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Maths

Question N.1:

Finite summations: $\Sigma_{k=0,n} 2^k = ?$

This is a particular case of geometric progression $1 + 2 + 4 + 8 + 16 + ...$
$\Sigma_{k=0,n} a^k = \frac{a^{n+1}-1}{a-1}$ for $a > 1$.

$\Sigma_{k=0,n} 2^k = 2^{n+1} - 1$

Question N.2:

Prove $\Sigma_{k=1,n}(k^2(k+1) - k(k-1)^2) = n^2(n+1)$

The idea here was to figure out **how to write a simple proof.**
The basic insight comes by remarking that the result we are looking for (i.e. the right hand side) is contained into the summation, more precisely, this is the first term with $k = n$. Then, the summation can be written as follows:
$n^2(n+1) + \Sigma_{k=1,n-1}(k^2(k+1) - \Sigma_{k=1,n} k(k-1)^2)$
Now, let us also remark that the second can be simplified since the first term is nul for $k = 1$:
$\Sigma_{k=2,n} k(k-1)^2)$
Now, let shift the indices in this sum (change k to $k' = k+1$):
$\Sigma_{k'=1,n-1}(k'+1)k'^2)$.
This concludes the proof since both summations are the same.

Question N.3:

Identities: $a^n - b^n = ?$
$(a+b)^n = ?$

$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + ... + b^{n-1})$.
The expression of question 1 above is obtained for $a = 2$ and $b = 1$.
The second expression is the classical Newton binomial formula.

Question N.4:

What are the values of $\Sigma_{k>0} \frac{1}{2^k}$ and $\Sigma_{k>0} \frac{1}{k}$?

The *limit* of the first sum is 1.
Obtaining a finite value for an infinite sum was a paradox until the infinitesimal calculus of Leibniz/Newton on the XVIIth century.

The second sum is unbounded (it goes to $+\infty$).

Question N.5:

Classify the functions:
$log(n)$, 2^n, $\sqrt{n}$, n^n, $log(log(n))$, n^3

They are all not decreasing functions, the hierarchy is the following:
for some rank n, we have $log(log(n)) \leq log(n) \leq \sqrt{n} \leq n^3 \leq 2^n \leq n^n$.

Question N.6:

Consider $T = 1 + 2 + 4 + ...$
Compute $2T = 2 + 4 + 8 + ... = T - 1$, thus $T = -1$.
What is wrong?

A sum of non-negative integers can not be negative.
What is status of ∞?
Think about the summation 1-1+1-1 ...

Question N.7:

Give the definition of the derivative. Describe briefly its geometric interpretation.

The derivative of a continuous function is expressed as $f'(x) = lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

Question N.8:

What are the derivative of the following functions: $f_1(x) = log(x)$, $f_2(x) = x^2 + 2x$, $f_3(x) = \frac{1}{x}$?

log and $\frac{1}{x}$ are not defined for all x. The derivatives are obtained by applying simple rules (additive, multiplicative, division and composition).
$f_1'(x) = \frac{1}{x}$
$f_2'(x) = 2x + 2$
$f_3'(x) = -\frac{1}{x^2}$

Question N.9:

Recall the interpretation of the integral of a function. Example for x^2 on 0..1

The integral of a positive function is the surface below the curve (and above the variable axis).
It is the limit of the Riemann's sums.
Example: $\frac{1}{3}$

Question N.10:

Consider a – continuous – function $f(x)$. Give a definition and an example for the following asymptotic notations: $O(f(x))$, $\Omega(f(x))$, $\Theta(f(x))$.

$f = O(g) \Leftrightarrow \exists C > 0\ \exists n_0 > 0\ \forall n > n_0\ f(n) \leq Cg(n)$ – upper bound

$f = \Omega(g) \Leftrightarrow g = O(f)$ – lower bound

$f = \Theta(g) \Leftrightarrow f = O(g)$ and $f = \Omega(g)$

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 3 & 1 \\ 0 & 5 & 2 \end{pmatrix}$$

Question N.11:

Compute the determinant of A.

The purpose of both questions (this one plus the following) is to check the basic knowledge in linear algebra and matrix manipulation.
$det(A) = 3 \times 2 \times 2 + 3 \times 5 - 2 \times 5 = 17$

Question N.12:

Compute A^2.

$$A = \begin{pmatrix} 4 & 5 & 4 \\ 15 & 14 & 1 \\ 15 & 25 & 9 \end{pmatrix}$$

Question N.13:

Write the number 2019 in basis 2 (binary).

This is a simple example of coding: $2048 = 1024 + 512 + 256 + 128 + 64 + 32 + 2 + 1$
thus, $(2019)_{10} = (11111100011)_2$

Question N.14:

Define the notion of *equivalence relation*.

A binary relation $\mathcal{R}$ is an equivalence relation if the three following properties are fulfilled:

Reflexivity: $x\mathcal{R}x$

Symmetry: if $x\mathcal{R}y$ then $y\mathcal{R}x$

Transitivity: if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$

Question N.15:

Prove that the following relation $\mathcal{R}$ between two pairs of integers (n_i, m_i): $(n_1, m_1)\mathcal{R}(n_2, m_2)$ iff $n_1 + m_2 = n_2 + m_1$ is an equivalence relation.
Give the equivalent class of $(n = 1, m = 0)$.

Intuitively, this relation reflects the geometrical argument that states that the two pairs of points (n_1, m_1) and (n_2, m_2) are equivalent iff the differences $n_1 - m_1$ and $n_2 - m_2$ are equal.
Equivalence classes.
Give the equivalence class for $(1, 0)$ of the previous relation.

Question N.16:

What is an order relation?
What is a total order?

A binary relation $\mathcal{R}$ is an equivalence relation if the three following properties are fulfilled:

Reflexivity: $x\mathcal{R}x$

Transitivity: if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$

A total order is obtained by adding the property of *anti-symmetry*:
if $x\mathcal{R}y$ then $y\mathcal{R}x$

Question N.17:

Express $\log_a(x)$ with logarithms in base b.

log are a very useful functions.
b is the base.

They are strongly linked with exponentials $f(x) = b^x$ where classical multiplication becomes addition:
$b^x \times b^y = b^{x+y}$
$b^{log_b(a)} = a$
$log_b(x \times y) = log_b(x) + log_b(x)$

$log_b(1) = 0$, this is a consequence of the definition, not by convention! Easy to prove.

Now, let answer the question: $\log_a(x) = \log_a(b). \log_b(x)$.

Question N.18:

Give another expression for $n^{\log_a(b)}$.

For a deep understanding of what is log: $n^{\log_a(b)} = b^{\log_a(n)}$

Question N.19:

Solve the following problem:
You are attending a cocktail party that is populated by n couples. In order to create a warm atmosphere, the host requests that each attendee shake the hand of every attendee that he/she does not know.
Prove that some two attendees shake the same number of hands.

This is an illustration of the so-called *pigeon hole principle*:
The number of persons that each attendee does not know belongs to the set $\{0, 1, ..., 2n - 2\}$. We obtain the result since there are $2n$ hand shakers.

Question N.20:

Solve the following problem:
An integer n is divisible by 9 if, and only if, the sum of the digits of its base-10 digits is divisible by 9.

Some of you checked this property on several numbers to figure out if this result is true: good starting point. However, checking on examples is never a (formal) proof!