Numbers

Types of Numbers:

- Natural Numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
- Whole Numbers: $\mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$
- Integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Rational Numbers: $\mathbb{Q} = \left\{ \frac{a}{b} \ : \ a, b \in \mathbb{Z} \ , \ b \neq 0 \right\}$. Rational Numbers comprise of fractions (includes all proper and improper fractions and mixed numbers). All terminating decimals (eg, 10.87) and recurring decimals (eg, $0.3\dot{7}\dot{1} = 0.3717171\ldots$) are rational numbers because these can all be expressed as fractions. All integers are rational numbers.
- Irrational Numbers: numbers that cannot be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Eg. π, $\sqrt{2}$, $\sqrt{5}$, $-4\sqrt{7}$, e, $2.75e$, etc. Irrational numbers are non-recurring decimals. Any non-zero rational number multiplied to an irrational number results in an irrational number. For example, $-\frac{3}{4}\pi$ is irrational.
- Perfect Squares: $\{1, 4, 9, 16, 25, \ldots\}$
- Perfect Cubes: $\{1, 8, 27, 64, \ldots\}$
- Prime Numbers: Positive integers at least 2 whose only positive divisors are 1 and itself. $\{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\}$

Equality and Inequality Symbols

Symbol	Meaning	Example
$=$	is equal to	$0.1 = \frac{1}{10}$
$\neq$	is not equal to	$0.11 \neq \frac{1}{10}$
$>$	is greater than	$0.1 > 0.01$
$\geqslant$	is greater than or equal to	$a \geqslant 5$
$<$	is less than	$0.05 < 5$
$\leqslant$	is less than or equal to	$b \leqslant 5$

Prime Factorization, HCF, LCM

- Example of prime factorization:

$$\begin{array}{r|c}
2 & 4356 \\ \hline
2 & 2178 \\ \hline
3 & 1089 \\ \hline
3 & 363 \\ \hline
11 & 121 \\ \hline
 & 11 \\ \hline
\end{array}$$

  Hence $4356 = 2^2 \times 3^2 \times 11^2$ (in index notation).

- Example of HCF and LCM using prime factorization:

  $4800 = 2^6 \times 3 \times 5^2$

  $5544 = 2^3 \times 3^2 \times 7 \times 11$

  $\text{HCF} = 2^3 \times 3$

  [take common prime factors and lowest power of each]

  $\text{LCM} = 2^6 \times 3^2 \times 5^2 \times 7 \times 11$

  [take all prime factors and highest power of each]

- Examples of square roots and cube roots using prime factorization:

$$54756 = 2^2 \times 3^4 \times 13^2$$
$$\sqrt{54756} = 2 \times 3^2 \times 13 = 234$$
$$1728 = 2^6 \times 3^3$$
$$\sqrt[3]{1728} = 2^2 \times 3 = 12$$

Approximation

Significant Figures

Rules of identifying number of significant digits:

1. All non-zero digits are significant.
2. Zeros between non-zero digits are significant.

   Eg. 302 (3 sf)

   Eg. 10.2301 (6 sf)
3. In a whole number, zeros after the last nonzero digit may or may not be significant. It depends on the estimation being made.

Eg.

$7436000 = 7000000$ (1 sf)
$7436000 = 7400000$ (2 sf)
$7436000 = 7440000$ (3 sf)
$7436000 = 7436000$ (4 sf)
$7436000 = 7436000$ (5 sf)
$7436000 = 7436000$ (6 sf)

4. In a decimal number, zeros before the 1st non-zero digit are not significant.

   Eg. 0.004 (1 sf)

   Eg. 0.07008 (4 sf)

5. In a decimal number, zeros after the last non-zero digit are significant.

   Eg. 6.40 (3 sf)

   Eg. 12.000 (5 sf)

   Eg. 20300.000 (8 sf)

   Eg. 0.0700800 (6 sf)

Decimal Place Rounding

Examples:
$0.7374 = 0.74$ (2 dp)
$58.301 = 58.30$ (2 dp)
$207.6296 = 207.630$ (3 dp)
$207.6296 = 207.63$ (2 dp)
$207.977 = 208.0$ (1 dp)
$207.977 = 207.98$ (2 dp)
$18.997 = 19.00$ (2 dp)

Standard Form

$\pm A \times 10^n$, where $1 \leq A < 10$ and n is an integer.

Examples:
$1350000 = 1.35 \times 10^6$
$0.000375 = 3.75 \times 10^{-4}$

Common Prefixes

10^{12}	trillion	tera	T
10^9	billion	giga	G
10^6	million	mega	M
10^3	thousand	kilo	k
10^{-3}	thousandth	milli	m
10^{-6}	millionth	micro	μ
10^{-9}	billionth	nano	n
10^{-12}	trillionth	pico	p

Indices

Rules of Indices:

Assume that a, b, m, n are non-zero.

$$a^0 = 1$$
$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$(ab)^n = a^n b^n$$
$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$
$$(a^n)^m = a^{nm}$$
$$a^{-n} = \frac{1}{a^n}$$
$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$$
$$a^{\frac{1}{n}} = \sqrt[n]{a}$$
$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Caution:
1. For indices (powers) that are not integers, the above rules of indices hold only for $a, b > 0$.
2. Likewise, if some of the indices are negative or there is division by either a or b, then the above rules hold only for $a, b > 0$.
3. $\sqrt{(-8)^2} = \sqrt{64} = 8$, NOT -8.
4. $\sqrt[3]{-27} = -3$.

Equalities of Indices:
1. If $a^m = a''$, then $m = n$.
2. If $a^m = b^m$, then $a = b$.

Percentage, Ratio, Rate

- To express a percentage as a fraction or decimal, divide by 100:

  $x\% = \frac{x}{100}$

  Eg, $23.5\% = \frac{23.5}{100} = \frac{235}{1000} = \frac{47}{200}$

  Eg, $401\% = \frac{401}{100} = 4.01$

- To express any number as a percentage, multiply it by 100%.

  Eg, $0.165 = 0.165 \times 100\% = 16.5\%$.

- Expressing a quantity A as a percentage of a quantity B:

  $\frac{A}{B} \times 100\%$

  Eg, Express 63.7 as a percentage of 98.

  Answer: $\frac{63.7}{98} \times 100\% = 65\%$

  In words, we say that 63.7 is 65% of 98.

- Increase or decrease a quantity by a given percentage:

  Eg, Increase 45 by 2.4%:

  Answer: $45 \times \left(1 + \frac{2.4}{100}\right) = 45 \times 1.024 = 46.08$

  Eg, Decrease 45 by 90%:

  Answer: $45 \times \left(1 - \frac{90}{100}\right) = 45 \times 0.1 = 4.5$

- Percentage Increase and Percentage Decrease:

  When a quantity increases, the percentage increase is

  $\frac{\text{final value } - \text{ initial value}}{\text{initial value}} \times 100\%$

  When a quantity dereases, the percentage decrease is

  $\frac{\text{initial (bigger) value } - \text{ final (smaller) value}}{\text{initial value}} \times 100\%$

  Percentage increase will always be > 0 if the quantity has increased.

  Percentage decrease will always be > 0 if the quantity has decreased.

  Percentage change is

  $\frac{\text{final value } - \text{ initial value}}{\text{initial value}} \times 100\%$

  regardless of whether the quantity has increased or decreased. Percentage change can be either positive or negative depending on whether the quantity has increased or decreased.

- When writing ratios such as $a : b$, a, b are positive integers. Always reduce ratios to the simplest form, eg, $10 : 6$ is to be reduced to $5 : 3$. The ratio $a : b$ expressed in fraction form is $\frac{a}{b}$.

  Eg, If 7 times of x is equal to 5 times of y, then $x : y = 5 : 7$ (note the switching of the order)

- We can use ratios to increase and decrease quantities. For example, if we increase a quantity x in the ratio $6 : 5$, the new quantity is $\frac{6}{5}x$; if we decrease a quantity x in the ratio $5 : 6$, the new quantity is $\frac{5}{6}x$.

- Various units of measurement:

  Mass:

  1 kg = 1000 g

  1 g = 1000 mg

  Length:

  1 km = 1000 m

  1 m = 100 cm

  1 cm = 10 mm

  Area:

  $1 \text{ km}^2 = 10^6 \text{ m}^2$

  $1 \text{ m}^2 = 10000 \text{ cm}^2$

  Volume:

  $1\ l = 1000$ ml

  $1 \text{ cm}^3 = 1$ ml

  $1 \text{ m}^3 = 10^6 \text{ cm}^3 = 1000\ l$

  Time:

  1 hr = 60 min

  1 min = 60 sec

- Distance = Speed $\times$ Time

- Average speed = (Total Distance) / (Total Time Taken)

- Conversion of units for speed:

  26km/h = 26000m/h = $\frac{26000}{3600}$m/s = $\frac{65}{9}$m/s

  35m/s = 0.035km/s = (0.035 × 3600)km/h = 126km/h

- Density = Mass / Volume.

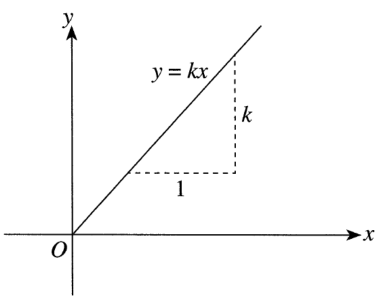
  Units are usually g/cm^3 or kg/m^3.

  1g/cm^3 = 1000kg/m^3.

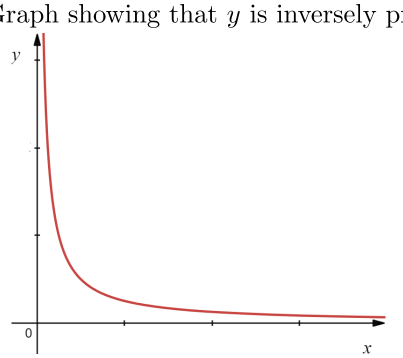
  Eg, 0.235g/cm^3 = 235kg/m^3.

Direct and Inverse Proportion

If y is directly proportional to x, then $y = kx$, where k is a constant and $k \neq 0$. The ratios $\frac{x}{y}$ and $\frac{y}{x}$ are constant. Furthermore, the graph on y against x (or of x against y) is a straight line through the origin.
Graph showing that y is directly proportional to x:



If y is inversely proportional to x, then $y = \frac{k}{x}$, where k is a constant and $k \neq 0$. The product xy is constant.
Graph showing that y is inversely proportional to x:



Map Scales

- Linear scale:

  1 : n means 1 unit length on map represents n units length on ground.

  Eg. 1 : 5000 means

  1 cm represents 5000 cm

  which implies 1 cm represents 50 m

  which implies 1 cm represents 0.05 km

- Representative Fraction (RF):

  If the linear scale is 1 : n, the RF is expressed as $\frac{1}{n}$.

  Eg, if 3 cm represents 6 m, then RF is $\frac{1}{200}$.

- Area Scale:

  If linear scale is 1 : 20000, then it means

  1 cm represents 20000 cm

  which implies 1 cm represents 0.2 km

  which implies 1^2 cm^2 represents $(0.2)^2$ km^2

  which implies 1 cm^2 represents 0.04 km^2

Number Patterns

Common number patterns:

- Constant difference

  Eg, $-5, -2, 1, 4, 7, 10, \ldots$

  The n^{th} term, denoted T_n, is given by

  $T_n = a + d(n-1)$

  where a is the first term and d is the common difference.

  For the sequence $-5, -2, 1, 4, 7, 10, \ldots$,

  $T_n = -5 + (n-1)(3) = 3n - 8$.

  Alternatively,

  $T_n = b + dn$

  where b is the term that would have come before the first term (ie, the "zeroth" term).

- Constant multiple (or common ratio)

  Eg, $3, 15, 75, 375, \ldots$

  $T_n = a \times r^{n-1}$

  where a is the first term, r is the common ratio, that is, r is the number that when multiplied a term gives the next term.

  For the sequence $3, 15, 75, 375, \ldots$

  $T_n = 3 \times 5^{n-1}$.

- Perfect squares and perfect cubes

  $1, 4, 9, 16, 25, \ldots \ : \ T_n = n^2$

  $1, 8, 27, 64, 125, \ldots \ : \ T_n = n^3$

  $2, 8, 18, 32, 50, \ldots \ : \ T_n = 2n^2$

  $3, 10, 29, 66, 127, \ldots \ : \ T_n = n^3 + 2$

Algebra

Simultaneous Linear Equations

Method 1: Elimination

$5x - 2y = 21 \quad$ —(1)

$2x - y = 8 \quad$ —(2)

$(1) \times 2 : 10x - 4y = 42 \quad$ —(3)

$(2) \times 5 : 10x - 5y = 40 \quad$ —(4)

(3) - (4) : $\quad y = 2$

Sub into (1): $5x - 2(2) = 21$

$5x - 4 = 21$

$5x = 25$

$x = 5$

Method 2: Substitution

$5x - 2y = 21 \quad$ —(1)

$2x - y = 8 \quad$ —(2)

From (2):

$y = 2x - 8 \quad$ —(3)

Sub (3) into (1): $5x - 2(2x - 8) = 21$

$5x - 4x + 16 = 21$

$x - 16 = 21$

$x = 5$

Sub into (2):

$2(5) - y = 8$

$10 - y = 8$

$y = 2$

Expansion

Eg.

$2p - 3(p + 1)$

$= 2p - 3p - 3$

$= -p - 3$

Eg.

$5x - (x + 1)(2x - 3)$

$= 5x - \left(2x^2 - 3x + 2x - 3\right)$

$= 5x - 2x^2 + 3x - 2x + 3$

$= -2x^2 + 6x + 3$

Factorization and Identities

Factorization of Quadratic Expressions:

$5x^2 + 9x - 2$

$\times$	$5x$	-1
x	$5x^2$	$-x$
2	$10x$	-2

$\therefore \quad 5x^2 + 9x - 2 = (5x - 1)(x + 2)$

Identities:

1. $(a + b)^2 = a^2 + 2ab + b^2$
2. $(a - b)^2 = a^2 - 2ab + b^2$
3. $(a + b)(a - b) = a^2 - b^2$

Common Factorisation Techniques:

- Common Factors

  Eg. $6a^3b - 2a^2b = 2a^2b(3a - 1)$

- Grouping

  Eg.

  $6p^2 - 3pq - 10ap + 5aq$

  $= 3p(2p - q) - 5a(2p - q)$

  $= (3p - 5a)(2p - q)$

- Using Difference of Two Squares

  Eg. $9a^2 - 1$

  $= (3a)^2 - (1)^2$

  $= (3a + 1)(3a - 1)$

Eg. $16a^4 - 81$
$= (4a^2 + 9)(4a^2 - 9)$
$= (4a^2 + 9)(2a + 3)(2a - 3)$

- Combination of methods:

  Eg. $3x^3 - 12xy^2 = 3x\left(x^2 - 4y^2\right)$ common factor
  $= 3x(x + 2y)(x - 2y)$ then diff. of 2 squares

  Always try common factor first

  Eg.

  $4 - p^2 + 6pq - 9q^2$
  $= 4 - (p^2 - 6pq + 9q^2)$
  $= (2)^2 - (p - 3q)^2$
  $= (2 + (p - 3q))(2 - (p - 3q))$
  $= (2 + p - 3q)(2 - p + 3q)$

Algebraic Fractions

Eg.
$$\frac{x+2}{3} - \frac{x-5}{2}$$
$$= \frac{2(x+2) - 3(x-5)}{6}$$
$$= \frac{2x + 4 - 3x + 15}{6}$$
$$= \frac{19 - x}{6}$$

Eg.
$$\frac{5}{x+1} - \frac{2}{x-3}$$
$$= \frac{5(x-3) - 2(x+1)}{(x+1)(x-3)}$$
$$= \frac{5x - 15 - 2x - 2}{(x+1)(x-3)}$$
$$= \frac{3x - 17}{(x+2)(x-5)}$$

Eg.
$$\frac{5}{3x} + \frac{2}{x}$$
$$= \frac{5}{3x} + \frac{6}{3x}$$
$$= \frac{5+6}{3x}$$
$$= \frac{11}{3x}$$

Eg.
$$\frac{3}{(x+2)^2} - \frac{4}{x+2}$$
$$= \frac{3}{(x+2)^2} - \frac{4(x+2)}{(x+2)^2}$$
$$= \frac{3 - 4(x+2)}{(x+2)^2}$$
$$= \frac{3 - 4x - 8}{(x+2)^2}$$
$$= \frac{-4x - 5}{(x+2)^2}$$

Eg.
$$\frac{7}{x^2 - 9} - \frac{1}{x-3}$$
$$= \frac{7}{(x+3)(x-3)} - \frac{1}{x-3}$$
$$= \frac{7}{(x+3)(x-3)} - \frac{x-3}{(x-3)^2}$$
$$= \frac{7 - (x+3)}{(x+3)(x-3)}$$
$$= \frac{7 - x - 3}{(x+3)(x-3)}$$
$$= \frac{4 - x}{(x+3)(x-3)}$$

Eg.
$$\frac{9}{x-5} + \frac{3}{5-x}$$
$$= \frac{9}{x-5} - \frac{3}{x-5}$$
$$= \frac{9-3}{x-5}$$
$$= \frac{6}{x-5}$$

Eg.

$$\frac{2x}{3y-8x}+\frac{11x}{80x-30y}$$
$$=\frac{2x}{3y-8x}+\frac{11x}{-10(3y-8x)}$$
$$=\frac{2x}{3y-8x}-\frac{11x}{10(3y-8x)}$$
$$=\frac{20x-11x}{10(3y-8x)}$$
$$=\frac{9x}{10(3y-8x)}$$

Eg.

$$\frac{6p^3}{7q}\div\frac{2p}{21q^2}$$
$$=\frac{6p^3}{7q}\times\frac{21q^2}{2p}\quad\text{[do cancelling]}$$
$$=9p^2q$$

Eg.

$$\frac{4pq^2+4pqr}{9pqr^2+9pq^2r}=\frac{4pq(q+r)}{9pqr(r+q)}$$
$$=\frac{4}{9r}$$

Eg.

$$\frac{5k^2-17k-12}{5k^2-10k-40}=\frac{(5k+3)(k-4)}{5\,(k^2-2k-8)}$$
$$=\frac{(5k+3)(k-4)}{5(k-4)(k+2)}$$
$$=\frac{5k+3}{5(k+2)}$$

Eg.

$$\frac{xy-z^2-xz+yz}{y^2-2yz+z^2}\div\frac{11}{2xz+x^2+z^2}$$
$$=\frac{xy-xz+yz-z^2}{y^2-2yz+z^2}\times\frac{2xz+x^2+z^2}{11}$$
$$=\frac{x(y-z)+z(y-z)}{(y-z)^2}\times\frac{(x+z)^2}{11}$$
$$=\frac{(x+z)(y-z)}{(y-z)^2}\times\frac{(x+z)^2}{11}$$
$$=\frac{(x+z)^3}{11(y-z)}$$

Inequalities

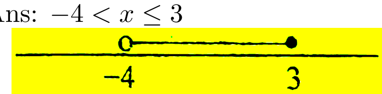
Inequality sign is reversed when both sides are multiplied or divided by a negative number.

Eg. $-3x+4\geq 12$
$-3x\geq 12-4$
$-3x\geq 8$
$x\leq -\frac{8}{3}$

Eg. $3(x-1)<4x+1\leq 7+2x$

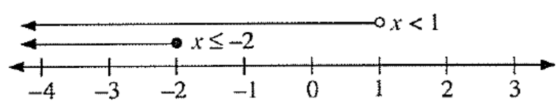
$3(x-1)<4x+1$	$4x+1\leq 7+2x$
$3x-3<4x+1$	$4x-2x\leq 7-1$
$3x-4x<1+3$	$2x\leq 6$
$-x<4$	$x\leq 3$
$x>-4$	

Ans: $-4<x\leq 3$



Eg. $5x+4\leq 3x<6-3x$

$5x+4\leq 3x$	$3x<6-3x$
$2x\leq -4$	$6x<6$
$x\leq -2$	$x<1$



Ans: $x\leq -2$.

Making Subject of Formula

Eg: Make a the subject

$$y=m(x-a)+b$$
$$y-b=m(x-a)$$
$$m(x-a)=y-b$$
$$x-a=\frac{y-b}{m}$$
$$a=x-\frac{y-b}{m}$$

Eg: Make x the subject

$$ax - by = 3 - 2x$$
$$ax + 2x = 3 + by$$
$$x(a+2) = by + 3$$
$$x = \frac{by+3}{a+2}$$

Eg: Make d the subject

$$T = 0.25\pi d^2$$
$$\pi d^2 = 4T$$
$$d^2 = \frac{4T}{\pi}$$
$$d = \pm\sqrt{\frac{4T}{\pi}}$$

Note the $\pm$ when taking square-roots in this type of question.

Eg: Make c the subject

$$d = \frac{8-c}{c+7}$$
$$d(c+7) = 8 - c$$
$$cd + 7d = 8 - c$$
$$cd + c = 8 - 7d$$
$$c(d+1) = 8 - 7d$$
$$c = \frac{8-7d}{d+1}$$

Eg: Make q the subject

$$5a = \sqrt{\frac{b^2}{q} - \frac{3c}{4}}$$
$$25a^2 = \frac{b^2}{q} - \frac{3c}{4}$$
$$\frac{b^2}{q} = 25a^2 + \frac{3c}{4}$$
$$\frac{b^2}{q} = \frac{100a^2 + 3c}{4}$$
$$\frac{q}{b^2} = \frac{4}{100a^2 + 3c}$$
$$q = \frac{4b^2}{100a^2 + 3c}$$

Eg: Make y the subject

$$\frac{x\left(yz - w^2\right)}{2} - \frac{y}{3} = 6y$$
$$3x\left(yz - w^2\right) - 2y = 36y$$
$$3xyz - 3w^2x - 2y = 36y$$
$$3xyz - 38y = 3w^2x$$
$$y(3xz - 38) = 3w^2x$$
$$y = \frac{3w^2x}{3xz - 38}$$