Predicates and Quantifiers

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Excercise

$$
\begin{aligned}
[(p \vee q) \wedge (p \rightarrow \neg r) \wedge r] \rightarrow q &\equiv \neg[(p \vee q) \wedge (\neg p \vee \neg r) \wedge r] \vee q \\
&\equiv \neg[(p \vee q) \wedge r \wedge (\neg p \vee \neg r)] \vee q \\
&\equiv \neg\{[(p \vee q) \wedge [(r \wedge \neg p) \vee (r \wedge \neg r)]\} \vee q \\
&\equiv \neg\{[(p \vee q) \wedge [(r \wedge \neg p) \vee \mathrm{F}]\} \vee q \\
&\equiv \neg[(p \vee q) \wedge r \wedge \neg p] \vee q \\
&\equiv \neg[\neg p \wedge (p \vee q) \wedge r] \vee q \\
&\equiv \neg\{[(\neg p \wedge p) \vee (\neg p \wedge q)] \wedge r]\} \vee q \\
&\equiv \neg\{[\mathrm{F} \vee (\neg p \wedge q)] \wedge r]\} \vee q \\
&\equiv \neg(\neg p \wedge q \wedge r) \vee q \\
&\equiv \neg(q \wedge \neg p \wedge r) \vee q \\
&\equiv \neg q \vee \neg(\neg p \wedge r) \vee q \\
&\equiv \neg q \vee q \vee \neg(\neg p \wedge r) \\
&\equiv \mathrm{T} \vee \neg(\neg p \wedge r) \\
&\equiv \mathrm{T}
\end{aligned}
$$

The proposition is a tautology, so the argument is valid.

Propositional Logic Solution?

- **If Socrates is a man and man is mortal, then Socrates is mortal.**

- This can be done by introducing propositional constants *SMN* (for "Socrates is a man"), *SML* (for "Socrates is mortal") and MRM (for “Man is mortal")

$$SMN \wedge \mathrm{MRM} \rightarrow SML$$

- *What is Man? All?* Therefore?*... Vague definition*

What's done, and next..

- Logic of sentences (**propositional logic**)
  - Propositional logic (**Zeroth-order logic**) deals with simple declarative propositions. It cannot adequately express the meaning of all statements in mathematics and in natural language.

- Logic of objects (**predicate logic**)
  - Distinguished from propositional logic by its use of quantified variables (**first-order logic**)

- While propositional logic deals with simple declarative propositions, first-order logic additionally covers predicates and quantification, that permit us to reason and explore relationships between objects.

Predicate logic

- The propositional logic is not powerful enough to represent all types of assertions that are used in computer science and mathematics. Example:
  - For all positive integers n, n^2+n+41 is prime.
  - There is an integer k that is both even and odd.

- In essence, these statements assert something about lots of simple propositions all at once. For instance,
  - 1st statement is asserting that 0^2+0+41 is prime, 1^2+1+41 is prime, and so on.
  - 2nd statement is saying that as k ranges over every possible integer, we will find at least one value for which the statement is satisfied.

Propositions?

Propositions?

- Why are previous examples considered to be propositions, while earlier we claimed that "$x+1=2$" was not?
- The reason is that in these two examples, there is an underlying "universe" that we are working in.
- The statements are then *quantified* over that universe.

Predicate Logic

- *Predicate logic* is an extension of propositional logic that permits concisely reasoning about whole *classes* of entities.

  E.g., “$x>1$”, “$x+y=10$”

- Such statements are neither true or false when the values of the variables are not specified.

1.6 Predicates and Quantifiers

Let's see the new type of logic: Predicate Logic

consider the statement: “x is greater than 3” - it has two parts:

variable x — (subject of a statement)

predicate — (refers to a property the subject of a statement can have)

<u>denotation</u>: *P(x)*: ”*x* is greater than 3”
- this kind of statement is neither true nor false when the value of variable is not specified.

P(x) reads as propositional function P at *x*

Once *x* is assigned a value, *P(x)* <u>becomes a proposition</u> that has a truth value.

Predicates and Quantifiers

- Example: Let *P(x)* denote the statement "x > 3." What are the truth values of *P*(4) and *P*(2)?

  Solution: *P*(4) – "4 > 3", *true*
  P(3) – "2 > 3", *false*

- Example: Let *Q*(x,y) denote the statement "x = y + 3." What are the truth values of the propositions *Q*(1,2) and *Q*(3,0)?

  Solution: *Q*(1,2) – "1 = 2 + 3" , *false*
  Q(3,0) – "3 = 0 + 3", *true*

Predicates and Quantifiers

- Example: Let *A(c,n)* denote the statement "Computer *c* is connected to network *n*", where *c* is a variable representing a computer and *n* is a variable representing a network. Suppose that the computer MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the values of A(MATH1, CAMPUS1) and A(MATH1, CAMPUS2)?

Solution: *A(MATH1, CAMPUS1) – "MATH1 is connect to CAMPUS1", false*

A(MATH1, CAMPUS2) – "MATH1 is connect to CAMPUS2", true

Predicates and Quantifiers

- A statement involving n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.
- A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \ldots, x_n)$, and P is also called a ***n*-place predicate** or a ***n*-ary predicate**.

Predicates and Quantifiers

Quantifiers

- Quantification: express the extent to which a predicate is true over a range of elements.
- **Universal quantification**: a predicate is true for every element under consideration
- **Existential quantification**: a predicate is true for one or more element under consideration
- A domain must be specified.

Universe of Discourse

- Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the **universe of discourse/domain**.

Predicates and Quantifiers

DEFINITION 1

The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **Universal Quantifier**. We read $\forall x P(x)$ as "for all $x P(x)$" or "for every $x P(x)$." An element for which $P(x)$ is false is called a **counterexample** of $x P(x)$.

Example: Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Solution: *Because $P(x)$ is true for all real numbers, the quantification is true.*

Predicates and Quantifiers

- A statement $\forall x P(x)$ is false, if and only if $P(x)$ is not always true where x is in the domain. One way to show that is to find a counterexample to the statement $\forall x P(x)$.
- Example: Let $Q(x)$ be the statement "x < 2". What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

  Solution: *$Q(x)$ is not true for every real numbers, e.g. $Q(3)$ is false. $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus the quantification is false.*

- $\forall x P(x)$ is the same as the conjunction

  $$P(x_1) \wedge P(x_2) \wedge \ldots. \wedge P(x_n)$$

Predicates and Quantifiers

- Example: What does the statement $\forall x N(x)$ mean if $N(x)$ is "Computer x is connected to the network" and the domain consists of all computers on campus?

Solution: *"Every computer on campus is connected to the network."*

Predicates and Quantifiers

DEFINITION 2

The *existential quantification* of $P(x)$ is the statement

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the **Existential Quantifier**.

- The existential quantification $\exists x P(x)$ is read as

  "There is an x such that $P(x)$," or

  "There is at least one x such that $P(x)$," or

  "For some x, $P(x)$."

Predicates and Quantifiers

- Example: Let *P(x)* denote the statement "*x* > 3". What is the truth value of the quantification $\exists xP(x)$, where the domain consists of all real numbers?

Solution: *"x > 3" is sometimes true – for instance when x = 4. The existential quantification is true.*

- $\exists xP(x)$ is false if and only if $P(x)$ is false for every element of the domain.
- Example: Let Q(x) denote the statement "x = x + 1". What is the true value of the quantification $\exists xQ(x)$, where the domain consists for all real numbers?

Solution: *Q(x) is false for every real number. The existential quantification is false.*

Predicates and Quantifiers

- If the domain is empty, $\exists x Q(x)$ is false because there can be no element in the domain for which $Q(x)$ is true.
- The existential quantification $\exists x P(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

Quantifiers		
Statement	***When True?***	***When False?***
$\forall x P(x)$	*xP(x) is true for every x.*	*There is an x for which xP(x) is false.*
$\exists x P(x)$	*There is an x for which P(x) is true.*	*P(x) is false for every x.*

∃

Predicates and Quantifiers

- Uniqueness quantifier $\exists!$ or $\exists_1$
  - $\exists!xP(x)$ or $\exists_1 P(x)$ states "There exists a unique x such that $P(x)$ is true."
- Quantifiers with restricted domains
  - Example: What do the following statements mean? The domain in each case consists of real numbers.
    - $\forall x < 0\ (x^2 > 0)$: For every real number x with $x < 0$, $x^2 > 0$. "The square of a negative real number is positive." It's the same as $\forall x(x < 0 \rightarrow x^2 > 0)$
    - $\forall y \neq 0\ (y^3 \neq 0)$: For every real number y with $y \neq 0$, $y^3 \neq 0$. "The cube of every non-zero real number is non-zero." It's the same as $\forall y(y \neq 0 \rightarrow y^3 \neq 0)$.
    - $\exists z > 0\ (z^2 = 2)$: There exists a real number z with $z > 0$, such that $z^2 = 2$. "There is a positive square root of 2." It's the same as $\exists z(z > 0 \land z^2 = 2)$:

Predicates and Quantifiers

- Precedence of Quantifiers
  - $\forall$ and $\exists$ have higher precedence than all logical operators.
  - E.g. $\forall x P(x) \vee Q(x)$ is the same as $(\forall x P(x)) \vee Q(x)$

Predicates and Quantifiers

Translating from English into Logical Expressions

- Example: Express the statement “Every student in this class has studied calculus” using predicates and quantifiers.

Solution:

If the domain consists of students in the class –

$$\forall x C(x)$$

where $C(x)$ is the statement “x has studied calculus.

If the domain consists of all people –

$$\forall x(S(x) \rightarrow C(x))$$

where $S(x)$ represents that person x is in this class.

Translating from English into Logical Expressions

Example 13:

Let $P(x)$ be the statement "x took a discrete math course", and $Q(x)$ be the statement "x knows the computer language Python". Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers and logical connectives. Let the domain for quantifiers consist of all students from Mathematics, CS, and Engineering majors.

a) There is a student who took a discrete math course.

$$\exists x P(x)$$

b) There is a student who took a discrete math course, but doesn't know Python.

$$\exists x\, (P(x) \land \neg Q(x))$$

c) Every student either took a discrete math course or knows the computer language Python.

$$\forall x\, (P(x) \lor Q(x))$$

d) No student took a discrete math course, but knows Python.

$$\neg \exists x (P(x) \land Q(x))$$

Predicates and Quantifiers

- Example: Consider these statements. The first two are called *premises* and the third is called the *conclusion*. The entire set is called an *argument*.

  "All lions are fierce."

  "Some lions do not drink coffee."

  "Some fierce creatures do not drink coffee."

Solution: Let $P(x)$ be "x is a lion."

$Q(x)$ be "x is fierce."

$R(x)$ be "x drinks coffee."

$\forall x(P(x) \rightarrow Q(x))$

$\exists x(P(x) \land \neg R(x))$

$\exists x(Q(x) \land \neg R(x))$

Examples:

Consider these statements, of which the first three are premises and the fourth is a valid conclusion.

"All hummingbirds are richly colored."
"No large birds live on honey."
"Birds that do not live on honey are dull in color."
"Hummingbirds are small."

Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a hummingbird," "x is large," "x lives on honey," and "x is richly colored," respectively. Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

Solution: We can express the statements in the argument as

$$\forall x(P(x) \to S(x)).$$
$$\neg\exists x(Q(x) \wedge R(x)).$$
$$\forall x(\neg R(x) \to \neg S(x)).$$
$$\forall x(P(x) \to \neg Q(x)).$$

Nested Quantifiers

If a predicate has more than one variable, each variable must be bound by a separate quantifier.

Definition

A logical expression with more than one quantifier is said to have **nested quantifiers**. The logical expression is a proposition if all the variables are bound.

Examples

$\forall x \exists y P(x, y)$	(x and y are both bound)	Proposition
$\forall x P(x, y)$	(x is bound and y is free)	Not a Proposition
$\exists y \exists z T(x, y, z)$	(y and z are bound, x is free)	Not a Proposition

Quantifiers of the Same Type

Consider a scenario where the domain is a group of people who are all working on a joint project. Define the predicate M to be:

$$M(x, y) : x \text{ sent an email to } y$$

and consider the proposition: $\forall x \forall y M(x, y)$.

The proposition can be expressed in English as:

$$\forall x \forall y M(x, y) \leftrightarrow \text{“Everyone sent an email to everyone.”}$$

Quantifiers of the Same Type

The statement $\forall x \forall y M(x, y)$ is true if for every pair, x and y, $M(x, y)$ is true. The universal quantifiers include the case that $x = y$, so if $\forall x \forall y M(x, y)$ is true, then everyone sent an email to everyone else and everyone sent an email to himself or herself.

The statement $\forall x \forall y M(x, y)$ is false if there is any pair, x and y, that causes $M(x, y)$ to be false. In particular, $\forall x \forall y M(x, y)$ is false even if there is a single individual who did not send himself or herself an email.

Quantifiers of the Same Type

Now consider the proposition: $\exists x \exists y M(x, y)$. The proposition can be expressed in English as:

$$\exists x \exists y M(x, y) \leftrightarrow \text{"There is a person who sent an email to someone."}$$

The statement $\exists x \exists y M(x, y)$ is true if there is a pair, x and y, in the domain that causes $M(x, y)$ to evaluate to true.

In particular, $\exists x \exists y M(x, y)$ is true even in the situation that there is a single individual who sent an email to himself or herself. The statement $\exists x \exists y M(x, y)$ is false if all pairs, x and y, cause $M(x, y)$ to evaluate to false.

Different Quantifiers

A quantified expression can contain both types of quantifiers as in: $\exists x \forall y M(x, y)$. The quantifiers are applied from left to right, so the statement $\exists x \forall y M(x, y)$ translates into English as:

$$\exists x \forall y M(x, y) \leftrightarrow \text{"There is a person who sent an email to everyone"}.$$

Switching the quantifiers changes the meaning of the proposition:

$$\forall x \exists y M(x, y) \leftrightarrow \text{"Every person sent an email to someone"}.$$

Different Quantifiers

In reasoning whether a quantified statement is true or false, it is useful to think of the statement as a two player game in which two players compete to set the statement's truth value.

One of the players is the "existential player" and the other player is the "universal player". The variables are set from left to right in the expression. The table below summarizes which variables are set by which player and the goal of each player:

Player	Action	Goal
Existential player	Selects values for existentially bound variables	Tries to make the expression true
Universal player	Selects values for universally bound variables	Tries to make the expression false

Example

If the predicate is true after all the variables are set then the quantified statement is true. If the predicate is false after all the variables are set, then the quantified statement is false.

Consider as an example the following quantified statement in which the domain is the set of all integers:

$$\forall x \exists y \; x + y = 0$$

The universal player first selects the value of x. Regardless of which value the universal player selects for x, the existential player can select y to be $-x$, which will cause the sum $x + y$ to be 0. Because the existential player can always succeed in causing the predicate to be true, the statement $\forall x \exists y \; x + y = 0$ is true.

Example

Switching the order of the quantifiers gives the following statement:

$$\exists x \forall y \; x + y = 0$$

Now, the existential player goes first and selects a value for x. Regardless of the value chosen for x, the universal player can select some value for y that causes the predicate to be false.

For example, if x is an integer, then $y = -x + 1$ is also an integer and $x + y = 1 \neq 0$. Thus, the universal player can always win and the statement $\exists x \forall y \; x + y = 0$ is false.

Example:

Nested Quantifiers

Note Title 12/18/2013

$P(x,y) \rightarrow$ "student x has taken class y"

Domain: x - all students in my class

y - all CS classes at MCC

$\exists x \exists y \; P(x,y)$

$\forall y \exists x \; P(x,y)$

$\forall x \exists y \; P(x,y)$

Example:

$\exists n \, \forall m (n^2 < m)$ Domain $\rightarrow$ Integers

$2^2 = 4 < m$ 3, 0, -3

Pick $m = -1$.

$n^2 < -1$ is false for all integers n.

False

Example:

$\underline{\forall n\ \exists m}\ (n^2 \geq m)$ $\qquad 2^2 = 4 \geq -10$ $\qquad n^2 \geq m$ $\qquad \underline{\exists n\ \forall m}\ (n^2 < m)$

Pick $m = -3$.

$n^2 \geq -3$ is true for all ints. n.

True

Example:

$\exists n \, \forall m \, (\boxed{n^2 > m})$

$2^2 = 4 > m$ 5, 6, 100

Pick ~~$m = 100$.~~ ←

~~$n^2 > 100$~~ ← $2^2 = 4 > m$

$3^2 = 9 > m$ 5, 6, 100

Pick $n = 2$ $m = 100$.

$2^2 > 100$

Pick $\boxed{m = n^2 + 1}$. ←

$n^2 > n^2 + 1$ ←

$0 > 1$ ← False for all int values of n.

Example:

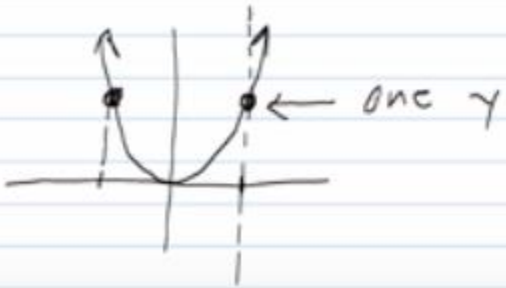
$\exists x \forall y \, (x^2 = y)$ Domain $\rightarrow \mathbb{R}$

Pick $y = -0.12$. $2^2 = y$

$x^2 = -0.12$

is false for any real value of x. ←

$y = x^2$



There exist some X so that for all real numbers, x^2 is equal to Y

Applications of Predicate Logic

- It is *the* formal notation for writing perfectly clear, concise, and unambiguous mathematical *definitions*, *axioms*, and *theorems* for *any* branch of mathematics.
- Supported by some of the more sophisticated *database query engines.*
- Basis for *automatic theorem provers* and many other Artificial Intelligence systems.