

# CS/ECE 374 A (Spring 2022)

## Homework 4 Solutions

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**Problem 4.1:** For each of the following languages, determine whether it is regular or not, and give a proof. To prove that a language is not regular, you should use the fooling set method. (To prove that a language is regular, you are allowed to use known facts about regular languages, e.g., closure properties, all finite languages are regular, ...)

- (a)  $\{x(110)^n x^R : x \in \{0,1\}^*, n \geq 1\}$
- (b)  $\{0^i 1^j 0^k : i+k \text{ is divisible by } 3, \text{ and } k \text{ is divisible by } j, \text{ and } i, j, k \geq 1\}$
- (c)  $\{yxx^R z : x, y, z \in \{0,1\}^*, |x| \geq 374\}$
- (d)  $\{y0^n 1^n 0^n z : y, z \in \{0,1\}^*, n \geq 374\}$

**Solution:** In each of the parts below, let  $L$  be the language in question.

- (a) We prove that  $L$  is **not regular** by the fooling set method.

Choose  $F = \{0^i : i \geq 0\}$ .

Let  $x$  and  $y$  be two arbitrary distinct strings in  $F$ .

Then  $x = 0^i$  and  $y = 0^j$  for some  $i \neq j$ .

Choose  $z = 110 \cdot 0^i$ .

Then  $xz = 0^i \cdot 110 \cdot 0^i = 0^i \cdot 110 \cdot (0^i)^R \in L$ .

On the other hand,  $yz = 0^i \cdot 110 \cdot 0^j \notin L$ , because  $i \neq j$  (in more detail:  $yz$  has only one occurrence of a substring of the form  $(110)^n$  with  $n \geq 1$ , and that substring has  $n = 1$ ; the part before the substring is  $0^i$  and the part after the substring is  $0^j$ , but  $0^i \neq (0^j)^R$  if  $i \neq j$ ).

Thus,  $F$  is a fooling set.

Since  $F$  is infinite,  $L$  cannot be regular.

- (b) We prove that  $L$  is **not regular** by the fooling set method.

Choose  $F = \{01^{3n-1} : n \geq 1\}$ .

Let  $x$  and  $y$  be two arbitrary distinct strings in  $F$ .

Then  $x = 01^{3m-1}$  and  $y = 01^{3n-1}$  for some  $m, n \geq 1$  with  $m \neq n$ . Without loss of generality, assume  $m < n$  (the other case is symmetric).

Choose  $z = 0^{3m-1}$ .

Then  $xz = 01^{3m-1}0^{3m-1} \in L$ , since  $1 + (3m - 1) = 3m$  is divisible by 3 and  $3m - 1$  is divisible by  $3m - 1$ .

On the other hand,  $yz = 01^{3n-1}0^{3m-1} \notin L$ , because  $3m - 1$  is not divisible by  $3n - 1$  since  $m < n$ .

Thus,  $F$  is a fooling set.

Since  $F$  is infinite,  $L$  cannot be regular.

[Note:  $F = \{0^n : n \geq 1\}$  won't work here.]

(c) We prove that  $L$  is **regular**.

By definition,  $L$  consists of all strings that contain  $xx^R$  as a substring for some string  $x$  of length at least 374, i.e., all strings that contain an even-length palindrome of length at least  $2 \cdot 374 = 748$ .

Observe that if a string  $w$  contains an even-length palindrome of length at least 748, i.e., a substring of the form  $a_\ell \cdots a_1 \cdot a_1 \cdots a_\ell$  with  $a_1, \dots, a_\ell \in \{0, 1\}$  and  $\ell \geq 374$ , then it must contain a palindrome of length exactly 748, namely,  $a_{374} \cdots a_1 \cdot a_1 \cdots a_{374}$ .

Let  $A$  be the set of all palindromes of length exactly 748. Since  $A$  is finite,  $A$  is regular. Since  $L = (0 + 1)^* A (0 + 1)^*$ , we conclude that  $L$  is also regular.

(d) We prove that  $L$  is **not regular** by the fooling set method.

Choose  $F = \{0^i : i \geq 374\}$ .

Let  $x$  and  $y$  be two arbitrary distinct strings in  $F$ .

Then  $x = 0^i$  and  $y = 0^j$  for some  $i, j \geq 374$  with  $i \neq j$ . Without loss of generality, assume  $i > j$  (the other case is symmetric).

Choose  $z = 1^i 0^i$ .

Then  $xz = 0^i 1^i 0^i \in L$ , since  $xz$  trivially contains  $0^i 1^i 0^i$  as a substring and  $i \geq 374$ .

On the other hand,  $yz = 0^j 1^i 0^i \notin L$  by the following argument: if  $yz$  is in  $L$ , then it contains a substring of the form  $0^n 1^n 0^n$  for some  $n \geq 374$ ; then the middle block  $1^n$  must be equal to  $1^i$ , and so  $n = i$ ; and the left block  $0^n$  must be contained in  $0^j$ , implying that  $i = n \leq j$ , which contradicts the  $i > j$  assumption.

Thus,  $F$  is a fooling set.

Since  $F$  is infinite,  $L$  cannot be regular.

**Problem 4.2:** Give a context-free grammar (CFG) for each of the following languages. You must provide explanation for how your grammar works, by describing in English what is generated by each non-terminal. (Formal proofs of correctness are not required.)

- (a) (30 pts)  $\{x(110)^n x^R : x \in \{0, 1\}^*, n \geq 1\}$
- (b) (30 pts)  $\{1^i 0^j 1^k : j = 2i + 3k, i, j, k \geq 0\}$
- (c) (40 pts)  $\{1^i 0^j 1^k : i + k \text{ is divisible by } 3 \text{ and } 0 \leq j \leq k\}$

**Solution:**

(a)

$$\begin{aligned} S &\rightarrow 0S0 \mid 1S1 \mid A \\ A &\rightarrow 110A \mid 110 \end{aligned}$$

Explanation:

- $A$  generates  $\{(110)^n : n \geq 1\}$ .
- $S$  generates  $\{x(110)^n x^R : x \in \{0, 1\}^*\}$ .

(b)

$$\begin{aligned} S &\rightarrow AB \\ A &\rightarrow 1A00 \mid \varepsilon \\ B &\rightarrow 000B1 \mid \varepsilon \end{aligned}$$

Explanation:

- $A$  generates all strings of the form  $1^i 0^{2i}$  ( $i \geq 0$ ).
- $B$  generates all strings of the form  $0^{3k} 1^k$  ( $k \geq 0$ ).
- $S$  generates all strings of the form  $1^i 0^{2i} \cdot 0^{3k} 1^k = 1^i 0^j 1^k$  with  $j = 2i + 3k$  ( $i, k \geq 0$ ), as desired.

(c)

$$\begin{aligned} S &\rightarrow A_0 B_0 A_0 \mid A_0 B_1 A_2 \mid A_0 B_2 A_1 \mid \\ &\quad A_1 B_0 A_2 \mid A_1 B_1 A_1 \mid A_1 B_2 A_0 \mid \\ &\quad A_2 B_0 A_1 \mid A_2 B_1 A_0 \mid A_2 B_2 A_2 \\ A_0 &\rightarrow 111A_0 \mid \varepsilon \\ A_1 &\rightarrow 1A_0 \\ A_2 &\rightarrow 11A_0 \\ B_0 &\rightarrow 000B_0 111 \mid \varepsilon \\ B_1 &\rightarrow 0B_0 1 \\ B_2 &\rightarrow 00B_0 11 \end{aligned}$$

Explanation: Observe that  $L = \{1^i 0^j 1^k : i+k \equiv 0 \pmod{3}, i \geq 0, k \geq j\} = \{1^i \cdot (0^j 1^j) \cdot 1^\ell : i+j+\ell \equiv 0 \pmod{3}, i, \ell \geq 0\}$ .

- for each  $p \in \{0, 1, 2\}$ :  $A_p$  generates all strings of the form  $1^i$  with  $i \equiv p \pmod{3}$ .
- for each  $q \in \{0, 1, 2\}$ :  $B_q$  generates all strings of the form  $0^j 1^j$  with  $j \equiv q \pmod{3}$ .
- $S$  generates  $L = \{1^i \cdot (0^j 1^j) \cdot 1^\ell : i+j+\ell \equiv 0 \pmod{3}, i, \ell \geq 0\}$ , since  $S$  goes to  $A_p B_q A_r$  over all combinations of  $p, q, r \in \{0, 1, 2\}$  with  $p+q+r \equiv 0 \pmod{3}$ .

[Note: There are other equivalent solutions. For example:

$$\begin{aligned} S &\rightarrow AC \mid 1AC_2 \mid 11AC_1 \\ A &\rightarrow 111A \mid \varepsilon \\ C_0 &\rightarrow 000C_0 111 \mid 00C_0 111 \mid 0C_0 111 \mid C_0 111 \mid \varepsilon \\ C_1 &\rightarrow 000C_1 111 \mid 00C_1 111 \mid 0C_1 111 \mid C_1 111 \mid 01 \mid 1 \\ C_2 &\rightarrow 000C_2 111 \mid 00C_2 111 \mid 0C_2 111 \mid C_2 111 \mid 0011 \mid 011 \mid 11 \end{aligned}$$

Here,  $A$  generates all strings of the form  $1^i$  with  $i \equiv 0 \pmod{3}$ , and  $C_p$  generates all strings of the form  $0^j 1^k$  with  $j \leq k$  and  $k \equiv p \pmod{3}$ .]