5/Mar/19



# :- Random Variable And Probability Distributions :-

→ Random Variable : Function that associates a real number with each element in sample space.

eg#1 Toss a coin thrice

{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Let X = No. of Heads

| | |
|---|---|
| 0 | 1 |
| 1 | 3 |
| 2 | 3 |
| 3 | 1 |

eg#2 4 red balls and 3 black balls. Selects two.

Total = 7 {RR, RB, BR, BB}

Let Y = Red

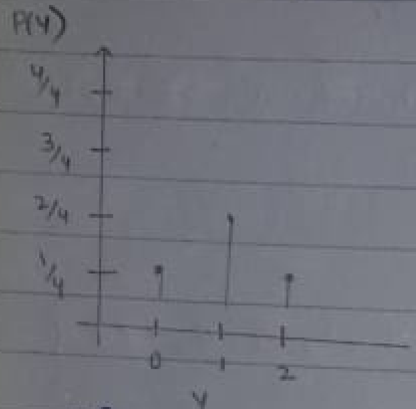
| y | s | | P(Y) |
|---|---|---|---|
| 0 | 1 | → BB | 1/4 |
| 1 | 2 | → RB, BR | 2/4 |
| 2 | 1 | → RR | + 1/4 |
| | | | 1 |

} Probability distribution.

Since 'Y' is discrete it is called Discrete Probability distribution (OR) Probability Mass Function.

* Probability density function → continuous variable

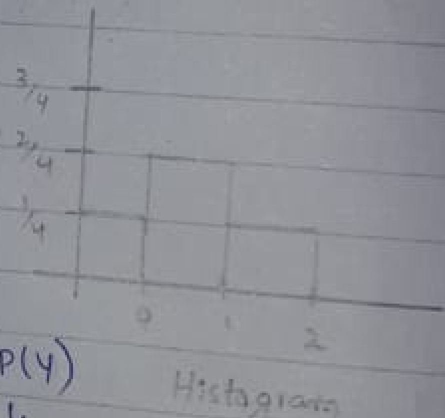
Can be represented graphically.




"Line graph"

eg#3

Defective = 3
Total = 20
Selection = 2
{DD, DI, ID, II}




| Y | | S | P(Y) |
|---|---|---|---|
| 0 | 1 → | II | 1/4 |
| 1 | 2 → | DI, ID | 2/4 |
| 2 | 1 → | DD | + 1/4 |
| | | | 1 |

→ Probability Mass Function:

1. $f(x) > 0$
2. $\sum_{m} f(x) = 1$
3. $P(X = x) = f(x)$

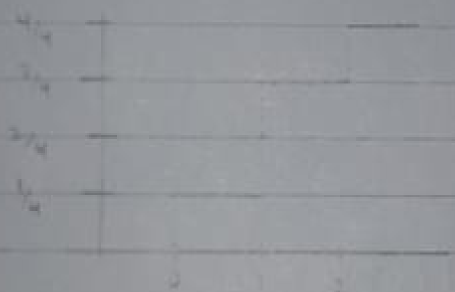
## → Cumulative Distribution Function :

step function

eg# 3

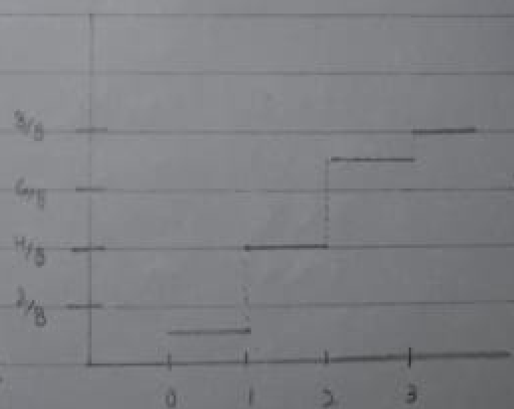
| d | P(Y) | F(Y) |
|---|---|---|
| 0 | $1/4$ | $1/4$ |
| 1 | $2/4$ | $3/4$ |
| 2 | $1/4$ | $4/4$ |

$d < 0 \rightarrow 0$
$0 \leq d < 1 \rightarrow 1/4$
$1 \leq d < 2 \rightarrow 3/4$
$d \geq 2 \rightarrow 4/4$



eg# 1 Toss a coin thrice.

| d | P(Y) | F(Y) |
|---|---|---|
| 0 | $1/8$ | $1/8$ |
| 1 | $3/8$ | $4/8$ |
| 2 | $3/8$ | $7/8$ |
| 3 | $1/8$ | $8/8$ |




$d<0 \rightarrow 0$
$0 \leq d < 1 \rightarrow 1/8$
$1 \leq d < 2 \rightarrow 4/8$
$2 \leq d < 3 \rightarrow 7/8$
$3 \leq d < 4 \rightarrow 8/8$

**Eg#01** PmF : $F(x) = (x/6)$ , $x = 1, 2, 3$

(i) Distribution Function and graph.

| $x$ | $P(x)$ | $F(x)$ |
|---|---|---|
| 1 | $1/6$ | $1/6$ |
| 2 | $2/6$ | $3/6$ |
| 3 | $3/6$ | $6/6$ |

$x < 0 \rightarrow 0$

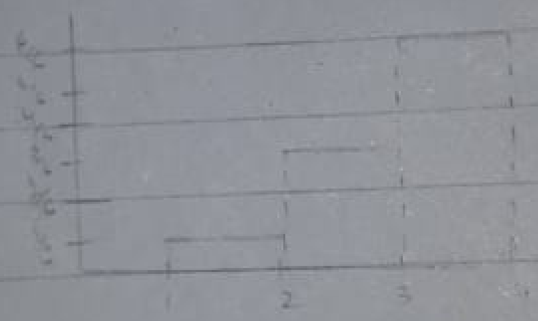
$x < 1 \rightarrow 0$

$1 \leq x < 2 \rightarrow 1/6$

$2 \leq x < 3 \rightarrow 3/6$

$3 \leq x < 4 \rightarrow 6/6$

Graph:



(ii)

$= F(4.5) - F(1.5)$

$P(a < x < b) = F(b) - F(a)$

$= 1 - 1/6$

$= 5/6$

$x < 1 \rightarrow 0$

$1 \leq x < 2 \rightarrow 1/6$

$2 \leq x \rightarrow 3/6$

$3 \leq x \rightarrow 1$

7/Mar/19



Eg #02

$P(x=2) - F(2)$

$F(x=2) = F(2) - F(1)$

$= 11/16 - 5/16$

| | | |
|---|---|---|
| $F(x) =$ | $0$ | $x<0$ |
| | $1/16$ | $0 \le x < 1$ |
| | $5/16$ | $1 \le x \le 2$ |
| | $15/16$ | $2 \le x < 3$ |
| | $1$ | $x \geq 4$ |

Eg #03

(i) $F(x) = (x+2)/5$ for $x = 1, 2, 3, 4, 5$

(ii) $F(x) = {}^4C_x / 25$ for $x = 0, 1, 2, 3, 4$

PmF = ?

// Function must be non-negative } Conditions of
// Sum = 1 } pmF.

| x | P(x) | P(y) |
|---|---|---|
| 0 | | $1/25$ |
| 1 | $3/5$ | $4/25$ |
| 2 | $4/5$ | $6/25$ |
| 3 | $5/5$ | $4/25$ |
| 4 | $6/5$ | $+\ 1/25$ |
| 5 | $+\ 7/5$ | $1/2$ |
| | $5$ | |

(i) and (ii) both cannot serve as pmf

$P(T) = w \qquad P(H) = 3w$

$x$ = No. of H

$P(1 \leq x \leq 3) = ?$

$S = \{HHH, HTH, THT, HHT, TTH, HTT, THH, TTT\}$

$\frac{9}{64} \quad \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{3}{4} \quad \frac{9}{64} \quad \frac{3}{64} \quad \frac{3}{64} \quad \frac{9}{64} \quad \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}$

| $x$ | $P(x)$ |
|---|---|
| 0 | $1/64$ |
| 1 | $9/64$ |
| 2 | $27/64$ |
| 3 | $27/64$ |
| | 1 |

$P(1 \leq x \leq 3) = 9/64 + 27/64 + 27/64$

$= 63/64$.

$0 \leq x < 1 \rightarrow 1/64$

$1 \leq x < 2 \rightarrow 9/64$

$2 \leq x < 3 \rightarrow 27/64$

$3 \leq x < 4 \rightarrow 27/64$

**Eg#05.** Two dice are rolled once

Pmf of sum of points = ?

c.d.f = ? Graph = ?

$Z = x + y$

| $Z$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(Z)$ | $1/36$ | $2/36$ | $3/36$ | $4/36$ | $5/36$ | $6/36$ | $5/36$ | $4/36$ | $3/36$ | | |
| $F(Z)$ | $1/36$ | $3/36$ | $6/36$ | $10/36$ | $15/36$ | $21/36$ | $26/36$ | $30/36$ | $33/36$ | | |

$$F(x) = \begin{cases} 0 & z < 2 \\ 1/36 & 2 \le z < 3 \\ 3/36 & 3 \le z < 4 \\ 6/36 & 4 \le z < 5 \\ 10/36 & 5 \le z < 6 \\ 15/36 & 6 \le z < 7 \\ 21/36 & 7 \le z < 8 \\ 26/36 & 8 \le z < 9 \\ 30/36 & 9 \le z < 10 \\ 33/36 & 10 \le z < 11 \\ 35/36 & 11 \le z < 12 \\ 36/36 & 12 \le z < 13 \end{cases}$$

Eg # 6. Fair coin is tossed until "head" appear

(a) P m F

1 , 2 , 3 , 4 , 5....

$1/2$ , $1/4$ , $1/8$ , $1/16$, $1/32$....

(b) $f(x) = (1/2)^x$

GP = $a/(1-r)$

$$\Sigma = \frac{1/2}{1 - 1/2}$$

$$F(x) = \sum_{x=1}^{n} f(x)$$

(c) $F(4) = f(4) + f(3) + f(2) + f(1)$

$= \frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \frac{1}{2}$

$= \frac{15}{16}$

(OR) $F(4) = \frac{31}{32} - \frac{1}{32} = \frac{15}{16}$

Eg#07

$F(x) = 1 - (\frac{1}{2})^{\wedge}(x+1)$ , for $x = 0, 1, 2, \cdots$

(a) $P(X = 3)$

$= F(3) - F(2) = [1 - (\frac{1}{2})^4] - [1 - (\frac{1}{2})^3] = \frac{1}{16}$

(b) $P(7 \leq x \leq 10)$

$= F(9) - F(6) = \frac{7}{2^{10}}$

(c) PMF

if $x_1 < x_2 \ldots < x_n$

$F(X) = F(x_1) - F(x_{2-1})$

$= 1 - (\frac{1}{2})^{x+1} - [1 - (\frac{1}{2})^{x+1-1}]$

∵ Single variable distribution



Eg # 08

3.5 (a) $f(x) = c(x^2+4)$ , for $x = 0,1,2,3$.

$$\sum_{x=0}^{3} c(x^2+4) = 1$$

$$f(0) + f(1) + f(2) + f(3) = 1$$

$$4c + 5c + 8c + 13c = 1$$

$$30c = 1$$

$$c = 1/30$$

(b) $f(x) = c\binom{2}{x}\binom{3}{3-x}$ for $x = 0,1,2$

$$\sum_{x=0}^{2} c\binom{2}{x}\binom{3}{3-x} = 1$$

$$f(0) + f(1) + f(2) = 1$$

$$c\binom{2}{0}\binom{3}{3-0} + c\binom{2}{1}\binom{3}{3-1} + c\binom{2}{2}\binom{3}{3-2} = 1$$

$$10c = 1$$

$$c = 1/10 .$$

Friday
8/Mar/19

(10)

→ Mathematical Expectation:

Eg#08 Find Expected value of $x$

| $x$ | $P(x)$ | $x \cdot P(x)$ | $x^2 P(x)$ |
|---|---|---|---|
| 0 | 4/30 | 0 | 0 |
| 1 | 5/30 | 5/30 | 5/30 |
| 2 | 8/30 | 16/30 | 32/30 |
| 3 | 13/30 | 39/30 | 117/30 |
| | | 60/30 = 2 | 154/30 |

Expected value of x

$$E(x) = \sum_{x=0}^{3} x\,P(x) = 2 = \text{Mean}$$

Expected value of $x^2$

$$E(x^2) = \sum x^2 P(x) = 154/30$$

Variance of Random value of $x$:

$$V(x) = E(x^2) - [E(x)]^2$$

$$= \frac{154}{30} - (2)^2$$

$$V(x) = 1.133$$

$0 < x \leq 1$ → Continuous

$0, 1, 2, 3$ → Discrete.

$E[\ ] = \Sigma[\ ]$

## Joint Probability Distribution:

The function $f(x,y)$ is a joint probability distribution or probability mass function of the discrete random variables X and Y if

1. $f(x,y) \geq 0$ for all $(x,y)$
2. $\sum_x \sum_y f(x,y) = 1$
3. $P(X=x, Y=y) = f(x,y)$

For any region A in xy plane
$P[(X,Y) \in A] = \sum \sum_A f(x,y)$

**#09** Total = 8, Blue = 3, Green = 3, Red = 2
Selected = 2

X = No. of Blue selected
Y = No. of Red selected

| X \ Y | 0 | 1 | 2 | Total |
|---|---|---|---|---|
| 0 | $^3C_2 / {^8C_2} = 3/28$ | $\frac{^3C_0\, ^2C_1\, ^3C_1}{^8C_2} = 6/28$ | $^2C_2 / {^8C_2} = 1/28$ | $10/28$ |
| 1 | $\frac{^3C_1\, ^2C_0\, ^3C_1}{^8C_2} = 9/28$ | $\frac{^3C_1\, ^2C_1\, ^3C_0}{^8C_2} = 6/28$ | 0 | $15/28$ |
| 2 | $\frac{^3C_2\, ^2C_0\, ^3C_0}{^8C_2} = 3/28$ | 0 | 0 | $3/28$ |
| Total | $15/28$ | $12/28$ | $1/28$ | $28/28$ |

Marginal ~~Probability~~ distribution of X

Marginal ~~Probability~~ distribution of Y

) Find joint probability function $f(x, y)$.

$f(0,0) = 3/28$ , $f(0,1) = 6/28$ , $f(0,2) = 1/28$

$f(1,0) = 9/28$ , $f(1,1) = 6/28$ , $f(2,0) = 3/28$

) $P[(x,y) \in A]$ where A is the region $\{(x,y) \mid y \leq 1\}$

$$= \frac{15}{28} + \frac{12}{28} = \frac{27}{28}$$

(OR) $f(0,0) + f(0,1) + f(1,0) + f(1,1) + f(2,0) = 27/28$

Marginal Distribution of $x$

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $f(x)$ | $10/28$ | $15/28$ | $3/28$ |

$g(x) = \sum_y f(x, y)$

Marginal Distribution of $y$

| $y$ | 0 | 1 | 2 |
|---|---|---|---|
| $f(y)$ | $15/28$ | $12/28$ | $1/28$ |

$h(y) = \sum_x f(x, y)$

→ Conditional Distribution:

$$f(y|x) = \frac{f(x,y)}{g(x)} \qquad f(x|y) = \frac{f(x,y)}{h(y)}$$

For independent: $f(y|x) = \frac{g(x)h(y)}{g(x)} = h(y)$

Eg#09 Find $P(X=0|Y=1)$ → Conditional Distribution?

$* \; P(X=0|Y=1) = \frac{f(x,y)}{h(y=1)} = \frac{3/14}{3/7} = 1/2$

$f(x,y)$

| $y \backslash x$ | 0 | 1 | 2 | Total |
|---|---|---|---|---|
| 0 | 3/28 | 9/28 | 3/28 | 15/28 |
| 1 | 3/14 | 3/14 | 0 | 3/7 |
| 2 | 1/28 | 0 | 0 | 1/28 |
| Total | 5/14 | 15/28 | 3/28 | 1 |

$* \; P(X=1|Y=0) = \frac{f(x,y)}{h(y=0)} = \frac{9/28}{15/28} = 3/5$

$* \; P(Y=0|X=1) = \frac{f(x,y)}{g(x)} = \frac{9/28}{15/28} = 3/5$

## Statistical Independence:

The random variable x and y are statistically independent if and only if

$$f(x,y) = g(x)\,h(x)$$

for all $(x,y)$ within their range

eg#9 Show $x$ & $y$ are statistical independent at $f(0,1)$

$$f(0,1) = g(x=0)\,h(y=1)$$
$$= 5/14 \cdot 3/7$$
$$3/14 \neq 15/98$$

hence x, y are not statistical independent

11/March/19

eg#10 Joint prob distribution of x and y = ?

$f(x,y) = (x+y)/30$, for $x = 0,1,2,3$
$y = 0,1,2$

| x \ y | 0 | 1 | 2 | Total |
|---|---|---|---|---|
| 0 | 0 | 1/30 | 2/30 | 3/30 |
| 1 | 1/30 | 2/30 | 3/30 | 6/30 |
| 2 | 2/30 | 3/30 | 4/30 | 9/30 |
| 3 | 3/30 | 4/30 | 5/30 | 12/30 |
| Total | 6/30 | 10/30 | 14/30 | 1 |

Find.

(a) $P(x \leq 2, y = 1) \Rightarrow f(0,1) + f(1,1) + f(2,1)$
$= 1/5$

(b) $P(x > 2, y \leq 1) \Rightarrow f(3,0) + f(3,1)$
$= 7/30$

(c) $P(x > y) \Rightarrow f(1,0) + f(2,1) + f(3,1) + f(3,2) +$
$f(2,0) + f(3,0) = 1/30 + 2/30 + \frac{3}{30} + 3/10 + 4/30 + 5/30 = 3/5$

(d) $P(x + y = 4) \Rightarrow f(2,2) + f(3,1)$
$= 8/30 = 4/15$

Q $X$ = No. of times machine will malfunction.
$Y$ = No. of times technician is called

| | | $x$ | | (b) |
|---|---|---|---|---|
| $f(x,y)$ | 1 | 2 | 3 | Marginal (Y) |
| $y$ 1 | 0.05 | 0.05 | 0.10 | 0.20 |
| 3 | 0.05 | 0.10 | 0.35 | 0.50 |
| (a) 5 | 0.00 | 0.20 | 0.10 | 0.30 |
| Marginal(X) | 0.10 | 0.35 | 0.55 | 1 |

(c) Find $P(Y=3 \mid X=2)$

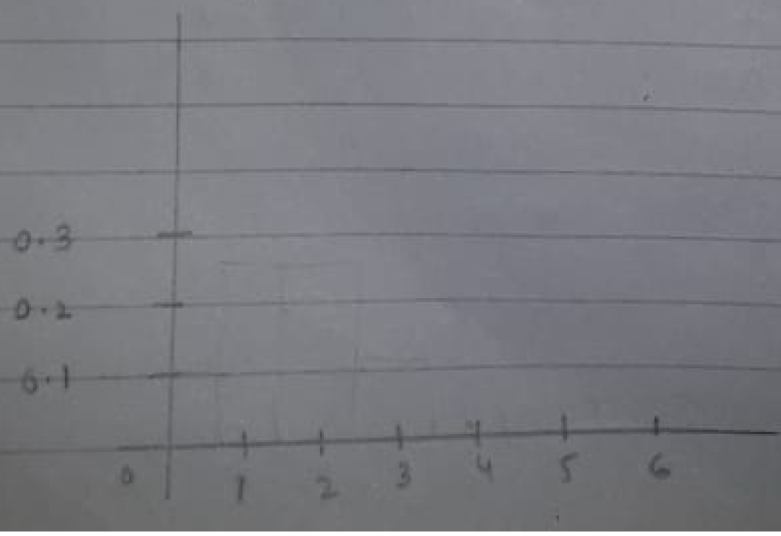
$$P(Y=3 \mid X=2) = \frac{f(3,2)}{g(x=2)} = \frac{0.10}{0.35}$$

**Q** $$f(x) = \frac{e^{-2}\, 2^{x}}{x!} \; ; \; x = 0, 1, 2, 3, 4, 5, 6$$

$f(x) = P(X = x)$

(a) PMF (b) Plot Graph of PMF (c) CDF

| $x$ | $f(x)$ | $F(x) = P(X \le x)$ |
|---|---|---|
| 0 | 0.135 | 0.135 |
| 1 | 0.270 | 0.405 |
| 2 | 0.270 | 0.675 |
| 3 | 0.180 | 0.855 |
| 4 | 0.090 | 0.945 |
| 5 | 0.036 | 0.981 |
| 6 | 0.012 | 0.993 |



→ Continuous Probability Distribution:

- Probability distribution of continuous variable is called Probability Density Function.

- $P(a<X\leq b) = P(a<x<b) + \underbrace{\boxed{P(X=b)}}_{\substack{\text{Probability at a particular} \\ \text{point is zero}}}^{=0} = P(a<x<b)$

- Solved by using Area under the curve.

Condition:

- Function must be non-negative $f(x) \geq 0$
- Upper & lower limit rkh k integrate kren toh answer one ana chahye. $\int_{-\infty}^{\infty} f(x)dx = 1$
- $P(a<x<b) = \int_a^b f(x)dx$

Eg#

$$f(x) = \frac{x^2}{3} \begin{cases} -1<x<2 \\ 0, \text{ else where} \end{cases}$$

(a) $\int_{-1}^{2} \frac{x^2}{3} = \frac{x^3}{3(3)}\Big|_{1}^{2} = \frac{1}{9}(8+1) = \cancel{9}/\cancel{9} = 1$

(b) $P(0\leq X\leq 1) = \int_0^1 \frac{x^2}{3} = \frac{1}{9}$

12/Mar/19



→ CDF for Continuous RV

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt, \text{ for } -\infty < x < \infty$$

$$P(a < X < b) = F(b) - F(a) \quad \text{and} \quad f(x) = dF(x)/dx$$

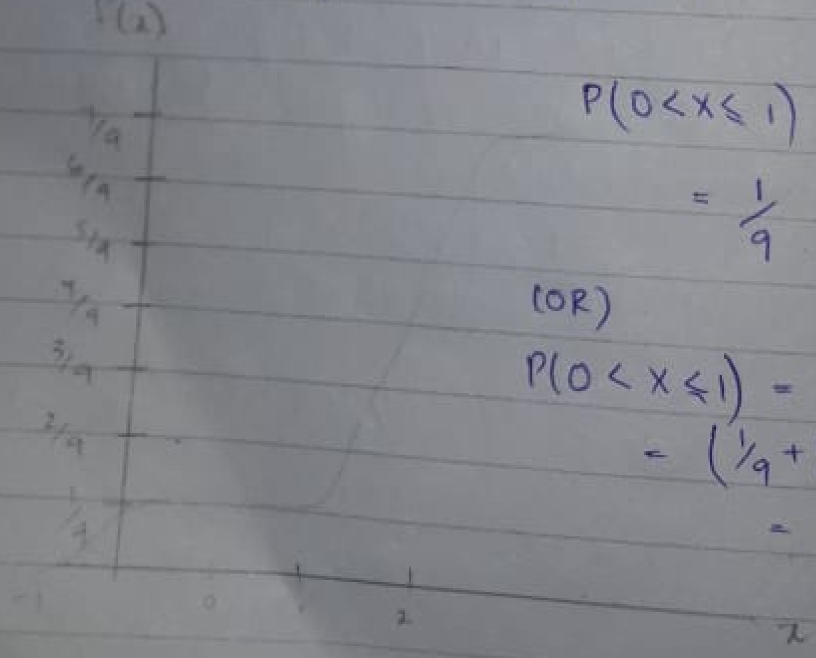
eg#02 $F(x) = ?$

$P(0 < X \leq 1) = ?$

$$f(x) = \begin{cases} x^2/3, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

$$P(X \leq x) = F(x) = \int_{-1}^{x} x^2/3\,dx$$

$$= x^3/9 \Big|_{-1}^{x} = \frac{x^3}{9} + \frac{1}{9}$$



$$P(0 < X \leq 1) = \int_0^1 x^2/3\,dx$$

$$= \frac{1}{9}$$

(OR)

$$P(0 < X \leq 1) = F(1) - F(0)$$

$$= \left(1/9 + 1/9\right) - \left(0/9 + 1/9\right)$$

$$= 1/9$$

Q#3 (a) $P(y) = \begin{cases} 5/8b & , \ 2/5\,b \le y \le 2b \\ 0 & , \text{ elsewhere} \end{cases}$

$$F(y) = \int_{2/5b}^{y} 5/8b \, dy$$

$$= 5y/8b \Big|_{2/5b}^{y} = \frac{5y}{8b} - \frac{5(2/5b)}{8b}$$

$$F(y) = 5y/8b - 1/4.$$

(b) $P(Y \le b)$

$$= F(b) - F(2/5b)$$

$$= \frac{5(b)}{8b} - \frac{1}{4} - \frac{5(2/5b)}{8b} + \frac{1}{4}$$

$$= 5/8 - 1/4. = 3/8$$

eg#04 $f(x) = \begin{cases} 20000/(x+100)^3 & , x > 0 \\ 0, & \text{elsewhere} \end{cases}$

(a) $P(X \ge 200)$

$$= \int_{200}^{\infty} \frac{20000}{(x+100)^3} dx = 20000 \int_{200}^{\infty} (x+100)^{-3} dx$$

$$= 20000 \frac{(x+100)^{-2}}{-2} \Big|_{200}^{\infty} = 1/9$$

(b) $P(80 < x < 120) = \int_{80}^{120} 20000(x+100)^{-3}\,dx = \frac{20000}{-2}\left(\frac{1}{(x+100)^2}\right)_{80}^{120}$

$= -10000\left(\frac{1}{(220)^2} - \frac{1}{(180)^2}\right) = \frac{1000}{9801}$

eq#05 $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 \le x < 2 \\ 0, & \text{elsewhere} \end{cases}$

(a) $P(X < 120) = P(X < 1.2)$

$= \int_0^1 x\,dx + \int_1^{1.2} (2-x)\,dx$

$= \frac{1}{2} + \frac{2}{5} - \frac{11}{50} = \frac{17}{25} = 0.68$

(b) $P(50 < x < 100) = P(0.5 < x < 1)$

$= \int_{0.5}^{1} x\,dx$

$= \left.\frac{x^2}{2}\right]_{0.5}^{1} = \frac{1}{2} - \frac{(0.5)^2}{2}$

$= 0.375$

eg#06 $F(x) = \begin{cases} 0 & x<0 \\ 1-e^{-8x} & x \geq 0 \end{cases}$

12/60

(a) $P(X<0.12) = 1 - e^{-8(0.12)}$
$= 0.617$

(b) $F'(x) = d/dx\, F(x)$
$= d/dx(1 - e^{-8x})$
$= 8e^{-8x}$

$= \int_0^{0.12} 8e^{-8x}\,dx.$

$= 0.617$

eg#07
$f(x) = K(3-x^2) \quad ; -1 \leq x \leq 1$
(a) $K = ?$
(b) $P(X < 1/2)$
(c) $P[|X| < 0.8]$ $= P(X < -0.8) + P(X \geq 0.8)$

(a) $\int_{-1}^{1} K(3-x^2)\,dx = 1 \Rightarrow K(3x - x^3/3)_{-1}^{1} = 1$
$K = 3/16$

(b) $\int_{-1}^{1/2} K(3-x^2)\,dx = \int_{-1}^{1/2} 3/16\,(3-x^2)\,dx = 99/128$

$= 3/16 \left( \int_{-1}^{1/2} 3dx - \int_{-1}^{1/2} x^2 dx \right) =$

82

14/Mar/19

$$P[|x|<0.8] = P(x>0.8) + P(x<-0.8)$$

$$= 1 - P(x<0.8) + P(x<-0.8)$$

$$= 1 - \int_{-1}^{0.8} \frac{3}{16}(3-x^2)dx + \int_{-1}^{-0.8} \frac{3}{16}(3-x^2)dx$$

$$= \frac{41}{250}$$

$$= 0.164$$

eg#08

$$f(x) = \begin{cases} 3x^{-4}, & x>1 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Valid density function - ?

$$= \int_{1}^{\infty} 3x^{-4}dx$$

$$= 3\left.\frac{x^{-3}}{-3}\right|_{1}^{\infty} = 1 \quad \text{hence valid}$$

(b) $F(x) = ?$ (c.d.f)

$$= \int_{1}^{x} 3x^{-4}dx$$

$$= x^{-3}\Big|_{1}^{x} = -x^{-3} + 1$$

$F(x) = 1 - x^{-3}$

(c) $P(x > 4)$ (using cdf)

$= 1 - P(x < 4)$

$= 1 - \int_1^4 3x^{-4}\,dx$

$= 1 - [1 - 4^{-3}]$

$= \frac{1}{64}$

$P(x<4) = F(4) - F(1)$

$P(x>4) = 1 - P(x<4)$

$= 1 - [F(4) - F(1)]$

$=$

(using pdf)

$P(x \cancel{>} 4) = \int_1^4 3x^{-4}\,dx$

$P(x < 4) = 1 - \int_1^4 3x^{-4}\,dx$

$= 1/64$

→ Joint Density Function for Continuous Variable:

1. $f(x,y) \geq 0 \quad \forall (x,y)$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

eg#09

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & , \text{elsewhere} \end{cases}$$

sol (a)

$$= \int_0^1 \int_0^1 \frac{2}{5}(2x+3y)\, dx\, dy$$

$$= \frac{2}{5} \int_0^1 (x^2 + 3xy)_0^1 \, dy$$

$$= \frac{2}{5} \int_0^1 (1 + 3y)\, dy$$

$$= \frac{2}{5} \left[ y + \frac{3y^2}{2} \right]_0^1$$

$$= \frac{2}{5}\left(1 + \frac{3}{2}\right)$$

$$= 1$$

hence it is joint density function.

(b) $P\left(0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}\right)$

sol

$$= \int_{1/4}^{1/2} \int_0^{1/2} \frac{2}{5}(2x+3y)\, dx\, dy$$

$$= \frac{2}{5} \int_{1/4}^{1/2} (x^2 + 3xy)_0^{1/2} \, dy$$

$$= \frac{2}{5} \int_{1/4}^{1/2} \left(\frac{1}{4} + \frac{3}{2}y\right) dy$$

$$= \frac{2}{5}\left(\frac{1}{4}y + \frac{3y^2}{4}\right)_{1/4}^{1/2}$$

$$= \frac{2}{5}\left(\frac{1}{8} + \frac{3}{16} - \frac{1}{16} - \frac{3}{64}\right)$$

$$= 0.08$$

(c) Marginal distribution of x:

(integrate joint distribution w.r.t y).

$$g(x) = \int_0^1 \frac{2}{5}(2x + 3y)\,dy$$

$$= \frac{2}{5}\left(2x + \frac{3}{2}\right) \quad \text{for} \quad 0 \leq x \leq 1$$

Marginal distribution of y:

$$h(y) = \int_0^1 \frac{2}{5}(2x+3y)\,dx$$

$$= \frac{2}{5}(1 + 3y) \quad \text{for} \quad 0 \leq y \leq 1$$

→ Marginal distribution of Continuous Variable:

Marginal distribution of X:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

Marginal distribution of Y:

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Eg #10

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & , 0 \le x \le 1 , 0 \le y \le 1 \\ 0 & , \end{cases}$$

Expected value of x

$$E(x) = \int_0^1 x \cdot g(x)\,dx$$

$$E(y) = \int_0^1 y \cdot h(y)\,dy$$

$$V(x) = E(x^2) - [E(x)]^2 \quad , \quad V(y) = E(y^2) - [E(y)]^2$$

$$E(x^2) = \int_0^1 x^2 g(x)\,dx \quad , \quad E(y^2) = \int_0^1 y^2 \cdot h(y)\,dy$$

Eg #11

$$f(x,y) = \begin{cases} 10xy^2 & , 0 < x < y < 1 \\ 0 , \end{cases}$$

(a) $\underbrace{g(x) = ? \quad , \quad h(y) = ?}_{\text{marginal}} \quad , \quad f(y|x) = ?$ → conditional

$$g(x) = \int_x^1 10xy^2\,dy = 10x \frac{y^3}{3} \Big|_x^1$$

$$g(x) = \frac{10}{3}x - \frac{10x^4}{3}$$

$$g(x) = \frac{10x}{3}(1 - x^3) . \quad \rightarrow \quad 0 < x < 1$$

$$h(y) = \int_0^y 10xy^2\,dx = 10y^2\left(\frac{x^2}{2}\right)_0^y$$

$$h(y) = 5y^4.$$

$$f(y|x) = \frac{f(x,y)}{g(x)}$$ → joint density, marginal density

$$= \frac{10xy^2}{\frac{10x}{3}(1-x^3)}$$

$$f(y|x) = \frac{3y^2}{1-x^3}$$

(b) $P(Y > \frac{1}{2} \mid X = 0.25)$

$$= \int_{1/2}^{1} \frac{3y^2}{1-(0.25)^3}\,dy = \frac{8}{9}$$

Friday
15/Mar/19
eg#12

$$P(x,y) = \begin{cases} x(1+3y^2)/4 & 0<x<2,\ 0<y<3 \\ 0, & \text{else where} \end{cases}$$

Find $g(x)$, $h(y)$, $f(x|y)$, $P(\frac{1}{4} < x < \frac{1}{2} \mid y = \frac{1}{3})$

(a) $g(x) = \int_0^3 x(1+3y^2)/4 \; dy$

$= x/4 \left(y + y^3\right)_0^3 = x/2 \rightarrow 0 < x < 2$

(b) $h(y) = \int_0^2 x(1+3y^2)/4 \; dx$

$= \frac{(1+3y^2)}{4} \int_0^2 x \, dx = \frac{(1+3y^2)}{4} \cdot \frac{x^2}{2} \Big]_0^2$

$= \frac{1+3y^2}{2} \rightarrow 0 < y < 3$

(c) $f(x|y) = f(x,y) / h(y)$

$= \frac{x\cancel{(1+3y^2)}/\cancel{4}}{\cancel{(1+3y^2)}/\cancel{2}}$

$f(x|y) = x/2$

(d) $P(1/4 < x < 1/2 \,|\, y = 1/3)$

$= \int_{1/4}^{1/2} x/2 \, dx$

$= 1/4 \left(1/4 - 1/16\right) = 3/64$

eg #13 $f(x,y) = \begin{cases} 2, & 0 < x \le y < 1 \\ 0 \end{cases}$

(a) Find if X & Y are independent ?

Condition of independence is

$$f(x,y) = g(x) \cdot h(x) \quad \rightarrow (A)$$

$$g(x) = \int_x^1 2\, dy = 2y \Big|_x^1 = 2(1-x)$$

$$h(y) = \int_0^y 2dx = 2x\Big|_0^y = 2y$$

Substitute in eq (A) $f(x,y) = 2(1-x) \cdot 2y$

$$2 \neq 4y(1-x)$$

So $x$ & $y$ are not independent

(b) $P\left(\frac{1}{4} < x < \frac{1}{2} \,\middle|\, y = \frac{3}{4}\right)$

$$f(x|y) = f(x,y) / h(y)$$

$$= \frac{2}{2y} = \frac{1}{y}$$

$$P\left(\tfrac{1}{4} < x < \tfrac{1}{2} \,\middle|\, y = \tfrac{3}{4}\right) = \int_{1/4}^{1/2} \frac{1}{y}\, dx$$

$$= \frac{1}{3}.$$

eg # 14 $X, Y, Z \rightarrow$ joint probability df

$$f(x,y,z) = \begin{cases} kxy^2z, & 0<x, y<1, 0<z<2 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find $k$:

$$\Rightarrow \int_0^2 \int_0^1 \int_0^1 kxy^2z \, dx \, dy \, dz = 1$$ b/c of jpdf

$$\Rightarrow \int_0^2 \int_0^1 \frac{ky^2z}{2} (x^2)_0^1 \, dy \, dz = 1$$

$$\Rightarrow \int_0^2 \int_0^1 ky^2z/2 \, dy \, dz = 1$$

$$\Rightarrow \int_0^2 kz/6 \, (y^3)_0^1 \, dz = 1$$

$$\Rightarrow k/12 \, (z^2)_0^2 = 1$$

$$\Rightarrow k/3 = 1 \quad \Rightarrow \boxed{k = 3}$$

(b) $P(X < 1/4, Y > 1/2, 1 < Z < 2)$

1 → given in question

$$= \int_1^2 \int_{1/2}^1 \int_0^{1/4} 3xy^2z \, dx \, dy \, dz$$

$$= \int_1^2 \int_{1/2}^1 3y^2z/2 \, (x^2)_0^{1/4} \, dy \, dz$$

$$= \int_1^2 \int_{1/2}^1 \frac{3}{32} y^2 z \, dy \, dz$$

$$= \int_1^2 \frac{1}{32} z \left(y^3\right)_{1/2}^{1} dz$$

$$= \frac{1}{32} \cdot \frac{7}{8} \int_1^2 z\, dz$$

$$= \frac{1}{32} \cdot \frac{7}{8} \cdot \frac{1}{2} \left(z^2\right)_1^2$$

$$= 0.0410156$$

(c) Marginal distribution of $g(x,y)$

$g(x,y) \rightarrow$ integral w.r.t 'z'

$$g(x,y) = \int_0^2 3xy^2z\, dz$$

$$= 6xy^2 \qquad 0 < x, y < 1$$

(d) $g(x,z) = \int_0^1 3xy^2z\, dy \quad = xz$

(e) $g(y,z) = \int_0^1 3xy^2z\, dx \quad = \frac{3}{2} y^2 z$

(f) $g(x) = \int_0^2 \int_0^1 3xy^2z\, dy\, dz$

Monday
18/Mar/19

(92)

eg#16 $f(x,y) = \begin{cases} 24xy & 0 \leq x, y \leq 1, \ x+y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$

where X = Turkish, Y = Domestic

ⓐ $P(X \geq 1/2) = \int_0^{1-x} g(x)\,dy \rightarrow$ Ⓐ

$x+y \leq 1$
$y \leq 1-x$

$g(x) = \int_0^{1-x} f(x,y)\,dy = \int_0^{1-x} 24xy\,dy$

$= \frac{24xy^2}{2}\Big|_0^{1-x}$

$g(x) = 12x(1-x)^2$

ghalat krwale hain

$P(X \geq 1/2) = \int_0^{1-x} 12x(1-x)^2\,dy$

$= 6x(1-x)^2 \cdot (y)_0^{1-x}$

$P(X \geq 1/2) = 6x(1-x)^3$

ⓑ $h(y) = \int_0^{1-y} f(x,y)\,dx$

$= \int_0^{1-y} 24xy\,dx = 12y\,(x^2)_0^{1-y}$

$h(y) = 12y(1-y)^2 \rightarrow 0 < y < 1$

) $P(X \leq 1/8 \mid y = 3/4)$

h/c conditional prob find krlen

$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{\cancel{24}\,xy}{\cancel{12}\cancel{y}(1-y)^2} = \frac{2x}{(1-y)^2}$

$$P(x \le 1/8 \mid y = 3/4) = \int_0^{1/8} \frac{2x}{(1-y)^2}\,dx = \int_0^{1/8} \frac{2x}{(1-3/4)^2}\,dx = \int_0^{1/8} 32x\,dx$$

$$= 16\,(x^2)\Big|_0^{1/8} = 16/64 = 1/4$$

eg#17 Expectation = ?

| X \$ | $P(X = x)$ | $x \cdot P(x)$ |
|---|---|---|
| 0\$ | $0.3 \times 0.6$ | |
| 1000\$ | $0.7 \times 0.6$ | |
| 1500\$ | $0.3 \times 0.4$ | |
| 2500\$ | $0.7 \times 0.4$ | |
| Total | | 1300\$ |

$$E(x) = \Sigma\, x \cdot P(x) = \$1300$$

eg#18 . Joint Expectation = ?

$$E(x, y) = \sum_{x=0}^{2} \sum_{y=0}^{2} xy\, f(x, y)$$

$$= (0)(0)(3/28) + (0)(1)(9/28) + (0)(2)(3/28) + 0$$
$$+ (1)(1)(3/14) + (1)(2)(0) + 0 + 0 + 0 .$$

$$E(x, y) = 3/14$$

$$E(x) = x\,g(x)$$
$$= 0\,(5/14) + 1\,(15/28) + 2\,(3/28) = 21/28$$

$E(y) = y \cdot h(y)$

$= 0(^{15}/_{28}) + 1(^{3}/_{7}) + 2(^{1}/_{28}) = ^{1}/_{2}$

- Co-variance of X & Y

Covariance $(X, Y) = E(X, Y) - E(X)E(Y)$

$= ^{3}/_{14} - ^{21}/_{28}(^{1}/_{2})$

$= -^{9}/_{56}$

.- Co-Relation $= \dfrac{\text{Co-variance}(x, y)}{\sigma x \, \sigma y}$

measure of linear association b/w x & y

$V(x) = \delta^2 x = E(x^2) - [E(x)]^2$

$V(y) = \delta^2 y = E(y^2) - [E(y)]^2$

19/Mar/19

eg#19 Find $E(Y/X)$, $E(XY)$, $E(X)$, $E(y)$, $V(x)$, $V(y)$, Covariance$(X, Y)$, Corelation.

$f(x,y) = \begin{cases} x(1+3y^2)/4 , & 0<x<2, 0<y<1 \\ 0 , & \text{elsewhere} \end{cases}$

Sol (i) $E(XY) = \sum_{x=0}^{2} \sum_{y=0}^{1} xy\, f(x,y)$

$= \int_0^1 \int_0^2 \frac{x^2 y(1+3y^2)}{4} \, dx\, dy \Rightarrow \int_0^1 \left( x^3y/_{12} + 3x^2y^2/_{12} \right) dy$

$E(X, Y) = ^{5}/_{6}$

(ii) $E(x) = \int_0^2 x \cdot g(x)\, dx$

$g(x) = \int_0^1 x(1+3y^2)/4\, dy = x/2$

$E(x) = \int_0^2 x \cdot x/2\, dx = \left.\frac{x^3}{6}\right]_0^2 = 8/6$

$E(x) = 4/3$

(iii) $E(y) = \int_0^1 y \cdot h(y)\, dy$

$h(y) = \int_0^2 x(1+3y^2)/4\, dx = \left( x^2/8 + 3x^2y^2/8 \right)^2$

$= 1/2 + 3/2\, y^2$

$E(y) = \int_0^1 y \cdot \left( 1/2 + 3/2\, y^2 \right) dy$

$= \int_0^1 \left( y/2 + 3/2\, y^3 \right) dy =$

$E(y) = 5/8$

(iv) $V(x) = \delta^2 x = E(x^2) - [E(x)]^2$

$= 2/9$

(v) $V(y) = \delta^2 y = E(y^2) - [E(y)]^2$

$= 73/960$

(vi) Covariance $(X, Y) = E(X, Y) - E(X)E(Y)$

$$= \frac{5}{6} - \frac{4}{3}\left(\frac{5}{8}\right)$$

Covariance $= 0$

hence $X, Y$ are not linearly related.

(vii) Correlation $= \dfrac{Cov(X, Y)}{\sigma_x \sigma_y} = 0$

## Discrete Probability Distribution.

* Bernoulli Process :
  1. Experiment consists of repeated trials.
  2. Each trial results in an outcome either 'success' or 'failure'
  3. Probability of success, remains constant from trial to trial.
  4. 

Binomial Distribution: Sampling with replacement

X no. of successes in n Bernoulli trials called a Binomial Random Variable

Probability distribution of discrete random variable is called Binomial Distribution

$B(n,p) = {}^{n}C_{r} \; p^{r}(1-p)^{n-r}$

Eg#01 Coin tossed 3 times. Probability of getting exactly 2 heads

$n = 3 \;,\; r = 2 \;,\; P = 1/2$

$P(x=2) = {}^{3}C_{2} \left(1/2\right)^{2} \left(1 - 1/2\right)^{2-1}$

$= 3/8$

25/March/19

Eq #05 A coin is tossed 3 times
Find $\bar{X}$ & Var for no of H.

$\bar{X} = np \Rightarrow 3(1/2) = 1.5$

$Var = npq \Rightarrow 3(1/2)(1/2) = 3/4 \Rightarrow 0.75$.

$S = \{HHH, HTH, THT, TTH, HHT, THH, HTT, TTT\}$

| $x$ | $P(X)$ | $xP(x)$ | $x^2P(x)$ |
|---|---|---|---|
| 0 | 1/8 | 0 | 0 |
| 1 | 3/8 | 3/8 | 3/8 |
| 2 | 3/8 | 6/8 | 12/8 |
| 3 | 1/8 | 3/8 | 9/8 |
| Total | 8/8 | 12/8 | 24/8 |

$E(x) = \sum x P(x) = 12/8 = 1.5$

$V(x) = E(x^2) - [E(x)]^2 = 24/8 - (12/8)^2 = 0.75$

* Multinomial Experiments:

(1) Trials in an experiment has more than two outcomes.

eg # 52 playing cards, rolling a dice.

* Multinomial Distribution:

$$P(X) = \frac{n!}{x_1! \cdot x_2! \cdots x_k!} \cdot p_1^{x_1} \cdot p_2^{x_2} \cdots p_k^{x_k}$$

$\because p_1 + p_2 + \cdots + p_k = 1$

$\because x_1 + x_2 + \cdots + x_k = n$

eg#08

leisure activities; movie 50%, dinner & play 30%, shopping 20%

3 movie, 1 dinner & play, 1 shopping

$$P(3,1,1) = \frac{5!}{3!\,1!\,1!} (0.50)^3 (0.30)^1 (0.20)^1$$

$$= 3/20 = 0.15$$

100

eg#09.

$$P(2,4,1,1) = \frac{8!}{2!\,4!\,1!\,1!}(0.3)^2(0.5)^4(0.15)^1(0.05)^1$$

$$= 0.035.$$

eg#10

$$P(3,3,1,2) = \frac{9!}{3!\,3!\,1!\,2!}(0.4)^3(0.2)^3(0.3)^1(0.1)^2$$

$$= 0.077.$$

* Hypergeometric Experiment:

Sampling without replacement

- Result can be Success or Failure.
- Probability of success changes on each trial.
- Successive trials are dependent.
- Experiment is repeated fixed number of time

$$P(X) = \frac{{}^{a}C_{x} \cdot {}^{b}C_{n-x}}{{}^{a+b}C_{n}}$$

eg#11. 10 apply for job.

3 are selected at random.

Probability that all 3 are college graduates?

$a = 5$ , $b = 5$ (non-graduate) , $a+b = 10$.
$x = 3$ , $n = 3$

$$P(X=3) = {}^5C_3 \cdot {}^5C_0 / {}^{10}C_3$$
$$= 1/12 = 0.08.$$

eg#12

| a | b | a+b |
|---|---|---|
| 2 | 8 | = 10 |
| ↓ | ↓ | ↓ |
| $x = 1$ | 4 $(n-x)$ | $n = 5$ |

eg#13

| a | b | a+b |
|---|---|---|
| 3 | 9 | 12 |
| | | ↓ |
| | | $n = 3$ |

$$P(\text{rejecting a lot}) = 1 - P(X=0)$$
$$= 1 - \left({}^3C_0 \cdot {}^9C_3 / {}^{12}C_3\right)$$
$$= 0.618.$$

Geometric Distribution:

- Outcomes → success / failure.
- Probability of success is same for each experiment.
- Each experiment is independent.
- Experiment is repeated a variable number of times

$$g(x;p) = pq^{x-1} \quad , x = 1, 2, 3.$$

Mean & variance

$$u = 1/p \quad , \quad \delta^2 = (1-p)/p^2$$

eg#14. On average, 1 in every 100 items is defective. Probability that fifth item inspected is first defective item found?

$$x = 5 \quad , \quad p = 1/100$$

$$P(x=5) = (0.01)(1-0.01)^{5-1}$$

$$= 9.60 \times 10^{-3}$$

eg#15 $P = 0.85$ , Probability → pass on 3rd trial

$$P(X=3) = (0.85)(1-0.85)^2$$
$$= 0.0191$$

eg#16 $P = 0.05$ , $X = 5$

$$P(X=5) = (0.05)(1-0.05)^4$$
$$= 0.040$$

* Poisson Distribution:

Frequency Data, Count Data

- Average number of successes.
- Time interval region is known.

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

∴ $X$ = no. of times event occur

$\lambda$ = mean of $X$ .

eg#17 Radioactive Particles

$\lambda = 4$ , $x = 6$

$$P(x=6) = \frac{+4^6 \times e^{-4}}{6!}$$

$= +0.104$

eg#18 Oil tankers arrival

$\lambda = 10$ , $x = 15$

$P(x \geq 15) = 1 - P(x \leq 15)$

$= 1 - 0.9513$

$= 0.0487.$

eg#19 Typographical Errors

$x = 3$

$\lambda = 200/500 = 0.4$

$$P(x=3) = e^{-0.4} \frac{(0.4)^3}{3!}$$

$= 0.0071$

eg#20

(a) $P(x \leq 3) = P(x=0) + \dots + P(x=3)$

(b) $P(x \geq 3) = 1 - P(x \leq 3)$

(c) $P(x \geq 5) = 1 - P(x \leq 4)$

eg # 21

$\lambda = np < 5 \rightarrow$ Binomial & Poisson are same.

Q. $2\% \rightarrow 200$

$P(x=5) = ?$

$\lambda = np = 200(0.02) = 4$

$= \frac{e^{-4} 4^5}{5!}$

March/19



# Miscellaneous Problems :-

1) 1.5 customers arrive per minute.

Poisson

a) Atmost 4 will arrive

$P(X \leq 4) = \sum_{x=0}^{4} e^{-1.5}(1.5)^x / \underline{x} = 0.98$

(b) At least 3 will arrive during an interval of 2 minutes.

$P(x \geq 3) = 1 - P(X \leq 2)$

$= 1 - \sum_{x=0}^{2} e^{-3} 3^x / \underline{x} = 0.576.$

(c) Exactly 15 will arrive during an interval of 6 minutes.

$P(X = 15) = e^{-6(1.5)}(6(1.5))^{15} / 15! = 0.019$

2) 5.12 $P(X<4) = \sum_{x=0}^{3} ({}^{9}C_x) p^x q^{n-x}$

$= \sum_{x=0}^{3} \binom{9}{x}(0.25)^x(0.75)^{9-x}$

$= 0.83.$

3) 5.3 (a) $f(x) = 1/10$ for $x = 1, \ldots, 10$

(b) $P(X<4) = 3/10$

4 0.9 calls per minute

(a) $P(X \geq 1) = 1 - P(X = 0) = 0.59$.

(b) $P(X = 2) = 0.2677$.

(c) $P(X \leq 2) = 0.30$.

5 5.100 2 LM, 1 AP, 2 = 5

$\downarrow$ 2 $\quad \downarrow$ 2

$P(LM) = ({}^2C_2) / {}^5C_2$

$P = ({}^2C_1)({}^1C_1) /$

6 5.97 (a) $1 - (0.2)^3 \rightarrow$

Binomial (b) $P(X = 1) = {}^3C_1 (0.8)^1 (0.2)^2$.

(c) $P(X \geq 2) = P(X = 2) + P(X = 3)$

$= 0.872$