Interdisciplinary postgraduate program «Optics and Vision»

Foundations of Modern Optics

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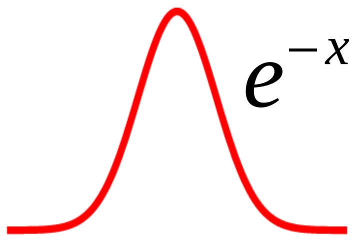
Licenses

5 Propagation of Gaussian Beams

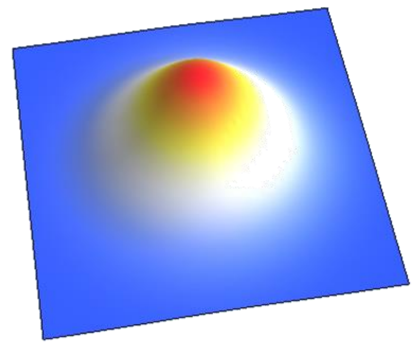
Description of a Gaussian Beam

The amplitude is described by a Gaussian distribution function

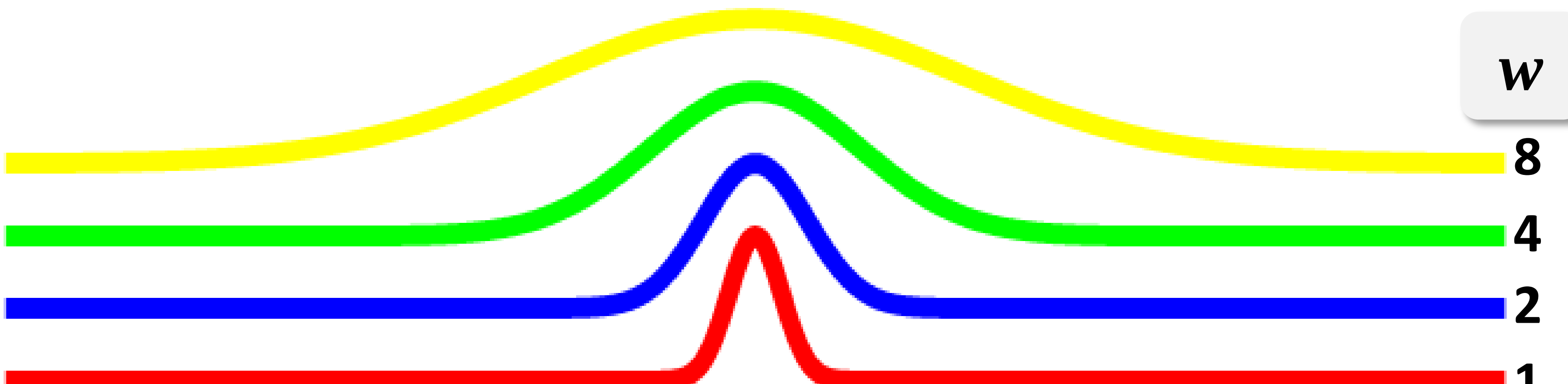
$$E(x, y; z) = E_o(z)\, e^{-\frac{x^2+y^2}{w^2(z)}}$$



$$e^{-x^2}$$

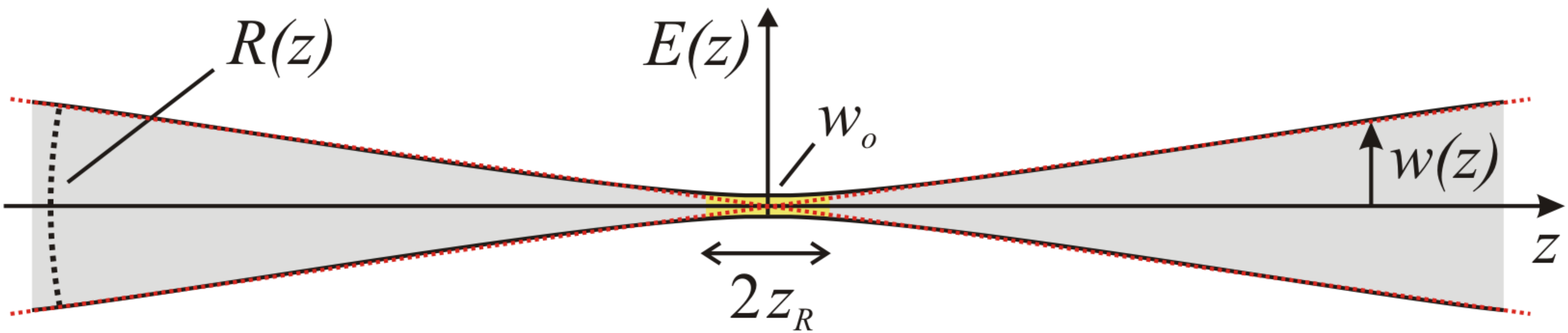


$$e^{-(x^2+y^2)}$$



Propagation of a Gaussian

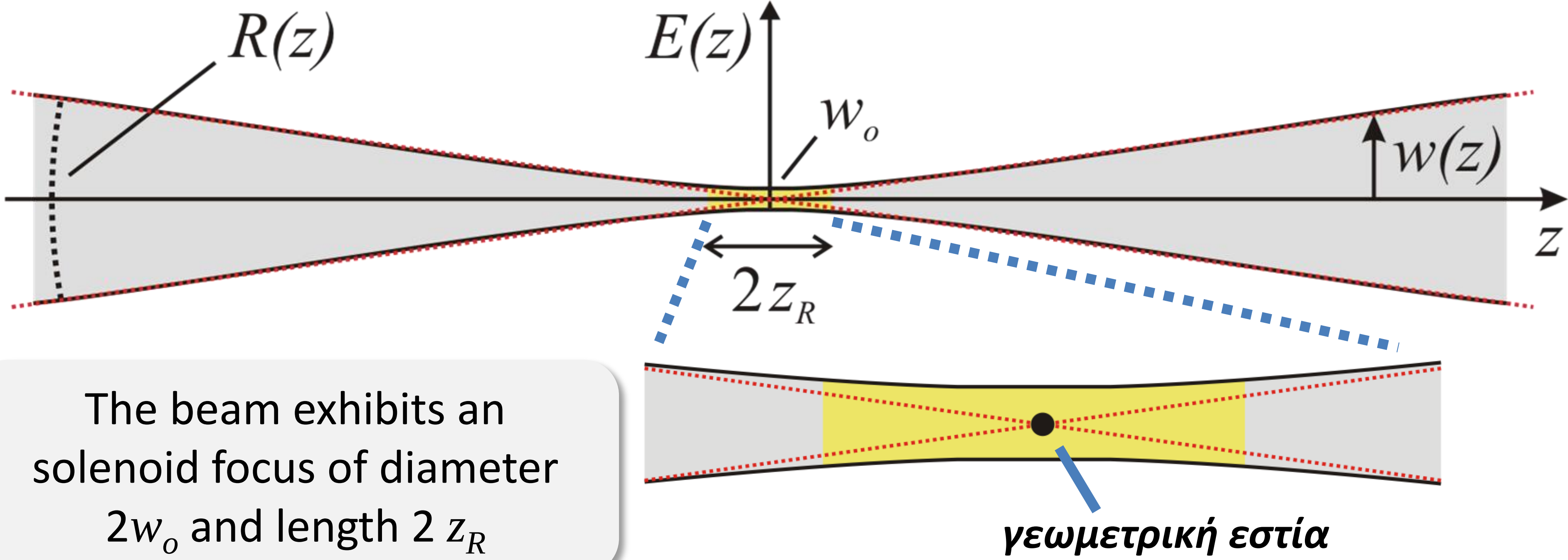
We can **analytically describe** the distribution of a Gaussian beam along it's propagation



phase terms

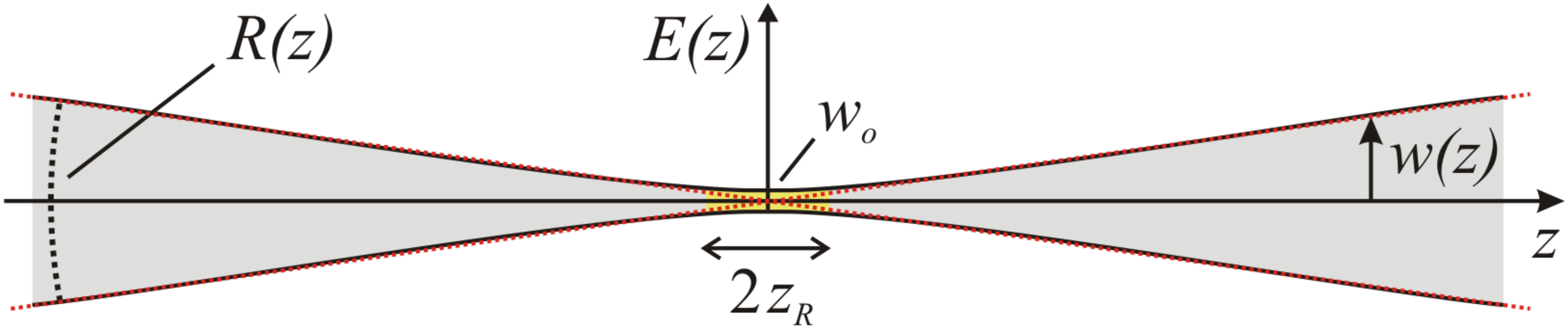
$$E(\mathbf{r}) = E_o \frac{w_o}{w(z)} \cdot e^{-\frac{x^2+y^2}{w^2(z)}} \cdot e^{-i\frac{\pi}{\lambda}\frac{x^2+y^2}{R(z)}} \cdot e^{-i[kz+\psi(z)]}$$

always Gaussian on a transverse plane



The beam exhibits an solenoid focus of diameter $2w_o$ and length $2\ z_R$

$w(z)$	semi-diameter of the beam *(beam waist)*
$R(z)$	*radius of curvature*
$\psi(z)$	Longitudinal phase distribution *(Gouy phase)*



Values of the Gaussian beam parameters along propagation

$$w(z) = w_o \sqrt{1 + \frac{z^2}{z_R^2}}$$

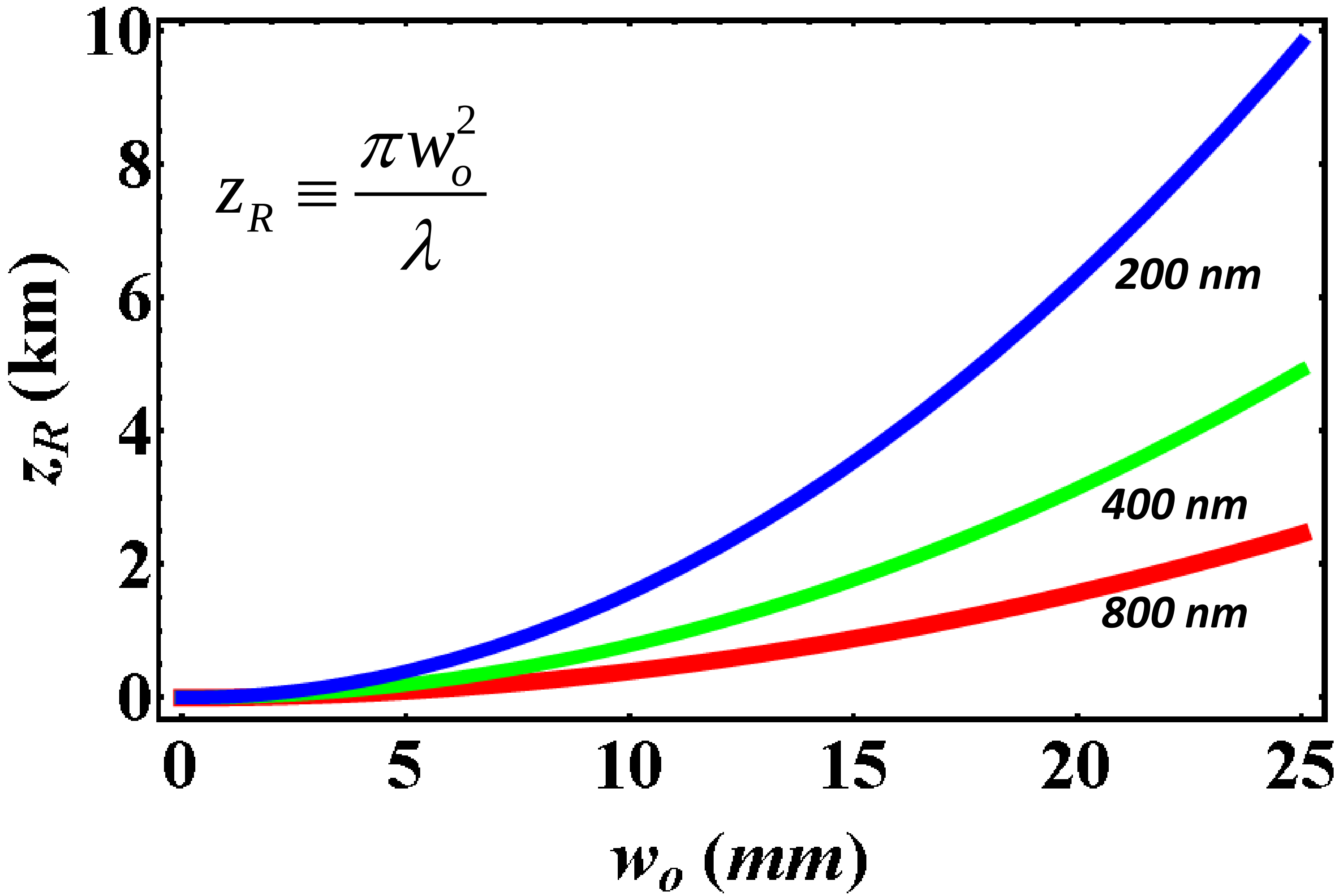
$$R(z) = z + \frac{z_R^2}{z}$$

$$\psi(z) = \arctan(\frac{z}{z_R})$$

Rayleigh length

$$z_R \equiv \frac{\pi w_o^2}{\lambda}$$

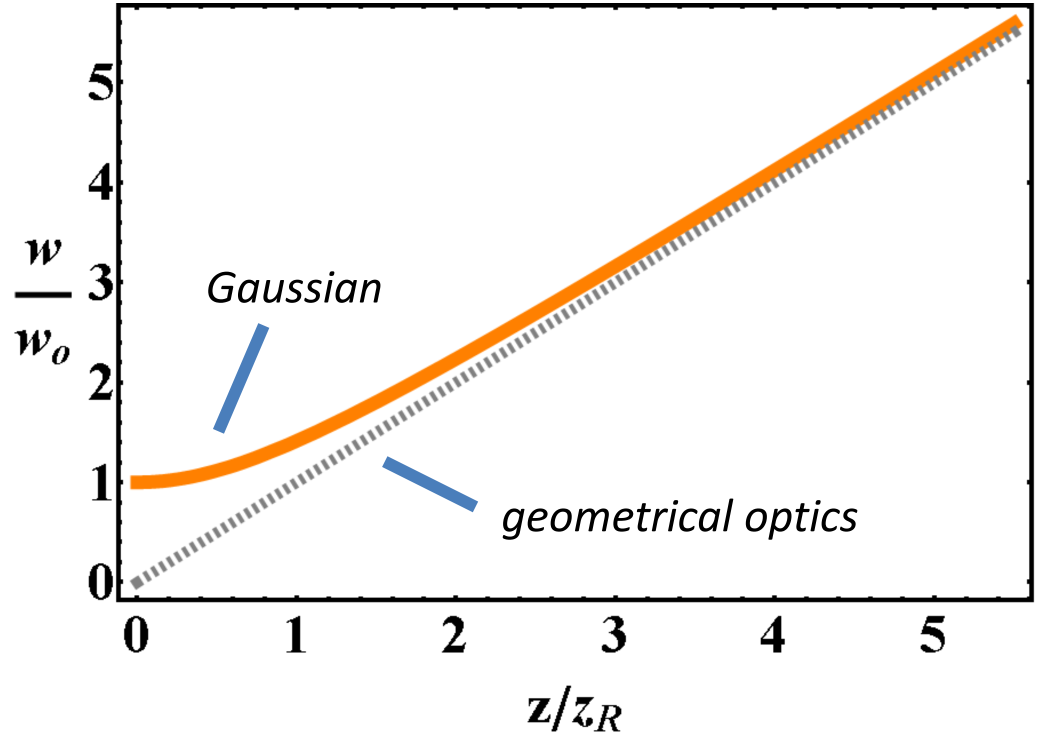
Distance z_R
(often called Rayleigh length)
equals the 1/2 of the
solenoid focus length



- Beams that are tightly focused (small w_o) expand more rapidly!

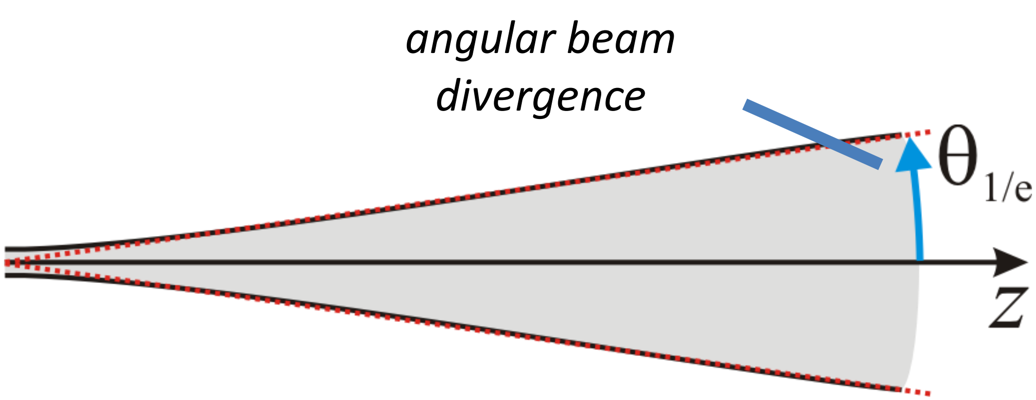
- By reducing the wavelength we "slow down" the beam spatial expansion

$$w(z) = w_o \sqrt{1 + \frac{z^2}{z_R^2}}$$



$$z \gg z_R \Rightarrow w(z) \cong \frac{w_o}{z_R} z$$

at large enough distances we can simply use geometrical optics

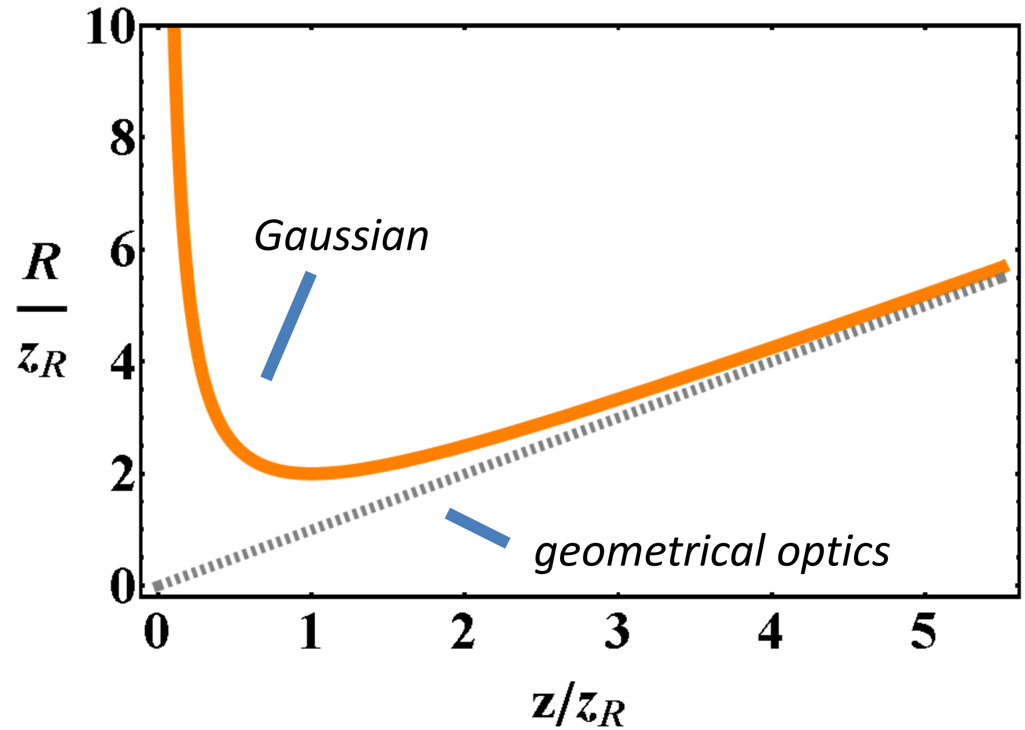


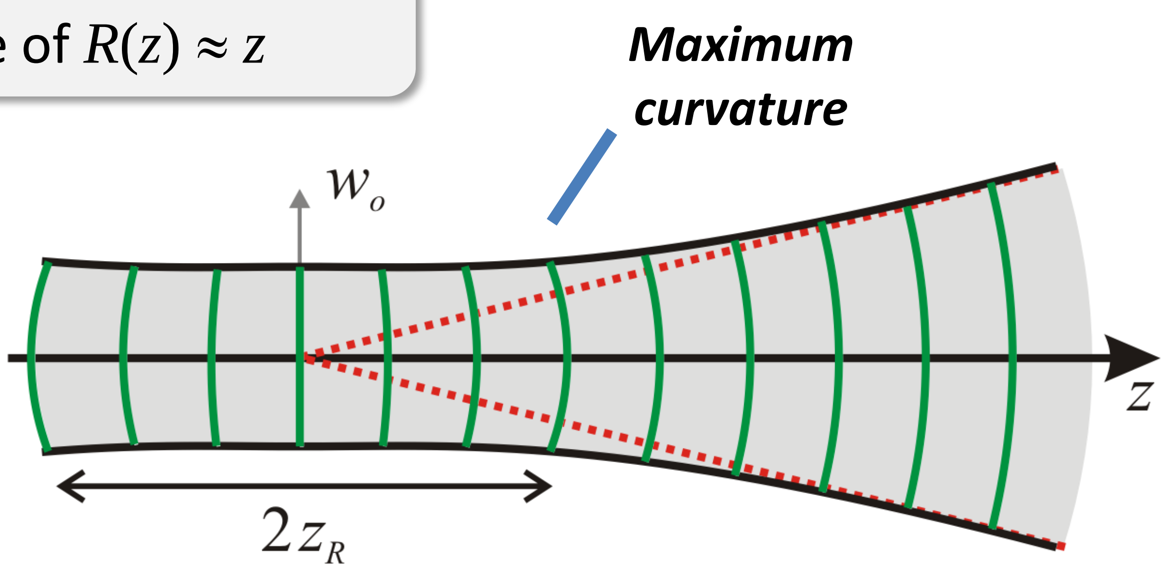
$$\theta_{1/e} \cong \frac{w_o}{z_R} = \frac{\lambda}{\pi w_o}$$

$$R(z) = z + \frac{z_R^2}{z}$$

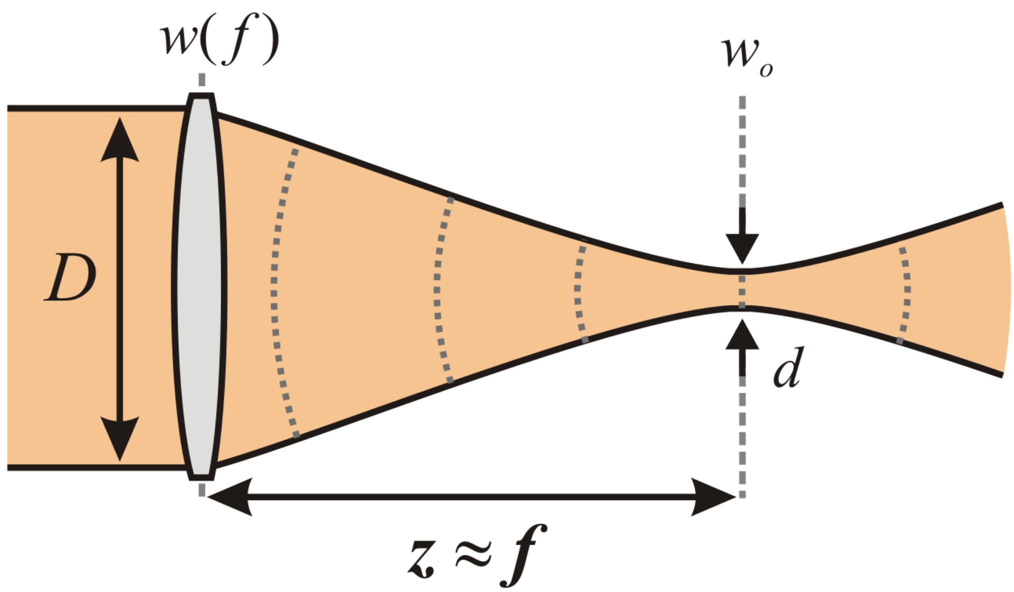
When $z \approx 0$, the wavefront is flat!

Away from the focus ($z >> z_R$) the wavefront is spherical with radius of curvature of $R(z) \approx z$





Typical focusing of a Gaussian beam*

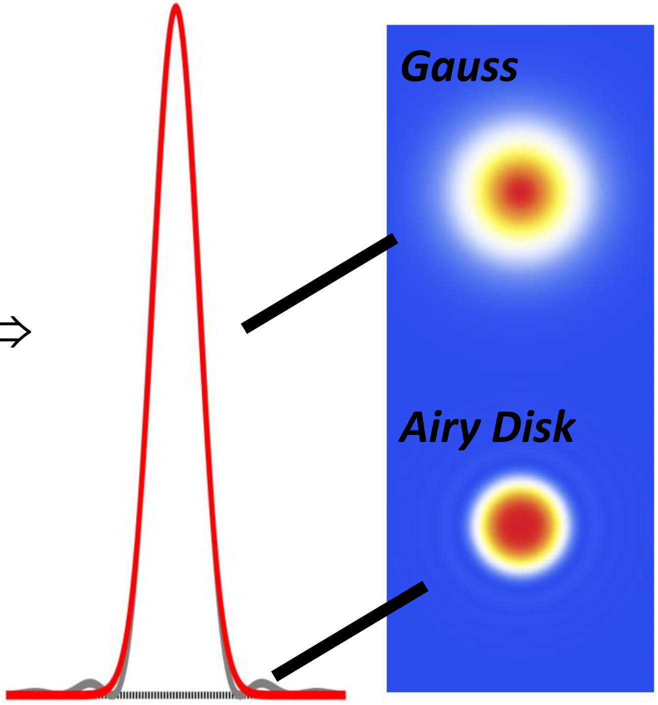


$$z_R \gg f$$

$$f \gg z'_R$$

$$D = 2w(f) = 2w_o\sqrt{1+\frac{f^2}{z'^2_R}} \cong \frac{2w_o}{z'_R} f = \frac{2\lambda}{\pi w_o} f \Rightarrow$$

$$\boxed{d = 2w_o \cong \frac{4\lambda}{\pi D} f}$$

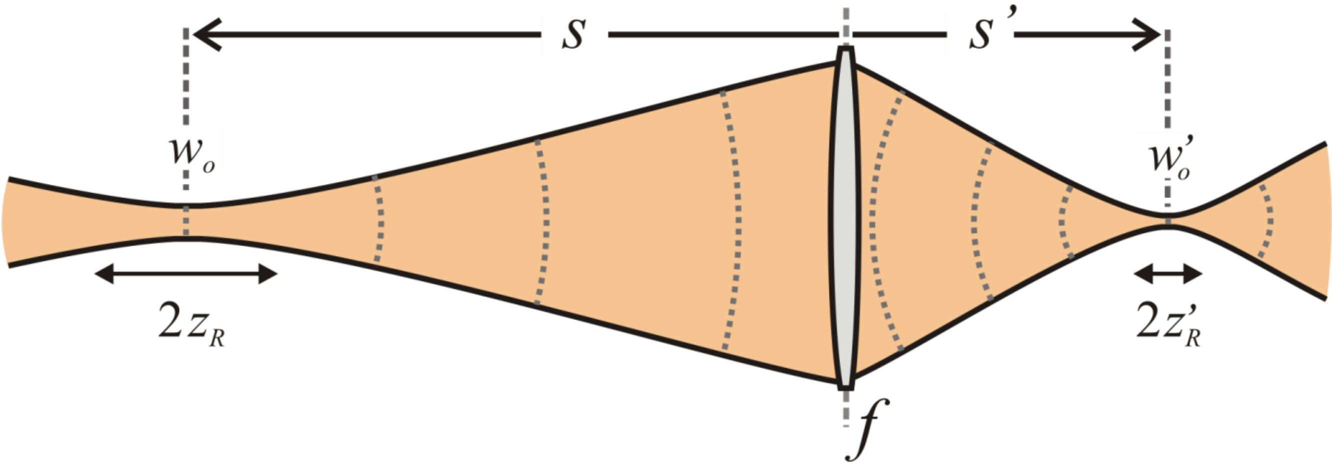


Σύγκριση εστίας Γκάους με κατανομή Airy

* *Idealized calculation. We assume that the input beam waist is on the lens entrance*

Imaging Gaussian beams

Imaging relations



We assume that the input Gaussian beam focus is the **object**, while the output Gaussian beam is the **image**

Geometrical optics

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$$

$$\frac{1}{s+\dfrac{z_R^2}{s-f}}+\frac{1}{s'}=\frac{1}{f}$$

input beam parameters

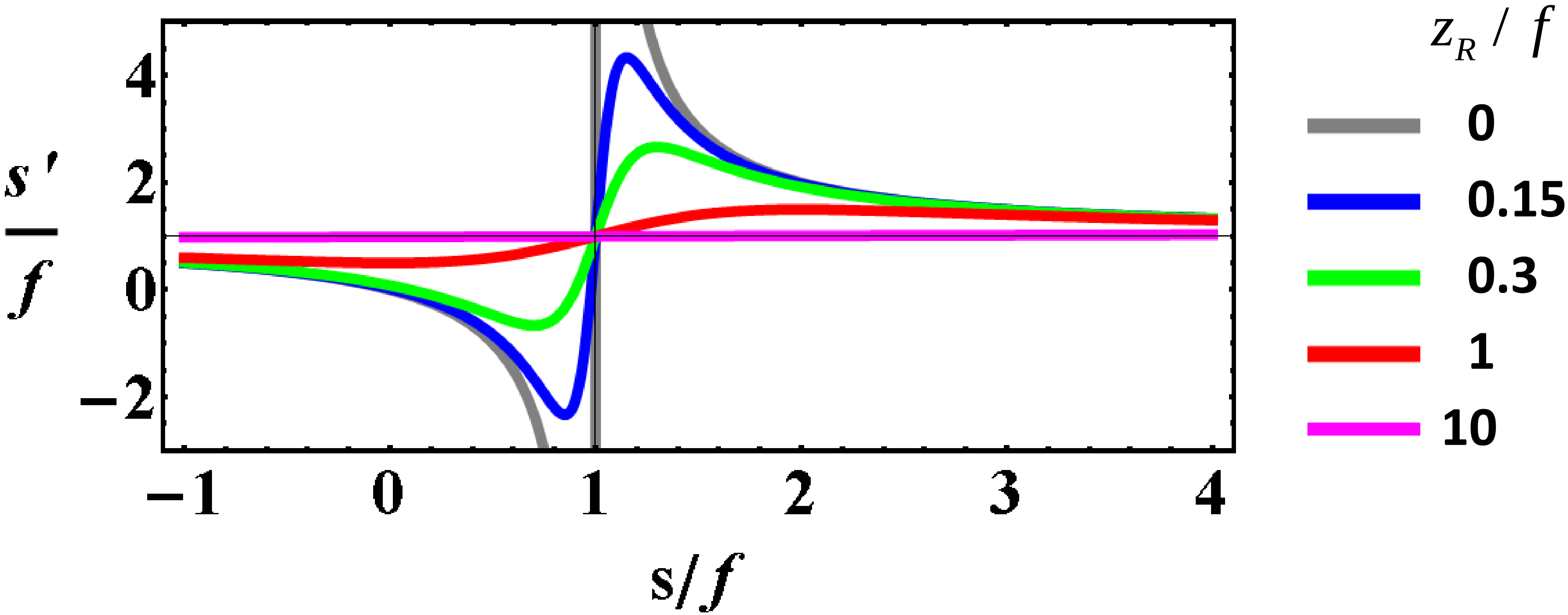
Gaussian beams

$$\frac{1}{s}+\frac{1}{s'+\dfrac{z'^2_R}{s'-f}}=\frac{1}{f}$$

output beam parameters

In contrast to the classical geometrical optics the imaging formulas are **not symmetric** to the input and output beam parameters

Position of the image



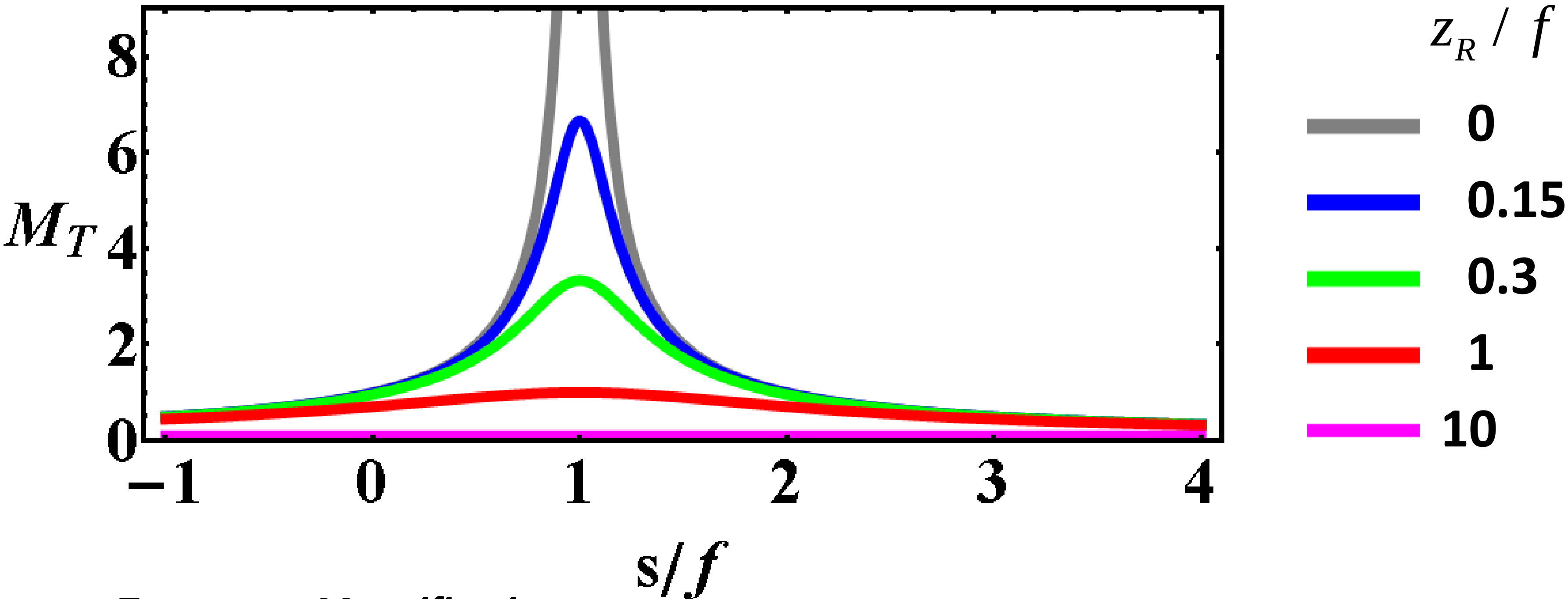
There is a **minimum** and a **maximum** distance for the image position in Gaussian beams

$$s = f + z_R \Rightarrow \boxed{s'_{\max} = f\left(1 + \frac{f}{2z_R}\right)}$$

The image is located at the maximum distance from the lens when $s = f + z_R$ and not at $s = f$

$$\boxed{s = f \Rightarrow s' = f}$$

Magnification



Transverse Magnification

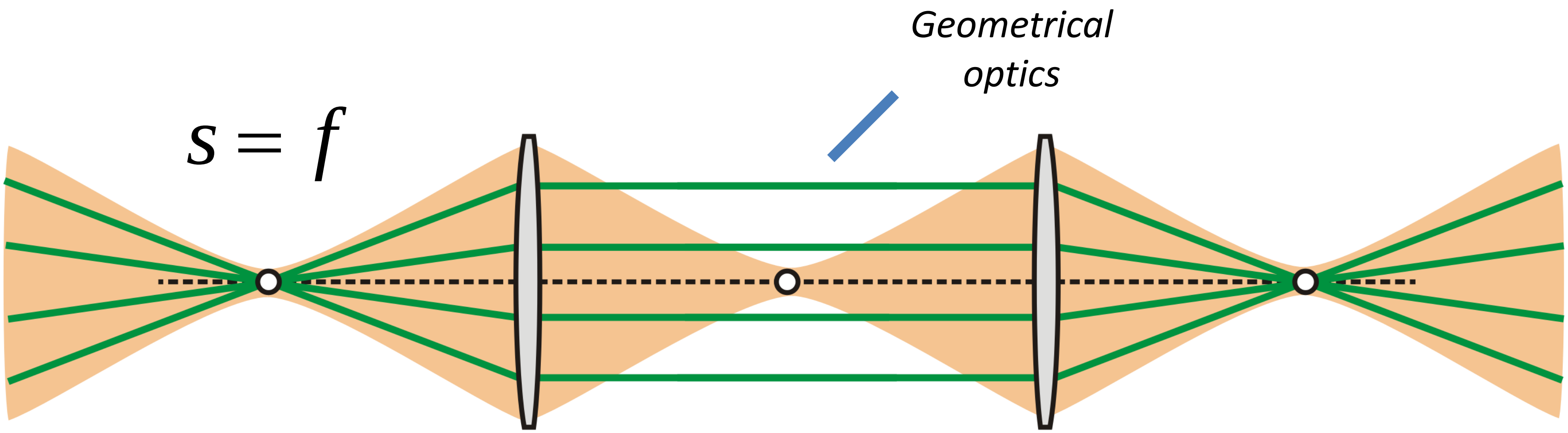
$$M_T \equiv \frac{w_o'}{w_o} = \frac{1}{\sqrt{(1-s/f)^2 + z_R^2/f^2}}$$

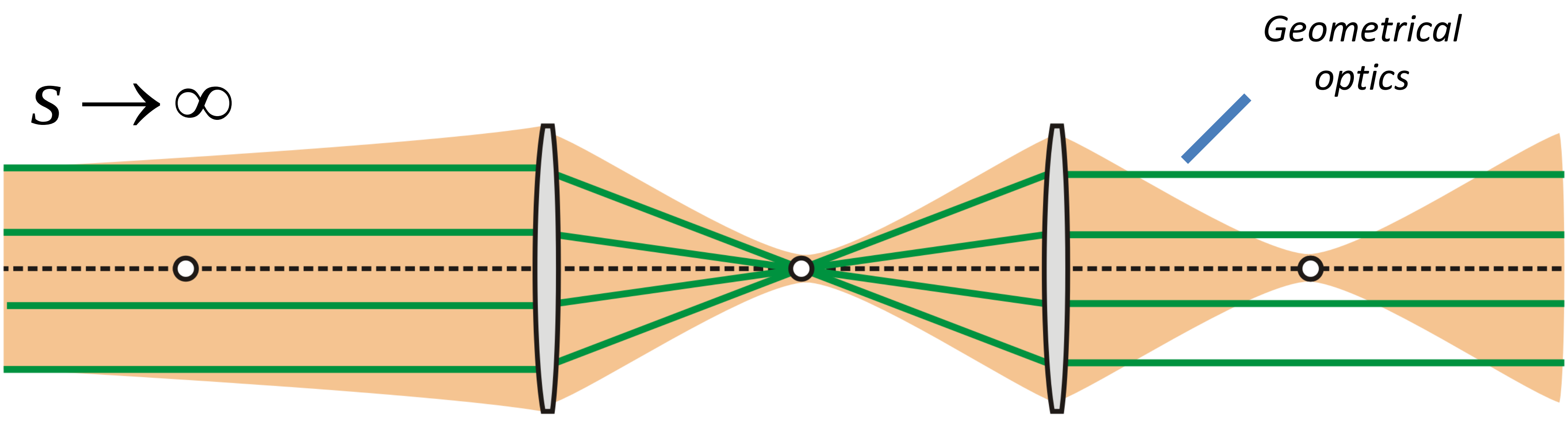
$$s = f \Rightarrow \boxed{M_T^{\max} = \frac{f}{z_R}}$$

Longitudinal magnification

$$M_L \equiv \frac{z_R'}{z_R} = M_T^2$$

Examples of peculiar behavior





Generic analysis

Normalization

it is advantageous to normalize all spatial values on the lens focal length

	input Gaussian beam	*output Gaussian beam*
distance from the lens	$\xi = \dfrac{s}{f}$	$\xi' = \dfrac{s'}{f}$
Rayleigh Length	$\zeta_R = \dfrac{z_R}{f} = \dfrac{\pi w_o^2}{f \cdot \lambda}$	$\zeta'_R = \dfrac{z'_R}{f} = \dfrac{\pi w_o'^2}{f \cdot \lambda}$

normalized distance of a plane lying at distance z

$$z_n = \frac{z}{f}$$

Normalized imaging equations

position of the output Gaussian Beam

$$\xi' = 1 + \frac{\xi - 1}{\zeta_R^2 + (\xi - 1)^2}$$

$$\zeta'_R = \frac{\zeta_R}{\zeta_R^2 + (\xi - 1)^2}$$

Rayleigh Length of the output Gaussian Beam

waist of the output Gaussian Beam

$$w'_o = \frac{w_o}{\sqrt{(\xi - 1)^2 + \zeta_R^2}}$$

Transverse Magnification

$$M_T \equiv \frac{w_o'}{w_o} = \frac{1}{\sqrt{(\xi - 1)^2 + \zeta_R^2}}$$

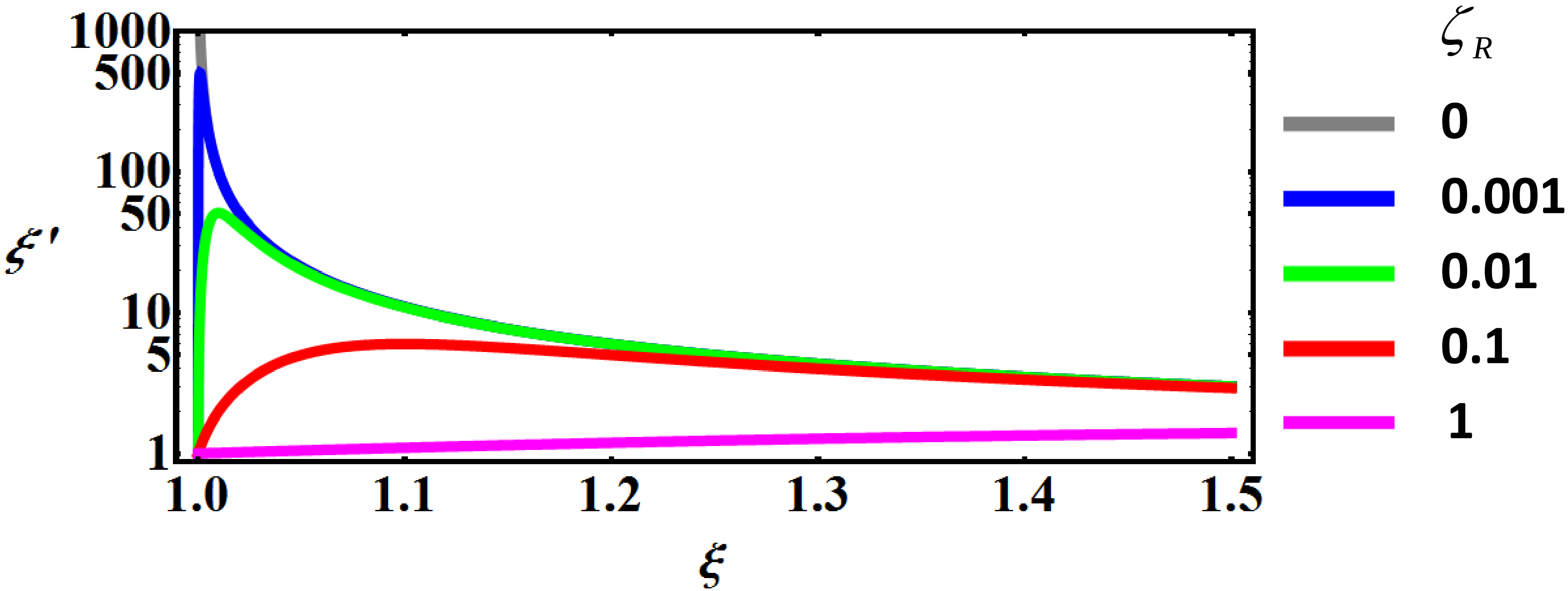
Longitudinal Magnification

$$M_L \equiv \frac{z_R'}{z_R} = \frac{\zeta_R'}{\zeta_R} = M_T^2 = \frac{1}{(\xi - 1)^2 + \zeta_R^2}$$

output beam semi-diameter w_{out} size at a distance z_n from the lens:

$$\frac{w_{out}^2(\xi, z_n)}{w_o^2} = 1 + \frac{\xi^2}{\zeta_R^2} + \frac{z_n^2}{\zeta_R^2}[\zeta_R^2 + (\xi - 1)^2] - 2\frac{z_n}{\zeta_R^2}[\zeta_R^2 + (\xi - 1)\xi]$$

Position of the image



There is a **minimum** and a **maximum** distance for the image position in Gaussian beams

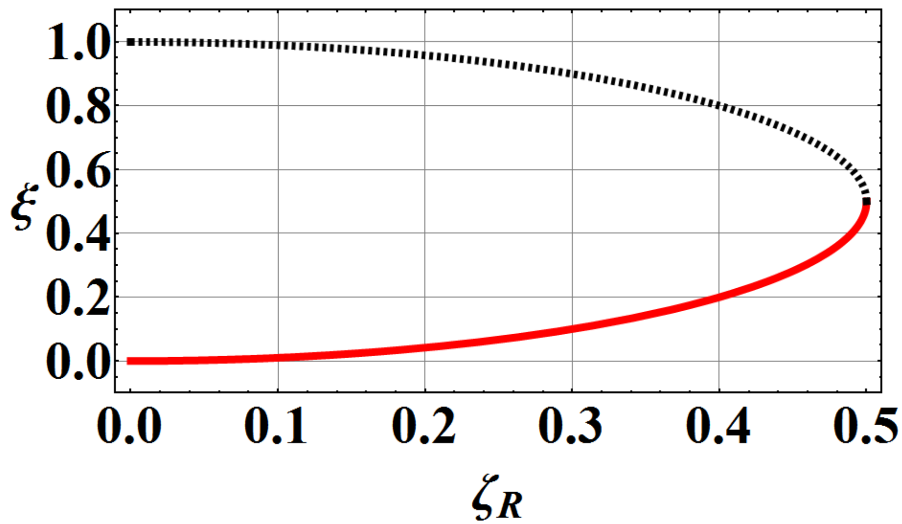
$$\xi = 1 \Rightarrow \xi' = 1$$

$$\xi = 1 + \zeta_R \Rightarrow \begin{cases} \xi'_{max} = 1 + \dfrac{1}{2\zeta_R} \\ \xi'_{min} = 1 - \dfrac{1}{2\zeta_R} \end{cases}$$

Image on the lens $\xi' = 0$

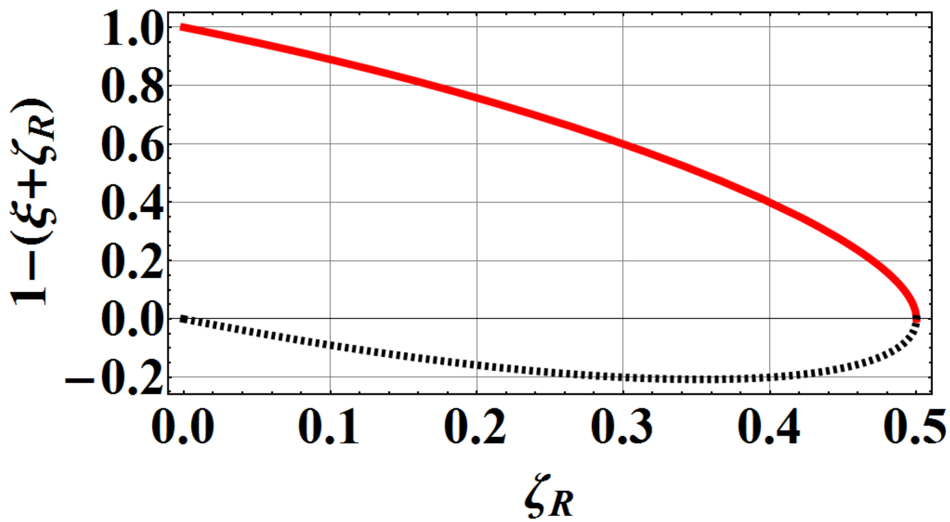
$$\xi' = 0 \Rightarrow \xi = \frac{1 \pm \sqrt{1 - 4\zeta_R^2}}{2}$$

There are two solutions in this problem that merge at $\zeta_R = 0.5$

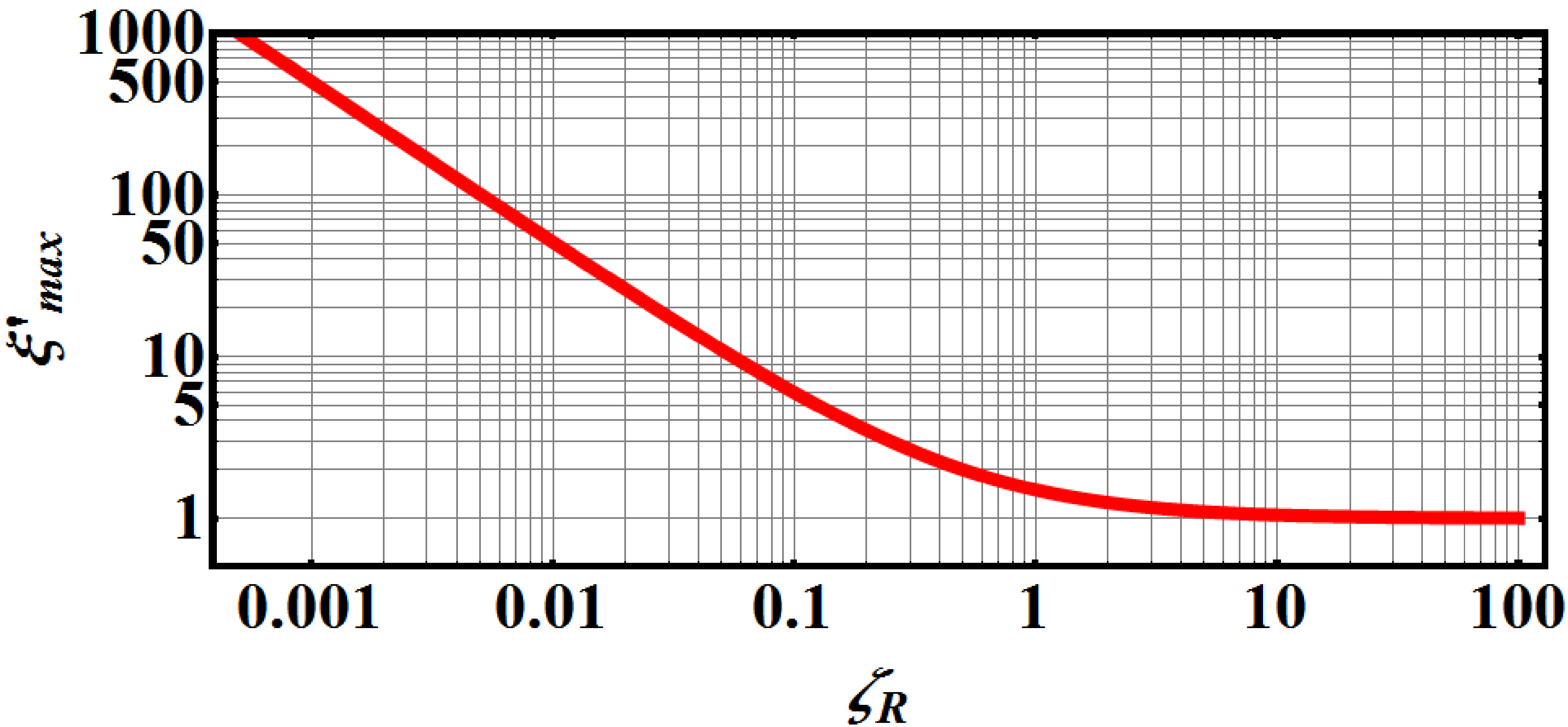


"Geometrical regime"

$$\zeta_R \to 0 \Rightarrow \begin{cases} \xi \to 0 \\ \xi \to 1 \end{cases}$$

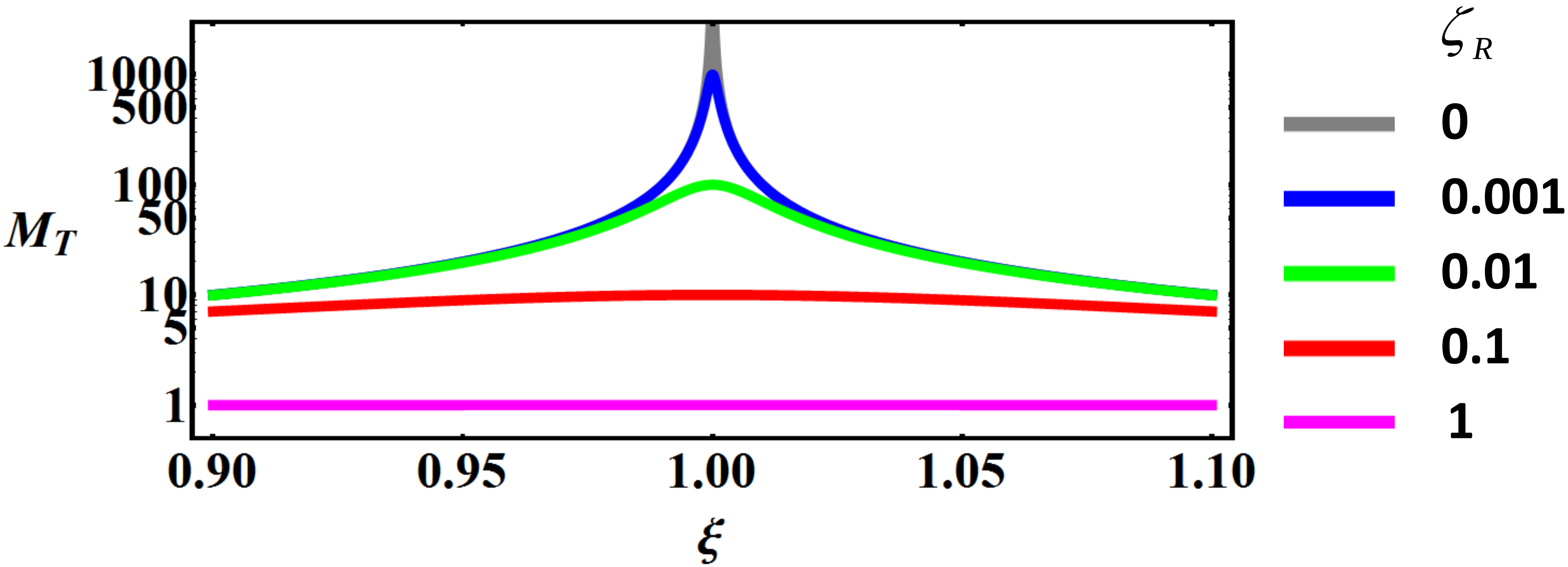


$$\xi'_{max} = 1 + \frac{1}{2\zeta_R}$$



maximal position of the image as a function of the normalized Rayleigh length of the input Gaussian Beam

Magnification



Transverse Magnification is maximized at

$$\xi = 1 \Rightarrow M_T^{\max} = \frac{1}{\zeta_R}$$

At what distance ξ is the output beam collimated?

*One safe criterion is that the output Gaussian beam is **collimated** when it's Rayleigh length is **maximized**.*

Since the only varying parameter is the distance of the input Gaussian beam to the lens, it is sufficient to estimate the maximum of the function ζ_R' in respect to ξ:

$$\frac{\partial \zeta'_R}{\partial \xi} = 0 \Rightarrow \boxed{\xi_{col} = 1}$$

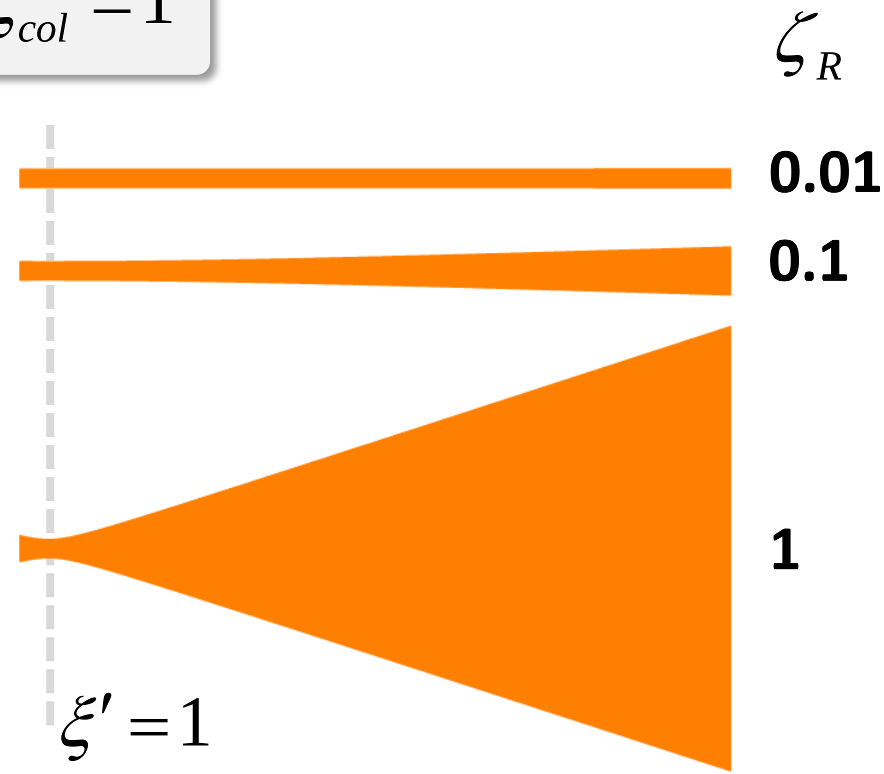
In this case:

$$\xi' = 1$$

$$\zeta'_R = \frac{1}{\zeta_R}$$

$$M_T = \frac{1}{\zeta_R}$$

$$M_L = \frac{1}{\zeta_R^2}$$

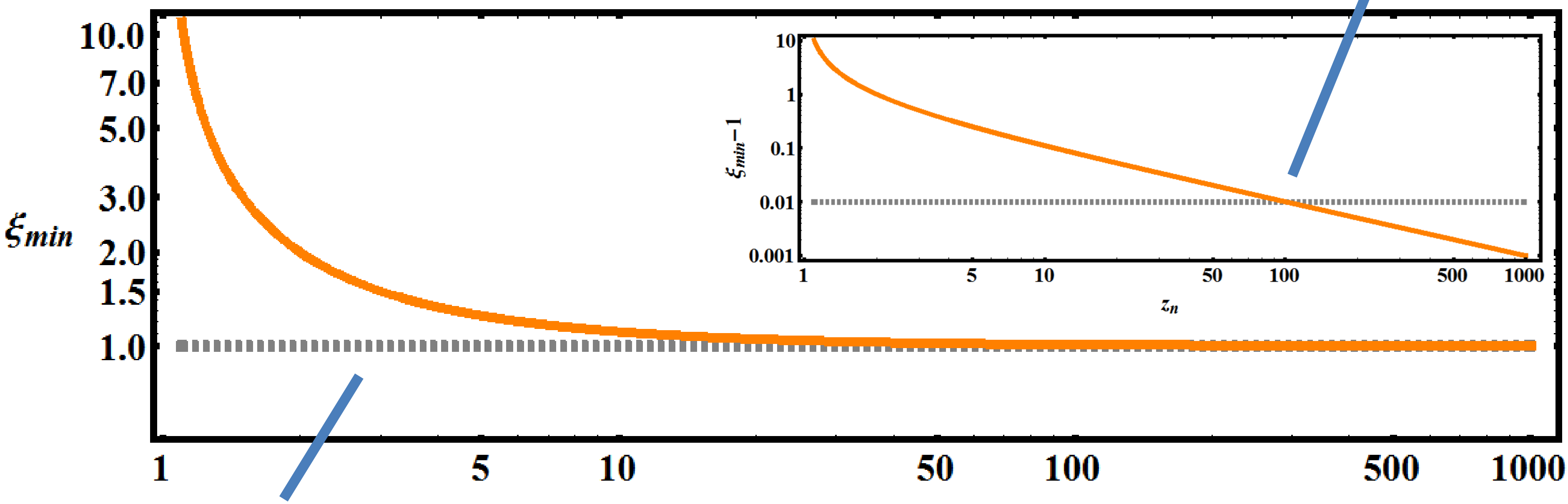


Minimum beam diameter at a distance z_n

Since the only varying parameter is the distance of the input Gaussian beam to the lens, it is sufficient to estimate the minimum of the function w_{out} in respect to ξ:

$$\frac{\partial w_{out}(\xi,\zeta_R;z_n)}{\partial \xi}=0 \Rightarrow \boxed{\xi_{min}=\frac{z_n}{z_n-1}}$$

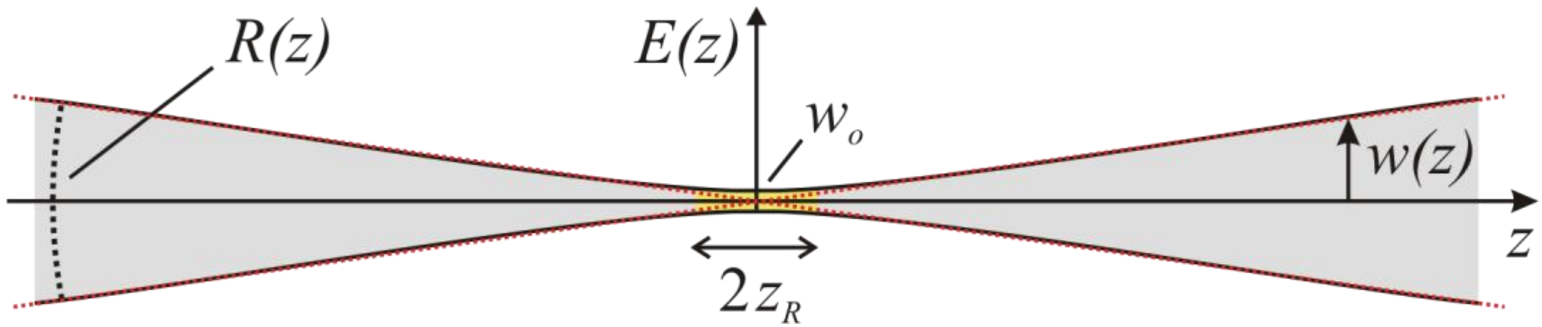
collimation error <1%



optimum collimation

ABCD matrix theory and Gaussian beams

Complex parameter *q*



$$E(\mathbf{r}) = E_o \frac{e^{-i[kz+\psi(z)]}}{w(z)} \cdot e^{-\frac{x^2+y^2}{w^2(z)} - i\frac{\pi}{\lambda}\frac{x^2+y^2}{R(z)}} \equiv E_o \frac{e^{-i[kz+\psi(z)]}}{w(z)} e^{-i\frac{\pi}{\lambda}\frac{x^2+y^2}{q(z)}}$$

$$\frac{1}{q(z)} \equiv \frac{1}{R(z)} - i\frac{\lambda}{\pi}\frac{1}{w^2(z)}$$

The complex parameter *q(z)* describes the Gaussian beam completely

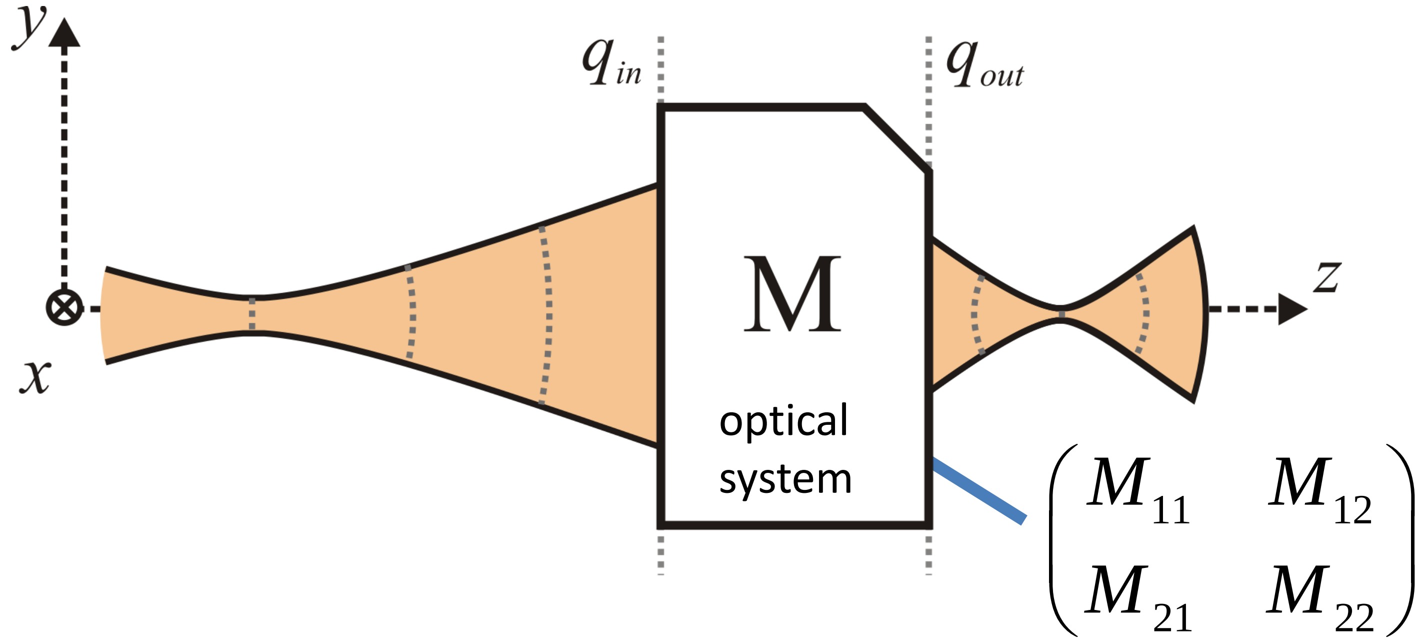
By replacing the known values of the $R(z)$, $w(z)$ parameters for the free space propagation we get:

$$\left.\begin{aligned} w(z) &= w_o\sqrt{1+\frac{z^2}{z_R^2}} \\ R(z) &= z+\frac{z_R^2}{z} \end{aligned}\right\} \Rightarrow \boxed{\frac{1}{q(z)} = \frac{1}{z+z_R^2/z} - i\frac{1}{z_R+z^2/z_R}}$$

$$\frac{1}{q(z)} = \frac{z_R+z^2/z_R - i(z+z_R^2/z)}{(z_R+z^2/z_R)(z_R+z^2/z_R)} \Rightarrow$$

$$\boxed{q(z) = z + i\,z_R}$$

Propagation though an optical system



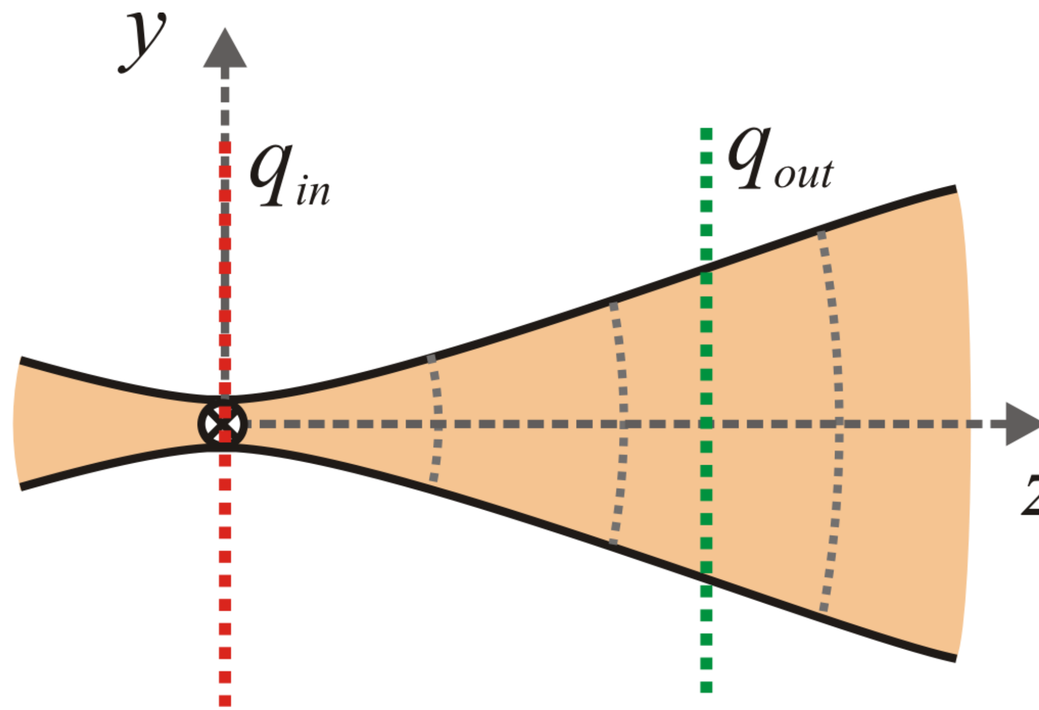
$$q_{out} \equiv \frac{M_{11}\, q_{in} + M_{12}}{M_{21}\, q_{in} + M_{22}}$$

The parameter q in the exit of an optical system is defined by the optical transfer matrix of the system M and it's value in the input of the system.

Application in free space propagation

From the ray matrix theory we know that free space propagation is described as a simple transport over z

$$M = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$



$$\left. \begin{aligned} q(z) &\equiv q_{out} \\ q(0) &= q_{in} = i\, z_R \end{aligned} \right\} \Rightarrow \boxed{q(z) = \frac{1 \cdot q(0) + z}{0 \cdot q(0) + 1} = z + i\, z_R}$$