# Numbers

- Natural Numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
- Whole Numbers: $\mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$
- Integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Rational Numbers: $\mathbb{Q} = \left\{\frac{a}{b} \ : \ a, b \in \mathbb{Z} \ , \ b \neq 0\right\}$. Rational Numbers comprise of fractions (includes all proper and improper fractions and mixed numbers). All terminating decimals (eg, 10.87) and recurring decimals (eg, $0.3\dot{7}\dot{1} = 0.3717171\ldots$) are rational numbers because these can all be expressed as fractions. All integers are rational numbers.
- Irrational Numbers: numbers that cannot be expressed in the form $\frac{a}{b}$, where $a$ and $b$ are integers and $b \neq 0$. Eg. $\pi$, $\sqrt{2}$, $\sqrt{5}$, $-4\sqrt{7}$, $e$, $2.75e$, etc. Irrational numbers are non-recurring decimals. Any non-zero rational number multiplied to an irrational number results in an irrational number. For example, $-\frac{3}{4}\pi$ is irrational.
- Perfect Squares: $\{1, 4, 9, 16, 25, \ldots\}$
- Perfect Cubes: $\{1, 8, 27, 64, \ldots\}$
- Prime Numbers: Positive integers at least 2 whose only positive divisors are 1 and itself. $\{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\}$
- Composite Numbers: Positive integers $\geq 4$ that are not prime, in other words, having positive divisors apart from 1 and itself.

# Equality and Inequality Symbols

| Symbol | Meaning | Example |
|---|---|---|
| $=$ | is equal to | $0.1 = \frac{1}{10}$ |
| $\neq$ | is not equal to | $0.11 \neq \frac{1}{10}$ |
| $>$ | is greater than | $0.1 > 0.01$ |
| $\geqslant$ | is greater than or equal to | $a \geqslant 5$ |
| $<$ | is less than | $0.05 < 5$ |
| $\leqslant$ | is less than or equal to | $b \leqslant 5$ |

# Prime Factorization, HCF, LCM

- Example of prime factorization:

$$\begin{array}{r|c} 2 & 4356 \\ \hline 2 & 2178 \\ \hline 3 & 1089 \\ \hline 3 & 363 \\ \hline 11 & 121 \\ \hline & 11 \\ \hline \end{array}$$

  Hence $4356 = 2^2 \times 3^2 \times 11^2$ (in index notation).

- Example of HCF and LCM using prime factorization:

  $4800 = 2^6 \times 3 \times 5^2$

  $5544 = 2^3 \times 3^2 \times 7 \times 11$

  $\text{HCF} = 2^3 \times 3$

  [take common prime factors and lowest power of each]

  $\text{LCM} = 2^6 \times 3^2 \times 5^2 \times 7 \times 11$

  [take all prime factors and highest power of each]

- Examples of square roots and cube roots using prime factorization:

$$\begin{aligned} 54756 &= 2^2 \times 3^4 \times 13^2 \\ \sqrt{54756} &= 2 \times 3^2 \times 13 = 234 \\ 1728 &= 2^6 \times 3^3 \\ \sqrt[3]{1728} &= 2^2 \times 3 = 12 \end{aligned}$$

# Approximation

## Significant Figures

Rules of identifying number of significant digits:

1. All non-zero digits are significant.
2. Zeros between non-zero digits are significant.

   Eg. 302 (3 sf)

   Eg. 10.2301 (6 sf)
3. In a whole number, zeros after the last nonzero digit may or may not be significant. It depends on the estimation being made.

Eg.

$7436000 = 7000000$ (1 sf)

$7436000 = 7400000$ (2 sf)

$7436000 = 7440000$ (3 sf)

$7436000 = 7436000$ (4 sf)

$7436000 = 7436000$ (5 sf)

$7436000 = 7436000$ (6 sf)

4. In a decimal number, zeros before the 1st non-zero digit are not significant.

   Eg. 0.004 (1 sf)

   Eg. 0.07008 (4 sf)

5. In a decimal number, zeros after the last non-zero digit are significant.

   Eg. 6.40 (3 sf)

   Eg. 12.000 (5 sf)

   Eg. 20300.000 (8 sf)

   Eg. 0.0700800 (6 sf)

### Decimal Place Rounding

Examples:

$0.7374 = 0.74$ (2 dp)

$58.301 = 58.30$ (2 dp)

$207.6296 = 207.630$ (3 dp)

$207.6296 = 207.63$ (2 dp)

$207.977 = 208.0$ (1 dp)

$207.977 = 207.98$ (2 dp)

$18.997 = 19.00$ (2 dp)

## Standard Form

$\pm A \times 10^n$, where $1 \leq A < 10$ and $n$ is an integer.

Examples:

$1350000 = 1.35 \times 10^6$

$0.000375 = 3.75 \times 10^{-4}$

## Common Prefixes

| | | | |
|---|---|---|---|
| $10^{12}$ | trillion | tera | T |
| $10^9$ | billion | giga | G |
| $10^6$ | million | mega | M |
| $10^3$ | thousand | kilo | k |
| $10^{-3}$ | thousandth | milli | m |
| $10^{-6}$ | millionth | micro | $\mu$ |
| $10^{-9}$ | billionth | nano | n |
| $10^{-12}$ | trillionth | pico | p |

## Indices

Rules of Indices:

Assume that $a, b, m, n$ are non-zero.

$$a^0 = 1$$

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$(a^n)^m = a^{nm}$$

$$a^{-n} = \frac{1}{a^n}$$

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Caution:

1. For indices (powers) that are not integers, the above rules of indices hold only for $a, b > 0$.
2. Likewise, if some of the indices are negative or there is division by either $a$ or $b$, then the above rules hold only for $a, b > 0$.
3. $\sqrt{(-8)^2} = \sqrt{64} = 8$, NOT $-8$.
4. $\sqrt[3]{-27} = -3$.

Equalities of Indices:

1. If $a^m = a''$, then $m = n$.
2. If $a^m = b^m$, then $a = b$.

## Percentage, Ratio, Rate

- To express a percentage as a fraction or decimal, divide by 100:

  $x\% = \frac{x}{100}$

  Eg, $23.5\% = \frac{23.5}{100} = \frac{235}{1000} = \frac{47}{200}$

  Eg, $401\% = \frac{401}{100} = 4.01$

- To express any number as a percentage, multiply it by 100%.

  Eg, $0.165 = 0.165 \times 100\% = 16.5\%$.

- Expressing a quantity $A$ as a percentage of a quantity $B$:

  $\frac{A}{B} \times 100\%$

  Eg, Express 63.7 as a percentage of 98.

  Answer: $\frac{63.7}{98} \times 100\% = 65\%$

  In words, we say that 63.7 is 65% of 98.

- Increase or decrease a quantity by a given percentage:

  Eg, Increase 45 by 2.4%:

  Answer: $45 \times \left(1 + \frac{2.4}{100}\right) = 45 \times 1.024 = 46.08$

  Eg, Decrease 45 by 90%:

  Answer: $45 \times \left(1 - \frac{90}{100}\right) = 45 \times 0.1 = 4.5$

- Percentage Increase and Percentage Decrease:

  When a quantity increases, the percentage increase is

  $\frac{\text{final value} - \text{initial value}}{\text{initial value}} \times 100\%$

  When a quantity dereases, the percentage decrease is

  $\frac{\text{initial (bigger) value} - \text{final (smaller) value}}{\text{initial value}} \times 100\%$

  Percentage increase will always be $> 0$ if the quantity has increased.

  Percentage decrease will always be $> 0$ if the quantity has decreased.

  Percentage change is

  $\frac{\text{final value} - \text{initial value}}{\text{initial value}} \times 100\%$

  regardless of whether the quantity has increased or decreased. Percentage change can be either positive or negative depending on whether the quantity has increased or decreased.

- When writing ratios such as $a : b$, $a, b$ are positive integers. Always reduce ratios to the simplest form, eg, $10 : 6$ is to be reduced to $5 : 3$. The ratio $a : b$ expressed in fraction form is $\frac{a}{b}$.

  Eg, If 7 times of $x$ is equal to 5 times of $y$, then $x : y = 5 : 7$ (note the switching of the order)

- We can use ratios to increase and decrease quantities. For example, if we increase a quantity $x$ in the ratio $6 : 5$, the new quantity is $\frac{6}{5}x$; if we decrease a quantity $x$ in the ratio $5 : 6$, the new quantity is $\frac{5}{6}x$.

- Various units of measurement:

  Mass:

  1 kg = 1000 g

  1 g = 1000 mg

  Length:

  1 km = 1000 m

  1 m = 100 cm

  1 cm = 10 mm

  Area:

  1 $\text{km}^2 = 10^6\ \text{m}^2$

  1 $\text{m}^2 = 10000\ \text{cm}^2$

  Volume:

  1 $l$ = 1000 ml

  1 $\text{cm}^3$ = 1 ml

  1 $\text{m}^3 = 10^6\ \text{cm}^3 = 1000\ l$

  Time:

  1 hr = 60 min

  1 min = 60 sec

- Distance = Speed × Time

- Average speed = (Total Distance) / (Total Time Taken)

- Conversion of units for speed:

  $26\text{km/h} = 26000\text{m/h} = \frac{26000}{3600}\text{m/s} = \frac{65}{9}\text{m/s}$

  $35\text{m/s} = 0.035\text{km/s} = (0.035 \times 3600)\text{km/h} = 126\text{km/h}$

- Density = Mass / Volume.

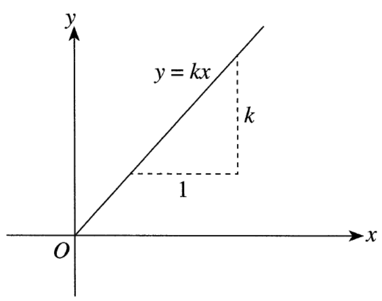
  Units are usually $\text{g/cm}^3$ or $\text{kg/m}^3$.

  $1\text{g/cm}^3 = 1000\text{kg/m}^3$.

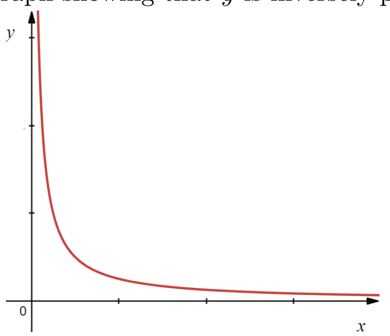
  Eg, $0.235\text{g/cm}^3 = 235\text{kg/m}^3$.

# Direct and Inverse Proportion

If $y$ is directly proportional to $x$, then $y = kx$, where $k$ is a constant and $k \neq 0$. The ratios $\frac{x}{y}$ and $\frac{y}{x}$ are constant. Furthermore, the graph on $y$ against $x$ (or of $x$ against $y$) is a straight line through the origin.
Graph showing that $y$ is directly proportional to $x$:




If $y$ is inversely proportional to $x$, then $y = \frac{k}{x}$, where $k$ is a constant and $k \neq 0$. The product $xy$ is constant.
Graph showing that $y$ is inversely proportional to $x$:




# Map Scales

- Linear scale:

  $1 : n$ means 1 unit length on map represents $n$ units length on ground.

  Eg. $1 : 5000$ means

  1 cm represents 5000 cm

  which implies 1 cm represents 50 m

  which implies 1 cm represents 0.05 km

- Representative Fraction (RF):

  If the linear scale is $1 : n$, the RF is expressed as $\frac{1}{n}$.

  Eg, if 3 cm represents 6 m, then RF is $\frac{1}{200}$.

- Area Scale:

  If linear scale is $1 : 20000$, then it means

  1 cm represents 20000 cm

  which implies 1 cm represents 0.2 km

  which implies $1^2$ cm$^2$ represents $(0.2)^2$ km$^2$

  which implies 1 cm$^2$ represents 0.04 km$^2$

# Number Patterns

Common number patterns:

- Constant difference

  Eg, $-5, -2, 1, 4, 7, 10, \ldots$

  The $n^{\text{th}}$ term, denoted $T_n$, is given by

  $T_n = a + d(n-1)$

  where $a$ is the first term and $d$ is the common difference.

  For the sequence $-5, -2, 1, 4, 7, 10, \ldots$,

  $T_n = -5 + (n-1)(3) = 3n - 8$.

  Alternatively,

  $T_n = b + dn$

  where $b$ is the term that would have come before the first term (ie, the "zeroth" term).

- Constant multiple (or common ratio)

  Eg, $3, 15, 75, 375, \ldots$

  $T_n = a \times r^{n-1}$

  where $a$ is the first term, $r$ is the common ratio, that is, $r$ is the number that when multiplied a term gives the next term.

  For the sequence $3, 15, 75, 375, \ldots$

  $T_n = 3 \times 5^{n-1}$.

- Perfect squares and perfect cubes

  $1, 4, 9, 16, 25, \ldots \ : \ T_n = n^2$

  $1, 8, 27, 64, 125, \ldots \ : \ T_n = n^3$

  $2, 8, 18, 32, 50, \ldots \ : \ T_n = 2n^2$

  $3, 10, 29, 66, 127, \ldots \ : \ T_n = n^3 + 2$

- Fibonacci Sequences

  Each term is the sum of the previous two terms:

  $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

  $-2, 3, 1, 4, 5, 9, 14, 23, 37, \ldots$

## Simultaneous Linear Equations

Method 1: Elimination

$5x - 2y = 21 \quad$ —(1)

$2x - y = 8 \quad$ —(2)

$(1) \times 2 : 10x - 4y = 42 \quad$ —(3)

$(2) \times 5 : 10x - 5y = 40 \quad$ —(4)

(3) - (4) : $\quad y = 2$

Sub into (1): $5x - 2(2) = 21$

$5x - 4 = 21 \ \Rightarrow \ 5x = 25 \ \Rightarrow \ x = 5$

Method 2: Substitution

$5x - 2y = 21 \quad$ —(1)

$2x - y = 8 \quad$ —(2)

From (2):

$y = 2x - 8 \quad$ —(3)

Sub (3) into (1): $5x - 2(2x - 8) = 21$

$5x - 4x + 16 = 21 \ \Rightarrow \ x - 16 \ = 21$

$x = 5$

Sub into (2):

$2(5) - y = 8 \ \Rightarrow \ 10 - y = 8 \ \Rightarrow \ y = 2$

## Inequalities

Inequality sign is reversed when both sides are multiplied or divided by a negative number.

Eg. $-3x + 4 \geq 12$

$-3x \geq 12 - 4$

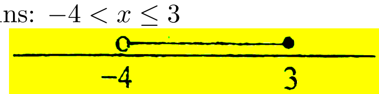
$-3x \geq 8$

$x \leq -\frac{8}{3}$

Eg. $3(x - 1) < 4x + 1 \leq 7 + 2x$

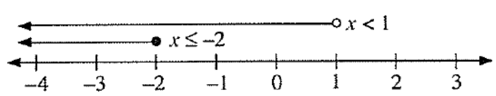
| | |
|---|---|
| $3(x-1) < 4x + 1$ | $4x + 1 \leq 7 + 2x$ |
| $3x - 3 < 4x + 1$ | $4x - 2x \leq 7 - 1$ |
| $3x - 4x < 1 + 3$ | $2x \leq 6$ |
| $-x < 4$ | $x \leq 3$ |
| $x > -4$ | |

Ans: $-4 < x \leq 3$




Eg. $5x + 4 \leq 3x < 6 - 3x$

| | |
|---|---|
| $5x + 4 \leq 3x$ | $3x < 6 - 3x$ |
| $2x \leq -4$ | $6x < 6$ |
| $x \leq -2$ | $x < 1$ |




Ans: $x \leq -2$.

## Expansion

Eg.

$2p - 3(p + 1)$

$= 2p - 3p - 3$

$= -p - 3$

Eg.

$5x - (x + 1)(2x - 3)$

$= 5x - \left(2x^2 - 3x + 2x - 3\right)$

$= 5x - 2x^2 + 3x - 2x + 3$

$= -2x^2 + 6x + 3$

# Factorization and Identities

Factorization of Quadratic Expressions:
$5x^2 + 9x - 2$

| $\times$ | $5x$ | $-1$ |
|---|---|---|
| $x$ | $5x^2$ | $-x$ |
| $2$ | $10x$ | $-2$ |

$\therefore \quad 5x^2 + 9x - 2 = (5x-1)(x+2)$

Identities:

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(a+b)(a-b) = a^2 - b^2$

Common Factorisation Techniques:

- Common Factors
  Eg. $6a^3b - 2a^2b = 2a^2b(3a-1)$
- Grouping
  Eg.
  $6p^2 - 3pq - 10ap + 5aq$
  $= 3p(2p-q) - 5a(2p-q)$
  $= (3p-5a)(2p-q)$
- Using Difference of Two Squares
  Eg. $9a^2 - 1$
  $= (3a)^2 - (1)^2$
  $= (3a+1)(3a-1)$
  Eg. $16a^4 - 81$
  $= (4a^2+9)(4a^2-9)$
  $= (4a^2+9)(2a+3)(2a-3)$
- Combination of methods:
  Eg. $3x^3 - 12xy^2 = 3x\left(x^2 - 4y^2\right)$ common factor
  $= 3x(x+2y)(x-2y)$ then diff. of 2 squares
  Always try common factor first
  Eg.
  $4 - p^2 + 6pq - 9q^2$
  $= 4 - (p^2 - 6pq + 9q^2)$
  $= (2)^2 - (p-3q)^2$
  $= (2 + (p-3q))(2 - (p-3q))$
  $= (2 + p - 3q)(2 - p + 3q)$

# Algebraic Fractions

Eg.
$$\frac{x+2}{3} - \frac{x-5}{2} = \frac{2(x+2)}{6} - \frac{3(x-5)}{6}$$
$$= \frac{2(x+2) - 3(x-5)}{6}$$
$$= \frac{2x + 4 - 3x + 15}{6}$$
$$= \frac{19-x}{6}$$

Eg.
$$\frac{5}{x+1} - \frac{2}{x-3}$$
$$= \frac{5(x-3)}{(x+1)(x-3)} - \frac{2(x+1)}{(x-3)(x+1)}$$
$$= \frac{5(x-3) - 2(x+1)}{(x+1)(x-3)}$$
$$= \frac{5x - 15 - 2x - 2}{(x+1)(x-3)}$$
$$= \frac{3x-17}{(x+2)(x-5)}$$

Eg.
$$\frac{5}{3x} + \frac{2}{x}$$
$$= \frac{5}{3x} + \frac{6}{3x}$$
$$= \frac{5+6}{3x}$$
$$= \frac{11}{3x}$$

Eg.
$$\frac{3}{(x+2)^2} - \frac{4}{x+2}$$
$$= \frac{3}{(x+2)^2} - \frac{4(x+2)}{(x+2)^2}$$
$$= \frac{3 - 4(x+2)}{(x+2)^2}$$
$$= \frac{3 - 4x - 8}{(x+2)^2}$$
$$= \frac{-4x-5}{(x+2)^2}$$

Eg.

$$\frac{7}{x^2-9} - \frac{1}{x-3}$$
$$= \frac{7}{(x+3)(x-3)} - \frac{1}{x-3}$$
$$= \frac{7}{(x+3)(x-3)} - \frac{x-3}{(x-3)^2}$$
$$= \frac{7-(x+3)}{(x+3)(x-3)}$$
$$= \frac{7-x-3}{(x+3)(x-3)}$$
$$= \frac{4-x}{(x+3)(x-3)}$$

Eg.

$$\frac{3x+2}{x^2-4} + \frac{1}{x+2} - \frac{2}{x-2}$$
$$= \frac{3x+2}{(x+2)(x-2)} + \frac{1}{x+2} - \frac{2}{x-2}$$
$$= \frac{3x+2+(x-2)-2(x+2)}{(x+2)(x-2)}$$
$$= \frac{3x+2+x-2-2x-4}{(x+2)(x-2)}$$
$$= \frac{2x-4}{(x+2)(x-2)}$$
$$= \frac{2(x-2)}{(x+2)(x-2)}$$
$$= \frac{2}{x+2}$$

Eg.

$$\frac{9}{x-5} + \frac{3}{5-x}$$
$$= \frac{9}{x-5} - \frac{3}{x-5}$$
$$= \frac{9-3}{x-5}$$
$$= \frac{6}{x-5}$$

Eg.

$$\frac{2x}{3y-8x} + \frac{11x}{80x-30y}$$
$$= \frac{2x}{3y-8x} + \frac{11x}{-10(3y-8x)}$$
$$= \frac{2x}{3y-8x} - \frac{11x}{10(3y-8x)}$$
$$= \frac{20x-11x}{10(3y-8x)}$$
$$= \frac{9x}{10(3y-8x)}$$

Eg.

$$\frac{4}{x^2-4} + \frac{1}{2-x}$$
$$= \frac{4}{(x+2)(x-2)} - \frac{1}{x-2}$$
$$= \frac{4-(x+2)}{(x+2)(x-2)}$$
$$= \frac{4-x-2}{(x+2)(x-2)}$$
$$= \frac{2-x}{(x+2)(x-2)}$$
$$= \frac{-x-2)}{(x+2)(x-2)}$$
$$= -\frac{1}{x+2}$$

Eg.

$$\frac{6p^3}{7q} \div \frac{2p}{21q^2}$$
$$= \frac{6p^3}{7q} \times \frac{21q^2}{2p} \quad \text{[ do cancelling ]}$$
$$= 9p^2q$$

Eg.

$$\frac{4pq^2+4pqr}{9pqr^2+9pq^2r} = \frac{4pq(q+r)}{9pqr(r+q)}$$
$$= \frac{4}{9r}$$

Eg.

$$\frac{5k^2-17k-12}{5k^2-10k-40} = \frac{(5k+3)(k-4)}{5\left(k^2-2k-8\right)}$$
$$= \frac{(5k+3)(k-4)}{5(k-4)(k+2)}$$
$$= \frac{5k+3}{5(k+2)}$$

Eg.

$$\frac{xy-z^2-xz+yz}{y^2-2yz+z^2} \div \frac{11}{2xz+x^2+z^2}$$
$$= \frac{xy-xz+yz-z^2}{y^2-2yz+z^2} \times \frac{2xz+x^2+z^2}{11}$$
$$= \frac{x(y-z)+z(y-z)}{(y-z)^2} \times \frac{(x+z)^2}{11}$$
$$= \frac{(x+z)(y-z)}{(y-z)^2} \times \frac{(x+z)^2}{11}$$
$$= \frac{(x+z)^3}{11(y-z)}$$

Eg.

$$\frac{\frac{1}{x}+1}{x+1}$$
$$= \left(\frac{1}{x}+1\right) \div (x+1)$$
$$= \frac{1+x}{x} \times \frac{1}{x+1}$$
$$= \frac{1}{x}$$

Eg.

$$\frac{1-\frac{1}{x}}{1-\frac{1}{x^2}}$$
$$= \left(1-\frac{1}{x}\right) \div \left(1-\frac{1}{x^2}\right)$$
$$= \left(\frac{x-1}{x}\right) \div \left(\frac{x^2-1}{x^2}\right)$$
$$= \frac{x-1}{x} \times \frac{x^2}{x^2-1^2}$$
$$= \frac{x-1}{x} \times \frac{x^2}{(x+1)(x-1)}$$
$$= \frac{x}{x+1}$$

# Making Subject of Formula

Eg: Make $a$ the subject

$$y = m(x-a)+b$$
$$y-b = m(x-a)$$
$$m(x-a) = y-b$$
$$x-a = \frac{y-b}{m}$$
$$a = x - \frac{y-b}{m}$$

Eg: Make $x$ the subject

$$ax-by = 3-2x$$
$$ax+2x = 3+by$$
$$x(a+2) = by+3$$
$$x = \frac{by+3}{a+2}$$

Eg: Make $d$ the subject

$$T = 0.25\pi d^2$$
$$\pi d^2 = 4T$$
$$d^2 = \frac{4T}{\pi}$$
$$d = \pm\sqrt{\frac{4T}{\pi}}$$

Note the $\pm$ when taking square-roots in this type of question.

Eg: Make $c$ the subject

$$d = \frac{8-c}{c+7}$$
$$d(c+7) = 8-c$$
$$cd+7d = 8-c$$
$$cd+c = 8-7d$$
$$c(d+1) = 8-7d$$
$$c = \frac{8-7d}{d+1}$$

Eg: Make $q$ the subject

$$
\begin{aligned}
5a &= \sqrt{\frac{b^2}{q} - \frac{3c}{4}} \\
25a^2 &= \frac{b^2}{q} - \frac{3c}{4} \\
\frac{b^2}{q} &= 25a^2 + \frac{3c}{4} \\
\frac{b^2}{q} &= \frac{100a^2 + 3c}{4} \\
\frac{q}{b^2} &= \frac{4}{100a^2 + 3c} \\
q &= \frac{4b^2}{100a^2 + 3c}
\end{aligned}
$$

Eg: Make $y$ the subject

$$
\begin{aligned}
\frac{x\left(yz - w^2\right)}{2} - \frac{y}{3} &= 6y \\
3x\left(yz - w^2\right) - 2y &= 36y \\
3xyz - 3w^2x - 2y &= 36y \\
3xyz - 38y &= 3w^2x \\
y(3xz - 38) &= 3w^2x \\
y &= \frac{3w^2x}{3xz - 38}
\end{aligned}
$$

Eg: Make $x$ the subject

$$
\begin{aligned}
p\sqrt{x} + q &= r + s\sqrt{x} \\
p\sqrt{x} - s\sqrt{x} &= r - q \\
\sqrt{x}(p - s) &= r - q \\
\sqrt{x} &= \frac{r - q}{p - s} \\
x &= \left(\frac{r - q}{p - s}\right)^2
\end{aligned}
$$

## Solving Quadratic Equations and Equations Involving Algebraic Fractions

Three methods of solving quadratic equations:

1. Factorisation

$$
\begin{aligned}
&3x^2 - 2x - 8 = 0 \\
&(3x + 4)(x - 2) = 0 \\
&3x + 4 = 0 \text{ or } x - 2 = 0 \\
&x = -\frac{4}{3} \text{ or } x = 2
\end{aligned}
$$

2. Using General Formula

$$
\begin{aligned}
&ax^2 + bx + c = 0 \\
&x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

Eg. $12x^2 - x - 25 = 0$

$$
\begin{aligned}
&a = 2, b = -1, c = -25 \\
&x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-25)}}{2(2)} \\
&x = \frac{1 \pm \sqrt{201}}{4} \\
&x = \frac{1 + \sqrt{201}}{4} \text{ or } \frac{1 - \sqrt{201}}{4} \\
&x = 3.79(\text{2dp}) \text{ or } -3.29(\text{2dp})
\end{aligned}
$$

3. Complete the Square

$$
\begin{aligned}
&x^2 - 8x + 13 = (x + p)^2 + q \\
&x^2 - 8x + 13 \\
&\quad = x^2 - 8x + \left(\frac{8}{2}\right)^2 - \left(\frac{8}{2}\right)^2 + 13 \\
&\quad = x^2 - 8x + 16 - 16 + 13 \\
&\quad = (x - 4)^2 - 3 \\
&\quad \text{To Solve} \quad x^2 - 8x + 13 = 0 \\
&\quad \text{First Change To } (x - 4)^2 - 3 = 0 \\
&(x - 4)^2 = 3 \\
&x - 4 = \pm\sqrt{3} \\
&x - 4 = \sqrt{3} \text{ or } x - 4 = -\sqrt{3} \\
&x = \sqrt{3} + 4 \text{ or } x = -\sqrt{3} + 4 \\
&x = 5.73(\text{2dp}) \text{ or } 2.27(\text{2dp})
\end{aligned}
$$

Equations involving algebraic fractions:

Eg:

$$\begin{aligned} 3x &= \frac{1}{x} - 4 \\ 3x^2 &= 1 - 4x \\ 3x^2 + 4x - 1 &= 0 \\ x &= \frac{-4 \pm \sqrt{4^2 - 4(3)(-1)}}{2(3)} \\ &= \frac{-4 \pm \sqrt{28}}{6} \end{aligned}$$

$$x = \frac{-4 + \sqrt{28}}{6} \quad \text{or} \quad x = \frac{-4 - \sqrt{28}}{6}$$

$$x = 0.22 \quad \text{or} \quad x = -1.55 (2 \text{ d.}p.)$$

Eg:

$$\begin{aligned} x + 1 &= \frac{20}{x+2} \\ (x+1)(x+2) &= 20 \\ x^2 + 3x + 2 &= 20 \\ x^2 + 3x - 18 &= 0 \\ (x+6)(x-3) &= 0 \end{aligned}$$

$$x + 6 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = -6 \quad \text{or} \quad x = 3$$

Eg:

$$\frac{2-x}{x+1} + \frac{1}{x-3} = \frac{3}{5}$$

$$5(2-x)(x-3) + 5(x+1) = 3(x+1)(x-3)$$

$$5\left(2x - 6 - x^2 + 3x\right) + 5x + 5 = 3\left(x^2 - 3x + x - 3\right)$$

$$5\left(-x^2 + 5x - 6\right) + 5x + 5 = 3\left(x^2 - 2x - 3\right)$$

$$-5x^2 + 25x - 30 + 5x + 5 = 3x^2 - 6x - 9$$

$$8x^2 - 36x + 16 = 0$$

$$2x^2 - 9x + 4 = 0$$

$$(2x-1)(x-4) = 0$$

$$2x - 1 = 0 \text{ or } x - 4 = 0$$

$$x = \frac{1}{2} \text{ or } x = 4$$

Eg:

$$\frac{2x+5}{x^2+4x+3} + \frac{2}{x+3} = 1$$

$$\frac{2x+5}{(x+1)(x+3)} + \frac{2}{x+3} = 1$$

$$\begin{aligned} 2x + 5 + 2(x+1) &= (x+1)(x+3) \\ 2x + 5 + 2x + 2 &= x^2 + 4x + 3 \\ 4x + 7 &= x^2 + 4x + 3 \\ x^2 - 4 &= 0 \\ x^2 &= 4 \\ x &= \pm\sqrt{4} \\ x &= 2 \text{ or } x = -2 \end{aligned}$$

Eg:

$$\frac{x^2+2}{(5x-4)(2x-1)} = \frac{1}{3}$$

$$\begin{aligned} 3\left(x^2 + 2\right) &= (5x-4)(2x-1) \\ 3x^2 + 6 &= 10x^2 - 13x + 4 \end{aligned}$$

$$7x^2 - 13x - 2 = 0$$

$$(7x+1)(x-2) = 0$$

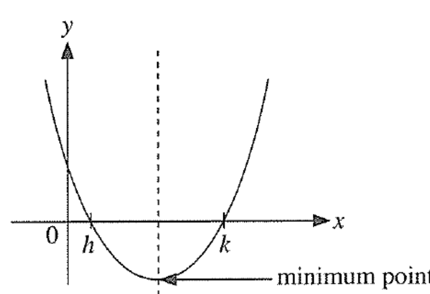
$$\therefore 7x + 1 = 0 \quad \text{or} \quad x - 2 = 0$$

$$x = -\frac{1}{7} \quad \text{or} \quad x = 2$$

# Quadratic Graphs

1. Suppose that a quadratic function $y = ax^2 + bx + c$ $(a \neq 0)$ can be factorized into $a(x-h)(x-k)$.

   Consider the case where $a > 0$. The graph is concave upwards (smiley face) and there is a minimum point.



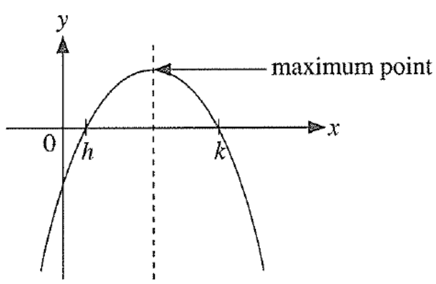
The graph intersects the $x$-axis at $(h, 0)$ and $(k, 0)$.

The line of symmetry is $x = \frac{h+k}{2}$.

The minimum point also occurs at $x = \frac{h+k}{2}$. To find the $y$-coordinate of the minimum point, substitute $x = \frac{h+k}{2}$ into the expression $y = ax^2 + bx + c$.

2. Suppose that a quadratic function $y = ax^2+bx+c$ $(a \neq 0)$ can be factorized into $a(x-h)(x-k)$.

   Consider the case where $a < 0$. The graph is concave downwards (sad face) and there is a maximum point.




   The graph intersects the $x$-axis at $(h, 0)$ and $(k, 0)$.

   The line of symmetry is $x = \frac{h+k}{2}$.

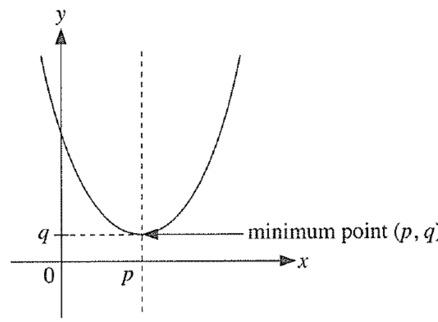
   The maximum point also occurs at $x = \frac{h+k}{2}$. To find the $y$-coordinate of the maximum point, substitute $x = \frac{h+k}{2}$ into the expression $y = ax^2 + bx + c$.

3. Suppose we complete the square quadratic function $y = ax^2 + bx + c$ $(a \neq 0)$ to obtain $y = a(x-p)^2 + q$.

   If $a > 0$, then the graph is smiley face, and has a minimum point at $(p, q)$. The minimum value of $y$ is $q$.

   The equation $ax^2 + bx + c = h$ has two solutions if $h > q$, one solution if $h = q$, and no solutions if $h < q$.

   The line of symmetry $x = p$.

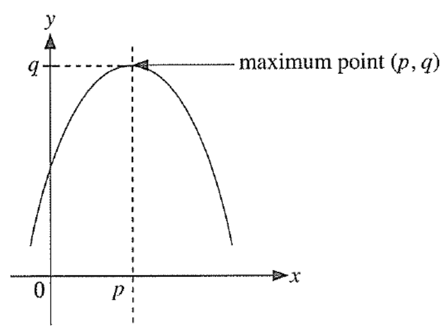



4. Suppose we complete the square quadratic function $y = ax^2 + bx + c$ $(a \neq 0)$ to obtain $y = a(x-p)^2 + q$.

   If $a < 0$, then the graph is sad face, and has a maximum point at $(p, q)$. The maximum value of $y$ is $q$.

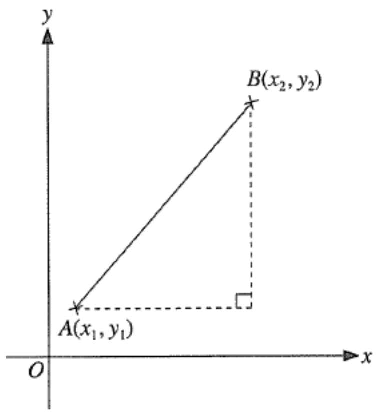
   The equation $ax^2 + bx + c = h$ has two solutions if $h < q$, one solution if $h = q$, and no solutions if $h > q$.

   The line of symmetry $x = p$.




# Coordinate Geometry

- The length of a line segment with end points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by
  $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
  or equivalently
  $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$




- The gradient of a line passing through $A\,(x_1, y_1)$ and $B\,(x_2, y_2)$ is given by

$$\frac{y_2 - y_1}{x_2 - x_1}$$

or equivalently

$$\frac{y_1 - y_2}{x_1 - x_2}.$$

- Collinear Points:

  If gradient of $AB$ = gradient of $BC$ = gradient of $AC$, we say that $A, B, C$ are collinear. This means that $A, B, C$ lie on the same straight line (not necessarily in the order $A - B - C$; it could also be in any order such as $B - C - A$.)

  Note: For collinear points, we only need to check the equality of one pair of gradients, for example, gradient of $AB$ = gradient of $BC$. Then automatically, gradient of $AB$ = gradient of $AC$ and gradient of $AC$ = gradient of $BC$ as well.

  When checking that gradient of $AB$ = gradient of $BC$, there must be a common point (in this case, $B$) in order to conclude that the points are collinear.

  If gradient of $AB$ = gradient of $CD$ (ie, no common points), then we cannot conclude any of the points are collinear.

- A horizontal line is parallel to the $x$-axis. If all the points on the line has $y$-coordinate equal to $b$, then the equation of the line is

$$y = b.$$

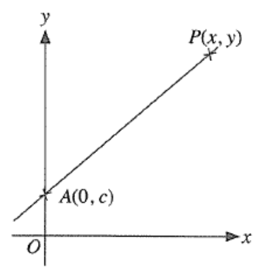
A vertical line is parallel to the $y$-axis. If all the points on the line has $x$-coordinate equal to $a$, then the equation of the line is

$$x = a.$$

If a line has a gradient $m$ and $y$-intercept $c$, then its equation is

$$y = mx + c.$$

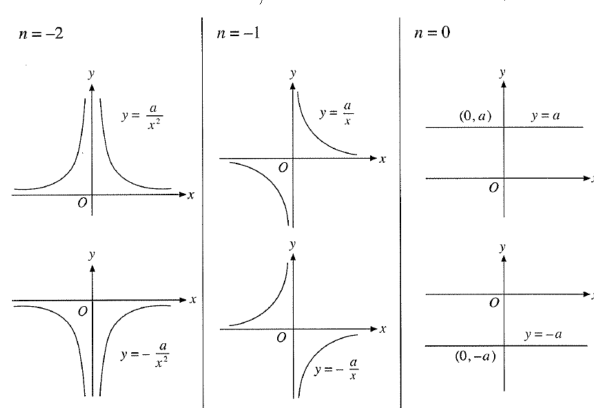
The equation $y = mx + c$ is known as the gradient-intercept form.

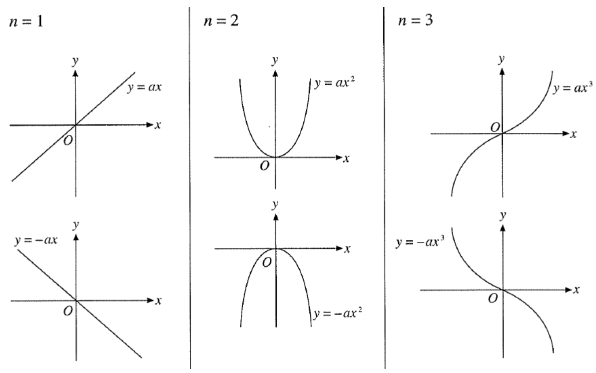



## Graphs and Functions

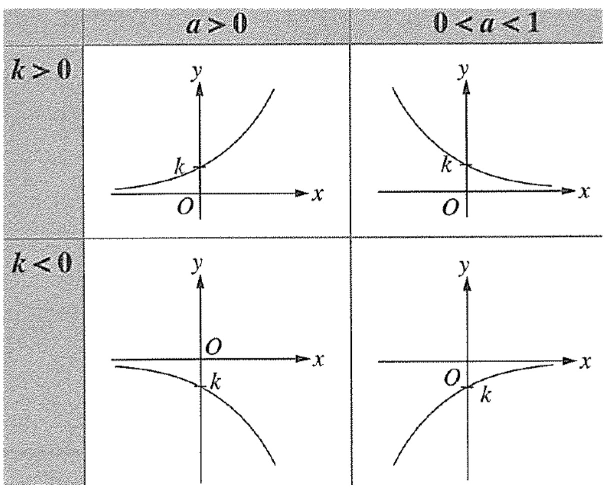
Below is a summary of the graphs of the form $y = ax^n$, where $n = -2, -1, 0, 1, 2$ and 3.

In the below sketches, we have assumed $a > 0$.

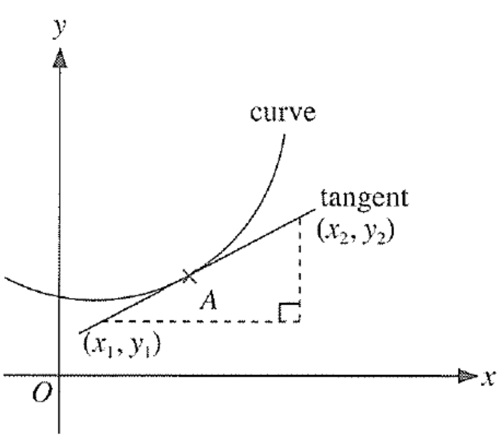





These are graphs of the form $y = ka^x$.



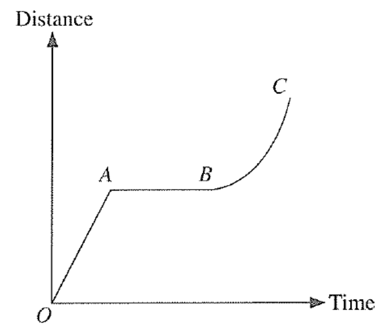

Tangent to graphs:




A straight line which touches a curve at a single point $A$ is called a tangent to the curve at the point $A$.

The gradient of the curve at the point $A$ is equal to the gradient of the tangent to the curve at $A$. In general, the gradient of the curve at any point is equal to the gradient of the tangent to the curve at that point.

Distance-Time Graphs:



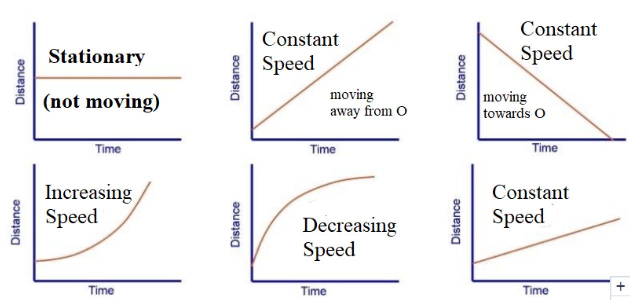

The gradient of a distance-time graph gives the speed of the object.

$OA$ is a straight line, i.e. speed of object = gradient of $OA$.

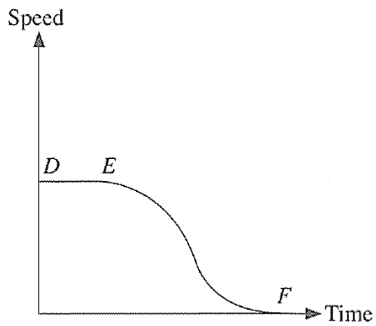
$AB$ is a horizontal line, i.e. speed of object = zero.

$BC$ is a curve, i.e. instantaneous speed of object = gradient of the tangent to the curve at a point.

Various types of distance-time graphs:



Speed-Time Graphs:




The gradient of a speed-time graph gives the acceleration of the object.

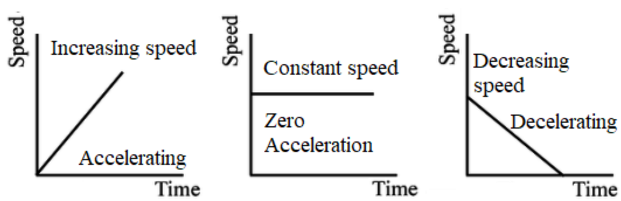
The area under a speed-time graph gives the distance travelled by the object.

$DE$ is a horizontal line, i.e. acceleration of object = zero.

$EF$ is a curve, i.e. instantaneous acceleration of object = gradient of tangent to the curve at a point.

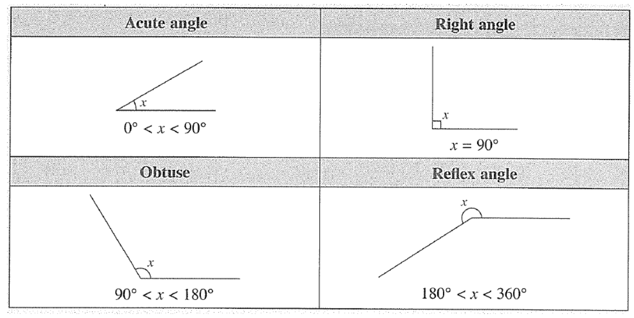
Total distance travelled = area under graph from $D$ to $F$

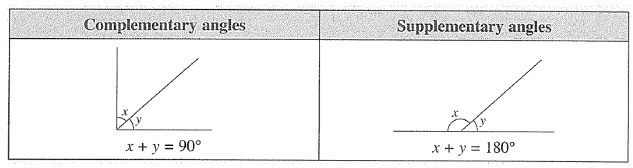
Various types of speed-time graphs:



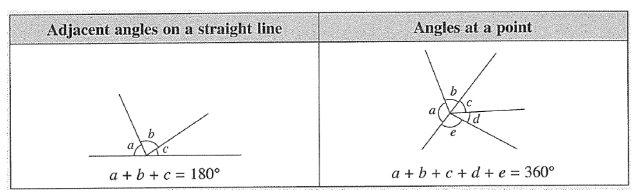

# Angles, Triangles, Polygons

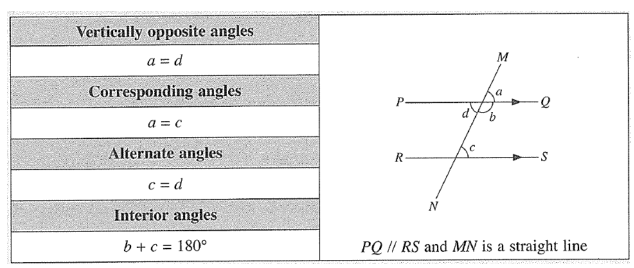
Types of angles:



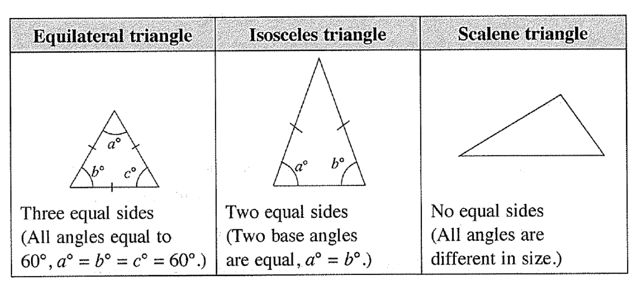



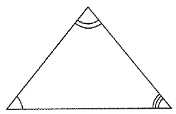
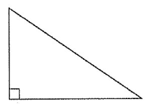
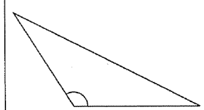
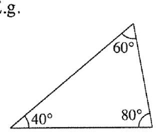
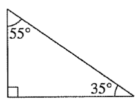
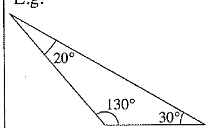

Geometric properties of parallel lines and angles:



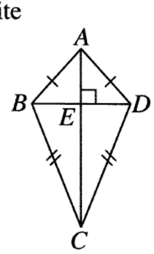

Types of triangles:



| Acute-angled triangle | Right-angled triangle | Obtuse-angled triangle |
| --- | --- | --- |
|  All three angles are acute. |  One right angle |  One obtuse angle |
| E.g.  | E.g.  | E.g.  |

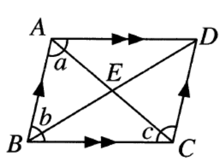
Types of quadrilaterals:

**Kite**



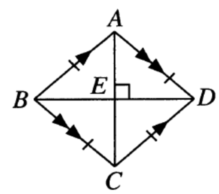

1. Two pairs of equal adjacent sides
2. Both $\triangle ABD$ and $\triangle BCD$ are isosceles, with $\angle ABD = \angle ADB$ and $\angle BDC = \angle DBC$
3. Diagonals $AC$ and $BD$ intersect each other at right angles at $E$
4. The longer diagonal $AC$ bisects the shorter diagonal $BD$

**Parallelogram**



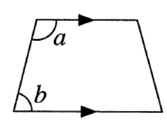

1. Opposite sides are parallel and are of equal length.
2. $\angle a + \angle b = 180°$ (int. $\angle$s )
3. Opposite angles are equal, i.e. $\angle a = \angle c$ (opp. $\angle$s of parallelogram)
4. Diagonals $AC$ and $BD$ bisect each other at $E$, i.e. $AE = EC$ and $BE = ED$, in other words, $E$ is the mid-point of $AC$ and $BD$.

**Rhombus**




1. Opposite sides are parallel.
2. All four sides are equal in length
3. $\angle ABC + \angle BCD = 180°$ (int. $\angle$s )
4. Opposite angles are equal, i.e. $\angle ABC = \angle ADC$ and $\angle BAD = \angle BCD$
5. The diagonals bisect the interior angles at the vertices, so that $\angle ABD = \angle CBD$ and $\angle ADB = \angle CDB$
6. Both $\triangle ABD$ and $\triangle BCD$ are isosceles, with $\angle ABD = \angle ADB$ and $\angle CBD = \angle CDB$
7. Diagonals $AC$ and $BD$ bisect each other at right angles, in other words, they are perpendicular bisectors of each other

**Trapezium**




1. One pair of parallel sides.
2. $\angle a + \angle b = 180°$ (int. $\angle$s)