Agenda

- Ch1.3 Predicates and Quantifiers
  - Predicates
  - Quantifiers
  - Quantifiers with Restricted Domains
  - Precedence of Quantifiers
  - Logical Equivalences Involving Quantifiers
  - Translation
- Ch1.4 Nested Quantifiers
  - Nested Quantifiers

Limitation of Propositional Logic

- Limitation 1:

p : John is a SCUT student

q : Peter is a SCUT student

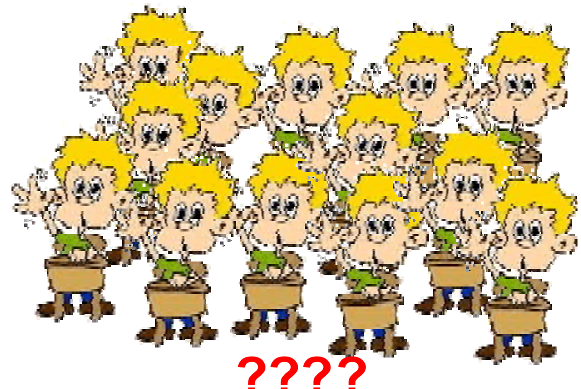
r : Mary is a SCUT student

- Try to represent them using propositional variable
  - However, these propositions are very similar
  - A more powerful type of logic named **Predicate Logic** will be introduced

Predicates

- **Predicate logic** is an extension of propositional logic that permits concisely reasoning about whole classes of entities

John Peter Mary



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- **Propositional Logic** treats simple propositions as atomic entities
- **Predicate Logic** distinguishes the subject of a sentence from its predicate

Predicates

- **Predicate** is a function of proposition
- Example:

Convention:
- lowercase variables denote objects
- UPPERCASE variables denote predicates

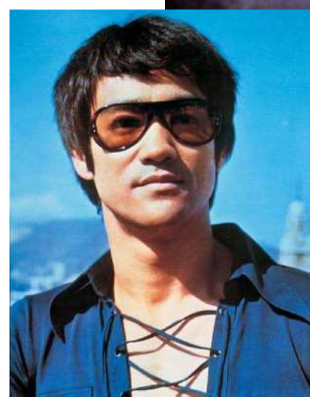
$$\overbrace{P(x)}^{\textbf{Propositional Function / Predicate}} : \quad \overbrace{x}^{\textbf{Variable}} \; \overbrace{\text{is greater than 3}}^{\textbf{Predicate}}$$

Refer to

- The truth value of proposition function can only be determined when the values of variables are known

Predicates

- Example:
  - $P(x)$: "$x > 3$"
  - What is **$P(4)$**? ✓
  - What is **$P(2)$**? ✗
  - $P(x)$: "x is a singer"
  - ***P*(Michael Jackson)**? ✓
  - ***P*(Bruce Lee)**? ✗

Predicates

- Propositional function can have more than one variables
- Example:
  - P(x, y): x + y = 7
    - P(2, 5) ✓
  - Q(x, y, z): x = y + z
    - Q(5, 2, 8) ✗

Predicates

- **General case**
  - A statement involving the n variables $x_1, x_2, \ldots, x_n$ can be denoted by

$$P(x_1, x_2, \ldots, x_n)$$

  - A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at the **n-tuple ($x_1, x_2, \ldots, x_n$)**
  - P is also called a ***n*-place predicate** or a ***n*-ary predicate**

Limitation of Propositional Logic

- Limitation 2:
  - Given

    P: "Every student in SCUT is clever"

    Q: "Peter is SCUT student"
  - What can we conclude?

    "Peter is clever"
  - Given

    P: "Peter cannot pass this Discrete Maths subject"

    Q: "Peter is a SCUT student"
  - What can we conclude?

    "At least one student in SCUT cannot pass this Discrete Maths subject"
  - **No rules of propositional logic** can conclude the truth of this statement

Limitation of Propositional Logic

- **Propositional Logic** does not adequately express the following meanings
  - Every, all, some, partial, at least one, one, etc

- A more powerful tool, **Quantifiers**, will be introduced

Quantifiers

- **Quantification** expresses the extent to which a predicate is true over a range of elements
- For example
  - Using Propositional Logic
    - p: Peter has iPhone
    - q: Paul has iPhone
    - r: Mary has iPhone
  - Using Predicate
    - P(x) : x has iPhone
    - P(Peter)
    - P(Paul)
    - P(Mary)

Assume our class only contains three students

- **Using Quantifier**
  - P(x) : x has iPhone
  - For all x, P(x) is true
  - Domain consists of all student in this class

Quantifiers

- Four aspects should be mentioned in Quantification
    1. Quantifier
       (e.g. all, some…)
    2. Variable
    3. Predicate
    4. Domain

P(x) : x has iPhone
For all x, P(x) is true
Domain consists of all student in this class

- The area of logic that deals with predicates and quantifiers is called the **Predicate Calculus**

Quantifiers

- **Universes of Discourse (U.D.s)**
    - Also called the **domain of discourse**
    - Refers to the collection of objects being discussed in a specific discourse
    - Example:
        - *P*(*x*) : "*x* breaths oxygen"
        - Domain consists of **humans**
          P(*x*) is true for all x?
        - Domain consists of **creatures**
          P(*x*) is true for all x?

Quantifiers

- Three types of quantification will be focused:
  - **Universal Quantification**
    - i.e. all, none
  - **Existential Quantification**
    - i.e. some, few, many
  - **Unique Quantification**
    - i.e. exactly one

Quantifiers

Universal Quantifiers (ALL)

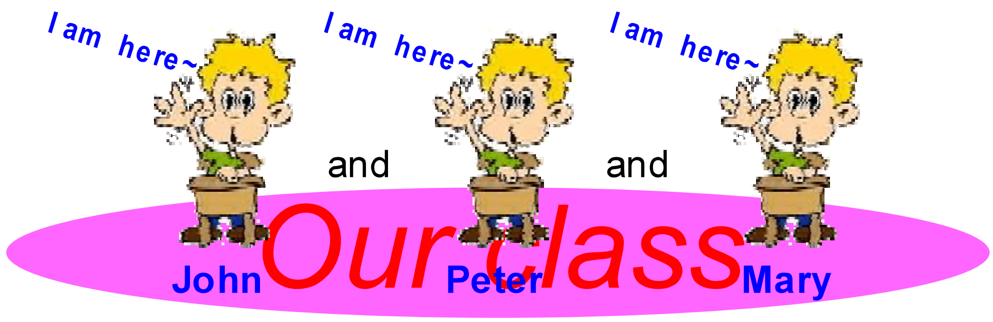
- Definition
  Universal quantification of $P(x)$ is the statement
  "$P(x)$ is true for all values of x in the domain"
- Notation: $\forall x\, P(x)$
  - $\forall$LL, reversed "A"
  - Read as
    - "for all x $P(x)$"
    - "for every x $P(x)$"
- Truth value
  - **True** when $P(x)$ is true for all x
  - **False** otherwise
    - An **element** for which $P(x)$ is **false** is called a counterexample

Universal Quantifiers

- When all of the elements in the universe of discourse can be listed one by one (discrete) (e.g. $x_1, x_2, \ldots, x_n$),

$$\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

- For example
  - **Our class** has three students: **John, Peter and Mary**
  - **Every student in our class has attended the class**



Existential Quantifiers (SOME)

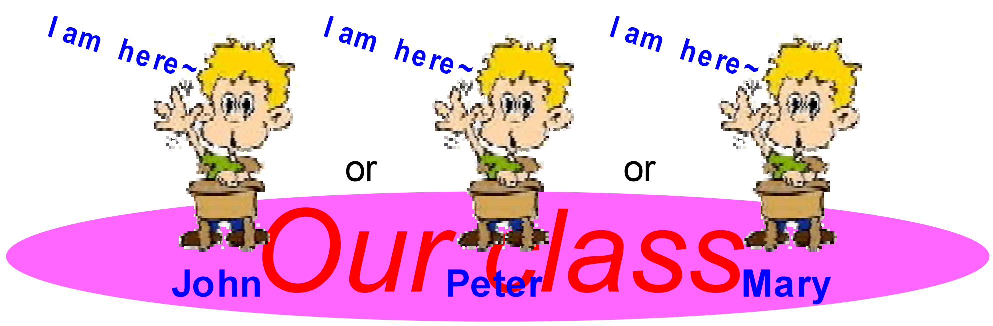
- Definition
  Existential quantification of $P(x)$ is the proposition
  "There exists an element x in the domain
  such that $P(x)$ is true"
- Notation: $\exists x\, P(x)$
  - ∃XIST, reversed "E"
  - Read as
    - "There is an x such that $P(x)$"
    - "There is at least one x such that $P(x)$"
    - "For some x $P(x)$"
- Truth value
  - **True** when $P(x)$ is false for all x
  - **False** otherwise

Existential Quantifiers

- When all of the elements in the universe of discourse can be listed one by one (discrete) (e.g. $x_1, x_2, \ldots, x_n$),

$$\exists x\, P(x) \equiv P(x_1) \lor P(x_2) \lor \ldots \lor P(x_n)$$

- For example
  - **Our class** has three students: **John, Peter and Mary**
  - **Any student in our class has attended the class**



Quantifiers

- Examples:
  - ***P*(*x*): *x*+1>*x***, U.D.s: the set of real number
    - $\forall x\, P(x)$? **True**
    - $\exists x\, P(x)$? **True**

    $P(x)$ is always true
  - ***Q*(*x*): *x*<2**, U.D.s: the set of real number
    - $\forall x\, Q(x)$? **False** $Q(y)$ is false when $y \geq 3$ (counterexamples)
    - $\exists x\, Q(x)$? **True** $Q(y)$ is true when $y < 2$
  - ***S*(*x*): 2*x*<*x***, U.D.s: the set of real positive number
    - $\forall x\, S(x)$? **False**
    - $\exists x\, S(x)$? **False**

    $S(x)$ is always false

Universal Quantifiers

- Examples:
  - ***P*(*x*): x^2<10,**
    U.D.s. the positive integer not exceeding 4
    - $\forall x\, P(x)$?

      $$\forall x\, P(x) \equiv P(1) \wedge P(2) \wedge P(3) \wedge P(4) \;\; \equiv \mathbf{F}$$
    - $\exists x\, P(x)$?

      $$\exists x\, P(x) \equiv P(1) \vee P(2) \vee P(3) \vee P(4) \;\; \equiv \mathbf{T}$$

$P(1)$ ✓ $P(2)$ ✓ $P(3)$ ✓ $P(4)$ ✗

counterexample

Quantifiers

- How can we prove the followings:
  - Universal quantification is true
  - Universal quantification is false
  - Existential quantification is true
  - Existential quantification is false

Finding one is ok (counterexample)

Need to consider ALL

Attend the class?

John Peter Mary

Statement	When true?	When false?
$\forall x\, P(x)$	P(x) is true for every x	There is an x for which P(x) is false.
$\exists x\, P(x)$	There is an x for which P(x) is true.	P(x) is false for every x.

Precedence of Quantifiers

- Recall,

Precedence	Operator	
1	$\neg$	NOT
2	$\wedge$	AND
3	$\vee$ $\oplus$	OR XOR
4	$\rightarrow$	Imply
5	$\leftrightarrow$	Equivalent

- $\forall$ and $\exists$ have higher precedence than all logical operators from proposition calculus

- Example
  - **$\forall x\ P(x) \wedge Q(x)$**

    $(\forall x P(x)) \wedge Q(x)$ ✓ $\forall x\ (P(x) \wedge Q(x))$ ✗

☺ Small Exercise ☺

- How to interpret the following expression:
  - $\forall x\ (P(x) \wedge \exists z\ Q(x,z) \rightarrow \exists y\ R(x,y)) \vee Q(x,y)$
  - $\forall x\ (\ P(x) \wedge \underbrace{(\exists z\ Q(x,z))} \rightarrow \underbrace{(\exists y\ R(x,y))}\ ) \vee Q(x,y)$
  - $\forall x\ (\ \underbrace{(P(x) \wedge \underbrace{(\exists z\ Q(x,z))})} \rightarrow \underbrace{(\exists y\ R(x,y))}\ ) \vee Q(x,y)$

Bound and Free Variable

- **Free Variable**: No any restriction
- **Bound Variable**: Some restrictions (quantifier or condition)
- Example:
  - $P(x)$: "$x > 3$" **Free Variable** Not Proposition
  - $P(x)$: "$x > 3$" and $x = 4$ **Bound Variable** Proposition
  - $\forall x\, P(x, y)$ ***x*:Bound Variable** **y:Free Variable** Not Proposition
- All the variables that occur in a quantifier must be bounded to turn it into a proposition
  - i.e. the truth value can be detemined
- Giving restrictions on a free variable is called ***blinding***

Scope

- The part of a logical expression to which a quantifier is applied is called the ***scope*** of this quantifier
- For example

$$\forall x\, \underbrace{(P(x) \wedge (\exists y\, \overbrace{Q(y)}^{\text{Scope of } \exists y}))}_{\text{Scope of } \forall x} \vee R(z)$$

☺ Small Exercise ☺

- $\forall x\ (\ (P(x) \wedge (\exists z\ Q(x,z))) \rightarrow (\exists y\ R(x,y))\ ) \vee Q(x,y)$
    - Scope of $\exists z$: $Q(x,z)$
    - Scope of $\exists y$: $R(x,y)$
    - Scope of $\forall x$: $P(x) \wedge \exists z\ Q(x,z) \rightarrow \exists y\ R(x,y)$
    - Free Variable: x, y in $Q(x,y)$
    - Bound Variable: x, y, z in the first component

☺ Small Exercise ☺

- $\forall x\ \exists x\ P(x)$ **Not a free variable**
    - Any problem?

      x is not a free variable in $\exists x\ P(x)$, therefore the $\forall x$ binding is not used
- $\forall x\ P(x) \wedge Q(x)$ **1st x is Bounded variable** **2nd x is Free variable**
    - Is x a free variable?

      The variable x in $Q(x)$ is outside of the *scope* of the $\forall x$ quantifier, and is therefore free
- $(\forall x\ P(x)) \wedge (\exists x\ Q(x))$ **Different variables**
    - Are x the same?
    - This is legal, because there are 2 different x

Recall....

- $\exists x\ (x^2>1)$
  - Domain of x is real number ✓
  - Domain of x is between -1 and 1 ✗
- $\forall x\ (x^2 \geqslant 1)$
  - Domain of x is integer ✗
  - Domain of x is positive integer ✓

Recall, the Equivalences

- Two propositions P and Q are logically equivalent if $P \leftrightarrow Q$ is a tautology
- $P \leftrightarrow Q$ means $(P\rightarrow Q) \wedge (Q\rightarrow P)$
  - $(P\rightarrow Q)$: Given P, Q is true
  - $(Q\rightarrow P)$: Given Q, P is true
- Therefore,
  if we want to show **$P \equiv Q$**,
  we can show $P\rightarrow Q$ and $Q\rightarrow P$