Week No. 04
Digital Logic Design

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BOOLEAN ALGEBRA

- Boolean algebra is the mathematics of digital systems. A basic knowledge of Boolean algebra is indispensable to the study and analysis of logic circuits.
- A variable is a symbol used to represent a logical quantity. Any single variable can have a 1 or a 0 value.
- The complement is the inverse of a variable and is indicated by a bar over the variable (overbar). For example,

  If $A = 1$, then $\overline{A} = 0$. If $A = 0$, then $\overline{A} = 1$.
- A literal is a variable or the complement of a variable.

Laws of Boolean Algebra

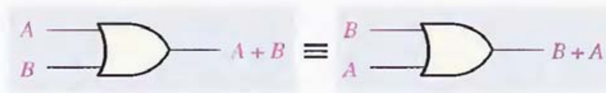
- Commutative Laws The commutative law of addition for two variables is written as

  A+B=B+A

- This law states that the order in which the variables are ORed makes no difference.

FIGURE 4–1

Application of commutative law of addition.



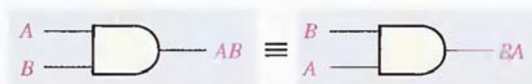
The *commutative law of multiplication* for two variables is

$$AB = BA$$

This law states that the order in which the variables are ANDed makes no difference. Figure 4–2 illustrates this law as applied to the AND gate.

FIGURE 4–2

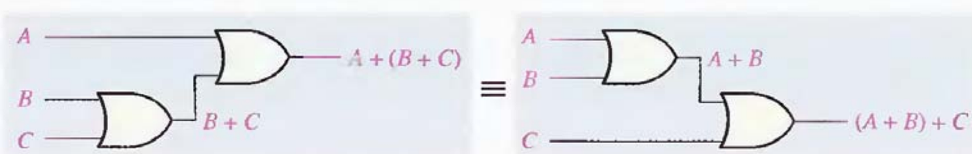
Application of commutative law of multiplication.



Associative Laws The *associative law of addition* is written as follows for three variables:

$$A + (B + C) = (A + B) + C$$

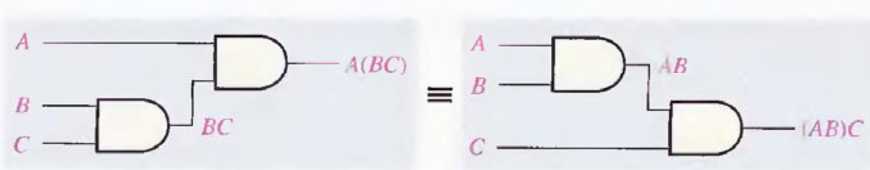
This law states that when ORing more than two variables, the result is the same regardless of the grouping of the variables. Figure 4–3 illustrates this law as applied to 2-input OR gates.



The *associative law of multiplication* is written as follows for three variables:

$$A(BC) = (AB)C$$

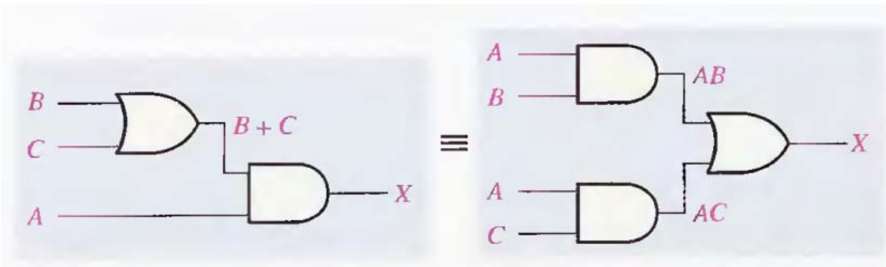
This law states that it makes no difference in what order the variables are grouped when ANDing more than two variables. Figure 4–4 illustrates this law as applied to 2-input AND gates.



Distributive Law The distributive law is written for three variables as follows:

$$A(B + C) = AB + AC$$

This law states that ORing two or more variables and then ANDing the result with a single variable is equivalent to ANDing the single variable with each of the two or more varibles and then ORing the products:



$X = A(B + C)$ $X = AB + AC$

1. $A + 0 = A$	7. $A \cdot A = A$
2. $A + 1 = 1$	8. $A \cdot \overline{A} = 0$
3. $A \cdot 0 = 0$	9. $\overline{\overline{A}} = A$
4. $A \cdot 1 = A$	10. $A + AB = A$
5. $A + A = A$	11. $A + \overline{A}B = A + B$
6. $A + \overline{A} = 1$	12. $(A + B)(A + C) = A + BC$

A, B, or *C* can represent a single variable or a combination of variables.

TABLE 4–1

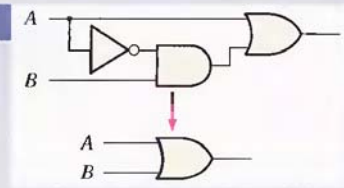
Basic rules of Boolean algebra.

Rule 11. $A + \overline{A}B = A + B$ This rule can be proved as follows:

$A + \overline{A}B = (A + AB) + \overline{A}B$	Rule 10: $A = A + AB$
$= (AA + AB) + \overline{A}B$	Rule 7: $A = AA$
$= AA + AB + A\overline{A} + \overline{A}B$	Rule 8: adding $A\overline{A} = 0$
$= (A + \overline{A})(A + B)$	Factoring
$= 1 \cdot (A + B)$	Rule 6: $A + \overline{A} = 1$
$= A + B$	Rule 4: drop the 1

A	B	$\overline{A}B$	$A + \overline{A}B$	$A + B$
0	0	0	0	0
0	1	1	1	1
1	0	0	1	1
1	1	0	1	1

equal

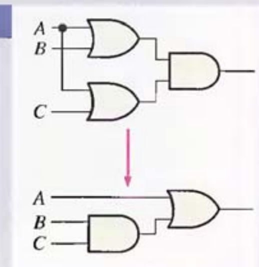


Rule 12. $(A + B)(A + C) = A + BC$ This rule can be proved as follows:

$$\begin{aligned}(A + B)(A + C) &= AA + AC + AB + BC && \text{Distributive law}\\ &= A + AC + AB + BC && \text{Rule 7: } AA = A\\ &= A(1 + C) + AB + BC && \text{Factoring (distributive law)}\\ &= A \cdot 1 + AB + BC && \text{Rule 2: } 1 + C = 1\\ &= A(1 + B) + BC && \text{Factoring (distributive law)}\\ &= A \cdot 1 + BC && \text{Rule 2: } 1 + B = 1\end{aligned}$$

A	B	C	$A + B$	$A + C$	$(A + B)(A + C)$	BC	$A + BC$
0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	1	1	1	1
1	0	0	1	1	1	0	1
1	0	1	1	1	1	0	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

equal



DEMORGAN'S THEOREMS

- DeMorgan's theorems provide mathematical verification of the equivalency of the NAND and negative-OR gates and the equivalency of the NOR and negative-AND gates.

The complement of two or more ANDed variables is equivalent to the OR of the complements of the individual variables.

The formula for expressing this theorem for two variables is

$$\overline{XY} = \overline{X} + \overline{Y} \qquad \text{Equation 4–6}$$

DeMorgan's second theorem is stated as follows:

The complement of a sum of variables is equal to the product of the complements of the variables.

Stated another way,

The complement of two or more ORed variables is equivalent to the AND of the complements of the individual variables.

The formula for expressing this theorem for two variables is

$$\overline{X + Y} = \overline{X}\,\overline{Y} \qquad \text{Equation 4–7}$$

Figure 4–15 shows the gate equivalencies and truth tables for Equations 4–6 and 4–7.

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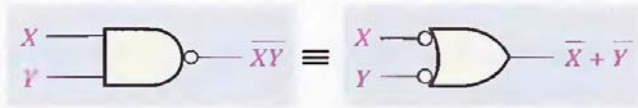
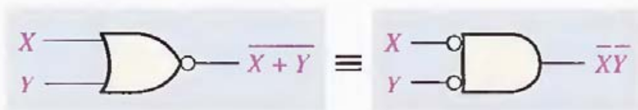


NAND Negative-OR

Inputs		Output	
X	Y	$\overline{XY}$	$\overline{X} + \overline{Y}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0



NOR Negative-AND

Inputs		Output	
X	Y	$\overline{X + Y}$	$\overline{X}\,\overline{Y}$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

EXAMPLE 4–3

Apply DeMorgan's theorems to the expressions $\overline{XYZ}$ and $\overline{X + Y + Z}$.

Solution

$$\overline{XYZ} = \overline{X} + \overline{Y} + \overline{Z}$$

$$\overline{X + Y + Z} = \overline{X}\,\overline{Y}\,\overline{Z}$$

Related Problem Apply DeMorgan's theorem to the expression $\overline{\overline{X} + \overline{Y} + \overline{Z}}$.

EXAMPLE 4–4

Apply DeMorgan's theorems to the expressions $\overline{WXYZ}$ and $\overline{W + X + Y + Z}$.

Solution

$$\overline{WXYZ} = \overline{W} + \overline{X} + \overline{Y} + \overline{Z}$$

$$\overline{W + X + Y + Z} = \overline{W}\,\overline{X}\,\overline{Y}\,\overline{Z}$$

Related Problem Apply DeMorgan's theorem to the expression $\overline{\overline{W}\,\overline{X}\,\overline{Y}\,\overline{Z}}$.

Applying DeMorgan's Theorems

The following procedure illustrates the application of DeMorgan's theorems and Boolean algebra to the specific expression

$$\overline{\overline{A + B\overline{C}} + D(\overline{E + \overline{F}})}$$

Step 1. Identify the terms to which you can apply DeMorgan's theorems, and think of each term as a single variable. Let $\overline{A + B\overline{C}} = X$ and $D(\overline{E + \overline{F}}) = Y$.

Step 2. Since $\overline{X + Y} = \overline{X}\,\overline{Y}$,

$$\overline{(\overline{A + B\overline{C}}) + (D(\overline{E + \overline{F}}))} = (\overline{\overline{A + B\overline{C}}})(\overline{D(\overline{E + \overline{F}})})$$

Step 3. Use rule 9 ($\overline{\overline{A}} = A$) to cancel the double bars over the left term (this is not part of DeMorgan's theorem).

$$(\overline{\overline{A + B\overline{C}}})(\overline{D(\overline{E + \overline{F}})}) = (A + B\overline{C})(\overline{D(\overline{E + \overline{F}})})$$

Step 4. Applying DeMorgan's theorem to the second term,

$$(A + B\overline{C})(\overline{D(\overline{E + \overline{F}})}) = (A + B\overline{C})(\overline{D} + (\overline{\overline{E + \overline{F}}}))$$

Step 5. Use rule 9 ($\overline{\overline{A}} = A$) to cancel the double bars over the $E + \overline{F}$ part of the term.

$$(A + B\overline{C})(\overline{D} + \overline{\overline{E + \overline{F}}}) = (A + B\overline{C})(\overline{D} + E + \overline{F})$$

The following three examples will further illustrate how to use DeMorgan's theorems.

EXAMPLE 4–5

Apply DeMorgan's theorems to each of the following expressions:

(a) $\overline{(A + B + C)D}$ **(b)** $\overline{ABC + DEF}$ **(c)** $\overline{A\overline{B} + \overline{C}D + EF}$

Solution

(a) Let $A + B + C = X$ and $D = Y$. The expression $\overline{(A + B + C)D}$ is of the form $\overline{XY} = \overline{X} + \overline{Y}$ and can be rewritten as

$$\overline{(A + B + C)D} = \overline{A + B + C} + \overline{D}$$

Next, apply DeMorgan's theorem to the term $\overline{A + B + C}$.

$$\overline{A + B + C} + \overline{D} = \overline{A}\,\overline{B}\,\overline{C} + \overline{D}$$

(b) Let $ABC = X$ and $DEF = Y$. The expression $\overline{ABC + DEF}$ is of the form $\overline{X + Y} = \overline{X}\,\overline{Y}$ and can be rewritten as

$$\overline{ABC + DEF} = (\overline{ABC})(\overline{DEF})$$

Next, apply DeMorgan's theorem to each of the terms $\overline{ABC}$ and $\overline{DEF}$.

$$(\overline{ABC})(\overline{DEF}) = (\overline{A} + \overline{B} + \overline{C})(\overline{D} + \overline{E} + \overline{F})$$

(c) Let $A\overline{B} = X$, $\overline{C}D = Y$, and $EF = Z$. The expression $\overline{A\overline{B} + \overline{C}D + EF}$ is of the form $\overline{X + Y + Z} = \overline{X}\,\overline{Y}\,\overline{Z}$ and can be rewritten as

$$\overline{A\overline{B} + \overline{C}D + EF} = (\overline{A\overline{B}})(\overline{\overline{C}D})(\overline{EF})$$

Next, apply DeMorgan's theorem to each of the terms $\overline{A\overline{B}}$, $\overline{\overline{C}D}$, and $\overline{EF}$.

$$(\overline{A\overline{B}})(\overline{\overline{C}D})(\overline{EF}) = (\overline{A} + B)(C + \overline{D})(\overline{E} + \overline{F})$$

Related Problem Apply DeMorgan's theorems to the expression $\overline{\overline{ABC} + D + E}$.

EXAMPLE 4–6

Apply DeMorgan's theorems to each expression:

(a) $\overline{\overline{(A + B)} + \overline{C}}$ **(b)** $\overline{(\overline{A} + B) + CD}$ **(c)** $\overline{(A + B)\overline{C}\,\overline{D} + E + \overline{F}}$

Solution

(a) $\overline{\overline{(A + B)} + \overline{C}} = (\overline{\overline{A + B}})\overline{\overline{C}} = (A + B)C$

(b) $\overline{(\overline{A} + B) + CD} = (\overline{\overline{A} + B})\overline{CD} = (\overline{\overline{A}}\,\overline{B})(\overline{C} + \overline{D}) = A\overline{B}(\overline{C} + \overline{D})$

(c) $\overline{(A + B)\overline{C}\,\overline{D} + E + \overline{F}} = (\overline{(A + B)\overline{C}\,\overline{D}})(\overline{E + \overline{F}}) = (\overline{A}\,\overline{B} + C + D)\overline{E}F$

Related Problem Apply DeMorgan's theorems to the expression $\overline{\overline{A}B(C + \overline{D}) + E}$.

EXAMPLE 4–7

The Boolean expression for an exclusive-OR gate is $A\overline{B} + \overline{A}B$. With this as a starting point, use DeMorgan's theorems and any other rules or laws that are applicable to develop an expression for the exclusive-NOR gate.

Solution Start by complementing the exclusive-OR expression and then applying DeMorgan's theorems as follows:

$$\overline{A\overline{B} + \overline{A}B} = (\overline{A\overline{B}})(\overline{\overline{A}B}) = (\overline{A} + \overline{\overline{B}})(\overline{\overline{A}} + \overline{B}) = (\overline{A} + B)(A + \overline{B})$$

Next, apply the distributive law and rule 8 ($A \cdot \overline{A} = 0$).

$$(\overline{A} + B)(A + \overline{B}) = \overline{A}A + \overline{A}\,\overline{B} + AB + B\overline{B} = \overline{A}\,\overline{B} + AB$$

The final expression for the XNOR is $\overline{A}\,\overline{B} + AB$. Note that this expression equals 1 any time both variables are 0s or both variables are 1s.

Related Problem Starting with the expression for a 4-input NAND gate, use DeMorgan's theorems to develop an expression for a 4-input negative-OR gate.

TABLE 4–5

Truth table for the logic circuit in Figure 4–16.

INPUTS				OUTPUT
A	B	C	D	$A(B + CD)$
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

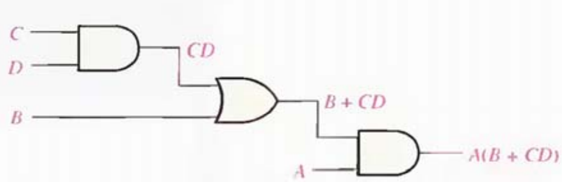


FIGURE 4–16

A logic circuit showing the development of the Boolean expression for the output.

Using Boolean algebra techniques, simplify this expression:

$$AB + A(B + C) + B(B + C)$$

Solution The following is not necessarily the only approach.

Step 1: Apply the distributive law to the second and third terms in the expression, as follows:

$$AB + AB + AC + BB + BC$$

Step 2: Apply rule 7 ($BB = B$) to the fourth term.

$$AB + AB + AC + B + BC$$

Step 3: Apply rule 5 ($AB + AB = AB$) to the first two terms.

$$AB + AC + B + BC$$

Step 4: Apply rule 10 ($B + BC = B$) to the last two terms.

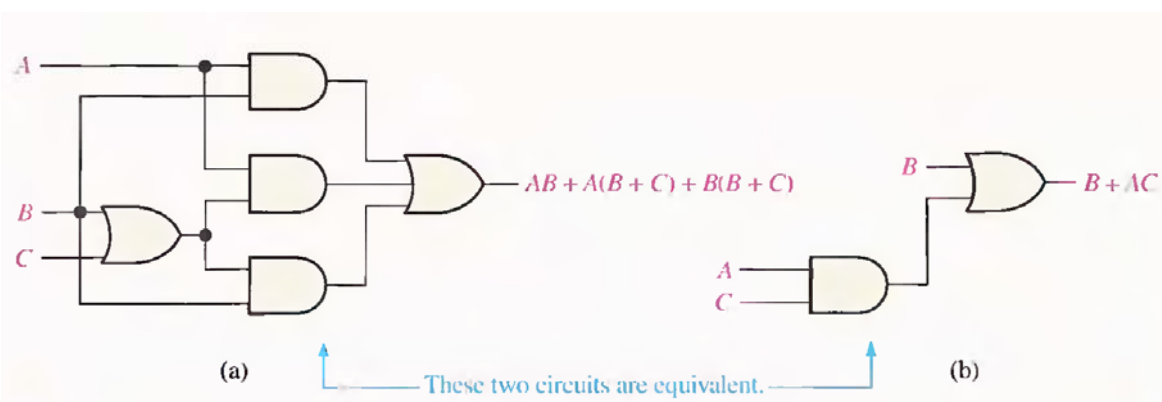
$$AB + AC + B$$

Step 5: Apply rule 10 ($AB + B = B$) to the first and third terms.

$$B + AC$$

At this point the expression is simplified as much as possible. Once you gain experience in applying Boolean algebra, you can often combine many individual steps.

Related Problem Simplify the Boolean expression $A\overline{B} + A(\overline{B + C}) + B(\overline{B + C})$.



E 4–7

Simplify the following Boolean expression:

$$[A\overline{B}(C + BD) + \overline{A}\,\overline{B}]C$$

Note that brackets and parentheses mean the same thing: the term inside is multiplied (ANDed) with the term outside.

Solution **Step 1:** Apply the distributive law to the terms within the brackets.

$$(A\overline{B}C + A\overline{B}BD + \overline{A}\,\overline{B})C$$

Step 2: Apply rule 8 ($\overline{B}B = 0$) to the second term within the parentheses.

$$(A\overline{B}C + A\cdot 0\cdot D + \overline{A}\,\overline{B})C$$

Step 3: Apply rule 3 ($A \cdot 0 \cdot D = 0$) to the second term within the parentheses.

$$(A\overline{B}C + 0 + \overline{A}\,\overline{B})C$$

Step 4: Apply rule 1 (drop the 0) within the parentheses.

$$(A\overline{B}C + \overline{A}\,\overline{B})C$$

Step 5: Apply the distributive law.

$$A\overline{B}CC + \overline{A}\,\overline{B}C$$

Step 6: Apply rule 7 ($CC = C$) to the first term.

$$A\overline{B}C + \overline{A}\,\overline{B}C$$

Step 7: Factor out $\overline{B}C$.

$$\overline{B}C(A + \overline{A})$$

Step 8: Apply rule 6 ($A + \overline{A} = 1$).

$$\overline{B}C\cdot 1$$

Step 9: Apply rule 4 (drop the 1).

$$\overline{B}C$$

Related Problem Simplify the Boolean expression $[AB(C + \overline{BD}) + \overline{AB}]CD$.

Simplify the following Boolean expression:

$$\overline{A}BC + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C + ABC$$

Solution **Step 1:** Factor BC out of the first and last terms.

$$BC(\overline{A} + A) + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C$$

Step 2: Apply rule 6 ($\overline{A} + A = 1$) to the term in parentheses, and factor $A\overline{B}$ from the second and last terms.

$$BC\cdot 1 + A\overline{B}(\overline{C} + C) + \overline{A}\,\overline{B}\,\overline{C}$$

Step 3: Apply rule 4 (drop the 1) to the first term and rule 6 ($C + C = 1$) to the term in parentheses.

$$BC + A\overline{B}\cdot 1 + \overline{A}\,\overline{B}\,\overline{C}$$

Step 4: Apply rule 4 (drop the 1) to the second term.

$$BC + A\overline{B} + \overline{A}\,\overline{B}\,\overline{C}$$

Step 5: Factor $\overline{B}$ from the second and third terms.

$$BC + \overline{B}(A + \overline{A}\,\overline{C})$$

Step 6: Apply rule 11 ($A + \overline{A}\,\overline{C} = A + \overline{C}$) to the term in parentheses.

$$BC + \overline{B}(A + \overline{C})$$

Step 7: Use the distributive and commutative laws to get the following expression:

$$BC + A\overline{B} + \overline{B}\,\overline{C}$$

Simplify the Boolean expression $AB\overline{C} + \overline{A}\,\overline{B}C + \overline{A}BC + \overline{A}\,\overline{B}\,\overline{C}$.

Simplify the following Boolean expression:

$$\overline{AB + AC} + \overline{A}\,\overline{B}C$$

Solution **Step 1:** Apply DeMorgan's theorem to the first term.

$$(\overline{AB})(\overline{AC}) + \overline{A}\,\overline{B}C$$

Step 2: Apply DeMorgan's theorem to each term in parentheses.

$$(\overline{A} + \overline{B})(\overline{A} + \overline{C}) + \overline{A}\,\overline{B}C$$

Step 3: Apply the distributive law to the two terms in parentheses.

$$\overline{A}\,\overline{A} + \overline{A}\,\overline{C} + \overline{A}\,\overline{B} + \overline{B}\,\overline{C} + \overline{A}\,\overline{B}C$$

Step 4: Apply rule 7 ($\overline{A}\,\overline{A} = \overline{A}$) to the first term, and apply rule 10 $[\overline{A}\,\overline{B} + \overline{A}\,\overline{B}C = \overline{A}\,\overline{B}(1 + C) = \overline{A}\,\overline{B}]$ to the third and last terms.

$$\overline{A} + \overline{A}\,\overline{C} + \overline{A}\,\overline{B} + \overline{B}\,\overline{C}$$

Step 5: Apply rule 10 $[\overline{A} + \overline{A}\,\overline{C} = \overline{A}(1 + \overline{C}) = \overline{A}]$ to the first and second terms.

$$\overline{A} + \overline{A}\,\overline{B} + \overline{B}\,\overline{C}$$

Step 6: Apply rule 10 $[\overline{A} + \overline{A}\,\overline{B} = \overline{A}(1 + \overline{B}) = \overline{A}]$ to the first and second terms.

$$\overline{A} + \overline{B}\,\overline{C}$$

Related Problem Simplify the Boolean expression $\overline{AB} + \overline{AC} + \overline{A}\,\overline{B}\,\overline{C}$.

STANDARD FORMS OF BOOLEAN EXPRESSIONS

- All Boolean expressions can be converted into either of two standard forms: the sum-of-products form or the product-of-sums form.

- Standardization makes the evaluation, simplification, and implementation of Boolean expressions much more systematic and easier.

The Sum-of-Products (SOp) Form:

When two or more product terms are summed by Boolean addition. the resulting expression is a sum-of-products (SOP). Some examples are

$$AB + ABC$$
$$ABC + CDE + \overline{B}C\overline{D}$$
$$\overline{A}B + \overline{A}B\overline{C} + AC$$

AND/OR Implementation of an SOP Expression

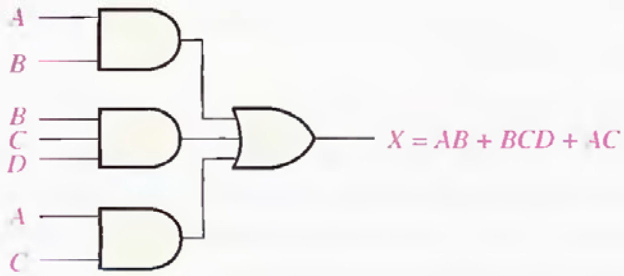


FIGURE 4–18

Implementation of the SOP expression $AB + BCD + AC$.

NAND/NAND Implementation of an SOP Expression

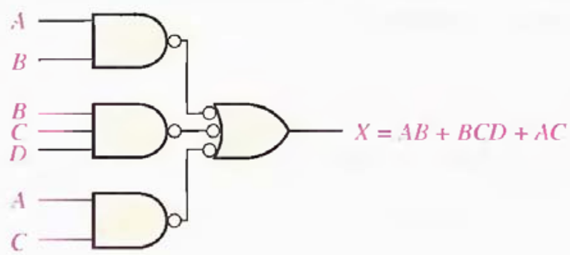


FIGURE 4–19

This NAND/NAND implementation is equivalent to the AND/OR in Figure 4–18.

Conversion of a General Expression to SOP Form

Any logic expression can be changed into SOP form by applying Boolean algebra techniques. For example, the expression $A(B + CD)$ can be converted to SOP form by applying the distributive law:

$$A(B + CD) = AB + ACD$$

EXAMPLE 4–12

Convert each of the following Boolean expressions to SOP form:

(a) $AB + B(CD + EF)$ **(b)** $(A + B)(B + C + D)$ **(c)** $\overline{(\overline{A + B}) + C}$

Solution

(a) $AB + B(CD + EF) = AB + BCD + BEF$

(b) $(A + B)(B + C + D) = AB + AC + AD + BB + BC + BD$

(c) $\overline{(\overline{A + B}) + C} = (\overline{\overline{A + B}})\overline{C} = (A + B)\overline{C} = A\overline{C} + B\overline{C}$

Related Problem Convert $\overline{A}B\overline{C} + (A + \overline{B})(B + \overline{C} + A\overline{B})$ to SOP form.