# TTK4135 Optimization and Control
# Lab Report

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**Abstract**

The purpose of the helicopter project in TTK4135 is to get experience combining optimization algorithms and control theory on a real world platform. The helicopter will be controlled using an optimal trajectory computed from a linearized model, both in open-loop and closed-loop with a linear quadratic regulator. The project also gives an introduction to hardware-in-the-loop (HIL) and automatic generation of code using MATLAB and QuaRC.

The main topics of this project are:

- Formulating and discretizing a state-space model for the helicopter
- Optimal control with linear inequality constraints
- Applying linear quadratic regulator (LQR) to the optimal trajectory
- Optimal control in two dimensions with non-linear inequality constraints

# Contents

# 1 Intro

## 1.1 Modelling and linearization

In this lab we will be using optimization theory to compute an optimal trajectory, with a corresponding input, for a small helicopter. We do this by deriving a non-linear model for the system dynamics, and linearizing around an equilibrium. This method is widely used in control theory and works well if the system operates close to the linearization point. We linearize once about a fixed equilibrium, but a possibility is to redo the linearization as new measurements of the system are made.

## 1.2 Optimal control, limitations without feedback

Optimization theory is used in many applications, from economy and production planning to control theory. In this lab we will be looking at the last application, and how it can be used to solve control theory problems. The main difference between optimal control and a conventional error-feedback controller, is the ability to "look into the future", as opposed to only using the current state.

While a conventional regulator will calculate an error and correct the input thereafter, a planned trajectory from an optimization problem will contain inputs for the complete horizon. However, straightforward application of the optimal input sequence will rarely give a good result. Modelling errors will cause the system to deviate from the optimal trajectory. It is therefore useful to include some form of output feedback.

## 1.3 Optimal control, trust the trajectory not the input

However, while the optimal sequence of inputs may not be perfect in open-loop, the planned trajectory for the states are often quite useful, even if the model have errors. Here it is possible to implement a traditional PID regulator, or as we will be doing in this project, implement a linear quadratic regulator (LQR).

The LQR can be both time variant and time invariant, dictated by the nature of the system. As mentioned in the linearizion part, the calculation of the LQR gain can and will be done by a computer, and will often be recalculated when re-linearizing. By tuning the LQR so that it prioritize to follow the trajectory of the states rather than following the optimal input, it is possible to compensate for modeling errors.

## 1.4 Optimal control with non-linear constraints, and thoughts on MPC

When controlling two degrees of freedom with non-linear inequality constraints with optimization theory, the time it takes to solve the optimization problem is quite long. When calculating the trajectory once and then executing the whole horizon this does not become a problem.

However when using MPC, which is to recalculate the horizon every time step and only use the first calculated input, this time becomes a problem. This calculation time can be improved by selecting a better solver, increase hardware specifications and do simplifications / linearizion. When achieving this on a low-power unit like a micro controller, FPGA, DSP or equivalent, a new world in control theory will open.

## 2 Problem Description

### 2.1 Hardware setup

The helicopter is fixed to a base and has an extended arm with motors attached at one end, and a balance weight at the other. We parametrize the helicopter state by three rotations:

- Travel ($\lambda$): At the base about the vertical axis
- Elevation ($e$): At the arm joint about a horizontal axis
- Pitch ($p$): At the head joint about the arm axis

These angles are measured by the encoders at each rotational joint. We can affect the motion by adjusting the voltage input to the DC motors. Propellers are attached to the motors, and the thrust generated is assumed to be proportional to the voltage applied, with identical motor constant for both motors. Figure (1) shows the helicopter and the relevant angles. Table (1) shows the parameters for the helicopter used in the report.

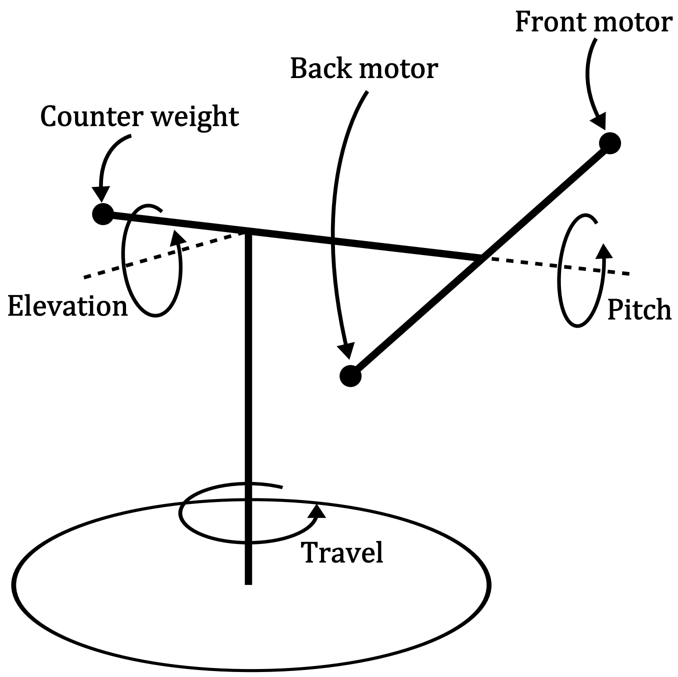


Figure 1: The helicopter hardware setup

### 2.2 Software setup

We use Simulink and Matlab to implement the control program, and interface with the helicopter through QuaRC and Realtime Workshop. See [Jesmani (2015)] for more information.

### 2.3 Model discussion

In order to compute an optimal control input, we need some measure of optimality. To this extent, we use a discrete-time state-space model, that describes the motion of our helicopter given an input sequence. The first step is to derive the equations of motions in continuous-time. We do this by considering torque balances about each rotation axis, and applying Newton's law. A full derivation is not interesting, so we refer to the exercise text [Jesmani (2015)] and list the resulting set of differential equations below.

$$\ddot{e} + K_3 K_{ed}\dot{e} + K_3 K_{ep} e = K_3 K_{ep} e_c \tag{1a}$$

$$\ddot{p} + K_1 K_{pd}\dot{p} + K_1 K_{pp} p = K_1 K_{pp} p_c \tag{1b}$$

$$\dot{\lambda} = r \tag{1c}$$

$$\dot{r} = -K_2 p \tag{1d}$$

The associated nomenclature is listed in table (2). This model deserves some discussion. First, a complete model would have (non-linear) trigonometric terms. But such a model is impractical for use in dynamic optimization schemes. Instead, we linearize the model by assuming that the pitch and elevation angles are both very close to zero.

The resulting model has the advantage of being compatible with highly efficient techniques that have been developed specifically for linear optimization problems. It is however a very simple model, and omits any interaction between elevation and pitch angle/travel. While there will always be modelling errors due to process noise, or inaccurate measurements/parameters, this simplification will affect the computed optimal control. We can dampen the effect from the error by keeping the system close to the linearization point, for instance by constraining the angles to be within a margin around equilibrium.

Second, the model incorporates a PID controller for the elevation, which is assumed to counteract the constant torque due to gravity, as well as a PD controller for the pitch. The result of this is that our goal in the optimization problem is to compute the setpoints, $e_c$ and $p_c$. Computing the appropriate voltages is left to the internal controllers.

Table 1: Parameters and values

| Symbol | Parameter | Value | Unit |
|---|---|---|---|
| $l_a$ | Distance from elevation axis to helicopter body | 0.63 | m |
| $l_h$ | Distance from pitch axis to motor | 0.18 | m |
| $K_f$ | Force constant motor | 0.20 | N/V |
| $J_e$ | Moment of inertia for elevation | 1.11 | $\mathrm{kg\,m^2}$ |
| $J_t$ | Moment of inertia for travel | 1.11 | $\mathrm{kg\,m^2}$ |
| $J_p$ | Moment of inertia for pitch | 0.045 | $\mathrm{kg\,m^2}$ |
| $m_h$ | Mass of helicopter | 1.42 | kg |
| $m_w$ | Balance weight | 1.80 | kg |
| $m_g$ | Effective mass of the helicopter | 0.025 | kg |
| $K_p$ | Force to lift the helicopter from the ground | 0.25 | N |

Table 2: Variables

| Symbol | Variable |
|---|---|
| $p$ | Pitch |
| $p_c$ | Setpoint for pitch |
| $\lambda$ | Travel |
| $r$ | Speed of travel |
| $r_c$ | Setpoint for speed of travel |
| $e$ | Elevation |
| $e_c$ | Setpoint for elevation |
| $V_f$ | Voltage, motor in front |
| $V_b$ | Voltage, motor in back |
| $V_d$ | Voltage difference, $V_f - V_b$ |
| $V_s$ | Voltage sum, $V_f + V_b$ |
| $K_{pp}, K_{pd}, K_{ep}, K_{ei}, K_{ed}$ | Controller gains |
| $T_g$ | Moment needed to keep the helicopter flying |

TODO: Variable notation for plots and stuff.

# 3 Repetition/Introduction to Simulink/QuaRC

This part is just a repetition, and therefore does not contain any important results. However, the part is important because it is used to test the hardware setup at the lab. This was done using a handed out script that made the helicopter hover at zero elevation and zero pitch. The helicopter suffered from minor drifting as a result of not perfectly tuned motor constants in the handed out script. This was expected and does not really introduce any problem since the drifting would be eliminated as soon as some form of feedback was implemented.

# 4 Optimal Control of Pitch/Travel without Feedback

In this section an optimal trajectory $x^*$ and a corresponding input sequence $u^*$ was calculated. The trajectory should bring the helicopter from an initial state to another predefined state. No feedback was used to correct for deviations from the calculated trajectory. Also, elevation was disregarded, that is, $e = 0$ was assumed for this part. The computation of the optimal trajectory was formulated as a discrete convex optimization problem.

## 4.1 The helicopter model on continuous-time state-space form

The system (1) describes the helicopter plant, with a basic control layer consisting of PID and PD controllers for elevation and pitch. The optimization layer gives the inputs to these regulators, as shown in figure (2).

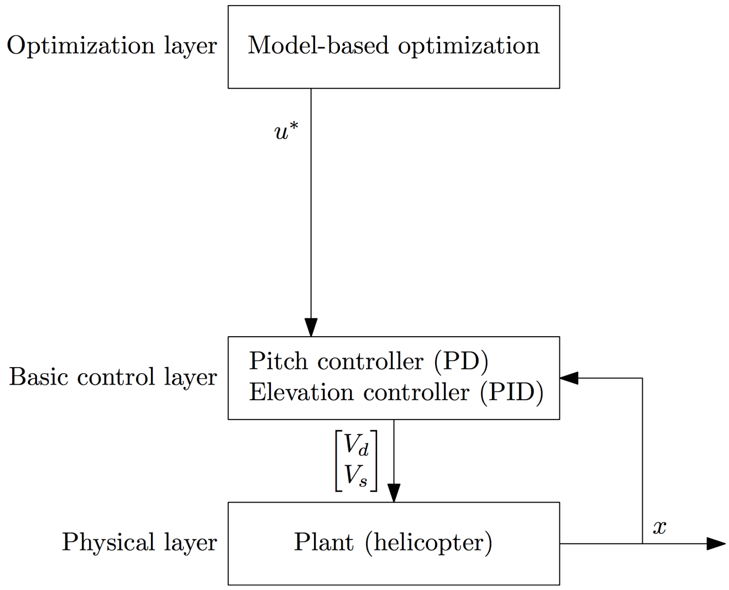


Figure 2: Control hierarchy

The system can be written on continuous-time state-space form:

$$\dot{x} = A_c x + B_c u \tag{2}$$

with $x = \begin{bmatrix} \lambda & r & p & \dot{p} \end{bmatrix}^T$ and $u = p_c$. The continuous-time system matrices for this model is:

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -K_2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -K_1 K_{pp} & -K_1 K_{pd} \end{bmatrix} \quad \text{and} \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ K_1 K_{pp} \end{bmatrix} \tag{3}$$

## 4.2 Discretization

The system dynamics was implemented as a sequence of constraints in the optimization problem, and the system model therefore had to be written in discrete-time state-space form,

$$x_{k+1} = A x_k + B u_k. \tag{4}$$

The model was discretized using forward-Euler with timestep $h$.

$$\dot{x}_k \approx \frac{x_{k+1} - x_k}{h} \tag{5}$$

Inserting this into (2),

$$\dot{x}_{k+1} \approx (I + hA_c)x_k + hB_c u_k \tag{6}$$

is obtained. A suitable approximation for the discrete-time matrices is then

$$A \approx I + hA_c \qquad \text{and} \qquad B \approx hB_c \tag{7}$$

These matrices are computed in Matlab, and are therefore not shown explicitly here.

### 4.3 Computation of optimal trajectory

An optimal trajectory can be generated by minimizing the cost function

$$\phi = \sum_{i=1}^{N} (\lambda_i - \lambda_f)^2 + qp_{ci}^2 \tag{8}$$

for some scalar weight $q \geq 0$ over the finite horizon of states and inputs

$$z = (x_1\ x_2\ \ldots\ x_N\ u_1\ u_2\ \ldots\ u_N)^T \tag{9}$$

This was done in Matlab using the function quadprog. The discrete-time system dynamics was implemented as equality constraints of the form $A_{\text{eq}}z = B_{\text{eq}}$, where $A_{\text{eq}}$ and $B_{\text{eq}}$ are given by the left- and right-hand side of the $N$ constraints

$$\begin{aligned} x_1 - Bu_0 &= Ax_0 \\ x_2 - Ax_1 - Bu_1 &= 0 \\ &\vdots \\ x_N - Ax_{N-1} - Bu_{N-1} &= 0 \end{aligned}$$

We would also like to constrain the system state and input to be within a range

$$x^{\min} \leq x_{t+1} \leq x^{\max} \tag{10}$$

$$u^{\min} \leq u_t \leq u^{\max} \tag{11}$$

for $t = 0 \ldots N-1$. Applying these constraints to all states and inputs in the solution horizon, we have

$$\begin{bmatrix} I \\ -I \end{bmatrix} z \leq \begin{bmatrix} \{x_{t+1}^{max}\} \\ \{u_t^{max}\} \\ \{x_{t+1}^{min}\} \\ \{u_t^{min}\} \end{bmatrix}_{t=0..N-1} \tag{12}$$

,

which can be implemented as an inequality constraint of the form $A_{\text{iq}}z \leq B_{\text{iq}}$. Solving the optimization problem with different weights $q$ did not lead to any significant differences in the trajectory because the model inaccuracies made the helicopter drift away from the desired state anyway.

The objective function (8) weigths the input relative to the state deviations using the weigth parameter $q$. The term $(\lambda_i - \lambda_f)^2$ is the state deviations. They are squared to priotitize the largest deviations. Note that the cost function (8) does not take into consideration that $\lambda_i$ plus some multiple of $2\pi$ describes the same physical orientation of the helicopter. For example, if the reference is 0 and $\lambda_i = 2\pi$, it will be regarded as a large error, even though the helicopter is infact in the desired orientation. A more optimal scheme would take this into consideration.

### 4.4 Implementation

Figure (3) shows the implementation with $q = 1$. Zeroes was added on both sides of the input sequense to give time to initialize and stabilize the helicopter. As can be seen from the figure, the helicopter does not reach its desired final state.

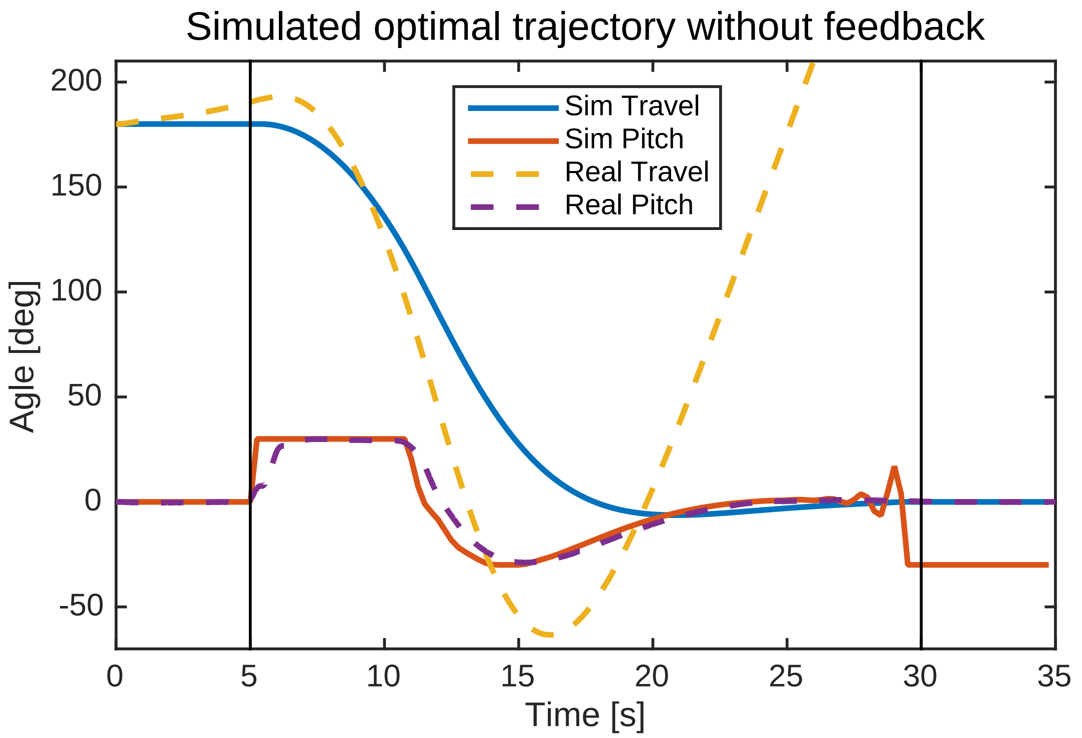


Figure 3: Plot of day 2

The deviation was caused by an imperfect model. A perfect model is impossible to construct, and without feedback, the model inaccuracies will lead to nonoptimal results in the real world. For example, the pitch-regulator in our model is fast enough to follow its input perfectly, and we are using a linear model that clearly does not correspond perfectly to the real helicopter.

## 5 Optimal Control of Pitch/Travel with Feedback (LQ)

This problem involves implementing an LQ controller for optimal control with feedback. We will calculate a gain matrix K using the LQ controller, implement feedback on the helicopter, and look into if MPC is a good alternative to LQ.

### 5.1 Calculating the gain matrix K

For calculating the gain matrix K we needed to use the matlab function dlqr, which is a linear-quadratic regulator design that minimiizes the cost function:

$$J = Sumx'Qx + u'Ru + 2 * x'Nu \tag{13}$$

. This function depend on the system matrices A and B aswell as Q and R, which we will need to choose an appropriate weight on. At first we choose Q and R as:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} 1 \end{bmatrix} \tag{14}$$

By trial and error we got the best result by penalizing the state for travel a lot more than the pitch. This gave us a new Q matrix on the form:

$$\mathbf{Q} = \begin{bmatrix} 50 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{15}$$

### 5.2 Implementing feedback on the helicopter

An implementation of feedback can be seen on the simulink diagram.

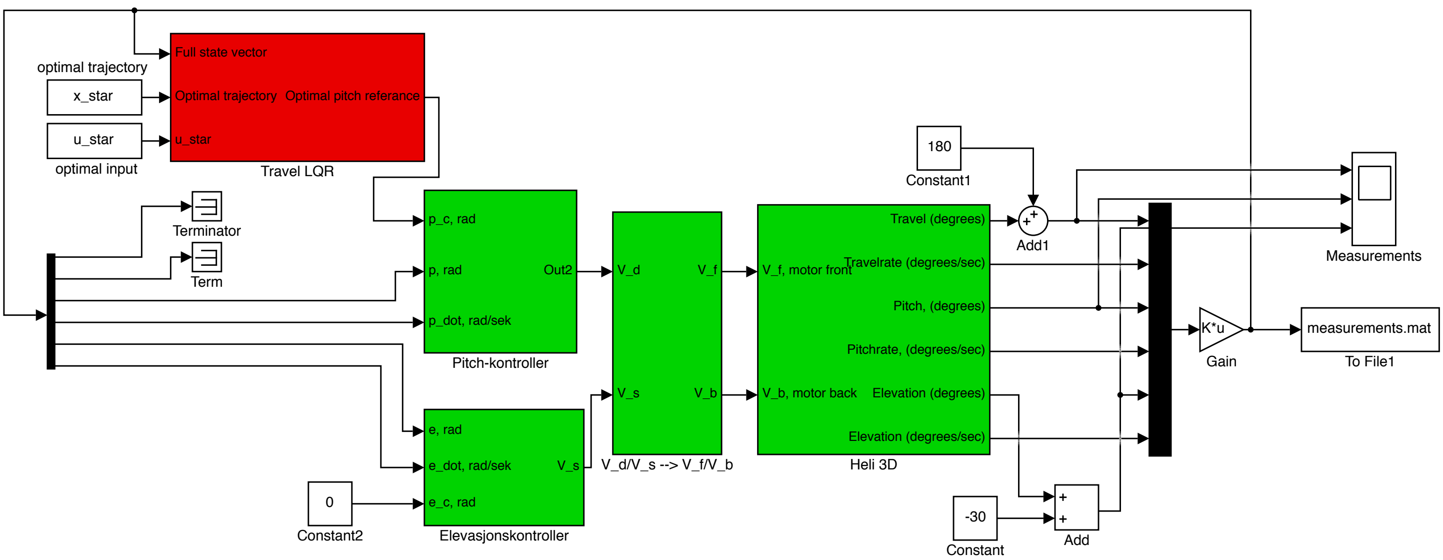


Figure 4: Simulink model with feedback

By using the gain matrix K calculated in last task, we see that by penalizing travel hard and pitch soft, the helicopter follows the travel trajectory closer than when weighting all states the same.

We also tried penalizing the pitch more than travel, which gave us a bad result. This because the optimal pitch reference trajectory is based on a bad model.

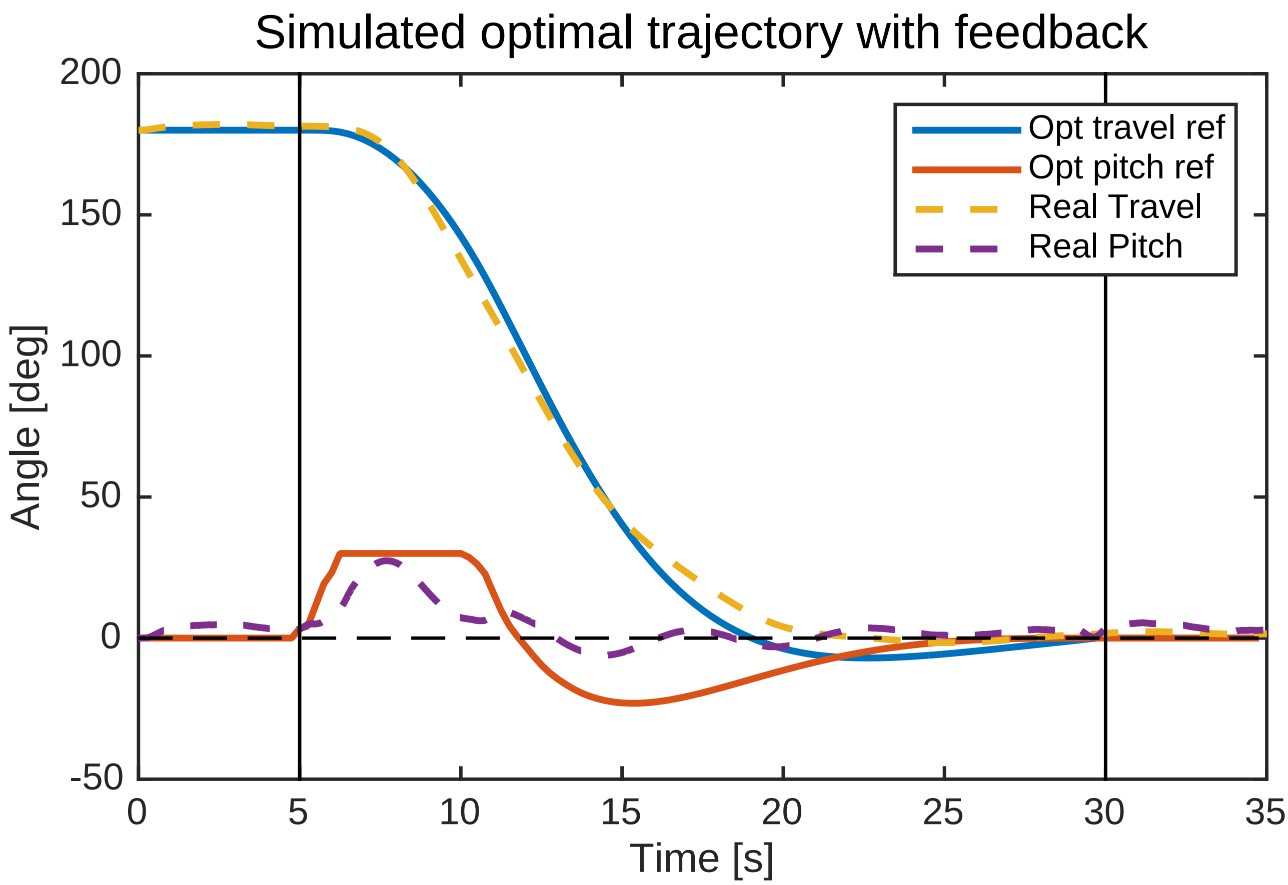


Figure 5: Plot of day 3 with Q = diag(50 1 1 1 )

### 5.3 Comparison between LQR and MPC

The way to implement an MPC controller would be to use the same procedure as for the LQR, but for every time step. Then using the first time step as the input u.

The advantages of using MPC instead of LQR; is that it gives us the possibility to have constraints in the regulator, can potentially produce a trajectory which performs its' task more cheaper and we get a implicit feedback with the use of MPC. The main disadvantage of using MPC is that the calculations are a lot heavier to perform.

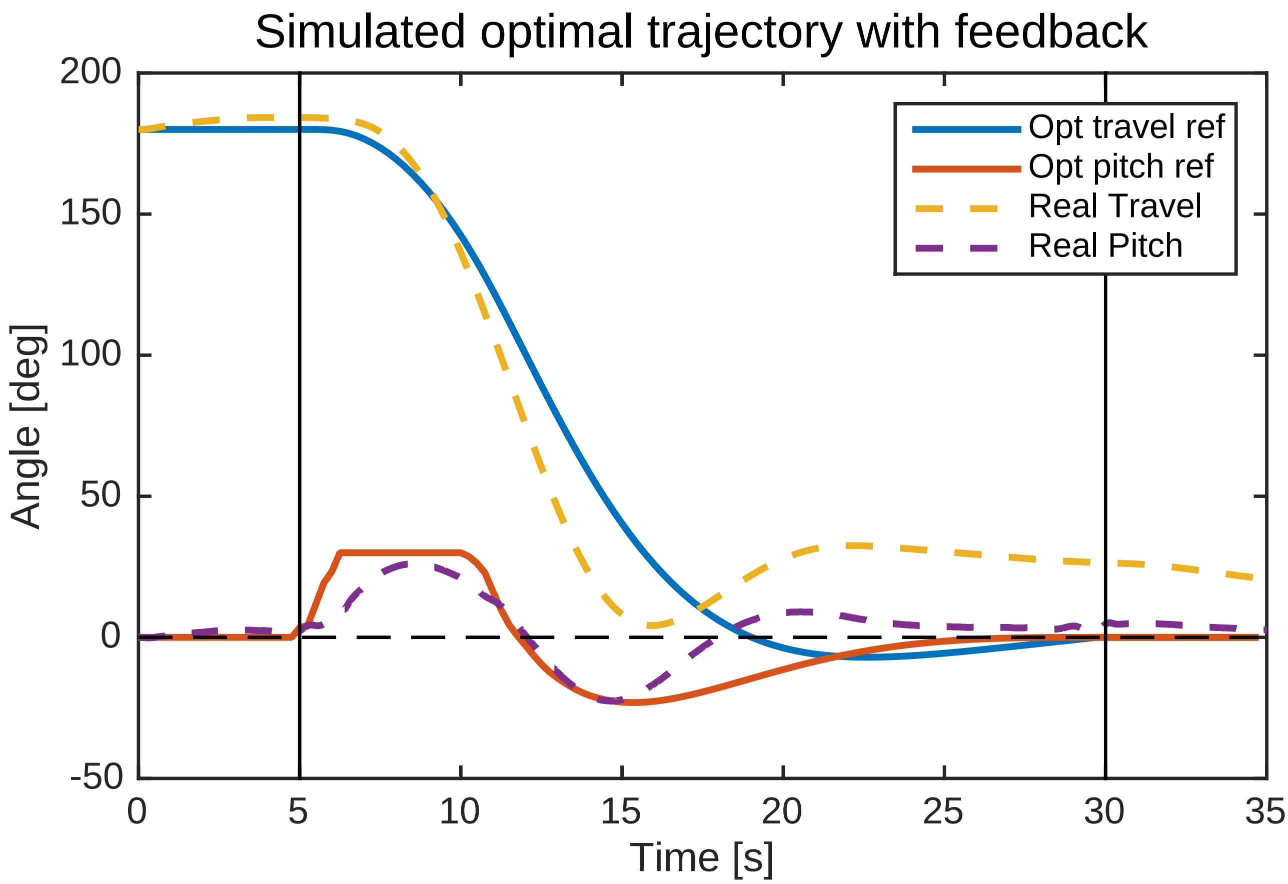


Figure 6: Plot of day 3 with Q = diag(1 1 10 1 )

# 6 Optimal Control of Pitch/Travel and Elevation with and without Feedback

In this section we extended our model to include the remaining states: elevation, $e$, and elevation rate, $\dot{e}$. We used a non-linear solver to compute an optimal trajectory in two dimensions, and additionally constrained the elevation to avoid a restriction shaped as a bell-curve.

## 6.1 State-space formulation

The state-vector was extended with the remaining states,

$$x = \begin{bmatrix} \lambda & r & p & \dot{p} & e & \dot{e} \end{bmatrix}^T \tag{16}$$

and the elevation setpoint was added to the input-vector,

$$u = \begin{bmatrix} p_c & e_c \end{bmatrix}^T \tag{17}$$

The system is on the usual state-space form (2), with

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -K_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -K_1 K_{pp} & -K_1 K_{pd} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -K_3 K_{ep} & -K_3 K_{ed} \end{bmatrix}}_{A_c} x + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ K_1 K_{pp} & 0 \\ 0 & 0 \\ 0 & K_3 K_{ep} \end{bmatrix}}_{B_c} u \tag{18}$$

## 6.2 Discretization

We discretized (18) using the same method as in section (4). That is, an approximation of the discrete-time state-space matrices was

$$A \approx I + hA_c \qquad \text{and} \qquad B \approx hB_c \tag{19}$$

where $I$ was the $6 \times 6$ identity matrix.

## 6.3 Modelling the restriction

A common application of optimal control is to implement restrictions, such as avoiding physical objects, as constraints in the optimization problem. Such restrictions can not be enforced when using only state-feedback controllers.

We wished to restrict the helicopter head to move above the bell-shaped curve

$$e_k \geq \alpha \exp(-\beta(\lambda_k - \lambda_t)^2) \tag{20}$$

for all timesteps $k$ of the solution horizon. Since this is a non-linear constraint, we could no longer use a QP solver. Instead, a non-linear solver was needed, and in this case, the MATLAB function fmincon was used with a SQP-type algorithm.

## 6.4 Objective function

For this assignment, the cost function is the same as in equation (8), but with an extra term for penalizing elevation. We chose to set up the cost function on a more general form:

$$\phi = \sum (x^T Q x + u^T R u) \tag{21}$$

Here, $Q$ and $R$ are diagonal matrices, with unity weight on $q_{travel}$, $r_{pitch}$ and $r_{elevation}$, so that it corresponds with the extended state and input vectors (16).

### 6.5 Results

For controlling the helicopter, we ended on a horizon of $N = 60$ timesteps, or 15 seconds. This gave an optimal trajectory that looked much the same as for assignment 3 for pitch and travel. In addition, an elevator trajectory just touching the top of the bell curve constraint was also calculated. The suggested horizon of $N = 40$ did not terminate in reasonable time, and was therefore replaced by a longer horizon giving reasonable results.

The solution without feedback acted in the same manner as it did in section (4). This can be seen in figure (??) !!!!!, the real trajectory of travel and elevation tried to follow the optimal solution, but the effect of not having a perfect model became more and more clear as time passed, and at the end of the horizon, the travel drifted away from the solution just like in section (4).

Using state feedback LQR, the helicopter managed to follow the calculated trajectory quite good after some tuning. As in section (5), it was appropriate to weight deviation(s) in travel (and here also elevation) more than deviation in pitch. When the regulator is working on an inaccurate model, it is for the task given more constructive to try to follow the travel and elevation trajectories rather than the ideal, linearized input trajectory. This input will, given the unlinearities of the model, not lead to an appropriate response for the trajectories. The result, with different weights on the states of $Q_{LQR}$ can be seen in the following figures.

One thing worth noticing is that the trajectory of elevation consequently fell below the constraint at the peak of the constraint. We believe this is due to the fact that our model suggests that there is no link between the model for elevation, and that of pitch. They are in fact linked, and because of this, the regulator doesn't manage to meet both demands perfectly at the same time, and we get a deviation. If the model would include these crossing terms, the result would probably be better. The problem with this approach is that it leads to a non linear objective function, which takes longer time to compute for the solver.

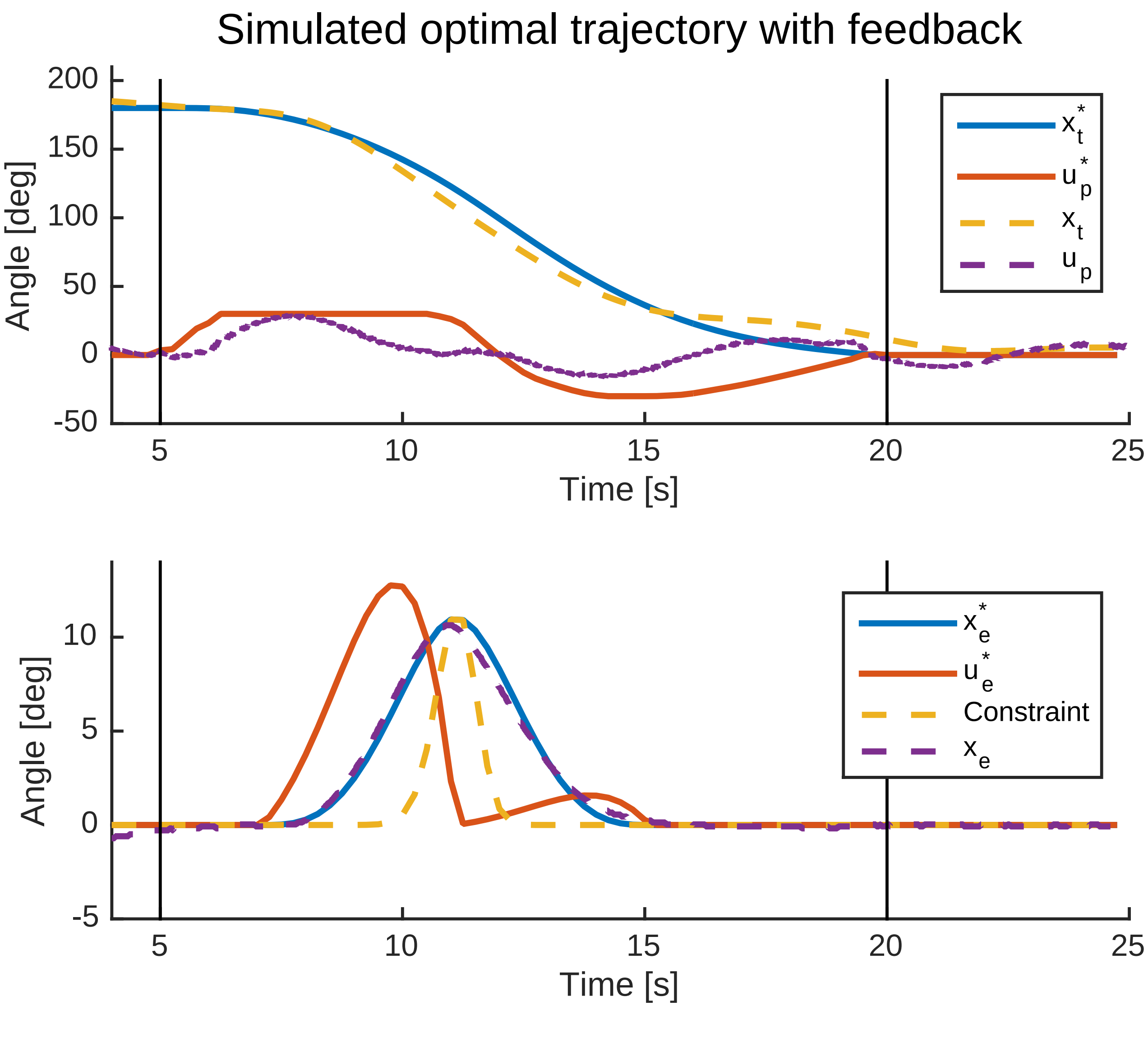


Figure 7: Plot day 4 closed loop Q=diag(20 1 1 1 30 10)

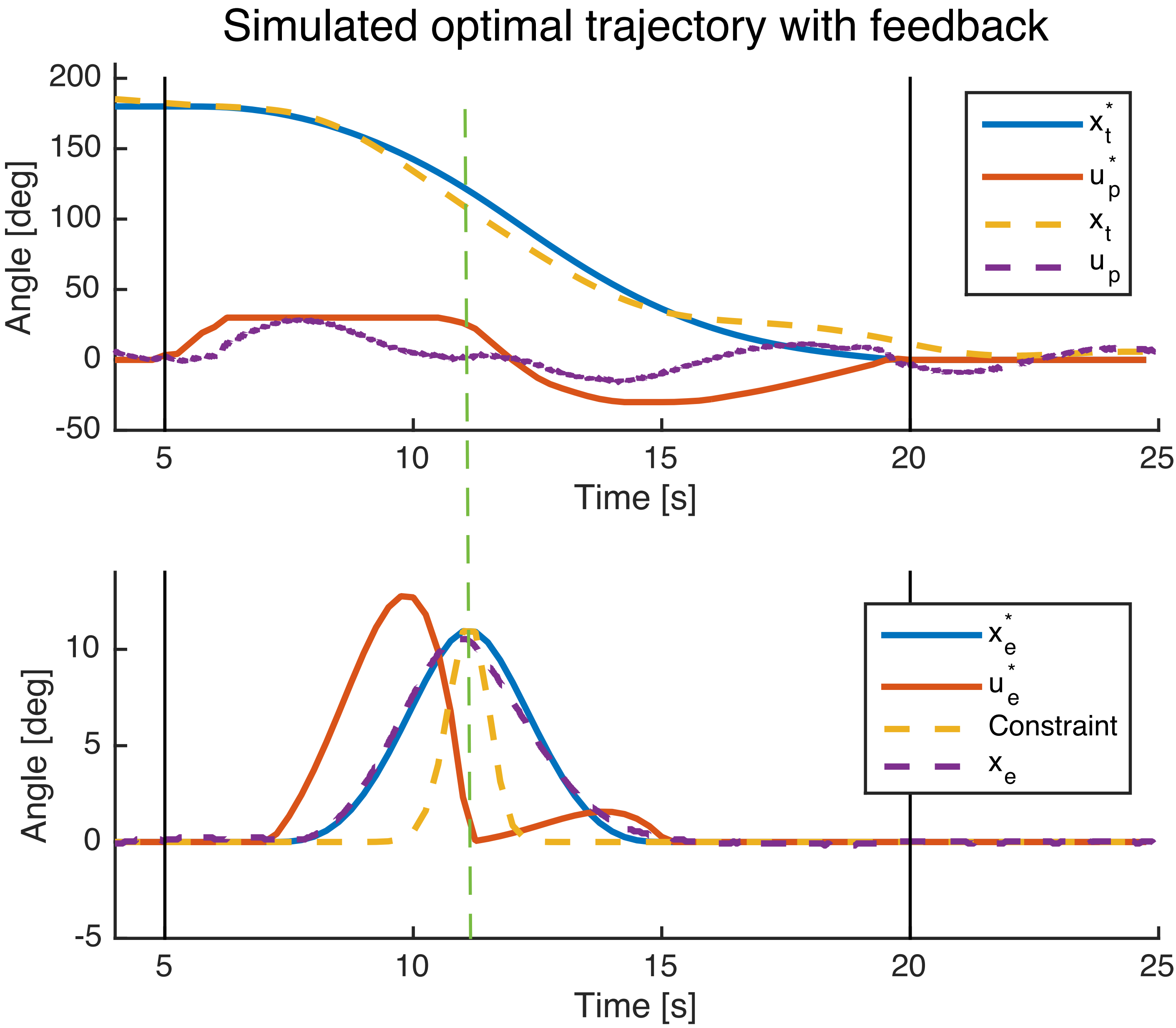


Figure 8: Coupling between pitch and elevation

# 7 Discussion

TODO: Se template

## 7.1 Optimal control without feedback

When controlling the helicopter with only the estimated optimal input sequence, the linearization and modeling errors becomes to large, and the helicopters behavior will deviate drastically from the estimated response.

## 7.2 Optimal control with feedback

While the open-loop configuration of optimal control was quite poor, the closed-loop configuration with a LQR was quite promising.

- Limit inputs - Avoid areas

## 7.3 MPC w/ implicit feedback

# 8 Conclusion

TODO: Se template

# A MATLAB Code

TODO: Se template

## A.1 plot_constraint.m

```
% Plot a figure with some Latex in the labels
l = linspace(70,170)*pi/180;
a = 0.2;
b = 20;
l_b = 2*pi/3;

e = a*exp(-b*(l-l_b).^2);

l_deg = l*180/pi;
e_deg = e*180/pi;

figure(1)
plot(l_deg,e_deg, 'LineWidth', 2)

handles(1) = xlabel('$\lambda$/degrees');
handles(2) = ylabel('$e$/degrees');
set(handles, 'Interpreter', 'Latex');
```

# B Simulink Diagrams

This section should contain your Simulink diagrams.

## B.1 A Simulink Diagram

# Bibliography

Jesmani, M. (2015). Ttk4135 optimization and control, helicopter lab.