Numbers

- Natural Numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
- Whole Numbers: $\mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$
- Integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Rational Numbers: $\mathbb{Q} = \left\{ \frac{a}{b} \ : \ a, b \in \mathbb{Z} \ , \ b \neq 0 \right\}$. Rational Numbers comprise of fractions (includes all proper and improper fractions and mixed numbers). All terminating decimals (eg, 10.87) and recurring decimals (eg, $0.3\dot{7}\dot{1} = 0.3717171\ldots$) are rational numbers because these can all be expressed as fractions. All integers are rational numbers.
- Irrational Numbers: numbers that cannot be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Eg. π, $\sqrt{2}$, $\sqrt{5}$, $-4\sqrt{7}$, e, $2.75e$, etc. Irrational numbers are non-recurring decimals. Any non-zero rational number multiplied to an irrational number results in an irrational number. For example, $-\frac{3}{4}\pi$ is irrational.
- Perfect Squares: $\{1, 4, 9, 16, 25, \ldots\}$
- Perfect Cubes: $\{1, 8, 27, 64, \ldots\}$
- Prime Numbers: Positive integers at least 2 whose only positive divisors are 1 and itself. $\{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\}$
- Composite Numbers: Positive integers ≥ 4 that are not prime, in other words, having positive divisors apart from 1 and itself.

Equality and Inequality Symbols

Symbol	Meaning	Example
$=$	is equal to	$0.1 = \frac{1}{10}$
$\neq$	is not equal to	$0.11 \neq \frac{1}{10}$
$>$	is greater than	$0.1 > 0.01$
$\geqslant$	is greater than or equal to	$a \geqslant 5$
$<$	is less than	$0.05 < 5$
$\leqslant$	is less than or equal to	$b \leqslant 5$

Prime Factorization, HCF, LCM

- Example of prime factorization:

$$\begin{array}{r|c} 2 & 4356 \\ \hline 2 & 2178 \\ \hline 3 & 1089 \\ \hline 3 & 363 \\ \hline 11 & 121 \\ \hline & 11 \\ \hline \end{array}$$

  Hence $4356 = 2^2 \times 3^2 \times 11^2$ (in index notation).

- Example of HCF and LCM using prime factorization:

  $4800 = 2^6 \times 3 \times 5^2$

  $5544 = 2^3 \times 3^2 \times 7 \times 11$

  $\text{HCF} = 2^3 \times 3$

  [take common prime factors and lowest power of each]

  $\text{LCM} = 2^6 \times 3^2 \times 5^2 \times 7 \times 11$

  [take all prime factors and highest power of each]

- Examples of square roots and cube roots using prime factorization:

$$54756 = 2^2 \times 3^4 \times 13^2$$
$$\sqrt{54756} = 2 \times 3^2 \times 13 = 234$$
$$1728 = 2^6 \times 3^3$$
$$\sqrt[3]{1728} = 2^2 \times 3 = 12$$

Approximation

Significant Figures

Rules of identifying number of significant digits:

1. All non-zero digits are significant.
2. Zeros between non-zero digits are significant.

   Eg. 302 (3 sf)

   Eg. 10.2301 (6 sf)
3. In a whole number, zeros after the last nonzero digit may or may not be significant. It depends on the estimation being made.

Eg.

$7436000 = 7000000$ (1 sf)

$7436000 = 7400000$ (2 sf)

$7436000 = 7440000$ (3 sf)

$7436000 = 7436000$ (4 sf)

$7436000 = 7436000$ (5 sf)

$7436000 = 7436000$ (6 sf)

4. In a decimal number, zeros before the 1st non-zero digit are not significant.

   Eg. 0.004 (1 sf)

   Eg. 0.07008 (4 sf)

5. In a decimal number, zeros after the last non-zero digit are significant.

   Eg. 6.40 (3 sf)

   Eg. 12.000 (5 sf)

   Eg. 20300.000 (8 sf)

   Eg. 0.0700800 (6 sf)

Decimal Place Rounding

Examples:

$0.7374 = 0.74$ (2 dp)

$58.301 = 58.30$ (2 dp)

$207.6296 = 207.630$ (3 dp)

$207.6296 = 207.63$ (2 dp)

$207.977 = 208.0$ (1 dp)

$207.977 = 207.98$ (2 dp)

$18.997 = 19.00$ (2 dp)

Standard Form

$\pm A \times 10^n$, where $1 \leq A < 10$ and n is an integer.

Examples:

$1350000 = 1.35 \times 10^6$

$0.000375 = 3.75 \times 10^{-4}$

Common Prefixes

10^{12}	trillion	tera	T
10^9	billion	giga	G
10^6	million	mega	M
10^3	thousand	kilo	k
10^{-3}	thousandth	milli	m
10^{-6}	millionth	micro	μ
10^{-9}	billionth	nano	n
10^{-12}	trillionth	pico	p

Indices

Rules of Indices:

Assume that a, b, m, n are non-zero.

$$a^0 = 1$$

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$(a^n)^m = a^{nm}$$

$$a^{-n} = \frac{1}{a^n}$$

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Caution:

1. For indices (powers) that are not integers, the above rules of indices hold only for $a, b > 0$.

2. Likewise, if some of the indices are negative or there is division by either a or b, then the above rules hold only for $a, b > 0$.

3. $\sqrt{(-8)^2} = \sqrt{64} = 8$, NOT -8.

4. $\sqrt[3]{-27} = -3$.

Equalities of Indices:

1. If $a^m = a''$, then $m = n$.

2. If $a^m = b^m$, then $a = b$.

Percentage, Ratio, Rate

- To express a percentage as a fraction or decimal, divide by 100:

  $x\% = \frac{x}{100}$

  Eg, $23.5\% = \frac{23.5}{100} = \frac{235}{1000} = \frac{47}{200}$

  Eg, $401\% = \frac{401}{100} = 4.01$

- To express any number as a percentage, multiply it by 100%.

  Eg, $0.165 = 0.165 \times 100\% = 16.5\%$.

- Expressing a quantity A as a percentage of a quantity B:

  $\frac{A}{B} \times 100\%$

  Eg, Express 63.7 as a percentage of 98.

  Answer: $\frac{63.7}{98} \times 100\% = 65\%$

  In words, we say that 63.7 is 65% of 98.

- Increase or decrease a quantity by a given percentage:

  Eg, Increase 45 by 2.4%:

  Answer: $45 \times \left(1 + \frac{2.4}{100}\right) = 45 \times 1.024 = 46.08$

  Eg, Decrease 45 by 90%:

  Answer: $45 \times \left(1 - \frac{90}{100}\right) = 45 \times 0.1 = 4.5$

- Percentage Increase and Percentage Decrease:

  When a quantity increases, the percentage increase is

  $\frac{\text{final value} - \text{initial value}}{\text{initial value}} \times 100\%$

  When a quantity dereases, the percentage decrease is

  $\frac{\text{initial (bigger) value} - \text{final (smaller) value}}{\text{initial value}} \times 100\%$

  Percentage increase will always be > 0 if the quantity has increased.

  Percentage decrease will always be > 0 if the quantity has decreased.

  Percentage change is

  $\frac{\text{final value} - \text{initial value}}{\text{initial value}} \times 100\%$

  regardless of whether the quantity has increased or decreased. Percentage change can be either positive or negative depending on whether the quantity has increased or decreased.

- When writing ratios such as $a : b$, a, b are positive integers. Always reduce ratios to the simplest form, eg, $10 : 6$ is to be reduced to $5 : 3$. The ratio $a : b$ expressed in fraction form is $\frac{a}{b}$.

  Eg, If 7 times of x is equal to 5 times of y, then $x : y = 5 : 7$ (note the switching of the order)

- We can use ratios to increase and decrease quantities. For example, if we increase a quantity x in the ratio $6 : 5$, the new quantity is $\frac{6}{5}x$; if we decrease a quantity x in the ratio $5 : 6$, the new quantity is $\frac{5}{6}x$.

- Various units of measurement:

  Mass:

  1 kg = 1000 g

  1 g = 1000 mg

  Length:

  1 km = 1000 m

  1 m = 100 cm

  1 cm = 10 mm

  Area:

  1 km^2 = 10^6 m^2

  1 m^2 = 10000 cm^2

  Volume:

  1 l = 1000 ml

  1 cm^3 = 1 ml

  1 m^3 = 10^6 cm^3 = 1000 l

  Time:

  1 hr = 60 min

  1 min = 60 sec

- Distance = Speed × Time

- Average speed = (Total Distance) / (Total Time Taken)

- Conversion of units for speed:

  $26\text{km/h} = 26000\text{m/h} = \frac{26000}{3600}\text{m/s} = \frac{65}{9}\text{m/s}$

  $35\text{m/s} = 0.035\text{km/s} = (0.035 \times 3600)\text{km/h} = 126\text{km/h}$

- Density = Mass / Volume.

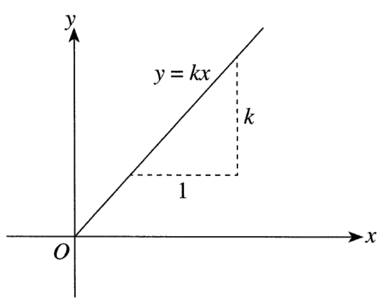
  Units are usually g/cm^3 or kg/m^3.

  $1\text{g/cm}^3 = 1000\text{kg/m}^3$.

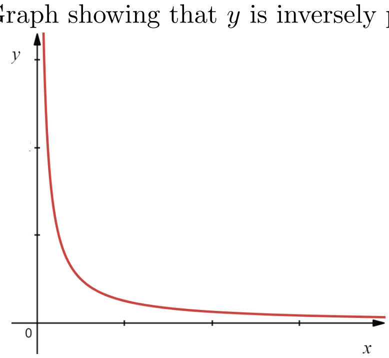
  Eg, $0.235\text{g/cm}^3 = 235\text{kg/m}^3$.

Direct and Inverse Proportion

If y is directly proportional to x, then $y = kx$, where k is a constant and $k \neq 0$. The ratios $\frac{x}{y}$ and $\frac{y}{x}$ are constant. Furthermore, the graph on y against x (or of x against y) is a straight line through the origin.
Graph showing that y is directly proportional to x:



If y is inversely proportional to x, then $y = \frac{k}{x}$, where k is a constant and $k \neq 0$. The product xy is constant.
Graph showing that y is inversely proportional to x:



Map Scales

- Linear scale:

  $1 : n$ means 1 unit length on map represents n units length on ground.

  Eg. $1 : 5000$ means

  1 cm represents 5000 cm

  which implies 1 cm represents 50 m

  which implies 1 cm represents 0.05 km

- Representative Fraction (RF):

  If the linear scale is $1 : n$, the RF is expressed as $\frac{1}{n}$.

  Eg, if 3 cm represents 6 m, then RF is $\frac{1}{200}$.

- Area Scale:

  If linear scale is $1 : 20000$, then it means

  1 cm represents 20000 cm

  which implies 1 cm represents 0.2 km

  which implies 1^2 cm^2 represents $(0.2)^2$ km^2

  which implies 1 cm^2 represents 0.04 km^2

Number Patterns

Common number patterns:

- Constant difference

  Eg, $-5, -2, 1, 4, 7, 10, \ldots$

  The n^{th} term, denoted T_n, is given by

  $T_n = a + d(n - 1)$

  where a is the first term and d is the common difference.

  For the sequence $-5, -2, 1, 4, 7, 10, \ldots$,

  $T_n = -5 + (n - 1)(3) = 3n - 8$.

  Alternatively,

  $T_n = b + dn$

  where b is the term that would have come before the first term (ie, the "zeroth" term).

- Constant multiple (or common ratio)

  Eg, $3, 15, 75, 375, \ldots$

  $T_n = a \times r^{n-1}$

  where a is the first term, r is the common ratio, that is, r is the number that when multiplied a term gives the next term.

  For the sequence $3, 15, 75, 375, \ldots$

  $T_n = 3 \times 5^{n-1}$.

- Perfect squares and perfect cubes

  $1, 4, 9, 16, 25, \ldots \ : \ T_n = n^2$

  $1, 8, 27, 64, 125, \ldots \ : \ T_n = n^3$

  $2, 8, 18, 32, 50, \ldots \ : \ T_n = 2n^2$

  $3, 10, 29, 66, 127, \ldots \ : \ T_n = n^3 + 2$

- Fibonacci Sequences

  Each term is the sum of the previous two terms:

  $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

  $-2, 3, 1, 4, 5, 9, 14, 23, 37, \ldots$

Simultaneous Linear Equations

Method 1: Elimination

$5x - 2y = 21 \quad$ —(1)

$2x - y = 8 \quad$ —(2)

$(1) \times 2 : 10x - 4y = 42 \quad$ —(3)

$(2) \times 5 : 10x - 5y = 40 \quad$ —(4)

(3) - (4) : $\quad y = 2$

Sub into (1): $5x - 2(2) = 21$

$5x - 4 = 21 \ \Rightarrow \ 5x = 25 \ \Rightarrow \ x = 5$

Method 2: Substitution

$5x - 2y = 21 \quad$ —(1)

$2x - y = 8 \quad$ —(2)

From (2):

$y = 2x - 8 \quad$ —(3)

Sub (3) into (1): $5x - 2(2x - 8) = 21$

$5x - 4x + 16 = 21 \ \Rightarrow \ x - 16 \ = 21$

$x = 5$

Sub into (2):

$2(5) - y = 8 \ \Rightarrow \ 10 - y = 8 \ \Rightarrow \ y = 2$

Inequalities

Inequality sign is reversed when both sides are multiplied or divided by a negative number.

Eg. $-3x + 4 \geq 12$

$-3x \geq 12 - 4$

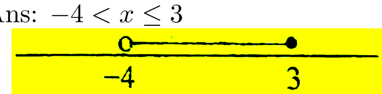
$-3x \geq 8$

$x \leq -\frac{8}{3}$

Eg. $3(x-1) < 4x + 1 \leq 7 + 2x$

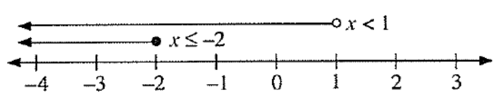
$3(x-1) < 4x + 1$	$4x + 1 \leq 7 + 2x$
$3x - 3 < 4x + 1$	$4x - 2x \leq 7 - 1$
$3x - 4x < 1 + 3$	$2x \leq 6$
$-x < 4$	$x \leq 3$
$x > -4$	

Ans: $-4 < x \leq 3$



Eg. $5x + 4 \leq 3x < 6 - 3x$

$5x + 4 \leq 3x$	$3x < 6 - 3x$
$2x \leq -4$	$6x < 6$
$x \leq -2$	$x < 1$



Ans: $x \leq -2$.

Expansion

Eg.

$2p - 3(p+1)$

$= 2p - 3p - 3$

$= -p - 3$

Eg.

$5x - (x+1)(2x-3)$

$= 5x - \left(2x^2 - 3x + 2x - 3\right)$

$= 5x - 2x^2 + 3x - 2x + 3$

$= -2x^2 + 6x + 3$

Factorization and Identities

Factorization of Quadratic Expressions:
$5x^2 + 9x - 2$

$\times$	$5x$	-1
x	$5x^2$	$-x$
2	$10x$	-2

$\therefore \quad 5x^2 + 9x - 2 = (5x-1)(x+2)$

Identities:

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(a+b)(a-b) = a^2 - b^2$

Common Factorisation Techniques:

- Common Factors
  Eg. $6a^3b - 2a^2b = 2a^2b(3a-1)$
- Grouping
  Eg.
  $6p^2 - 3pq - 10ap + 5aq$
  $= 3p(2p-q) - 5a(2p-q)$
  $= (3p-5a)(2p-q)$
- Using Difference of Two Squares
  Eg. $9a^2 - 1$
  $= (3a)^2 - (1)^2$
  $= (3a+1)(3a-1)$
  Eg. $16a^4 - 81$
  $= (4a^2+9)(4a^2-9)$
  $= (4a^2+9)(2a+3)(2a-3)$
- Combination of methods:
  Eg. $3x^3 - 12xy^2 = 3x\left(x^2 - 4y^2\right)$ common factor
  $= 3x(x+2y)(x-2y)$ then diff. of 2 squares
  Always try common factor first
  Eg.
  $4 - p^2 + 6pq - 9q^2$
  $= 4 - (p^2 - 6pq + 9q^2)$
  $= (2)^2 - (p-3q)^2$
  $= (2 + (p-3q))(2 - (p-3q))$
  $= (2 + p - 3q)(2 - p + 3q)$

Algebraic Fractions

Eg.
$$\frac{x+2}{3} - \frac{x-5}{2} = \frac{2(x+2)}{6} - \frac{3(x-5)}{6}$$
$$= \frac{2(x+2) - 3(x-5)}{6}$$
$$= \frac{2x + 4 - 3x + 15}{6}$$
$$= \frac{19-x}{6}$$

Eg.
$$\frac{5}{x+1} - \frac{2}{x-3}$$
$$= \frac{5(x-3)}{(x+1)(x-3)} - \frac{2(x+1)}{(x-3)(x+1)}$$
$$= \frac{5(x-3) - 2(x+1)}{(x+1)(x-3)}$$
$$= \frac{5x - 15 - 2x - 2}{(x+1)(x-3)}$$
$$= \frac{3x-17}{(x+2)(x-5)}$$

Eg.
$$\frac{5}{3x} + \frac{2}{x}$$
$$= \frac{5}{3x} + \frac{6}{3x}$$
$$= \frac{5+6}{3x}$$
$$= \frac{11}{3x}$$

Eg.
$$\frac{3}{(x+2)^2} - \frac{4}{x+2}$$
$$= \frac{3}{(x+2)^2} - \frac{4(x+2)}{(x+2)^2}$$
$$= \frac{3 - 4(x+2)}{(x+2)^2}$$
$$= \frac{3 - 4x - 8}{(x+2)^2}$$
$$= \frac{-4x-5}{(x+2)^2}$$

Eg.

$$\frac{7}{x^2-9}-\frac{1}{x-3}$$
$$=\frac{7}{(x+3)(x-3)}-\frac{1}{x-3}$$
$$=\frac{7}{(x+3)(x-3)}-\frac{x-3}{(x-3)^2}$$
$$=\frac{7-(x+3)}{(x+3)(x-3)}$$
$$=\frac{7-x-3}{(x+3)(x-3)}$$
$$=\frac{4-x}{(x+3)(x-3)}$$

Eg.

$$\frac{3x+2}{x^2-4}+\frac{1}{x+2}-\frac{2}{x-2}$$
$$=\frac{3x+2}{(x+2)(x-2)}+\frac{1}{x+2}-\frac{2}{x-2}$$
$$=\frac{3x+2+(x-2)-2(x+2)}{(x+2)(x-2)}$$
$$=\frac{3x+2+x-2-2x-4}{(x+2)(x-2)}$$
$$=\frac{2x-4}{(x+2)(x-2)}$$
$$=\frac{2(x-2)}{(x+2)(x-2)}$$
$$=\frac{2}{x+2}$$

Eg.

$$\frac{9}{x-5}+\frac{3}{5-x}$$
$$=\frac{9}{x-5}-\frac{3}{x-5}$$
$$=\frac{9-3}{x-5}$$
$$=\frac{6}{x-5}$$

Eg.

$$\frac{2x}{3y-8x}+\frac{11x}{80x-30y}$$
$$=\frac{2x}{3y-8x}+\frac{11x}{-10(3y-8x)}$$
$$=\frac{2x}{3y-8x}-\frac{11x}{10(3y-8x)}$$
$$=\frac{20x-11x}{10(3y-8x)}$$
$$=\frac{9x}{10(3y-8x)}$$

Eg.

$$\frac{4}{x^2-4}+\frac{1}{2-x}$$
$$=\frac{4}{(x+2)(x-2)}-\frac{1}{x-2}$$
$$=\frac{4-(x+2)}{(x+2)(x-2)}$$
$$=\frac{4-x-2}{(x+2)(x-2)}$$
$$=\frac{2-x}{(x+2)(x-2)}$$
$$=\frac{-x-2)}{(x+2)(x-2)}$$
$$=-\frac{1}{x+2}$$

Eg.

$$\frac{6p^3}{7q}\div\frac{2p}{21q^2}$$
$$=\frac{6p^3}{7q}\times\frac{21q^2}{2p}\quad \text{[do cancelling]}$$
$$=9p^2q$$

Eg.

$$\frac{4pq^2+4pqr}{9pqr^2+9pq^2r}=\frac{4pq(q+r)}{9pqr(r+q)}$$
$$=\frac{4}{9r}$$

Eg.
$$\begin{aligned}\frac{5k^2-17k-12}{5k^2-10k-40} &= \frac{(5k+3)(k-4)}{5\left(k^2-2k-8\right)}\\ &= \frac{(5k+3)(k-4)}{5(k-4)(k+2)}\\ &= \frac{5k+3}{5(k+2)}\end{aligned}$$

Eg.
$$\begin{aligned}&\frac{xy-z^2-xz+yz}{y^2-2yz+z^2} \div \frac{11}{2xz+x^2+z^2}\\ &= \frac{xy-xz+yz-z^2}{y^2-2yz+z^2} \times \frac{2xz+x^2+z^2}{11}\\ &= \frac{x(y-z)+z(y-z)}{(y-z)^2} \times \frac{(x+z)^2}{11}\\ &= \frac{(x+z)(y-z)}{(y-z)^2} \times \frac{(x+z)^2}{11}\\ &= \frac{(x+z)^3}{11(y-z)}\end{aligned}$$

Eg.
$$\begin{aligned}&\frac{\frac{1}{x}+1}{x+1}\\ &= \left(\frac{1}{x}+1\right) \div (x+1)\\ &= \frac{1+x}{x} \times \frac{1}{x+1}\\ &= \frac{1}{x}\end{aligned}$$

Eg.
$$\begin{aligned}&\frac{1-\frac{1}{x}}{1-\frac{1}{x^2}}\\ &= \left(1-\frac{1}{x}\right) \div \left(1-\frac{1}{x^2}\right)\\ &= \left(\frac{x-1}{x}\right) \div \left(\frac{x^2-1}{x^2}\right)\\ &= \frac{x-1}{x} \times \frac{x^2}{x^2-1^2}\\ &= \frac{x-1}{x} \times \frac{x^2}{(x+1)(x-1)}\\ &= \frac{x}{x+1}\end{aligned}$$

Making Subject of Formula

Eg: Make a the subject
$$\begin{aligned}y &= m(x-a)+b\\ y-b &= m(x-a)\\ m(x-a) &= y-b\\ x-a &= \frac{y-b}{m}\\ a &= x-\frac{y-b}{m}\end{aligned}$$

Eg: Make x the subject
$$\begin{aligned}ax-by &= 3-2x\\ ax+2x &= 3+by\\ x(a+2) &= by+3\\ x &= \frac{by+3}{a+2}\end{aligned}$$

Eg: Make d the subject
$$\begin{aligned}T &= 0.25\pi d^2\\ \pi d^2 &= 4T\\ d^2 &= \frac{4T}{\pi}\\ d &= \pm\sqrt{\frac{4T}{\pi}}\end{aligned}$$

Note the $\pm$ when taking square-roots in this type of question.

Eg: Make c the subject
$$\begin{aligned}d &= \frac{8-c}{c+7}\\ d(c+7) &= 8-c\\ cd+7d &= 8-c\\ cd+c &= 8-7d\\ c(d+1) &= 8-7d\\ c &= \frac{8-7d}{d+1}\end{aligned}$$

Eg: Make q the subject

$$5a = \sqrt{\frac{b^2}{q} - \frac{3c}{4}}$$
$$25a^2 = \frac{b^2}{q} - \frac{3c}{4}$$
$$\frac{b^2}{q} = 25a^2 + \frac{3c}{4}$$
$$\frac{b^2}{q} = \frac{100a^2 + 3c}{4}$$
$$\frac{q}{b^2} = \frac{4}{100a^2 + 3c}$$
$$q = \frac{4b^2}{100a^2 + 3c}$$

Eg: Make y the subject

$$\frac{x\left(yz - w^2\right)}{2} - \frac{y}{3} = 6y$$
$$3x\left(yz - w^2\right) - 2y = 36y$$
$$3xyz - 3w^2x - 2y = 36y$$
$$3xyz - 38y = 3w^2x$$
$$y(3xz - 38) = 3w^2x$$
$$y = \frac{3w^2x}{3xz - 38}$$

Eg: Make x the subject

$$p\sqrt{x} + q = r + s\sqrt{x}$$
$$p\sqrt{x} - s\sqrt{x} = r - q$$
$$\sqrt{x}(p - s) = r - q$$
$$\sqrt{x} = \frac{r - q}{p - s}$$
$$x = \left(\frac{r - q}{p - s}\right)^2$$

Solving Quadratic Equations and Equations Involving Algebraic Fractions

Three methods of solving quadratic equations:

1. Factorisation

$$3x^2 - 2x - 8 = 0$$
$$(3x + 4)(x - 2) = 0$$
$$3x + 4 = 0 \text{ or } x - 2 = 0$$
$$x = -\frac{4}{3} \text{ or } x = 2$$

2. Using General Formula

$$ax^2 + bx + c = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Eg. $12x^2 - x - 25 = 0$

$$a = 2, b = -1, c = -25$$
$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-25)}}{2(2)}$$
$$x = \frac{1 \pm \sqrt{201}}{4}$$
$$x = \frac{1 + \sqrt{201}}{4} \text{ or } \frac{1 - \sqrt{201}}{4}$$
$$x = 3.79(2\text{dp}) \text{ or } -3.29(2\text{dp})$$

3. Complete the Square

$$x^2 - 8x + 13 = (x + p)^2 + q$$
$$x^2 - 8x + 13$$
$$= x^2 - 8x + \left(\frac{8}{2}\right)^2 - \left(\frac{8}{2}\right)^2 + 13$$
$$= x^2 - 8x + 16 - 16 + 13$$
$$= (x - 4)^2 - 3$$

To Solve $x^2 - 8x + 13 = 0$

First Change To $(x - 4)^2 - 3 = 0$

$$(x - 4)^2 = 3$$
$$x - 4 = \pm\sqrt{3}$$
$$x - 4 = \sqrt{3} \text{ or } x - 4 = -\sqrt{3}$$
$$x = \sqrt{3} + 4 \text{ or } x = -\sqrt{3} + 4$$
$$x = 5.73(2\text{dp}) \text{ or } 2.27(2\text{dp})$$

Equations involving algebraic fractions:

Eg:

$$\begin{aligned} 3x &= \frac{1}{x} - 4 \\ 3x^2 &= 1 - 4x \\ 3x^2 + 4x - 1 &= 0 \\ x &= \frac{-4 \pm \sqrt{4^2 - 4(3)(-1)}}{2(3)} \\ &= \frac{-4 \pm \sqrt{28}}{6} \end{aligned}$$

$$x = \frac{-4 + \sqrt{28}}{6} \quad \text{or} \quad x = \frac{-4 - \sqrt{28}}{6}$$

$$x = 0.22 \quad \text{or} \quad x = -1.55(2 \text{ d.}p.)$$

Eg:

$$\begin{aligned} x + 1 &= \frac{20}{x+2} \\ (x+1)(x+2) &= 20 \\ x^2 + 3x + 2 &= 20 \\ x^2 + 3x - 18 &= 0 \\ (x+6)(x-3) &= 0 \end{aligned}$$

$$x + 6 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = -6 \quad \text{or} \quad x = 3$$

Eg:

$$\frac{2-x}{x+1} + \frac{1}{x-3} = \frac{3}{5}$$

$$5(2-x)(x-3) + 5(x+1) = 3(x+1)(x-3)$$

$$5\left(2x - 6 - x^2 + 3x\right) + 5x + 5 = 3\left(x^2 - 3x + x - 3\right)$$

$$5\left(-x^2 + 5x - 6\right) + 5x + 5 = 3\left(x^2 - 2x - 3\right)$$

$$-5x^2 + 25x - 30 + 5x + 5 = 3x^2 - 6x - 9$$

$$8x^2 - 36x + 16 = 0$$

$$2x^2 - 9x + 4 = 0$$

$$(2x-1)(x-4) = 0$$

$$2x - 1 = 0 \text{ or } x - 4 = 0$$

$$x = \frac{1}{2} \text{ or } x = 4$$

Eg:

$$\frac{2x+5}{x^2+4x+3} + \frac{2}{x+3} = 1$$

$$\frac{2x+5}{(x+1)(x+3)} + \frac{2}{x+3} = 1$$

$$\begin{aligned} 2x + 5 + 2(x+1) &= (x+1)(x+3) \\ 2x + 5 + 2x + 2 &= x^2 + 4x + 3 \\ 4x + 7 &= x^2 + 4x + 3 \\ x^2 - 4 &= 0 \\ x^2 &= 4 \\ x &= \pm\sqrt{4} \\ x &= 2 \text{ or } x = -2 \end{aligned}$$

Eg:

$$\frac{x^2+2}{(5x-4)(2x-1)} = \frac{1}{3}$$

$$\begin{aligned} 3\left(x^2 + 2\right) &= (5x-4)(2x-1) \\ 3x^2 + 6 &= 10x^2 - 13x + 4 \end{aligned}$$

$$7x^2 - 13x - 2 = 0$$

$$(7x+1)(x-2) = 0$$

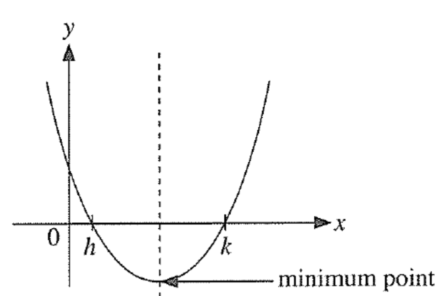
$$\therefore 7x + 1 = 0 \quad \text{or} \quad x - 2 = 0$$

$$x = -\frac{1}{7} \quad \text{or} \quad x = 2$$

Quadratic Graphs

1. Suppose that a quadratic function $y = ax^2 + bx + c$ $(a \neq 0)$ can be factorized into $a(x-h)(x-k)$.

   Consider the case where $a > 0$. The graph is concave upwards (smiley face) and there is a minimum point.



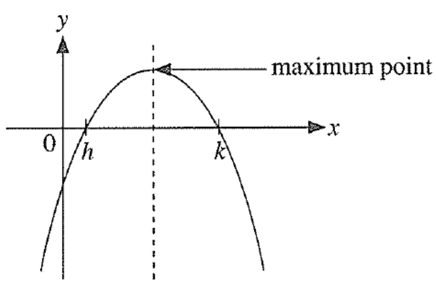
The graph intersects the x-axis at $(h, 0)$ and $(k, 0)$.

The line of symmetry is $x = \frac{h+k}{2}$.

The minimum point also occurs at $x = \frac{h+k}{2}$. To find the y-coordinate of the minimum point, substitute $x = \frac{h+k}{2}$ into the expression $y = ax^2 + bx + c$.

2. Suppose that a quadratic function $y = ax^2 + bx + c$ $(a \neq 0)$ can be factorized into $a(x-h)(x-k)$.

   Consider the case where $a < 0$. The graph is concave downwards (sad face) and there is a maximum point.

   The graph intersects the x-axis at $(h, 0)$ and $(k, 0)$.

   The line of symmetry is $x = \frac{h+k}{2}$.

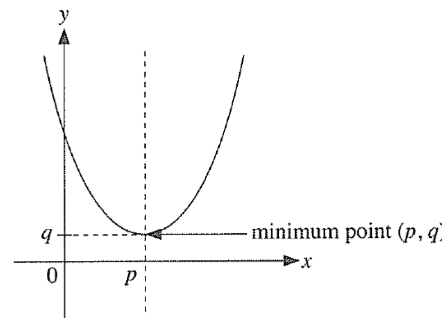
   The maximum point also occurs at $x = \frac{h+k}{2}$. To find the y-coordinate of the maximum point, substitute $x = \frac{h+k}{2}$ into the expression $y = ax^2 + bx + c$.

3. Suppose we complete the square quadratic function $y = ax^2 + bx + c$ $(a \neq 0)$ to obtain $y = a(x-p)^2 + q$.

   If $a > 0$, then the graph is smiley face, and has a minimum point at (p, q). The minimum value of y is q.

   The equation $ax^2 + bx + c = h$ has two solutions if $h > q$, one solution if $h = q$, and no solutions if $h < q$.

   The line of symmetry $x = p$.

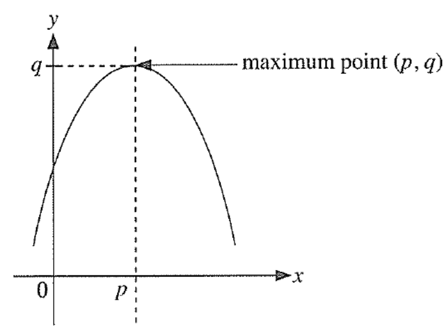
   

4. Suppose we complete the square quadratic function $y = ax^2 + bx + c$ $(a \neq 0)$ to obtain $y = a(x-p)^2 + q$.

   If $a < 0$, then the graph is sad face, and has a maximum point at (p, q). The maximum value of y is q.

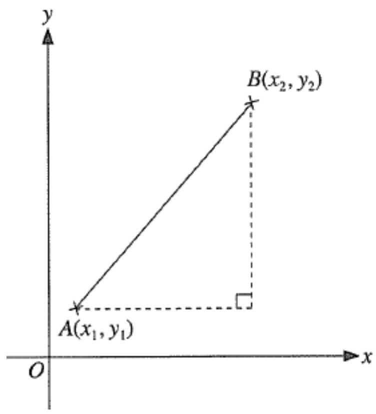
   The equation $ax^2 + bx + c = h$ has two solutions if $h < q$, one solution if $h = q$, and no solutions if $h > q$.

   The line of symmetry $x = p$.

Coordinate Geometry

- The length of a line segment with end points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by
  $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
  or equivalently
  $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



- The gradient of a line passing through $A\left(x_1, y_1\right)$ and $B\left(x_2, y_2\right)$ is given by

$$\frac{y_2 - y_1}{x_2 - x_1}$$

or equivalently

$$\frac{y_1 - y_2}{x_1 - x_2}.$$

- Collinear Points:

  If gradient of AB = gradient of BC = gradient of AC, we say that A, B, C are collinear. This means that A, B, C lie on the same straight line (not necessarily in the order $A - B - C$; it could also be in any order such as $B - C - A$.)

  Note: For collinear points, we only need to check the equality of one pair of gradients, for example, gradient of AB = gradient of BC. Then automatically, gradient of AB = gradient of AC and gradient of AC = gradient of BC as well.

  When checking that gradient of AB = gradient of BC, there must be a common point (in this case, B) in order to conclude that the points are collinear.

  If gradient of AB = gradient of CD (ie, no common points), then we cannot conclude any of the points are collinear.

- A horizontal line is parallel to the x-axis. If all the points on the line has y-coordinate equal to b, then the equation of the line is

$$y = b.$$

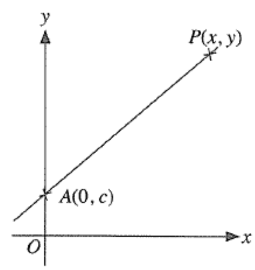
  A vertical line is parallel to the y-axis. If all the points on the line has x-coordinate equal to a, then the equation of the line is

$$x = a.$$

  If a line has a gradient m and y-intercept c, then its equation is

$$y = mx + c.$$

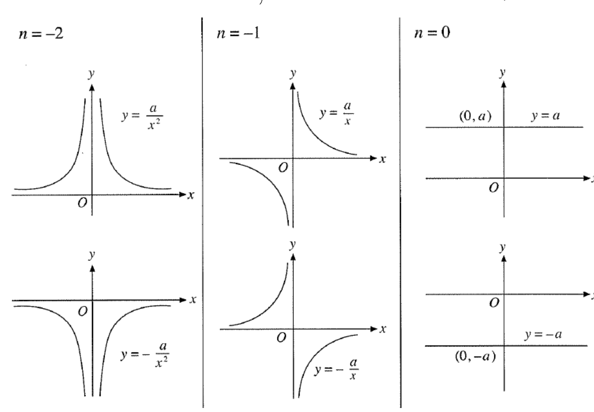
  The equation $y = mx + c$ is known as the gradient-intercept form.

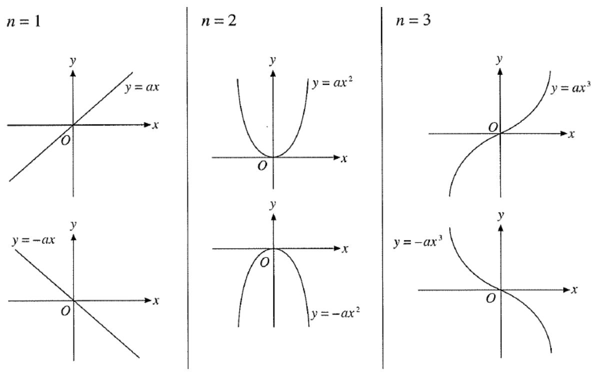


Graphs and Functions

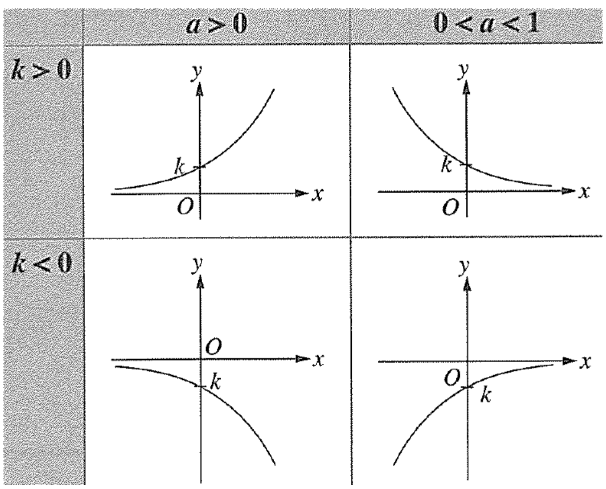
Below is a summary of the graphs of the form $y = ax^n$, where $n = -2, -1, 0, 1, 2$ and 3.

In the below sketches, we have assumed $a > 0$.

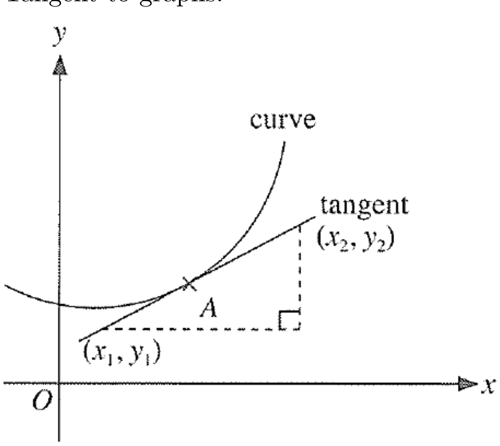




These are graphs of the form $y = ka^x$.



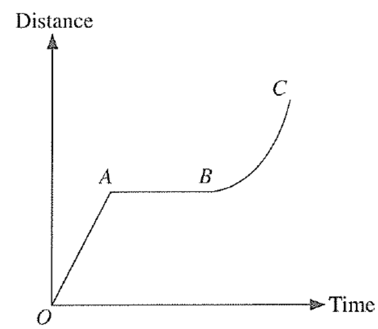
Tangent to graphs:



A straight line which touches a curve at a single point A is called a tangent to the curve at the point A.

The gradient of the curve at the point A is equal to the gradient of the tangent to the curve at A. In general, the gradient of the curve at any point is equal to the gradient of the tangent to the curve at that point.

Distance-Time Graphs:



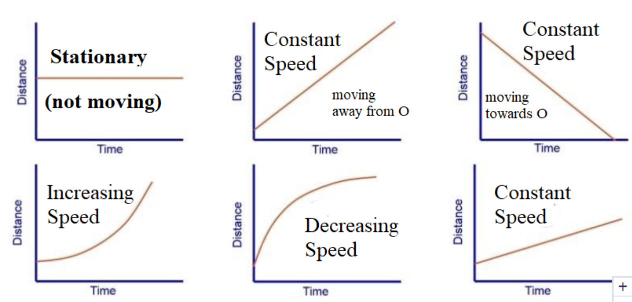
The gradient of a distance-time graph gives the speed of the object.

OA is a straight line, i.e. speed of object = gradient of OA.

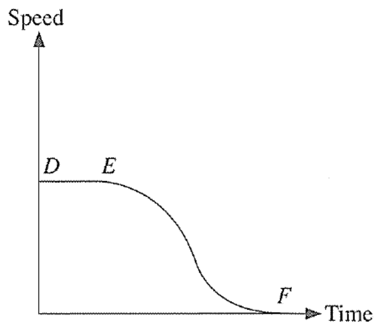
AB is a horizontal line, i.e. speed of object = zero.

BC is a curve, i.e. instantaneous speed of object = gradient of the tangent to the curve at a point.

Various types of distance-time graphs:



Speed-Time Graphs:



The gradient of a speed-time graph gives the acceleration of the object.

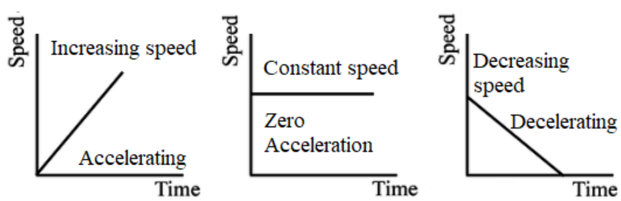
The area under a speed-time graph gives the distance travelled by the object.

DE is a horizontal line, i.e. acceleration of object = zero.

EF is a curve, i.e. instantaneous acceleration of object = gradient of tangent to the curve at a point.

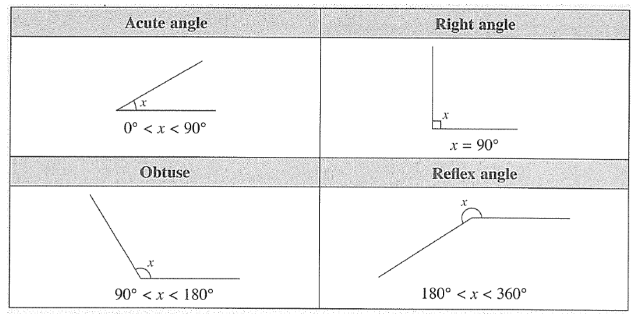
Total distance travelled = area under graph from D to F

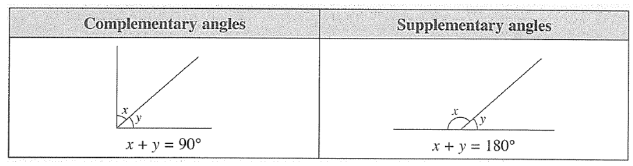
Various types of speed-time graphs:



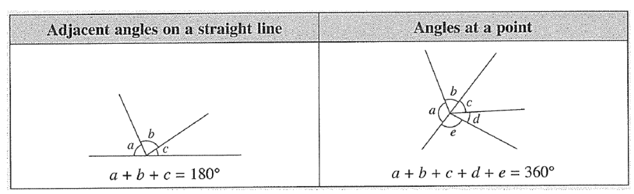
Angles, Triangles, Polygons

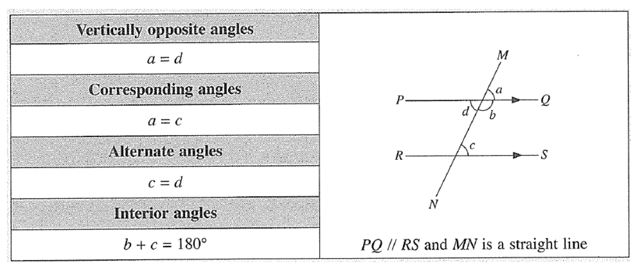
Types of angles:



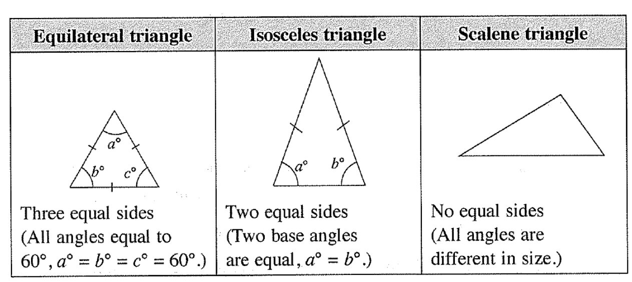


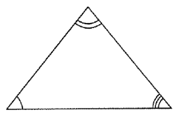
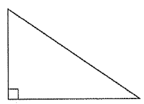
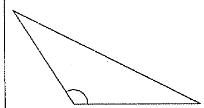
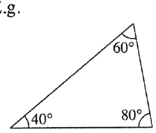
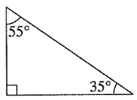
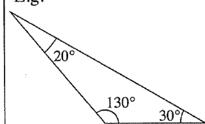
Geometric properties of parallel lines and angles:





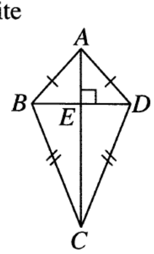
Types of triangles:



Acute-angled triangle	Right-angled triangle	Obtuse-angled triangle
 All three angles are acute.	 One right angle	 One obtuse angle
E.g. 	E.g. 	E.g. 

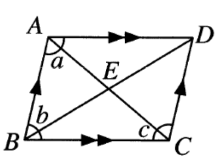
Types of quadrilaterals:

Kite



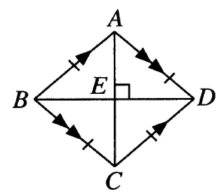
1. $AB = AD$ and $CB = CD$
2. Both $\triangle ABD$ and $\triangle BCD$ are isosceles, with $\angle ABD = \angle ADB$ and $\angle BDC = \angle DBC$
3. Diagonals AC and BD intersect each other at right angles at E
4. The longer diagonal AC bisects the shorter diagonal BD

Parallelogram



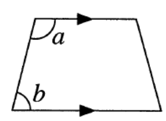
1. Opposite sides are parallel and are of equal length.
2. $\angle a + \angle b = 180°$ (int. $\angle$s)
3. Opposite angles are equal, i.e. $\angle a = \angle c$ (opp. $\angle$s of parallelogram)
4. Diagonals AC and BD bisect each other at E, i.e. $AE = EC$ and $BE = ED$, in other words, E is the mid-point of AC and BD.

Rhombus



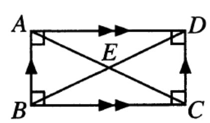
1. Opposite sides are parallel.
2. All four sides are equal in length
3. $\angle ABC + \angle BCD = 180°$ (int. $\angle$s)
4. Opposite angles are equal, i.e. $\angle ABC = \angle ADC$ and $\angle BAD = \angle BCD$
5. The diagonals bisect the interior angles, so that $\angle ABD = \angle CBD$ and $\angle ADB = \angle CDB$
6. Both $\triangle ABD$ and $\triangle BCD$ are isosceles, with $\angle ABD = \angle ADB$ and $\angle CBD = \angle CDB$
7. Diagonals AC and BD bisect each other at right angles, in other words, they are perpendicular bisectors of each other

Trapezium



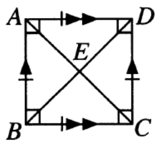
1. One pair of parallel sides.
2. $\angle a + \angle b = 180°$ (int. $\angle$s)
3. Note that the pair of opposite parallel sides may not be same length
4. Also, it may be that none of the interior angles are equal

Rectangle



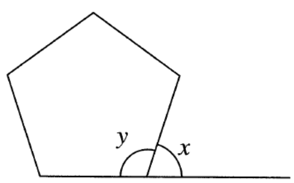
1. Same properties as parallelogram.
2. In addition, the diagonals AC and BD are of equal length (in a parallelogram, the diagonals may not be of equal length)
3. Each interior angle is 90°.
4. $\triangle AED$, $\triangle AEB$, $\triangle BEC$ and $\triangle CED$ are isosceles triangles.

Square



1. Same properties as rhombus.
2. In addition, the diagonals AC and BD are of equal length (in a rhombus, the diagonals may not be of equal length)
3. Each interior angle is $90°$.
4. Each of the four triangles $\triangle AED$, $\triangle AEB$, $\triangle BEC$ and $\triangle CED$ are isosceles right-angled triangles whose base angles are each $45°$.

Angle properties of polygons:

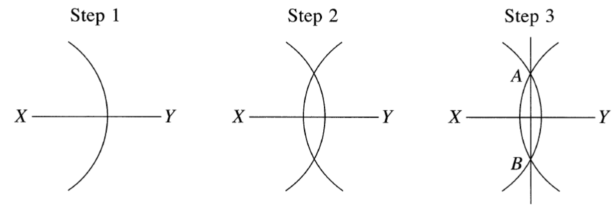


1. In a polygon, the sum of an interior angle and its corresponding exterior angle is $180°$, i.e. $\angle x + \angle y = 180°$.
2. The sum of exterior angles of an n-sided polygon is $360°$.
3. In the case of a n-sided regular polygon, each exterior angle, $\angle x = \frac{360°}{n}$.
4. The sum of interior angles of an n-sided polygon is $(n-2) \times 180°$.
5. In the case of a n-sided regular polygon, each interior angle, $\angle y = \frac{(n-2)\times 180°}{n}$.

No. of sides (***n***)	Name of Polygon
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
9	Nonagon
10	Decagon

Geometric Constructions

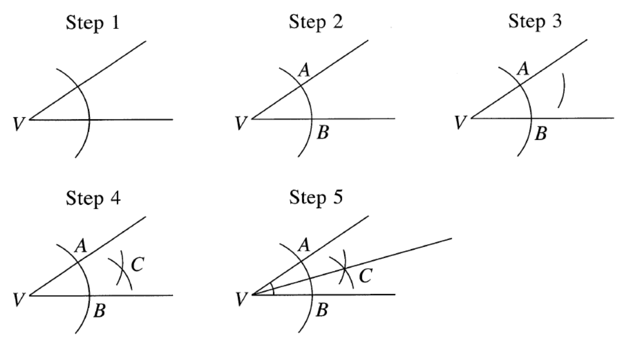
Constructing Perpendicular Bisector of a given line segment XY:



1. Put your compass on point X and set it to be more than half the length of line segment XY. Draw an arc.
2. Without adjusting the length of your compass, put it on point Y. Draw another arc of the same radius to cut the first arc above and below the line segment XY.
3. Label the points of intersection of the two arcs as A and B respectively. Draw a line passing through A and B. The line AB is the perpendicular bisector of the line segment XY.

Any point M on the perpendicular bisector of the line segment XY is equidistant from the points X and Y, i.e. $XM = YM$.

Constructing Angle Bisector:



1. Place your compass on point V and draw an arc that crosses both sides of the angle. Label the two intersection points as A and B.
2. Place your compass on point A and draw an arc of any suitable radius between the two sides of the angle.

3. Without adjusting the length of your compass, place it on point B and draw another arc of the same radius as the arc drawn in Step 2. Label the point where the two arcs meet as C.

4. Draw a straight line through V and C. The line VC is the angle bisector of $\angle V$ where $\angle AVC = \angle BVC$.

Any point P on the angle bisector of $\angle AVB$ is equidistant from the two lines VA and VB.

Radian Measure, Perimeter and Area of Plane Figures, Arc Length and Area of Sector

Convert between radian and degrees:

π radian $= 180°$

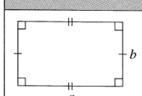
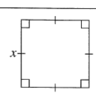
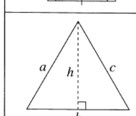
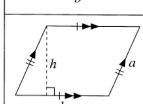
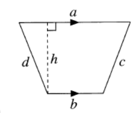
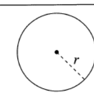
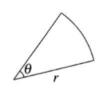
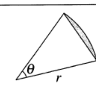
1 radian $= \frac{180°}{\pi}$

$1° = \frac{\pi}{180°}$ radian

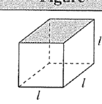
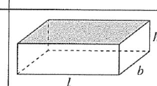
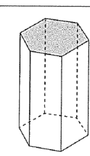
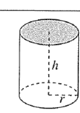
Examples:

2.1 radian $= 2.1 \times \frac{180°}{\pi} = 120.3°$ (to 1 d.p.)

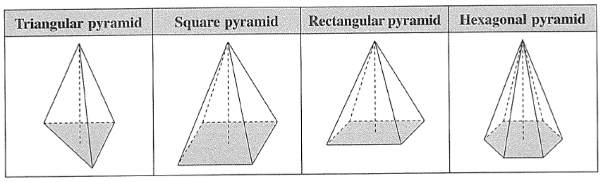
$30° = \frac{\pi}{180°} \times 30° = \frac{\pi}{6}$ rad

Diagram	Geometrical Figure	Formulae	
		Area (square units)	Perimeter (units)
	Rectangle	$a \times b$	$2(a + b)$
	Square	x^2	$4x$
	Triangle	$\frac{1}{2} \times b \times h$	$(a + b + c)$
	Parallelogram	$b \times h$	$2(a + b)$
	Trapezium	$\frac{1}{2}(a + b) \times h$	$(a + b + c + d)$
	Circle	πr^2	$2\pi r$
	Sector	$\frac{\theta}{360°} \times \pi r^2$ where θ is in degrees	Arc length $= \frac{\theta}{360°} \times 2\pi r$ where θ is in degrees Perimeter $= \frac{\theta}{360°} \times 2\pi r + 2r$
		$\frac{1}{2}r^2\theta$ where θ is in radians	Arc length $= r\theta$ where θ is in radians Perimeter $= r\theta + 2r$
	Segment	$\frac{1}{2}r^2(\theta - \sin\theta)$ where θ is in radians	

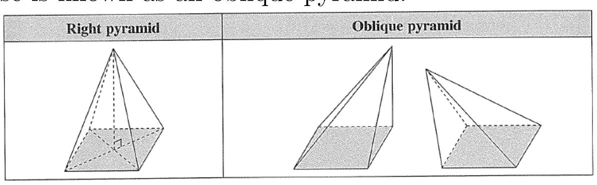
Surface Area and Volume of Solids

Name	Figure	Volume	Total Surface Area
Cube		l^3	$6l^2$
Cuboid		lbh	$2(lb + lh + bh)$
Prism		Area of cross section × height or Area of cross section × length	Total area of lateral faces + 2 × base area = perimeter of base × height + 2 × base area
Closed cylinder		$\pi r^2 h$	$2\pi r^2 + 2\pi rh$

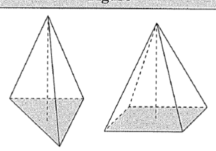
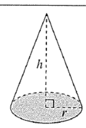
A pyramid is a solid with a polygonal base and triangles as its slanted faces. Each corner point of a pyramid is called a vertex. The vertex of the pyramid that is above the plane of the polygonal base is known as the apex of the pyramid. The apex is also the point where all the slanted triangular faces meet.

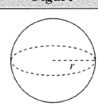
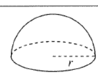


A pyramid whose apex is vertically above the centre of its base is known as a right pyramid. A pyramid whose apex is not vertically above the centre of its base is known as an oblique pyramid.



Surface Area and Volume formulae for Pyramids, Cones, Spheres, Hemispheres:

Name	Figure	Volume	Total Surface Area
Pyramid	 and any other solid with a polygonal base and triangles as its slanted faces	$\frac{1}{3} \times$ base area $\times$ height	Total area of all faces
Cone		$\frac{1}{3}\pi r^2 h$	$\pi r^2 + \pi r l$

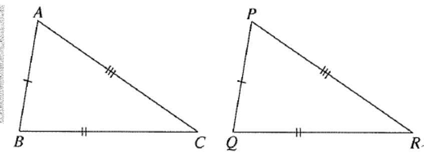
Name	Figure	Volume	Total Surface Area
Sphere		$\frac{4}{3}\pi r^3$	$4\pi r^2$
Hemisphere		$\frac{2}{3}\pi r^3$	$2\pi r^2 + \pi r^2 = 3\pi r^2$

Congruency and Similarity

SSS Congruency Test:

Two triangles are congruent if the **three sides** of one triangle is equal to the corresponding three sides of the other triangle.

If $AB = PQ$,
$BC = QR$ and
$AC = PR$ then
$\triangle ABC \equiv \triangle PQR$.

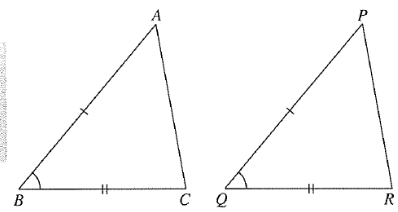


This is known as the **SSS** rule. (side, side, side)

SAS Congruency Test:

Two triangles are congruent if **two sides** and the **included angle** of one triangle are equal to two sides and the included angle of the other triangle.

If $AB = PQ$,
$BC = QR$ and
$\angle B = \angle Q$ then
$\triangle ABC \equiv \triangle PQR$.

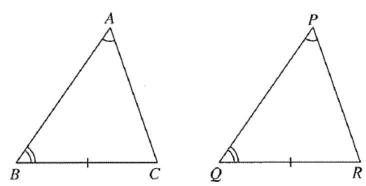


This is known as the **SAS** rule. (side, angle, side)

AAS Congruency Test:

Two triangles are congruent if **two angles** and a **side** of one triangle are equal to two angles and a side of the other triangle.

If $\angle A = \angle P$,
$\angle B = \angle Q$ and
$BC = QR$ then
$\triangle ABC \equiv \triangle PQR$.

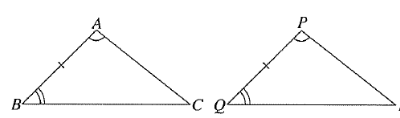


This is known as the **AAS** rule. (angle, angle, side)

ASA Congruency Test:

Two triangles are congruent if **two angles** and the **included side** of one triangle are equal to two angles and the included side of the other triangle.

If $\angle A = \angle P$,
$\angle B = \angle Q$ and
$AB = PQ$ then
$\triangle ABC \equiv \triangle PQR$.



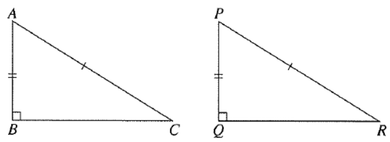
This is known as the **ASA** rule. (angle, side, angle)

(Often, ASA and AAS are regarded as the same test)

RHS Congruency Test:

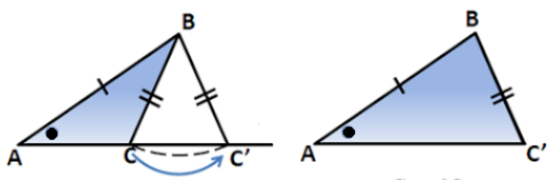
Two triangles are congruent if the **hypotenuse** and **one side** of one right-angled triangle are equal to the hypotenuse and one side of the other right-angled triangle.

If $\angle B = \angle Q = 90°$
$AC = PR$ and
$AB = PQ$ then
$\triangle ABC \equiv \triangle PQR$.



This is known as the **RHS** rule. (right angle, hypotenuse, side)

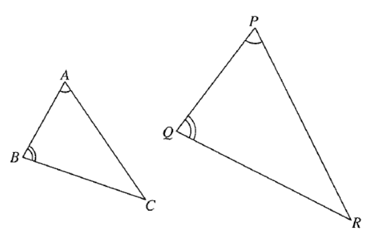
NOTE: Two sides and **non-included** angle, SSA, is not a valid test of congruency.



AA Similarity Test:

Two triangles are similar if **two of their angles are equal.**

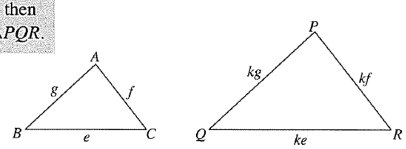
If $\angle A = \angle P$ and
$\angle B = \angle Q$ then
$\triangle ABC$ is similar to $\triangle PQR$.



SSS Similarity Test:

Two triangles are similar if **all three corresponding sides are proportional/ in the same ratio.**

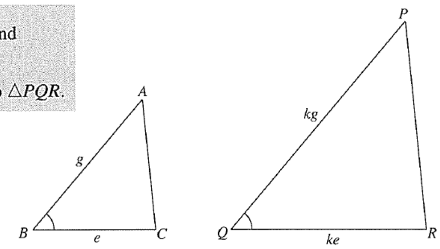
If $\frac{PQ}{AB} = \frac{QR}{BC} = \frac{PR}{AC} = k$,
where k is a constant, then
$\triangle ABC$ is similar to $\triangle PQR$.



SAS Similarity Test:

Two triangles are similar if **two of their sides are proportional and the included angle is equal.**

If $\frac{PQ}{AB} = \frac{QR}{BC} = k$ and
$\angle B = \angle Q$, then
$\triangle ABC$ is similar to $\triangle PQR$.



Areas of Similar Figures:

If two figures are similar, then the ratio of their areas is equal to the square of the ratio of any two corresponding sides of the two figures. If A_1 and A_2 denote the areas of two similar figures and l_1 and l_2 denote their corresponding lengths, then

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$$

Surface Areas and Volumes of Similar Solids:

The ratio of the volumes of two similar solids is equal to the cube of the ratio of any two corresponding lengths of the two solids. If V_1 and V_2 denote the volumes of the two similar solids and l_1 and l_2 denote their corresponding lengths, then

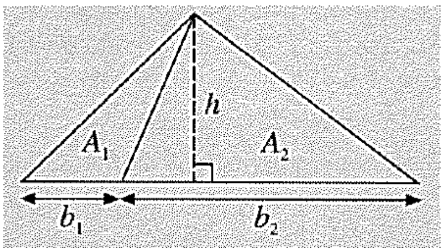
$$\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$$

If A_1 and A_2 denote the surface areas of two similar solids and l_1 and l_2 denote their corresponding lengths, then

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$$

The following technique is a very common problem solving technique that is used in geometry and vectors problems, not just within the topic of congruency and similarity. The ratio of the areas of two triangles that have a **common vertical height** is equal to the ratio of the bases of the two triangles.

$$\frac{A_1}{A_2} = \frac{b_1}{b_2}$$



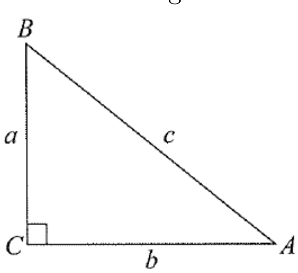
NOTE that we do **NOT** square the ratio of the length of the bases because we are **NOT** using the properties of similar triangles. There are no similar figures here.

Pythagoras' Theorem and its converse, Trigonometry, Applications of Trigonometry

Pythagoras' Theorem:
For a right-angled $\triangle ABC$,

$$a^2 + b^2 = c^2$$

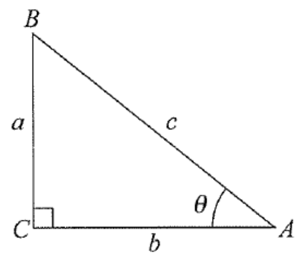
where c is the length of the hypotenuse.



Converse of Pythagoas' Theorem:
If in a $\triangle ABC$, we have $a^2 + b^2 = c^2$, then it follows that $\triangle ABC$ is a right-angled triangle with $\angle BCA = 90°$

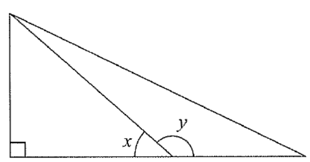
Trigonometric Ratios of an Acute Angle:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$$



Obtuse Angles:

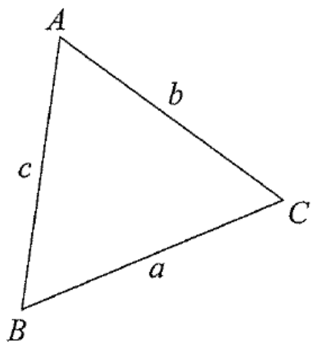
$$\begin{aligned} &\text{If } x + y = 180° \text{ , then} \\ &\sin y = \sin(180° - x) = \sin x \\ &\cos y = \cos(180° - x) = -\cos x \\ &\tan y = \tan(180° - x) = -\tan x \end{aligned}$$



Sine Rule:

For any triangle,

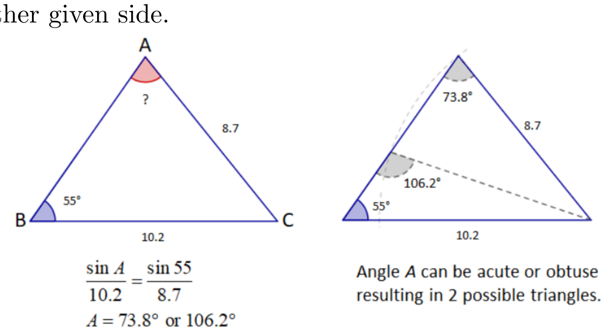
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
or, equivalently,
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Sine Rule can be used if the following are given:
(i) Two angles and a side, OR
(ii) Two sides and an angle opposite one of the given sides, that is two sides and a **non-included** angle.

If a set of given information using Sine Rule gives two possible sets of solutions, it is said to be ambiguous.

Ambiguous case will occur when
(a) two sides and a non-included angle are given AND
(b) the given angle is acute, AND
(c) the side opposite the given angle is less than the other given side.



No ambiguous case exists when
(a) the given angle is obtuse OR
(b) the side opposite the given angle is greater than the other given side.

Cosine Rule:

For $\triangle ABC$,

$$c^2 = a^2 + b^2 - 2ab\cos C \Leftrightarrow \cos C = \frac{a^2+b^2-c^2}{2ab}$$
$$b^2 = a^2 + c^2 - 2ac\cos B \Leftrightarrow \cos B = \frac{a^2+c^2-b^2}{2ac}$$
$$a^2 = b^2 + c^2 - 2bc\cos A \Leftrightarrow \cos A = \frac{b^2+c^2-a^2}{2bc}$$

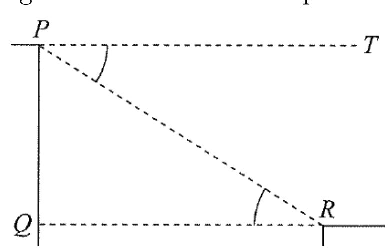
Cosine Rule can be used if the following are given:
(i) Two sides and an **included** angle, or
(ii) Three sides all given but no angles given

Area of Triangle:

The area of $\triangle ABC$ is given by

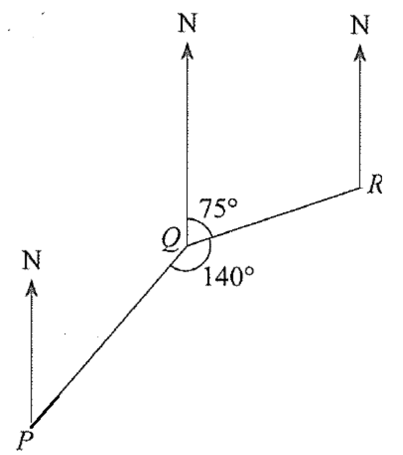
$$\frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$$

Angles of Elevation and Depression:



Assume that QR is level ground, and PT is a line parallel to QR.
$\angle PRQ$ is known as the angle of elevation of P from R.
$\angle TPR$ is known as the angle of depression of R from P.
By alternate angles,
Angle of elevation of P from R, $\angle PRQ$ = Angle of depression of R from P, $\angle TPR$.

Bearings:



To find the bearing of R from Q:
Step 1: Draw a north line at point Q.
Step 2: Draw a line from point Q to point R.
Step 3: Measure the angle from the north line at Q to the line QR in a clockwise direction.

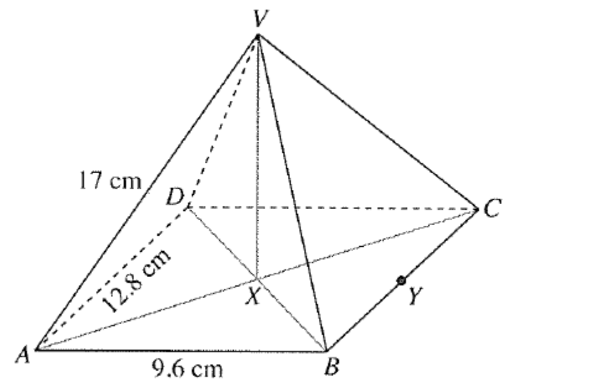
For example, in the above diagram, the bearing of R from Q is $075°$.

Bearing of R from Q: Imagine standing at Q and looking at R.

1. Bearing is measured from the North in a clockwise direction.
2. Bearing ranges from $0°$ to $360°$.
3. Bearing is written as a 3-digit number in degrees, e.g. $060°, 290°, 009°, 032.1°$ (to 1 decimal place if necessary).

In the above diagram,
(a) Bearing of P from Q is $215°$.
(b) Bearing of Q from P is $035°$.
(b) Bearing of Q from R is $255°$.

Three-Dimensional Problems:



The diagram shows a pyramid $VABCD$ with a rectangular base. V is vertically above X . X is the point of intersection of the diagonals AC and BD and Y is the midpoint of BC. Given that $AB = 9.6$ cm, $AD = 12.8$ cm and $VA = 17$ cm, calculate
(a) VX,
(b) VY,
(c) $\angle VAX$,
(d) $\angle XVY$,
(e) the volume of the pyramid $VABCD$.

Ans:
(a)

$$AC^2 = 12.8^2 + 9.6^2 = 256$$
$$AC = \sqrt{256} = 16 \text{ cm}$$
$$AX = \frac{1}{2}AC = \frac{1}{2} \times 16 = 8 \text{ cm}$$

$$VX^2 + 8^2 = 17^2$$
$$VX^2 = 225$$
$$VX = \sqrt{225} = 15 \text{ cm}$$

(b)
$XY = 9.6 \div 2 = 4.8$ cm
$$VY^2 = 15^2 + 4.8^2 = 248.04$$
$$VY = \sqrt{248.04}$$
$$= 15.749 = 15.7 \text{ cm (3 sig. fig.)}$$

(c)
In $\triangle VAX$,
$$\cos \angle VAX = \frac{8}{17}$$
$$\angle VAX = 61.9° (1 \text{ d.p. })$$

(d)

In $\triangle VYX$

$\tan\angle XVY = \dfrac{4.8}{15}$

$\angle XVY = 17.7^\circ$ (1 d.p.)

(e)

Volume of pyramid

$$
\begin{aligned}
&= \frac{1}{3} \times \text{ Base area } \times \text{ Height} \\
&= \frac{1}{3} \times (12.8 \times 9.6) \times 15 \\
&= 614.4 \text{ cm}^2
\end{aligned}
$$