ELASTICITY

Assignment #01

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DATE: April 3, 2022

ELASTICITY ASSIGNMENT 1

Chapter 1 Preliminaries

Exercises

1.1 For the given matrix/vector pairs, compute the following quantities .....

(a) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}, b_i = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Solution:-

$$a_{ii} = a_{11} + a_{22} + a_{33}$$
$$= 1 + 4 + 1 = 6 \text{ (scalar)}$$

$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$$
$$= (1)(1) + (1)(1) + (1)(1) + (0)(0) + (4)(4) + (2)(2) + (0)(0) + (1)(1) + (1)(1)$$
$$= 1 + 1 + 1 + 0 + 16 + 4 + 0 + 1 + 1$$
$$= 25 \text{ (scalar)}$$

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3$$
$$= \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \text{ (vector)}$$

$$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$$
$$= 1 + 0 + 2 + 0 + 0 + 0 + 0 + 0 + 4 = 7 \text{ (scalar)}$$

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \text{ (matrix)}$$

$b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3 = 1 + 0 + 4 = 5$ (Scalar)

(b) $a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$ $\quad b_i = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

Solution:-

$a_{ii} = a_{11} + a_{22} + a_{33}$

$= 1 + 2 + 2 = 5$ (scalar)

$a_{ij} a_{ij} = a_{11} a_{11} + a_{12} a_{12} + a_{13} a_{13} + a_{21} a_{21} + a_{22} a_{22} + a_{23} a_{23} + a_{31} a_{31} + a_{32} a_{32} + a_{33} a_{33}$

$= 1 + 4 + 0 + 0 + 4 + 1 + 0 + 16 + 4 = 30$ (scalar)

$a_{ij} a_{jk} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{bmatrix}$ (matrix)

$a_{ij} b_j = a_{i1} + a_{i2} b_2 + a_{i3} b_3$

$= \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}$ (vector)

$a_{ij} b_i b_j = a_{11} b_1 b_1 + a_{12} b_1 b_2 + a_{13} b_1 b_3 + a_{21} b_2 b_1 + a_{22} b_2 b_2 + a_{23} b_2 b_3 + a_{31} b_3 b_1 + a_{32} b_3 b_2 + a_{33} b_3 b_3$

$= 4 + 4 + 0 + 0 + 2 + 1 + 0 + 4 + 2$

$= 17$ (scalar)

$b_i b_j = \begin{bmatrix} b_1 b_1 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2 b_2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3 b_3 \end{bmatrix}$

$= \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ (matrix)

$b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3$

$= 4 + 1 + 1$

$= 6$ (scalar)

(c) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $b_i = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Solution:-

$a_{ii} = a_{11} + a_{22} + a_{33}$

$= 1 + 0 + 4$

$= 5$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1 + 1 + 1 + 1 + 0 + 4 + 0 + 1 + 16 = 25$ (scalar)

$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

$= \begin{bmatrix} 2 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & 4 & 18 \end{bmatrix}$ (matrix)

$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3$

$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ (vector)

$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$

$= 1 + 1 + 0 + 1 + 0 + 0 + 0 + 0 + 0$

$= 3$ (scalar)

$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (matrix)

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3$

$= 1 + 1 + 0 = 2$ (scalar)

1.2 Use the decomposition result (1.2.10) to express - - - - - of Section 1.2

(a)

$$a_{ij} = \frac{1}{2}(a_{ij}+a_{ji}) + \frac{1}{2}(a_{ij}-a_{ji})$$

$$= \frac{1}{2}\left(\begin{bmatrix}1&1&1\\0&4&2\\0&1&1\end{bmatrix} + \begin{bmatrix}1&0&0\\1&4&1\\1&2&1\end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix}1&1&1\\0&4&2\\0&1&1\end{bmatrix} - \begin{bmatrix}1&0&0\\1&4&1\\1&2&1\end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix}2&1&1\\1&8&3\\1&3&2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0&1&1\\-1&0&1\\-1&-1&0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}2&2&2\\0&8&4\\0&2&2\end{bmatrix} = \begin{bmatrix}1&1&1\\0&4&2\\0&1&1\end{bmatrix}$$

Clearly $a(ij)$ and $a[ij]$ satisfy the appropriate conditions

(b)

$$a_{ij} = \frac{1}{2}(a_{ij}+a_{ji}) + \frac{1}{2}(a_{ij}-a_{ji})$$

$$= \frac{1}{2}\left(\begin{bmatrix}1&2&0\\0&2&1\\0&4&2\end{bmatrix} + \begin{bmatrix}1&0&0\\2&2&4\\0&1&2\end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix}1&2&0\\0&2&1\\0&4&2\end{bmatrix} - \begin{bmatrix}1&0&0\\2&2&4\\0&1&2\end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix}2&2&0\\2&4&5\\0&5&4\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0&2&0\\-2&0&-3\\0&3&0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}2&4&0\\0&4&2\\0&8&4\end{bmatrix} = \begin{bmatrix}1&2&0\\0&2&1\\0&4&2\end{bmatrix}$$

Clearly $a(ij)$ and $a[ij]$ satisfy the appropriate conditions

(c)

$$a_{ij} = \frac{1}{2}(a_{ij}+a_{ji}) + \frac{1}{2}(a_{ij}-a_{ji})$$

$$= \frac{1}{2}\left(\begin{bmatrix}1&1&1\\1&0&2\\0&1&4\end{bmatrix} + \begin{bmatrix}1&1&0\\1&0&1\\1&2&4\end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix}1&1&1\\1&0&2\\0&1&4\end{bmatrix} - \begin{bmatrix}1&1&0\\1&0&1\\1&2&4\end{bmatrix}\right)$$

$$a_{(ij)} = \frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 4 \\ 0 & 2 & 8 \end{bmatrix}$$

$$a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

Clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions

1.3 If a_{ij} is symmetric .... from Exercise 1.2

$$a_{ij} b_{ij} = -a_{ji} b_{ji}$$
$$a_{ij} b_{ij} = -a_{ij} b_{ij}$$
$$2 a_{ij} b_{ij} = 0$$
$$\Rightarrow a_{ij} b_{ij} = 0$$

(a) $a_{(ij)} a_{[ij]} = \frac{1}{4} \operatorname{tr}\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$

(b) $a_{(ij)} a_{[ij]} = \frac{1}{4} \operatorname{tr}\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}^T\right) = 0$

(c) $a_{(ij)} a_{[ij]} = \frac{1}{4} \operatorname{tr}\left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}\right) = 0$

1.4

Explicitly verify the following properties ......

Solution:-

$$\delta_{ij} a_j = \delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3$$

$$= \begin{bmatrix} \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 \\ \delta_{21}a_1 + \delta_{22}a_2 + \delta_{23}a_3 \\ \delta_{31}a_1 + \delta_{32}a_2 + \delta_{33}a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i$$

$$\delta_{ij}a_{jk} = \begin{bmatrix} \delta_{11}a_{11} + \delta_{12}a_{21} + \delta_{13}a_{31} & \delta_{11}a_{12} + \delta_{12}a_{22} + \delta_{13}a_{32} & \delta_{11}a_{13} + \delta_{12}a_{23} + \delta_{13}a_{33} \\ \delta_{21}a_{11} + \delta_{22}a_{21} + \delta_{23}a_{31} & \delta_{21}a_{12} + \delta_{22}a_{22} + \delta_{23}a_{32} & \delta_{21}a_{13} + \delta_{22}a_{23} + \delta_{23}a_{33} \\ \delta_{31}a_{11} + \delta_{32}a_{21} + \delta_{33}a_{31} & \delta_{31}a_{12} + \delta_{32}a_{22} + \delta_{33}a_{32} & \delta_{31}a_{13} + \delta_{32}a_{23} + \delta_{33}a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij}$$

1.5 Formally expand the expression ---- $\det[a_{ij}]$

Solution:-

$$\det(a_{ij}) = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}$$

$$= \varepsilon_{123}a_{11}a_{22}a_{33} + \varepsilon_{231}a_{12}a_{23}a_{31} + \varepsilon_{312}a_{13}a_{21}a_{32} + \varepsilon_{321}a_{13}a_{22}a_{31}$$
$$+ \varepsilon_{132}a_{11}a_{23}a_{32} + \varepsilon_{213}a_{12}a_{21}a_{33}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}$$
$$- a_{12}a_{21}a_{33}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Scanned with CamScanner

1.6

Determine the components of vectors ..... in Example 1.2.

Solution:-

45^0 rotation about x_1-axis

$$\Rightarrow Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

(a)

$$b'_i = Q_{ij} b_j$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 4 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

(b)

$$b'_i = Q_{ij} b_j$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$a'_{ij} = \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 4.5 & -1.5 \\ 0 & 1.5 & -0.5 \end{bmatrix}$$

(c)

$$b'_i = Q_{ij} b_j$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2}/2 & 3.5 & 2.5 \\ -\sqrt{2}/2 & 1.5 & 0.5 \end{bmatrix}$$

1.7

Consider the two dimensional coordinate system

- - - - - - - - in the rotated polar coordinates

Solution:-

$$Q_{ij} = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) \\ \cos(x_2', x_1) & \cos(x_2', x_2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \cos(90^\circ - \theta) \\ \cos(90^\circ + \theta) & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$b_i' = Q_{ij} b_j$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1\cos\theta + b_2\sin\theta \\ -b_2\sin\theta + b_2\cos\theta \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$= \begin{bmatrix} a_{11}\cos^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\sin^2\theta & a_{12}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{21}\sin^2\theta \\ a_{21}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{12}\sin^2\theta & a_{11}\sin^2\theta - (a_{21}+a_{12})\sin\theta\cos\theta + a_{22}\cos^2\theta \end{bmatrix}$$

1.8

show that the second order tensor .... isotropic second-order tensor

Solution:-

$$a\, \delta_{ij}' = Q_{ip} Q_{jq}\, a\, \delta_{pq}$$

$$= a\, Q_{ip} Q_{jq}$$

$$= a\, \delta_{ij}$$

1.9

The most general form of a 4th ------ the general transformation given by. (1.5.1)5

$= \alpha' \delta'_{ij} \delta'_{kl} + \beta' \delta'_{ik} \delta'_{jl} + \gamma' \delta'_{il} \delta'_{jk}$

$= Q_{im} Q_{jn} Q_{kp} Q_{lq} (\alpha \delta_{mn} \delta_{pq} + \beta \delta_{mp} \delta_{nq} + \gamma \delta_{mq} \delta_{np})$

$= \alpha Q_{im} Q_{jm} Q_{kp} Q_{lp} + \beta Q_{im} Q_{jn} Q_{km} Q_{ln} + \gamma Q_{im} Q_{jn} Q_{kn} Q_{lm}$

$= \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$

1.10

For the fourth order isotropic tensor ----- the following symmetry $C_{ijkl} = C_{klij}$

$C_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$

$= \alpha \delta_{ij} \delta_{kl} + \beta (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$

$= \alpha \delta_{kl} \delta_{ij} + \beta (\delta_{ki} \delta_{lj} + \delta_{kj} \delta_{li})$

$= C_{klij}$

1.11

Show that the fundamental ---- by relations (1.6.5)

Solution:-

If $a = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$I_a = a_{ii} = \lambda_1 + \lambda_2 + \lambda_3$

$II_a = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{vmatrix}$

$= \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3$

$$III_a = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2\lambda_3$$

1·12

Determine the invariants - - - - at the appropriate diagonal form

(a) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$I_a = -1$, $II_a = -2$, $III_a = 0$

Characteristic Equation is

$$-\lambda^3 - \lambda^2 + 2\lambda = 0$$

$$\lambda(-\lambda^2 + \lambda - 2) = 0$$

$$\lambda(\lambda+2)(\lambda-1) = 0$$

Roots $\Rightarrow$ $\lambda_1 = -2$, $\lambda_2 = 0$, $\lambda_3 = 1$

$\lambda_1 = -2$ Case:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$$n_1 + n_2 = 0$$

$$n_3 = 0$$

$$n_1 = -n_2$$

$$n_1 = \pm \frac{\sqrt{2}}{2}$$

$$n = \frac{+\sqrt{2}}{2}(-1, 1, 0)$$

$\lambda_2 = 0$ case

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$$-n_1 + n_2 = 0$$
$$n_3 = 0$$

$$n_1 = n_2$$
$$n_1 = n_2 = \pm\sqrt{2}/2$$
$$n = \pm\sqrt{2}/2\,(1, 1, 0)$$

$\lambda_3 = 1$ case

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$$-2n_1 + n_2 = 0$$
$$n_1 - 2n_2 = 0$$
$$n_1 = n_2 = 0$$
$$n_3 = 1$$

$$n = \pm(0, 0, 1)$$

The rotation matrix is given by

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2}/2 \end{bmatrix}$$

and

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq}.$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2}/2 \end{bmatrix}^T$$

$$a_{ij} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$a_{ij} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$I_a = -4$, $II_a = 3$, $III_a = 0$

characteristic equation is

$$-\lambda^3 - 4\lambda^2 - 3\lambda = 0$$

$$\lambda(\lambda^2 + 4\lambda + 3) = 0$$

$$\lambda(\lambda+3)(\lambda+1) = 0$$

Roots $\Rightarrow \lambda_1 = -3$, $\lambda_2 = -1$, $\lambda_3 = 0$

$\lambda_1 = -3$ case

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$$n_1 + n_2 = 0$$
$$n_3 = 0$$

$$n_1 = -n_2$$
$$n_2 = \pm\sqrt{2}/2$$

$$n = \pm \frac{\sqrt{2}}{2}(-1, 1, 0)$$

$\lambda_2 = -1$ Case

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$-n_1 + n_2 = 0$

$n_3 = 0$

$n_1 = n_2$

$n_1 = \pm \frac{\sqrt{2}}{2}$

$n = \pm \frac{\sqrt{2}}{2}(1, 1, 0)$

$\lambda_3 = 0$ case

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$-2n_1 + n_2 = 0$

$n_1 - 2n_2 = 0$

$n_1 = 0, n_2 = 0 \quad n_3 = 1$

$n = \pm(0, 0, 1)$

The rotation matrix is given by

$$Q_{ij} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$ and

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$I_a = -2$, $II_a = 0$, $III_a = 0$

Characteristic equation is

$-\lambda^3 - 2\lambda^2 = 0$

$\lambda^2(\lambda + 2) = 0$

Roots $\Rightarrow \lambda_1 = -2, \lambda_2 = \lambda_3 = 0$

$\lambda_1 = -2$ case

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$n_1 + n_2 = 0$

$n_3 = 0$

$n_1 = -n_2$

$n_1 = \pm\sqrt{2}/2$

$n = \pm\sqrt{2}/2(-1, +1, 0)$

$\lambda_2 = \lambda_3 = 0$ case

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$-n_1 + n_2 = 0 \quad \Rightarrow n_1 = n_2$

$n_1^2 + n_2^2 + n_3^2 = 0$

$n_3^2 = (1 - 2n_1^2)$

$n = \pm(K, K, \sqrt{1-2K^2})$

For arbitrary k, and thus directions are not uniquely determined. For convenience we may choose $k = \sqrt{2}/2$ and 0 to get

$n = \pm\sqrt{2}/2(1, 1, 0)$ and $n = \pm(0, 0, 1)$

The rotation matrix is given by.

$$Q_{rj} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \text{ and.}$$

$$a_{rj} = Q_{ip} Q_{jq} a_{pq}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1.14

Calculate the quantities $\nabla \cdot u$, $\nabla \times u$, $\nabla^2 u$, $tr(\nabla u)$ for the following cartesian vector fields.

(a) $u = x_1 e_1 + x_1 x_2 e_2 + 2x_1 x_2 x_3 e_3$

$$\nabla \cdot u = u_{1,1} + u_{2,2} + u_{3,3}$$

$$= 1 + x_1 + 2x_1 x_2$$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_2 x_2 & 2x_1 x_2 x_3 \end{vmatrix}$$

$$= 2x_1 x_3 e_1 - 2x_2 x_3 e_2 + x_2 e_3$$

$$\nabla^2 u = 0e_1 + 0e_2 + 0e_3 = 0$$

$$\nabla u = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 2x_2x_3 & 2x_1x_3 & 2x_1x_2 \end{bmatrix}$$

$$tr(\nabla u) = 1 + x_1 + 2x_1x_2$$

(b) $u = x_1^2 e_1 + 2x_1 x_2 e_2 + x_3^3 e_3$

$$\nabla \cdot u = u_{1,1} + u_{2,2} + u_{3,3}$$
$$= 2x_1 + 2x_1 + 3x_3^2$$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1^2 & 2x_1x_2 & x_3^2 \end{vmatrix}$$

$$\nabla \times u = 0 x_1 - 0 e_2 + 2x_2 e_3$$

$$\nabla^2 u = 2e_1 + 0e_2 + 6x_3 e_3 = 0$$

$$\nabla u = \begin{bmatrix} 2x_1 & 0 & 0 \\ 2x_2 & 2x_2 & 0 \\ 0 & 0 & 3x_3^2 \end{bmatrix}$$

$$tr(\nabla u) = 4x_1 + 3x_3^2$$

(c) $u = x_2^2 e_1 + 2x_2 x_3 e_2 + 4x_1^2 e_3$

$$\nabla \cdot u = u_{1,1} + u_{2,2} + u_{3,2}$$
$$= 0 + 2x_3 + 0$$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2^2 & 2x_2x_3 & 4x_1^2 \end{vmatrix}$$

$$= 2x_2 e_1 - 8x_1 e_2 - 2x_2 e_3$$

$\nabla^2 u = 2e_1 + 0e_2 + 8e_3 = 0$

$$\nabla u = \begin{bmatrix} 0 & 2x_2 & 0 \\ 0 & 2x_3 & 2x_2 \\ 8x_1 & 0 & 0 \end{bmatrix}$$

$tr(\nabla u) = 3x_3$

1.15

The dual vector - - - - - to get $a_{jk} = -\varepsilon_{ijk} a_i$

$$a_i = -\frac{1}{2} \varepsilon_{ijk} a_{jk}$$

$$\varepsilon_{imn} a_i = -\frac{1}{2} \varepsilon_{ijk} \varepsilon_{imn} a_{jk}$$

$$= -\frac{1}{2} \begin{vmatrix} \delta_{ii} & \delta_{im} & \delta_{in} \\ \delta_{ji} & \delta_{jm} & \delta_{jn} \\ \delta_{ki} & \delta_{km} & \delta_{kn} \end{vmatrix} a_{jk}$$

$$= -\frac{1}{2} (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) a_{jk}$$

$$= -\frac{1}{2} (a_{mn} - a_{nm})$$

$$= -\frac{1}{2} (a_{mn} + a_{mn})$$

$$= -a_{mn}$$

$$\therefore a_{jk} = -\varepsilon_{ijk} a_i$$

1.16

Using index notation explicitly verify the vector identities

(a)

$$\nabla(\phi\psi) = (\phi\psi)_{,k} = \phi\psi_{,k} + \phi_{,k}\psi = \nabla\phi\psi + \phi\nabla\psi$$

$$\nabla^2(\phi\psi) = (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k} = \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k} + \phi_{,kk}\psi$$

$$= \phi_{,kk}\psi + \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k}$$

$$= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi\cdot\nabla\psi$$

$\nabla\cdot(\phi u) = (\phi u_k)_{,k} = \phi u_{k,k} + \phi_{,k}u_k = \nabla\phi\cdot u + \phi(\nabla\cdot u)$

(b)

$\nabla\times(\phi u) = \varepsilon_{ijk}(\phi u_k)_{,j} = \varepsilon_{ijk}(\phi u_{k,j} + \phi_{,j}u_k) = \varepsilon_{ijk}\phi_{,j}u_k + \phi\varepsilon_{ijk}u_{k,j}$
$= \nabla\phi\times u + \phi(\nabla\times u)$

$\nabla\cdot(u\times v) = (\varepsilon_{ijk}u_j v_k)_{,i} = \varepsilon_{ijk}(u_j v_{k,i} + u_{j,i}v_k) = v_k\varepsilon_{ijk}u_{j,i} + u_j\varepsilon_{ijk}v_{k,i}$
$= v\cdot(\nabla\times u) - u\cdot(\nabla\times u)$

$\nabla\times\nabla\phi = \varepsilon_{ijk}(\phi_{,k})_{,j} = \varepsilon_{ijk}\phi_{,kj} = 0$ because of symmetry and antisymmetry in jk

$\nabla\cdot\nabla\phi = (\phi_{,k})_{,k} = \phi_{,kk} = \nabla^2\phi$

(c) $\nabla\cdot(\nabla\times u) = (\varepsilon_{ijk}u_{k,j})_{,i} = \varepsilon_{ijk}u_{k,ji} = 0$ because of symmetry and antisymmetry in ij

$\nabla\times(\nabla\times u) = \varepsilon_{mni}(\varepsilon_{ijk}u_{k,j})_{,n} = \varepsilon_{imn}\varepsilon_{ijk}u_{k,jn}$
$= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})u_{k,jn} = u_{n,nm} - u_{m,nn}$
$= \nabla(\nabla\cdot u) - \nabla^2 u$

$u\times(\nabla\times u) = \varepsilon_{ijk}u_j(\varepsilon_{kmn}u_{n,m}) = \varepsilon_{kij}\varepsilon_{kmn}u_j u_{n,m}$
$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})u_j u_{n,m} = u_n u_{n,i} - u_m u_{i,m}$
$= \frac{1}{2}\nabla(u\cdot u) - u\cdot\nabla u$

1.17

Extend the results found in - - - - - cylindrical coordinate system

Solution:-

cylindrical coordinates: $\xi^1 = r,\ \xi^2 = \theta,\ \xi^3 = z$

$(ds)^2 = (dr)^2 + (r\,d\theta)^2 + (dz)^2 \Rightarrow h_1 = 1,\ h_2 = r,\ h_3 = 1$

$\hat{e}_r = \cos\theta\, e_1 + \sin\theta\, e_2$, $\hat{e}_\theta = -\sin\theta\, e_1 + \cos\theta\, e_2$, $\hat{e}_z = e_3$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta, \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r, \frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_\theta}{\partial r} = \frac{\partial \hat{e}_r}{\partial z} = \frac{\partial \hat{e}_z}{\partial \theta} = \frac{\partial \hat{e}_z}{\partial z} = 0$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\nabla f = \hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_z \frac{\partial f}{\partial z}$$

$$\nabla \cdot u = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \times u = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)\hat{e}_z$$

1.18

For the spherical coordinate system ....

.... $h_1 = 1, h_2 = R, h_3 = R\sin\phi$

Solution:-

Spherical coordinates: $\xi^1 = R$, $\xi^2 = \phi$, $\xi^3 = \theta$

$x^1 = \xi^1 \sin\xi^2 \cos\xi^3, x^2 = \xi^1 \sin\xi^2 \sin\xi^3, x^3 = \xi^1 \cos\xi^2$

Scale factors.

$$(h_1)^2 = \frac{\partial x^k}{\partial \xi^1}\frac{\partial x^k}{\partial \xi^1} = (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi = 1 \Rightarrow h_1 = 1$$

$$(h_2)^2 = \frac{\partial x^k}{\partial \xi^2}\frac{\partial x^k}{\partial \xi^2} = R^2 \Rightarrow h_2 = R.$$

$$(h_3)^2 = \frac{\partial x^k}{\partial \xi^3}\frac{\partial x^k}{\partial \xi^3} = R^2\sin^2\phi \Rightarrow h_3 = R\sin\phi$$

Unit vectors:

$\hat{e}_R = \cos\theta\sin\phi\, e_1 + \sin\theta\sin\phi\, e_2 + \cos\phi\, e_3$

$\hat{e}_\phi = \cos\theta\cos\phi\, e_1 + \sin\theta\cos\phi\, e_2 - \sin\phi\, e_3$

$\hat{e}_\theta = -\sin\theta\, e_1 + \cos\theta\, e_2$

$$\frac{\partial \hat{e}_R}{\partial R} = 0, \quad \frac{\partial \hat{e}_R}{\partial \phi} = \hat{e}_\phi, \quad \frac{\partial \hat{e}_R}{\partial \theta} = \sin\phi\, \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\phi}{\partial R} = 0, \quad \frac{\partial \hat{e}_\phi}{\partial \phi} = -\hat{e}_R, \quad \frac{\partial \hat{e}_\phi}{\partial \theta} = \cos\phi\, \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\theta}{\partial R} = 0, \quad \frac{\partial \hat{e}_\theta}{\partial \phi} = 0, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos\phi\, \hat{e}_\phi$$

$$\nabla = \hat{e}_R \frac{\partial}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial}{\partial \phi} + \hat{e}_\theta \frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}$$

$$\nabla f = \hat{e}_R \frac{\partial f}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial f}{\partial \phi} + \hat{e}_z \frac{1}{R\sin\phi}\frac{\partial f}{\partial \theta}$$

$$\nabla\cdot u = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}\left(R^2\sin\phi\, u_R\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(R\sin\phi\, u_\phi\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \theta}\left(R u_\theta\right)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 u_R\right) + \frac{1}{R\sin\phi}\left(\sin\phi\, u_\phi\right) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}\left(u_\theta\right)$$

$$\nabla^2 f = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}\left(R^2\sin\phi\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial \theta}\right)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$$

$$\nabla\times u = \left(\frac{1}{R^2\sin\phi}\left[\frac{\partial}{\partial \phi}\left(R\sin\phi\, u_\theta\right) - \frac{\partial}{\partial \theta}\left(R u_\phi\right)\right]\right)\hat{e}_R + \left(\frac{1}{R\sin\phi}\left[\frac{\partial}{\partial \theta}(u_R) - \frac{\partial}{\partial R}\left(R\sin\phi\, u_\theta\right)\right]\right)\hat{e}_\phi + \left(\frac{1}{R}\frac{\partial}{\partial R}\left[(R u_\phi) - \frac{\partial}{\partial \phi}(u_R)\right]\right)\hat{e}_\theta$$

$$= \left[\frac{1}{R^2\sin\phi}\left(\frac{\partial}{\partial \phi}\left(\sin\phi\, u_\theta\right) - \frac{\partial u_\phi}{\partial \theta}\right)\right]\hat{e}_R + \left[\frac{1}{R\sin\phi}\frac{\partial u_R}{\partial \theta} - \frac{1}{R}\left(R u_\theta\right)\right]\hat{e}_\phi$$

$$+ \left[\frac{1}{R}\left(\frac{\partial}{\partial R}\left(R u_\phi\right) - \frac{\partial u_R}{\partial \phi}\right)\right]\hat{e}_\theta$$

Verify that the alternator e_{ijk} has the property that
$$e_{ijk} = Q_{ip} Q_{jq} Q_{kr} e_{pqr}$$ for all proper orthogonal matrices $[Q]$

$$e_{ijk} \neq Q_{ip} Q_{jq} Q_{kr} e_{pqr}$$

For this the alternator is not an isotropic 3-tensor

Transform strain displacement relation from Cartesian to cylindrical and spherical coordinates.

Cylindrical coordinates:-

$$u_x = u_r \cos\theta - u_\theta \sin\theta$$
$$u_y = u_r \sin\theta + u_\theta \cos\theta$$
$$u_z = u_z$$

Derivatives of $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ where $r = \sqrt{x^2+y^2}$, $\theta = \arctan\left(\frac{y}{x}\right)$ is given by

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

It follows that

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} + \left(\cos\theta \frac{\partial}{\partial r}\right)\left(-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta \frac{\partial}{\partial r}\right)$$

$$\frac{\partial^2}{\partial x^2} = \cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} - \cos\theta \sin\theta\left[\frac{-1}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right]$$
$$+ \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial\theta\partial r}$$

$$= \cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta} - \frac{1}{r}\sin\theta\cos\theta$$
$$\frac{\partial^2}{\partial r\partial\theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial\theta\partial r} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r}$$

$$\frac{\partial^2}{\partial x^2} = \cos^2\theta\frac{\partial^2}{\partial r^2} + \sin^2\theta\left[\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right] + 2\sin\theta\cos\theta$$
$$\left[\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right]$$

Likewise

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta\frac{\partial^2}{\partial r^2} + \cos^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial r^2}{\partial\theta^2}\right) - 2\sin\theta\cos\theta$$
$$\left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right)$$

$$e_{xx} = \frac{\partial u_x}{\partial x} = \cos\theta\frac{\partial}{\partial r}(u_r\cos\theta - u_\theta\sin\theta) - \frac{\sin\theta}{r}\left(\frac{\partial}{\partial\theta}(u_r\cos\theta -\right.$$
$$\left.u_\theta\sin\theta)\right)$$

$$= \frac{\partial u_r}{\partial r}\cos^2\theta - \frac{\partial u_\theta}{\partial r}\sin\theta\cos\theta - \frac{\partial u_r}{\partial\theta}\frac{\sin\theta\cos\theta}{r} + \frac{u_r}{r}\sin^2\theta$$
$$+ \frac{\partial u_\theta}{\partial\theta}\frac{\sin^2\theta}{r} + \frac{u_\theta}{r}\sin\theta\cos\theta$$

$$= \frac{\partial u_r}{\partial r}\cos^2\theta + \left(\frac{u_\theta}{r} - \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\sin\theta\cos\theta.$$
$$+ \left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\sin^2\theta$$

$$e_{yy} = \frac{\partial u_y}{\partial y} = \sin\theta\frac{\partial}{\partial r}(u_r\sin\theta + u_\theta\cos\theta) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}(u_r\sin\theta + u_\theta\cos\theta)$$

$$e_{xy} = 2\left(\frac{\partial x}{\partial y} + \frac{\partial y}{\partial x}\right)$$

Thus

$$e_{rr} = \frac{\partial u_r}{\partial r},\quad e_{\theta\theta} = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial\theta}\right),\quad e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

and

$$e_{zz} = \frac{\partial u_z}{\partial z}$$

Suppose the basis $\{e_1', e_2', e_3'\}$ is obtained by rotating basis $\{e_1, e_2, e_3\}$ through angle θ about unit vector e_3. Write out the rule for 2-tensors explicitly

$$e_1' = \cos\theta\, e_1 + \sin\theta\, e_2$$
$$e_2' = -\sin\theta\, e_1 + \cos\theta\, e_2$$
$$e_3' = e_3$$

$$[Q] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A' = [Q][A][Q^T]$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11}\cos\theta + A_{12}\sin\theta & -A_{11}\sin\theta + A_{12}\cos\theta & A_{13} \\ A_{21}\cos\theta + A_{22}\sin\theta & -A_{21}\sin\theta + A_{22}\cos\theta & A_{23} \\ A_{31}\cos\theta + A_{32}\sin\theta & -A_{31}\sin\theta + A_{32}\cos\theta & A_{33} \end{bmatrix}$$

$$A' = \begin{bmatrix} A_{11}\cos^2\theta + A_{22}\sin^2\theta + (A_{12}+A_{21})\sin\theta\cos\theta & A_{12}\cos^2\theta - A_{21}\sin^2\theta + (A_{22}-A_{11})\cos\theta\sin\theta & A_{13}\cos\theta + A_{23}\sin\theta \\ A_{21}\cos^2\theta - A_{12}\sin^2\theta + (A_{22}-A_{11})\sin\theta\cos\theta & A_{22}\cos^2\theta + A_{11}\sin^2\theta - (A_{12}+A_{21})\cos\theta\sin\theta & A_{23}\cos\theta - A_{13}\sin\theta \\ A_{31}\cos\theta + A_{32}\sin\theta & A_{32}\cos\theta - A_{31}\sin\theta & A_{33} \end{bmatrix}$$

Using half angle identities: $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$, $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$$

$$= \begin{bmatrix} \left(\frac{A_{11}+A_{22}}{2}\right) + \left(\frac{A_{11}-A_{22}}{2}\right)\cos 2\theta + \left(\frac{A_{12}+A_{21}}{2}\right)\sin 2\theta & \left(\frac{A_{12}-A_{21}}{2}\right) + \left(\frac{A_{12}+A_{21}}{2}\right)\cos 2\theta + \left(\frac{A_{22}-A_{11}}{2}\right)\sin 2\theta & A_{13}\cos\theta + A_{23}\sin\theta \\ \left(\frac{A_{21}-A_{12}}{2}\right) + \left(\frac{A_{21}+A_{12}}{2}\right)\cos 2\theta + \left(\frac{A_{22}-A_{11}}{2}\right)\sin 2\theta & \left(\frac{A_{22}+A_{11}}{2}\right) + \left(\frac{A_{22}-A_{11}}{2}\right)\cos 2\theta - \left(\frac{A_{12}+A_{21}}{2}\right)\sin 2\theta & A_{23}\cos\theta - A_{13}\sin\theta \\ A_{31}\cos\theta + A_{32}\sin\theta & A_{32}\cos\theta - A_{31}\sin\theta & A_{33} \end{bmatrix}$$

By comparing, we get

$$A'_{11} = \frac{A_{11}+A_{22}}{2} + \frac{A_{11}-A_{22}}{2}\cos 2\theta + \frac{A_{12}+A_{21}}{2}\sin 2\theta$$

$$A'_{12} = \frac{A_{12}-A_{21}}{2} + \frac{A_{12}+A_{21}}{2}\cos 2\theta + \frac{A_{22}-A_{11}}{2}\sin 2\theta$$

$$A'_{13} = A_{13}\cos\theta + A_{23}\sin\theta$$

$$A'_{21} = \frac{A_{21}-A_{12}}{2} + \frac{A_{21}+A_{12}}{2}\cos 2\theta + \frac{A_{22}-A_{11}}{2}\sin 2\theta$$

$$A'_{22} = \frac{A_{22}+A_{11}}{2} + \frac{A_{22}-A_{11}}{2}\cos 2\theta - \frac{A_{12}+A_{21}}{2}\sin 2\theta$$

$$A'_{23} = A_{23}\cos\theta - A_{13}\sin\theta$$

$$A'_{31} = A_{31}\cos\theta + A_{32}\sin\theta$$

$$A'_{32} = A_{32}\cos\theta - A_{31}\sin\theta$$

$$A'_{33} = A_{33}$$

In the special case when [A] is symmetric in addition $A_{13} = A_{23} = 0$, so nine equations simplify to

$$A'_{11} = \frac{A_{11}+A_{22}}{2} + \frac{A_{11}-A_{22}}{2}\cos 2\theta + A_{12}\sin 2\theta$$

$$A'_{22} = \frac{A_{11}+A_{22}}{2} - \frac{A_{11}-A_{22}}{2}\cos 2\theta + A_{12}\sin 2\theta$$

$$A'_{12} = -\frac{A_{11}-A_{22}}{2}\sin 2\theta$$

together with $A'_{13} = A'_{23} = 0$ and $A'_{33} = A_{33}$

They are well known equations underlying the Mohr's circle for transforming 2-tensors in 2D.