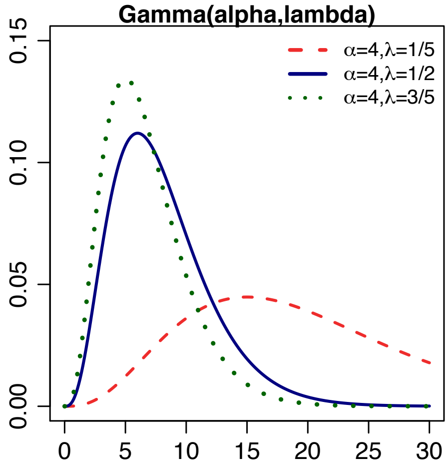
- the parameters $\alpha > 0$ and $\lambda > 0$ are called shape and scale parameters, respectively

$\rightsquigarrow$ therefore, if $X \sim \mathrm{Gamma}(\alpha, 1)$, then $X/\lambda \sim \mathrm{Gamma}(\alpha, \lambda)$

- the Gamma distribution is a generalization of the exponential density with a mode at some strictly positive m value
- it includes the exponential as a special case and can be very skewed, to being almost a bell-shaped density
- we will show that it arises, naturally, as the density of the sum of a number of independent exponential random variables
- the CDF of the Gamma distribution does not exist in explicit form, therefore the inverse method cannot be used for variate generation
- in Bayesian analysis is a natural conjugate prior for the standard deviation of a normal distribution

Gamma distribution in R





- keeping λ fixed, the maximum of the peak moves to the right with increasing values of α
- a similar behavior can be seen by keeping α fixed, and increasing λ to higher values

Sum of variables with an exponential distribution

- let's suppose we have n independent variables $T_j \sim \mathrm{Exp}(\lambda)$, with $j = 1, \dots, n$
- we build $Y_n = \sum_{j=1}^{n} T_j$
- it can be proved that $Y_n \sim \mathrm{Gamma}(n, \lambda)$

Proof

- T_j are all independent for $t < 1/\lambda$

$$
\begin{aligned}
E[\exp(Y_n\, t)] &= E[\exp((T_1 + T_2 + \dots T_n) t)] \\
&= E[\exp(T_1\, t) \exp(T_2\, t) \dots \exp(T_n\, t)] \\
&= E[\exp(T_1\, t)] E[\exp(T_2\, t)] \dots E[\exp(T_n\, t)] \\
&= \prod_{j=1}^{n} (1 - \lambda t)^{-1} \\
&= (1 - \lambda t)^{-n}
\end{aligned}
$$

- therefore $Y_n \sim \mathrm{Gamma}(n, \lambda)$

The Beta distribution

- a random variable X follows a beta distribution, $X \sim \mathrm{Beta}(\alpha, \beta)$, if the pdf is

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1} \quad \text{with } 0 \le x \le 1 \text{ and } \alpha, \beta > 0$$

⇢ beta densities appear in the study of the median of a sample of random points $\sim \mathcal{U}\,(0,1)$

⇢ let's generate n points $X_j \sim \mathrm{Beta}(\alpha,\beta)$ and assume they are ordered $X_1, X_2, \ldots, X_n$ with $X_{j+1} > X_j$

⇢ if $n = 2k+1$ (n is odd), the median is X_{k+1}

⇢ if $n = 2k$ (n is even), the median is $(X_k + X_{k+1})/2$

⇉ the median of $2n+1$ random numbers $\sim \mathcal{U}(0,1)$ is $\sim \mathrm{Beta}\,(n+1, n+1)$

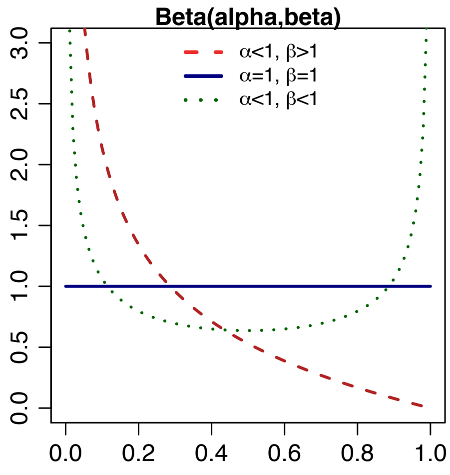
- the expected value and variance are

$$E[X] = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad Var(x) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

- the central moments are

$$E\,[X^n] = \frac{\Gamma(\alpha+n)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)\Gamma(\alpha)}$$

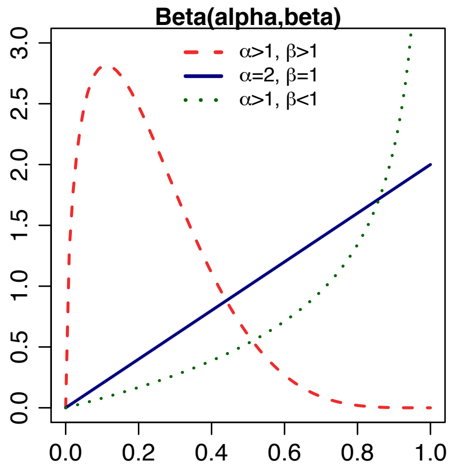
Beta distribution in R



- a beta distribution can be
  - ↱ increasing $\alpha > 1$ and $\beta < 1$
  - ↱ decreasing $\alpha < 1$ and $\beta > 1$
  - ↱ symmetric unimodal $\alpha = \beta$
  - ↱ asymmetric unimodal $\alpha \neq \beta$
  - ↱ U-shaped $\alpha < 1$ and $\beta < 1$

- it cannot be bimodal: it cannot have two local maxima in the interval $[0,1]$

- note that $\mathrm{Beta}(1,1)$ is simply $\mathcal{U}(0,1)$

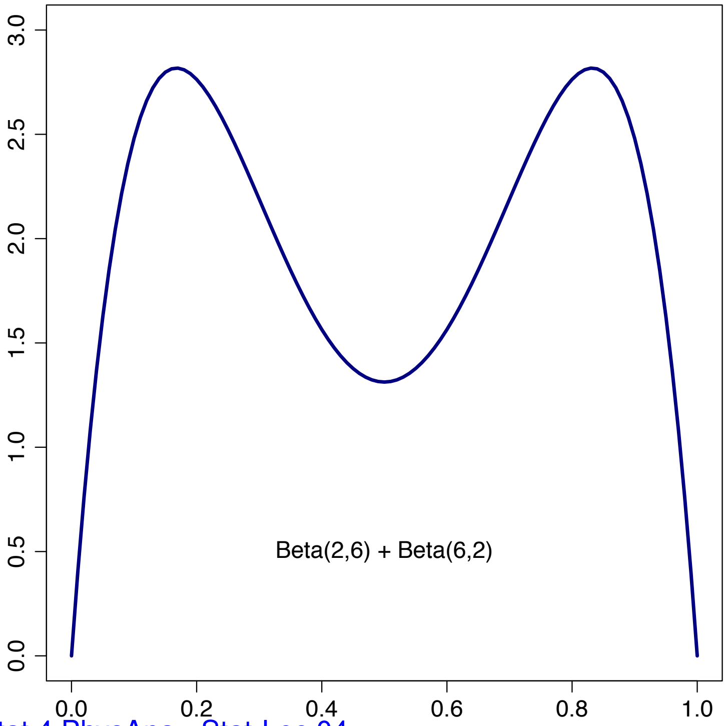


Example: mixture of Beta distributions

- a beta distribution can have only one mode in $[0, 1]$
- in some cases we have to model a random variable that exhibit two modes, for some physical reason

- this can be done by mixing two beta distributions

$$\begin{aligned} f(x) &= \frac{1}{2}\mathrm{Beta}(2,6) + \frac{1}{2}\mathrm{Beta}(6,2) \\ &= \frac{1}{2}\left[42x^5(1-x)\right] \\ &+ \frac{1}{2}\left[42x(1-x)^5\right] \\ &= 21\ x\ (1-x)\ \left[x^4 + (1-x^4\right] \end{aligned}$$



The Normal distribution

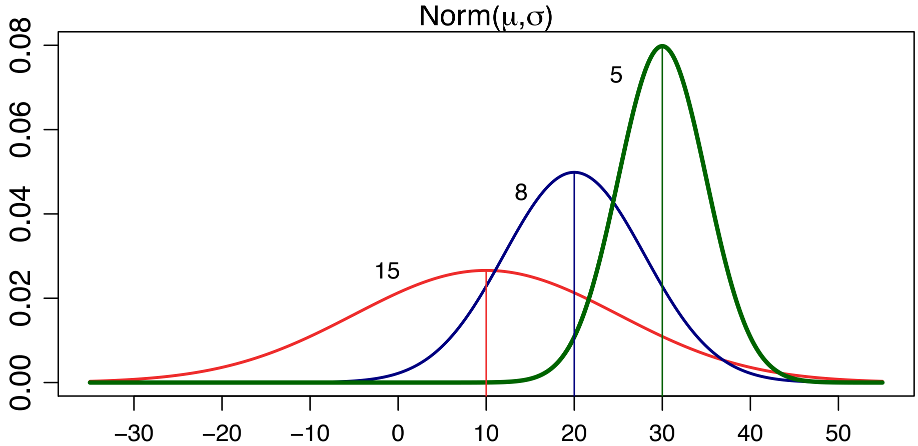
- a random variable X follows a normal distribution, $X \sim \mathrm{N}(\mu, \sigma^2)$, if the pdf is

$$f(x) = \frac{1}{\sigma\sqrt{2\ \pi}}\ \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

--> where μ can be any real number and $\sigma > 0$ and teh distribution is called standard normal

- if $\mu = 0$ and $\sigma = 1$, it is called a standard normal distribution $X \sim \mathrm{N}(0, 1)$
- the expected value and variance are

$$E[X] = \mu \quad \text{and} \quad Var(x) = \sigma^2$$



The sum of independent normal variables

- the standard normal distribution is symmetric and unimodal about the mean, μ
- keeping σ^2 fixed, and changeing μ the normal distribution only gets shifted to a new center
- mantaining μ fixed and increasing σ^2, the distribution becomes more spread out about the same mean value

Theorem

- let $X_1, X_2, \ldots, X_n$, independent random variables with $X_j \sim \mathrm{N}(\mu_j, \sigma_j^2)$
- we build $Y_n = \sum_{j=1}^{n} X_j$
- it can be proved that

$$Y_n \sim \mathrm{Norm}\left(\sum_{j=1}^{n} \mu_j, \sum_{j=1}^{n} \sigma_j^2\right)$$

Still on the sum of independent normal variables

Corollary

- given n random variables, all following the same $\mathrm{N}(\mu, \sigma^2)$ distribution
- them

$$\overline{X} = \frac{\sum X_j}{n} \sim \mathrm{Norm}\left(\mu, \frac{\sigma^2}{n}\right)$$

$\Longmapsto$ the distribution of X gets more concentrated around the mean value μ as n increases, because the variance, σ^2, decreases with n

Theorem

- any linear combination of independent normal variables is also normal

$$\sum_{j=1}^{n} a_j X_j \sim \mathrm{Norm}\left(\sum_{j=1}^{n} a_j \mu_j, \sum_{j=1}^{n} a_j^2 \sigma_j^2\right)$$

Limit Theorems

- let $X_1, X_2, \ldots, X_n$ independent random variables from the same distribution with mean μ and variance σ^2
- we define $S_n = \sum_{j=1}^{n} X_j$
- since X_j are independent and identically distributed,

  $E[S_n] = nE[X_j] = n\mu$ and $\mathrm{Var}(S_n) = n\mathrm{Var}(X_j) = n\sigma^2$
- the following theorems apply

Strong Law of Large Numbers

$$P\left(\lim_{n\to\infty} \frac{S_n}{n} = \mu\right) = 1$$

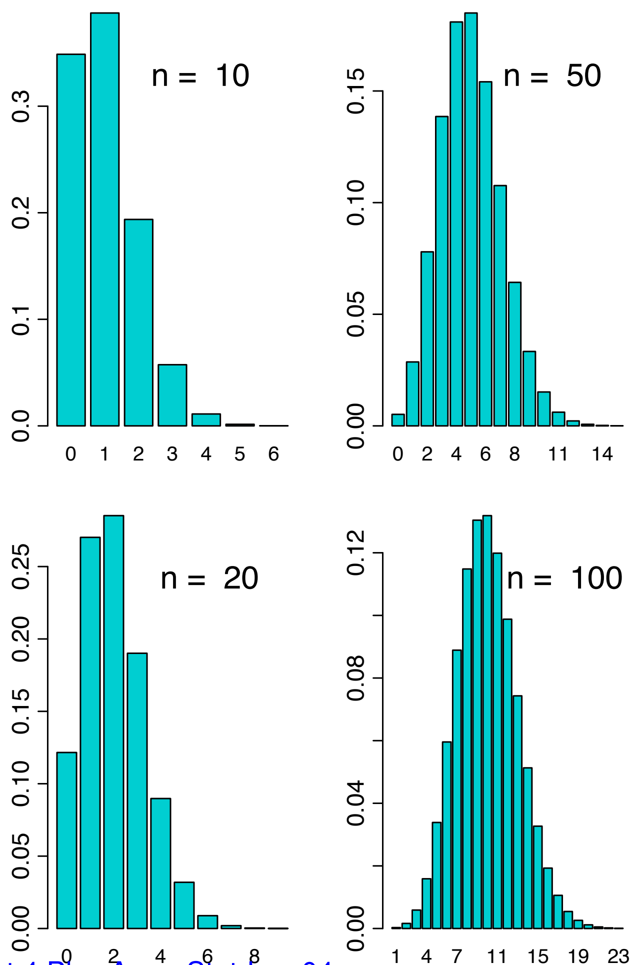
Central Limit Theorem

$$\lim_{n\to\infty} P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right) = \Phi(x)$$

Where $\Phi(x)$ is the CDF of the standard normal distribution

CLT for Binomial distribution

- $X \sim \mathrm{Binom}(n, p)$ with $p = 0.1$
- the histogram is rather skewed for the small n values
- as n increases, it gets less skewed, and for the largest value, $n = 100$, the histogram looks bell-shaped, centered between 10 and 11, resembling a normal density curve
- indeed, the binomial distribution, $\mathrm{Binom}(n, p)$ can be well approximated by $\mathrm{Norm}\left(np, np(1-p)\right)$, for any fixed p, when n is large



CLT for Gamma distribution

- the sum of variables distributed according to $\mathrm{Gamma}(\alpha, \lambda)$ is again a gamma distribution
- the CLT tell us that when the number of terms in the sum is large, the resulting gamma distribution should be approximately normal
- the larger alpha, the less skewed the distribution of the individual terms is ➜ the smaller n has to be to get a good normal approximation