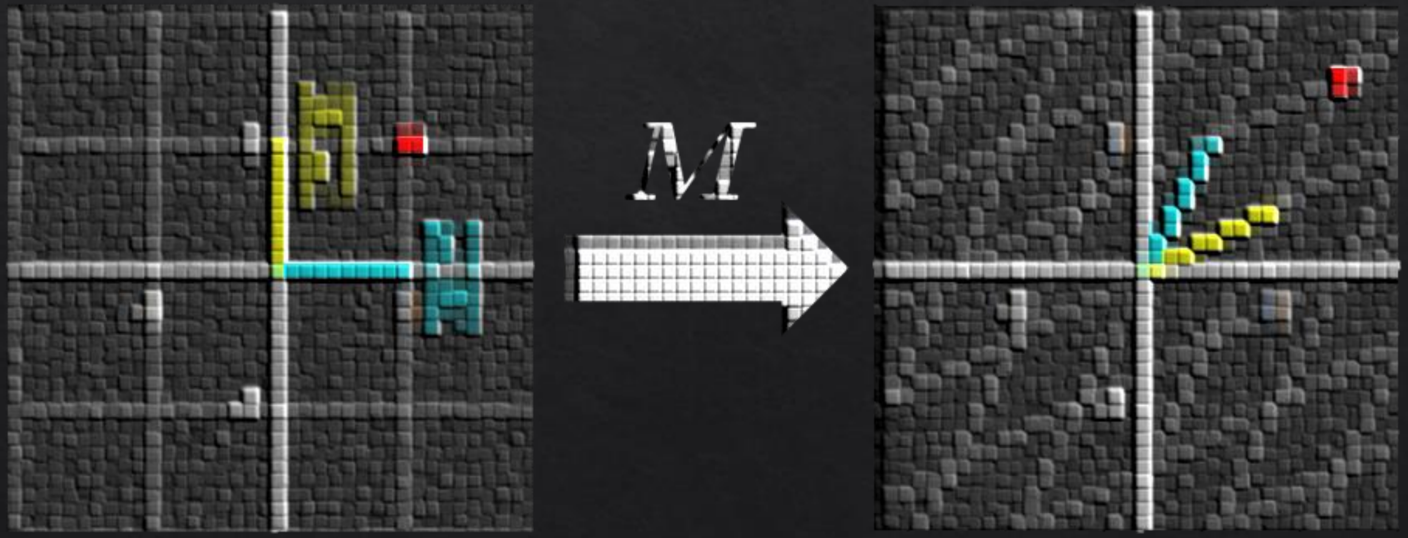
Essential Linear Algebra

Machine Learning Methods – Lecture 1



June 2021

fixel your pixels

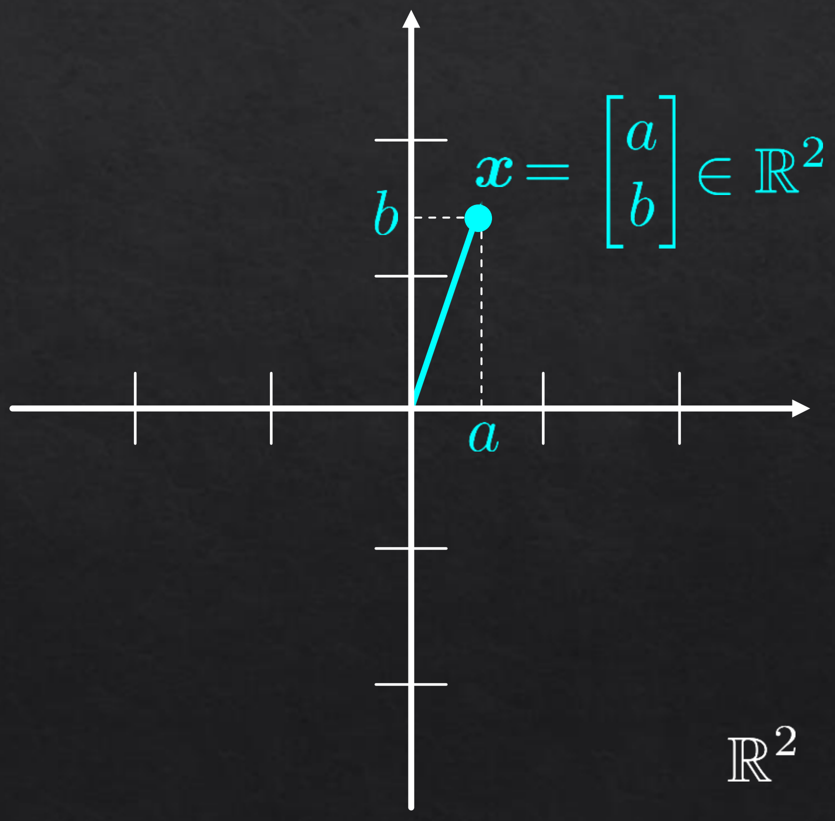
fixelalgorithms.gitlab.io

Or Yair

Vectors

- This is the real line $\mathbb{R}$:

- Points in $\mathbb{R}$ are called scalars, e.g. $\alpha, \beta \in \mathbb{R}$.
- This is $\mathbb{R}^2$:
- Points in $\mathbb{R}^2$ are called vectors, e.g. $\boldsymbol{x} \in \mathbb{R}^2$.



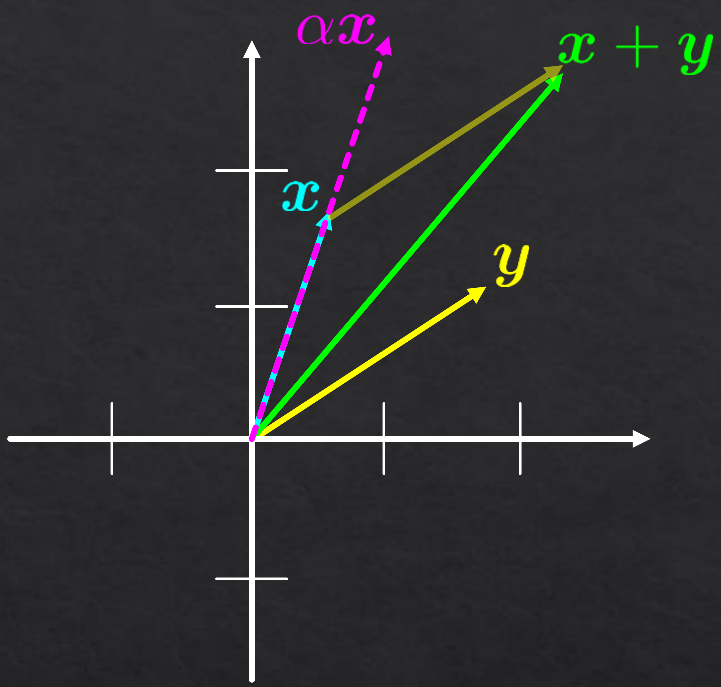
- $\boldsymbol{y} \in \mathbb{R}^4$ is a vector (point) in $\mathbb{R}^4$:

$$\boldsymbol{y} = \begin{bmatrix} c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}[1] \\ \boldsymbol{y}[2] \\ \boldsymbol{y}[3] \\ \boldsymbol{y}[4] \end{bmatrix}$$

y_i might be confusing

Vector Space

- Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$.
- $\mathbb{R}^d$ is a vector space:
  - Closure under addition: $\boldsymbol{x} + \boldsymbol{y} \in \mathbb{R}^d$.
  - Closure under scalar multiplication: $\alpha \boldsymbol{x} \in \mathbb{R}^d$.
  - Overall: $\underbrace{\alpha \boldsymbol{x} + \beta \boldsymbol{y}}_{\text{linear combination}} \in \mathbb{R}^d$ for all $\alpha, \beta \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$



- The sphere $\mathcal{S}$ in not a vector space:

$\boldsymbol{p}_1, \boldsymbol{p}_2 \in \mathcal{S}$

$\boldsymbol{p}_1 + \boldsymbol{p}_2 \notin \mathcal{S}$

no closure under addition

vector addition:

$$\boldsymbol{x} + \boldsymbol{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

vector by scalar multiplication:

$$\alpha \boldsymbol{x} = \alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}$$

Distance Function – Motivation

- Consider the following three images:

$\boldsymbol{I}_1$

$\boldsymbol{I}_2$

$\boldsymbol{I}_3$

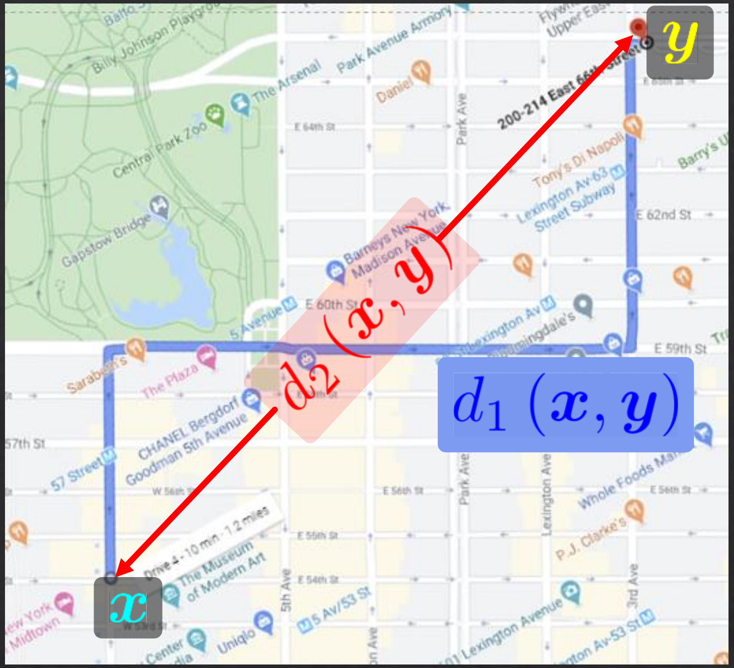
- Is $\boldsymbol{I}_2$ more similar to $\boldsymbol{I}_1$ or $\boldsymbol{I}_3$?
- Formally, we ask:

$$d\left(\boldsymbol{I}_2, \boldsymbol{I}_1\right) \overset{?}{\gtrless} d\left(\boldsymbol{I}_2, \boldsymbol{I}_3\right)$$

d is a distance function (metric)

- Answer: it depends on the metric d.

Metric (distance) Function(s)



- A metric on $\mathbb{R}^d$ is a function:

$$d(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$$

such that, for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^d$:

1. $d(\boldsymbol{x}, \boldsymbol{y}) = 0 \iff \boldsymbol{x} = \boldsymbol{y}$ (identity)
2. $d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{y}, \boldsymbol{x})$ (symmetry)
3. $d(\boldsymbol{x}, \boldsymbol{z}) \leq d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z})$ (triangle inequality)
4. $d(\boldsymbol{x}, \boldsymbol{y}) \geq 0$ (non-negativity) $\leftarrow$ implied by the others

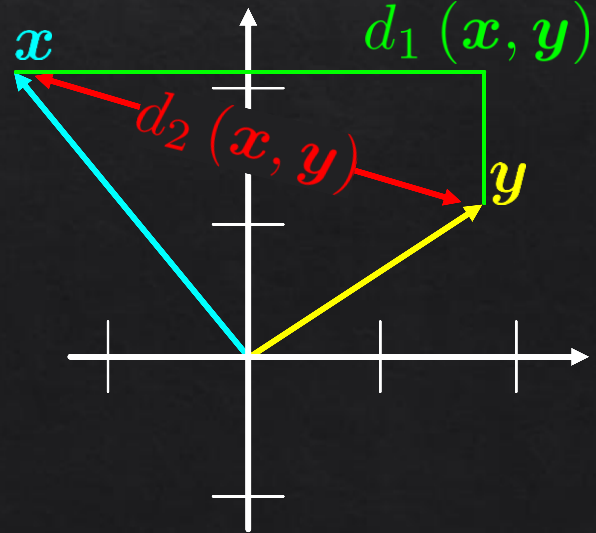
Examples

Euclidean distance:

$$d_2(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}$$

Manhattan distance:

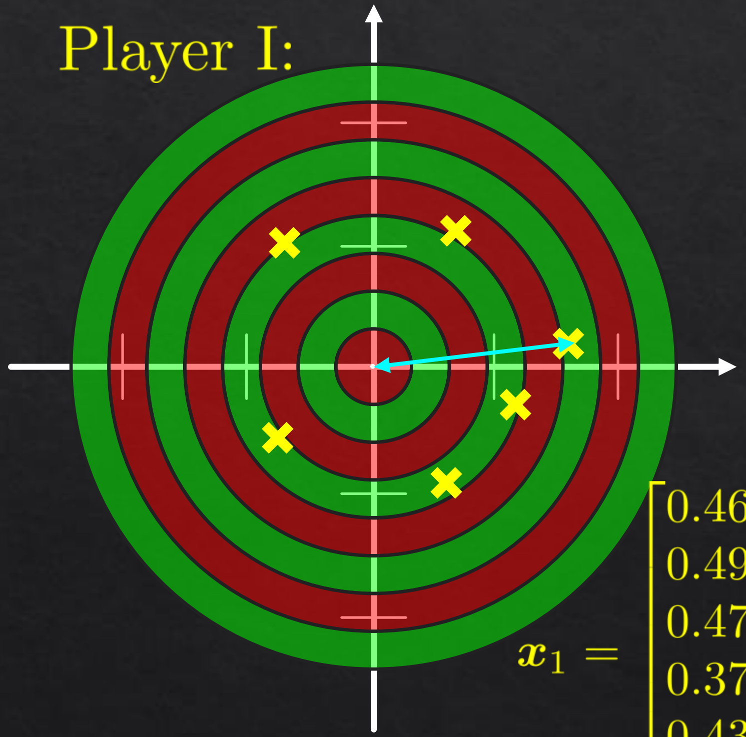
$$d_1(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{d} |x_i - y_i|$$



L^p Norm – Motivation

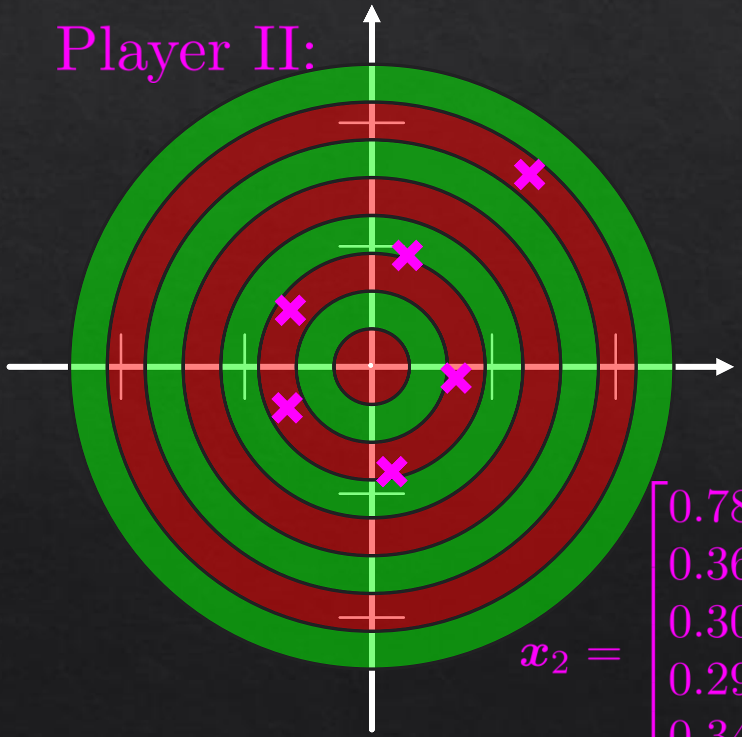
- Consider the game of darts.
- The (negative) score of each throw is the distance from the origin. (Euclidean distance)
- Consider the results of 3 players: (6 darts each)

Player I:



$$\boldsymbol{x}_1 = \begin{bmatrix} 0.46 \\ 0.49 \\ 0.47 \\ 0.37 \\ 0.43 \\ 0.61 \end{bmatrix}$$

Player II:



$$\boldsymbol{x}_2 = \begin{bmatrix} 0.78 \\ 0.36 \\ 0.30 \\ 0.29 \\ 0.34 \\ 0.27 \end{bmatrix}$$

Player III:

$$\boldsymbol{x}_3 = \begin{bmatrix} 0.88 \\ 0.74 \\ 0.14 \\ 0.15 \\ 0.04 \\ 0.15 \end{bmatrix}$$

Which player is the best? which $\boldsymbol{x}_i$ is the "smallest"?

The Norm Operator

$d(\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$

- The norm $\|\cdot\| : \mathbb{R}^d \to [0,\infty)$ is a function which satisfies:
  1. $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{0}$ (positive definite)
  2. $\|\alpha \boldsymbol{x}\| = |\alpha| \cdot \|\boldsymbol{x}\|$ (absolutely homogeneous)
  3. $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ (triangle inequality)
  4. $\|\boldsymbol{x}\| \geq 0$ (non-negativity) $\leftarrow$ implied by the others

Examples

The L^2 (Euclidean) norm:

$$\|\boldsymbol{x}\|_2^2 = \sum_{i=1}^{d} x_i^2$$

The L^∞ norm:

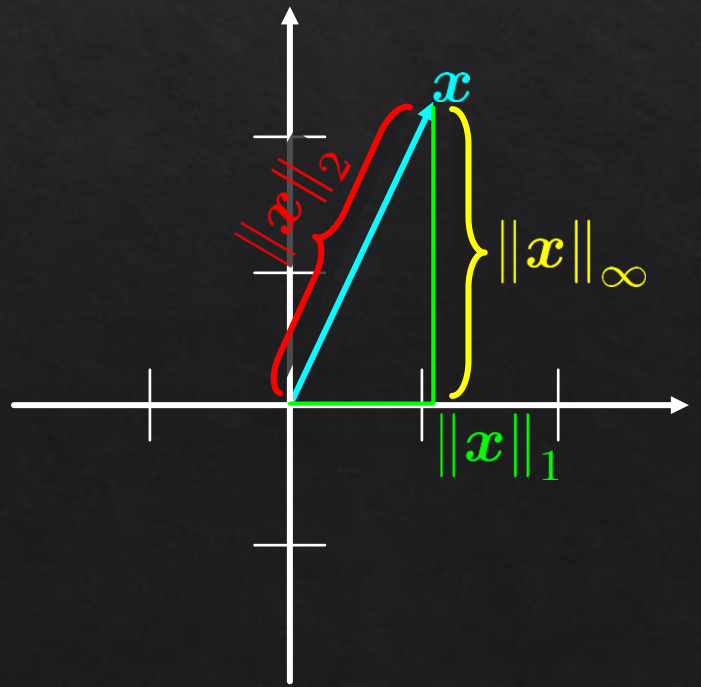
$$\|\boldsymbol{x}\|_\infty = \max_i |x_i|$$

The L^1 norm:

$$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{d} |x_i|$$

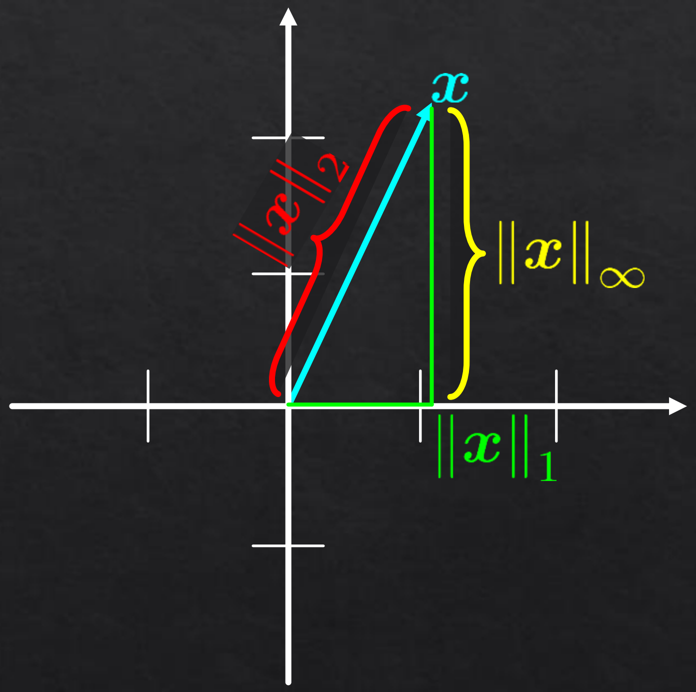
The L^p norm $(1 \leq p \leq \infty)$:

$$\|\boldsymbol{x}\|_p^p = \sum_{i=1}^{d} |x_i|^p$$



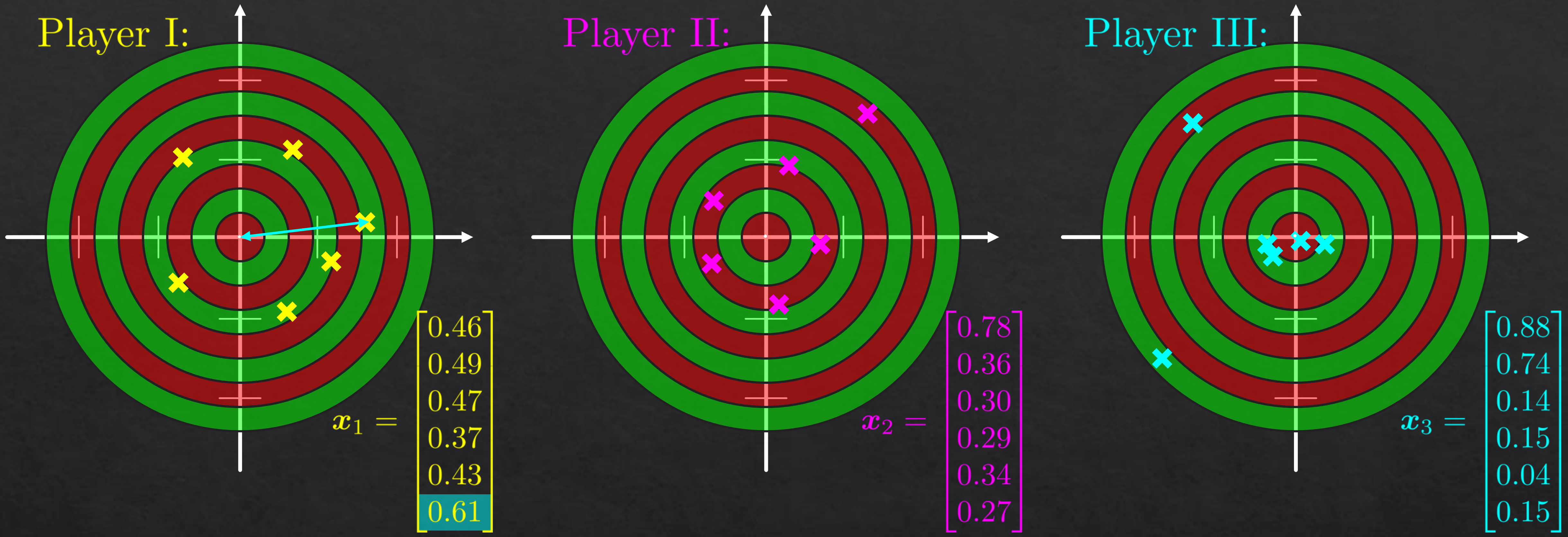
L^p Norm Example

Exercise Plot all vectors $\boldsymbol{x} \in \mathbb{R}^2$ such that $\|\boldsymbol{x}\|_p \leq 1$. (for $p \in \{1, 2, \infty\}$)



[L^p Norm]

L^1 vs L^2 vs L^∞



low sensitivity to outliers $\rightarrow \|\cdot\|_1$

high sensitivity to outliers $\rightarrow \|\cdot\|_\infty$

	x_1	x_2	x_3
$\|\cdot\|_1$	2.86	2.35	2.10
$\|\cdot\|_2$	1.18	1.05	1.18
$\|\cdot\|_\infty$	0.61	0.78	0.88

Metric Induced by Norm

metric:

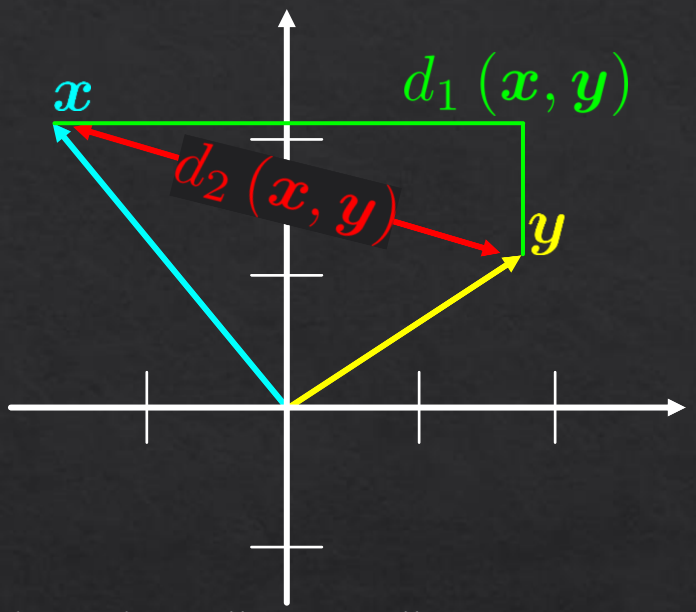
$d : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$

1. $d(\boldsymbol{x}, \boldsymbol{y}) = 0 \iff \boldsymbol{x} = \boldsymbol{y}$
2. $d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{y}, \boldsymbol{x})$
3. $d(\boldsymbol{x}, \boldsymbol{z}) \leq d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z})$

norm:

$\|\cdot\| : \mathbb{R}^d \to [0, \infty)$

1. $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{0}$
2. $\|\alpha \boldsymbol{x}\| = |\alpha| \cdot \|\boldsymbol{x}\|$
3. $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$



$d_p(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|_p, \quad p \geq 1$

- Given a norm $\|\cdot\|$, the induced metric d is given by:

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\| \quad \text{induced metric}$$

- Not every metric is induced by a norm:
  - The geodesic distance: $d_g(\boldsymbol{p}_1, \boldsymbol{p}_2)$.
  - $d_{\mathrm{LE}}(x, y) = |\log(x) - \log(y)|, \quad x, y > 0$

Norm – Center of Mass Example

- Consider the set of points $\{x_i\}_{i=1}^{7}$ on the real line $\mathbb{R}$:

- Find the center of mass $\mu_2 \in \mathbb{R}$ which is given by:

$$\mu_2 = \arg\min_{\mu \in \mathbb{R}} \underbrace{\sum_{i=1}^{7} (x_i - \mu)^2}_{=: f(\mu)}$$

comparing the derivative to zero →

$$f'(\mu_2) = 0$$

$$-2 \sum_{i=1}^{7} (x_i - \mu_2) = 0$$

$$\mu_2 = \frac{1}{7} \sum_{i=1}^{7} x_i$$

$$\|\cdot\| : \mathbb{R}^d \to [0, \infty)$$

$$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{d} |x_i|$$

$$\|\boldsymbol{x}\|_2^2 = \sum_{i=1}^{d} x_i^2$$

$$\|\boldsymbol{x}\|_\infty = \max_i |x_i|$$

- Note that:

$$f(\mu) = \sum_{i=1}^{7} (x_i - \mu)^2 = \left\| \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_7 \end{bmatrix} - \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix} \right\|_2^2 = \|\boldsymbol{x} - \mathbf{1}\mu\|_2^2$$

μ_2 minimizes the L^2 norm

$\Downarrow$

we can minimize other norms

Norm – Center of Mass Example

- Consider the set of points $\{x_i\}_{i=1}^{7}$ on the real line $\mathbb{R}$:

$$f(\mu) = \sum_{i=1}^{7} (x_i - \mu)^2 = \left\| \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_7 \end{bmatrix} - \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix} \right\|_2^2 = \|\boldsymbol{x} - \mathbf{1}\mu\|_2^2$$

$$\|\cdot\| : \mathbb{R}^d \to [0, \infty)$$

$$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{d} |x_i|$$

$$\|\boldsymbol{x}\|_2^2 = \sum_{i=1}^{d} x_i^2$$

$$\|\boldsymbol{x}\|_\infty = \max_i |x_i|$$

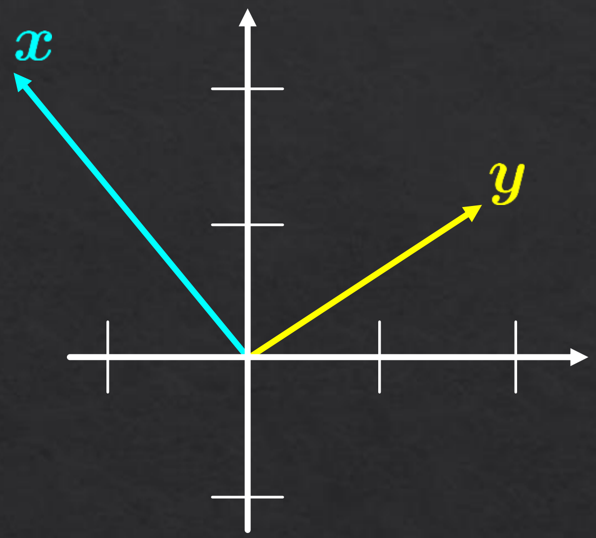
we can minimize other norms

L^2: $$\mu_2 = \arg\min_{\mu \in \mathbb{R}} \|\boldsymbol{x} - \mathbf{1}\mu\|_2^2 = \frac{1}{7}\sum_{i=1}^{7} x_i$$

L^1: $$\mu_1 = \arg\min_{\mu \in \mathbb{R}} \|\boldsymbol{x} - \mathbf{1}\mu\|_1 = \text{median}\left(\{x_i\}_{i=1}^{7}\right)$$

L^∞: $$\mu_\infty = \arg\min_{\mu \in \mathbb{R}} \|\boldsymbol{x} - \mathbf{1}\mu\|_\infty = \frac{1}{2}(x_1 + x_7)$$

Inner Product – Definition



- An inner product on $\mathbb{R}^d$ is a function

$$\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

such that, for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^d$:

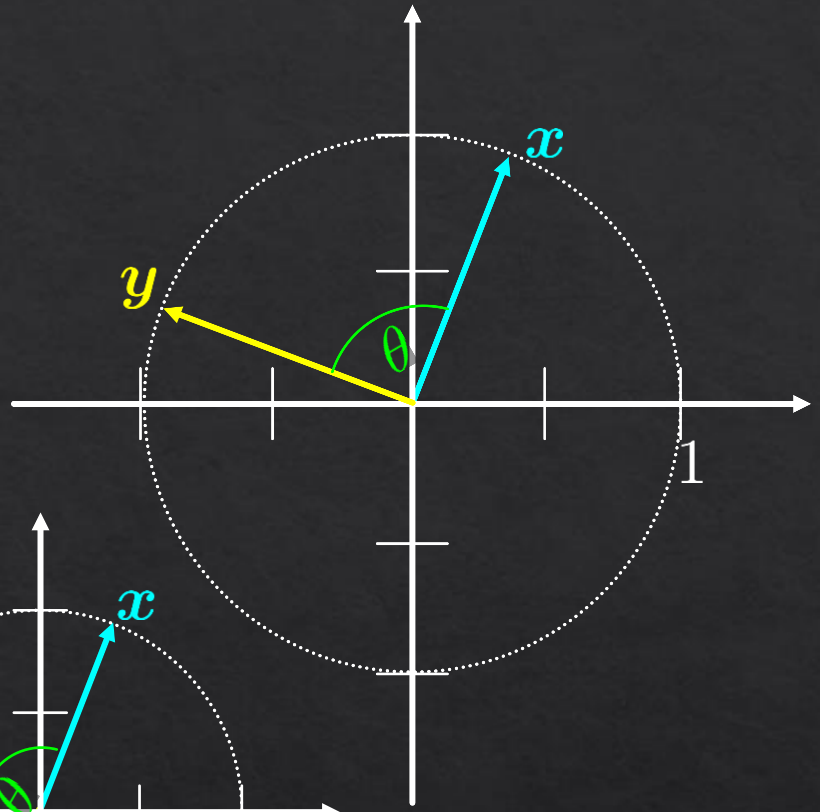
1. $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$ (symmetry)
2. $\langle \alpha \boldsymbol{x} + \beta \boldsymbol{y}, \boldsymbol{z} \rangle = \alpha \langle \boldsymbol{x}, \boldsymbol{z} \rangle + \beta \langle \boldsymbol{y}, \boldsymbol{z} \rangle$ (linearity in the first argument)
3. $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$ (positive definite)
4. $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \iff \boldsymbol{x} = \mathbf{0}$ $\leftarrow$ implied by the others

Example

The Euclidean inner product:

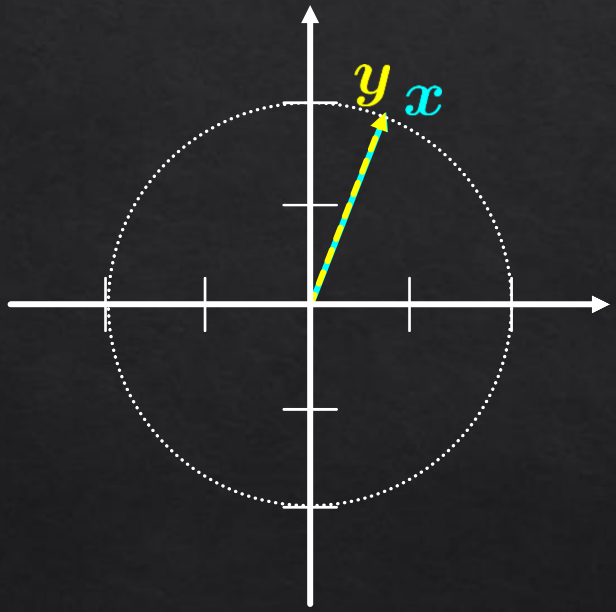
$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{bmatrix} \right\rangle = \sum_{i=1}^{d} x_i y_i$$

Euclidean Inner Product – Intuition

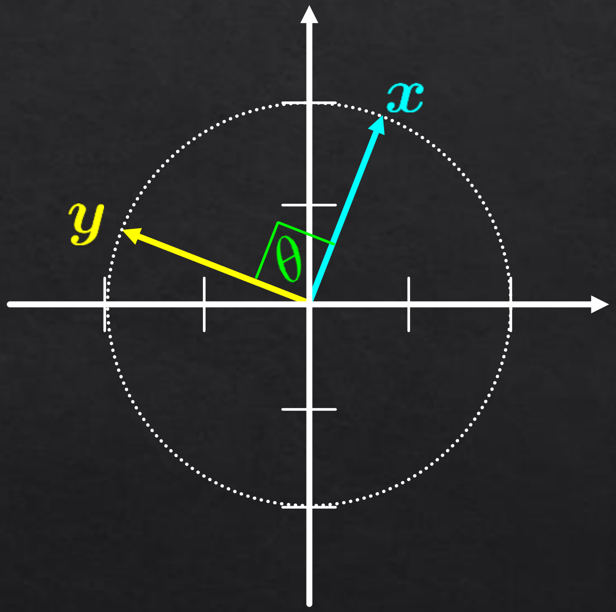


- Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be unit vectors ($\|\boldsymbol{x}\|_2 = \|\boldsymbol{y}\|_2 = 1$).
- Let $\theta \in [0, 2\pi]$ be the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$.
- Then:

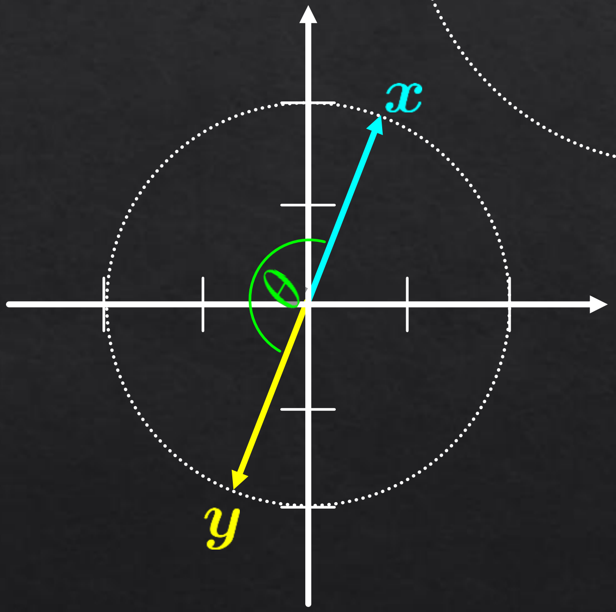
$$\implies \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \cos(\theta)$$



$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 1$



$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$



$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -1$

- For general vectors, we have: $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \|\boldsymbol{x}\|_2 \, \|\boldsymbol{y}\|_2 \cos(\theta)$

Norm Induced Inner Product

- Given an inner product $\langle \cdot, \cdot \rangle$,
  the induced norm $\|\cdot\|$ is given by:

$$\|\boldsymbol{x}\|^2 = \langle \boldsymbol{x}, \boldsymbol{x} \rangle$$ induced norm

The Euclidean inner product:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{d} x_i y_i$$

induces

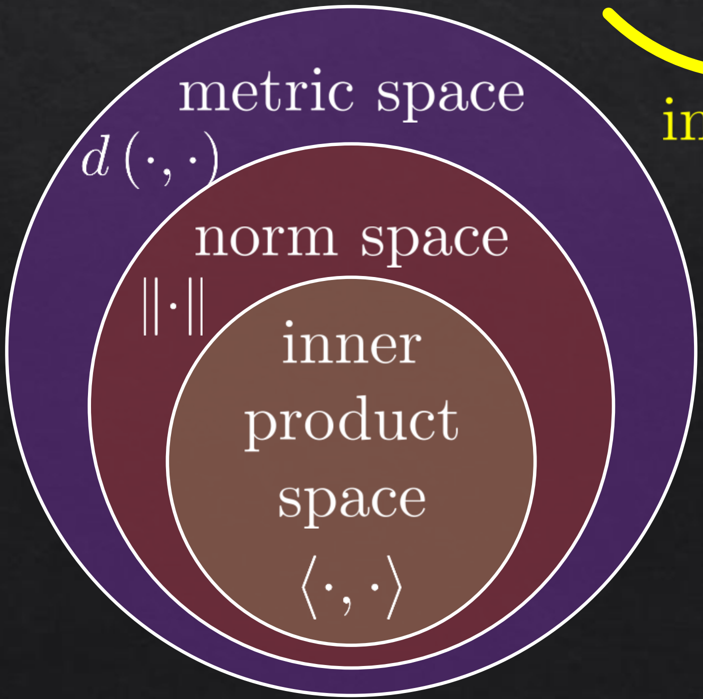
The L^2 (Euclidean) norm:

$$\|\boldsymbol{x}\|_2^2 = \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \sum_{i=1}^{d} x_i^2$$

The L^1 norm:

$$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{d} |x_i|$$

L^1 is not induced by an inner prodcut



norm:

$\|\cdot\| : \mathbb{R}^d \to [0, \infty)$

1. $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{0}$
2. $\|\alpha \boldsymbol{x}\| = |\alpha| \cdot \|\boldsymbol{x}\|$
3. $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$

inner product:

$\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$

1. $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$
2. $\langle \alpha \boldsymbol{x}, \boldsymbol{z} \rangle = \alpha \langle \boldsymbol{x}, \boldsymbol{z} \rangle$
4. $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{0}$

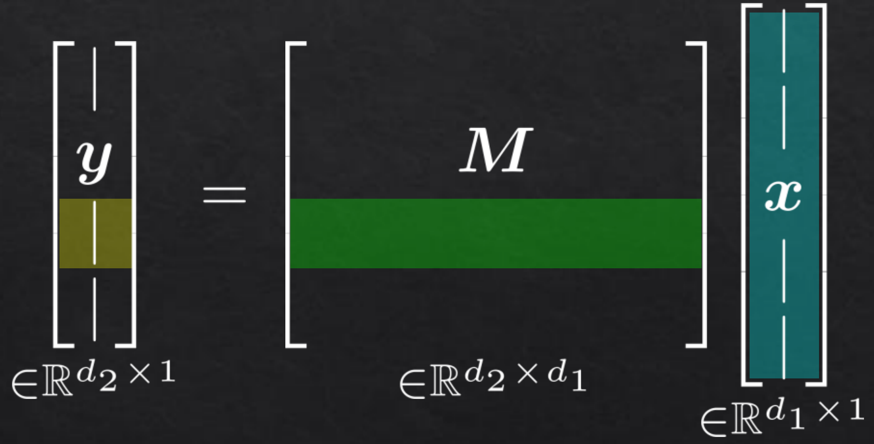
Matrices and Vector Multiplication

- Matrices are points in $\mathbb{R}^{d_1 \times d_2}$.

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \in \mathbb{R}^{3\times 2}, \quad \boldsymbol{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}, \quad \boldsymbol{A}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{23} \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

"Tall and skinny" square "short and fat"

- Matrix by vector multiplication:

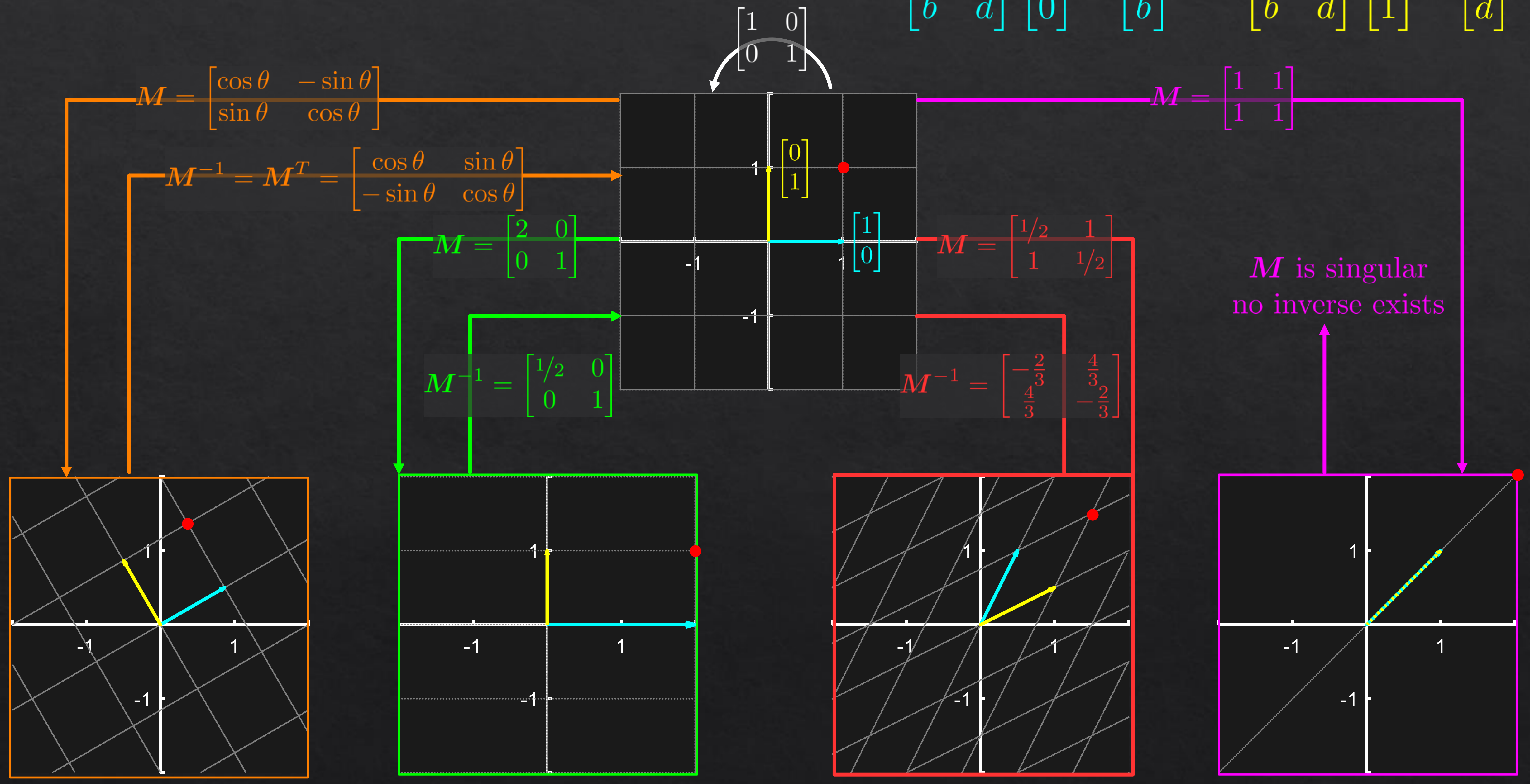


$$y_i = \sum_{j=1}^{d_1} M_{ij} x_j$$

- Matrix by vector multiplication can be viewed as:
  1. A linear operator.
  2. A linear system of equations.
  3. Multiple inner products.
  4. Linear combination of column vectors.

(Square) Matrix as Linear Operator

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$



Linear System of Equations.

- Consider the following linear (affine) function:

$$y_i = f\left(x_i\right) = a x_i + b$$

- Two measurements are given (x_1, y_1) and (x_2, y_2).
- Find a and b.
- The two points satisfy the following <u>linear</u> set of equations:

$$\begin{cases} y_1 = a x_1 + b \\ y_2 = a x_2 + b \end{cases} \implies \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}}_{\boldsymbol{y} = \boldsymbol{X}\boldsymbol{a}}$$

- Multiply by $\boldsymbol{X}^{-1}$ (from the left)*:

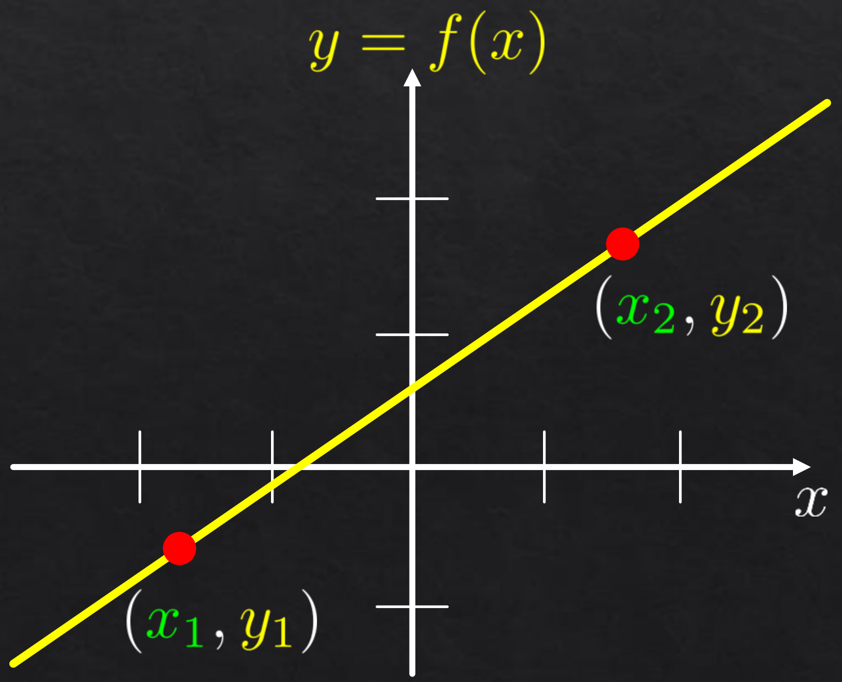
$$\implies \boldsymbol{X}^{-1}\boldsymbol{y} = \boldsymbol{X}^{-1}\boldsymbol{X}\boldsymbol{a}$$

$$\boldsymbol{X}^{-1}\boldsymbol{y} = \boldsymbol{a}$$

(*) – assuming $\boldsymbol{X}$ is not singular $(x_1 \neq x_2)$.

$$\underbrace{\begin{bmatrix} | \\ \boldsymbol{y} \\ | \end{bmatrix}}_{\in\mathbb{R}^{d_2\times 1}} = \underbrace{\begin{bmatrix} & \boldsymbol{M} & \\ & & \end{bmatrix}}_{\in\mathbb{R}^{d_2\times d_1}} \underbrace{\begin{bmatrix} | \\ \boldsymbol{x} \\ | \end{bmatrix}}_{\in\mathbb{R}^{d_1\times 1}}$$

$$y_i = \sum_{j=1}^{d_1} M_{ij} x_j$$



Multiple Inner Products

- The Euclidean inner product between $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{d\times 1}$:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{bmatrix} \right\rangle = \sum_{i=1}^{d} x_i y_i$$

$$\underset{\in \mathbb{R}^{d_2\times 1}}{\begin{bmatrix} | \\ \boldsymbol{y} \\ | \end{bmatrix}} = \underset{\in \mathbb{R}^{d_2\times d_1}}{\begin{bmatrix} & \boldsymbol{M} & \\ & & \end{bmatrix}} \underset{\in \mathbb{R}^{d_1\times 1}}{\begin{bmatrix} | \\ \boldsymbol{x} \\ | \end{bmatrix}}$$

$$y_i = \sum_{j=1}^{d_1} M_{ij} x_j$$

- In matrix form, we have:

$$\implies \boxed{\boldsymbol{y}^T \boldsymbol{x} = \begin{bmatrix} — & \boldsymbol{y}^T & — \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{x} \\ | \end{bmatrix} = \langle \boldsymbol{x}, \boldsymbol{y} \rangle}$$

- Let:

$$\begin{bmatrix} | \\ \boldsymbol{x} \\ | \end{bmatrix} \in \mathbb{R}^{d\times 1}, \quad \boldsymbol{M} = \begin{bmatrix} — & \boldsymbol{m}_1^T & — \\ — & \boldsymbol{m}_2^T & — \\ & \vdots & \\ — & \boldsymbol{m}_N^T & — \end{bmatrix} \in \mathbb{R}^{N\times d} \implies \begin{bmatrix} — & \boldsymbol{m}_1^T & — \\ — & \boldsymbol{m}_2^T & — \\ & \vdots & \\ — & \boldsymbol{m}_N^T & — \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{x} \\ | \end{bmatrix} = \begin{bmatrix} \langle \boldsymbol{x}, \boldsymbol{m}_1 \rangle \\ \langle \boldsymbol{x}, \boldsymbol{m}_2 \rangle \\ \vdots \\ \langle \boldsymbol{x}, \boldsymbol{m}_N \rangle \end{bmatrix}$$

Linear Combination of Column Vectors

$$\underbrace{\begin{bmatrix} | \\ \boldsymbol{y} \\ | \end{bmatrix}}_{\in \mathbb{R}^{d_2 \times 1}} = \underbrace{\begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix}}_{\in \mathbb{R}^{d_2 \times d_1}} \underbrace{\begin{bmatrix} | \\ \boldsymbol{x} \\ | \end{bmatrix}}_{\in \mathbb{R}^{d_1 \times 1}}$$

$$y_i = \sum_{j=1}^{d_1} M_{ij} x_j$$

- Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^d$.
- Let

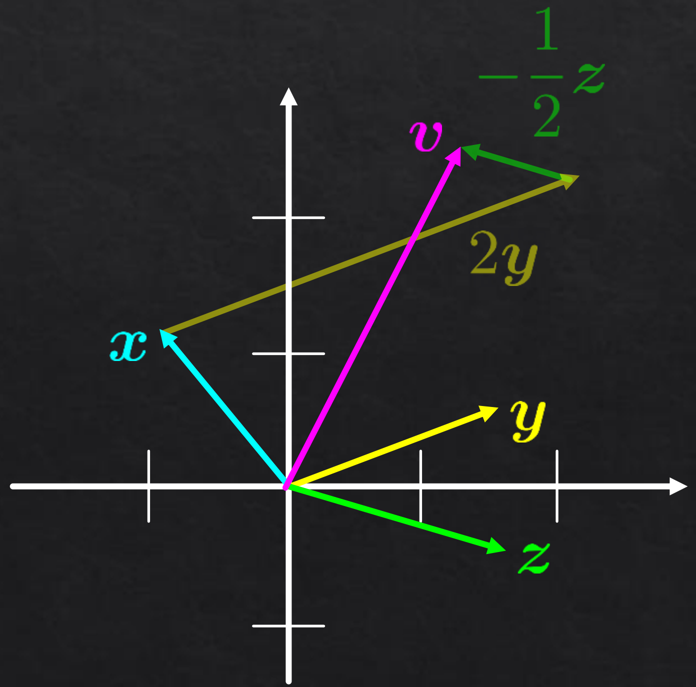
$$\boldsymbol{M} = \begin{bmatrix} | & | & | \\ \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} \\ | & | & | \end{bmatrix} \in \mathbb{R}^{d \times 3}, \quad \boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3$$

- Any linear combination of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ can be written as:

$$\boldsymbol{M}\boldsymbol{a} = \begin{bmatrix} | & | & | \\ \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} \\ | & | & | \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1\boldsymbol{x} + a_2\boldsymbol{y} + a_3\boldsymbol{z}$$

Example

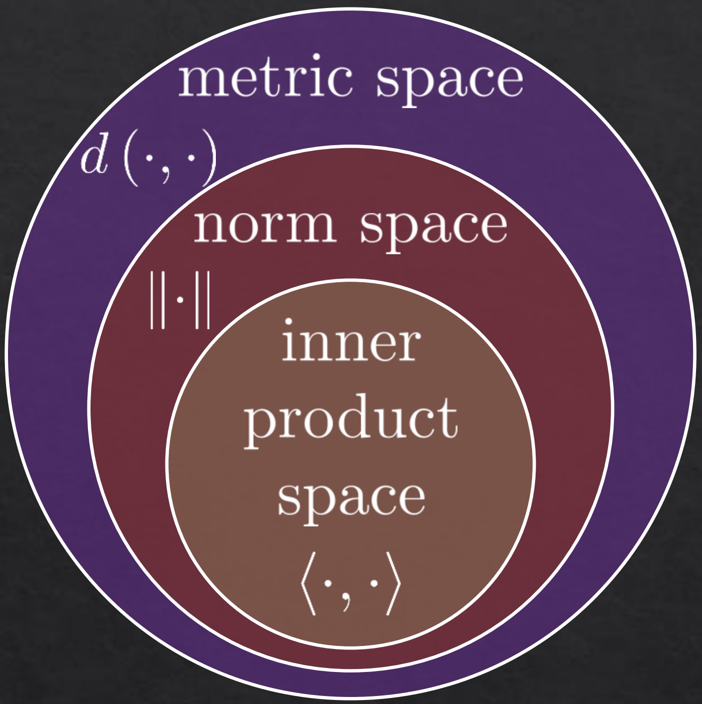
$$\boldsymbol{v} = \begin{bmatrix} | & | & | \\ \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} \\ | & | & | \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -\frac{1}{2} \end{bmatrix} = 1\boldsymbol{x} + 2\boldsymbol{y} - \frac{1}{2}\boldsymbol{z}$$

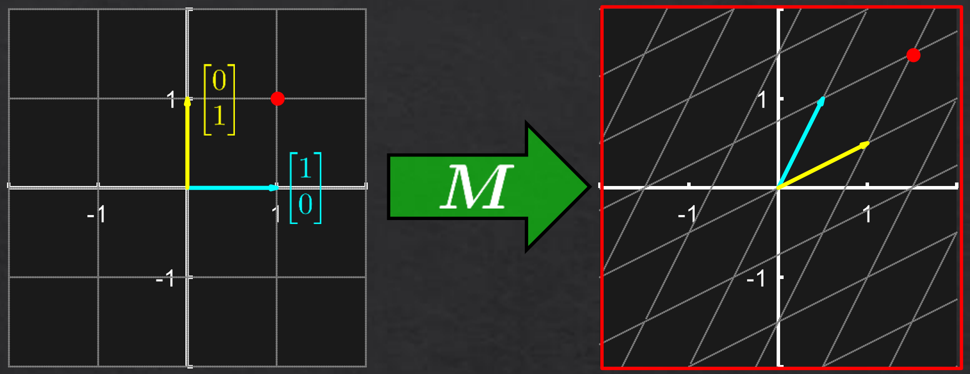


Other Important Topics

- Linearity
- Span and basis
- Change of basis
- Determinant
- Pseudo-Inverse
- Orthogonality
- Projections
- Eigendcomposition
- Singular value decomposition

Questions





$$\begin{cases} y_1 = a x_1 + b \\ y_2 = a x_2 + b \end{cases} \implies \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}}_{\boldsymbol{y} = \boldsymbol{X}\boldsymbol{a}}$$

$$\begin{bmatrix} - & \boldsymbol{m}_1^T & - \\ - & \boldsymbol{m}_2^T & - \\ & \vdots & \\ - & \boldsymbol{m}_N^T & - \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{x} \\ | \end{bmatrix} = \begin{bmatrix} \langle \boldsymbol{x}, \boldsymbol{m}_1 \rangle \\ \langle \boldsymbol{x}, \boldsymbol{m}_2 \rangle \\ \vdots \\ \langle \boldsymbol{x}, \boldsymbol{m}_N \rangle \end{bmatrix}$$

$$\underset{\in \mathbb{R}^{d_2 \times 1}}{\begin{bmatrix} | \\ \boldsymbol{y} \\ | \end{bmatrix}} = \underset{\in \mathbb{R}^{d_2 \times d_1}}{\begin{bmatrix} & \boldsymbol{M} & \end{bmatrix}} \underset{\in \mathbb{R}^{d_1 \times 1}}{\begin{bmatrix} | \\ \boldsymbol{x} \\ | \end{bmatrix}}$$

$$y_i = \sum_{j=1}^{d_1} M_{ij} x_j$$

$$\boldsymbol{M}\boldsymbol{a} = \begin{bmatrix} | & | & | \\ \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} \\ | & | & | \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1 \boldsymbol{x} + a_2 \boldsymbol{y} + a_3 \boldsymbol{z}$$

fixel your pixels

fixelalgorithms.gitlab.io