UNIVERSITÉ Grenoble Alpes

Programming-Language Semantics and Compiler Design / Sémantique des Langages de Programmation et Compilation

Tutorials / Travaux Dirigés

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CONTENTS

SERIES

ONE

NATURAL OPERATIONAL SEMANTICS OF LANGUAGE WHILE

In this exercise series, we consider language While and its natural operational semantics where configurations belong to $\mathsf{Stm} \times \mathsf{State} \cup \mathsf{State}$, i.e., are of the form (S, σ) or σ.

1.1 Properties of Arithmetical and Boolean Expressions

1.1.1 Free Variables

The set of free variables of an arithmetical or a Boolean expression is defined to be the set of variables occurring in it.

Exercise 1 — Locating free variables

Indicate the free variables in the following arithmetical expressions.

1. $x + 1$
2. $3 * x + y$

Exercise 2 — Free variables of arithmetical expressions

1. Define, in a compositional manner, a function $FV : \mathsf{Aexp} \to 2^{\mathsf{Vars}}$ that computes the *free variables* for an arithmetic expression.
2. Prove that only the free variables of an arithmetical expression may influence its value.

Exercise 3 — Free variables of Boolean expressions

1. Define, in a compositional manner, a function $FV : \mathsf{Bexp} \to 2^{\mathsf{Vars}}$ that computes the *free variables* of Boolean expressions.
2. Prove that only the free variables of a Boolean expression may influence its value.

1.1.2 Substitution

In the lecture, we define the notion of substitution for a state.

Exercise 4 — Computing some substitutions

Indicate the values of variables in the following states, where $\sigma = [x \mapsto 1, y \mapsto 2, z \mapsto 3]$:

1. $\sigma[x \mapsto y + z]$
2. $\sigma[x \mapsto 2 * x + y, y \mapsto x]$

Substituting a_0 to y in an arithmetical expressions a consists in replacing every occurrence of y by a_0, and is noted $a[y \mapsto a_0]$.

Exercise 5 — Defining Substitution for arithmetical expressions

We consider the arithmetical expressions defined in the course lecture.

1. Define formally substitution for arithmetical expressions.
2. Prove that the defined substitution is correct.

Exercise 6 — Defining substitution for Boolean expressions

Define substitution for Boolean expressions where variables are replaced by arithmetical expressions.

1.2 Natural Operational Semantics of (Pure) Language While

Exercise 7 — Computing some (simple) derivation trees

Consider the state σ_0 which maps all variables but x and y to 0, maps x to 5, and y to 7.

1. Give a derivation tree for the following statements on σ_0:
   a) $(x := y; x := z); y := z$,
   b) $(z := x; x := y); y := z$.
2. Give a derivation tree for the following statements:
   a) `if` $x + y \geq 3$ `then` $y := x$ `else` $x := y$ `fi`,
   b) `if` $y \geq x$ `then` $y := x$ `else` $x := y$ `fi`.

Exercise 8 — Computing some derivation trees

Consider the empty state $[\,]$[1]. Give a derivation tree for the following statement:

$$y := 1 : x := 2; \texttt{while}\ x \geq 1\ \texttt{do}\ (y := y * x; x := x - 1)\ \texttt{od}.$$

Exercise 9 — Computing some (simple) derivation trees

Consider state σ_0 where x has value 17 and y has value 5.

1. Construct a derivation tree for the following statement when executed on σ_0:

$$z := 0; \texttt{while}\ y \leq x\ \texttt{do}\ (z := z + 1; x := x - y)\ \texttt{od}$$

Exercise 10 — Execution terminates or loops?

For each of the following statements (where x designates a variable of type $\mathbb{Z}$), argue whether:

- its execution loops in every state, or
- its execution stops in every state, or
- there are states from which the execution terminates, and some from which it does not.

1. `while` $1 \leq x$ `do` $(y := y * x;\ x := x - 1)$ `od`
2. `while true do skip od`
3. `while` $\neg(x = 1)$ `do` $(y := y * x;\ x := x - 1)$ `od`

Exercise 11 — Execution terminates or loops? - proofs

Prove your answers for the previous exercise.

Exercise 12 — Operational semantics for arithmetical and Boolean expressions

[1] Recall that a state is a partial function for variable names Vars to $\mathbb{Z}$. The empty state is thus the partial function undefined for every variable name.

The purpose of this exercise is to define an alternative definition to the semantics of arithmetical and Boolean expressions. The semantics given in the lecture course is a declarative semantics defined inductive on the syntax of arithmetical expressions. In this exercise, we want to define an operational semantics.

1. Define an operational semantics for the set of arithmetical expressions **AExp**. The semantics should have two kinds of configurations:
   - (a, σ) denoting that a has to be evaluated in state σ, and
   - z denoting the final value (an element of $\mathbb{Z}$).

   That is the set of configurations of the underlying transition system is

   $$\mathit{Configurations} = (\mathsf{Aexp} \times \mathsf{State}) \cup \mathbb{Z}.$$

2. Prove that the semantics of a defined in this way is equivalent to the one defined in the lecture course (i.e., equivalent to the semantics defined by the inductive function $\mathcal{A}$).
3. Define an operational semantics for the set of Boolean expressions Bexp.
4. Prove that the meaning of a defined in this way is equivalent to the one defined in the lecture course (i.e., equivalent to the meaning defined by the inductive function $\mathcal{A}$).

Exercise 13 — A "subset" of language While

We consider the language defined by the following BNF:

$$S ::= x := a \mid \texttt{skip} \mid S_1; S_2 \mid \texttt{if } b \texttt{ then } S_1 \texttt{ else } S_2 \texttt{ fi}$$

1. What can we say about termination of programs written in this language?
2. Prove your statement.

Exercise 14 — Determinism of the natural operational semantics of language While

Prove that the natural operational semantics of language While is deterministic.

Exercise 15 — Associativity of sequential composition

1. Prove that, for all statements S_1, S_2, S_3, the following statements are semantically equivalent:
   - $S_1; (S_2; S_3)$, and
   - $(S_1; S_2); S_3$.

   You may use the fact that the semantics is deterministic.
2. Prove that, in general, $S_1; S_2$ is not semantically equivalent to $S_2; S_1$.

1.3 Natural Operational Semantics of Extensions of Language While

Exercise 16 — Extending language While with construct `repeat` ... `until` ...

We want to add the following statement to language While:

$$\texttt{repeat } S \texttt{ until } b$$

The informal semantics of this construct is that statement S should be executed until the Boolean condition b becomes true.

1. Provide the semantics rules in order to define `repeat` S `until` b without using the while b do $\cdots$ od construction.

2. Prove that the following statements are semantically equivalent:
   - `repeat` S `until` b, and
   - S; `if` b `then skip else` (`repeat` S `until` b) `fi`.
3. We want to prove that statement `repeat` S `until` b does not add expressiveness to language While. To do so, give a function which transforms every program with the statement `repeat` S `until` b into a program in language While. Is the given transformation computable? Compare the size of a program and its image resulting of the transformation.

Exercise 17 — Extending language While with construct `for ... from ... to ... do ...`

We want to add another iterative construct to language While. Consider the statement

`for` x `from` a_1 `to` a_2 `do` S.

where the first expression a_1 is the initial value that x is assigned to, the second expression a_2 is the limit that x should be assigned to. Moreover, the "step" of the loop is fixed.

The purpose of this exercise is to extend the semantics of language While by providing appropriate rules for this construct (and without using the `while` $\cdots$ `do` $\cdots$ `od` construct). You may need to assume that you have an "inverse" to $\mathcal{N}$, so that there is a numeral for each number that may arise during computation. There are several alternatives depending on what is allowed for S. Examples of criteria are:

- Evaluation of a_1 and a_2 are done once (at the beginning) or each time the loop body is executed.
- Evaluation of a_1 and a_2 are done each time the loop body is executed.

1. Provide semantics rules for an alternative where the first criterion holds.
2. Provide semantics rules for for an alternative where the second criterion holds.
3. Consider the state σ which maps x to 5. Evaluate the following statement in σ:

   $$y := 1; \texttt{for } z \texttt{ from } 1 \texttt{ to } x \texttt{ do } y := y + x; x := x - 1.$$

SERIES

TWO

NATURAL OPERATIONAL SEMANTICS OF LANGUAGES BLOCK AND PROC

In this exercise series, we consider language Block and Proc and its natural operational semantics.

2.1 Natural Operational Semantics of Language Block

Exercise 18 — Computing the semantics of a program in Block

Compute the semantics of the following program on $\sigma_0 = [x \mapsto 0]$:

```
begin  var y := 1;
       (x := 1;
       begin var x := 2 ; y := x + 1 end
       x := y + x)
end
```

2.2 Natural Operational Semantics of Language Proc

Exercise 19 — Completing the semantics of Proc with static links

Complete the semantics of Proc with blocks and procedures with static links for variables and procedures.

Exercise 20 — Computing the semantics of a program

Let us consider the following code snippet seen in the lecture course:

```
begin  var x := 0;
       proc p is x := x * 2 end;
       proc q is call p end;
       begin
           var x := 5;
           proc p is x := x + 1 end;
           call q; y := x;
       end;
end
```

Assume that variables x and y have been previously assigned, before entering this code, to x_0 and y_0 respectively. Moreover, assume that there is no procedure declared before entering this code snippet (i.e., the procedure environment is empty). Compute the semantics of this program according to the following three variants:

1. Dynamic links for procedures and variables.
2. Static links for procedures and dynamic link for variables.
3. Static links for procedures and variables.

Exercise 21 — Computing the semantics of a program

Let us consider the following program seen in the lecture course:

```
begin  var x := 2;
       proc p is x := 0 end;
       proc q is begin var x := 1; proc p is call p end; call p; end; end
       call q;
end
```

Compute the semantics of this program according to the following variants.

1. With static links for procedures, dynamic links for variables, and the recursive call rule,
2. With static links for procedures, dynamic links for variables and the non-recursive call rule.

Exercise 22 — Procedures as variables

We add the following statement to language While with blocks and procedures:

$\mathsf{Stm} ::= p := S.$

Give a semantics to this language. Your semantics should be conservative wrt. the semantics of language While.

1. Consider the semantics with dynamic scopes for variables and procedures.
2. Consider the semantics with static scopes for variables and procedures.

Exercise 23 — Procedures with parameters

We modify the syntax of procedures to allow parameters:

$$
\begin{array}{lcl}
S & \in & \mathsf{Stm} \\
S & ::= & x := a \mid \texttt{skip} \mid S_1; S_2 \\
 & & \mid \texttt{if } b \texttt{ then } S_1 \texttt{ else } S_2 \texttt{ fi} \mid \texttt{while } b \texttt{ do } S \texttt{ od} \\
 & & \mid \texttt{begin } D_V\ D_P\ ;\ S \texttt{ end} \\
 & & \mid \texttt{call } p(a_1, a_2) \\
D_V & ::= & \texttt{var } x := a;\ D_V \mid \epsilon \\
D_P & ::= & \texttt{proc } p(x_1, x_2) \texttt{ is } S \texttt{ end};\ D_P \mid \epsilon
\end{array}
$$

We are interested in the semantics with *static link for procedures and dynamic link for variables.*

1. Modify the semantics of procedure declaration and call in order to obtain a semantics with call-by-value.
2. Same question with call-by-reference.
3. Same question with call by result.
4. Same question with call by value-result.

Exercise 24 — Completing a program

We are interested in the semantics with static links for variables and procedures. We extend language Proc with a write command: *write* x prints out the value of $\sigma \circ \rho(x)$, but variable environment and storage function are left unchanged.

We consider the following program:

```
begin
        var x := 2;
        begin
                        var y := 7;
                        begin
                                        var x := 5;
                                        var y := 0;
                                        call p;
                        end
        end
end
```

1. Place the instructions `proc` p `is` $x := x * y$ `end`; and *write* x so that the value 14 appears on screen.
2. Justify your answer by computing the output using the rules.

SERIES

THREE

STRUCTURAL OPERATIONAL SEMANTICS

In this series we start from the structural operational semantics of language While as seen in the course.

Exercise 25 — Computing some derivation sequences

Give a derivation sequence and the associated derivation trees for the following statements on the following states:

1. $(z := x; x := y); y := z$ on $[x \mapsto 2, y \mapsto 5, z \mapsto 7]$,
2. $(y := 1;$ `while` $\neg(x = 1)$ `do` $(y := y * x; x := x - 1)$ `od` on $[x \mapsto 5, y \mapsto 7]$,
3. $z := 0;$ `while` $(y \leq x)$ `do` $(z := z + 1; x := x - y)$ `od` on $[x \mapsto 17, y \mapsto 5]$.

Exercise 26 — Some properties of the structural semantics

Prove the following claims:

1. If $(S_1, \sigma) \Rightarrow^k \sigma'$ then $(S_1; S_2, \sigma) \Rightarrow^k (S_2; \sigma')$.
   That is the execution of S_1 is not influenced by the statement following it.
2. If $(S_1; S_2, \sigma) \Rightarrow^k \sigma''$ then there exists σ' and k_1 such that: $(S_1, \sigma) \Rightarrow^{k_1} \sigma'$ and $(S_2, \sigma') \Rightarrow^{k-k_1} \sigma''$.
   That is, if it takes k steps to execute $S_1; S_2$ in σ, then there exists an integer k_1 s.t. it takes k_1 steps to execute S_1 in σ to yield a final configuration σ'. Then executing S_2 in σ' takes $k - k_1$ steps.

Exercise 27 — A Structural Semantics for Blocks

1. Define a structural operational semantics for the language Block.
2. Apply it to the following program:

begin
  D_{V_0} [`var` $x := 1;$
  S_0 [$x := 7$
    S_1 [begin
      `var` $y := 8;$
      $y := x$
      end]
    ]
end

Exercise 28 — Determinism of the structural operational semantics

1. Show that the structural operational semantics of While is deterministic.
2. Deduce that there is exactly one derivation sequence starting in a configuration (S, σ).

3. Argue that a statement S of While cannot both terminate and loop on a state σ and hence it can not both be always terminating and always looping.

Exercise 29 — Equivalence between the natural and the structural semantics

Prove the following claims.

1. If $(S, \sigma) \to \sigma'$ then $(S, \sigma) \Rightarrow^* \sigma'$.
   It means that if the execution of a statement terminates in a given state in the natural semantics, then it will terminate in the same state in the structural semantics.
2. If $(S, \sigma) \Rightarrow^k \sigma'$ then $(S, \sigma) \to \sigma'$.
   It means that if the execution of a statement terminates in a given state in the operational semantics, then it will terminate in the same state in the operational semantics.
3. Deduce that the natural semantics is equivalent to the structural semantics.

Exercise 30 — Equivalence between some statements

1. Show that the two following statements are semantically equivalent (You may use the fact that the semantics is deterministic.):
   - S; `skip`, and
   - S.
2. Same question for
   - `repeat` S `until` b, and
   - S; `while` $\neg b$ `do` S `od`.

Exercise 31 — Associativity of sequential composition in the structural semantics

1. Prove that, for all statements S_1, S_2, S_3, the following statements are semantically equivalent:
   - $S_1; (S_2; S_3)$
   - $(S_1; S_2); S_3$

   You may use any result of the previous exercises.
2. Prove that, in general, $S_1; S_2$ is not semantically equivalent to $S_2; S_1$.

SERIES

FOUR

AXIOMATIC SEMANTICS - HOARE LOGIC

Exercise 32

Prove that the following Hoare triples are valid.

1. $\{x = 0\}\ x := x + 1; x := x + 1\ \{x = 2\}$
2. $\{x > 0\}\ y := 1\ \{x = x * y\}$
3. $\{x > a\}\ x := x + 1\ \{x > a + 1\}$
4. $\{x = 4 * a + 4\}\ a := a + 2\ \{x = 4 * a - 4\}$
5. $\{x > a\}\ x := x + 1; x := x + x\ \{x > 2a + 2\}$
6. $\{x \geq 0\}$ `if` $x \geq 0$ `then` $y := 8$ `else` $y := 9$ `fi` $\{y = 8\}$

Exercise 33

Prove that the following Hoare triples are valid.

1. $\{y \geq 0\}\ S\ \{z = y!\}$, where S is:

```
x := y ;
z := 1 ;
while 1 < x do
   z := z * x ;
   x := x - 1
od
```

2. $\{x \geq 0\}\ S\ \{x = 2 \times a + b\}$, where S is:

```
a := x div 2 ;
if even(x) then
   b := 0
else
   b := 1
fi
```

3. $\{a \geq 0 \wedge b > 0\}\ S\ \{a = b \times q + r \wedge r \geq 0 \wedge r < b\}$, where S is:

```
q := 0 ;
r := a ;
while b <= r do
  q := q + 1 ;
  r := r - b ;
od
```

4. $\{b \geq 0\}\ S\ \{p = b^2\}$, where S is:

```
c := b ;
p := 0 ;
while c > 0 do
  p := p + b ;
  c := c - 1 ;
od
```

5. $\{n \geq 1\}\ S\ \{p = m \times n\}$, where S is:

```
p := 0 ;
c := 1 ;
while c <= n do
  p := p + m ;
  c := c + 1
od
```

6. $\{n \geq 0\}\ S\ \{x = \mathsf{Fib}(n)\}$, where S is:

```
x := 0 ;
y := 1 ;
c := n ;
while c > 0 do
  h := y ;
  y := x + y ;
  c := c - 1
od
```

Exercise 34

Prove that the following Hoare triples are valid.

1. $\{\texttt{True}\}\ S\ \{y1 * y1 \leq x \wedge (y1+1) * (y1+1) > x\}$, where the loop invariant is $y1 * y2 \leq x$ and S is:

```
y1 := 0 ; y2 := 1 ; y3 := 1 ;
while y3 <= x do
   y1 := y1+1 ;
   y2 := y2+2 ;
   y3 := y1+y2 ;
   y := x + y ;
   c := c - 1
od
```

2. $\{\texttt{True}\}\ S\ \{p = |x - y|\}$, where S is:

```
a := x ;
b := y ;
if a> b then
   p := a-b
else
   p := b-a
fi
```

Exercise 35

Let S be the following program:

```
u := 0;
while x>1 do
   if even(x) then
      x := x / 2;
      y := y * 2
   else
      x := x - 1;
      u := u + y
   fi
od;
y := y + u
```

1. Use Hoare logic to show that $\vdash \{x = x_0 \wedge y = y_0 \wedge x_0 > 0\}\ S\ \{y = x_0 * y_0\}$

Exercise 36

We consider the pre-condition and the post-condition in the incomplete Hoare triple below:

$$\{x = n \wedge y = m \wedge m \geq 0 \wedge n \geq 0\}\ \ldots\ \{z = n * m\}$$

1. Give a program S which satisfies the specification and which uses only addition and subtraction:
2. Demonstrate using Hoare logic that your program satisfies the specification.

Exercise 37

We consider the pre-condition and the post-condition in the incomplete Hoare triple below:

$$\{x = n \wedge y = m \wedge m \geq 0 \wedge n \geq 0\}\ S\ \{z = n^m\}$$

1. Give a program that satisfies the specification and which uses only addition and subtraction.
2. Demonstrate using Hoare logic that your program satisfies the specification.

Exercise 38

We consider the statement

$$\texttt{for}\ i\ \texttt{from}\ 1\ \texttt{to}\ n\ \texttt{do}\ S$$

where n is the denotation of a natural number in $\mathbb{N}$ and S is a statement in which i is not modified.

1. Recall the operational semantics rule for this construct.
2. Give a rule in axiomatic semantics (Hoare logic) for this construct.

Exercise 39

We add the statement:

`repeat` S `until` b

to the While language.

1. Give an inference rule for `repeat` S `until` b.
2. Demonstrate the correctness of your rule.
3. Is the rule complete? (prove your answer)

Exercise 40

1. Demonstrate that the predicate transformer wlp is:
   a) Monotone wrt. the implication.
   b) $\text{wlp}(S, P \wedge Q) \iff (\text{wlp}(S, P) \wedge \text{wlp}(S, Q)$.
2. Does the following hold ?
   a) $\text{wlp}(S, P \vee Q) \iff (\text{wlp}(S, P) \vee \text{wlp}(S, Q))$.
   b) $\text{wlp}(S, \neg P)) \iff \neg\, \text{wlp}(S, P)$

Exercise 41

We consider the following predicate transformer:

$$\text{post}(S, P) = \{\sigma' \mid \exists \sigma \cdot \sigma \models P \wedge (S, \sigma) \to \sigma'\}$$

1. Prove that: $\{P\}\ S\ \{Q\}$ iff $\text{post}(S, P) \Rightarrow Q$.
2. Prove that: $\vdash \{P\}\ S\ \{\text{post}(S, P)\}$.
3. Deduce that the Hoare logic is complete.

SERIES

FIVE

SEMANTIC ANALYSIS - TYPING

Exercise 42 — Program correctly typed or not?

Consider the environment $\Gamma = [x_1 \mapsto \texttt{Int}, x_2 \mapsto \texttt{Int}, x_3 \mapsto \texttt{Bool}]$. Indicate whether the following programs are correctly typed or not.

1. Program 1:

   $x_1 := 3;$
   `while` $\neg x_3$ `do`
   $x_1 := x_2 + 1;$
   $x_3 := x_3$ `and true`
   `od`

2. Program 2:
   $x_1 := 3 * x_1 + 1;$
   `if` x_2 `and` $\neg x_3$ `then`
   $x_1 := x_2 + 1$
   `else`
   $x_1 := x_2;$
   `fi`

Exercise 43 — Sequential or Collateral evaluation in declarations

Consider the sequence of variable declarations $D_V =$ `var` $x_1 := 3;$ `var` $x_2 := 2 * x_1 + 1;$ `var` $x_3 :=$ `true` and the initial environment $\Gamma_V = [\,]$.

1. Compute the resulting environment by updating Γ_V with D_V using *sequential* evaluation.
2. Compute the resulting environment by updating Γ_V with D_V using *collateral* evaluation.

Exercise 44 — Program correctly typed or not?

Indicate whether the following programs are correctly typed or not (in the empty environment) using the two modes of variable declarations (sequential and collateral).

1. ```
   begin
      var x := 5 ;
      var y := x + 5 ;
      x := y + 4
   end
   ```

2. ```
   begin
      var x := 5 ;
      begin
         var x := x < 3 ;
         var y := x > 3;
         x := not x
      end ;
   x := 4
   end
   ```

3. ```
   begin
      var x := 5 ;
      var y := y + x ;
      x := y
   end
   ```

Exercise 45 — Program correctly typed or not?

We consider the following program.

```
begin
   var x := 2 ;
```
```

```
   var y := true ;
   var z = x + 2 ;
   begin
      var u := x + 6 ;
      var v := not y ;
      x := u (* observation point 1 *)
   end
   y := false (* observation point 2 *)
end
```

1. Give the value of the environment at the two observation points, using the two modes of variable declarations (sequential and collateral).

Exercise 46 — Adding a typing rule for a new construct

We are interested in the construct/expression "a_1 ? a_2 : a_3" which is available in C or Java. The informal semantics of this construct is as follows: if a_1 is true then the value of this expression is a_2 else the value is a_3.

1. Complete the abstract syntax of expressions to support this construct.
2. Give typing rules for this construct.

Exercise 47 — Introducing floats and type conversion

We want to add the type `Float` to the While language.

1. Complete the abstract syntax and the type system to support the type `Float` where no conversion is allowed between `Int` and `Float`.
2. Complete the type system to allow *implicit* conversion from `Int` to `Float`.
3. Complete the abstract syntax and the type system to allow the *explicit* conversion from `Int` to `Float` through an appropriate type conversion operator.

Exercise 48 — Typing rules for the for and repeat constructs

We add two new statements to the While language (introduced in the lecture session):

- a "repeat" statement: `repeat` S `until` b
- a "for" statement: `for` x `from` e_1 `to` e_2 `do` S

1. Give the typing rule(s) associated to the "repeat" statement.
2. Give the typing rule(s) associated to the "for" statement. You will distinguish between two cases:
   - the "for" statement *declares* the variable x (like in Ada or Java), the scope of this new variable is S ;
   - the "for" statement *does not declare* the variable x (like in C), and therefore x has to exist in the current environment.

Exercise 49 — Other forms of variable declarations

Modify the type system seen in the course when variable declarations can take the following additional forms.

1. `var` x : t
2. `var` x := e : t

Exercise 50 — Type-checking a program

We consider the type system seen in the course and the following program.

```
begin
  var x := 3
  proc p is x := x + 1
  proc q is call p
  begin
    proc p is x := x + 5
    call q
    call p
  end
  call p
end
```

1. Determine whether this program is correctly type in the case of *static* binding for variables and procedures.
2. Determine whether this program is correctly type in the case of *dynamic* binding for variables and procedures.

Exercise 51 — Mutually-recursive procedures

We consider the programs below.

```
begin
   proc p is
        call p ;
   call p ;
end
```

```
begin
   proc p1 is
        call p2 ;
   proc p2 is
        call p1 ;
   call p1 ;
end
```

1. Show that, with the type system defined so far for the Proc language, the programs are *incorrect*.
2. Modify this type system to take into account such *(mutually) recursive* procedures. Verify that these programs are now correct with the new type system.
   Clue. Each sequence of procedure declaration should be analyzed twice: a first time to build its associated local environment, and a second time to check its correctness with respect to this local environment.

Exercise 52 — Correctly-initialized variables

A variable is said to be *correctly initialized* if it is never *used* before being assigned with an expression containing only correctly initialized variables. Let us consider for instance the following program:

```
x := 0 ; y := 2 + x ; z := y + t ; u := 1 ; u := w ; v := v+1 ;
```

In this program:

- `x` and `y` are correctly initialized ;
- `z` is not correctly initialized (because `t` is not correctly initialized); `u` is not correctly initialized (because `w` is not correctly initialized); and `v` is not correctly initialized (because `v` is not correctly initialized).

Some compilers, such as `javac`, reject programs that contain non-correctly initialized variables. We want to define in this exercise a type system which formalizes this check. To do so, we consider the following judgments:

- an environment is simply a set V of correctly initialized variables;
- $V \vdash e$ means that "in the environment V, expression e is correct (it does not contain non correctly initialized variables)";

- $V \vdash S \mid V'$ means that "in the environment V, statement S is correct and produces the new environment V' ".

1. Give the corresponding type system for the While language (without blocks nor procedures).
2. Apply the type system to the following code snippet, using $\Gamma = \emptyset$:
   a) $x := 1; \texttt{if } x = 0 \texttt{ then } y := x + 1 \texttt{ else } y := x - 1 \texttt{ fi}$
   b) $x := 1; \texttt{if } x = 0 \texttt{ then } x := x + 1 \texttt{ else } y := x - 1 \texttt{ fi}$,
   c) $x := 1;\ \texttt{while } x \leq 10 \texttt{ do } y := x + y\ ;\ x := x + 1$.
3. Show (on an example) that, similarly to `javac`, your type system may reject programs that would be correct at run-time.

Exercise 53 — Procedures with one parameters

We consider the following modified abstract syntax where procedures can have one parameter:

$$\begin{array}{lll} Dp & ::= & \texttt{proc } p\ (y\ :\ t)\ \texttt{is } S\ ;Dp \mid \ldots \\ S & ::= & \ldots \mid \texttt{call } x\ (e) \end{array}$$

1. Modify the type system to handle procedures with one parameter.
2. Use the extended type system to prove that the following program is correctly typed.

```
begin
  var x := 3
  proc p (u : int) is x := u + 1
  begin
    var x := true
    proc p (u : bool) is not u
    call p (x)
  end
  call p (x)
end
```

Exercise 54 — Considering functions

We extend language Proc to handle procedures that return value, aka functions. This entails that functions can be called within expressions.

1. Extend the abstract grammar of Proc.
2. Extend the type system of Proc accordingly.

Exercise 55 — Adding parameters to procedures in the type system

We aim at extending the While language to add *parameters* to procedures. We shall proceed in several steps.

1. Consider only `in` parameters.
2. Consider only `out` parameters.
3. Consider both `in` and `out` parameters.
4. Take into account the extra rule (inspired from the Ada language), stating that:
   - `out` parameters cannot appear in right-hand side of an assignment;
   - `in` parameters cannot appear in left-hand side of an assignment.
5. Show that, in this last case, your type system may *reject* correct programs because of this rule. How could you solve this problem?

Exercise 56 — Sub-typing and dynamic types

We extend language While by introducing the notion of *sub-typing* through the following syntax for blocks, where t is a **type identifier** and `extends` means "is a sub-type of" (like in Java):

$$\begin{array}{rcl} S & ::= & \cdots \mid \texttt{begin}\ D_T\ ;\ D_V\ ;\ S\ \texttt{end} \\ D_T & ::= & \texttt{type}\ t\ \texttt{extends}\ B_T; D_T \mid \varepsilon \\ B_T & ::= & \texttt{Top} \mid \texttt{Int} \mid \texttt{Bool} \mid t \end{array}$$

We aim to define a type system for this language which reflects the usual notion of sub-typing, namely:

- The sub-typing relation is a partial order $\sqsubseteq$ whose greatest element is `Top`. It can be formalized by a *type hierarchy* $(X, \sqsubseteq)$, where X is a set of declared types (including the predefined types `Top`, `Int` and `Bool`).

- A value of type t_2 can be assigned to a variable of type t_1 whenever $t_2 \sqsubseteq t_1$. The converse is false.

1. Propose a type system which takes these rules into account. Judgments could be of the form:

   - $(X, \sqsubseteq), \Gamma \vdash S$, meaning that "in the environment Γ and with the type hierarchy $(X, \sqsubseteq)$, the statement S is well-typed" ;
   - $(X, \sqsubseteq), \Gamma \vdash e : t$, meaning that "in the environment Γ and with the type hierarchy $(X, \sqsubseteq)$, the expression e is well-typed and of type t" ;
   - $(X, \sqsubseteq) \vdash D_T \mid (X', \sqsubseteq')$, meaning that "type declaration D_T is correct within the type hierarchy $(X, \sqsubseteq)$ and produces the type hierarchy $(X', \sqsubseteq')$" ;
   - $(X, \sqsubseteq), \Gamma \vdash D_V \mid \Gamma_l$, meaning that "in the environment Γ and with the type hierarchy $(X, \sqsubseteq)$, the variable declaration D_V is correct and produces the environment Γ_l".

2. Show that the following program is rejected by your type system:

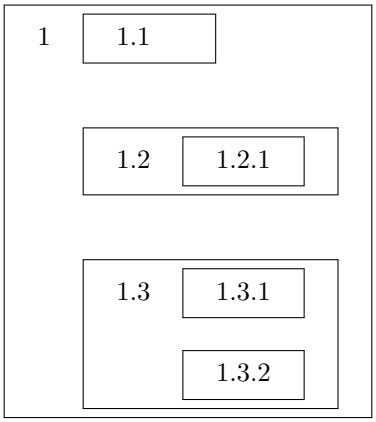
```
begin
    type t extends Int ;
    var x1 : Int ;
    var x2 : t ;
    var x3 : Int ;
    x1 := x2 ;
    x3 := x1 ;
    x2 := x3
end
```

3. This question requires to be familiar with the natural operational semantics of While. Although rejected by your type system, the previous program is perfectly safe (it does not violate the informal sub-typing rules). However, its correctness can only be ensured at run-time, by introducing a notion of *dynamic type* to each identifier. This dynamic type corresponds to the actual type value held by this identifier at each program step (contrarily to the *static type*, the one *declared* for this variable).

   Rewrite the (natural) operational semantics of the While language to take into account this notion of *dynamic type* and perform the type-checking at run-time. You can extend the configurations with a (dynamic) environment ρ which associates its dynamic type to each identifier.

Exercise 57 — Nested blocks and global environment

To define the type system of Block (possibly with nested blocks, but without procedures), we propose a notion of *global* environment in which each identifier is *uniquely* defined. More precisely, we assume a hierarchical numbering of blocks:



An environment now associates a type to a **pair** in $\mathsf{Vars} \times \mathbb{N}^*$, and a statement is type-checked **within** a given block.

1. Define the corresponding judgments and type system[1].

Exercise 58 — Static vs Dynamic Type system

We consider the following While program (with a command 'write'):

```
begin  var x := 2;
       var y := 1;
       proc p is x := x + y end;
       begin var y := true;
             call p;
             write x;
       end;
end;
```

1. According to the static semantics for variables and procedures, what does this program write?
2. Is this program well-typed in the static semantics type system?
3. According to the dynamic semantics for variables and procedures, what happens with this program?
4. Is this program well-typed in the dynamic semantics type system? We deduce that even if a program is well-typed in the static type system, it does not matter when we want to execute it with a dynamic semantics!
5. Propose a modification of this program so that it is well-typed in the dynamic semantics type system, and it displays a Boolean.
6. (optional) If you master the static-dynamic semantics, you can try to exhibit a program which is well-typed in the dynamic type system but not in the static-dynamic type system. You can use a second procedure `q`.

[1] $\mathbb{N}^*$ denotes the set of finite words over $\mathbb{N}$.

SERIES

SIX

INTERMEDIATE-CODE OPTIMIZATION USING DATA-FLOW ANALYSIS

6.1 Defined and Used variables

We recall the syntax of the While language:

$$\begin{array}{lcl} S & ::= & x := a \mid \texttt{skip} \mid S; S \mid \texttt{if } b \texttt{ then } S \texttt{ else } S \mid \texttt{while } b \texttt{ do } S \\ a & ::= & n \mid x \mid a + a \\ b & := & \texttt{true} \mid \texttt{false} \mid a = a \mid a \leq a \mid \neg b \mid b \wedge b \end{array}$$

6.1.1 Defining assigned and used variables in statements

We want to define the functions `Def` and `Use` which associate, to each syntactic construct, the set of assigned and used variables respectively. Assigned variables are the variables appearing in the left-hand side of an assignment. Used variables are the variables appearing on the right-hand side of assignments or in expressions.

Exercise 59 — A formal definition of `Def` and `Use`

We want to define formally functions `Def` and `Use`.

1. What is the signature of these functions?
2. Consider conditional statements built using the `if` ... `then` S_1 `else` S_2 `fi` construct. What are the possible situations (in terms of definition and use) for a variable depending on S_1 and S_2?
3. Based on the observation made in the previous question, one needs two definitions of the `Def` and `Use` functions. Give formal definitions for these functions.

Exercise 60 — Applying `Def` and `Use`

1. Apply functions `Def` and `Use` to: `if` $a < b$ `then` $c := d - y$ `else` $y := e - x$ `fi` .

6.1.2 Adding an operator $\|$ for parallel execution

We extend the syntax of statements in the following way:

$$S ::= \cdots \mid S \parallel S.$$

Exercise 61 — Parallelism in Natural Operational Semantics

We consider operator $\|$ for parallelizing statements.

1. Recall the natural operational semantics of this operator.

In the following, we consider the While language extended with the $\|$ operator and its natural operational semantics.

Exercise 62 — Applying the parallelism operator

Applying natural operational semantics rules extended with the rules for $\|$, compute the obtained state(s) by executing the following commands on the initial state $\sigma_0 = [x \mapsto 1, y \mapsto 1]$.

1. $x := x + 1 \parallel y := y + 1$
2. $x := y + 1 \parallel y := x + 2$

Exercise 63 — A sufficient condition so that parallelism does not influence computation

Give a sufficient condition such that the two previously defined rules yield the same result. That is, when we evaluate $S_1 \parallel S_2$ in a state σ, then the obtained state σ_2 is independent of the order of evaluation of S_1 and S_2.

Hint: One can use the functions `Def` and `Use`, previously defined.

Exercise 64

Propose a unique semantic rule that evaluates $S_1 \parallel S_2$ on the state σ. This rules should propose an evaluation of S_1 and S_2 on state σ supposing that the previous condition holds.

Exercise 65

1. Apply the previously defined semantic rules to the statement in Exercise 62.

6.2 Preliminaries

6.2.1 Lattice, recursive equations, fix-points

Exercise 66

We consider set $E = \{a, b, c\}$.

1. Draw the lattice $(2^E, \subseteq)$, where 2^E denotes the powerset of E.
2. For each of the following subsets of E, indicate whether it contains a maximum or not, what is the set of its upper-bounds, what is its least upper bound (when it exists).
   1. $\{\{a\}, \{b\}\}$
   2. $\{\{a, b\}, \{b, c\}, \{a, c\}\}$
   3. $\{\{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$
   4. 2^E

Exercise 67

For a set X endowed with an order relation $\leq$, we say that function f is monotonic if $\forall x, y : x \leq y \implies f(x) \leq f(y)$. We consider the following functions:

- "complement" function: $F1(x) = 2^{\{a,b,c\}} \setminus x$
- "union with $\{b, c\}$" function: $F_2(x) = x \cup \{b, c\}$
- "intersection with $\{b, c\}$" function: $F_3(x) = x \cap \{b, c\}$
- function $F_4 = F_3 \circ F_2$.

1. Indicate which functions are monotonic on $(2^{\{a,b,c\}}, \subseteq)$.
2. We are now interested in recursive equations of the form $X = F(X)$ defined on $(2^{\{a,b,c\}}, \subseteq)$. Give (without proof) the set of solutions of these equations for F_1, F_2, F_3 and F_4.
3. Compute the least solution (i.e., the least fix-point) of equation $X = F_2(X)$ by successively computing $F_2(\ldots F_2(F_2(\emptyset)))$.

4. Compute the greatest solution (i.e., the greatest fix-point) of equation $X = F_3(X)$ by successively computing $F_3(\ldots F_3(F_3(\{a, b.c\})))$.
5. Are the computations possible for equation $X = F_4(X)$? And for equation $X = F_1(X)$?

6.2.2 Control-flow Graph (CFG)

Exercise 68

1. Draw the CFG of the following program.

```
x := 3 ;
while (x < 10) {
   y := x+1 ;
   if (y<5) {
      z := 2*x ;
      y := y-1 ;
     }
     else {
        z := y + 1
     }
     x := x+1 ;
 }
 y := y + z ;
```

Exercise 69

We consider the following 3-address code sequence:

```
1.  a := 1
2.  b := 2
3.  e := a+b
4.  d := c-a
5.  if a+b>0 goto 11
6.  d := b*d
7.  goto 8
8.  d := a+b
9.  e := e+1
10. goto 3
11. b := a+b
12. e := c-a
13. if c > 3 goto 3
14. c := a+b
15. b := a-d
end
```

1. Split this sequence into basic blocks, and draw the resulting control flow graph.

6.3 Available Expressions

Exercise 70

We consider the program in Exercise 69 and its CFG.

1. Give the set of data-flow equations for computing *available expressions*.
2. Solve these equations.
3. Suppress *redundant computations*.

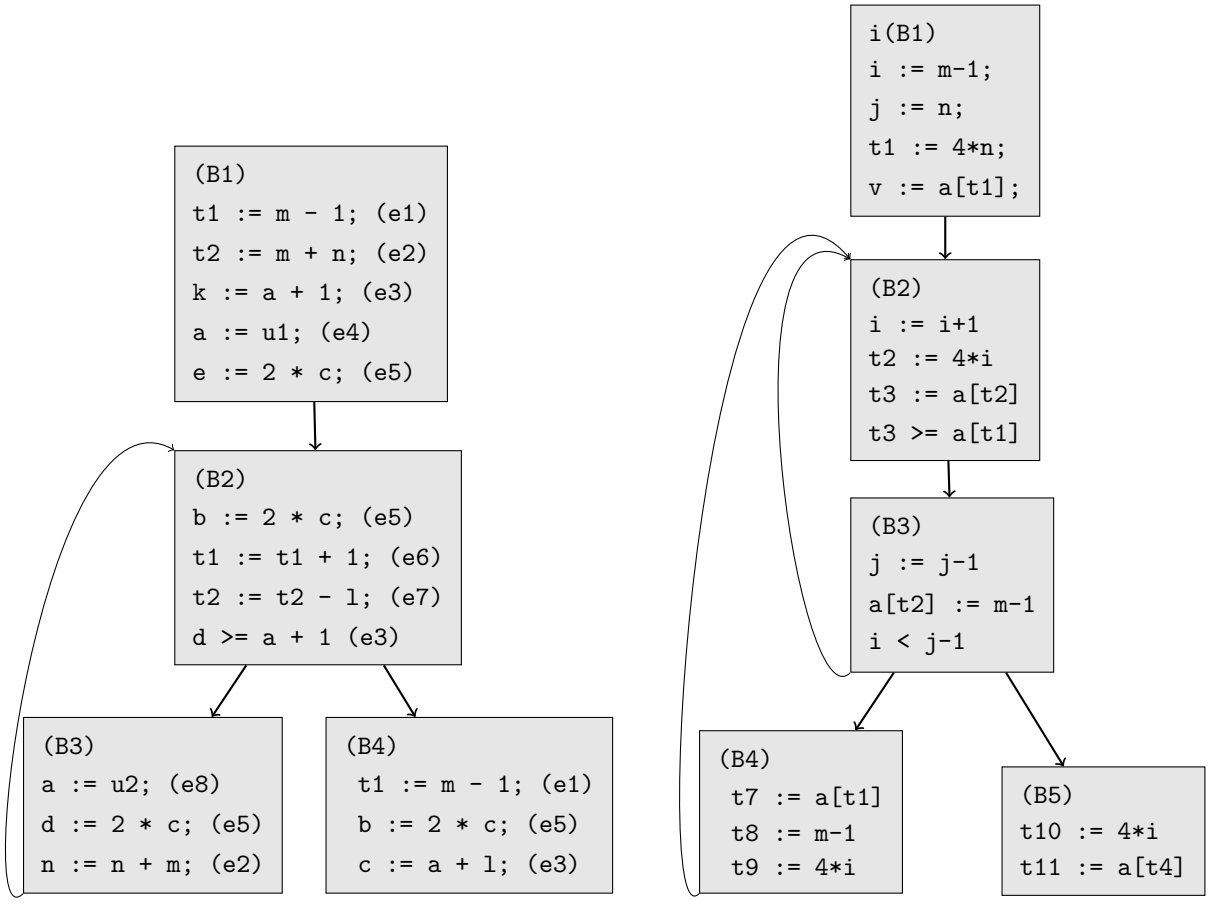


Figure 6.1: CFGs for Exercise 71

Exercise 71

For each of the CFGs in Figure 6.1:

1. Give the set of data-flow equations for computing *available expressions*.
2. Solve these equations.
3. Suppress *redundant computations*.

Exercise 72

Suppress redundant computations in the following program:

```
     a := 5
     c := 1
L1: if c>a goto L2
     c := c+c
     goto L1
L2: a:= c-a
     c:=0
```

6.4 Live Variables

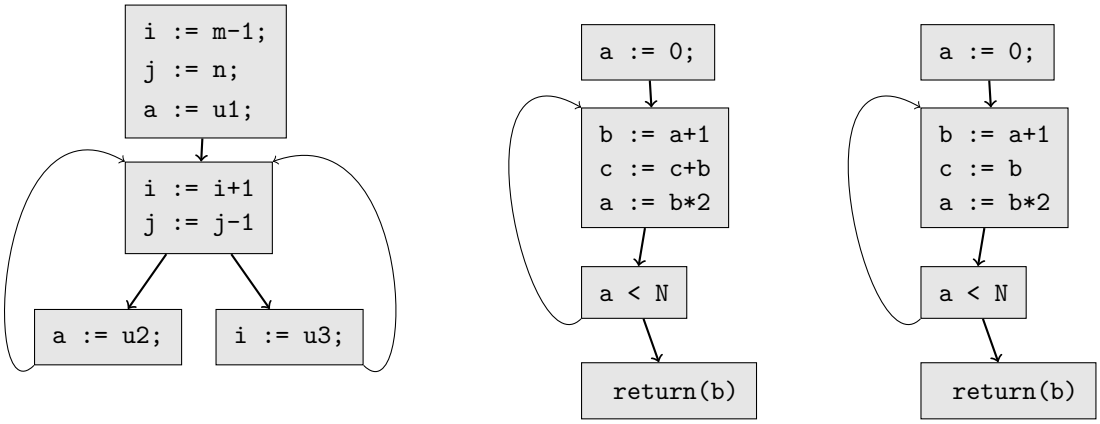


Figure 6.2: CFGs for Exercise 74

Exercise 73

We consider the following program:

```
while d>0 do {
   a := b+c
   d := d-b
   e := a+f
   if e > 0
       {f := a-d ; b := d+f}
   else
       {e := a-c}
   b := a+c
}
```

1. Write the 3-address code sequence corresponding to this program.
2. Split this sequence into basic blocks, and draw the resulting control flow graph.
3. Give the set of data-flow equations for computing *live variables*.
4. Solve these equations.
5. Suppress *useless assignments*.

Exercise 74

For each of the CFGs depicted in Fig. 6.2 (obtained after some intermediate code generation):

1. Give the set of data-flow equations for computing *live variables*.
2. Solve these equations.
3. Suppress *useless assignments*.

6.5 Constant Propagation

Exercise 75

We consider the CFGs in Figure 6.3.

1. Modify these CFGs by performing constant propagation.

Exercise 76

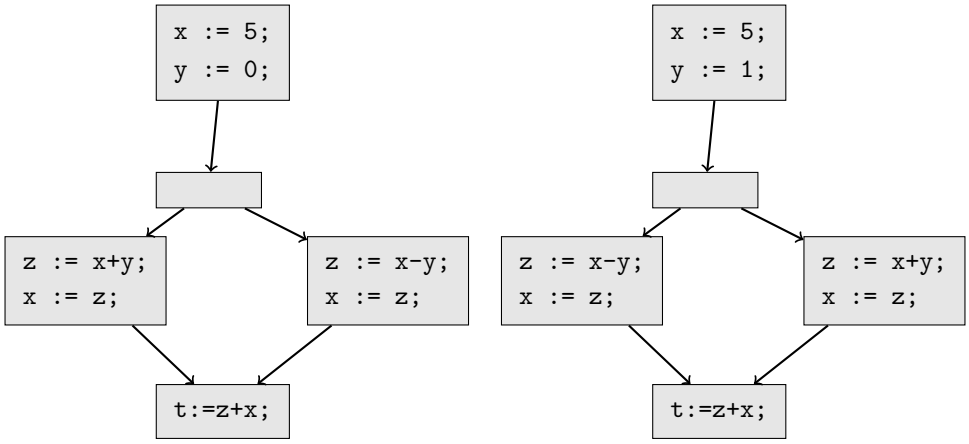


Figure 6.3: CFGs for Exercise 75

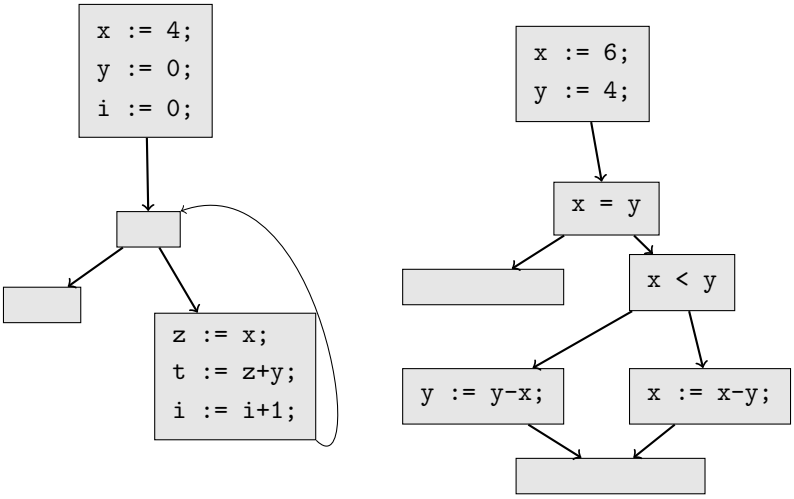


Figure 6.4: CFGs for Exercise 76

We consider the CFGs in Figure 6.4.

1. Modify these CFGs by performing constant propagation.

SERIES

SEVEN

CODE GENERATION (OF ASSEMBLY CODE)

Exercise 77

We consider the following program.

```
main(c) {
   int x, y, z;
   x = 5;
   y = 1;
   z = x + y;
}
```

1. Determine the symbol table (integers are coded on 4 bytes).
2. Generate assembly code.

Exercise 78

Let us consider the following program with its symbol table.

```
var x, y : int ;
while (not x = y) do
   if x > y then x := x - y else y := y - x fi
od
```

Environment

Procedure	Shifting
main	x → 4
	y → 8

1. Generate code.

Exercise 79

1. Extend the code generation function to handle the following construct:

   ```
   for (var = lbound; var < ubound; step) {
       S(variable)
   }
   ```

   where:

   - `lbound` and `ubound` are respectively the lower and upper bounds, given by arithmetical expressions,
   - `S(var)` is a statement that depends on `variable`, and
   - step is an expression that is added to `variable` at the end of each iteration.

2. For the program below, generate assembly code.

   ```
   int t1[10];
   int i;
   for(i=0;i<10;i++) { t1[i] = 0 }
   ```

3. For the program below, generate assembly code.

```
int t1[10][10] ;
int i,j ;
for(i=0; i<10; i++) {
  for(j=0; j<10; j++)
    t1[i][j] = 0;
 }
```

Exercise 80

Extend the code generation function seen for While.

1. To consider statements of the form `repeat` S `until` b ,
2. To consider Boolean expressions of the form `b1 xor b2`,
3. To consider arithmetical expressions of the form `b ? e1 : e2`.

Exercise 81

Let us consider the two following programs with their symbol tables. For each of them, generate code.

```
begin
 var x : int ; var y : int ;
 procedure p is
    y := x ;
 end
 x := 1 ;
 call p ;
end
```

Environment

Procedure	Shifting
main	x $\rightarrow$ 4 y $\rightarrow$ 8
main.p	

Exercise 82

We consider the following program.

```
begin
 var x : int ; var y : int ; var z : int :
 procedure p is
  begin var x : int ; var t : int :
    x := 4 ;
    y := 5 ;
    z := x + y ;
  end
  x := 5;
  call p ;
end
```

Environment

Procedure	Shifting
main	x $\mapsto$ 4 y $\mapsto$ 8 z $\mapsto$ 12
main.p	x $\mapsto$ 4 t $\mapsto$ 8

1. Generate the corresponding assembly code.

Exercise 83

Let us consider the following program:

```
var x : int ; var y : int ; var z : int ; var a : int ; var b : int ; var c : int ;
c := 4 ;
x := a+b+c;
y := a * b + 2 * c ;
```

1. Generate 3-address code and then assembly code by supposing an unlimited number of registers.
2. Generate 3-address code and then assembly code by supposing that we have only 4 registers R_1, R_2, R_3, R_4.

Exercise 84

We consider the `C` program in Figure 7.1.

1. Build the environment (symbol table)

```
void main() {
  int *x1 ;
  void p1() {
    int x2 ;
    void p2(int *x) { x=&x2; }
    void q2(int *x) { p2(x1); }
    q2(x1);
  }
  x1=malloc(4);
  *x1=1 ;
  p1();
}
```

Figure 7.1: Program for Exercise 84

2. Build the call-tree.
3. Generate code.

Exercise 85

For the `C` program in Figure 7.2.

1. Build the environment (symbol table)
2. Build the call-tree.
3. Generate code.

Exercise 86

We consider the program in Figure 7.3.

1. Draw the execution stack when executing `g3` by representing only the dynamic and static chaining.
2. We consider procedure `f1`. Give the code snippet that performs `y = h2(32)`.
3. We consider procedure `f1`. Give the code snippet that performs `return y`.
4. We consider procedure `f2`. Give the code snippet that performs `x = 11`
5. We consider procedure `f2`. Give the code snippet that performs `y=y+ f3(p)`
6. We consider procedure `f3`. Give the code snippet that performs `return p(y+x)`.
7. We consider procedure `g3`. Give the code snippet that performs `return p`.
8. We consider procedure `g2`. Give the code snippet that performs `z = x+y`.
9. We consider procedure `g2`. Give the code snippet that performs `return ((g3(g2))(x-1))`.

Exercise 87

We consider the program in Figure 7.4.

1. Represent the symbol table.
2. Give the call tree and the call stack by only displaying the environment of each procedure.
3. Generate code for this program.

```
int main () {
   int x1;
   void P1 () { /*static nesting 1*/
      int x2 ; int y2 ; int z2 ;
      int Y (int x) {
        printf ("%d\n", x+1); return (x+1);
      } /* Y */
      void R2 ( int b3, int p(int) ) { /*static nesting 2*/
        int y3 ;
        void R3 ( int p(int) ) {
           y2 = 2 ;
           x2 = y2+x1+b3;
           y3 = 1+x2 ;
           b3 = p(x1);
        } /* R3 */
        R3(p);
      } /* R2 */
      void P2 ( int a, int p (int) ) { /*static nesting 2*/
        int x3;
        void P3 ( int p(int) ) {
             x3 = p(x1);
             R2(x3,Y);
        } /* P3 */;
         x2 = 1+a ;
         P3(p);
      } /* P2 */
      int X (int x) {
         printf ("%d\n", x); return x;
      } /* X */
      void Q ()  { /*static nesting 2*/
         int z3;
         y2 = 2 ;
         z2 = 3 ;
         z3 = x2 + y2 ;
         P2(z3,X);
      } /* Q */
   x1 = 11 ;
   Q();
   } /* P1 */
   P1();
} /* prog */
```

Figure 7.2: Program for Exercise 85

```
#include <stdio.h>
typedef int (*intprocint) (int);
typedef int (*intprocintint) (int, int);
typedef int (*intprocvoid) (void);

main(){
  int x;
  int f1 () {
    int x ;
    int y ;

    int f2(int x,intprocint p) {
      int f3 (intprocint p){ return p(y+x); }
      x = 11;
      y = y + f3(p);
      return y;
    }

    int g2(int y) {
      int z ;
      intprocint g3(intprocint p){ return (p) ; }
      z = x + y;
      if (x>0) return ((g3(g2))(x-1));
    }

    int h2(int z){
      y = f2(3,g2);
      return(z+y);
    }
    x = 33;
    y = h2(32);
    return y;
  }
  x = 0;
  x = f1();
}
```

Figure 7.3: Program for Exercise 86

```
int main () {
   var x1;
   procedure p1 is
      begin
      procedure p2 (x, y) is begin z := x + y end
      z := 0
      call p2 (z + 1, 3)
      end
   procedure p3 (x) is
      begin
      var z
      call p1 ();
      z := z + x
      end
   call p3 (42)
} /* prog */
```

Figure 7.4: Program for Exercise 87

APPENDIX

A

REMINDERS

A.1 Lexicon, Syntax, and Semantics

Exercise 88 — Compilation errors

Give examples of code snippets (in any programming language) that produce the following errors.

1. A lexical error.
2. A syntactic error.
3. A static-semantics error.
4. A dynamic-semantics error.

A.2 Grammars and Abstract syntax

Exercise 89 — From languages to grammars

We consider vocabulary $V = \{a, b, c\}$. For each of the following languages, propose a grammar.

1. $L_1 = a^* \cdot b \cdot c^*$
2. $L_2 = \{a^n \cdot b \cdot c^n \mid n \geq 0\}$
3. $L_3 = \{a^n \cdot b \cdot c^m \mid 0 < n < m\}$
4. $L_4 = \{w \mid w \text{ is a palindrome}\}$

Exercise 90 — Automata and grammars for regular languages

We consider regular language $L_0 = a^* \cdot b^*$. Consider grammar G_0 defined by the following production rules:

$$S \to A \cdot B; \quad A \to a \cdot A \mid \epsilon; \quad B \to b \cdot B \mid \epsilon$$

where S is the axiom describing language L_0. The finite-state automaton $A_0 = (Q, V, \delta, q_0, F)$ recognizes this language:

$$\begin{array}{rcl} Q & = & \{q_0, q_1\} \text{ is the set of states} \\ V & = & \{a, b\} \text{ is the input alphabet} \\ q_0 & \in & Q \text{ is the initial state} \\ F & = & \{q_1\} \text{ is the set of accepting states} \\ \delta & = & \{\ (q_0, a, q_0), (q_0, \epsilon, q_1), (q_1, b, q_1)\ \} \text{ is the transition relation} \end{array}$$

1. Give a derivation of grammar G_0 for string *aaaabb*.
2. Give the sequence of transitions of automaton A_0 allowing to recognize the string *aabb*.

3. Give the sequence of configurations of automaton A_0 allowing to recognize the string *aabb*.

Exercise 91 — Context-free languages and grammars

We consider the context-free language $L1 = \{a^n \cdot b^n \mid n \geq 0\}$. Consider grammar G_1 defined by the following production rules: $S \to a\ S\ b$ and $S \to \epsilon$, where S is the axiom describing language L_1. The push-down automaton $A_1 = (Q, V, \Gamma, \Delta, Z, q_0, F)$ recognizes this language.

$$
\begin{array}{rcl}
Q & = & \{q_0, q_1, q_2, q_3\} \text{ is the set of states} \\
V & = & \{a, b\} \text{ is the input alphabet} \\
\Gamma & = & \{Z, A\} \text{ is the stack alphabet} \\
Z & \in & \Gamma \text{ is the initial stack symbol} \\
q_0 & \in & Q \text{ is the initial state} \\
F & = & \{q_3\} \text{ is the set of accepting states} \\
\Delta & = & \{\ (q_0, a, Z) \to (q_1, Z\ A), \\
 & & (q_1, a, u) \to (q_1, \omega, u\ A), \\
 & & (q_1, b, u\ A) \to (q_2, u), \\
 & & (q_2, b, u\ A) \to (q_2, u), \\
 & & (q_2, \epsilon, Z) \to (q_3, \epsilon)\ \} \\
 & & \Delta \text{ is the set of transitions, with } \omega \in V^* \text{ and } u \in \Gamma^*
\end{array}
$$

1. Give the derivation of grammar G_1 for string *aabb*.
2. Give the sequence of configurations of automaton A_1 allowing to recognize string *aabb*.

Exercise 92 — Grammars for arithmetic expressions

We want to define a grammar purposed to describe *arithmetic expressions* built from binary operators for substraction ($-$) and multiplication ($*$) and for which operands are integers (e).

1. We first consider grammar G_0 with the following production rules:
   $Z \longrightarrow E \qquad E \longrightarrow E - E \qquad E \longrightarrow e$
   Using sequence "10 - 2 - 3", show that this grammar is ambiguous.
2. To eliminate the ambiguity of G_0, we consider grammar G_1, defined as follows:
   $Z \longrightarrow E \qquad E \longrightarrow E - T \qquad E \longrightarrow T \qquad T \longrightarrow e$
   Draw the derivation tree corresponding to the example from the previous question. We say here that subtraction is "left associative".
3. How G_1 should be modified to introduce a multiplication operator such that:
   - multiplication is left associative;
   - multiplication has precedence over subtraction.

   Justify your answer by constructing syntactic trees for the following expressions: "$10 - 2 * 3$", "$10 * 2 - 3$".
4. We want to allow to put a sub-expression between parenthesis so as to write e.g., "$(10 - 2) * 3$" or "$10 * (2 - 3)$". Modify the previous grammar. Build the syntactic tree for the expression given as examples.
5. We add the unary minus operator (noted $-$), that has precedence over $-$ and $*$. Modify the previous grammar accordingly. Build the syntactic tree for "$10 * -2 - -3$".

Exercise 93 — Grammars and ambiguity

We consider the following grammar that describes statements from a programming language:

$$
\begin{aligned}
Z &\longrightarrow I \\
I &\longrightarrow \text{if } e \text{ then } I \\
I &\longrightarrow \text{if } e \text{ then } I \text{ else } I \\
I &\longrightarrow a
\end{aligned}
$$

1. Show that this grammar is ambiguous: find a sentence from the language that leads to two different derivation trees.
2. Propose a solution to make this grammar non-abiguous (by for instance modifying the described language).
3. Is it possible to make this grammar non-ambiguous without modifying the described language?

Exercise 94 — From derivation trees to machine code

Let G be a grammar describing assignments where terminals `c` and `i` designate an integer constant and an identifier, respectively.

```
Z -> A ;              E -> E + T | T        F -> c | i | (E)
A -> i := E           T -> T * F | F
```

We consider the two following assignements where `x` and `y` are identifiers:
(1) $x := 5 + x * 2 + y;$ (2) $y := (5 + x) * 2 + y;$

1. Build the derivations trees corresponding to assignements (1) and (2).
2. We consider a processor with registers noted `Ri` and with the following instruction set:
   - `LD Ri,op` loads the value of `op` in register `Ri`, `op` denotes a variable or a constant,
   - `ST Ri,x` gives the value contained in register `Ri` to variable `x`,
   - `ADD Ri,op1,op2` puts the sum `op1+op2` in register `Ri`,
   - `MULT Ri,op1,op2` puts the product `op1*op2` in register `Ri`,
     where `op1` denotes a register and `op2` denotes a register or a constant.

   Write the sequence of instructions corresponding to the traduction of (1) and (2).
3. We want to generate code for the machine from derivation trees. In the sequence proposed in the previous question, to which node should the instructions of the machine be associated with? Decorate the derivation trees of (1) and (2) with the sequences obtained: each instruction should decorate a node from the tree, certain nodes should not be decorated.
4. Inspiring from the result of question 3, draw simplified versions of trees (1) and (2). A simplified tree is an abstract tree containing only decorated nodes.
5. Propose an abstract syntax for G (in the form of a grammar). Propose several ways to write this grammar.

Exercise 95 — Statement for - concrete/abstract grammar

The following code snippets give examples of "for" statements in ADA and C :

```
for i in 1..N loop            for (i=0 ; i<N ; i++) {
   -- statements              /* statements */
end loop ;                    }
```

1. Propose a concrete syntax (even a simplified one) for statement "for" for each language.
2. Propose a corresponding abstract syntax.

APPENDIX

B

A SIMPLE COMPILER – COMPILER ARCHITECTURE

B.1 Introduction

The logical steps of a compiler usually consists of:

1. Lexical Analysis
2. Syntactic Analysis
3. Semantic Analysis
4. Intermediate Code Generation
5. Code Optimization
6. (Target) Code Generation.

B.1.1 A language, its grammar

Program	::=	Block
Block	::=	`begin` Declaration_list ; Statement_list `end`
Declaration_list	::=	Declaration_list ; Declaration | Declaration
Declaration	::=	`Idf` := Expression : Type
Type	::=	`integer`
Statement_list	::=	Statement_List ; Statement | Statement
Statement	::=	`Idf` := Expression | `skip`
		| `if` Expression `then` Statement `else` Statement `endif`
		| `if` Expression `then` Statement `endif` |
		`while` Expression `do` Statement `done`
Expression	::=	Disjunction
Disjunction	::=	Conjunction | Disjunction `or` Conjunction
Conjunction	::=	Comparison | Conjunction `and` Comparison
Comparison	::=	Relation | Relation `=` Relation
Relation	::=	Sum | Sum $<$ Sum
Sum	::=	Term | Sum $+$ Term | Sum $-$ Term
Term	::=	Factor | Factor $*$ Term
Factor	::=	`Not` Factor| Denotation | `Idf` | '('Expression')'

B.1.2 Abstract Grammar

$$
\begin{array}{lll}
S & ::= & x := a \mid \texttt{skip} \mid S; S \mid \texttt{if}\ b\ \texttt{then}\ S\ \texttt{else}\ S \mid \texttt{while}\ b\ \texttt{do}\ S \\
a & ::= & n \mid a + a \mid a - a \mid a * a \mid x \\
b & := & \texttt{true} \mid \texttt{false} \mid a = a \mid a \leq a \mid \neg b \mid b \wedge b
\end{array}
$$

B.2 Lexical Analysis

A scanner (lexical analyzer) has as input a character string and outputs a pair lexical class, element of the class.

Exercise 96 — **Lexical analysis**

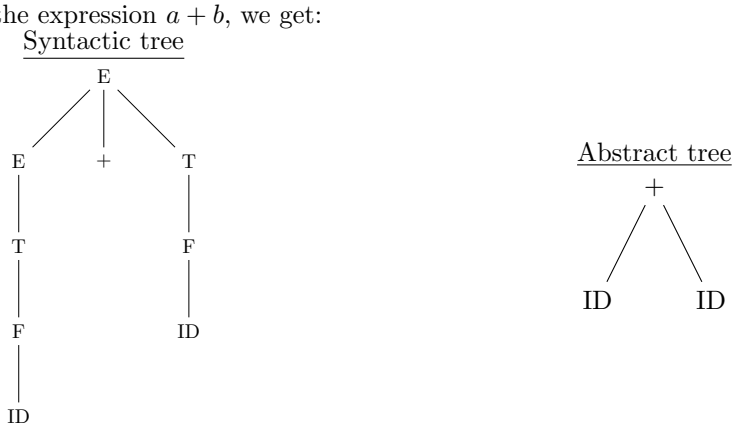
Determine the lexical classes and give them a specification as an automaton or a regular expression.

B.3 Syntactic Analysis

A parser (syntactic analyzer) has as input a flow of lexical classes and outputs an abstract tree described by the abstract grammar. The syntactic tree is obtained by rule derivation.of the abstract grammar. The abstract tree is obtained from the derivation tree: it should contain operations as nodes and IDs as leaves. When reducing, the derivation tree to a syntax tree, terminals are "lifted" in the tree. You will use the following grammar for expressions:

$$\begin{aligned} E &\to E + T \mid E - T \mid T \\ T &\to T * F \mid F \\ F &\to ID \mid NUM \end{aligned}$$

For instance, for the expression $a + b$, we get:



Exercise 97 — **Derivation trees and AST**

Build the derivation trees and the abstract syntax trees corresponding to the expressions below.

1. $a + b - c$
2. $a - b + c$
3. $a + b * c$

B.4 Type checking

Type checking is performed on the abstract tree.

Exercise 98 — **Rules for type checking**

1. Give informally the rules for type checking. Rules should be given on the abstract grammar.

B.5 Code Generation

As input to code generation, we have an abstract tree. As output, we have a sequence of statements. Considering the abstract syntax of Section B.1.2, we define the following generation functions:

- A function that takes as input a statement S, and that produces a sequence of machine statements.
- A function that takes an arithmetical expression, and that produces a sequence of machine statements and the register that contains the result.
- A function that takes as input a Boolean expression and that produces a sequence of machine statements and the register that contains the result.

B.5.1 The M Machine

In the following, we describe a machine with registers. The arithmetical and logical operations update the condition codes. The arithmetical and logical operations, noted `OPER`, are `ADD, SUB, AND, OR`.
General registers are noted `Ri`. The `PC` register designates the program counter, the register `FP` designates the basis of local environment, and the register `SP` the head of the stack. Statements and addresses are coded on 4 bytes, the stack pointer designates the last occupied slot. We push on the stack by decrementing by 4 the stack pointer et we pop on the stack by incrementing by 4 the register `SP`. Integers are encoded on 4 bytes.
To each variable of a program is associated an address of the form `FP - shift`, where `shift` is computed and inserted in the symbol table.
In this exercise, we are interested in elementary operations, which syntax is given below. We will see branching operations and procedure calls later in lecture course on code generation.

`OPER` Ri, Rj, Rk Ri is the destination register

`OPER` Ri, Rk, val Ri is the destination register

`LD` Ri, [addr]

`ST` Ri, [addr]

Addresses are described as follows:
addr ::= Ri + Rj | Ri + val | Ri | val

Exercise 99 — Generating 3-address code

Give the sequence of code that corresponds to the following expressions or sequences of statements. We suppose having the following symbol table.

identifier	shifting
x	4
y	8
z	12
u	16
v	20
w	24

1. x-y+z
2. x := 3
3. x := 3 ; y := x+2 ;
4. x := 3 ; y :=2 ; z:= x + y
5. x := 3 ; y := 2 ; z := x + y u := x + y+ z ; v := z + y ; w := y + v ;

B.5.2 Local optimizations

Exercise 100 — Optimizing code

Minimize the number of registers used in the previous code sequences. For this purpose, we can be interested in:

- the minimal number of register to evaluate an expression;
- the analysis of active variables to minimize the `LD` and `ST` operations that can be avoided;
- a bound of the total number of registers and the backup policy.

The backup policy is in place when the number of registers is not sufficient in order to perform the computation. In that case, the value of a register can be "backed up" in the local environment (after the variables of the local environment). That is for the symbol table mentioned below, the first free slot is located at [FP-28].

B.6 Global Optimizations Independent from the Machine

Exercise 101 — Global optimizations

Let us consider the following program written in an intermediate representation:

```
(1) a := 1          (4) d := a+b          (7) f := a+b
(2) c := a+b        (5) c := c+2          (8) c := c-1
(3) if c>0 goto 7   (6) goto 9            (9) g := d+c
```

1. Build the control-flow graph.
2. Suppress redundant computations.
3. Suppress the assignments that can be avoided, while supposing that:
   - no variable is used outside the code snippet,
   - g is used outside the code snippet.
4. Perform constant propagation.

APPENDIX

C

PROVABLY-CORRECT IMPLEMENTATION

C.1 The Abstract Machine AM

The purpose of these exercises is to get familiar with the instructions and the semantics of AM.

Exercise 102 — Computing some executions of AM

1. Compute the execution of

   $$\mathsf{push\text{-}1} \cdot \mathsf{fetch}(x) \cdot \mathsf{add} \cdot \mathsf{store}(x)$$

   on an initial memory $m = [x \mapsto 3]$.
2. Compute the execution of $\mathsf{loop}(\mathsf{True}, \mathsf{noop})$ in any memory m.
3. Compute the execution of $\mathsf{fetch}(x) \cdot \mathsf{fetch}(y) \cdot \mathsf{le} \cdot \mathsf{branch}(\mathsf{push\text{-}1} \cdot \mathsf{add}, \mathsf{noop})$ in an initial memory $m = [x \mapsto 2, y \mapsto 1]$.

Exercise 103 — Abstracting machine code

What is the function computed by this code?

$$\begin{array}{l}\mathsf{push\text{-}0} \cdot \mathsf{store}(z) \cdot \mathsf{fetch}(x) \cdot \mathsf{store}(r)\\ \mathsf{loop}(\mathsf{fetch}(r) \cdot \mathsf{fetch}(y) \cdot \mathsf{le},\\ \quad \mathsf{fetch}(y) \cdot \mathsf{fetch}(r) \cdot \mathsf{sub} \cdot \mathsf{store}(r) \cdot\\ \quad \mathsf{push\text{-}1} \cdot \mathsf{fetch}(z) \cdot \mathsf{add} \cdot \mathsf{store}(z)\\ )\end{array}$$

C.2 Properties of AM

The purpose of these exercises is to exhibit some properties of the abstract machine. These properties are similar to the properties we exhibited for the Structural Operational Semantics and shall be used later to prove the correctness of code generation.

Exercise 104

Prove that code and stack contents can be extended.
Formally: $\forall c_1, c_2, c \in \textbf{Code}, \forall s_1, s_2, s \in \textbf{Stack}, \forall m_1, m_2 \in \mathsf{State}$:

$$(c_1, s_1, m_1) \triangleright^k (c_2, s_2, m_2) \text{ implies } (c_1 \cdot c, s_1 \cdot s, m_1) \triangleright^k (c_2 \cdot c, s_2 \cdot s, m_2)$$

Exercise 105 — Code can be composed and decomposed

Prove that the code can be composed and decomposed.
Formally: $\forall c_1, c_2 \in \textbf{Code}, \forall s_1, s_2 \in \textbf{Stack}, \forall m_1, m_2 \in \mathsf{State}$:

$(c_1 \cdot c_2, s, m) \triangleright^k (\epsilon, s_2, m_2)$ implies $\exists k' \in \mathbb{N}, (\epsilon, s', m') \in \textbf{Config} : (c_1, s, m) \triangleright^{k'} (\epsilon, s', m') \wedge (c_2, s', m') \triangleright^{k-k'} (\epsilon, s_2, m_2)$

Exercise 106 — Relation $\triangleright$ is deterministic

Prove that the relation $\triangleright$ is deterministic.
Formally: $\forall c, c_1, c_2 \in \textbf{Code}, \forall s_1, s_2 \in \textbf{Stack}, \forall m_1, m_2 \in \mathsf{State}$:

$$(c, s, m) \triangleright (c_1, s_1, m_1) \wedge (c, s, m) \triangleright (c_2, s_2, m_2) \text{ implies } (c_1, s_1, m_1) = (c_2, s_2, m_2)$$

C.3 Code generation

The purpose of these exercises is to define completely code generation for all possible statements of the While language. Similarly to what was done before when giving an operational semantics to the While language, we proceed in several steps: defining code generation for arithmetical expressions, for Boolean expressions and statements.

Exercise 107 — Code generation for arithmetical expressions

We are interested in the code generation function for arithmetical expressions.

1. Give the signature of this function.
2. Give the complete definition of the function.

Exercise 108 — Generating code for some arithmetical expressions

Generate the code for the following arithmetical expressions by applying the corresponding generation function.

1. $x + 1$
2. $2 * x$

Exercise 109 — $\mathcal{CA}$ and associativity

In the lecture and exercises, we have seen that the semantic function for arithmetical expressions is associative, i.e., $\mathcal{A}[(a_1 + (a_2 + a_3)] = \mathcal{A}[(a_1 + a_2) + a_3]$, for any $a_1, a_2, a_3 \in \mathsf{Aexp}$.

1. Show that it is not the case that $\mathcal{CA}[(a_1 + (a_2 + a_3)] = \mathcal{CA}[(a_1 + a_2) + a_3]$.
2. Show that $\mathcal{CA}[(a_1 + (a_2 + a_3)]$ and $\mathcal{CA}[(a_1 + a_2) + a_3]$ behave in the same manner.

Exercise 110 — Code generation for Boolean expressions

We are interested in the code generation function for Boolean expressions.

1. Give the signature of this function.
2. Give the complete definition of the function.

Exercise 111 — Generating code for some Boolean expressions

Generate the code for the following Boolean expressions by applying the corresponding generation functions.

1. $x \leq 2$
2. $2 * x = 5 * y$

Exercise 112 — Code generation for statements

We are interested in the code generation function for statements.

1. Give the signature of this function.
2. Give the complete definition of the function.

Exercise 113 — Generating code for a program

Give the target code obtained when translating the *factorial program.*

$y := 1; \texttt{while } \neg(x = 1) \texttt{ do } y := y * x; x := x - 1 \texttt{ od}$

Exercise 114 — Generating code for a statement

Use the code generation functions to generate code for the following statement:

$z := 0; \texttt{while } y \leq x \texttt{ do } z := z + 1; x := x - y \texttt{ od}$

Exercise 115 — Generating code for the `repeat until` construct

Extend the code generation function of statement to generate code for the `repeat until` construct. The definition of the code generation function should remain compositional and the instruction set of the machine should remain the same.

C.4 Correctness of code generation

The purpose of these exercises is to show that the code generation for the abstract machine is correct. It is performed in several steps.
Reminder from the course: in the following, for the sake of simpler notations, we assume that the memory component of AM and the state in the operational semantics are merged, i.e., we do not make any distinction between m and σ.

Exercise 116 — Correctness for Arithmetical expressions

Prove that the code generation function as defined in Exercise 107 is correct. Formally:

$\forall a \in \mathsf{Aexp} : (\mathcal{CA}[a], \epsilon, \sigma) \triangleright^* (\epsilon, \mathcal{A}[a]\sigma, \sigma)$

Exercise 117 — Correctness for Boolean expressions

Prove that the code generation function as defined in Exercise 110 is correct. Formally:

$\forall b \in \mathsf{Bexp} : (\mathcal{CB}[b], \epsilon, \sigma) \triangleright^* (\epsilon, \mathcal{B}[b]\sigma, \sigma)$

Exercise 118 — Correctness for Statements

Prove that the code generation function as defined in Exercise 112 is correct. The proof amounts to proving the two following claims.

1. $\forall S \in \mathsf{Stm}, \forall \sigma, \sigma' \in \mathsf{State}$: $(S, \sigma) \to \sigma'$ implies $(\mathcal{CS}[S], \epsilon, \sigma) \triangleright^* (\epsilon, \epsilon, \sigma')$
2. $\forall S \in \mathsf{Stm}, \forall \sigma, \sigma' \in \mathsf{State}, \forall k \in \mathbb{N}$: $(\mathcal{CS}[S], \epsilon, \sigma) \triangleright^k (\epsilon, e, \sigma')$ implies $(S, \sigma) \to \sigma'$ and $e = \epsilon$

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