**Max-Planck-Institute**
**for Gravitational physics**
`Albert-Einstein-Institute`

# Testing the stability of overtone model

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Final Presentation

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# Testing general relativity with GWs

Gravitational wave (GW) observations are fast increasing along with the coming of future GW detectors and a wider frequency band. Up to date, the second part of the third observing run (O3b)[1] of the LIGO-Virgo collaboration has added 35 new GW candidates in total to the previous catalog (GWTC1-2).
These observations, especially the binary black hole cases, place unprecedented constraints on general relativity in its strong regime, in which the post-merger phase in particular provides one of the most promising channels for such studies.

[1] R. Abbott et al. "GWTC-3: Compact Binary Coalescences Observed by LIGO and Virgo During the Second Part of the Third Observing Run". In: (Nov. 2021). arXiv: 2111.03606 [gr-qc].

# Tests of the no-hair theorem

The black hole no-hair theorem in general relativity (GR) implies that the final state of an uncharged black hole merger, and the associated quasi-normal mode (QNM) spectrum are fully and uniquely determined by the values of the final mass and spin.

There are several approaches for testing the no-hair theorem with GWs:

- Inspiral-merger-ringdown (IMR) consistency test.
- Black hole spectroscopy.
  - i) Fundamental tone plus one overtone.
  - ii) Dominant angular mode plus another angular mode.

While there are several attempts have been made[2], for near equal-mass-ratio non-spinning binaries, i) becomes a promising possibility.

[2] Collin D. Capano et al. "Observation of a multimode quasi-normal spectrum from a perturbed black hole". In: (May 2021). arXiv: 2105.05238 [gr-qc], Maximiliano Isi and Will M. Farr. "Analyzing black-hole ringdowns". In: (July 2021). arXiv: 2107.05609 [gr-qc].

# Black hole perturbation theory

In the black hole perturbation theory, Teukolsky equation[3] is usually used to describe the perturbation fields (scalar, vector, tensor, ...) solutions to the Einstein field equation in Kerr metric.

$$
\begin{aligned}
&\left[\frac{\left(r^2+a^2\right)^2}{\Delta}-a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2}+\frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi}+\left[\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)\\
&-2s\left[\frac{a(r-M)}{\Delta}+\frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi}-2s\left[\frac{M\left(r^2-a^2\right)}{\Delta}-r-ia\cos\theta\right]\frac{\partial\psi}{\partial t}+\left(s^2\cot^2\theta-s\right)\psi=4\pi\Sigma T.
\end{aligned}
\tag{1}
$$

In above formula, $\psi$ denotes the linear perturbation field quantity, and the source term $T$ is written in Newman-Penrose notation.

[3] Saul A. Teukolsky. "Rotating Black Holes: Separable Wave Equations for Gravitational and Electromagnetic Perturbations". In: *Physical Review Letters* 29.16 (Oct. 1972). URL: https://link.aps.org/doi/10.1103/PhysRevLett.29.1114.

# Overtone models

In the late time of an binary black holes coalescence event, we can assume that gravitational waves follow the wave solution of Teukolsky equation:

$$ {}_sX = \sum_{l \geq s, m, n} \mathcal{A}_{lmn} \exp\left[ -\iota\omega_{lmn}(t - t_r) - \frac{t - t_r}{\tau_{lmn}} \right] {}_sY_{lm}. \qquad (2) $$

The subscript in the left-side of terms $s$ denotes the spin-weighted factor, which is $-2$ in our cases since it describe the outgoing gravitational waves. The right-side subscript $n$ in the equation is the so called "overtone index", which accounts for the n-th excitations of a given mode. And the other two ($l$ and $m$) are angular indices coming from decomposition of spheroidal harmonics. The amplitudes satisfy $\mathcal{A}_{lmn} = A_{lmn} e^{\iota\phi_{lmn}}$.

# Quasi-normal modes (QNMs)

Theoretically, such waves will have a damped-sinusoidal form, and the complex frequency will be: $\omega_{lmn} = \omega_R + i\omega_I$ where the $\omega_R$ is frequency, and $\tau_{lmn} = -1/\omega_I$ is damping time, they are mode-dependent values of the waves as shown in the formula (2), which are the so called "Quasi-normal modes" as illustrated in Fig. 1.

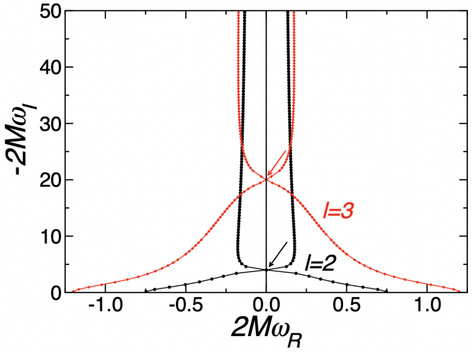


Figure 1: QNM diagram.

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[4] Emanuele Berti, Vitor Cardoso, and Andrei O Starinets. "Quasinormal modes of black holes and black branes". In: *Classical and Quantum Gravity* 26.16 (2009), p. 163001.

# Instability of overtone models

In modified gravity theories, the QNM values are different from the GR solutions to some degrees(suxh as MOG[5]), hence distinguishable ringdown signals are required to test GR as well as constrain modified theories of gravity.
While in previous work, the considerations on the instability inside overtone models are often absent. Therefore, we investigate the uncertainties of overtone models in both the:

- QNM values' deviation to Kerr spectrum.
- Alternative forms of damped sinusoids.

[5] Luciano Manfredi, Jonas Mureika, and John Moffat. "Quasinormal modes of modified gravity (MOG) black holes". In: *Physics Letters B* (2018).

# The fractional deviation to Kerr spectrum

As shown in the new wave solution below, the QNM deviations assigned on one specific tone are characterized in the perturbed factors $\alpha_{lmn}$ and $\beta_{lmn}$. They could also represent the QNM spectrum of some modified gravity.

$$ {}_sX = \sum_{l \geq s, m, n} \mathcal{A}_{lmn} \exp\left[ -\imath\omega_{lmn}(1+\alpha_{lmn})(t-t_r) - \frac{t-t_r}{\tau_{lmn}(1+\beta_{lmn})} \right] {}_sY_{lm}. \quad (3) $$

Using our novel grid methods, we can then present the results in an unprecedented way while providing extra information by combining the two most important quantities in current ringdown analysis.

# 2-order adaptive grid

The fit performance is quantified as:

$$\mathcal{M} = 1 - \frac{\langle h_{\mathrm{NR}} \mid h_x \rangle}{\sqrt{\langle h_{\mathrm{NR}} \mid h_{\mathrm{NR}} \rangle \langle h_x \mid h_x \rangle}} \tag{4}$$

where $\langle f \mid g \rangle = \int_{t_0}^{t_f} f(t) g^*(t) dt$.
The mass/spin consistency is measured by the bias factor $\epsilon$:

$$\epsilon = \sqrt{\left(\frac{\delta M_f}{M}\right)^2 + \delta a_f^2}, \tag{5}$$

where $\delta M_f = M_f^{\mathrm{fit}} - M_f^{\mathrm{true}}$ and $\delta a_f = a_f^{\mathrm{fit}} - a_f^{\mathrm{true}}$.

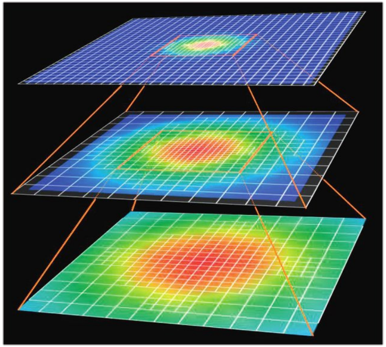

Figure 2: Illustration of the adaptive grid.

# Fit starting time

In the equations (2) and (5), we can see there is a reference time $t_r$ which indicates when will the linear perturbation apply, we fix it to be $t_r = t_{peak} = 0$ since . Also, in our epsilon analysis, we will fix the fit starting time to be $t_0 = t_{peak} = 0$ to avoid the fluctuations at later times.

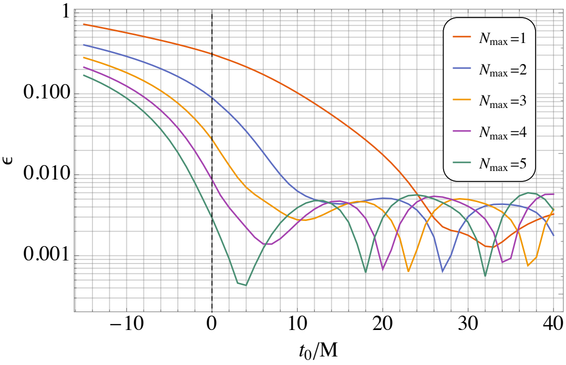


Figure 3: $t_0 - \epsilon$ plot of overtone models.

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[6] Pierre Mourier et al. "Quasinormal modes and their overtones at the common horizon in a binary black hole merger". In: *Physical Review D* 103.4 (2021), p. 044054.

# Mismatches plots

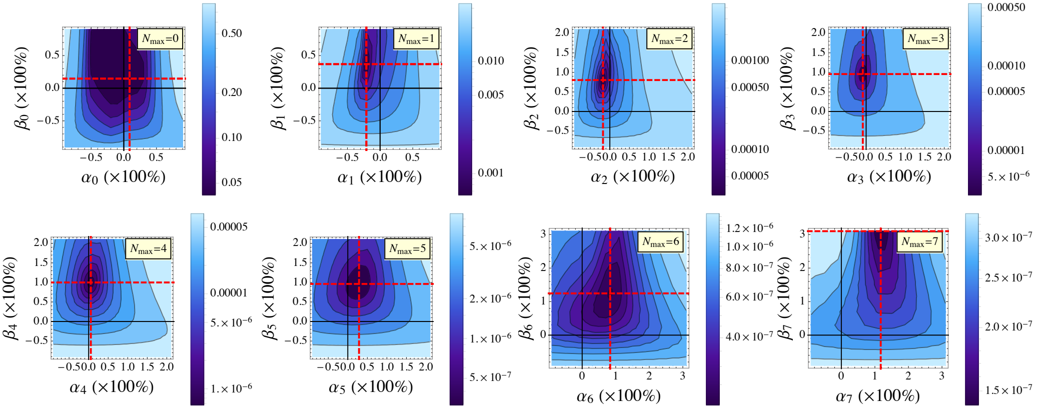


Figure 4: Density plot of mismatch in different perturbed highest overtone models.

Similar patterns are present in the mismatch plots.

# Epsilon plots

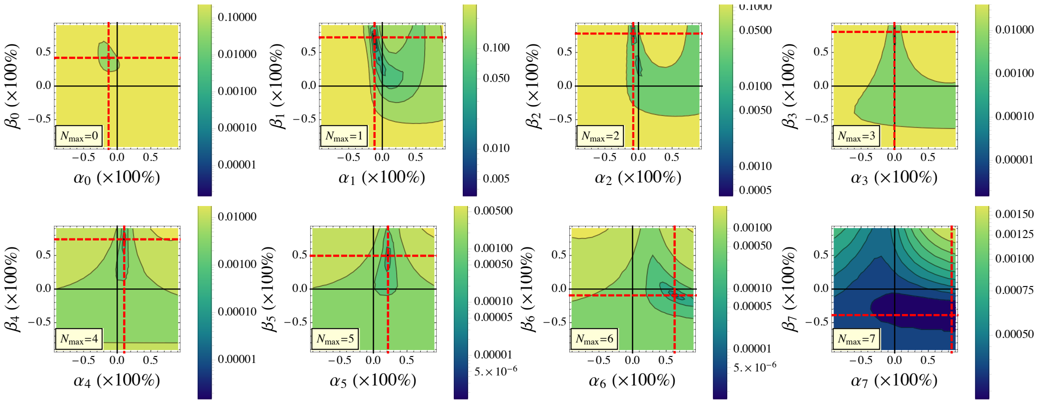


Figure 5: Density plot of $\epsilon$ in different perturbed highest overtone models.

The crossover points of red dashed lines correspond to the lowest epsilon cases.

# Mismatch and epsilon

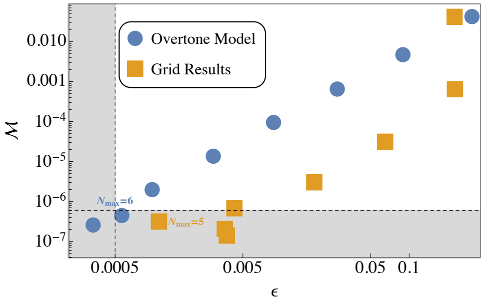


Figure 6: Epsilon-mismatch plot.

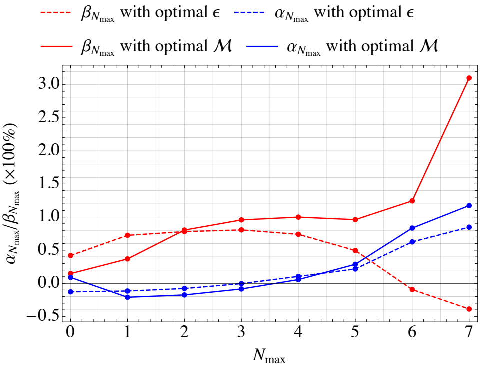


Figure 7: Best $\alpha$ and $\beta$ values in the epsilon and mismatch analysis.

# Modified overtone models

Apart from the perturbed modes, we can also consider the another approach:

- $\mathbf{PLM}\,(N_{\max}) : \mathbf{OM}\,(N_{\max} - 1) + x_p t^{-\gamma}$
- $\mathbf{PLCM}\,(N_{\max}) : \mathbf{OM}\,(N_{\max} - 2) + x_p t^{-\gamma} \cos(\omega t + \phi)$
- $\mathbf{EM}\,(N_{\max}) : \mathbf{OM}\,(N_{\max} - 1) + x_p \exp\left(-\frac{t}{\tau}\right)$
- $\mathbf{PLM}^*\,(N_{\max}) : \mathbf{OM}\,(N_{\max} - 1) \times (1 + x_p t^{-\gamma})$
- $\mathbf{TCTM}\,(N_{\max}) : \mathbf{OM}\,(t) \rightarrow \mathbf{OM}\left(Ae^{-t/\tau}\right)$

The above modified models are also based on the overtone models (**OM**), but modify the highest modes to alternative forms of damped sinusoids. All of these new models are believed to be capable of depicting the trends how the waveform amplitude vary with time as the original model does

# Comparison of models

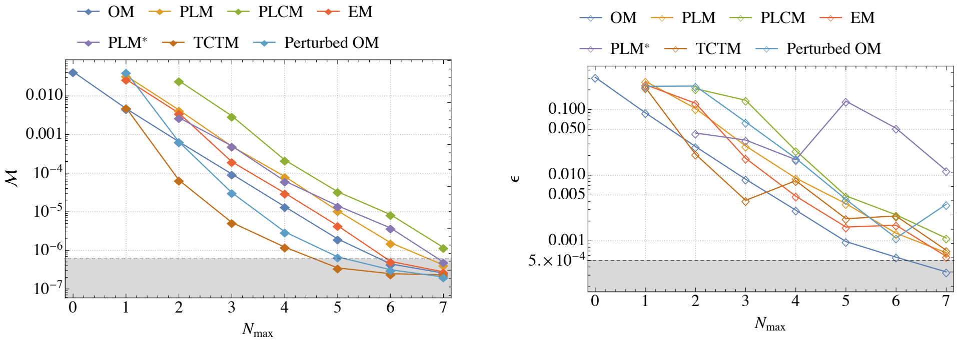


Figure 8: In the left, right panel, we show the comparison of mismatch, epsilon evolution with $N_{max} \in [0, 7]$ in models have the same number of free parameters, respectively. The computation was done on the BBH:0305 waveform data from SXS catalogue.

# The uncertainties on deviation factors

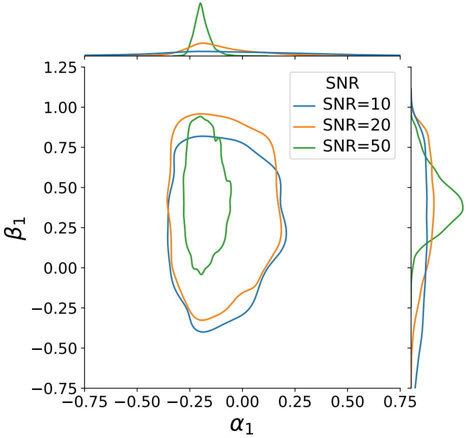


Figure 9: 90% credible contours of the mismatches in injections with different SNR.

We use the information saved in our computation of the 2-dimension $\alpha_N$ / $\beta_N$ grids by marginalizing the mismatch distribution. Then, injections with different SNR values could be tested and inserted to the form with the assumption of Gaussian likelihood which is widely used in this field.

The model is a 2 tone model in which deviations are allowed in $\alpha_1$ / $\beta_1$ 2-dimension plane with injection SNR=10, 20, 50.

# 0, 1 tone perturbed overtone model

We also investigate the case which deviations are allowed in 2 tones (i.e. fundamental tone plue one overtone) model. In the analysis, we use different SNR as injections, with higher one of which the distribution would get thinner, and hence better error range can be obtained.

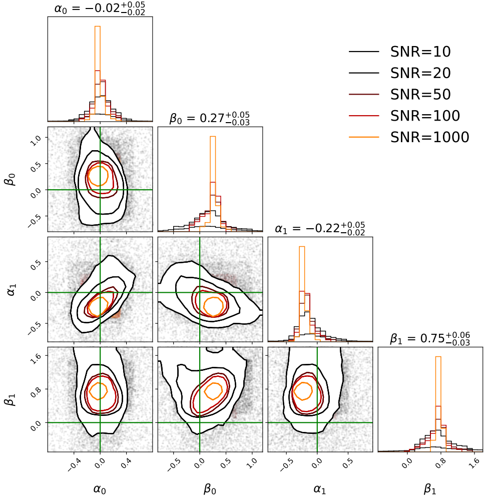


Figure 10: Corner plot of the posterior of 4 deviations.

# Stability of the fundamental tones

We can also look into the cases which we modify only the fundamental tone. The best-fit deviation factor $\alpha_0$ and $\beta_0$ will decrease with the overtone numbers.

This means that the lower tones are able to be well constraint, but we still need a high overtone model to ensure it.

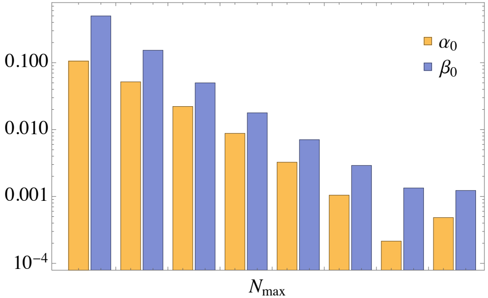


Figure 11: Best-fit $\alpha_0$ and $\beta_0$ absolute values varied with tones, they are taken averaged from 5 waveforms in SXS catalog.

# Conclusions

- When doing black hole spectroscopy, the higher overtones are not only important, but also necessary to be considered. Otherwise, the analysis could be totally wrong since we cannot discern GR and modified gravity through the fitting performance in the match-filtering process.
- Our results of the 0, 1 tone perturbed model with the optimal fractional deviation factors: $\alpha_0 = -0.04$, $\beta_0 = 0.08$, $\alpha_1 = -0.24$, $\beta_1 = 0.48$ also give a new reference for the realistic gravitational waves testing of GR by providing the intrinsic uncertainty ranges of the overtone models in 2 tone black hole spectroscopic analysis without assuming specific priors.

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Thank you for listening!

Acknowledgement: Thanks for the very nice and helpful instructions from my two supervisors: Xisco and Pierre, and the remote support from AEI Atlas computing.

# Appendix

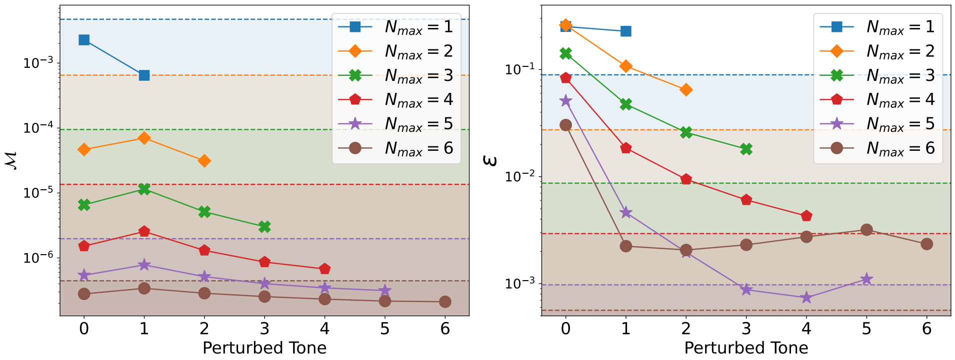


Figure 12: The left panel shows the lowest mismatch variation in regard to different perturbed tones in overtone models with $N_{max} \in [0, 7]$. While the right panel shows the lowest epsilon variation. The colored region in both panels stand for the value space of either mismatch or epsilon values that are lower than those of the "unperturbed" overtone models with the corresponding $N_{max}$.