Chapter No. 04

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THE KARNAUGH MAP

- A Karnaugh map provides a systematic method for simplifying Boolean expressions and produce the simplest SOP or POS expression possible, known as the minimum expression.

- Karnaugh map is an array of cells in which each cell represents a binary value of the input variables.

- Karnaugh maps can be used for expressions with two, three, four, and five variables.

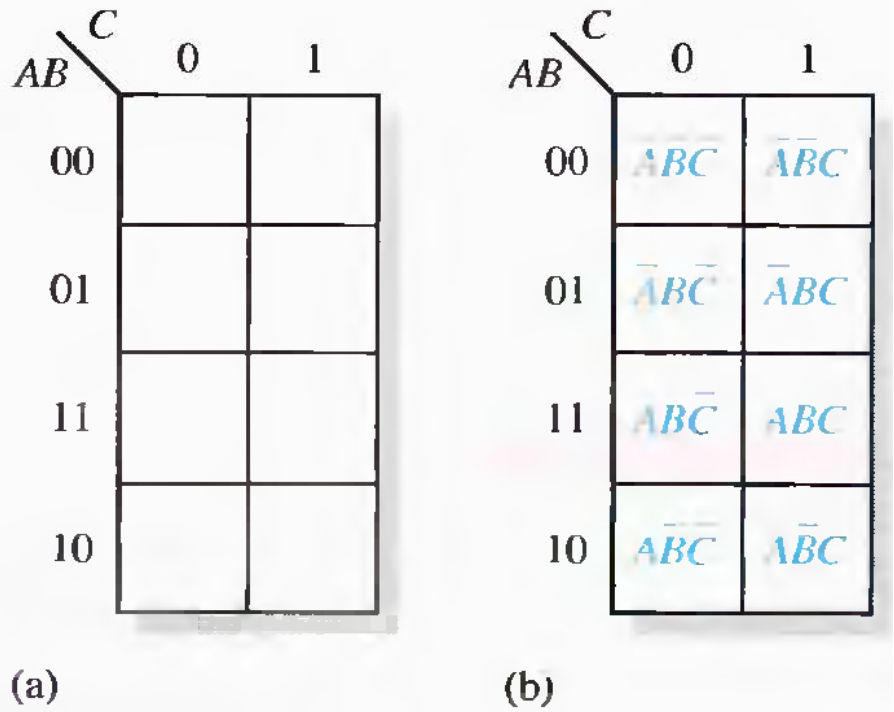
The 3-Variable Karnaugh Map

- For three variables, the number of cells is $2^3 = 8$. For four variables, the number of cells is $2^4 = 16$.

- The value of a given cell is the binary values of A and B at the left in the same row combined with the value of C at the top in the same column.

FIGURE 4–21

A 3-variable Karnaugh map showing product terms.



The 4-Variable Karnaugh Map

- The 4-variable Karnaugh map is an array of sixteen cells.
- The value of a given cell is the binary values of A and B at the left in the same row combined with the binary values of C and D at the top in the same column.

Mapping a Standard SOP Expression

- For an SOP expression in standard form, a 1 is placed on the Karnaugh map for each product term in the expression.
- Each 1 is placed in a cell corresponding to the value of a product term.
- For example, for the product term $A\overline{B}C$, , a 1 goes in the 101 cell on a 3-variable map.

- The following steps and the illustration in Figure 4-24 show the mapping process:
  - Step 1. Determine the binary value of each product term in the standard SOP expression.
  - Step 2. As each product term is evaluated, place a 1 on the Karnaugh map in the cell having the same value as the product term.

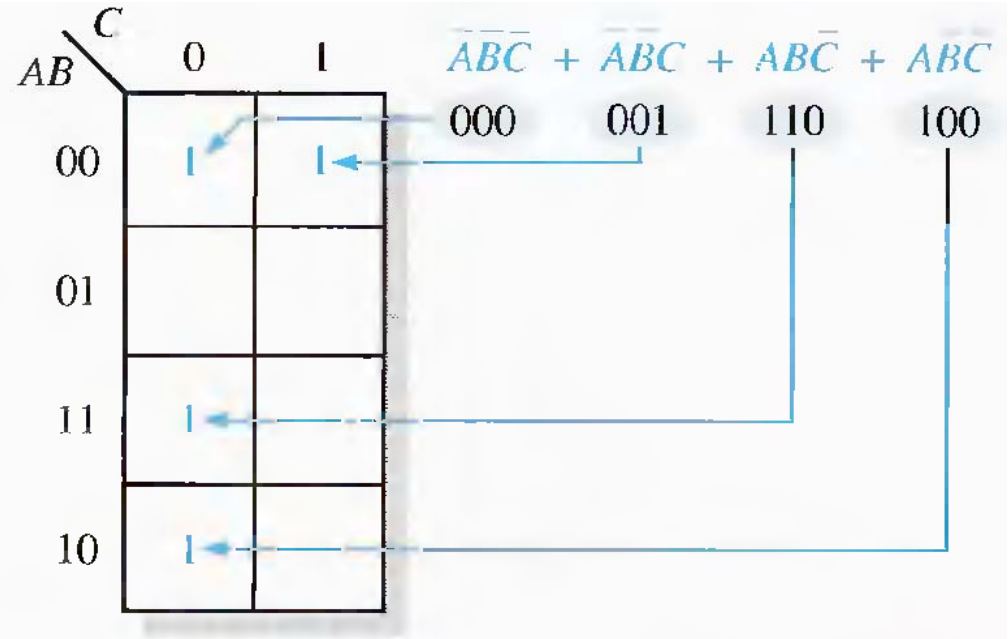


FIGURE 4–24

Example of mapping a standard SOP expression.

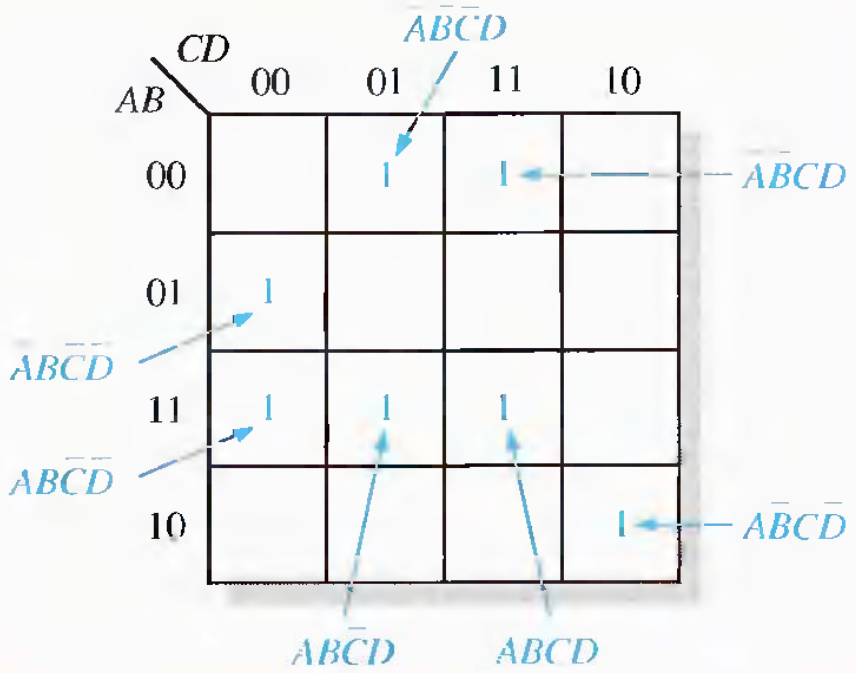
Map the following standard SOP expression on a Karnaugh map:

$$\overline{A}\,\overline{B}CD + \overline{A}B\overline{C}\,\overline{D} + AB\overline{C}D + ABCD + AB\overline{C}\,\overline{D} + \overline{A}\,\overline{B}\,\overline{C}D + A\overline{B}C\overline{D}$$

Solution Evaluate the expression as shown below. Place a 1 on the 4-variable Karnaugh map in Figure 4–26 for each standard product term in the expression.

$$\overline{A}\,\overline{B}CD + \overline{A}B\overline{C}\,\overline{D} + AB\overline{C}D + ABCD + AB\overline{C}\,\overline{D} + \overline{A}\,\overline{B}\,\overline{C}D + A\overline{B}C\overline{D}$$
$$0011 \quad 0100 \quad 1101 \quad 1111 \quad 1100 \quad 0001 \quad 1010$$

▶ **FIGURE 4–26**



Karnaugh Map Simplification of SOP Expressions

The process that results in an expression containing the fewest possible terms with the fewest possible variables is called **minimization.** After an SOP expression has been mapped, a minimum SOP expression is obtained by grouping the 1s and determining the minimum SOP expression from the map.

Grouping the 1s You can group 1s on the Karnaugh map according to the following rules by enclosing those adjacent cells containing 1s. The goal is to maximize the size of the groups and to minimize the number of groups.

1. A group must contain either 1, 2, 4, 8, or 16 cells, which are all powers of two. In the case of a 3-variable map, $2^3 = 8$ cells is the maximum group.
2. Each cell in a group must be adjacent to one or more cells in that same group. but all cells in the group do not have to be adjacent to each other.
3. Always include the largest possible number of 1s in a group in accordance with rule 1.
4. Each 1 on the map must be included in at least one group. The 1s already in a group can be included in another group as long as the overlapping groups include noncommon 1s.

EXAMPLE 4–27

Determine the product terms for each of the Karnaugh maps in Figure 4–32 and write the resulting minimum SOP expression.

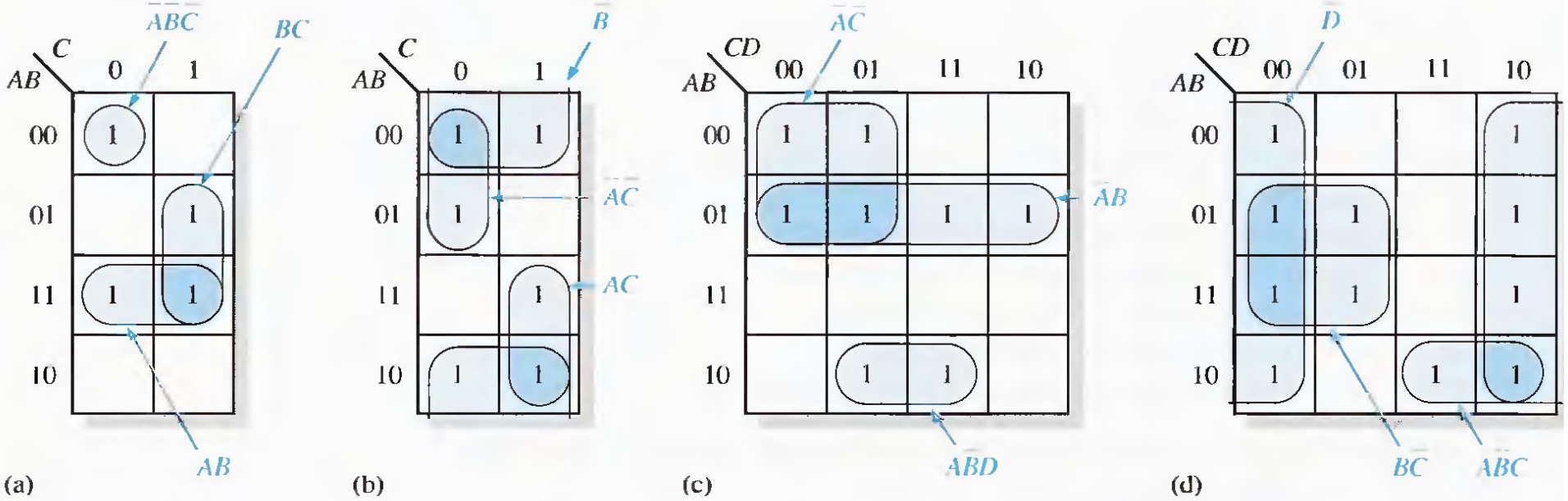


FIGURE 4–32

Solution The resulting minimum product term for each group is shown in Figure 4–32. The minimum SOP expressions for each of the Karnaugh maps in the figure are

(a) $AB + BC + \overline{A}\,\overline{B}\,\overline{C}$ (b) $\overline{B} + \overline{A}\,\overline{C} + AC$

(c) $A\overline{B} + \overline{A}\,\overline{C} + A\overline{B}D$ (d) $\overline{D} + A\overline{B}C + B\overline{C}$

Determining the Minimum SOP Expression from the Map

1. Group the cells that have 1's. Each group of cells containing 1's creates one product term composed of all variables that occur in only one form (either uncomplemented or complemented) within the group. Variables that occur both uncomplemented and complemented within the group are eliminated. These are called contradictory variables.

2. Determine the minimum product term for each group.
   - a. For a 3-variable map:
     (1) A 1-cell group yields a 3-variable product term
     (2) A 2-cell group yields a 2-variable product term
     (3) A 4-cell group yields a 1-variable term
     (4) An 8-cell group yields a value of 1 for the expression

b. For a 4-variable map:

(1) A 1-cell group yields a 4-variable product term

(2) A 2-cell group yields a 3-variable product term

(3) A 4-cell group yields a 2-variable product term

(4) An 8-cell group yields a 1-variable term

(5) A 16-cell group yields a value of 1 for the expression

3. When all the minimum product terms are derived from the Karnaugh map, they are summed to form the minimum SOP expression.

Use a Karnaugh map to minimize the following standard SOP expression:

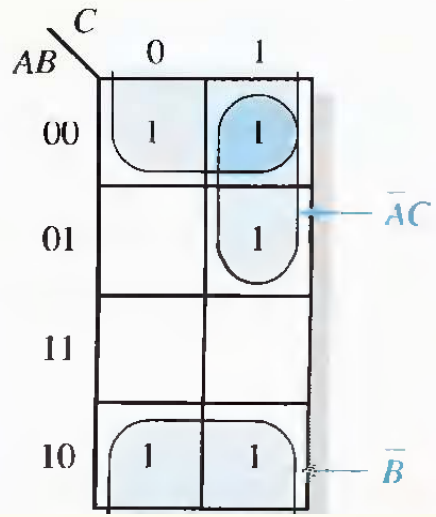
$$A\overline{B}C + \overline{A}BC + \overline{A}\,\overline{B}C + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C}$$

Solution The binary values of the expression are

$$101 + 011 + 011 + 000 + 100$$

Map the standard SOP expression and group the cells as shown in Figure 4–33.

FIGURE 4–33



$$\overline{B} + \overline{A}C$$

Keep in mind that this minimum expression is equivalent to the original standard expression.

Related Problem Use a Karnaugh map to simplify the following standard SOP expression:

$$X\overline{Y}Z + XY\overline{Z} + \overline{X}YZ + \overline{X}Y\overline{Z} + X\overline{Y}\,\overline{Z} + XYZ$$

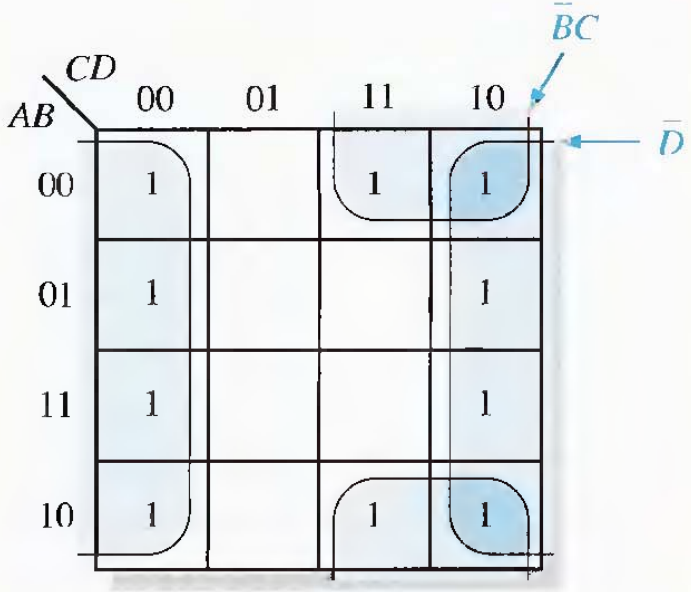
Use a Karnaugh map to minimize the following SOP expression:

$$\overline{B}\,\overline{C}\,\overline{D} + \overline{A}B\overline{C}\,\overline{D} + AB\overline{C}\,\overline{D} + \overline{A}\,\overline{B}CD + A\overline{B}CD + \overline{A}\,\overline{B}C\overline{D} + \overline{A}BC\overline{D} + ABC\overline{D} + A\overline{B}C\overline{D}$$

Solution The first term $\overline{B}\,\overline{C}\,\overline{D}$ must be expanded into $A\overline{B}\,\overline{C}\,\overline{D}$ and $\overline{A}\,\overline{B}\,\overline{C}\,\overline{D}$ to get the standard SOP expression, which is then mapped; and the cells are grouped as shown in Figure 4–34.

► **FIGURE 4–34**

$$\overline{D} + \overline{B}C$$



"Don't Care" Conditions

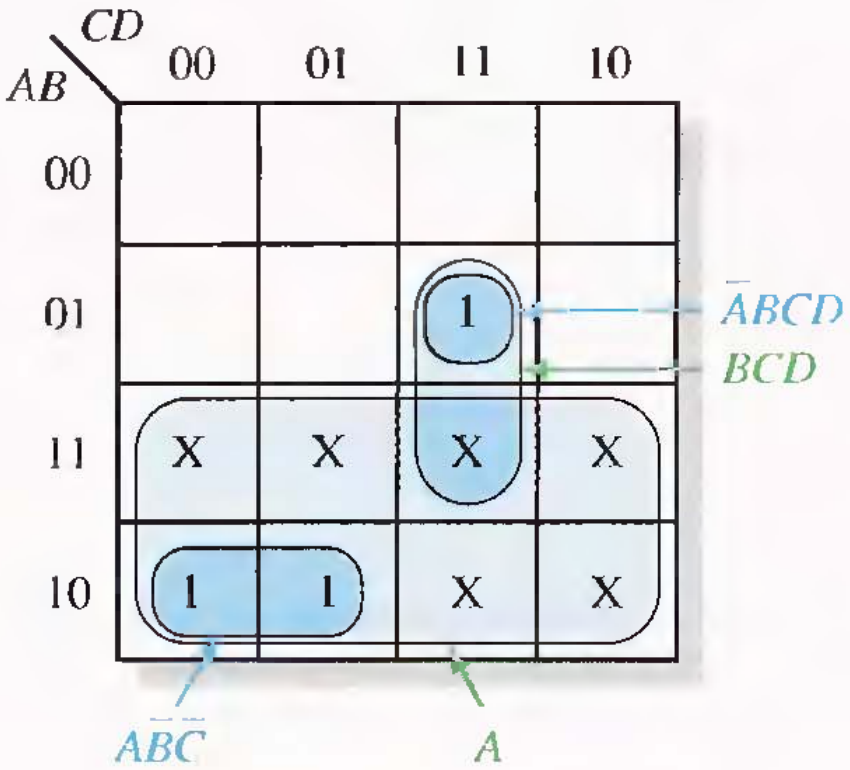
- Sometimes a situation arises in which some input variable combinations are not allowed.
- For example, in the BCD code, there are six invalid combinations: 1010, 1011, 1100, 1101, 1110, and 1111. Since these unallowed states will never occur in an application involving the BCD code, they can be treated as "don't care" terms with respect to their effect on the output.
- The "don't care" terms can be used to advantage on the Karnaugh map.
- Figure 4-36 shows that for each "don't care" term, an X is placed in the cell.
- When grouping the 1's, the Xs can be treated as 1's to make a larger grouping or as 0's if they cannot be used to advantage. The larger a group, the simpler the resulting term will be.

Inputs				Output
A	B	C	D	Y
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	X
1	0	1	1	X
1	1	0	0	X
1	1	0	1	X
1	1	1	0	X
1	1	1	1	X

(a) Truth table

◀ FIGURE 4–36

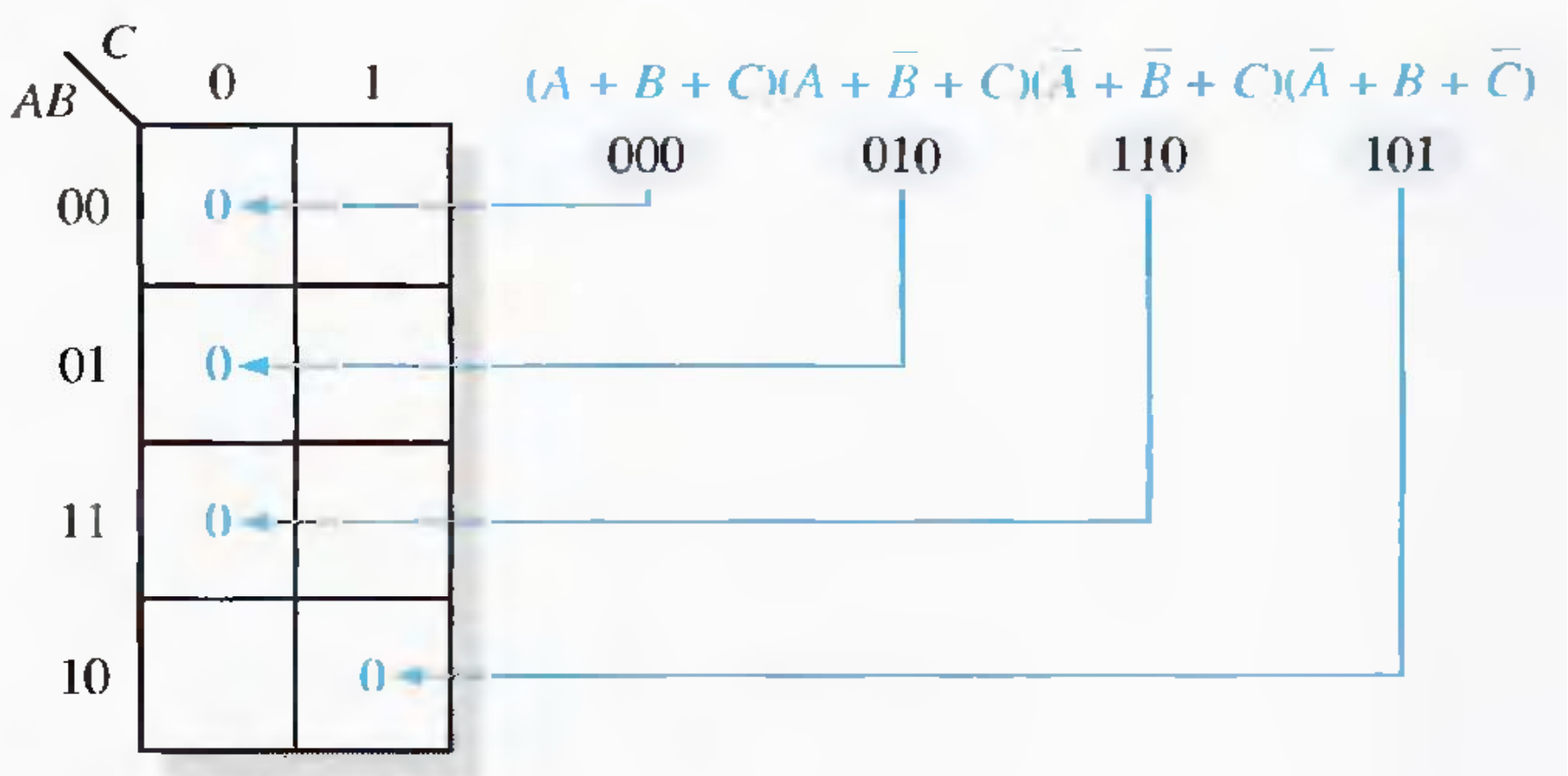
Example of the use of "don't care" conditions to simplify an expression.



(b) Without "don't cares" $Y = A\bar{B}\bar{C} + \bar{A}BCD$
With "don't cares" $Y = A + BCD$

KARNAUGH MAP POS MINIMIZATION

- With POS expressions, 0's representing the standard sum terms are placed on the Karnaugh map instead of 1's.



EXAMPLE 4–30

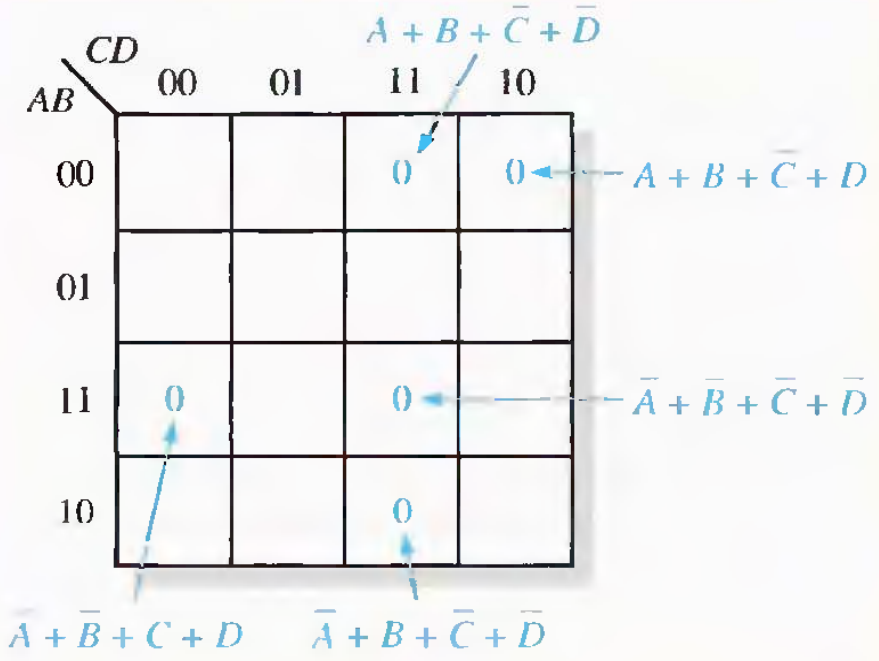
Map the following standard POS expression on a Karnaugh map:

$$(\overline{A} + \overline{B} + C + D)(\overline{A} + B + \overline{C} + \overline{D})(A + B + \overline{C} + D)(\overline{A} + \overline{B} + \overline{C} + \overline{D})(A + B + \overline{C} + \overline{D})$$

Solution Evaluate the expression as shown below and place a 0 on the 4-variable Karnaugh map in Figure 4–38 for each standard sum term in the expression.

$$(\overline{A} + \overline{B} + C + D)(\overline{A} + B + \overline{C} + \overline{D})(A + B + \overline{C} + D)(\overline{A} + \overline{B} + \overline{C} + \overline{D})(A + B + \overline{C} + \overline{D})$$
$$1100 \qquad 1011 \qquad 0010 \qquad 1111 \qquad 0011$$

▶ **FIGURE 4–38**



Use a Karnaugh map to minimize the following standard POS expression:

$$(A + B + C)(A + B + \overline{C})(A + \overline{B} + C)(A + \overline{B} + \overline{C})(\overline{A} + \overline{B} + C)$$

Also, derive the equivalent SOP expression.

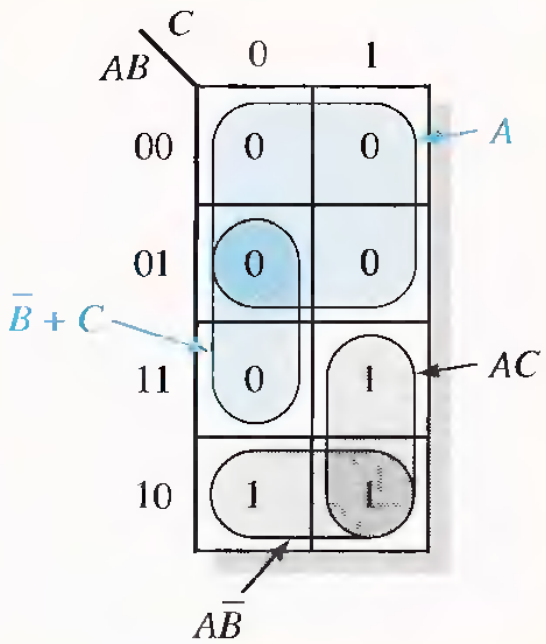
Solution The combinations of binary values of the expression are

$$(0 + 0 + 0)\,(0 + 0 + 1)\,(0 + 1 + 0)\,(0 + 1 + 1)\,(1 + 1 + 0)$$

Map the standard POS expression and group the cells as shown in Figure 4–39.

FIGURE 4–39

$$A(\overline{B} + C)$$



Grouping the 1s as shown by the gray areas yields an SOP expression that is equivalent to grouping the 0s.

$$AC + A\overline{B} = A(\overline{B} + C)$$

Use a Karnaugh map to minimize the following POS expression:

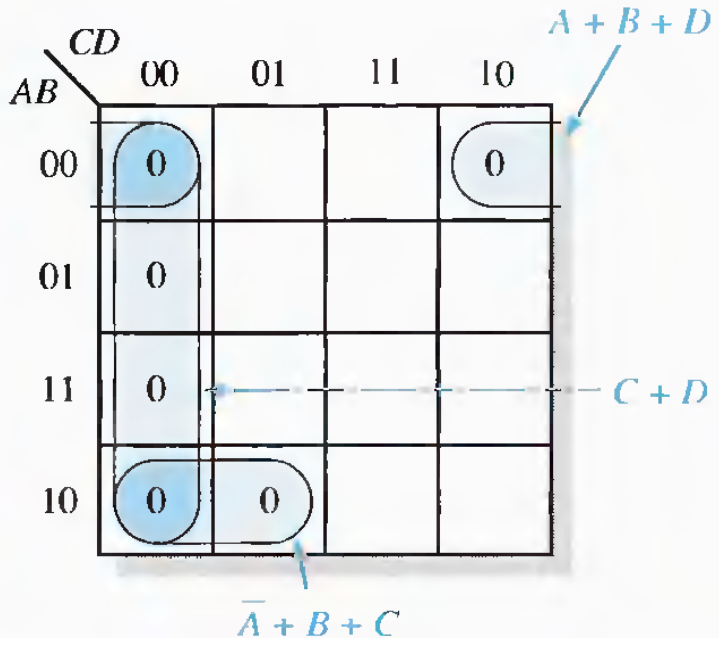
$$(B + C + D)(A + B + \overline{C} + D)(\overline{A} + B + C + \overline{D})(A + \overline{B} + C + D)(\overline{A} + \overline{B} + C + D)$$

Solution The first term must be expanded into $\overline{A} + B + C + D$ and $A + B + C + D$ to get a standard POS expression, which is then mapped; and the cells are grouped as shown in Figure 4–40. The sum term for each group is shown and the resulting minimum POS expression is

$$(C + D)\,(A + B + D)\,(\overline{A} + B + C)$$

Keep in mind that this minimum POS expression is equivalent to the original standard POS expression.

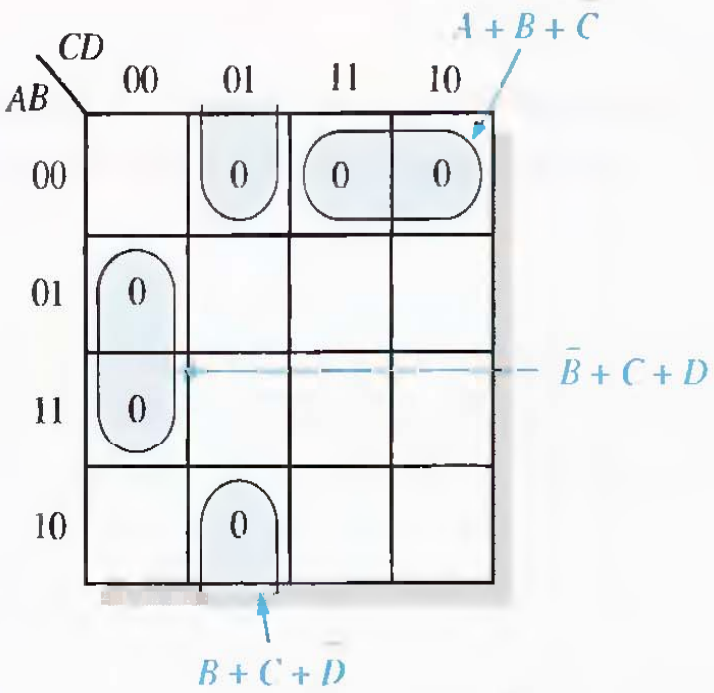
FIGURE 4–40



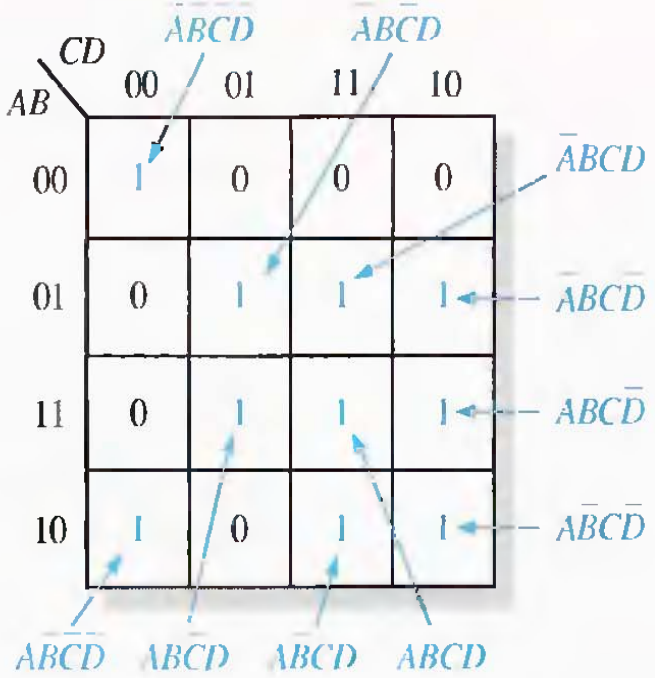
Converting Between POS and SOP Using the Karnaugh Map

Using a Karnaugh map, convert the following standard POS expression into a minimum POS expression, a standard SOP expression, and a minimum SOP expression.

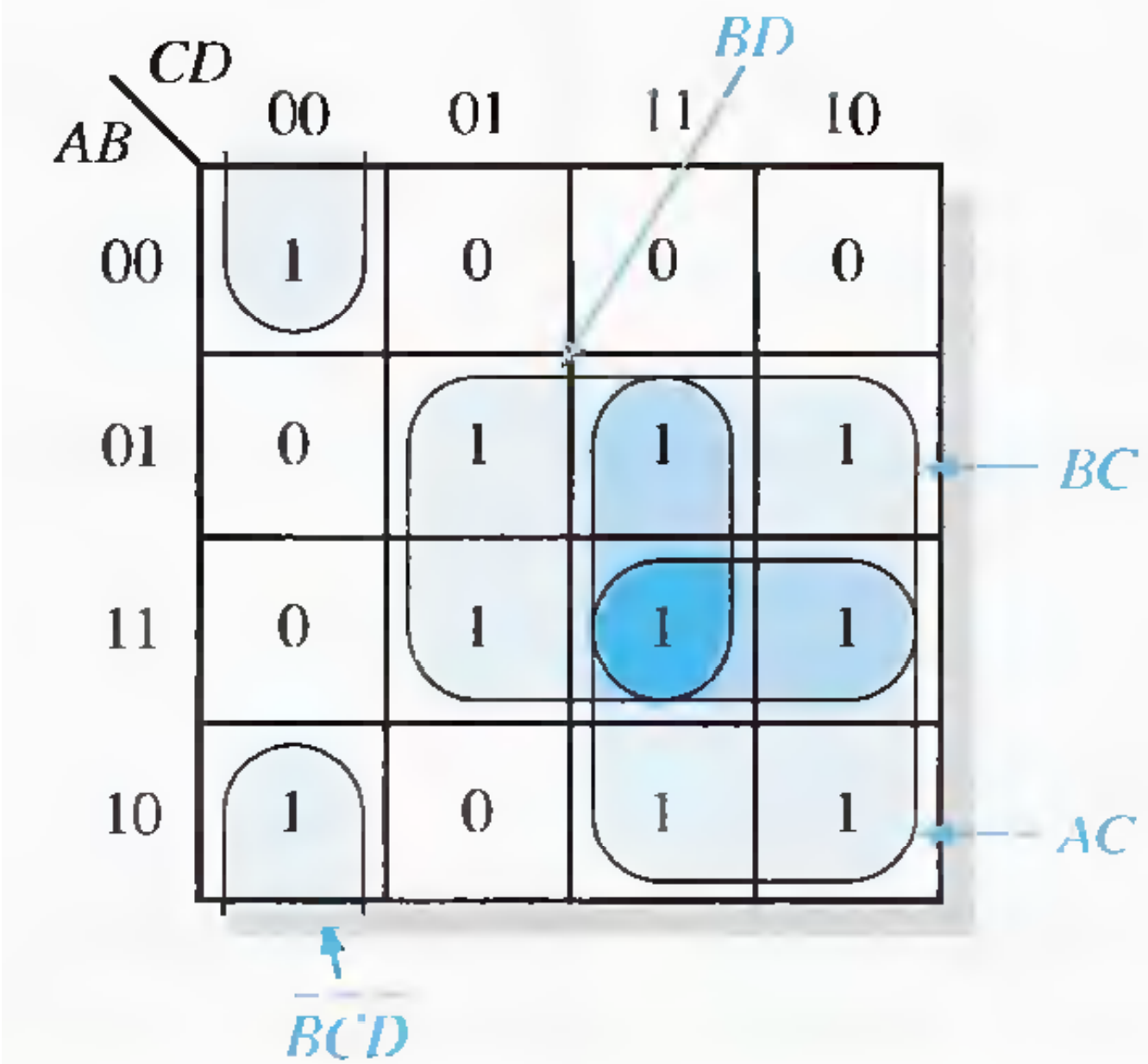
$$(\overline{A} + \overline{B} + C + D)(A + \overline{B} + C + D)(A + B + C + \overline{D})$$
$$(A + B + \overline{C} + \overline{D})(\overline{A} + B + C + \overline{D})(A + B + \overline{C} + D)$$



(a) Minimum POS: $(A + B + C)(\overline{B} + \overline{C} + D)(B + C + \overline{D})$



(b) Standard SOP:
$\overline{A}\overline{B}\overline{C}\overline{D} + \overline{A}\overline{B}\overline{C}D + \overline{A}B\overline{C}D + \overline{A}BC\overline{D} + ABC\overline{D} + A\overline{B}C\overline{D} +$
$A\overline{B}\overline{C}\overline{D} + AB\overline{C}D + A\overline{B}CD + ABCD$



(c) Minimum SOP: $AC + BC + BD + \overline{B}\overline{C}\overline{D}$

FIVE-VARIABLE KARNAUGH MAPS

A Karnaugh map for five variables (*ABCDE*) can be constructed using two 4-variable maps with which you are already familiar. Each map contains 16 cells with all combinations of variables *B*, *C*, *D*, and *E*. One map is for $A = 0$ and the other is for $A = 1$, as shown in Figure 4–42.

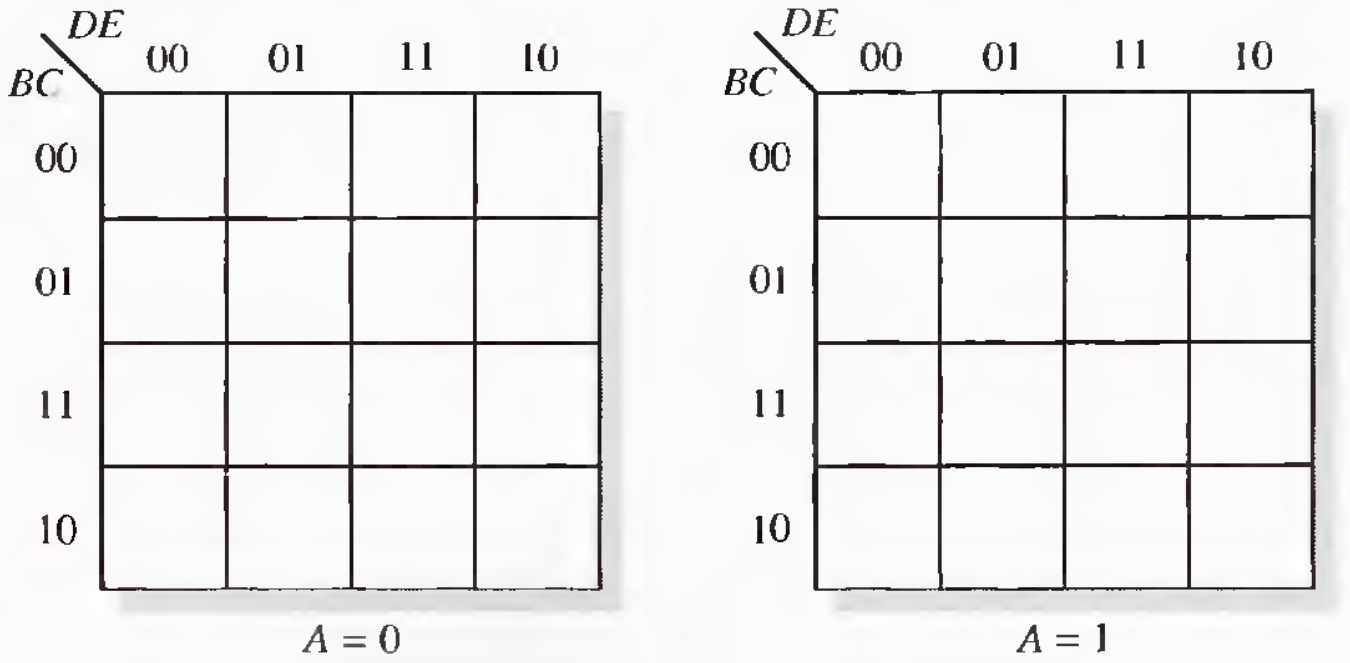


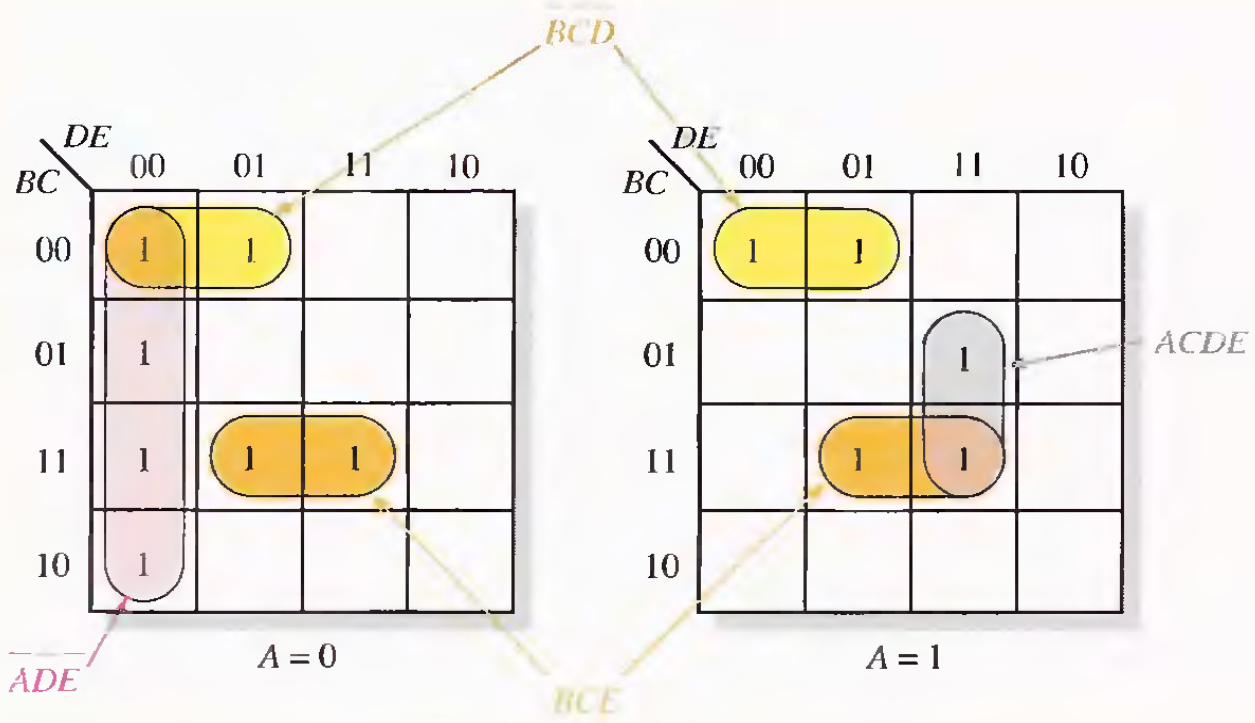
FIGURE 4–42
A 5-variable Karnaugh map.

Use a Karnaugh map to minimize the following standard SOP 5-variable expression:

$$X = \overline{A}\,\overline{B}\,\overline{C}\,\overline{D}\,\overline{E} + \overline{A}\,\overline{B}\,\overline{C}\,\overline{D}\,E + \overline{A}\,\overline{B}\,C\,\overline{D}\,\overline{E} + \overline{A}\,\overline{B}\,C\,\overline{D}\,E + \overline{A}\,B\,\overline{C}\,\overline{D}\,\overline{E} + \overline{A}\,B\,\overline{C}\,\overline{D}\,E + \overline{A}\,B\,C\,\overline{D}\,\overline{E}$$
$$+ \overline{A}\,B\,C\,\overline{D}\,E + A\,\overline{B}\,\overline{C}\,\overline{D}\,\overline{E} + A\,\overline{B}\,\overline{C}\,\overline{D}\,E + A\,\overline{B}\,C\,\overline{D}\,\overline{E} + A\,B\,C\,\overline{D}\,\overline{E} + A\,B\,C\,\overline{D}\,E + A\,\overline{B}\,C\,\overline{D}\,E$$

Map the SOP expression. Figure 4–44 shows the groupings and their corresponding terms. Combining the terms yields the following minimized SOP expression:

$$X + \overline{A}\,\overline{D}\,\overline{E} + \overline{B}\,\overline{C}\,\overline{D} + BCE + ACDE$$



Related Problem Minimize the following expression:

$$Y = \overline{A}\,\overline{B}\,\overline{C}\,\overline{D}\,\overline{E} + \overline{A}\,\overline{B}\,\overline{C}\,\overline{D}\,E + \overline{A}\,\overline{B}\,C\,\overline{D}\,\overline{E} + \overline{A}\,\overline{B}\,C\,\overline{D}\,E + \overline{A}\,B\,\overline{C}\,\overline{D}\,E + \overline{A}\,B\,C\,\overline{D}\,\overline{E} + \overline{A}\,B\,C\,\overline{D}\,E + \overline{A}\,B\,\overline{C}\,\overline{D}\,E$$
$$+ \overline{A}\,\overline{B}\,\overline{C}\,D\,\overline{E} + \overline{A}\,\overline{B}\,C\,D\,\overline{E} + \overline{A}\,B\,C\,D\,\overline{E} + \overline{A}\,B\,\overline{C}\,D\,\overline{E} + A\,\overline{B}\,\overline{C}\,D\,\overline{E} + A\,\overline{B}\,C\,D\,\overline{E} + A\,B\,C\,D\,\overline{E} + A\,B\,\overline{C}\,D\,\overline{E}$$