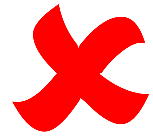
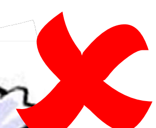
Recall…

- John is a cop. John knows first aid. Therefore, all cops know first aid



Recall…

- Some students work hard to study. Some students fail in examination. So, some work hard students fail in examination.



Argument

p: It rains
q: The floor is wet

	Argument	Argument Form
Premise / Hypothesis	If it rains, the floor is wet	p
	It rains	$p \to q$
Conclusion	∴ The floor is wet	$\therefore$ q

therefore

- **Argument** in propositional logic is a sequence of propositions
  - **Premises / Hypothesis:** All except the final proposition
  - **Conclusion:** The final proposition
- **Argument form** represents the argument by variables

Argument: Valid?

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

- Given an argument, where
  - $p_1, p_2, \ldots, p_n$ be the premises
  - q be the conclusion
- **The argument is valid** when $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to q$ is a tautology
  - When all premises are true, the conclusion should be true
  - When not all premises are true, the conclusion can be either true or false

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Focus on this case
Check if it happens

Argument

- Example:

Argument is valid

$p \to q$	If it rains, the floor is wet
p	It rains
q $\therefore$	The floor is wet

$$\boxed{(\ p \land (p \to q)\ ) \to q}$$

Tautology

Need to check if the conclusion is true or not

Must be true

p	q	$p \to q$	$p \land (p \to q)$	$(p \land (p \to q)) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Rules of Inference

- How to show an argument is valid?
  - **Truth Table**
    - May be tedious when the number of variables is large
  - **Rules of Inference**
    - Firstly establish the validity of some relatively simple argument forms, called **rules of inference**
    - These rules of inference can be used as building blocks to construct more complicated valid argument forms

Rules of Inference

- **Modus Ponens**
  - Affirm by affirming

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore\ q \end{array}$$

- **Modus Tollens**
  - Deny by denying

$$\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore\ \neg p \end{array}$$

Rules of Inference

- **Hypothetical Syllogism**

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$$

- **Disjunctive Syllogism**

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore\ q \end{array}$$

Rules of Inference

- **Addition**

$$\frac{p}{\therefore\ p \vee q}$$

- **Simplification**

$$\frac{p \wedge q}{\therefore\ p}$$

- **Conjunction**

$$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore\ p \wedge q}$$

Rules of Inference

- **Resolution**

p = T	p = F
q =	q =
r =	r =

$$\frac{\begin{array}{c} p \vee q \\ \neg p \vee r \end{array}}{\therefore\ q \vee r}$$

- Example
  - I go to swim or I play tennis
  - I do not go to swim or I play football
  - Therefore, I play tennis or I play football

Rules of Inference ($\rightarrow$)

Modus Ponens	$((p \rightarrow q) \wedge (p)) \rightarrow \mathbf{q}$
Modus Tollens	$((\neg q) \wedge (p \rightarrow q)) \rightarrow \mathbf{\neg p}$
Hypothetical Syllogism	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (\mathbf{p} \rightarrow \mathbf{r})$
Disjunctive Syllogism	$((p \vee q) \wedge (\neg p)) \rightarrow \mathbf{q}$
Addition	$(p) \rightarrow \mathbf{p} \vee \mathbf{q}$
Simplification	$((p) \wedge (q)) \rightarrow \mathbf{p}$
Conjunction	$((p) \wedge (q)) \rightarrow (\mathbf{p} \wedge \mathbf{q})$
Resolution	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (\mathbf{q} \vee \mathbf{r})$

Rules of Equivalence ($\leftrightarrow$)

- Recall…

Identify Laws	$p \wedge T \equiv p$ $p \vee F \equiv p$
Domination Laws	$p \vee T \equiv T$ $p \wedge F \equiv F$
Idempotent Laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
Negation Laws	$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$
Double Negation Law	$\neg(\neg p) \equiv p$
Commutative Laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
Associative Laws	$p \vee (q \vee r) \equiv (p \vee q) \vee r$ $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
Distributive Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Absorption Laws	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
De Morgan's Laws	$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Comparison between Inference and Equivalence

- **Inference ($p \rightarrow q$)**
  - Meaning:
    If p, then q
  - $p \rightarrow q$ does not mean $q \rightarrow p$
  - Either inference or equivalence rules can be used
  - $p \leftrightarrow q$ implies $p \rightarrow q$
  - $\Rightarrow$ is used in proof
- **Equivalence ($p \leftrightarrow q$)**
  - Meaning:
    p is equal to q
  - $p \leftrightarrow q$ mean $q \leftrightarrow p$
  - Only equivalence rules can be used
  - $p \leftrightarrow q$ can be proved by showing $p \rightarrow q$ and $q \rightarrow p$
  - $\Leftrightarrow$ is used in proof
- Equivalence ($\leftrightarrow$) is a **more restrictive** relation than Inference ($\rightarrow$)

Using Rules of Inference

- Example 1:
  - **Given:**
    - It is not sunny this afternoon and it is colder than yesterday.
    - We will go swimming only if it is sunny
    - If we do not go swimming, then we will take a canoe trip
    - If we take a canoe trip, then we will be home by sunset
  - Can these propositions lead to the **conclusion** "We will be home by sunset“ ?

Let p: It is sunny this afternoon
q: It is colder than yesterday
r: We go swimming
s: We take a canoe trip
t: We will be home by sunset

$\neg p \wedge q$	It is not sunny this afternoon and it is colder than yesterday
$r \to p$	We will go swimming only if it is sunny
$\neg r \to s$	If we do not go swimming, then we will take a canoe trip
$s \to t$	If we take a canoe trip, then we will be home by sunset
t	We will be home by sunset

Using Rules of Inference

Hypothesis:

$\neg p \wedge q$
$r \to p$
$\neg r \to s$
$s \to t$

Conclusion:

t

Step		Reason
1.	$\neg p \wedge q$	Premise
2.	$\neg p$	Simplification using (1)
3.	$r \to p$	Premise
4.	$\neg r$	Modus tollens using (2) and (3)
5.	$\neg r \to s$	Premise
6.	s	Modus ponens using (4) and (5)
7.	$s \to t$	Premise
8.	t	Modus ponens using (6) and (7)

Therefore, the propositions can lead to the conclusion
We will be home by sunset

Using Rules of Inference

- Or, another presentation method:

Hypothesis:

$\neg p \land q$

$r \to p$

$\neg r \to s$

$s \to t$

Conclusion:

t

$\boxed{(\neg p \land q)} \land (r \to p) \land (\neg r \to s) \land (s \to t)$

$\Rightarrow \boxed{\neg p \land (r \to p)} \land (\neg r \to s) \land (s \to t)$ By Simplification

$\Rightarrow \boxed{\neg r \land (\neg r \to s)} \land (s \to t)$ By Modus Tollens

$\Rightarrow \boxed{s \land (s \to t)}$ By Modus Ponens

$\Rightarrow t$ By Modus Ponens

☺ Small Exercise ☺

- **Given:**
  - If you send me an e-mail message, then I will finish writing the program
  - If you do not send me an e-mail message, then I will go to sleep early
  - If I go to sleep early, then I will wake up feeling refreshed
- Can these propositions lead to the **conclusion** "If I do not finish writing the program, then I will wake up feeling refreshed."

Let p: you send me an e-mail message
q: I will finish writing the program
r: I will go to sleep early
s: I will wake up feeling refreshed

$p \rightarrow q$
- If you send me an e-mail message, then I will finish writing the program

$\neg p \rightarrow r$
- If you do not send me an e-mail message, then I will go to sleep early

$r \rightarrow s$
- If I go to sleep early, then I will wake up feeling refreshed

---

$\neg q \rightarrow s$
- If I do not finish writing the program, then I will wake up feeling refreshed

☺ Small Exercise ☺

Hypothesis:

$p \rightarrow q$
$\neg p \rightarrow r$
$r \rightarrow s$

Conclusion:

$\neg q \rightarrow s$

Step		Reason
1.	$p \rightarrow q$	Premise
2.	$\neg q \rightarrow \neg p$	Contrapositive of (1)
3.	$\neg p \rightarrow r$	Premise
4.	$\neg q \rightarrow r$	Hypothetical Syllogism using (2) and (3)
5.	$r \rightarrow s$	Premise
6.	$\neg q \rightarrow s$	Hypothetical Syllogism using (4) and (5)

Therefore, the propositions can lead to the conclusion
If I do not finish writing the program,
then I will wake up feeling refreshed

☺ Small Exercise ☺

- Or, another presentation method:

Hypothesis:

$p \rightarrow q$

$\neg p \rightarrow r$

$r \rightarrow s$

Conclusion:

$\neg q \rightarrow s$

$$\boxed{(p \rightarrow q)} \wedge (\neg p \rightarrow r) \wedge (r \rightarrow s)$$

$$\Leftrightarrow \boxed{(\neg q \rightarrow \neg p) \wedge (\neg p \rightarrow r)} \wedge (r \rightarrow s)$$ Contrapositive

$$\Rightarrow \boxed{(\neg q \rightarrow r) \wedge (r \rightarrow s)}$$ By Hypothetical Syllogism

$$\Rightarrow (\neg q \rightarrow s)$$ By Hypothetical Syllogism

Using Rules of Inference

Fallacies

- **Are the following arguments correct?**
  - **Example 1 (Fallacy of affirming the conclusion)**

    Hypothesis
    - If you success, you work hard
    - You work hard

    Conclusion
    - You success

$$\begin{array}{l} p \rightarrow q \\ q \\ \hline \therefore\ p \end{array}$$

  - **Example 2 (Fallacy of denying the hypothesis)**

    Hypothesis
    - If you success, you work hard
    - You do not success

    Conclusion
    - You do not work hard

$$\begin{array}{l} p \rightarrow q \\ \neg p \\ \hline \therefore\ \neg q \end{array}$$

Rules of Inference for Quantifiers

- **Universal Instantiation**

$$\frac{\forall x\, P(x)}{\therefore P(a)}$$

where a is a particular member of the domain

- **Existential Instantiation**

$$\frac{\exists x\, P(x)}{\therefore P(c) \text{ for some element } c}$$

- **Universal Generalization**

$$\frac{P(b) \text{ for an arbitrary b}}{\therefore \forall x\, P(x)}$$

Be noted that b that we select must be an arbitrary, and not a specific

- **Existential Generalization**

$$\frac{P(d) \text{ for some element } d}{\therefore \exists x\, P(x)}$$

Rules of Inference for Quantifiers

- Example 1
  - **Given**
    - Everyone in this discrete mathematics class has taken a course in computer science
    - Marla is a student in this class
  - These premises imply the **conclusion** "Marla has taken a course in computer science"

Let	DC(x):	x studies in discrete mathematics
	CS(x):	x studies in computer science
	Domain of x:	student

$\forall x\ (DC(x) \rightarrow CS(x))$

- Everyone in this discrete mathematics class has taken a course in computer science

DC(Marla)

- Marla is a student in this class

CS(Marla)

- Marla has taken a course in computer science

Rules of Inference for Quantifiers

Premise:

$\forall x\ (DC(x) \rightarrow CS(x))$
DC(Marla)

Conclusion:

CS(Marla)

Step		Reason
1.	$\forall x\ (DC(x) \rightarrow CS(x))$	Premise
2.	DC(Marla) $\rightarrow$ CS(Marla)	Universal Instantiation from (1)
3.	DC(Marla)	Premise
4.	CS(Marla)	Modus ponens using (2) and (3)

Therefore, the propositions can lead to the conclusion Marla has taken a course in computer science

Using Rules of Inference for Quantifiers

- Or, another presentation method:

Premise:

$\forall x\, (DC(x) \rightarrow CS(x))$
$DC(Marla)$

Conclusion:

$CS(Marla)$

$\boxed{\forall x\, (DC(x) \rightarrow CS(x))} \wedge DC(Marla)$

By Universal Instantiation

$\Rightarrow \boxed{(DC(Marla) \rightarrow CS(Marla)) \wedge DC(Marla)}$

$\Rightarrow CS(Marla)$ By Modus ponens

☺ Small Exercise ☺

- **Given**
  - A student in this class has not read the book
  - Everyone in this class passed the first exam
- These premises imply the **conclusion** "Someone who passed the first exam has not read the book"

Let		
	C(x):	x in this class
	RB(x):	x reads the book
	PE(x):	x passes the first exam
	Domain of x:	any person

$\exists x\ (C(x) \land \neg RB(x))$

- A student in this class has not read the book

$\forall x\ (C(x) \rightarrow PE(x))$

- Everyone in this class passed the first exam

$\exists x\ (PE(x) \land \neg RB(x))$

- Someone who passed the first exam has not read the book

We cannot define the domain as student in this class since the conclusion means anyone

☺ Small Exercise ☺

Premise:

$\exists x\ (C(x) \land \neg RB(x))$

$\forall x\ (C(x) \rightarrow PE(x))$

Conclusion:

$\exists x\ (PE(x) \land \neg RB(x))$

Step		**Reason**
1.	$\exists x\ (C(x) \land \neg RB(x))$	Premise
2.	$C(a) \land \neg RB(a)$	Existential Instantiation from (1)
3.	$C(a)$	Simplification from (2)
4.	$\forall x\ (C(x) \rightarrow PE(x))$	Premise
5.	$C(a) \rightarrow PE(a)$	Universal Instantiation from (4)
6.	$PE(a)$	Modus ponens from (3) and (5)
7.	$\neg RB(a)$	Simplification from (2)
8.	$PE(a) \land \neg RB(a)$	Conjunction from (6) and (7)
9.	$\exists x\ (PE(x) \land \neg RB(x))$	Existential Generalization from (8)

Therefore, the propositions can lead to the conclusion Someone who passed the first exam has not read the book

☺ Small Exercise ☺

- Or, another presentation method:

$(\exists x\ (C(x) \land \neg RB(x))) \land (\forall x\ (C(x) \rightarrow PE(x)))$

$\Rightarrow C(a) \land \neg RB(a) \land (\forall x\ (C(x) \rightarrow PE(x)))$ By Existential Instantiation

$\Rightarrow C(a) \land \neg RB(a) \land (C(a) \rightarrow PE(a))$ By Universal Instantiation

$\Rightarrow PE(a) \land \neg RB(a)$ By Modus ponens

$\Rightarrow \exists x\ (PE(x) \land \neg RB(x))$ By Existential Generalization

Premise:
$\exists x\ (C(x) \land \neg RB(x))$
$\forall x\ (C(x) \rightarrow PE(x))$

Conclusion:
$\exists x\ (PE(x) \land \neg RB(x))$

Combining Rules of Inference

- The rules of inference of Propositions and Quantified Statements can be combined
  - **Universal Modus Ponens**

$\forall x\ (P(x) \rightarrow Q(x))$
$P(a)$, **where a is a particular element in the domain**

$\therefore\ Q(a)$

$(\forall x\ (P(x) \rightarrow Q(x))) \land (P(a))$

By **Universal** Instantiation
$\Rightarrow (P(a) \rightarrow Q(a)) \land (P(a))$

$\Rightarrow Q(a)$ By **Modus Ponens**

  - **Universal Modus Ponens**

$\forall x\ (P(x) \rightarrow Q(x))$
$\neg Q(a)$, **where a is a particular element in the domain**

$\therefore\ \neg P(a)$

$(\forall x\ (P(x) \rightarrow Q(x))) \land (\neg Q(a))$

By **Universal** Instantiation
$\Rightarrow (P(a) \rightarrow Q(a)) \land (\neg Q(a))$

$\Rightarrow \neg P(a)$ By **Modus Tollens**