Graphs

Design and Analysis of Algorithms

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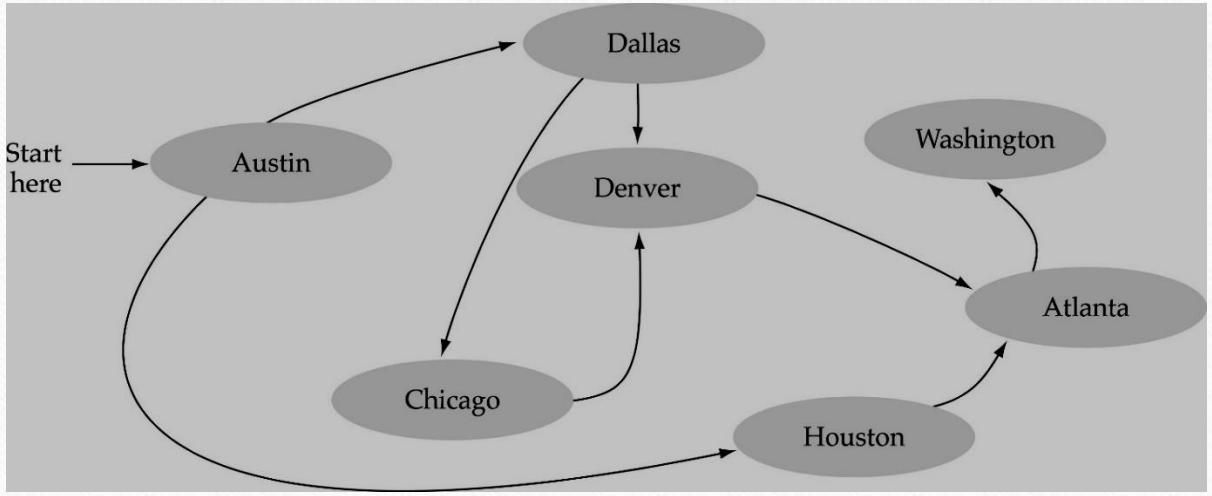
What is a graph?

- A data structure that consists of a set of nodes (*vertices*) and a set of edges that relate the nodes to each other
- The set of edges describes relationships among the vertices



Formal definition of graphs

- A graph *G* is defined as follows:

  $G=(V,E)$

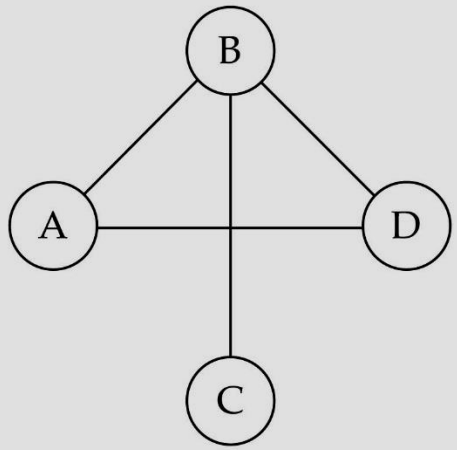
  $V(G)$: a finite, nonempty set of vertices

  $E(G)$: a set of edges (pairs of vertices)

Directed vs. undirected graphs

- When the edges in a graph have no direction, the graph is called *undirected*

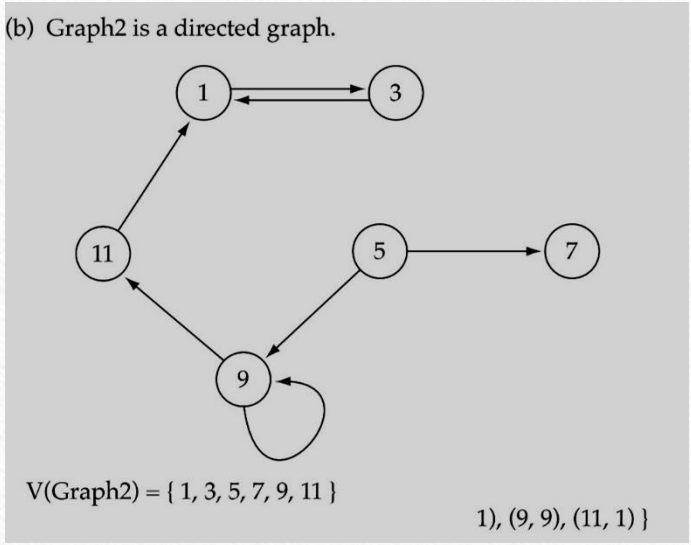
ected graph.



V(Graph1) = { A, B, C, D }
E(Graph1) = { (A, B), (A, D), (B, C), (B, D) }

Directed vs. undirected graphs (cont.)

- When the edges in a graph have a direction, the graph is called *directed* (or *digraph*)

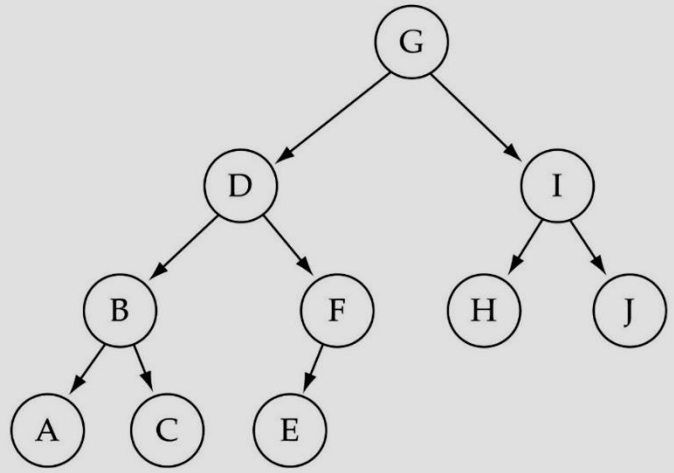


E(Graph2) = {(1,3) (3,1) (5,9) (9,11) (5,7)

Trees vs graphs

- Trees are special cases of graphs!!

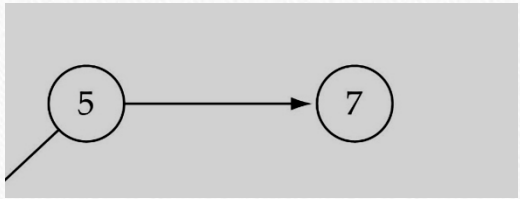
(c) Graph3 is a directed graph.



V(Graph3) = { A, B, C, D, E, F, G, H, I, J }
E(Graph3) = { (G, D), (G, J), (D, B), (D, F) (I, H), (I, J), (B, A), (B, C), (F, E) }

Graph terminology

- Adjacent nodes: two nodes are adjacent if they are connected by an edge



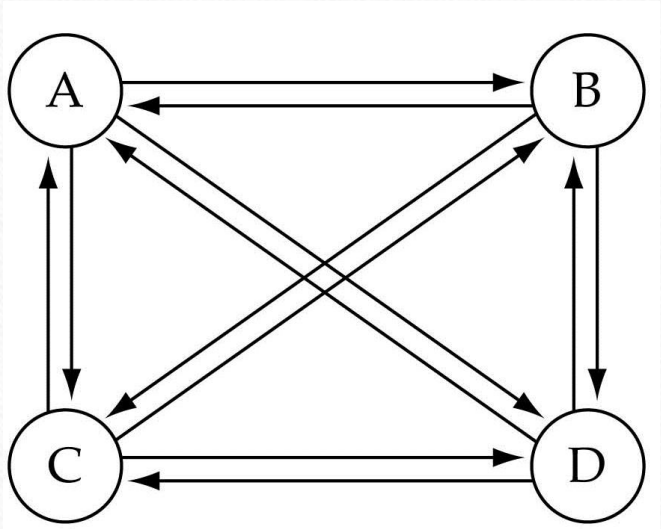
5 is adjacent to 7
7 is adjacent from 5

- Path: a sequence of vertices that connect two nodes in a graph
- Complete graph: a graph in which every vertex is directly connected to every other vertex

Graph terminology (cont.)

- What is the number of edges in a complete directed graph with N vertices? $O(N^2)$

  *N * (N-1)*



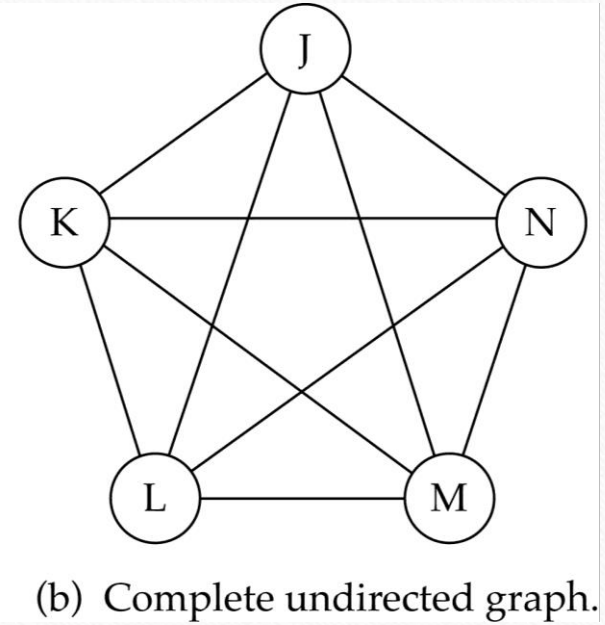
(a) Complete directed graph.

Graph terminology (cont.)

- What is the number of edges in a complete undirected graph with N vertices?

  $O(N^2)$

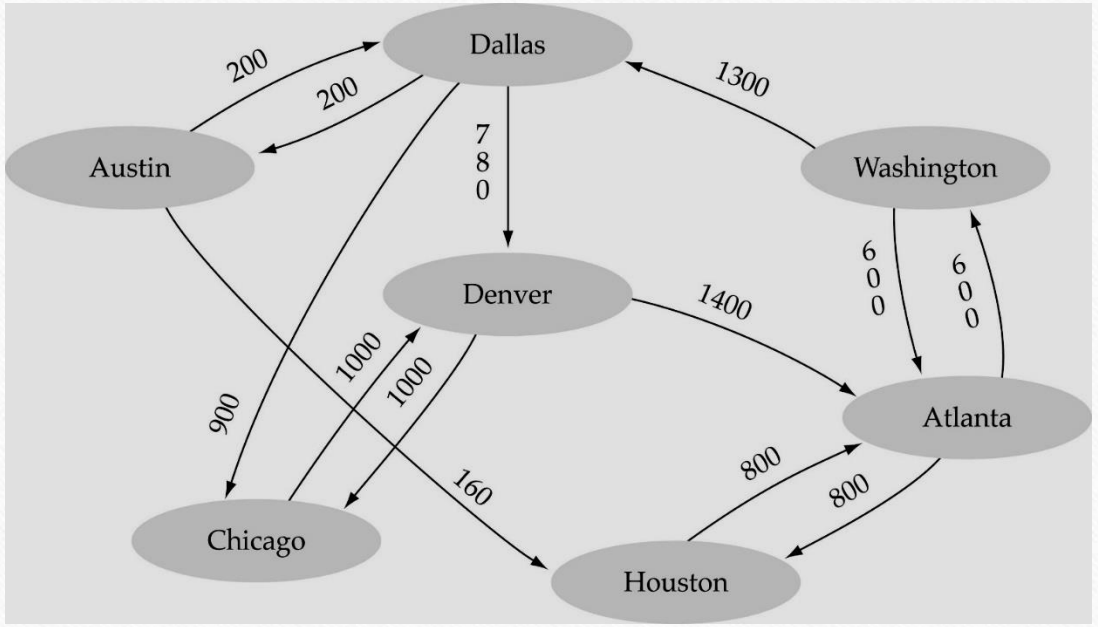
  *N * (N-1) / 2*



(b) Complete undirected graph.

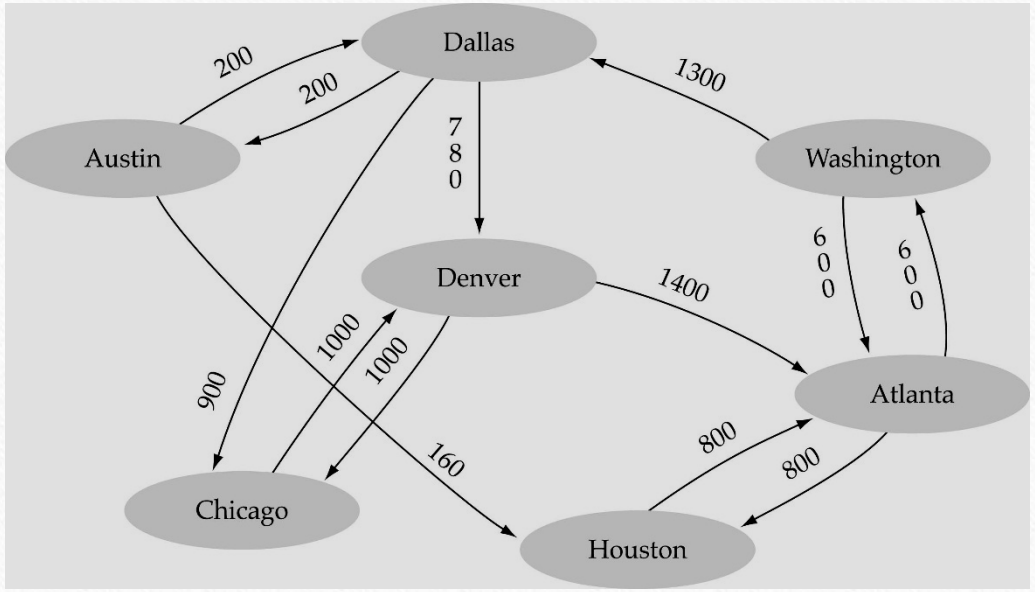
Graph terminology (cont.)

- Weighted graph: a graph in which each edge carries a value



Graph implementation

- Array-based implementation
  - A 1D array is used to represent the vertices
  - A 2D array (adjacency matrix) is used to represent the edges



Array-based implementation

graph
.numVertices 7
.vertices

[0]	"Atlanta "
[1]	"Austin "
[2]	"Chicago "
[3]	"Dallas "
[4]	"Denver "
[5]	"Houston "
[6]	"Washington"
[7]	
[8]	
[9]	

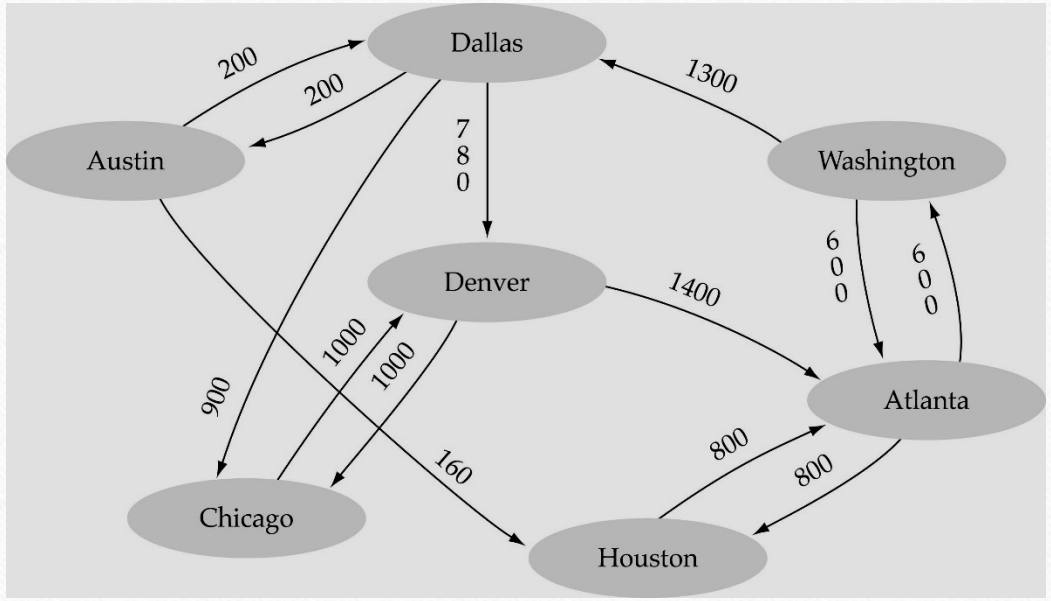
.edges

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
[0]	0	0	0	0	0	800	600	•	•	•
[1]	0	0	0	200	0	160	0	•	•	•
[2]	0	0	0	0	1000	0	0	•	•	•
[3]	0	200	900	0	780	0	0	•	•	•
[4]	1400	0	1000	0	0	0	0	•	•	•
[5]	800	0	0	0	0	0	0	•	•	•
[6]	600	0	0	1300	0	0	0	•	•	•
[7]	•	•	•	•	•	•	•	•	•	•
[8]	•	•	•	•	•	•	•	•	•	•
[9]	•	•	•	•	•	•	•	•	•	•

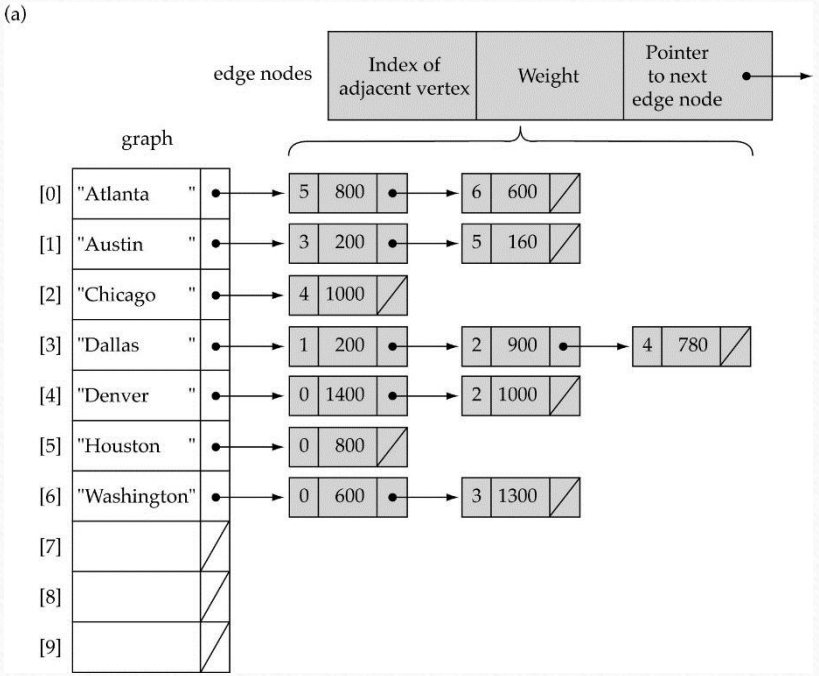
(Array positions marked '•' are undefined)

Graph implementation (cont.)

- Linked-list implementation
  - A 1D array is used to represent the vertices
  - A list is used for each vertex *v* which contains the vertices which are adjacent from *v* (adjacency list)



Linked-list implementation



Adjacency matrix vs. adjacency list representation

- **Adjacency matrix**
  - Good for dense graphs --$|E| \sim O(|V|^2)$
  - Memory requirements: $O(|V| + |E|) = O(|V|^2)$
  - Connectivity between two vertices can be tested quickly
- **Adjacency list**
  - Good for sparse graphs -- $|E| \sim O(|V|)$
  - Memory requirements: $O(|V| + |E|) = O(|V|)$
  - Vertices adjacent to another vertex can be found quickly

Graph specification based on adjacency matrix representation

```
const int NULL_EDGE = 0;

template<class VertexType>
class GraphType {
  public:
    GraphType(int);
    ~GraphType();
    void MakeEmpty();
    bool IsEmpty() const;
    bool IsFull() const;
    void AddVertex(VertexType);
    void AddEdge(VertexType, VertexType, int);
    int WeightIs(VertexType, VertexType);
    void GetToVertices(VertexType, QueType<VertexType>&);
    void ClearMarks();
    void MarkVertex(VertexType);
    bool IsMarked(VertexType) const;
  private:
    int numVertices;
    int maxVertices;
    VertexType* vertices;
    int **edges;
    bool* marks;
};
```

```
template<class VertexType>
GraphType<VertexType>::GraphType(int maxV)
{
 numVertices = 0;
 maxVertices = maxV;
 vertices = new VertexType[maxV];
 edges = new int[maxV];
 for(int i = 0; i < maxV; i++)
   edges[i] = new int[maxV];
 marks = new bool[maxV];
}

template<class VertexType>
GraphType<VertexType>::~GraphType()
{
 delete [] vertices;
 for(int i = 0; i < maxVertices; i++)
   delete [] edges[i];
 delete [] edges;
 delete [] marks;
}
```


(continues)

```
void GraphType<VertexType>::AddVertex(VertexType vertex)
{
 vertices[numVertices] = vertex;

 for(int index = 0; index < numVertices; index++) {
   edges[numVertices][index] = NULL_EDGE;
   edges[index][numVertices] = NULL_EDGE;
 }

 numVertices++;
}

template<class VertexType>
void GraphType<VertexType>::AddEdge(VertexType fromVertex,
                    VertexType toVertex, int weight)
{
 int row;
 int column;

 row = IndexIs(vertices, fromVertex);
 col = IndexIs(vertices, toVertex);
 edges[row][col] = weight;
}
```


(continues)

```
template<class VertexType>
int GraphType<VertexType>::WeightIs(VertexType fromVertex,
                                    VertexType toVertex)
{
 int row;
 int column;

 row = IndexIs(vertices, fromVertex);
 col = IndexIs(vertices, toVertex);
 return edges[row][col];
}
```

Graph searching

- *Problem*: find a path between two nodes of the graph (e.g., Austin and Washington)
- *Methods*: Depth-First-Search (DFS) or Breadth-First-Search (BFS)

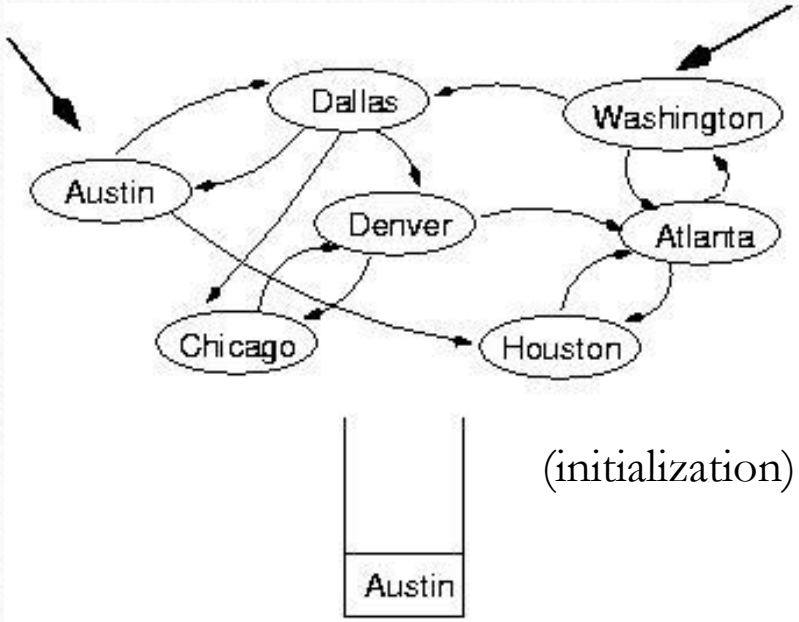
Depth-First-Search (DFS)

- What is the idea behind DFS?
  - Travel as far as you can down a path
  - Back up *as little as possible* when you reach a "dead end" (i.e., next vertex has been "marked" or there is no next vertex)
- DFS can be implemented efficiently using a *stack*

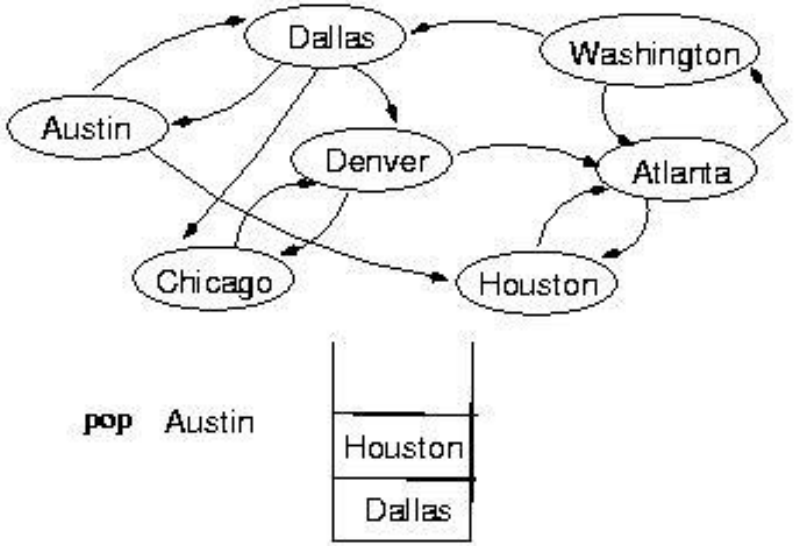
Depth-First-Search (DFS) *(cont.)*

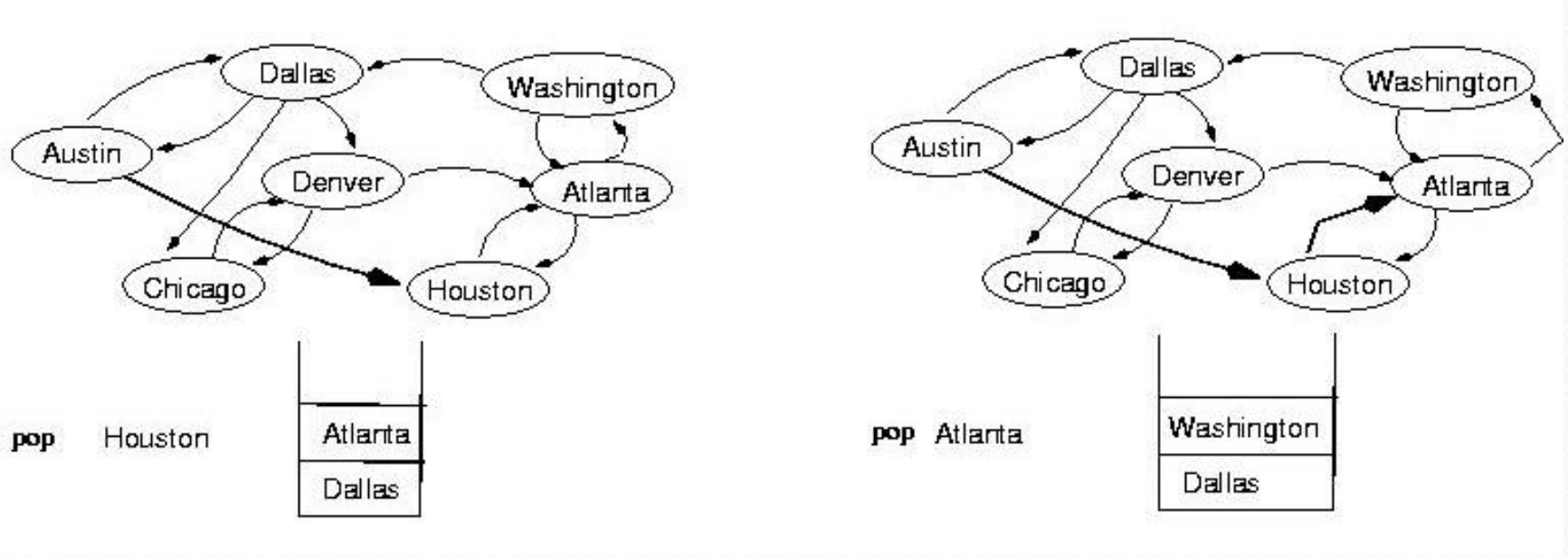
```
Set found to false
stack.Push(startVertex)
DO
  stack.Pop(vertex)
  IF vertex == endVertex
    Set found to true
  ELSE
    Push all adjacent vertices onto stack
WHILE !stack.IsEmpty() AND !found

IF(!found)
  Write "Path does not exist"
```



(initialization)

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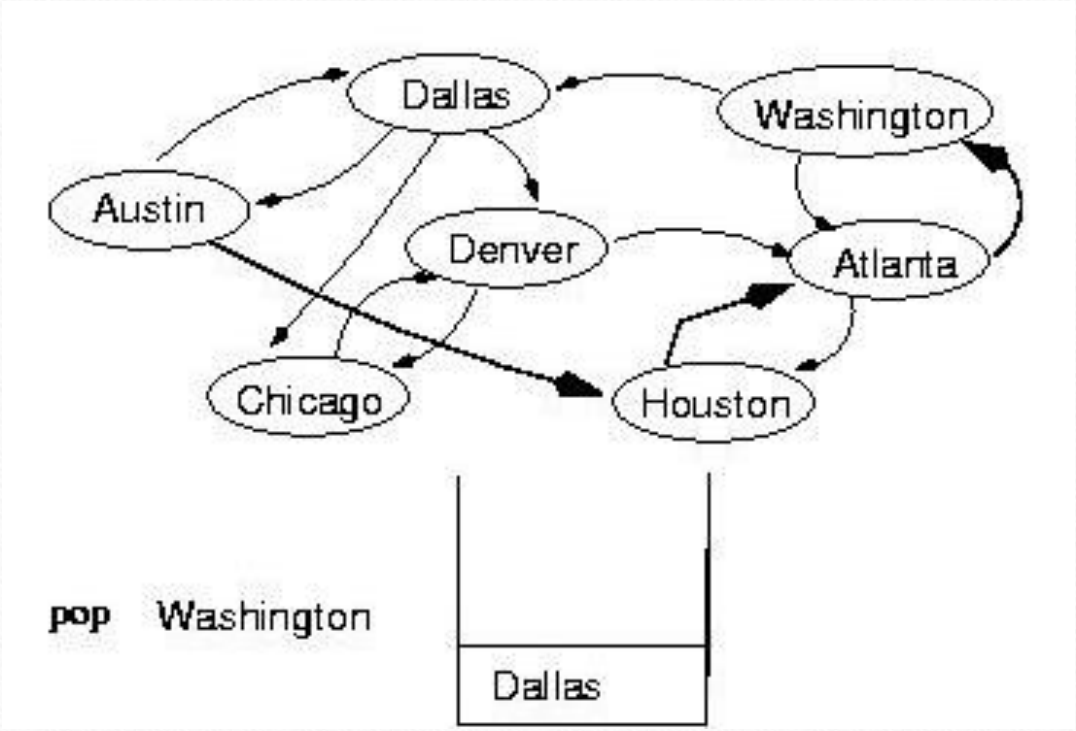
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```
template <class ItemType>
void DepthFirstSearch(GraphType<VertexType> graph, VertexType startVertex, VertexType endVertex)
{
 StackType<VertexType> stack;
 QueType<VertexType> vertexQ;

 bool found = false;
 VertexType vertex;
 VertexType item;

 graph.ClearMarks();
 stack.Push(startVertex);
 do {
   stack.Pop(vertex);
   if(vertex == endVertex)
     found = true;
```


(continues)

```
 else {
    if(!graph.IsMarked(vertex)) {
      graph.MarkVertex(vertex);
      graph.GetToVertices(vertex, vertexQ);

      while(!vertexQ.IsEmpty()) {
        vertexQ.Dequeue(item);
        if(!graph.IsMarked(item))
          stack.Push(item);
      }
    }
  } while(!stack.IsEmpty() && !found);

  if(!found)
    cout << "Path not found" << endl;
}
```


(continues)

```
template<class VertexType>
void GraphType<VertexType>::GetToVertices(VertexType vertex,
                          QueTye<VertexType>& adjvertexQ)
{
 int fromIndex;
 int toIndex;

 fromIndex = IndexIs(vertices, vertex);
 for(toIndex = 0; toIndex < numVertices; toIndex++)
   if(edges[fromIndex][toIndex] != NULL_EDGE)
     adjvertexQ.Enqueue(vertices[toIndex]);
}
```

Breadth-First-Searching (BFS)

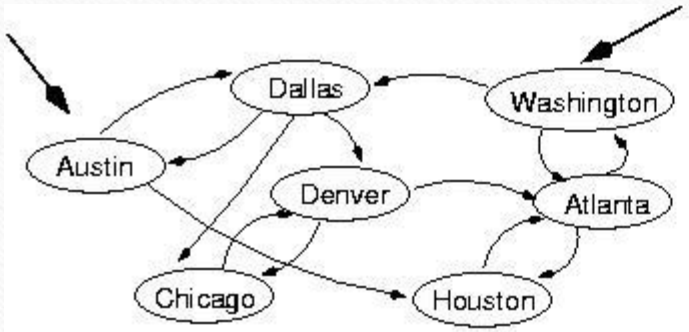
- What is the idea behind BFS?
  - Look at all possible paths at the same depth before you go at a deeper level
  - Back up *as far as possible* when you reach a "dead end" (i.e., next vertex has been "marked" or there is no next vertex)

Breadth-First-Searching (BFS) (cont.)

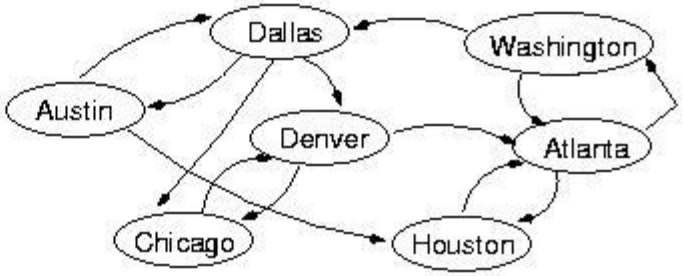
- BFS can be implemented efficiently using a *queue*

```
Set found to false
queue.Enqueue(startVertex)
DO
  queue.Dequeue(vertex)
  IF vertex == endVertex
    Set found to true
  ELSE
    Enqueue all adjacent vertices onto queue
WHILE !queue.IsEmpty() AND !found
```

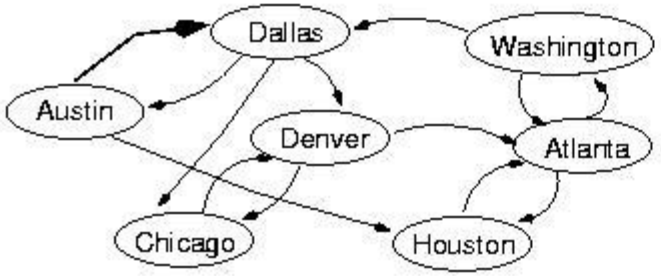
- Should we mark a vertex when it is enqueued or when it is dequeued ?



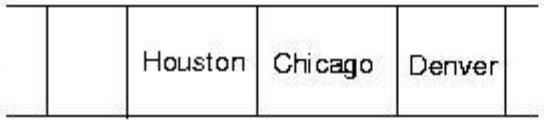
(initialization)

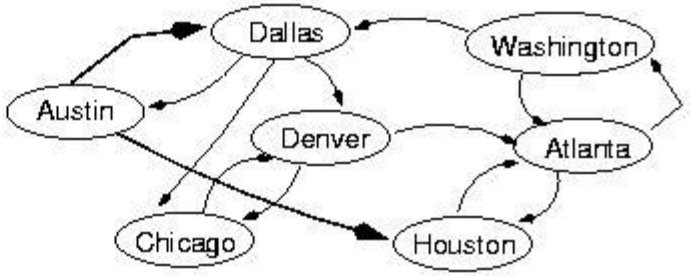


dequeue Austin

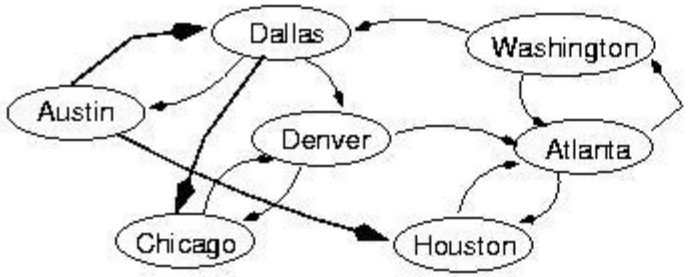


dequeue Dallas





dequeue Houston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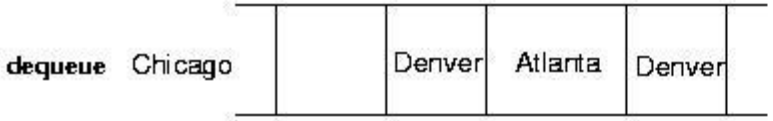
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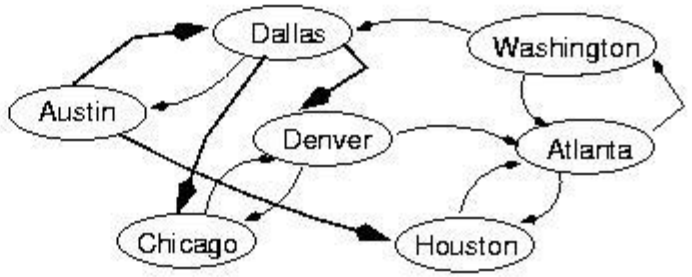
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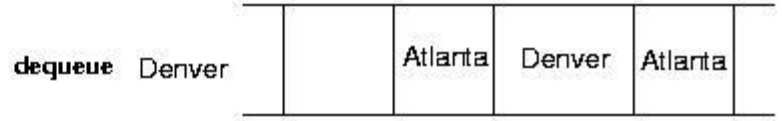
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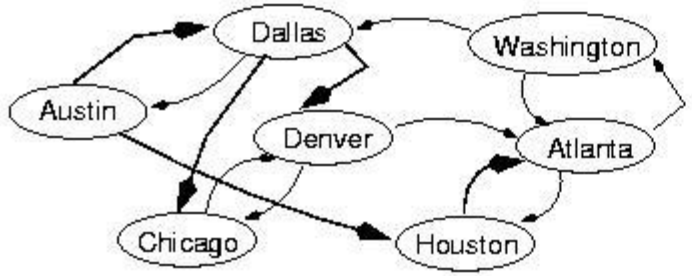
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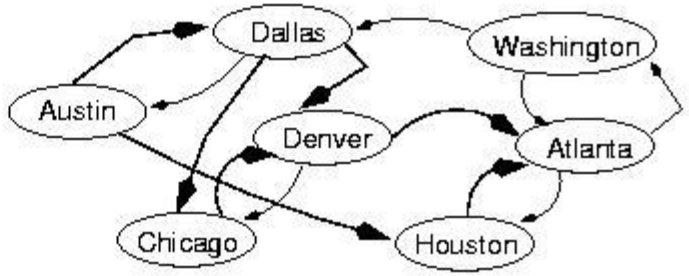
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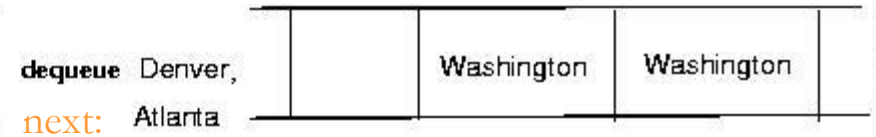
dequeue Denver,
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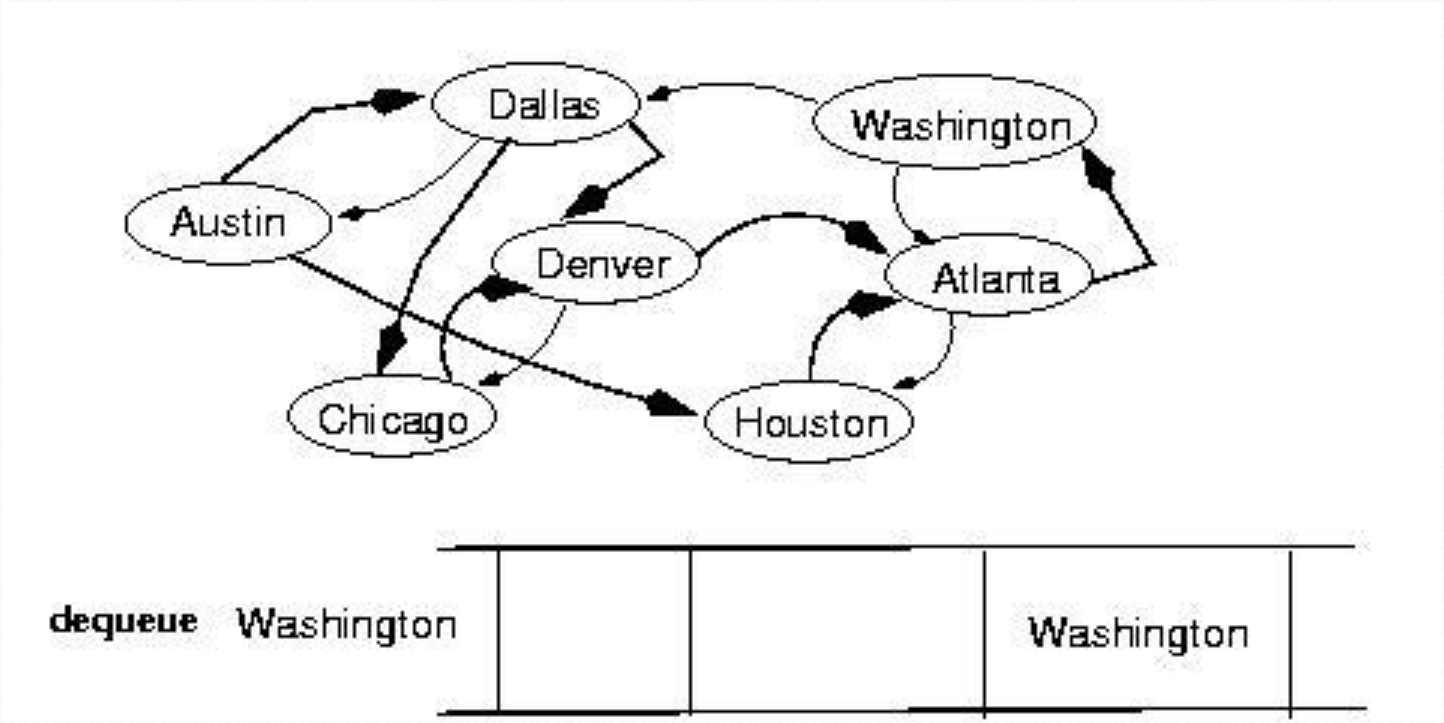
Washington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```
template<class VertexType>
void BreadthFirtsSearch(GraphType<VertexType> graph, VertexType startVertex, VertexType endVertex);
{
 QueType<VertexType> queue;
 QueType<VertexType> vertexQ;//

 bool found = false;
 VertexType vertex;
 VertexType item;

 graph.ClearMarks();
 queue.Enqueue(startVertex);
 do {
   queue.Dequeue(vertex);
   if(vertex == endVertex)
     found = true;
```


(continues)

```
 else {
    if(!graph.IsMarked(vertex)) {
      graph.MarkVertex(vertex);
      graph.GetToVertices(vertex, vertexQ);

      while(!vertxQ.IsEmpty()) {
        vertexQ.Dequeue(item);
        if(!graph.IsMarked(item))
          queue.Enqueue(item);
      }
    }
  }
 } while (!queue.IsEmpty() && !found);

 if(!found)
   cout << "Path not found" << endl;
}
```