1. In each case, show that $E \supseteq F$ is Galois, find the lattice of intermediate fields and find a primitive element for each intermediate field.

   a) $E = \mathbb{Q}(e^{2\pi i/5})$, $F = \mathbb{Q}$.

   b) $E = \mathbb{Q}(e^{2\pi i/7})$, $F = \mathbb{Q}$.

   c) $E = \mathbb{Q}(i, \sqrt{3})$, $F = \mathbb{Q}$.

   d) $E = \mathbb{Z}_2(u)$, where u is a root of $x^4 + x + 1$ and $F = \mathbb{Z}_2$.

   e) $E = \mathbb{Q}(\sqrt[4]{2}, i)$ and $F = \mathbb{Q}$.

2. Show that a finite Galois extension $E \supseteq F$ has a finite number of intermediate fields.

3. Here you are asked to prove Lemma 2.11 of the Lecture notes. Let $E \supseteq F$ be a field extension and $\mathscr{F}$ and $\mathscr{H}$ be defined as in the notes. For $K, L \in \mathscr{F}$ and $H, I \in \mathscr{H}$ one has

   i) $H \subseteq H^{**}$

   ii) $K \subseteq K^{**}$

   iii) $K \subseteq L$ implies $K^* \supseteq L^*$ and $H \subseteq I$ implies $H^* \supseteq I^*$

   iv) $H^{***} = H^*$ and $K^{***} = K^*$.

4. Let $E \supseteq F$ be fields and consider the correspondence mapping $*$ from the lecture.

   a) Show that $*\colon \mathscr{H} \to \mathscr{F}$ is onto if and only if $K^{**} = K$ for each $K \in \mathscr{F}$.

   b) Show that $*\colon \mathscr{F} \to \mathscr{H}$ is onto if and only if $H^{**} = H$ for each $H \in \mathscr{H}$.

5. Let $L \supseteq K$ be a finite Galois extension, and let $G =$Gal$(L : K)$ be its Galois group. For an element $a \in L$, we denote by $orb_G(a) = \{\sigma(a) \mid \sigma \in G\}$ the orbit of a by G. Also, we denote by $Stab_G(a) = \{\sigma \in G \mid \sigma(a) = a\}$ the stabilizer of a in G. Show that:

   $$p(x) = \prod_{a_i \in orb_G(a)} (x - a_i)$$

   is an irreducible polynomial in K[x] (hence in particular it is the minimal polynomial of a).

   Deduce that if $Stab_G(a) = \{\mathrm{id}\}$, then a is a primitive element for $L \supseteq K$.

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6. Let $a_1, ..., a_n$ be pairwise co-prime non-square integers. Consider the extension $\mathbb{Q}(\sqrt{a_1}, ..., \sqrt{a_n}) \supseteq \mathbb{Q}$ and let G be its Galois group.

    - Compute the Galois group G.
      Hint: you can use results from last week's exercise sheet!
    - Show that $\sum_{i=1}^{n} \sqrt{a_i}$ is a primitive element. [Hint: Use previous exercise.]
    - Show that $1 + \sqrt{2} + \sqrt{3} + ... + \sqrt{N} \notin \mathbb{Q}$.