Lecture 1 – Maths for Computer Science
Proof techniques

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Lecture notes MoSIG1

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Context

The main idea of this preliminary lecture is to **introduce the methodology to prove some results in Discrete Mathematics** (in the field of combinatorics, summations, counting, basic number theory).
We will show how to handle simple (and also some less easy) results with basic tools that do not require any sophisticated background in Maths.

A subsequent goal is to strengthen the intuition while *doing* Maths.

The holy grail of Mathematics: proving theorems

Schema of classical proofs.

- A *proof* is a sequence of *statements*.
  - The first statement must be an axiom or another proved theorem.
  - Each subsequent statement must be either an axiom or the result of applying a rule of inference to the statements that are already present in the sequence.
- A *theorem* is the last statement of a proof.

Within this formalism: a theorem is any assertion that is proved.

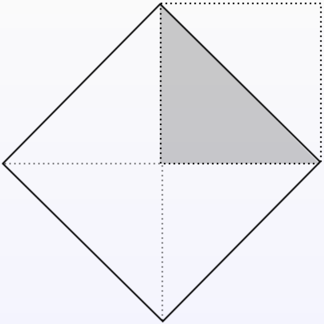
A difficulty is that assertions often require some modeling to be turned into mathematical statements.

Brief overview of proving techniques

- Contradiction *contradictio in contrarium*
- Induction
- Geometric proofs
- Combinatoric proofs
- Bijections between sets and Pigeon holes
- Unconventional proofs. All means are good!
- Proofs by computers
- *Fubini* (double counting principle).

Proof by contradiction

Let prove that $\sqrt{2}$ is irrational.



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Thus, $2.q^2 = p^2$.
p^2 is *even* (divisible by 2) then p is also even (the square of an odd number is odd). This means that $p = 2m$ for some positive integer m, which allows us to rewrite:
$2.q^2 = 4.m^2$, after simplification: $q^2 = 2.m^2$.
Thus, q must be even.
Both q and p have a common factor (2), which contradicts the assumption that they both share no common prime divisor.

Proof by induction

Proving that a statement *P(n)* involving integer n is true.

- **Basis.** Solve the statement for the small values of n.
- **Induction step.** Prove the statement for n assuming it is correct for $k \leq n - 1$.

Proposition. $\forall n$, the nth perfect square is the sum of the first n odd integers. Symbolically:

$$n^2 \;=\; 1 \;+\; 3 \;+\; 5 \;+\cdots+\; (2n-3) \;+\; (2n-1)$$

Proof. For every positive integer m, let $\mathbf{P}(m)$ denote the assertion

$$m^2 \;=\; 1+3+5+\cdots+(2m-1).$$

Let us proceed according to the standard format of an inductive argument.

- **Basis.** Because $1 \cdot 1 = 1$, proposition $\mathbf{P}(1)$ is true.
- **Induction step.** Let us assume, for the sake of induction, that assertion $\mathbf{P}(m)$ is true for all positive integers strictly smaller than n.

Consider now the summation

$$1 + 3 + 5 + \cdots + (2n-3) + (2n-1)$$

Because $\mathbf{P}(n-1)$ is true, we know that

$$\begin{aligned} 1 + 3 + \cdots + (2n-1) &= \big(1 + 3 + \cdots + (2n-3)\big) + (2n-1) \\ &= \big(1 + 3 + \cdots + (2(n-1)-1)\big) + (2n-1) \\ &= (n-1)^2 + (2n-1) \end{aligned}$$

By direct calculation, we now find that

$$(n-1)^2 + (2n-1) \;=\; (n^2 - 2n + 1) + (2n-1) \;=\; n^2.$$

The Principle of (Finite) Induction tells us that $\mathbf{P}(n)$ is true for all integer n.

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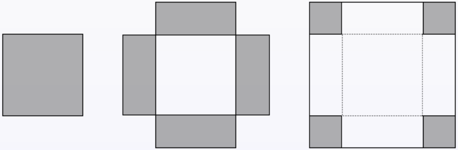

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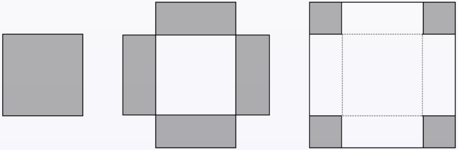

The surface of the cross is equal to the right hand side.
Adding the 4 little squares in the border leads to a total surface of $39 + 4\frac{25}{4} = 64$, which is the square of 8.
We finally deduce x: $8 - 2\frac{5}{2} = 3$.

Pigeon's holes and relations between sets.

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You are attending a party that hosts n couples. In order to create a nice social atmosphere, the hosts requests that each attendees shake the hand of every person that he/she does not know.
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Here, the boxes are the number of times someone shake hands.
The persons are the pigeons. There are $2n$ persons at the party.
The number of people that each attendee does not known is $\{0, 1, ..., 2n - 2\}$ which contains $2n - 1$ elements.

All means are good.

The problem of friends and strangers at a party.
Assertion In any gathering of six people, at least one of the following assertions is true.

A. There is a group of three people who know each other.
B. There is a group of three people none of whom knows either of the others.

Where (and how) to begin?!?

If we cannot reduce the provable world to sequences of assertions, then what is our goal?
Using evocative terms, the french mathematician René Thom tells us.

Est rigoureuse toute démonstration, qui, chez tout lecteur suffisamment instruit et préparé, suscite un état d'évidence qui entraîne l'adhésion.

Proof by computers.

The 4-colors theorem (which was a famous conjecture).
Coloring planar graphs using no more than 4 colors.

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Easy to color a planar graph in 6 colors.

Preliminary: coloring in 6

Proposition. Every planar graph G is 6-colorable.

Proof (sketch)

1. Remove from graph G a vertex v of smallest degree d_v, together with all its incident edges
   We guarantee that $d_v \leq 5$.
2. inductively color the vertices of the graph left after the removal of v (denoting the smaller graph by G').
   For planar graphs, we use an inductive assumption that can be colored with ≤ 6 colors.
3. Reattach v via its d_v edges and then color v.
   Note that the coloring guarantee in this result allows us to use $d_v + 1$ colors to color G. Because v has degree d_v, it can have no more than d_v neighboring vertices in G', so our access to $d_v + 1$ colors guarantees that we can successfully color v.

Extensions: coloring in 4

intermediate step: coloring in 5 colors.

For 4 colors, the initial proof needed to check the property on 1478 basic configurations!
It needs a computer.

Other unconventional ways to prove

The informal idea is to establish a one-to-one correspondence between elements of a set (integers).

Fubini's principle[1]:

Enumerate the elements of a set by two different methods, one leading to an evidence.

[1] Guido Fubini 1879-1943

Triangular numbers

Definition:
Triangular numbers are defined as the sum of the n first integers: $\Delta_n = \sum_{k=1}^{n} k$.

There exist many proofs for this result, the simplest one is obtained in writing this sum forward and backward and gathering the terms two by two as follows:

$$\begin{aligned} 2.\Delta_n &= \boxed{\begin{matrix} 1 \\ n \end{matrix}} + \boxed{\begin{matrix} 2 \\ n-1 \end{matrix}} + \dots + \boxed{\begin{matrix} n \\ 1 \end{matrix}} \\ &= (n+1) + (n+1) + \dots + (n+1) \end{aligned}$$

$2\Delta_n$ is n times $n+1$, thus, $\Delta_n = \frac{(n+1).n}{2}$

Another way of looking at this process (1)

Use the Fubini's principle.

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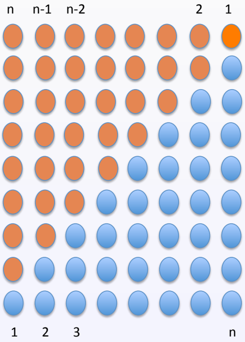
Δ_n is represented by piles of bullets arranged as a triangle. Putting two copies up side down gives the n by $n+1$ rectangle.



Sum of two consecutive triangular numbers

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Computing the first ranks leads us to an evidence: $\Delta_1 + \Delta_0 = 1$, then, 4, 9, 16, 25, 36, ...

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This result can be directly obtained using a geometric pattern: