Lecture 1 – Maths for Computer Science Multiple ways for solving a problem

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Lecture notes MoSIG1

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Context

The purpose of this lecture is to show multiple ways for solving the same problem.

We take the sum of squares as a more complete case study.

Various ways to solve the sum of squares

Definition:

Sum of the n first squares:
$\square_n = \sum_{k=1}^{n} k^2$.

Method 1: determine the asymptotic behavior

Very rough analysis:
as $k^2 \leq n^2 \ \forall k \leq n$, $\square_n \leq \sum_{k=1}^{n} n^2 = n^3$.

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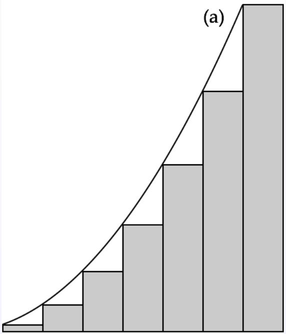
as $k^2 \leq n^2 \; \forall k \leq n$, $\square_n \leq \sum_{k=1}^{n} n^2 = n^3$.
A slightly more precise analysis is: $\square_n \leq c\frac{n^3}{3}$



In other words, it is in $O(\frac{n^3}{3})$.

asymptotic behavior

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asymptotic behavior

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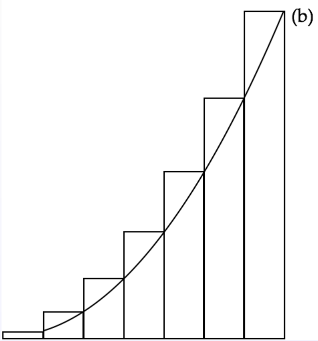
$\square_n \geq c' \frac{n^3}{3}$



It is in $\Omega(\frac{n^3}{3})$, thus, it is $\Theta(\frac{n^3}{3})$

Method 2: by induction

Compute the first ranks:

n	0	1	2	3	4	5	6	7	8	9	10
n^2	0	1	4	9	16	25	36	49	64	81	100
Sn	0	1	5	14	30	55	91	140	204	285	385

Guess the expression (or take it in a book):
$\square_n = \frac{n(n+1)(2n+1)}{6}$

Strong induction

- Basis: $\square_1 = \frac{(2\times 3)}{6} = 1^2$
- Assume $\square_n = \frac{n(n+1)(2n+1)}{6}$

  Compute $\square_{n+1} = \square_n + (n+1)^2$

  $= (n+1)\frac{n(2n+1)}{6} + n + 1$

  $= (n+1)\frac{2n^2+n+6n+6}{6}$

  $= \frac{(n+1)(n+2)(2n+3)}{6}$

Method 3: undetermined coefficients

Let write $\square_n = \alpha_0 + \alpha_1 n + \alpha_2 n^2 + \alpha_3 n^3$

$\square_0 = \alpha_0 = 0$
$\square_1 = \alpha_1 + \alpha_2 + \alpha_3 = 1$
$\square_2 = 2\alpha_1 + 4\alpha_2 + 8\alpha_3 = 5$
$\square_3 = 3\alpha_1 + 9\alpha_2 + 27\alpha_3 = 14$

Method 3: undetermined coefficients

Let write $\Box_n = \alpha_0 + \alpha_1 n + \alpha_2 n^2 + \alpha_3 n^3$

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$\alpha_1 = \frac{1}{6}$, $\alpha_1 = \frac{1}{2}$ and $\alpha_1 = \frac{1}{3}$

Thus, $\Box_n = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}$

Method 4: perturb the sum

Developing two ways to compute $C_n = \sum_{k=1}^{n} k^3$ allows to express $\square_n$.

1. $C_{n+1} = 1 + \sum_{k=2}^{n+1} k^3$
   $= 1 + \sum_{k=1}^{n}(k+1)^3$
   $= 1 + \sum_{k=1}^{n}(k^3 + 3k^2 + 3k + 1)$
   $= 1 + C_n + 3\square_n + 3\Delta_n + n$
2. $C_{n+1} = (n+1)^3 + \sum_{k=1}^{n} k^3 = (n+1)^3 + C_n$
   $= n^3 + 3n^2 + 3n + 1 + C_n$

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Let now equal both expression to deduce $\square_n$.

$$1 + 3\square_n + 3\frac{n^2+n}{2} + n = n^3 + 3n^2 + 3n + 1$$

$$3\square_n = n^3 + 3n^2 + 2n - 3\frac{n^2+n}{2} = n^3 + \frac{3n^2}{2} + \frac{n}{2}$$

Method 5: expand and contract the sum

$\square_n = \sum_{k=1}^{n} k^2$

$= \sum_{k=1}^{n} \sum_{i=1}^{k} k$

$= 1 + (2+2) + (3+3+3) + (4+4+4+4) + \ldots + (n+n+\ldots+n)$

Method 5: expand and contract the sum

$$\Box_n = \sum_{k=1}^{n} k^2$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{k} k$$

$$= 1 + (2+2) + (3+3+3) + (4+4+4+4) + ... + (n+n+...+n)$$

$$= (1+2+....+n) + (2+3+...+n) + (3+4+...+n) + ... + n$$

$$= \sum_{k=0}^{n-1} (\Delta_n - \Delta_k)$$

$$= n.\Delta_n - \sum_{k=1}^{n-1} \Delta_k$$

$$\Box_n = \frac{n^2(n+1)}{2} - \sum_{k=1}^{n-1} \frac{k^2}{2} - \frac{1}{2}\Delta_{n-1}$$

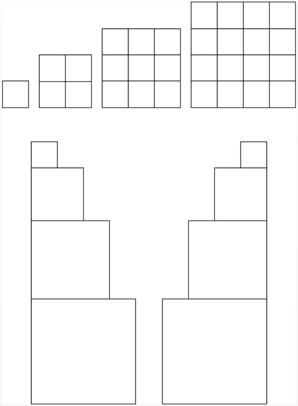
$$\Box_n = \frac{n^2(n+1)}{2} - \frac{1}{2}(\Box_n - n^2) - \frac{n(n-1)}{4}$$

$$\frac{3}{2}\Box_n = \frac{1}{2}(n^3 + n^2 + n^2 - \frac{n^2-n}{2})$$

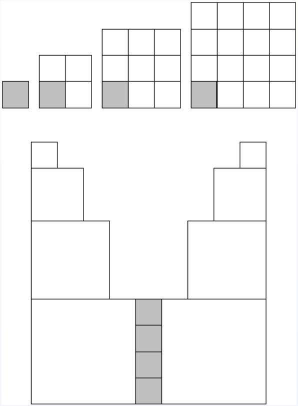
$$\Box_n = \frac{1}{3}(n^3 + \frac{3}{2}n^2 + \frac{n}{2})$$

Method 6: graphical proof

Consider 3 copies of the sum represented by unit squares.



Graphical proof



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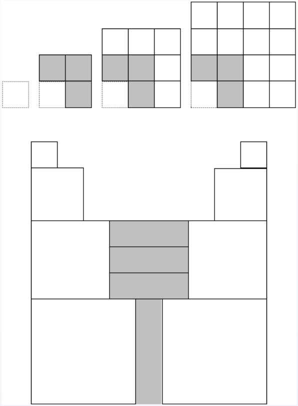

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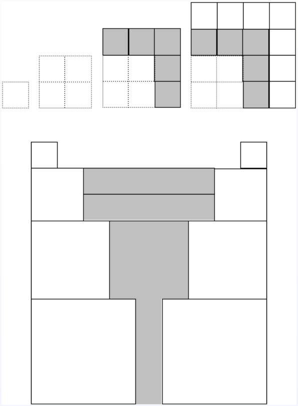

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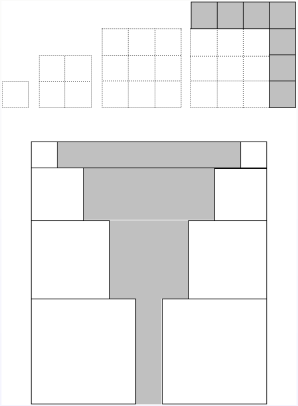

Graphical proof

Conclusion: The area of the 3 sums is equal to a big rectangle $2n+1$ by $\Delta_n = \frac{n(n+1)}{2}$.

Thus, $3\square_n = \frac{(2n+1)n(n+1)}{2}$