Foundations of Modern Optics

Solved exercises in Optics

Dimitris Papazoglou

Materials Science and Technology Department

University of Crete

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Dimitris Papazoglou dpapa@materials.uoc.gr

Introduction

This small collection of exercises is the result of teaching Optics in undergraduate and post graduate level over the last 14 years.

They comprise a good set of examples in Optics, covering from basic concepts, such as the description of an electromagnetic wave, up to the design of optical systems. In the field of Geometric Optics, special focus is given to optical matrix theory, a useful tool for analyzing the optical behavior of composite optical systems.

Each exercise is analytically solved, with supporting figures and graphs when necessary. The solutions provided are not unique since, especially in optical system design problems, alternative solutions exist.

Dimitris Papazoglou

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Wave description

1) Assuming that $f(z)$ is a known function is $f(z \pm \upsilon t)$ a solution of the wave equation? *(assume that υ is constant)*

Solution

We set $z' = z \pm \upsilon t$ thus:

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z'}\frac{\partial z'}{\partial z} = \frac{\partial f}{\partial z'} \Rightarrow \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial z'^2}$$

Respectively for the partial derivative with respect to time:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial z'}\frac{\partial z'}{\partial t} = \pm\upsilon \frac{\partial f}{\partial z'} \Rightarrow \frac{\partial^2 f}{\partial t^2} = \upsilon^2 \frac{\partial^2 f}{\partial z'^2} \Rightarrow \quad \frac{\partial^2 f}{\partial z'^2} = \frac{1}{\upsilon^2}\frac{\partial^2 f}{\partial t^2}$$

Equating the above leads to the wave equation:

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial z'^2} = \frac{1}{\upsilon^2}\frac{\partial^2 f}{\partial t^2} \Rightarrow \quad \frac{\partial^2 f}{\partial z^2} - \frac{1}{\upsilon^2}\frac{\partial^2 f}{\partial t^2} = 0$$

We can extend the above solution to the three-dimensional space by proving that starting from a known function $f(\mathbf{r})$ we can construct a propagating wave with the transformation $f(\mathbf{r} \pm \boldsymbol{\upsilon} t)$ where $\boldsymbol{\upsilon}$ is the velocity of propagation.

Indeed by setting $\mathbf{r}' = \mathbf{r} \pm \boldsymbol{\upsilon} t$ we can write:

$$x' = x \pm \upsilon_x t, \quad y' = y \pm \upsilon_y t, \quad z' = z \pm \upsilon_z t$$

$$\nabla f(\mathbf{r} \pm \boldsymbol{\upsilon} t) = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z'}\frac{\partial z'}{\partial z}\hat{\mathbf{z}} = \frac{\partial f}{\partial x'}\hat{\mathbf{x}} + \frac{\partial f}{\partial y'}\hat{\mathbf{y}} + \frac{\partial f}{\partial z'}\hat{\mathbf{z}} \Rightarrow$$

$$\nabla^2 f(\mathbf{r} \pm \boldsymbol{\upsilon} t) = \frac{\partial^2 f}{\partial x'^2}\hat{\mathbf{x}} + \frac{\partial^2 f}{\partial y'^2}\hat{\mathbf{y}} + \frac{\partial^2 f}{\partial z'^2}\hat{\mathbf{z}}$$

Now applying for the partial derivative with respect to time:

$$\frac{\partial f(\mathbf{r} \pm \boldsymbol{\upsilon} t)}{\partial t} = \nabla f(\mathbf{r} \pm \boldsymbol{\upsilon} t) \cdot \frac{\partial \mathbf{r}'}{\partial t} = \pm \boldsymbol{\upsilon} \cdot \nabla f(\mathbf{r} \pm \boldsymbol{\upsilon} t) \Rightarrow \frac{\partial^2 f(\mathbf{r} \pm \boldsymbol{\upsilon} t)}{\partial t^2} = |\boldsymbol{\upsilon}|^2 \nabla^2 f(\mathbf{r} \pm \boldsymbol{\upsilon} t)$$

By combining the above equations we get:

$$\frac{1}{|\mathbf{\upsilon}|^2}\frac{\partial^2 f}{\partial t^2} = \nabla^2 f(\mathbf{r} \pm \mathbf{\upsilon} t) = \nabla^2 f \Rightarrow \qquad \nabla^2 f - \frac{1}{|\mathbf{\upsilon}|^2}\frac{\partial^2 f}{\partial t^2} = 0$$

2) Specify whether the following functions describe waves, and if so what is the direction and propagation velocity?

i) $$\Psi(\mathbf{r},t) = \frac{\sin^2[2\pi(x+2z-2pt)]}{1+(x+2z-3pt)^2}$$

ii) $$\Psi(\mathbf{r},t) = \sin^2[\frac{10^9}{m}y + \frac{6}{\sec}\cdot t]$$

Solution

i) It does not describe a wave. (it cannot be described as a function of a single variable by a $z \pm \upsilon t$ transformation)

ii)
$$\Psi(\mathbf{r},t) = \sin^2[\frac{10^9}{m}y + \frac{6}{\sec}\cdot t] = \sin^2[\frac{10^9}{m}(y + 6\cdot 10^{-9}\frac{m}{\sec}\cdot t)] \Rightarrow$$
$$\Psi(\mathbf{r},t) = \sin^2[\frac{10^9}{m}y'], \quad y' \equiv y + 6\cdot 10^{-9}\frac{m}{\sec}\cdot t$$

Thus it describes a wave propagating along the $-y$ direction with a υ = 6 10^{-9} m/s, propagation velocity

3) Estimate the phase (in a 0, 2π range) of a harmonic plane wave light beam after propagating in 500.5 μm in vacuum and 2 *mm* in a medium with refractive index n=3/2 *(Assume that the initial phase is φ =0. vacuum wavelength: λ_o = 1 μm)*

Solution:

The general description of a plane wave at t=0 is:

$$\psi(\mathbf{r},0) = \Psi_o \cos(n k_o z)$$

thus the accumulated phase after propagating though a series of media of refractive index n_i and thickness d_i is

$$\varphi_{tot} = k_o \sum_i n_i d_i \equiv 2\pi(OPL)/\lambda_o$$

Finally we get

$$\frac{\varphi_{tot}}{2\pi} = \frac{500.5\mu m}{1\mu m} + \frac{2000\mu m}{\frac{1\mu m}{3/2}} = 500.5 + 3000 = 3500.5 \Rightarrow$$

$$\varphi_{tot} = \overbrace{3500 \cdot 2\pi}^{trivial} + 0.5 \cdot 2\pi \Rightarrow \Delta\varphi_{tot} = \pi$$

4) Analytically describe the following harmonic fields *(λ_o is the vacuum wavelength, and n is the refractive index. Assume that the light velocity in vacuum is c $\cong$ 3 10^8 m/sec)*:

i) $\mathbf{k} // \hat{\mathbf{x}} + \hat{\mathbf{y}}, \;\; \lambda_o = 1 \; \mu m, \;\; n = 1.5$

ii) $\mathbf{k} // \hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}, \;\; \lambda_o = 0.5 \; \mu m, \;\; n = 2$

Solution:

i) The propagation speed υ, the wavelength λ and the wavevector $\mathbf{k}$ are described as:

$$\upsilon = \frac{c}{n} \cong 2 \cdot 10^8 m/s, \;\; \lambda = \frac{\lambda_o}{n} = \frac{2}{3} 10^{-6} m, \;\; \mathbf{k} = \frac{2\pi}{\lambda} \frac{1}{\sqrt{2}} (\hat{\mathbf{x}} + \hat{\mathbf{y}}) = n \frac{2\pi}{\lambda_o} \frac{1}{\sqrt{2}} (\hat{\mathbf{x}} + \hat{\mathbf{y}}),$$

Thus the harmonic wave will be described by:

$$\Psi(\mathbf{r}, t) = Cos(\mathbf{k} \cdot \mathbf{r} - \omega t) = Cos[\frac{2\pi}{\lambda} (\hat{\mathbf{k}} \cdot \mathbf{r} - \upsilon t)] = Cos[3\pi \cdot 10^6 m^{-1} (\frac{x + y}{\sqrt{2}} - 2 \cdot 10^8 \frac{m}{s} t)]$$

ii) The propagation speed υ, the wavelength λ and the wavevector $\mathbf{k}$ are described as:

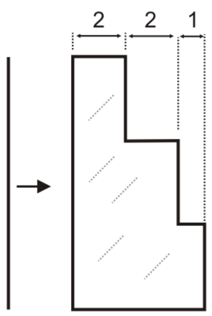
$$\upsilon = \frac{c}{n} \cong \frac{3}{2} 10^8 m/s, \;\; \lambda = \frac{\lambda_o}{n} = \frac{1}{4} 10^{-6} m, \;\; \mathbf{k} = \frac{2\pi}{\lambda} \frac{1}{\sqrt{3}} (\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}) = n \frac{2\pi}{\lambda_o} \frac{1}{\sqrt{3}} (\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}),$$

Thus the harmonic wave will be described by:

$$\Psi(\mathbf{r}, t) = Cos(\mathbf{k} \cdot \mathbf{r} - \omega t) = Cos[\frac{2\pi}{\lambda} (\hat{\mathbf{k}} \cdot \mathbf{r} - \upsilon t)] = Cos[8\pi \cdot 10^6 m^{-1} (\frac{x - y + z}{\sqrt{3}} - \frac{3}{2} 10^8 \frac{m}{s} t)]$$

5) A monochromatic, plane wave propagates through a glass block shown in the figure. Calculate and draw the geometrical shape of the wavefront after the exit from the glass block. Refractive index of glass n = 3/2.

(all distances are in mm, Ignore diffraction and internal reflection effects)

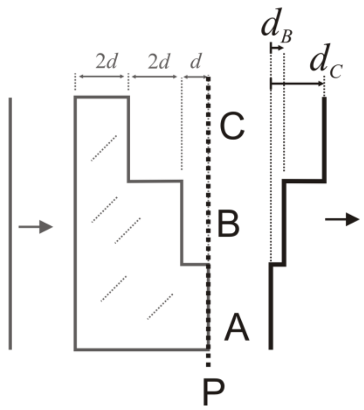


Solution:

According to the figure bellow the optical path in the exit plane P for each section (A, B, C) is given by:

$$(OPL)_A : \ (2d + 2d + d) \cdot n = 5 \cdot n \cdot d$$
$$(OPL)_B : \ (2d + 2d) \cdot n + d = (4n+1) \cdot d$$
$$(OPL)_C : \ 2d \cdot n + 3d = (2n+3) \cdot d$$

(d = 1 mm)

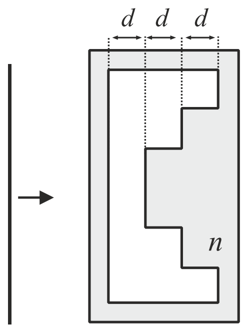


The wavefront will "deform" at the exit, in such a way so that each section will have propagated the same optical path. Thus according to the figure:

$$d_B = (OPL)_A - (OPL)_B = 5 \cdot n \cdot d - (4n+1) \cdot d = (n-1)d = 0.5\,mm$$
$$d_C = (OPL)_A - (OPL)_C = 5 \cdot n \cdot d - (2n+3) \cdot d = 3(n-1)d = 1.5\,mm$$

6) A monochromatic, plane wave propagates through a glass block shown in the figure. Calculate and draw the geometrical shape of the wavefront after the exit from the glass block.

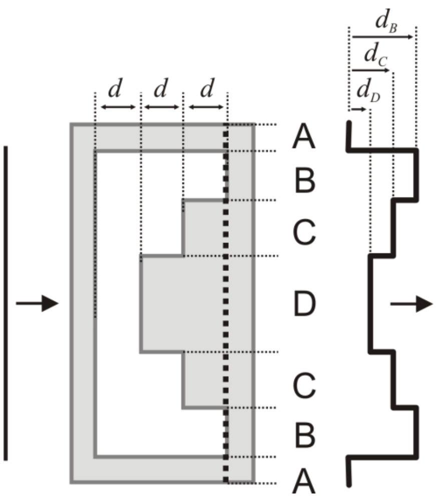
(Ignore diffraction and internal reflection effects)



Solution:

According to the following figure the optical path for each wavefront section (A, B, C) at the exit plane P is described by:

$$(OPL)_A:\ 3\cdot n\cdot d$$
$$(OPL)_B:\ 3\cdot d$$
$$(OPL)_C:\ 2d+n\cdot d=(n+2)\cdot d$$
$$(OPL)_D:\ d+2n\cdot d=(2n+1)\cdot d$$
$$(OPL)_A>(OPL)_D>(OPL)_C>(OPL)_B$$



The wavefront will “deform” at the exit, in such a way so that each section will have propagated the same optical path. Thus according to the figure:

$$d_B=(OPL)_A-(OPL)_B=3n\cdot d-3d=3(n-1)d$$
$$d_C=(OPL)_A-(OPL)_C=3n\cdot d-(n+2)\cdot d=2(n-1)d$$
$$d_D=(OPL)_A-(OPL)_D=3n\cdot d-(2n+1)\cdot d=(n-1)d$$

Dimitris Papazoglou dpapa@materials.uoc.gr

Refractive index

7) Prove that the known dispersion equation

$$n^2 = 1 + \frac{Nq^2}{\varepsilon_o m_e} \frac{1}{\omega_o^2 - \omega^2} \qquad \text{(a)}$$

in the visible spectral range, and for transparent media, reduces to the Cauchy approximation for the refractive index:

$$n = B + \frac{C}{\lambda_o^2} \qquad \text{(b)}$$

Solution

We can rewrite (a) as: $n^2 = 1 + \frac{Nq^2}{\varepsilon_o m_e \omega_o^2}(1 - \frac{\omega^2}{\omega_o^2})^{-1}$

In transparent media the resonance frequency ω_o lies in the ultraviolet so for the visible range we can safely assume that $\omega \ll \omega_o$ and approximate the frequency related term:

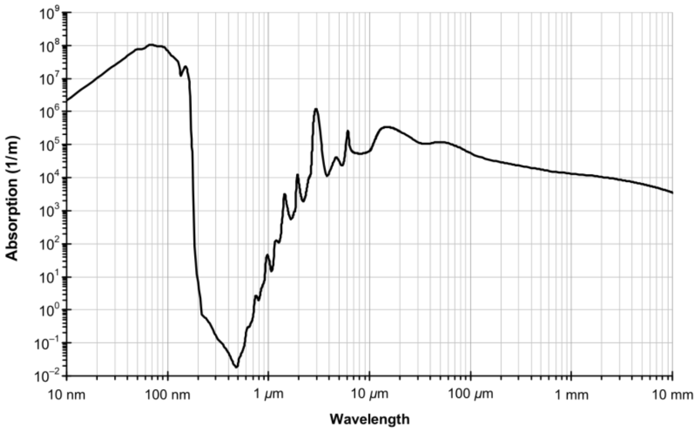
$$\omega \ll \omega_o \Rightarrow (1 - \frac{\omega^2}{\omega_o^2})^{-1} \cong 1 + \frac{\omega^2}{\omega_o^2}$$

Based on the above, we can rewrite (a):

$$\left.\begin{array}{c} n^2 \cong 1 + \frac{Nq^2}{\varepsilon_o m_e \omega_o^2}(1 + \frac{\omega^2}{\omega_o^2}) = 1 + \frac{Nq^2}{\varepsilon_o m_e \omega_o^2} + \frac{Nq^2}{\varepsilon_o m_e \omega_o^2}\frac{\omega^2}{\omega_o^2} \\ \omega = k_o c = \frac{2\pi c}{\lambda_o} \end{array}\right\} \Rightarrow n^2 \cong \overbrace{1 + \frac{Nq^2}{\varepsilon_o m_e \omega_o^2}}^{B} + \underbrace{\frac{Nq^2}{\varepsilon_o m_e \omega_o^2}\frac{4\pi c^2}{\omega_o^2}}_{C}\frac{1}{\lambda_o^2} \Rightarrow$$

$$n^2 \cong B + \frac{C}{\lambda_o^2}$$

8) Estimate imaginary part of the refractive index and the penetration depth (in meters and in wavelengths) for light propagating in water for for $\lambda_o = 0.5\ \mu m$. *(assume pure H_2O)*



"Absorption spectrum of liquid water" by Kebes at English Wikipedia. Licensed under CC BY-SA 3.0 via Commons - https://commons.wikimedia.org/wiki/File:Absorption_spectrum_of_liquid_water.png#/media/File:Absorption_spectrum_of_liquid_water.png

Solution

As we know the absorption coefficient a is proportional to the imaginary part of the refractive index κ:

$$a(\lambda_o) = 2\kappa(\lambda_o)\,k_o = \frac{4\pi\,\kappa(\lambda_o)}{\lambda_o} \Rightarrow \kappa(\lambda_o) = \frac{\lambda_o\,a(\lambda_o)}{4\pi}.$$

Since according to bibliography $a_{\mathrm{H_2O}}(0.5\,\mu m) \simeq 0.02\,m^{-1}$ then:

$$\kappa_{\mathrm{H_2O}}(0.5\,\mu m) = \frac{0.5\cdot 10^{-6}\,m \cdot 0.02\,m^{-1}}{4\pi} = \frac{2.5}{\pi}10^{-9} \cong 7.96\cdot 10^{-10}$$

On the other hand, the penetration depth d is the inverse of the absorption coefficient $d(\lambda_o) = \dfrac{1}{\alpha(\lambda_o)}$ thus:

$$d_{\mathrm{H_2O}}(0.5\,\mu m) \simeq \frac{1}{0.02\,m^{-1}} = 50\,m = 10^8\,\lambda_o$$

Polarization

9) Analytically describe the following E/M plane waves *(propagation in vacuum)*

(λ_o is the vacuum wavelength, while the vacuum light velocity is $c \cong 3\ 10^8$ m/sec)

i) Right handed circularly polarized (RCPL), $\mathbf{k} // \hat{\mathbf{x}},\ \ \lambda_o = 1\ \mu m,\ \ n = 1.5$

ii) Linearly polarized (LP), $\mathbf{E} \perp \hat{\mathbf{z}},\ \ \mathbf{k} // \hat{\mathbf{x}} + \hat{\mathbf{y}},\ \ \lambda_o = 0.5\ \mu m,\ \ n = 2$

iii) Left handed circularly polarized (LCPL), $\mathbf{k} // \hat{\mathbf{x}} + \hat{\mathbf{y}},\ \ \lambda_o = 1\ \mu m,\ \ n = 1.5$

Solution

The general description of a harmonic, plane E/M wave is the following:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_o \cos[\mathbf{k}\cdot\mathbf{r} - \omega t] = \mathbf{E}_o \cos\left[2\pi\left(\frac{1}{\lambda}\hat{\mathbf{k}}\cdot\mathbf{r} - \nu t\right)\right]] = \mathbf{E}_o \cos\left[\frac{2\pi}{\lambda_o}\left(n\hat{\mathbf{k}}\cdot\mathbf{r} - ct\right)\right]]$$

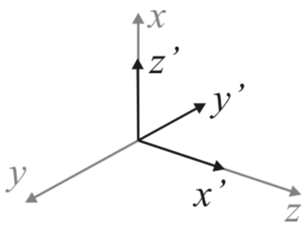
where $\mathbf{E}_o$ is the polarization, $\hat{\mathbf{k}}$ is a unitary vector parallel to the wavevector $\mathbf{k}$, λ_o is the vacuum wavelength, ω is the angular frequency, n is the refractive index of the medium and c is light velocity in vacuum. According to this notation, the propagation speed, the refractive index and the wavelength in vacuum is given by relations:

$$\upsilon = \lambda\nu,\quad n = \frac{c}{\upsilon} = \frac{c}{\lambda\nu},\ \ \lambda_o = n\cdot\lambda$$

i) Thus a wave that propagates along $\hat{\mathbf{x}}$ direction will be described by $\hat{\mathbf{k}} = \hat{\mathbf{x}}$. To describe polarization we should define a new right handed coordinate system in which the new z' axis will be parallel to the wavevector: $\hat{\mathbf{z}}' = \hat{\mathbf{k}}$.

Thus as shown in the figure:

$$\hat{\mathbf{x}}' = \hat{\mathbf{z}}$$
$$\hat{\mathbf{y}}' = -\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}}' = \hat{\mathbf{x}},$$
$$\hat{\mathbf{k}} = \hat{\mathbf{z}}'$$



In this new coordinate system the right handed circularly polarized wave will be described by:

$$\mathbf{E}(\mathbf{r},t)=\frac{E_o}{\sqrt{2}}(\hat{\mathbf{x}}'-i\,\hat{\mathbf{y}}')\cos\left[2\pi\left(\frac{1}{\lambda}\hat{\mathbf{k}}\cdot\mathbf{r}-\nu t\right)\right]=\frac{E_o}{\sqrt{2}}(\hat{\mathbf{x}}'-i\,\hat{\mathbf{y}}')\cos\left[\frac{2\pi}{\lambda_o}\left(n\hat{\mathbf{k}}\cdot\mathbf{r}-c\,t\right)\right]$$

Transforming back to the original (x, y, z) coordinate system:

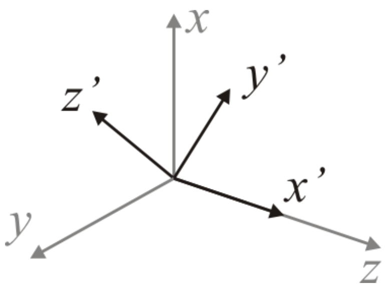
$$\mathbf{E}(\mathbf{r},t)=\frac{E_o}{\sqrt{2}}(\hat{\mathbf{z}}+i\,\hat{\mathbf{y}})\cos\left[2\pi 10^6 m^{-1}\left(1.5\,x-3\cdot 10^8\,\frac{m}{\sec}t\right)\right]$$

ii) A wave that propagates along $\hat{\mathbf{x}}+\hat{\mathbf{y}}$ direction will be described by $\hat{\mathbf{k}}=\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}})$.

To describe polarization we should define a new right handed coordinate system in which the new z' axis will be parallel to the wavevector: $\hat{\mathbf{z}}'=\hat{\mathbf{k}}$.

Thus as shown in the figure:

$$\hat{\mathbf{x}}'=\hat{\mathbf{z}}$$
$$\hat{\mathbf{y}}'=\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}})$$
$$\hat{\mathbf{z}}'=\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}}),$$
$$\hat{\mathbf{k}}=\hat{\mathbf{z}}'$$



Since the wave is linearly polarized with $\mathbf{E}\perp\hat{\mathbf{z}}$ we can choose: $\mathbf{E}//\hat{\mathbf{y}}'$. I In this new coordinate system the linearly polarized wave will be described by::

$$\mathbf{E}(\mathbf{r},t)=E_o\,\hat{\mathbf{y}}'\cos\left[2\pi\left(\frac{1}{\lambda}\hat{\mathbf{k}}\cdot\mathbf{r}-\nu t\right)\right]=E_o\,\hat{\mathbf{y}}'\cos\left[\frac{2\pi}{\lambda_o}\left(n\hat{\mathbf{k}}\cdot\mathbf{r}-c\,t\right)\right]$$

Transforming back to the original (x, y, z) coordinate system:

$$\mathbf{E}(\mathbf{r},t)=\frac{E_o}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}})\cos\left[4\pi 10^6 m^{-1}\left(\sqrt{2}\,(x+y)-3\cdot 10^8\,\frac{m}{\sec}t\right)\right]$$

iii) Since the propagation direction is again $\hat{\mathbf{x}}+\hat{\mathbf{y}}$ we will use the same coordinate system as in (ii). Thus in this case we have $\hat{\mathbf{k}}=\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}})$ and the wave will be described by:

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}(\mathbf{r},t)=\frac{E_o}{\sqrt{2}}(\hat{\mathbf{x}}'+i\,\hat{\mathbf{y}}')\cos\left[2\pi\left(\frac{1}{\lambda}\hat{\mathbf{k}}\cdot\mathbf{r}-\nu t\right)\right]]=\frac{E_o}{\sqrt{2}}(\hat{\mathbf{x}}'+i\,\hat{\mathbf{y}}')\cos\left[\frac{2\pi}{\lambda_o}\left(n\hat{\mathbf{k}}\cdot\mathbf{r}-c\,t\right)\right]$$

Transforming back to the original (*x*, *y*, *z*) coordinate system:

$$\mathbf{E}(\mathbf{r},t)=\frac{E_o}{2}[i(\hat{\mathbf{x}}-\hat{\mathbf{y}})+\sqrt{2}\,\hat{\mathbf{z}})]\cos\left[2\pi 10^6 m^{-1}\left(\frac{1.5}{\sqrt{2}}(x+y)-3\cdot 10^8\frac{m}{\sec}t\right)\right]$$

10) Analytically describe a right handed circularly polarized (RCPL) harmonic plane wave that propagates along $\mathbf{k}//\hat{\mathbf{x}}-\hat{\mathbf{y}}$ direction.

Solution

The general description of a harmonic, plane E/M wave is the following:

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_o\cos[\mathbf{k}\cdot\mathbf{r}-\omega t]=\mathbf{E}_o\cos\left[2\pi\left(\frac{1}{\lambda}\hat{\mathbf{k}}\cdot\mathbf{r}-\nu t\right)\right]]$$

where $\mathbf{E}_o$ is the polarization, $\hat{\mathbf{k}}$ is a unitary vector parallel to the wavevector $\mathbf{k}$, λ is the wavelength in the medium, ν is the frequency,. According to this notation, the propagation speed, the refractive index and the wavelength in vacuum is given by:

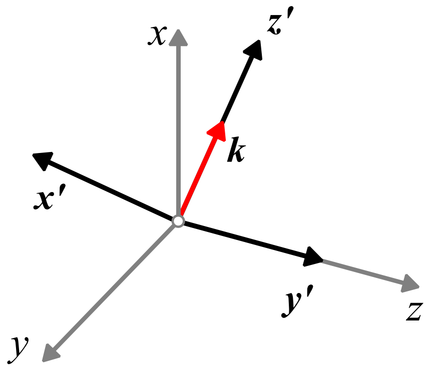
$$\upsilon=\lambda\nu,\quad n=\frac{c}{\upsilon}=\frac{c}{\lambda\nu},\ \lambda_o=n\cdot\lambda$$

A wave that propagates along $\hat{\mathbf{x}}-\hat{\mathbf{y}}$ direction will be described by $\hat{\mathbf{k}}=\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}})$.

To describe polarization we should define a new right handed coordinate system in which the new z' axis will be parallel to the wavevector: $\hat{\mathbf{z}}'=\hat{\mathbf{k}}$.

Thus as shown in the figure:

$$\hat{\mathbf{x}}' = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}})$$
$$\hat{\mathbf{y}}' = \hat{\mathbf{z}}$$
$$\hat{\mathbf{z}}' = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}}-\hat{\mathbf{y}}),$$
$$\hat{\mathbf{k}} = \hat{\mathbf{z}}'$$



In this new coordinate system the right handed circularly polarized wave will be described by:

$$\mathbf{E}(\mathbf{r},t) = E_o(\hat{\mathbf{x}}' - i\,\hat{\mathbf{y}}')\cos\left[2\pi\left(\frac{1}{\lambda}\hat{\mathbf{k}}\cdot\mathbf{r} - \nu t\right)\right]] = E_o(\hat{\mathbf{x}}' - i\,\hat{\mathbf{y}}')\cos\left[2\pi\left(\frac{1}{\lambda}z' - \nu t\right)\right]]$$

Transforming back to the original (x, y, z) coordinate system::

$$\mathbf{E}(\mathbf{r},t) = E_o[\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}}) - i\,\hat{\mathbf{z}}]\cos\left[2\pi\left(\frac{1}{\lambda}\hat{\mathbf{k}}\cdot\mathbf{r} - \nu t\right)\right]]$$
$$= \frac{E_o}{\sqrt{2}}[(\hat{\mathbf{x}}+\hat{\mathbf{y}}) - i\sqrt{2}\,\hat{\mathbf{z}}]\cos\left[2\pi\left(\frac{1}{\lambda}\frac{x-y}{\sqrt{2}} - \nu t\right)\right]]$$

11) Identify the polarization state, the propagation direction and the refractive index for the following E/M waves *(assume SI units, the vacuum light velocity is $c \simeq 3\ 10^8$ m/sec)*

i) $\mathbf{E}(\mathbf{r},t) = \hat{\mathbf{z}}\cos\left\{4\pi[\frac{1}{\sqrt{2}}\cdot 10^6(x+y) - 10\cdot 10^{13}t]\right\}$

ii) $\mathbf{E}(\mathbf{r},t) = (\hat{\mathbf{y}} + i\,\hat{\mathbf{z}})\cos[2\pi(2\cdot 10^6 x - 4\cdot 10^{14}t)]$

Solution

The general description of a harmonic, plane E/M wave is the following:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_o\cos[\mathbf{k}\cdot\mathbf{r} - \omega t] = \mathbf{E}_o\cos\left[2\pi\left(\frac{1}{\lambda}\hat{\mathbf{k}}\cdot\mathbf{r} - \nu t\right)\right]] = \mathbf{E}_o\cos\left[\frac{2\pi}{\lambda_o}\left(n\hat{\mathbf{k}}\cdot\mathbf{r} - ct\right)\right]]$$

i) Applying the general description we get:

$$\mathbf{E}(\mathbf{r},t)=\hat{\mathbf{z}}\cos\left\{4\pi[\frac{1}{\sqrt{2}}\cdot 10^{6}(x+y)-10\cdot 10^{13}t]\right\}\Rightarrow\begin{cases}\mathbf{E}_o=\hat{\mathbf{z}}\\ \hat{\mathbf{k}}\cdot\mathbf{r}=\frac{1}{\sqrt{2}}(x+y)\Rightarrow\hat{\mathbf{k}}=\frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}})\\ \frac{1}{\lambda}=2\cdot 10^{6}m^{-1}\Rightarrow\lambda=0.5\cdot 10^{-6}m=0.5\mu m\\ \nu=10\cdot 10^{13}Hz\Rightarrow\upsilon=\lambda\nu=0.5\cdot 10^{8}\frac{m}{s}\Rightarrow\\ n=\frac{c}{\upsilon}=\frac{3\cdot 10^{8}m/s}{0.5\cdot 10^{8}m/s}=6\\ \lambda_o=n\cdot\lambda=6\cdot 0.5\mu m=3\mu m\end{cases}$$

According to the above, this is a linearly polarized (parallel to the $\hat{\mathbf{z}}$ direction) wave, propagating along the $(\hat{\mathbf{x}}+\hat{\mathbf{y}})$ direction, while the refractive index of the medium is n = 6 and the vacuum wavelength is λ_o = 3μm

ii) Applying the general description we get

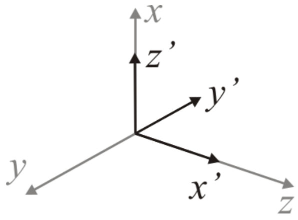
$$\mathbf{E}(\mathbf{r},t)=(\hat{\mathbf{y}}+i\,\hat{\mathbf{z}})\cos[2\pi(2\cdot 10^{6}x-4\cdot 10^{14}t)]\Rightarrow\begin{cases}\mathbf{E}_o=\hat{\mathbf{y}}+i\,\hat{\mathbf{z}}\\ \hat{\mathbf{k}}\cdot\mathbf{r}=x\Rightarrow\hat{\mathbf{k}}=\hat{\mathbf{x}}\\ \frac{1}{\lambda}=2\cdot 10^{6}m^{-1}\Rightarrow\lambda=0.5\cdot 10^{-6}m=0.5\mu m\\ \nu=4\cdot 10^{14}Hz\Rightarrow\upsilon=\lambda\nu=2\cdot 10^{8}\frac{m}{s}\Rightarrow\\ n=\frac{c}{\upsilon}=\frac{3\cdot 10^{8}m/s}{2\cdot 10^{8}m/s}=1.5\\ \lambda_o=n\cdot\lambda=1.5\cdot 0.5\mu m=0.75\mu m\end{cases}$$

According to the above, this is wave, propagates along the $\hat{\mathbf{x}}$ direction, while the refractive index of the medium is n = 1.5 and the vacuum wavelength is λ_o = 0.75 μm.

To describe the polarization we should define a new right handed coordinate system in which the new z' axis will be $\hat{\mathbf{z}}'\equiv\hat{\mathbf{k}}=\hat{\mathbf{x}}$.

Thus as shown in the figure:

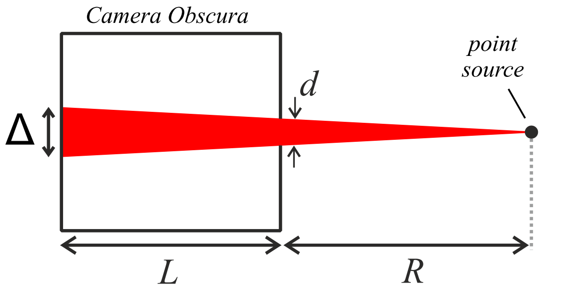
$$\hat{\mathbf{x}}' = \hat{\mathbf{z}}$$
$$\hat{\mathbf{y}}' = -\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}}' = \hat{\mathbf{x}},$$
$$\hat{\mathbf{k}} = \hat{\mathbf{z}}'$$



In this new coordinate system the polarization vector is described as $\mathbf{E}_o = -\hat{\mathbf{y}}' + i\,\hat{\mathbf{x}}' = i\,(\hat{\mathbf{x}}' + i\,\hat{\mathbf{y}}')$ thus we conclude that the wave is left handed circularly polarized light.

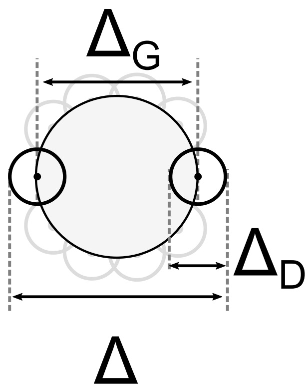
Resolution

12) What is the optimum pinhole diameter for a camera obscura?



Solution

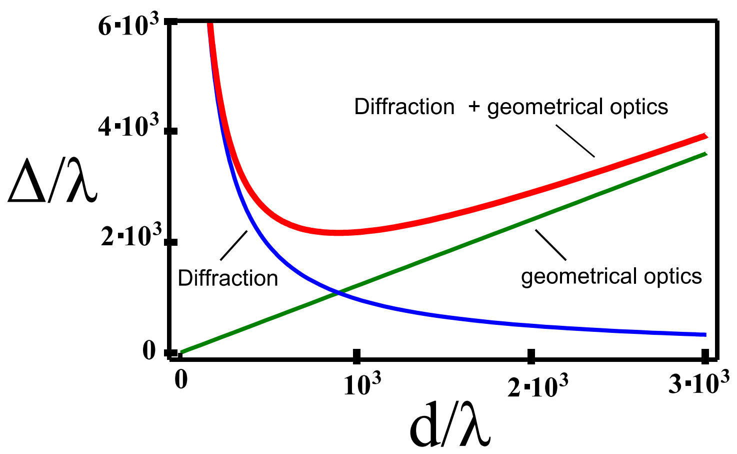
Each point from the object will be imaged to a disc of diameter Δ. We can estimate the size of this disk by taking into account that this is a result of a combination of geometrical imaging and diffraction. Thus the diameter that Δ will be approximately the sum of two factors the diameter Δ_G of a "geometrical" disk image and the diameter Δ_D of an Airy disk resulting from diffraction:



$$\Delta = \Delta_\Gamma + \Delta_\Pi = \frac{R+L}{R} d + 2.44 \cdot \frac{\lambda L}{d} \cong d + 2.44 \cdot \frac{\lambda L}{d}, \qquad (R \gg L)$$

We will get the minimum disk size when:

$$\frac{\partial \Delta}{\partial d} = 0 \Rightarrow 1 - 2.44 \cdot \frac{\lambda L}{d^2} \cong 0 \Rightarrow d \cong \sqrt{2.44 \lambda L}$$



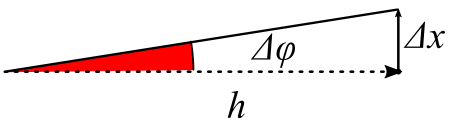
Disk diameter as a function of the pinhole diameter (wavelength units)

Dimitris Papazoglou dpapa@materials.uoc.gr

13) Hubble telescope is in orbit around earth and it is equipped with a primary mirror of 2.4 *m* diameter. What is the minimum crater diameter that this telescope can in principle detect on the surface of the Moon?

(observation wavelength λ_o *= 500 nm, Hubble- Moon distance ~*$4{\cdot}10^5$ *km)*

Solution



The angular resolving power of a telescope $\Delta\varphi$ can be estimated by: $\Delta\varphi \cong 1.22\dfrac{\lambda_o}{D}$

where λ_o is the wavelength and D the primary mirror diameter. This angular resolution corresponds objects that at a distance *h* are Δx apart. From the geometry of the problem we have:

$$\Delta x \cong h\cdot\Delta\varphi = 1.22\frac{\lambda_o}{D}h$$

In our case this is $\Delta x \cong 1.22\cdot 500\cdot 10^{-9}m\cdot 4\cdot 10^{8}m/2.4m = 101.67m$.

Thus the minimum crater diameter that Hubble can resolve on the Moon's surface is ~ 102 *m*:

14) A satellite that is in orbit around earth is equipped with a photographic lens of *f* = 50 *mm* focal distance and #F = 2. The image from the lens is projected on a CCD sensor with 6 μm pixel size. Calculate the resolving power of the whole imaging system assuming that the orbit height is *h* = 100 *km*. Can this be improved?

(observation wavelength λ_o *= 550 nm)*

Solution

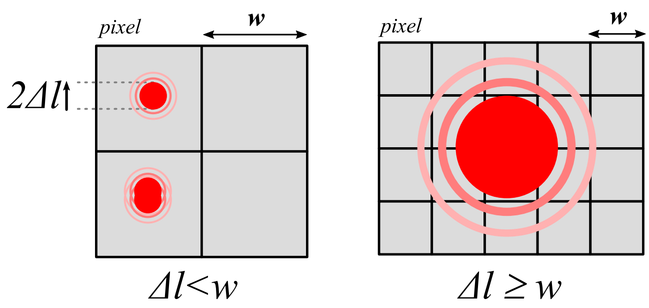
According to Rayleigh criterion the spatial resolution of a lens on the image plane is:

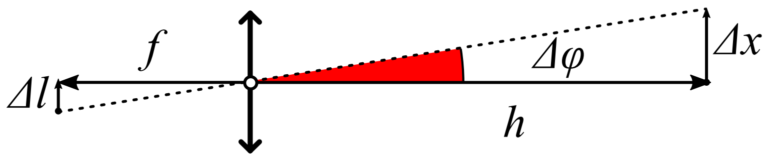
$$\Delta l \cong 1.22\frac{\lambda_o}{D}f = 1.22\lambda_o(\#F)$$

where λ_o is the wavelength, *D* is the entrance pupil diameter, *f* is the focal length and $(\#F) \equiv \dfrac{f}{D}$ is the F-Number of the optical system. This spatial resolution Δl corresponds to the minimum separation between the images of two bright point objects that are marginally resolved on the image plane, i.e. on the CCD sensor. For the optical system provided this resolution is:

$$\Delta l \cong 1.22\, 0.55\mu m \cdot 2 = 1.34\,\mu m$$

Since Δl is smaller than the pixel size w the spatial resolution of the imaging system is limited by the sensor and not the optical system angular resolution. Thus in this system the spatial resolution in the image plane is $\Delta l_s \cong w = 6\,\mu m$.





At a distance h the imaging system will be capable of resolving two bright point sources separated by a distance Δx:

$$\Delta x \cong h \cdot \frac{\Delta l_s}{f} = 100\,km\frac{6\,\mu m}{50\,mm} = 12\,m$$

It is obvious that in the imaging system the optical system resolution is not well matched to the CCD sensor pixel size. This results in poor resolution on the earth surface. A way to solve this and improve the overall system performance is to change the optical system.

For example, we can relax the resolution of the optical system to match to the CCD sensor size. In this case we will get a new F-Number:

$$\Delta l' = 6\,\mu m \Rightarrow (\#F)' \cong \frac{\Delta l'}{1.22\,\lambda_o} = \frac{6\mu m}{1.22 \cdot 0.55\mu m} = 8.9$$

Assuming that the diameter of the entrance pupil remains the same we get that the focal length of our new lens should be:

$$f' = \frac{(\#F)'}{(\#F)} f \cong \frac{8.9}{2} \cdot 50\,mm \cong 222\,mm$$

Although the imaging system resolution is the same as previously ($6\,\mu m$) the resolving power on the earth surface is enhanced by a ~ x4.5 factor since:

$$\Delta x' \cong h \cdot \frac{\Delta l'}{f'} = 100\,km \frac{6\,\mu m}{222\,mm} = 2.7\,m$$

On the other hand we should note that the image illumination is reduced by ~ 20 times since the *#F* is increased.

Geometrical Optics

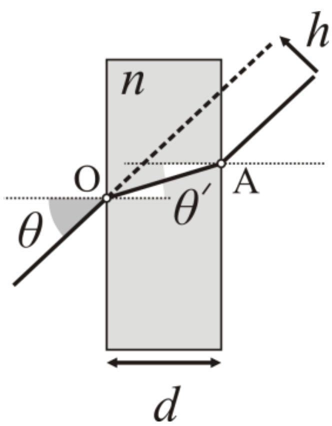
15) Estimate the parallel displacement of an optical ray that propagates through a glass plate with thickness *d* and refractive index *n*.

(assume that the glass plate in surrounded by air, $n_{air} = 1$)

Solution

$$\left.\begin{aligned} h=(OA)\sin(\theta-\theta') \Rightarrow h=\frac{d}{\cos\theta'}\sin(\theta-\theta') \\ \sin(\theta-\theta')=\sin\theta\cos\theta'-\cos\theta\sin\theta' \end{aligned}\right\}$$

$$\Rightarrow h=d(\sin\theta-\cos\theta\tan\theta') \quad (1)$$



From Snell's law we know that $\sin\theta'=\frac{1}{n}\sin\theta$. Likewise $\tan\theta'=\sin\theta'(1-\sin^2\theta')^{-1/2}$ so we can write:

$$\tan\theta'=\frac{1}{n}\frac{\sin\theta}{\sqrt{1-\frac{\sin^2\theta}{n^2}}}=\frac{\sin\theta}{\sqrt{n^2-\sin^2\theta}}$$

By replacing in eq. (1) we get: $h=d\sin\theta(1-\frac{\cos\theta}{\sqrt{n^2-\sin^2\theta}})$

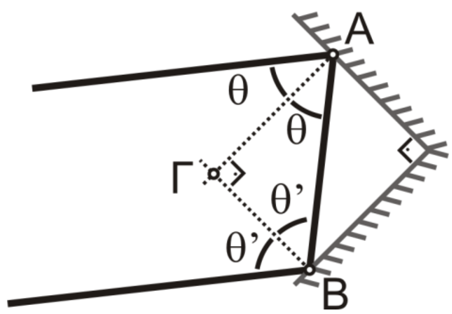
16) Prove that after two reflections from both sides of a corner reflector an optical ray exits parallel to its initial direction

Solution

It is sufficient to prove that $2\theta+2\theta'=180^\circ$.

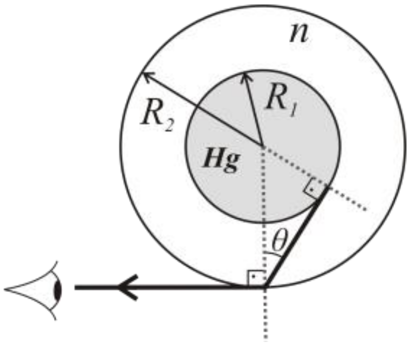
Indeed since the mirrors are perpendicular to each other we get from the *(ABΓ)* triangle that:

$$\theta+\theta'=90^\circ \Rightarrow 2\theta+2\theta'=180^\circ$$



17) A hollow glass tube contains Hg. If R_1, R_2 are respectively the inner and outer radius of curvature of the cylinder and n = 3/2 is the index of refraction of glass prove that above a certain value of the R_1/R_2 ratio Hg appears, for by an observer facing tangentially to the cylinder, to fill the whole cylinder volume.

Solution



According to the figure, for the cylinder to appear completely filled with Hg, the rays that exit tangentially should originate from the Hg.

Assuming that θ_c is the critical angle, we get from Snell's law: $\sin\theta_c = \frac{1}{n}\sin 90^o = \frac{1}{n}$.

On the other hand from the geometry of the problem (see figure) $R_1 = R_2 \sin\theta \Rightarrow R_1 = \frac{R_2}{n}$.

For values of $R_1 < \frac{R_2}{n}$ the rays that exit tangentially do not originate from the Hg. For Hg to appear as if it fills up the whole cylinder we should have $R_1 \geq \frac{R_2}{n} \Rightarrow \frac{R_1}{R_2} \geq \frac{2}{3}$

18) Prove that an optical system composed by two thin lenses in contact is achromatic if the following condition is fulfilled:

$$V_1 f_1^Y + V_2 f_2^Y = 0$$

where $V_1 = (n_1^Y - 1)/(n_1^B - n_1^R)$, $V_2 = (n_2^Y - 1)/(n_2^B - n_2^R)$ are the Abbe numbers of the optical materials the lenses are composed of and f_1^Y, f_2^Y their focal lengths.

($Y \rightarrow$ 589 nm, $B \rightarrow$ 486 nm, $R \rightarrow$ 656 nm)

Solution

Using lens maker's formula for Y (yellow), we can estimate the geometric parameters ρ :

$$\left.\begin{aligned} \frac{1}{f_1^Y} = (n_1^Y - 1)\rho_1,\ \ \rho_1 \equiv (\frac{1}{R_1^1} - \frac{1}{R_2^1}) \\ \frac{1}{f_2^Y} = (n_2^Y - 1)\rho_2,\ \ \rho_2 \equiv (\frac{1}{R_1^2} - \frac{1}{R_2^2}) \end{aligned}\right\} \Rightarrow \begin{aligned} \rho_1 = \frac{1}{f_1^Y}\frac{1}{(n_1^Y - 1)} \\ \rho_2 = \frac{1}{f_2^Y}\frac{1}{(n_2^Y - 1)} \end{aligned}$$

Since the system is achromatic the total focal length in the R (red) and B (blue) are equal:

$$f_{tot}^{R} = f_{tot}^{B} \Rightarrow \frac{1}{f_{tot}^{R}} = \frac{1}{f_{tot}^{R}}$$

Using the lens maker's formulas and replacing the known (from Y) geometric parameters ρ_1, ρ_2 we get :

$$(n_1^R - 1)\rho_1 + (n_2^R - 1)\rho_2 = (n_1^B - 1)\rho_1 + (n_2^B - 1)\rho_2 \Rightarrow$$
$$(n_1^B - n_1^R)\rho_1 + (n_2^B - n_2^R)\rho_2 = 0 \Rightarrow \frac{(n_1^B - n_1^R)}{(n_1^Y - 1)}\frac{1}{f_1^Y} + \frac{(n_2^B - n_2^R)}{(n_2^Y - 1)}\frac{1}{f_2^Y} = 0 \Rightarrow$$
$$\frac{1}{V_1 f_1^Y} + \frac{1}{V_2 f_2^Y} = 0 \Rightarrow V_1 f_1^Y + V_2 f_2^Y = 0$$

19) Calculate the radii of curvature of a bi-convex lens made of glass with Abbe number V_1 and a plano-concave lens made of glass with Abbe number V_2 that are in contact so as to form an achromatic system with an effective focal length f_o

Solution

For a lens system composed by thin lenses in contact to be achromatic the following condition must hold:

$$V_1 f_1^Y + V_2 f_2^Y = 0 \qquad (1)$$

Furthermore, the effective focal length f_o of such a system is given by:

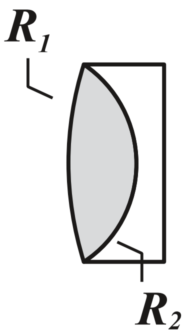
$$\frac{1}{f_o^Y} = \frac{1}{f_1^Y} + \frac{1}{f_2^Y} \qquad (2)$$

From eq. (1) we have $f_1^Y = -\frac{V_2}{V_1} f_2^Y$, and by replacing in eq. (2) we get :

$$\frac{1}{f_o^Y} = -\frac{V_1}{V_2}\frac{1}{f_2^Y} + \frac{1}{f_2^Y} = (1-\frac{V_1}{V_2})\frac{1}{f_2^Y} \Rightarrow$$

$$f_2^Y = \frac{V_2 - V_1}{V_2} f_o^Y \Rightarrow f_1^Y = -\frac{V_2 - V_1}{V_1} f_o^Y$$

To estimate the radii of curvature of the lenses we use the lens maker's formula for each lens. Thus for the 2nd plano-concave lens we get:



$$\frac{1}{f_2^Y} = (n_2^Y - 1)(\frac{1}{R_2} - \frac{1}{\infty}) \Rightarrow R_2 = (n_2^Y - 1) f_2^Y \Rightarrow R_2 = (n_2^Y - 1)\frac{V_2 - V_1}{V_2} f_o^Y$$

On the other hand for the 1st bi-convex lens:

$$\left.\begin{array}{l} \dfrac{1}{f_1^Y} = (n_1^Y - 1)(\dfrac{1}{R_1} - \dfrac{1}{R_2}) \Rightarrow \dfrac{1}{R_1} = \dfrac{1}{n_1^Y - 1}\dfrac{1}{f_1^Y} + \dfrac{1}{R_2} \\ \dfrac{1}{R_2} = \dfrac{1}{n_2^Y - 1}\cdot\dfrac{V_2}{V_2 - V_1}\dfrac{1}{f_o^Y}, \quad \dfrac{1}{f_1^Y} = -\dfrac{V_1}{V_2 - V_1}\dfrac{1}{f_o^Y} \end{array}\right\} \Rightarrow \frac{1}{R_1} = [1 - \frac{n_2^Y - 1}{n_1^Y - 1}\cdot\frac{V_1}{V_2}]\frac{1}{n_2^Y - 1}\frac{V_2}{V_2 - V_1}\frac{1}{f_o^Y} \Rightarrow$$

$$R_1 = [1 - \frac{n_2^Y - 1}{n_1^Y - 1}\cdot\frac{V_1}{V_2}]^{-1}(n_2^Y - 1)\frac{V_2 - V_1}{V_2} f_o^Y = [1 - \frac{n_2^Y - 1}{n_1^Y - 1}\cdot\frac{V_1}{V_2}]^{-1} R_2$$

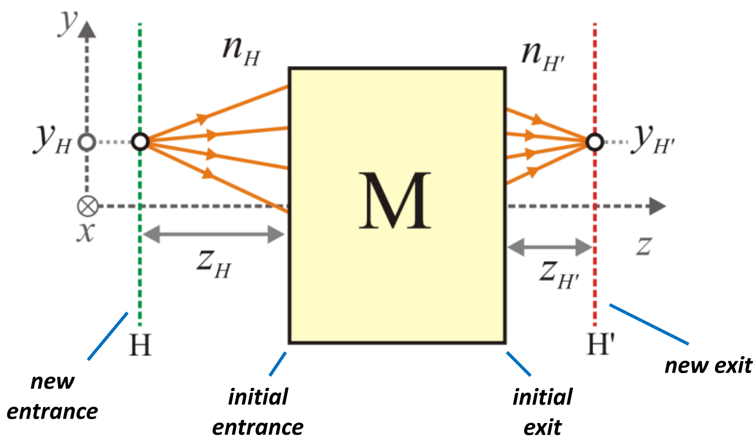
Matrix Theory

Simple systems

20) Prove that for an optical system described by a 2x2 matrix $\mathcal{M}$ the reduced distances of its principal planes *H, H'* from the entrance and exit of the system are

$$Z_H = \frac{1-M_{22}}{M_{21}},\ Z_{H'} = \frac{1-M_{11}}{M_{21}}$$

Solution



We define a new system with a new entrance and exit located at distances z_H and $z_{H'}$ respectively from the original ones. The matrix that describes this new system is:

$$\begin{bmatrix} y_{H'} \\ n_{H'}\beta_{H'} \end{bmatrix} = \begin{bmatrix} 1 & Z_{H'} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \cdot \begin{bmatrix} 1 & Z_H \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_H \\ n_H\beta_H \end{bmatrix}$$

where $Z_H \equiv \frac{z_H}{n_s}$, $Z_{H'} \equiv \frac{z_{H'}}{n_{s'}}$. For the new entrance and exit to be principal planes the matrix of the new system should be equal to a simple matrix of a diopter:

$$\begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & Z_{H'} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \cdot \begin{bmatrix} 1 & Z_H \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} M_{11} + Z_{H'}M_{21} & Z_H M_{11} + M_{12} + Z_{H'}\left(Z_H M_{21} + M_{22}\right) \\ M_{21} & M_{22} + Z_H M_{21} \end{bmatrix} \Rightarrow$$

$$\begin{cases} P \equiv -M_{21} \\ Z_H \equiv \dfrac{1-M_{22}}{M_{21}}, \quad Z_{H'} \equiv \dfrac{1-M_{11}}{M_{21}} \end{cases}$$

21) Calculate the all primary points (principal planes, optical power, foci, nodal points), of a thick plano-convex lens of thickness d and refractive index n that is surrounded by optical media of refractive index n_1, n_2

(assume that the radii of curvature of the lens surfaces are respectively R_1, $R_2=+\infty$))

Solution

The matrix describing a thick lens is

$$M=\begin{pmatrix}1 & 0\\ -P_2 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & D\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ -P_1 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ -P_2 & 1\end{pmatrix}\cdot\begin{pmatrix}1-DP_1 & D\\ -P_1 & 1\end{pmatrix}$$
$$=\begin{pmatrix}1-DP_1 & D\\ -(P_1+P_2-DP_1P_2) & 1-DP_2\end{pmatrix}$$

Where $P_1=\dfrac{n-n_1}{R_1}$, $P_2=\dfrac{n_2-n}{R_2}=0$, $D=\dfrac{d}{n}$.

Applying to M we get: $M=\begin{pmatrix}1-DP_1 & D\\ -P_1 & 1\end{pmatrix}$

The primary points of the optical system are then:

Optical power: $P_{tot}=P_1=\dfrac{n-n_1}{R_1}$

Principal planes:
$$Z_H\equiv\frac{1-M_{22}}{M_{21}}=\frac{1-1}{-P_1}=0,$$
$$Z_{H'}\equiv\frac{1-M_{11}}{M_{21}}=\frac{1-(1-DP_1)}{-P_1}=-D=-\frac{d}{n}$$

Foci:
$$f\equiv\frac{n_1}{P_{tot}}=\frac{n_1}{P_1}=\frac{n_1}{n-n_1}R_1,$$
$$f'\equiv\frac{n_2}{P_{tot}}=\frac{n_2}{P_1}=\frac{n_2}{n-n_1}R_1$$

Nodal points: $\ell_N\equiv-\ell_{N'}\equiv\dfrac{n_1-n_2}{n_1}f=\dfrac{n_1-n_2}{n_1}\dfrac{n_1}{n-n_1}R_1=\dfrac{n_1-n_2}{n-n_1}R_1$

22) Prove that the effective focal length of an optical system composed by N thin lenses with focal distances $f_1, f_2, \ldots, f_N$ that are in contact is given by:

$$\frac{1}{f_{tot}} = \frac{1}{f_1} + \frac{1}{f_2} + \ldots + \frac{1}{f_N}$$

Solution

This system is described by the matrix:

$$M_{tot} = \begin{pmatrix} 1 & 0 \\ -P_N & 1 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} 1 & 0 \\ -P_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix}, \quad P_i = \frac{1}{f_i}, i = 1, 2, \ldots N$$

Since for any combination P_i, P_j we get: $\begin{pmatrix} 1 & 0 \\ -P_j & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -[P_i + P_j] & 1 \end{pmatrix}$, we can iteratively repeat for the total matrix M_{tot} and write

$$M_{tot} = \begin{pmatrix} 1 & 0 \\ -\sum_{i=1}^{N} P_N & 1 \end{pmatrix} \Rightarrow P_{tot} = \sum_{i=1}^{N} P_i \Rightarrow \frac{1}{f_{tot}} = \frac{1}{f_1} + \frac{1}{f_2} + \ldots + \frac{1}{f_N}$$

23) Prove that in a telescopic optical system $(P_{tot} = 0)$ the transverse and the longitudinal magnification are constant and depend only from the angular magnification M_a of the system.

Solution

The matrix that describes a telescopic system is in general: $M = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{21} \end{pmatrix}$.

From the definition of ray vectors in matrix theory we know:

$$n_{s'}\beta_{s'} = 0 \cdot y_s + M_{22} \cdot n_s \beta_s \Rightarrow \frac{n_{s'}\beta_{s'}}{n_s \beta_s} = M_{22}$$

where n_s, $n_{s'}$ are respectively the refractive indices of the media in the entrance and the exit of the optical system. In the case where those media are the same this simplifies to:

$$n_s = n_{s'} \Rightarrow M_{22} = \frac{\beta_{s'}}{\beta_s} \equiv M_a$$

Thus in this case M_{22} is equal to the angular magnification M_a. Taking into account that the determinant of any matrix that describes an optical system equals to 1 we can also estimate the M_{11} matrix element:

$$\det(M) = 1 \Rightarrow M_{11} \cdot M_{22} - 0 \cdot M_{12} = 0 \Rightarrow M_{11} = \frac{1}{M_{22}}$$

Finally, the matrix that describes the telescopic system can be rewritten as:

$$M = \begin{pmatrix} 1/M_a & M_{12} \\ 0 & M_a \end{pmatrix}$$

Let's now focus on how such a system images an object. Assuming that an object is located at a distance *s* from the entrance of the telescopic system, and that we further displace by a distance *s'* from the exit, we get for the matrix of this new "system":

$$M' = \begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1/M_a & M_{12} \\ 0 & M_a \end{pmatrix} \cdot \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1/M_a & M_{12} + s/M_a \\ 0 & M_a \end{pmatrix} =$$
$$= \begin{pmatrix} 1/M_a & M_{12} + s/M_a + M_a s' \\ 0 & M_a \end{pmatrix}$$

For the image of the object to be located at a distance *s'* the M'_{12} element of the matrix M' should equal to zero.

$$M'_{12} \equiv 0 \Rightarrow M_{12} + s/M_a + M_a s' = 0 \Rightarrow s' = -\frac{M_{12}}{M_a} - \frac{1}{M_a^2} s$$

Clearly the position of the image depends linearly on the position of the object. Using now the definition of longitudinal magnification we get:

$$M_L = \frac{ds'}{ds} = -\frac{1}{M_a^2} = const$$

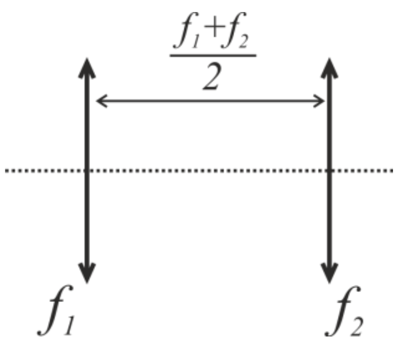
The longitudinal magnification is constant and independent from the position of the object. Since imaging between entrance and exit is fulfilled ($M'_{12} \equiv 0$) the matrix M' is further simplified to: $M' = \begin{pmatrix} 1/M_a & 0 \\ 0 & M_a \end{pmatrix}$. Using again the definition of ray vectors in matrix theory we get that the transverse magnification $M_T = M'_{11}$:

$$y_{s'} = M'_{11} \cdot y_s + 0 \cdot n_s \beta_s \Rightarrow M_T \equiv \frac{y_{s'}}{y_s} = M'_{11} = \frac{1}{M_a}$$

Thus in this case transverse magnification is the inverse of the angular: $M_T = M_a^{-1}$.

24) A Huygens type eyepiece consists of two thin lenses of focal distance f_1 and f_2 respectively. Estimate the position of all primary points (principal planes, optical power, foci, nodal points), as well as the back focal length (BFL) of the optical system.

(assume that the optical system is in air. In a Huygens eyepiece $d=(f_1+f_2)/2$)



Solution

If we define $D \equiv \dfrac{d}{n_{air}} = \dfrac{f_1+f_2}{2}$, $P_1 = \dfrac{1}{f_1}$, $P_2 = \dfrac{1}{f_2}$, the matrix M of the optical system is described as:

$$M = \begin{pmatrix} 1 & 0 \\ -P_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -P_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1-DP_1 & D \\ -P_1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-DP_1 & D \\ -(P_1+P_2-DP_1P_2) & 1-DP_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1-\dfrac{f_2}{f_1} & f_1+f_2 \\ -(\dfrac{1}{f_1}+\dfrac{1}{f_2}) & 1-\dfrac{f_1}{f_2} \end{pmatrix}$$

The positions of the principal planes are[1]:

[1] The distances Z_H, $Z_{H'}$ generally refer to reduced and not actual lengths z_H, $z_{H'}$. In this specific example n_s=$n_{s'}$=1 so that $Z_H = z_H$, $Z_{H'} = z_{H'}$

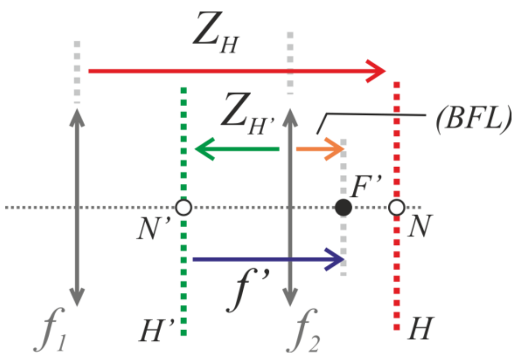
$$Z_H \equiv \frac{1-M_{22}}{M_{21}} = \frac{1-\frac{1}{2}(1-\frac{f_1}{f_2})}{-\frac{1}{2}(\frac{1}{f_1}+\frac{1}{f_2})} = -f_1, \qquad Z_{H'} \equiv \frac{1-M_{11}}{M_{21}} = \frac{1-\frac{1}{2}(1-\frac{f_2}{f_1})}{-\frac{1}{2}(\frac{1}{f_1}+\frac{1}{f_2})} = -f_2$$

The total optical power, the positions of the foci and the nodal points as well as the back focal length are then described by:

$$P_{tot} \equiv -M_{21} = \frac{1}{2}(\frac{1}{f_1}+\frac{1}{f_2}), \quad l_N \equiv \frac{n_s - n_{s'}}{P_{tot}} = 0 = -l_{N'}$$

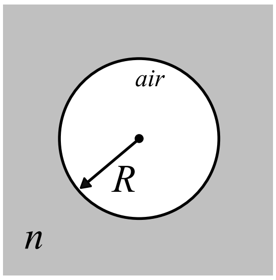
$$f \equiv \frac{n_s}{P_{tot}} = \frac{n_{air}}{P_{tot}} = 2\frac{f_1 f_2}{f_1+f_2}, \quad f' \equiv \frac{n_{s'}}{P_{tot}} = 2\frac{f_1 f_2}{f_1+f_2} = f$$

$$\text{(BFL)} \equiv f' + z_{H'} = 2\frac{f_1 f_2}{f_1+f_2} - f_2$$



25) Calculate all primary points (principal planes, optical power, foci, nodal points) of an air bubble of radius R = 5 *mm* trapped in glass. Can we use it's whole diameter to image a collimated beam?

(refractive indices: glass n = 3/2, air n_a =1)



Solution

The configuration is similar to a thick lens so the matrix that describes this optical system can be written as:

$$M = \begin{pmatrix} 1 & 0 \\ -P_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix} = \begin{pmatrix} 1-DP_1 & D \\ -(P_1+P_2-DP_1P_2) & 1-DP_2 \end{pmatrix}$$

where $P_1 = \frac{1-n}{R}$, $P_2 = \frac{n-1}{-R}$, $\Rightarrow P_1 = P_2 \equiv P$, $D = 2R$ and $R = 5mm, n = 1.5$. Thus the matrix M is simplified to

$$M = \begin{pmatrix} 1-2RP & 2R \\ -2P(1-RP) & 1-2RP \end{pmatrix} = \begin{pmatrix} 2 & 10\,mm \\ 300\,m^{-1} & 2 \end{pmatrix}$$

The primary points of the system are:

$$Optical\ power:\ P_{tot} = 2P(1-RP) = 2\frac{1-n}{R}(1-R\frac{1-n}{R}) = \frac{2n(1-n)}{R} = -0.3\,mm^{-1}$$

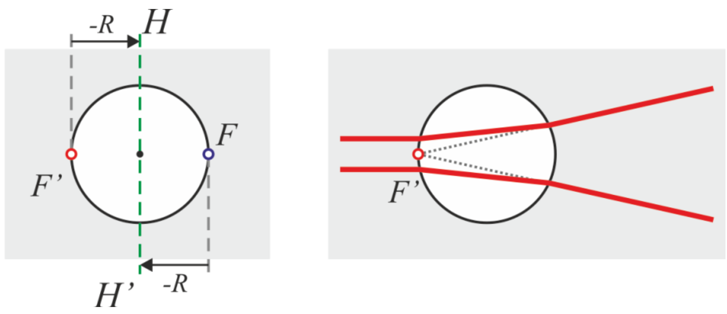
$$Principal\ planes:\ M_{11} = M_{22} \Rightarrow Z_H = Z_{H'}$$

$$Z_H = Z_{H'} \equiv \frac{1-M_{22}}{M_{21}} = \frac{2RP}{-P_{tot}} = -\frac{R}{n} \Rightarrow z_H = z_{H'} = -R = -5\,mm$$

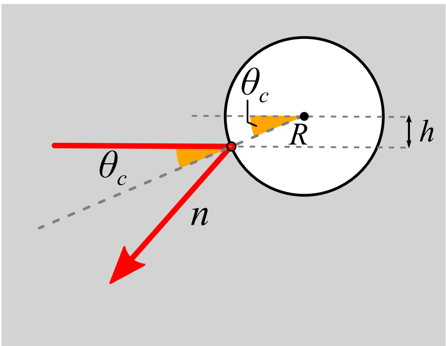
$$Effective\ focal\ length:\ f = f' \equiv \frac{n}{P_{tot}} = \frac{R}{2(1-n)} = -5\,mm$$

$$Back\ focal\ length:\ (BFL) \equiv f' + z_{H'} = -R - R = -2R = -10\,mm$$

So the bubble behaves like a diverging lens.



We cannot use the whole diameter of the bubble to image a collimated beam because as rays intersect the glass air interface, at a distance h from the optical axis, the incidence angle is increased and at some point we reach the critical angle of total internal reflection (TIR) θ_c.



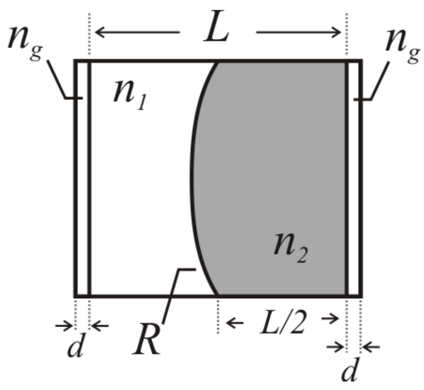
To avoid TIR the incidence angle should be

$$\theta < \theta_c \Rightarrow \sin\theta < \sin\theta_c = \frac{1}{n} \Rightarrow \frac{h}{R} < \frac{1}{n} \Rightarrow h < \frac{R}{n}$$

So the maximum usable diameter is $d_{\max} = \dfrac{2R}{n} \cong 6.7\,mm$

26) Calculate all primary points (principal planes, optical power, foci, nodal points) of the optical system as a function of the variable radius of curvature R. Estimate the total variation of optical power that we can achieve with this system.

(assume n = 3/2, n_1 =1.33, n_2=1.43, L=20 mm, d=1 mm, $|R| \geq 100\,mm$)



Solution

The matrix that describes the system is:

$$M = \begin{pmatrix} 1 & D_g \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D_2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D_g \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & D_2 + D_g \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D_1 + D_g \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & D_2 + D_g \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D_1 + D_g \\ -P & 1 - P(D_1 + D_g) \end{pmatrix}$$
$$= \begin{pmatrix} 1 - P(D_2 + D_g) & D_1 + D_g + (D_2 + D_g)[1 - P(D_1 + D_g)] \\ -P & 1 - P(D_1 + D_g) \end{pmatrix}$$

where:

$$P = \frac{n_2 - n_1}{R} = \frac{1.43 - 1.33}{R} = \frac{0.1}{R},$$
$$D_g = \frac{d}{n_g} = \frac{1mm}{1.5} = 0.67\,mm,\; D_1 = \frac{L/2}{n_1} = \frac{10\,mm}{1.33} \cong 7.52\,mm,\; D_2 = \frac{L/2}{n_2} = \frac{10\,mm}{1.43} \cong 6.99\,mm$$

According to the above the total power of the optical system is:

$$P_{tot} = P = \frac{0.1}{R},$$

Furthermore, the effective focal length and the principal plane positions are:

$$f_{tot} = \frac{n_{air}}{P_{tot}} = 10R$$
$$Z_H \equiv \frac{1 - M_{22}}{M_{21}} = \frac{(D_1 + D_g)P}{-P} = -(D_1 + D_g) \cong -(0.67\,mm + 7.52\,mm) = -8.19\,mm$$
$$Z_{H'} \equiv \frac{1 - M_{11}}{M_{21}} = \frac{(D_2 + D_g)P}{-P} = -(D_2 + D_g) \cong -(0.67\,mm + 6.99\,mm) = -7.66\,mm$$

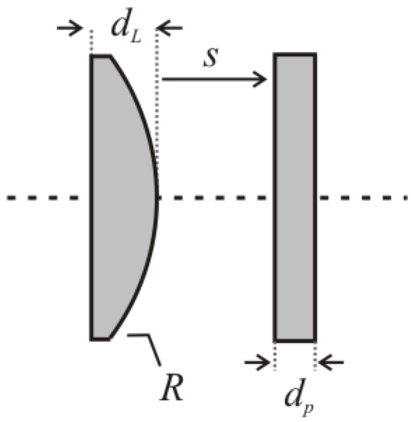
Since the entrance and the exit of the optical system are in the same medium (air) the nodal points coincide with the principal planes. Finally, the radius of curvature R is variable in this system and taking into account that $|R| \geq 100\,mm$ the corresponding variation in the total optical power is

$$\frac{0.1}{-0.1m} \leq P_{tot} \leq \frac{0.1}{0.1m} \Rightarrow -1m^{-1} \leq P_{tot} \leq 1m^{-1}$$

Thus the total variation in optical power is 2 diopters. The effective focal length respectively takes values in the range $|f_{tot}| \geq 1m$

27) Estimate the displacement of the back focus of a planoconvex lens after a glass plate of thickness d_p is inserted at a distance s from the lens, as shown in the figure. Does this displacement depend on s?

(assume that the refractive index of the lens and the glass plane are equal to n, the refractive index of air is 1 and that s≥0)



Solution

If we define $D_L = \dfrac{d_L}{n}$, $D_P = \dfrac{d_P}{n}$,$P_L = \dfrac{1-n}{-|R|}$, the matrix M of the optical system is described as:

$$M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D_p \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_L & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D_L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & s+D_p+x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_L & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D_L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & s+D_p+x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_L & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D_L \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & s+D_p+x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D_L \\ -P_L & 1-D_L P_L \end{pmatrix} = \begin{pmatrix} 1-(s+D_p+x)P_L & D_L + (s+D_p+x)(1-D_L P_L) \\ -P_L & 1-D_L P_L \end{pmatrix}$$

To find the back focus position we must set the exit of our optical system to be a focal plane. In this case M_{11}=0 thus:

$$M_{11} = 1-(s+D_p+x)P_L \equiv 0 \Rightarrow x = \frac{1}{P_L} - D_p - s$$

The back focus is now at a distance *F'* from the lens:

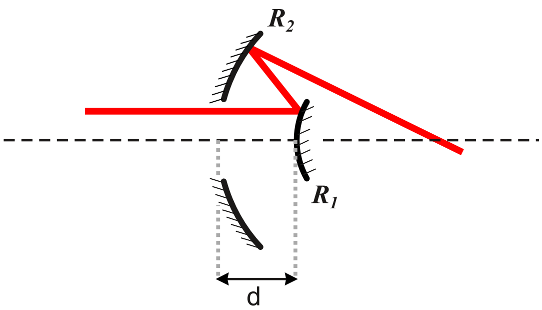
$$F' = s + d_p + x = s + d_p + \frac{1}{P_L} - \frac{d_p}{n} - s = \frac{1}{P_L} + \frac{n-1}{n} d_p$$

By setting $d_P \to 0$ we can find the initial position of the focus before the glass plate was inserted:

$$F = \frac{1}{P_L} \Rightarrow \Delta F = F' - F = \frac{n-1}{n} d_p$$

From the above it is obvious that the focus position does not depend on *s*.

28) A Schwarzschild microscope objective is comprised by two concentric mirrors $d = |R_2| - |R_1|$. Calculate all primary points (principal planes, optical power, foci, nodal points) and the back focal length of this system.



Solution

The matrix that describes the optical system is:

$$M = \begin{pmatrix} 1 & 0 \\ -P_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{-d}{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 - d\,P_1 & d \\ -[P_1 + P_2 - d\,P_1 P_2] & 1 - d\,P_2 \end{pmatrix}$$

where $P_1 = \frac{-1-1}{|R_1|} = -\frac{2}{|R_1|}$, $P_2 = \frac{1-(-1)}{|R_2|} = \frac{2}{|R_2|}$

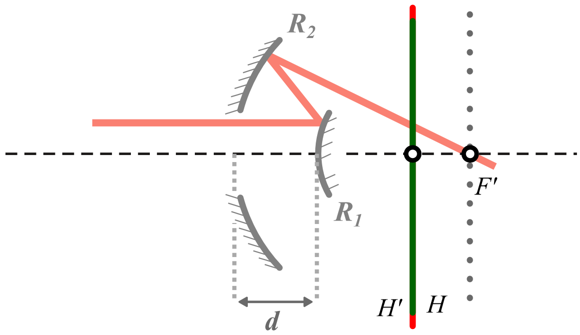
from the above we can estimate the total optical power of the system:

$$\left.\begin{aligned} P_{tot} &= P_1 + P_2 - d\,P_1 P_2 = 2\left(\frac{1}{|R_2|} - \frac{1}{|R_1|}\right) + \frac{4d}{|R_1||R_2|} \\ d &= |R_2| - |R_1| \end{aligned}\right\} \Rightarrow P_{tot} = 2\frac{|R_2| - |R_1|}{|R_1||R_2|}$$
,

Thus the effective focal distance and the principal planes positions are:

$$f_{tot} = \frac{1}{P_{tot}} = \frac{1}{2}\frac{|R_1||R_2|}{|R_2| - |R_1|}$$

$$Z_H = \frac{1 - M_{22}}{M_{21}} = \frac{d\,P_2}{-P_{tot}} = -|R_1|, \quad Z_{H'} = \frac{1 - M_{11}}{M_{21}} = \frac{d\,P_1}{-P_{tot}} = |R_2|$$



It is clear that the principal planes coincide in space, intersecting the common center of the two spherical surfaces.

The back focal length in a mirror system is measured not from the last element in the optical matrix layout but from the element which is actually nearest to the focus so in this case it is calculated by:

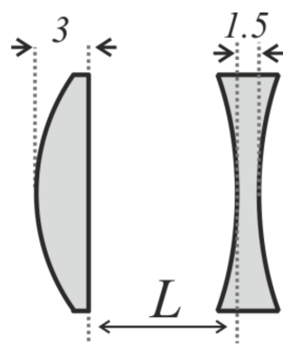
$$\begin{aligned}(BFL)^* &= (BFL) - d = f_{tot} + n_{s'} Z_{H'} - d \\ &= \frac{1}{2}\frac{|R_1||R_2|}{|R_2| - |R_1|} + |R_2| - (|R_2| - |R_1|) \\ &= \frac{1}{2}\frac{|R_1||R_2|}{|R_2| - |R_1|} + |R_1|\end{aligned}$$

Dimitris Papazoglou dpapa@materials.uoc.gr

Optical system design

29) Calculate the distance L so that the optical system shown in the figure is telescopic. What is the angular magnification in this case? The optical power of the lenses is +12 m^{-1} and -18 m^{-1} respectively.

(the second lens is bi-concave, with the same radii of curvature, all lenses are made of glass of refractive index n = 3/2, and all distances are in mm).



Solution

We fist calculate the matrix that describes the 2nd bi-convex lens:

$$M_{L2} = \begin{pmatrix} 1 & 0 \\ -P_2' & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D_2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_2' & 1 \end{pmatrix} = \begin{pmatrix} 1-D_2P_2' & D_2 \\ -[2P_2'-D_2P_2'^2] & 1-D_2P_2' \end{pmatrix}$$

where $P_2' = \frac{n-1}{-|R|}$, $\mathrm{D}_2 = \frac{d_2}{n} = \frac{1.5\,mm}{1.5} = 1\,mm$. From the total optical power of the lens we can estimate the power P_2' of each diopter and consequently all the elements of the matrix M_{L2} .

$$2P_2' - D_2P_2'^2 = -18\,m^{-1} \Rightarrow P_2' = -8.96\,m^{-1} \Rightarrow M_{L2} = \begin{pmatrix} 1.0089 & 1\,mm \\ 18\,m^{-1} & 1.0089 \end{pmatrix}$$

Since the 1st lens is plano-convex the optical power of its first diopter is equal to the optical power of the whole lens so the matrix that describes the whole optical system can be written as:

$$\begin{aligned} M_{tot} &= M_{L2} \cdot \begin{pmatrix} 1 & L+D_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-D_2P_2' & D_2 \\ -[2P_2'-D_2P_2'^2] & 1-D_2P_2' \end{pmatrix} \cdot \begin{pmatrix} 1 & L+D_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-D_2P_2' & D_2 \\ -[2P_2'-D_2P_2'^2] & 1-D_2P_2' \end{pmatrix} \cdot \begin{pmatrix} 1-(L+D_1)P_1 & L+D_1 \\ -P_1 & 1 \end{pmatrix} \\ &\cong \begin{pmatrix} 1.0089 & 1\,mm \\ 18\,m^{-1} & 1.0089 \end{pmatrix} \cdot \begin{pmatrix} 0.976-12\,m^{-1}L & L+2\,mm \\ -12\,m^{-1} & 1 \end{pmatrix} \\ &\cong \begin{pmatrix} 0.97-12.1\,m^{-1}L & L+3\,mm \\ -216\,m^{-2}L+5.46\,m^{-1} & 1.04+18\,m^{-1}L \end{pmatrix} \end{aligned}$$

where $P_1 = 12\,m^{-1}$, $D_1 = \frac{d_1}{n} = \frac{3mm}{1.5} = 2\,mm$. The condition for the system to be telescopic is $M_{tot(2,1)} = 0$ thus:

$$M_{tot(2,1)} = 0 \cong -216\,m^{-2}L + 5.46\,m^{-1} \Rightarrow L \cong 25.3\,mm$$

By applying this L value to the matrix of the system we get the matrix of the telescopic system:

$$M_{tel} \cong \begin{pmatrix} 2/3 & 28.5\,mm \\ 0 & 3/2 \end{pmatrix}$$

In a telescopic system the angular magnification is $M_\gamma \equiv M_{tel(2,2)}$, thus in our case $M_\gamma = 3/2$

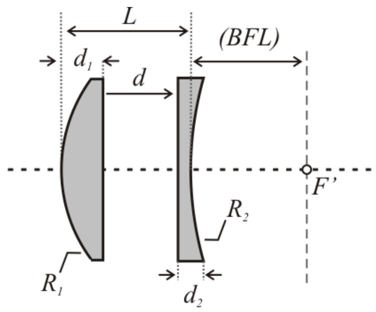
30) Design an optical system with effective focal distance *f* = 125 *mm* and *(BFL)* = 50 *mm*. The thickness of the lenses that you are going use should not be less than *1.5 mm*, while the total length of the system should not exceed 50 *mm*.
(refractive index of glass n = 3/2*).*

Solution

The system under design is telephoto since *f* >(BFL). This means that it should be comprised by a converging lens that is followed by a diverging lens. Let's assume that we use the optical system shown in the figure

We define the following:

$$D_1 = \frac{d_1}{n},\; D_2 = \frac{d_2}{n},\; L = d_1 + d_2 + d,$$
$$P_1 = \frac{n-1}{R_1},\; P_2 = \frac{1-n}{R_2},\; R_1, R_2 > 0$$



The total optical matrix of the system will then be:

$$M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D_2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 - L'P_1 - xP_{tot} & L' + x(1 - L'P_2) \\ -P_{tot} & 1 - L'P_2 \end{pmatrix}$$

where: $L' \equiv D_2 + d + D_1$, $P_{tot} \equiv (P_1 + P_2 - L'P_1P_2)$ and x is an arbitrary displacement. The condition for the exit to be a focal plane i.e. $x = (BFL)$ is: $M_{11} = 0$. Thus be replacing we get:

$$M_{11} \equiv 0 \Rightarrow 1 - L'P_1 - (BFL)P_{tot} = 0 \quad \textbf{(1)}$$

According the specifications we know: $f = \frac{1}{P_{tot}} = 125\,mm \Rightarrow P_{tot} = 8\,m^{-1}$. Applying this to eq. (1) we get:

$$L'P_1 = 1 - (BFL)P_{tot} = 1 - 0.05\,m \cdot 8m^{-1} = 1 - 0.4 \Rightarrow P_1 = \frac{0.6}{L'} \quad \textbf{(2)}$$

On the other side from the specs of the total length system we get:

$$L = d_1 + d_2 + d,\ L' = \frac{d_1 + d_2}{n} + d \Rightarrow L' = L - \frac{n-1}{n}(d_1 + d_2) \quad \textbf{(3)}$$

Assuming that $d_1 \equiv 3mm,\ d_2 \equiv 1.5\,mm \Rightarrow \frac{n-1}{n}(d_1 + d_2) = 1.5\,mm$ thus by replacing in eq. (3) and taking into account the specs limitation of the total length $(L \le 50\,mm)$ of the optical system we get:

$$L' \le 50\,mm - 1.5\,mm = 48.5\,mm \quad \textbf{(4)}$$

From eqs. (4),(2) we can estimate the optical power of the 1st lens:

$$P_1 \ge \frac{0.6}{0.0485} m^{-1} = 12.37\,m^{-1} \Rightarrow P_1 \equiv 13\,m^{-1} \Rightarrow L' = 46.15\,mm \Rightarrow L = 47.65\,mm \quad \textbf{(5)}$$

Likewise for the 2nd lens:

$$\begin{gathered} P_{tot} = (P_1 + P_2 - L'P_1P_2) \Rightarrow P_2 = \frac{P_{tot} - P_1}{1 - L'P_1} = \frac{P_{tot} - P_1}{(BFL)P_{tot}} \Rightarrow \\ P_2 = \frac{8\,m^{-1} - 13\,m^{-1}}{0.4} = -12.5\,m^{-1} \end{gathered} \quad \textbf{(6)}$$

From eq. (6) and the definitions of P_1, P_2 we can estimate the radii of curvature:

$$R_1 = \frac{n-1}{P_1} = \frac{0.5}{13\,m^{-1}} = 38.5\,mm,\ R_2 = \frac{1-n}{P_2} = \frac{-0.5}{-12.5\,m^{-1}} = 40\,mm$$

The distance between the lenses is: $d = L - d_1 - d_2 = 47.65\,mm - 3\,mm - 1.5\,mm = 43.15\,mm$

Finally the optical system specifications are:

$$R_1 = 38.5\,mm,\ d_1 = 3\,mm,\ d = 43.15\,mm,\ R_2 = 40\,mm,\ d_2 = 1.5\,mm$$

31) Design an optical system with effective focal distance *f* and back focal length *(BFL) = f/c* where *c* > 0. The total length of the system should not exceed (BFL)

Solution

Since *f* > *(BFL)* the optical system is telephoto. Thus it can be constructed by a combination of a converging lens followed by a diverging one. Since there is no limitation in the lens thickens we will design the system using thin lenses. So we define the following:

$$P_{tot} = \frac{1}{f},(BFL) = \frac{f}{c},\ d = m\cdot(BFL) = m\cdot\frac{f_{\varepsilon\nu}}{c},\ (m \le 1),\ P_1 \equiv \frac{1}{f_1},\ P_2 \equiv \frac{1}{f_2}$$

Based on the above the matrix that describes the optical system is

$$M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - d\,P_1 - x\,P_{tot} & x + d(1 - P_2 x) \\ -P_{tot} & 1 - d\,P_2 \end{pmatrix}$$

where $P_{tot} = P_1 + P_2 - d\,P_1 P_2 = P_o$, and *x* is a parameter. By assuming that the system exit is a focal plane (back focal plane) we get:

$$x = (BFL) \Rightarrow M_{11} = 0 \Rightarrow 1 - d\,P_1 - (BFL)P_{tot} = 0$$

Solving for P_1 we can estimate the optical power of the 1st length:

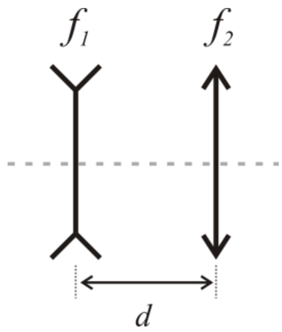
$$P_1 = \frac{1 - (BFL)P_{tot}}{d} = \frac{1 - \frac{f}{c}\cdot\frac{1}{f}}{m\cdot\frac{f}{c}} = \frac{c-1}{m\cdot f} \Rightarrow f_1 = \frac{m\cdot f}{c-1}$$

Using the power of the 1st lens we can estimate the power of the 2nd lens:

$$P_2 = \frac{P_{tot} - P_1}{1 - d\,P_1} = \frac{\dfrac{1}{f} - \dfrac{c-1}{m \cdot f}}{\dfrac{f}{c} \cdot \dfrac{1}{f}} = \frac{c\,(m-c+1)}{m \cdot f} \Rightarrow f_2 = \frac{1}{P_2} = \frac{m}{c\,(m-c+1)}\,f$$

32) Design an optical system with effective focal distance f = 15 *mm* and back focal length *(BFL)* = 50 *mm* which can achieve an angular resolution of $\Delta\varphi \le 5''$. The total length of the system should not exceed 20 *mm*.
(ignore any optical aberrations and assume that the optical system will operate in the visible λ_o = 500 *nm*, $1'' = 1^\circ / 3600$)

Solution



Since f < *(BFL)* the optical system is inverse-telephoto. Thus it can be constructed by a combination of a diverging lens followed by a converging one. Since there is no limitation in the lens thickens we will design the system using thin lenses. In this context we define the following:

$$P_{tot} = \frac{1}{f} = \frac{1}{15\,mm} \cong 66.7\,m^{-1},\;\; d = 20\,mm,\; P_1 \equiv \frac{1}{f_1} < 0,\; P_2 \equiv \frac{1}{f_2} > 0$$

Based on the above the matrix that describes the optical system is

$$M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - d\,P_1 - x\,P_{tot} & x + d\,(1 - P_2 x) \\ -P_{tot} & 1 - d\,P_2 \end{pmatrix}$$

where $P_{tot} = P_1 + P_2 - d\,P_1 P_2 = 66.7\,m^{-1}$ and x is a parameter. By assuming that the system exit is a focal plane (back focal plane) we get:

$$x = (BFL) \Rightarrow M_{11} = 0 \Rightarrow 1 - d\,P_1 - (BFL)P_{tot} = 0$$

Solving for P_1 we can estimate the optical power of the 1st length:

$$P_1 = \frac{1-(BFL)P_{tot}}{d} = \frac{1-50\,mm\,\dfrac{1}{15\,mm}}{20\,mm} \cong -116.7\,m^{-1} \Rightarrow f_1 = \frac{1}{P_1} \cong -8.6\,mm$$

Using the power of the 1st lens we can estimate the power of the 2nd lens:

$$P_2 = \frac{P_{tot}-P_1}{1-d\,P_1} = \frac{P_{tot}-P_1}{(BFL)\,P_{tot}} \Rightarrow P_2 \cong 55\,m^{-1} \Rightarrow f_1 = \frac{1}{P_2} \cong 18.2\,mm$$

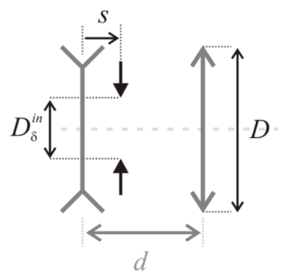
With the above complies to the imaging specifications. The angular resolution $\Delta\varphi$ of an optical system, if we ignore optical aberrations, is can be estimated by the Rayleigh criterion:

$$\Delta\varphi = 1.22\frac{\lambda_o}{D_{in}}$$

where D_{in} is the diameter of the entrance pupil. According to the specifications $\Delta\varphi \leq 5'' \cong 24.2\cdot 10^{-6}\,rad$. Using this value we can estimate the corresponding diameter of the entrance pupil:

$$D_{in} \geq 1.22\frac{\lambda_o}{\Delta\varphi} = 1.22\frac{500\cdot 10^{-6}\,mm}{24.2\cdot 10^{-6}} \cong 25.2\,mm$$

Assuming that both lenses comprising the system have the same diameter D we can identify the aperture stop. Essentially we need to calculate the position and the diameter of the image of the second lens at the object space. Assuming that this image is located at a distance s from the 1st lens we get:



$$M_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-d\,P_1 & d+s(1-d\,P_1) \\ -P_1 & 1-s\,P_1 \end{pmatrix}$$

In order for the entrance and the exit of this optical sub-system to be conjugate planes (object $\leftrightarrow$ image) the following condition should be fulfilled:

$$M_d^{(1,2)} \equiv 0 \Rightarrow d + s(1-d\,P_1) = 0 \Rightarrow s = \frac{d}{d\,P_1 - 1} \cong -6\,mm$$

From the above it is clear that the image of the 2nd lens is located at the right of the 1st lens with a transverse magnification that can be calculated from:

$$M_d^{(1,2)} \equiv 0 \Rightarrow M_T = M_d^{(1,1)} = 1 - d\,P_1 \cong 3.33$$

We have to note here that according to the matrix M_d we have used to describe the optical subsystem this magnification corresponds to the ratio $M_T = D / D_o^{in}$ where D is the diameter of the lens, and D_o^{in} is the diameter of its image in the object space. So finally: $D_o^{in} = D / M_T = D / 3.33 < D$. Since the 2nd lens is the aperture stop and in order to comply to the specifications:

$$D_o^{in} \geq 25.2\,mm \Rightarrow \frac{D}{3.33} \geq 25.2\,mm \Rightarrow D \geq 84\,mm$$

In conclusion the total system parameters are:

$$f_1 \cong -8.6\,mm, \quad f_2 = 18.2\,mm, \quad d = 20\ mm, \ \ D \geq 84\,mm$$

33) A typical matrix describing the optical system of the human eye is:

$$M_{eye} = \begin{pmatrix} 0.7 & 3.8mm \\ -60\,m^{-1} & 1.1 \end{pmatrix}$$

a) Estimate the optical power change of the eye when we are underwater without wearing a mask.

b) When we wear a mask while underwater are the objects magnified?

(refractive indices: *n_{air}=1, n_{water}=1.33, n_{cornea}=1.4, $n_{vitreous}$=1.34, n_{glass}=1.5, cornea radius of curvature R_{cornea}=8 mm)*

Solution

a) The total matrix (M_{eye}) that describes the human eye can be decomposed to the effect of the cornea (M_{cornea}) and the rest of the optical system (M_{rem}):

$$M_{eye} = M_{rem} \cdot M_{cornea} = M_{rem} \cdot \begin{pmatrix} 1 & 0 \\ -P_k & 1 \end{pmatrix}$$

Solving for the unknown matrix M_{rem} we get

$$M_{rem} = M_{eye} \cdot M_{cornea}^{-1} = \begin{pmatrix} 0.7 & 3.8mm \\ -60\,m^{-1} & 1.1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ P_k & 1 \end{pmatrix} = \begin{pmatrix} 0.89 & 3.8mm \\ -5\,m^{-1} & 1.1 \end{pmatrix}$$

$$P_k = \frac{n_{cornea} - 1}{R_{cornea}} = \frac{1.4 - 1}{8\,mm} = 50\,m^{-1}$$

when entering the water there is no practical change in the "internal" optical system described by the matrix (M_{rem}). On the other hand there is a considerable change in the optical power of the cornea and the corresponding matrix (M_{cornea}):

$$P_k^{water} = \frac{n_{cornea} - n_{water}}{R_{cornea}} = \frac{1.4 - 1.33}{8\,mm} = 8.75\,m^{-1} \Rightarrow M_{cornea}^{water} = \begin{pmatrix} 1 & 0 \\ -P_k^{water} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -8.75\,m^{-1} & 1 \end{pmatrix}$$

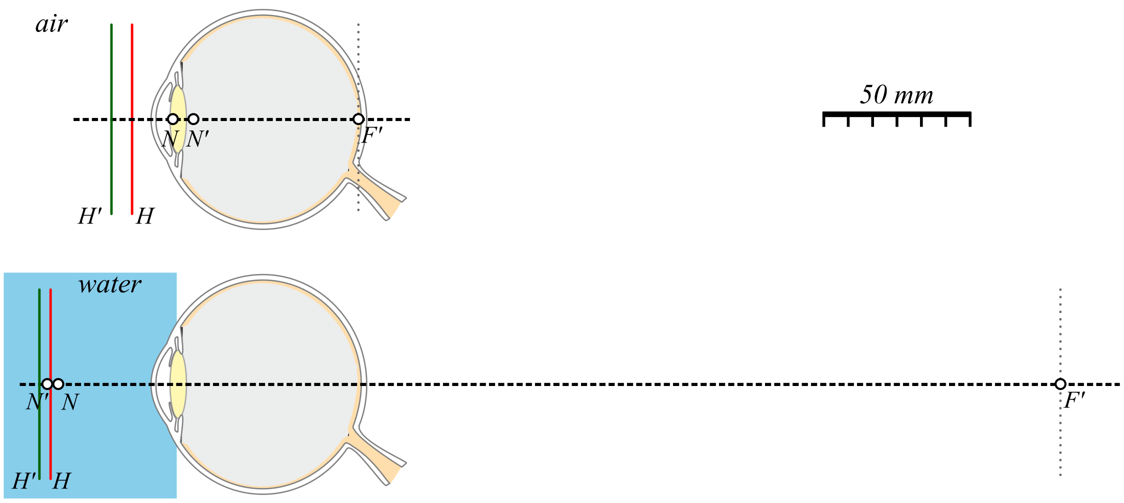
Replacing back to the total matrix that describes the eye we finally get that underwater M_{eye} becomes:

$$M_{eye}^{water} = M_{rem} \cdot M_{cornea}^{water} = \begin{pmatrix} 0.89 & 3.8mm \\ -5\,m^{-1} & 1.1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -8.75\,m^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 0.86 & 3.8\,mm \\ -14.6\,m^{-1} & 1.1 \end{pmatrix}$$

It is obvious that underwater the optical power of the eye is dramatically reduced from 60 m^{-1} to 14 m^{-1}, so we have a loss of 46 diopters!

Using the above we can estimate the effective focal length, the principal planes positions and the nodal points for each case:

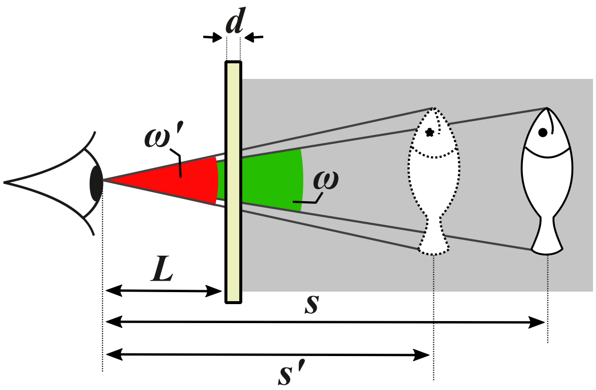
naked eye	*underwater*
$Z_H = \frac{1 - M_{eye}^{(2,2)}}{M_{eye}^{(2,1)}} \cong 1.7\,mm,$	$Z_H^{water} = \frac{1 - M_{eye}^{water(2,2)}}{M_{eye}^{water(2,1)}} \cong 6.8\,mm$
$Z_{H'} = \frac{1 - M_{eye}^{(1,1)}}{M_{eye}^{(2,1)}} \cong -5\,mm,$	$Z_{H'}^{water} = \frac{1 - M_{eye}^{water(1,1)}}{M_{eye}^{water(2,1)}} \cong -9.8\,mm$
$f'_{eff} = \frac{n_{vitreous}}{-M_{eye}^{(1,2)}} \cong 22.3\,mm,$	$f'^{water}_{eff} = \frac{n_{vitreous}}{-M_{eye}^{water(1,2)}} \cong 91.6\,mm$
$l_N = -l'_N = \frac{n_{air} - n_{vitreus}}{-M_{eye}^{(1,2)}} \cong -5.7mm$	$l_N^{water} = -l'^{Water}_N = \frac{n_{water} - n_{vitreus}}{-M_{eye}^{(1,2)}} \cong -0.7mm$



Principal points of the human eye in air and underwater.

As it is clear while underwater besides the dramatic drop in optical power (-46 m^{-1}) we also experience a foci shift due to the displacement of the principal planes. Assuming that in normal conditions the image is sharply focused on the cornea, then the back focal length is $(BFL) = f'_{eff} + n_{vitreus} Z_{H'} = 22.3\,mm - 1.34 \cdot 5\,mm = 15.6\,mm$. When we are underwater this changes to $(BFL)^{water} = f'^{water}_{eff} + n_{vitreus} Z^{water}_{H'} = 91.6\,mm - 1.34 \cdot 9.8\,mm = 78.5\,mm$, thus it has shifted by 62.9 *mm* instead of $f'^{water}_{eff} - f'_{eff} = 69.3\,mm$ one would expect if only the optical power change is taken into account.

b) Wearing a mask underwater we avoid this drop in optical power, and our optical system stays intact. On the other hand, since light is refracted from water to glass and then air we observe an apparent lifting of the image. Using a rough approximation (ignoring the effect of the glass window) we have an apparent shift of the image towards our eye by:



$$\frac{s'+L}{s+L} \cong \frac{s'}{s} = \frac{n_{air}}{n_{water}} = \frac{1}{1.33}, \quad (s, s' \gg L)\,.$$

This results to an angular magnification of 33 %

$$M_a \equiv \tan\omega' / \tan\omega = s/s' = 1.33$$

In a more accurate approach, we can estimate the matrix M_{mask} that describes the propagation of light from the object located at a distance s from our eye up to its image located at a distance s' from our eye (see figure):

$$M_{mask} = \begin{pmatrix} 1 & \dfrac{-s'}{n_{air}} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \dfrac{L}{n_{air}} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \dfrac{d}{n_{mask}} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \dfrac{s-(L+d)}{n_{water}} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{-s'+L}{n_{air}} + \dfrac{d}{n_{mask}} + \dfrac{s-L-d}{n_{water}} \\ 0 & 1 \end{pmatrix}$$

where d is the mask thickness, n_{mask} the refractive index of the mask's glass, and L is the distance of the mask glass from our eye. The condition for s' to be the position of the image (entrance & exit are conjugate planes) is $M_{mask}^{(1,2)} \equiv 0$ thus:

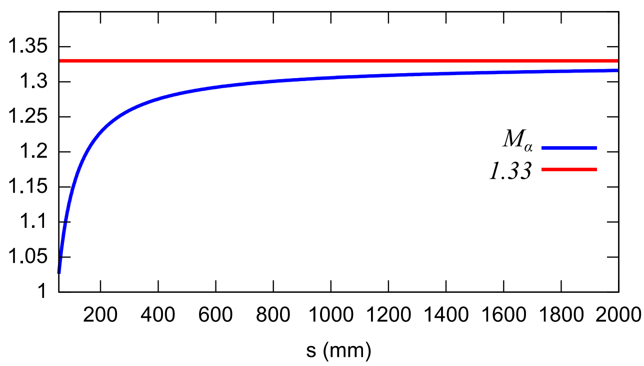
$$\frac{-s'+L}{n_{air}} + \frac{d}{n_{mask}} + \frac{s-L-d}{n_{water}} \equiv 0 \Rightarrow s' = \frac{n_{air}}{n_{water}} s + \frac{n_{water} - n_{air}}{n_{water}} L + \frac{n_{air}(n_{water} - n_{mask})}{n_{mask} \cdot n_{water}} d$$

For typical values $n_{water} = 1.33, n_{air} = 1, n_{mask} = 1.5, d = 5\,mm, L = 50\,mm$ we get:

$$s' = \frac{1}{1.33} s + 12.08\,mm$$

Since $s > L + d \Rightarrow s > 55\,mm$ then:

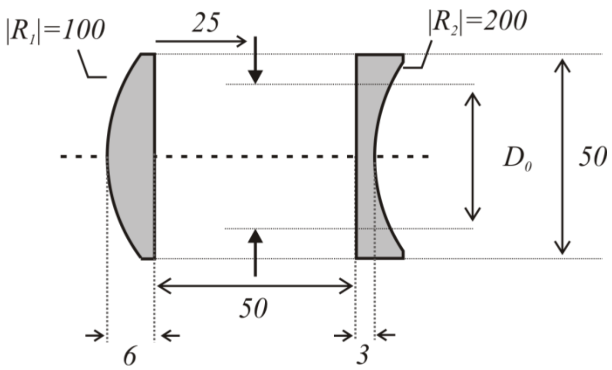
$$M_a \equiv \frac{\tan\omega'}{\tan\omega} = \frac{s}{s'} = \frac{1.33}{1 + \dfrac{16.07\,mm}{s}} > 1.03$$



Dimitris Papazoglou dpapa@materials.uoc.gr

Apertures and illumination

34) Calculate the diameter of the variable iris of the optical system shown in the figure so that #F ≈ 8.
(assume an object at infinity, all distances are in mm, all lenses are made of glass of refractive index n = 3/2*).*



Solution

We define the following:

$$P_1 = \frac{n-1}{R_1} \;, P_2 = \frac{1-n}{R_2},$$
$$D_1 = \frac{d_1}{n},\; D_2 = \frac{d_2}{n} \;, L' = L + D_1 + D_2,$$
$$L = 50\,mm,\; d_1 = 6\,mm,\; d_2 = 3\,mm, s = 25\,mm$$

The total matrix tha describes the system is:

$$M = \begin{pmatrix} 1 & 0 \\ -P_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L' \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 - L'P_1 & L' \\ -(P_1 + P_2 - L'P_1P_2) & 1 - L'P_2 \end{pmatrix} = \begin{pmatrix} 0.72 & 56\,mm \\ -3.2\,m^{-1} & 1.14 \end{pmatrix}$$

Thus the effective focal length of the system is:

$$f = f' = \frac{1}{P_1 + P_2 - L'P_1P_2} = \frac{1}{3.2}\,m = 312.5\,mm$$

The *#F* of an optical system is defined as the ratio of the effective focal distance to the diameter of the entrance pupil $\#F \equiv f / D_{in}$. In order to estimate the diameter of the entrance pupil we must first identify the aperture stop of the system. To do this we must find the position of the image of the iris, and the 2nd lens, in the object space. Assuming that this image is located at a distance *x* from the first lens we can write:

$$M_d = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1-(s+D_1)P_1 & s+D_1+x[1-(s+D_1)P_1] \\ -P_1 & 1-xP_1 \end{pmatrix}$$

For the entrance and the exit of the optical system described by M_d to be conjugate planes (object $\leftrightarrow$image) $M_d^{(1,2)} \equiv 0$ thus:

$$s+D_1+x[1-(s+D_1)P_1]=0 \Rightarrow$$
$$x=(s+D_1)\cdot[1-(s+D_1)P_1]^{-1}=-33.92\,mm$$

From the above it is clear that the image of the variable iris is located at the right of of the 1st lens with transverse magnification:

$$M_d^{(1,2)} \equiv 0 \Rightarrow M_T = M_d^{(1,1)} = 1-(s+D_1)P_1 = 0.855$$

We should note here that according to the matrix we have used this magnification refers to the ratio $M_T = D_\delta / D_{in}$ where D_δ is the iris diameter and D_{in} the desired diameter of the entrance pupil. Thus since by specs *#F* = 8 the diameter of the entrance pupil* should be :

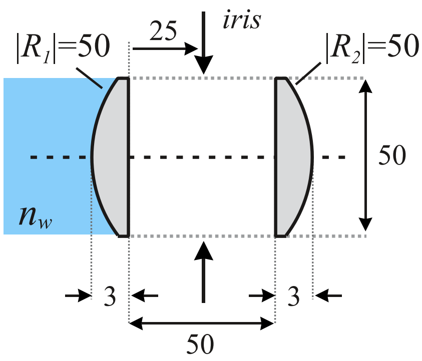
$$(\#F) \equiv \frac{f}{D_{in}} \Rightarrow D_{in} = \frac{f}{(\#F)} = \frac{312.5\,mm}{8} \cong 39.1\,mm$$

To achieve this we should reduce the diameter of the iris to:

$$D_\delta = M_T \cdot D_{in} \cong 0.855 \cdot 39.1\,mm \cong 33.4\,mm$$

* *One can easily check that following the same procedure the 2nd lens is not the aperture stop. When the iris is fully open the 1st lens becomes the aperture stop. In this case the #F of the system is:* $\#F_o \equiv f / D^1_{lens} = 312.5\,mm / 50\,mm = 6.25$

35) Calculate the diameter of the variable iris of the optical system shown in the figure so that the image illumination drops by a factor of 4.
(assume an object at infinity, all distances are in mm, all lenses are made of glass of refractive index $n = 3/2$, $n_w = 1.33$*).*



Solution

The image illumination is inversely proportional to the square of the F-Number *(#F)* of the optical system so to reduce the image illumination by a factor of 4 one should double the *(#F)*. To do that we must first calculate the *(#F)* of the optical system. We define the following:

$$L = 50\,mm,\ d_1 = d_2 = 3\,mm,\ s = 25\,mm,\ s_2 = 50\,mm$$
$$P_1 = \frac{n - n_w}{|R|} = \frac{1.5 - 1.33}{50\,mm} = 3.4\,m^{-1},\ P_2 = \frac{1-n}{-|R|} = \frac{1-1.5}{-50\,mm} = 10\,m^{-1},$$
$$D_1 = \frac{d_1}{n} = 2\,mm,\ D_2 = \frac{d_2}{n} = 2mm,\ \ D \equiv D_1 = D_2$$

Thus the matrix describing the optical system is:

$$M = \begin{pmatrix} 1 & 0 \\ -P_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} 1 & 0 \\ -P_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2D+L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix} = \begin{pmatrix} 1-(2D+L)P_1 & 2D+L \\ -[P_1 + P_2 - (2D+L)P_1P_2] & 1-(2D+L)P_2 \end{pmatrix} =$$
$$\cong \begin{pmatrix} 0.82 & 54\,mm \\ -11.6\,m^{-1} & 0.46 \end{pmatrix}$$

The optical power of the is $P_{tot} = P_1 + P_2 - (2D+L)P_1P_2 = 11.6\,m^{-1}$, thus the front and back effective focal distances are:

$$f = \frac{n_w}{P_{tot}} = \frac{1.33}{11.6}m \cong 115\,mm,\quad f' = \frac{1}{P_{tot}} = \frac{1}{11.6}m \cong 86.5\,mm$$

The *#F* of an optical system is defined as the ratio of the effective front focal distance to the diameter of the entrance pupil $\#F \equiv f / D_{in}$. In order to estimate the diameter of the entrance pupil we must first identify the aperture stop of the system. To do this we must find the position of the image of the iris, and the 2nd lens, in the object space. Assuming that this image is located at a distance x ($X \equiv x / n_w$) from the first lens we can write:

$$
\begin{aligned}
M_d &= \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & s+D \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1-(D+s)P_1 & X+(D+s)(1-X\,P_1) \\ -P_1 & 1-X\,P_1 \end{pmatrix}
\end{aligned}
$$

For the entrance and the exit of the optical system described by M_d to be conjugate planes (object ↔image) $M_d^{(1,2)} \equiv 0$ thus:

$$
M_d^{(1,2)} \equiv 0 \Rightarrow X + (D+s)(1 - X\,P_1) = 0 \Rightarrow x = n_w \frac{D+s}{(D+s)P_1 - 1} \cong -39.54\,mm
$$

From the above it is clear that the image of the variable iris is located at the right of of the 1st lens with transverse magnification:

$$
M_d^{(1,2)} \equiv 0 \Rightarrow M_T = M_d^{(1,1)} = 1-(D+s)\,P_1 \cong 0.91
$$

We should note here that according to the matrix we have used this magnification refers to the ratio $M_T = D_\delta / D_\delta^{in}$ where D_δ is the iris diameter and D_δ^{in} the diameter of the image of the iris in the object space. So finally:

$$
D_\delta^{in} = \frac{D_\delta}{M_T} = \frac{50\,mm}{0.91} \cong 54.9\,mm
$$

We follow the same procedure for the 2nd lens. In this case we can simply use the same formulas by replacing the distance s with s_2 (the distance of the 2nd lens from the 1st). Thus the position of the image of the 2nd lens in the object space is:

$$
x' = n_w \frac{D+s_2}{(D+s_2)P_1 - 1} \cong -84\,mm
$$

While its diameter is:

$$D_{2nd}^{in} = \frac{D_{2nd}}{1 - s_2 P_1} = \frac{50\,mm}{0.82} = 60.7\,mm$$

From the above it is clear that the smallest aperture in the object space is that of the 1st lens so this is also the aperture stop of the system when the iris is fully open. Likewise the *(#F)* of the system is:

$$(\#F) \equiv \frac{f}{D_{in}} = \frac{115\,mm}{50\,mm} = 2.3$$

By reducing the diameter of the entrance pupil by a factor of 2 we will double the *(#F)* and thus reduce in the image illumination by a factor of 4. We will achieve this by setting the iris as the aperture stop so that its image, the new entrance pupil, with have a diameter:

$$D_{\delta}^{in\prime} = \frac{50\,mm}{2} = 25\,mm$$

Likewise the diameter of the iris should reduce to $D_{\delta}^{in\prime} = M_T \cdot D_{in} \cong 0.91 \cdot 25\,mm \cong 22.7\,mm$

Gaussian beams

36) A telescopic system with an angular magnification M_γ is illuminated by a Gaussian beam of waist w_{in}, and Rayleigh range z_{in}. Calculate the beam waist and Rayleigh range in the exit.

Solution

The matrix that describes a telescopic system (when the optical media in the entrance and exit are the same) can be generally described as:

$$M = \begin{pmatrix} M_a^{-1} & M_{12} \\ 0 & M_a \end{pmatrix},$$

where M_a is the angular magnification of the telescopic system.

By assuming a new entrance at a position x_o form the entrance and a new exit at a position x_o' from the exit of the telescopic system the matrix transforms to:

$$\begin{aligned} M' &= \begin{pmatrix} 1 & x_o' \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} M_\gamma^{-1} & M_{12} \\ 0 & M_\gamma \end{pmatrix} \cdot \begin{pmatrix} 1 & x_o \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & x_o' \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} M_\gamma^{-1} & x_o / M_\gamma + M_{12} \\ 0 & M_\gamma \end{pmatrix} \\ &= \begin{pmatrix} M_\gamma^{-1} & \dfrac{x_o}{M_\gamma} + M_{12} + M_\gamma x_o' \\ 0 & M_\gamma \end{pmatrix} \end{aligned}$$

As we know when a Gaussian beam waist with q-factor $q_{in} = i\, z_R^{in}$ is located at the entrance of an optical system described by a matrix M' the corresponding q-factor q_{out} in the exit is:

$$\frac{1}{q_{out}} \equiv \frac{M_{21}'\, q_{in} + M_{22}'}{M_{11}'\, q_{in} + M_{12}'} = \frac{0 \cdot \left(i\, z_R^{in}\right) + M_\gamma}{M_\gamma^{-1} \cdot \left(i\, z_R^{in}\right) + \dfrac{x_o}{M_\gamma} + M_{12} + M_\gamma x_o'}$$

$$= \frac{\left(\dfrac{x_o}{M_\gamma} + M_{12} + M_\gamma x_o' - M_\gamma^{-1} \cdot i\, z_R^{in}\right) M_\gamma}{\left(\dfrac{x_o}{M_\gamma} + M_{12} + M_\gamma x_o'\right)^2 + \left(\dfrac{z_R^{in}}{M_\gamma}\right)^2} = \frac{x_o + M_{12} M_\gamma + M_\gamma^2 x_o'}{A} - i\,\frac{z_R^{in}}{A}$$

where $A \equiv \left(\dfrac{x_o}{M_\gamma} + M_{12} + M_\gamma x_o'\right)^2 + \left(\dfrac{z_R^{in}}{M_\gamma}\right)^2$.

The Gaussian beam will have a waist at the exit of the system M' if $\mathrm{Re}(\frac{1}{q_{out}}) \equiv 0$:

$$at\ waist: \ \mathrm{Re}[\frac{1}{q_{out}}] \equiv 0 \Rightarrow x_o + M_{12}M_\gamma + M_\gamma^2 x_o' = 0 \Rightarrow x_o' = -\frac{M_{12}}{M_\gamma} - \frac{x_o}{M_\gamma^2}$$

From the definition of the q-factor we know that at the waist:

$$at\ waist: \ \frac{1}{q_{out}} \equiv \frac{1}{i\, z_R^{out}} \equiv -i\frac{\lambda}{\pi}\frac{1}{w_{out}^2}$$

where z_R^{out}, w_{out} are respectively the Rayleigh length and the beam waist at the exit. By replacing for the value of x_o' we get:

$$at\ waist\ \ A = (\frac{z_R^{in}}{M_\gamma})^2 \Rightarrow \mathrm{Im}[\frac{1}{q_{out}}] \equiv -\frac{1}{z_R^{out}} = -\frac{z_R^{in}}{(\frac{z_R^{in}}{M_\gamma})^2} = -\frac{M_\gamma^2}{z_R^{in}} \Rightarrow$$

$$z_R^{out} = \frac{z_R^{in}}{M_\gamma^2} \Rightarrow w_{out} = \frac{w_{in}}{|M_\gamma|}$$

It is clear that for a telescopic system the Gaussian beam parameters z_R^{out}, w_{out} in the exit of the system depend only on the angular magnification and not on the position of the waist in th entrance.

37) A Gaussian beam is focused by a thin lens of focal distance *f*. Estimate the diameter of the focus when the waist of the Gaussian beam is located on the lens.

Solution

Assuming that the beam propagates by a distance *x* after the lens the matrix can be written as:

$$M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{x}{f} & x \\ -\frac{1}{f} & 1 \end{pmatrix},$$

As we know when a Gaussian beam waist with q-factor $q_{in} = i\, z_R^{in}$ is located at the entrance of an optical system described by a matrix M the corresponding q-factor q_{out} in the exit is:

$$\frac{1}{q_{out}} \equiv \frac{M_{21}\, q_{in} + M_{22}}{M_{11}\, q_{in} + M_{12}} = \frac{-\frac{1}{f}\cdot\left(i\, z_R^{in}\right)+1}{(1-\frac{x}{f})\cdot\left(i\, z_R^{in}\right)+x}$$

$$= \frac{(1-i\frac{z_R^{in}}{f})\cdot[x-i(1-\frac{x}{f})z_R^{in}]}{[(1-\frac{x}{f})z_R^{in}]^2+x^2} = \frac{x-(1-\frac{x}{f})\frac{(z_R^{in})^2}{f}}{A} - i\frac{z_R^{in}}{A}$$

where $A \equiv [(1-\frac{x}{f})z_R^{in}]^2+x^2$.

The Gaussian beam will have a waist at the exit of the system M if $\mathrm{Re}(\frac{1}{q_{out}}) \equiv 0$:

$$at\ \ waist: \ \mathrm{Re}[\frac{1}{q_{out}}] \equiv 0 \Rightarrow x-(1-\frac{x}{f})\frac{(z_R^{in})^2}{f} = 0 \Rightarrow x = \frac{f}{1+\left(f/z_R^{in}\right)^2}$$

From the definition of the q-factor we know that at the waist:

$$at\ \ waist: \ \frac{1}{q_{out}} \equiv \frac{1}{i\, z_R^{out}} \equiv -i\frac{\lambda}{\pi}\frac{1}{w_{out}^2}$$

where z_R^{out}, w_{out} are respectively the Rayleigh length and the beam waist at the exit. By replacing for the value of x we get, using $(1-\frac{x}{f})z_R^{in} = x\frac{f}{z_R^{in}}$:

$$at\ \ waist\ \ A = (\frac{z_R^{in}}{M_\gamma})^2 \Rightarrow \mathrm{Im}[\frac{1}{q_{out}}] \equiv -\frac{1}{z_R^{out}} = \frac{-z_R^{in}}{[(1-\frac{x}{f})z_R^{in}]^2+x^2} = \frac{-z_R^{in}}{x^2\left(f/z_R^{in}\right)^2+x^2} \Rightarrow$$

$$z_R^{out} = \frac{f^2}{1+\left(f/z_R^{in}\right)^2}\frac{1}{z_R^{in}} = \frac{f^2}{1+\left(z_R^{in}/f\right)^2} z_R^{in} \Rightarrow$$

$$\frac{\pi w_{out}^2}{\lambda} = \frac{f^2}{1+\left(f/z_R^{in}\right)^2}\frac{\lambda}{\pi w_{in}^2} \Rightarrow w_{out} = \frac{\lambda}{\pi}\frac{f}{w_{in}}\frac{1}{\sqrt{1+\left(f/z_R^{in}\right)^2}}$$

When the Raleigh range of the input beam is much larger than the focal length this is simplified to:

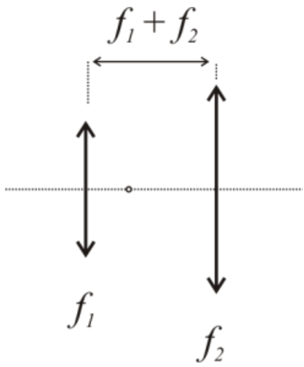
$$z_R^{out} \cong \frac{1}{z_R^{in}}, \qquad w_{out} \cong \frac{\lambda}{\pi}\frac{f}{w_{in}}, \quad z_R^{in} \gg f$$

38) The optical system shown in the figure is illuminated by a Gaussian beam. Assuming that the waist of the beam in the entrance is on the 1st lens (f_1) calculate

a) The position of the waist at the exit of the optical system

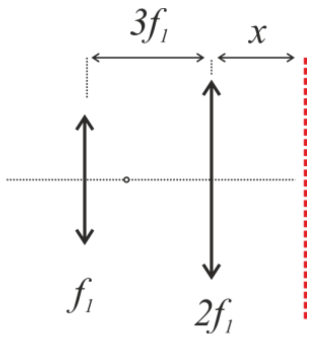
b) The relative change in the waist and Rayleigh range parameters

($f_2 = 2 f_1$).



Solution

Taking into account that $f_2 = 2 f_1$ and assuming that we further propagate by a distance x after the exit of the system the matrix that describes the propagation of the Gaussian beam is:



$$M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{2f_1} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & f_1 + f_2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} -2 & 3f_1 - \dfrac{x}{2} \\ 0 & -\dfrac{1}{2} \end{pmatrix}$$

As we know when a Gaussian beam waist with q-factor $q_{in} = i z_R^{in}$ is located at the entrance of an optical system described by a matrix M the corresponding q-factor q_{out} in the exit is:

$$\frac{1}{q_{out}} \equiv \frac{M_{21} q_{in} + M_{22}}{M_{11} q_{in} + M_{12}} = \frac{0 \cdot (i z_{in}) - \dfrac{1}{2}}{-2 \cdot (i z_{in}) + 3 f_1 - \dfrac{x}{2}}$$

$$= \frac{1}{x - 6 f_1 + 4i z_{in}} = \frac{x - 6 f_1}{(x - 6 f_1)^2 + 16 z_{in}^2} - \frac{4i z_{in}}{(x - 6 f_1)^2 + 16 z_{in}^2}$$

The Gaussian beam will have a waist at the exit of the system M if $\mathrm{Re}(\dfrac{1}{q_{out}}) \equiv 0$:

$$at\ \ waist: \ \mathrm{Re}[\frac{1}{q_{out}}] \equiv 0 \Rightarrow \frac{x - 6 f_1}{(x - 6 f_1)^2 + 16 z_{in}^2} = 0 \Rightarrow x = 6 f_1$$

Thus the waist is located at distance $6 f_1$ from the 2nd lens From the definition of the q-factor we know that at the waist:

$$at\ waist: \quad \frac{1}{q_{out}} \equiv \frac{1}{i\, z_R^{out}} \equiv -i\frac{\lambda}{\pi}\frac{1}{w_{out}^2}$$

where z_R^{out}, w_{out} are respectively the Rayleigh length and the beam waist at the exit. By replacing for the value of $x = 6 f_1$ we get:

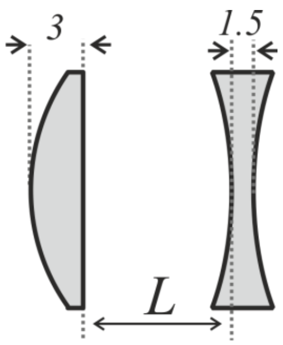
$$\frac{1}{q_{out}} \equiv -i\frac{\lambda}{\pi}\frac{1}{w_{out}^2} = -\frac{i}{4z_{in}} \Rightarrow -i\frac{\lambda}{\pi}\frac{1}{w_{out}^2} = -\frac{i}{4\dfrac{\pi w_{in}^2}{\lambda}} \Rightarrow$$

$$w_{out} = 2w_{in} \Rightarrow z_{out} = 4z_{in}$$

39) The optical system described in exercise (29), tuned to operate as a telescope, is illuminated by a Gaussian beam. Assuming that the waist of the beam in the entrance is on the 1st lens calculate:

a) The position of the waist at the exit of the optical system

b) The relative change in the waist and Rayleigh range parameters



Solution

Using the solution of exercise (29) and assuming that we further propagate by a distance x after the exit of the system, the matrix that describes the propagation of the Gaussian beam is:

$$M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2/3 & D \\ 0 & 3/2 \end{pmatrix} = \begin{pmatrix} 2/3 & D + 3x/2 \\ 0 & 3/2 \end{pmatrix}$$

where $D \equiv 28.5\,mm$. As we know when a Gaussian beam waist with q-factor $q_{in} = i\, z_R^{in}$ is located at the entrance of an optical system described by a matrix M the corresponding q-factor q_{out} in the exit is:

$$\frac{1}{q_{out}} \equiv \frac{M_{21}\, q_{in} + M_{22}}{M_{11}\, q_{in} + M_{12}} = \frac{0\cdot\left(i\, z_{in}\right) + \frac{3}{2}}{\frac{2}{3}\cdot\left(i\, z_{in}\right) + D + 3x/2}$$

$$= \frac{1}{x + \frac{2}{3}D + \frac{4}{9}i\, z_{in}} = \frac{x + \frac{2}{3}D}{(x + \frac{2}{3}D)^2 + \frac{16}{81}z_{in}^2} - \frac{\frac{4}{9}i\, z_{in}}{(x + \frac{2}{3}D)^2 + \frac{16}{81}z_{in}^2}$$

The Gaussian beam will have a waist at the exit of the system M if $\mathrm{Re}(\frac{1}{q_{out}}) \equiv 0$:

$$at\ \ waist:\ \mathrm{Re}[\frac{1}{q_{out}}] \equiv 0 \Rightarrow \frac{x + \frac{2}{3}D}{(x + \frac{2}{3}D)^2 + \frac{16}{81}z_{in}^2} = 0 \Rightarrow x = -\frac{2}{3}D$$

Thus the waist is located at distance $-2D/3 \cong -19\,mm$ from the 2nd lens. From the definition of the q-factor we know that at the waist:

$$at\ \ waist:\ \ \frac{1}{q_{out}} \equiv \frac{1}{i\, z_R^{out}} \equiv -i\frac{\lambda}{\pi}\frac{1}{w_{out}^2}$$

where z_R^{out}, w_{out} are respectively the Rayleigh length and the beam waist at the exit. By replacing for the value of $x = -2D/3$ we get:

$$\frac{1}{q_{out}} \equiv -i\frac{\lambda}{\pi}\frac{1}{w_{out}^2} = -\frac{9}{4}\frac{i}{z_{in}} \Rightarrow -i\frac{\lambda}{\pi}\frac{1}{w_{out}^2} = -\frac{9}{4}\frac{i}{\frac{\pi w_{in}^2}{\lambda}} \Rightarrow w_{out} = \frac{2}{3}w_{in} \Rightarrow z_{out} = \frac{4}{9}z_{in}$$