Numbers

- Natural Numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
- Whole Numbers: $\mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$
- Integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Rational Numbers: $\mathbb{Q} = \left\{\frac{a}{b} \ : \ a, b \in \mathbb{Z} \ , \ b \neq 0\right\}$. Rational Numbers comprise of fractions (includes all proper and improper fractions and mixed numbers). All terminating decimals (eg, 10.87) and recurring decimals (eg, $0.3\dot{7}\dot{1} = 0.3717171\ldots$) are rational numbers because these can all be expressed as fractions. All integers are rational numbers.
- Irrational Numbers: numbers that cannot be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Eg. π, $\sqrt{2}$, $\sqrt{5}$, $-4\sqrt{7}$, e, $2.75e$, etc. Irrational numbers are non-recurring decimals. Any non-zero rational number multiplied to an irrational number results in an irrational number. For example, $-\frac{3}{4}\pi$ is irrational.
- Perfect Squares: $\{1, 4, 9, 16, 25, \ldots\}$
- Perfect Cubes: $\{1, 8, 27, 64, \ldots\}$
- Prime Numbers: Positive integers at least 2 whose only positive divisors are 1 and itself. $\{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\}$
- Composite Numbers: Positive integers ≥ 4 that are not prime, in other words, having factors apart from 1 and itself.

Equality and Inequality Symbols

Symbol	Meaning	Example
$=$	is equal to	$0.1 = \frac{1}{10}$
$\neq$	is not equal to	$0.11 \neq \frac{1}{10}$
$>$	is greater than	$0.1 > 0.01$
$\geqslant$	is greater than or equal to	$a \geqslant 5$
$<$	is less than	$0.05 < 5$
$\leqslant$	is less than or equal to	$b \leqslant 5$

Prime Factorization, HCF, LCM

- Example of prime factorization:

$$\begin{array}{r|c} 2 & 4356 \\ \hline 2 & 2178 \\ \hline 3 & 1089 \\ \hline 3 & 363 \\ \hline 11 & 121 \\ \hline & 11 \\ \hline \end{array}$$

  Hence $4356 = 2^2 \times 3^2 \times 11^2$ (in index notation).

- Example of HCF and LCM using prime factorization:

  $4800 = 2^6 \times 3 \times 5^2$

  $5544 = 2^3 \times 3^2 \times 7 \times 11$

  $\text{HCF} = 2^3 \times 3$

  [take common prime factors and lowest power of each]

  $\text{LCM} = 2^6 \times 3^2 \times 5^2 \times 7 \times 11$

  [take all prime factors and highest power of each]

- Examples of square roots and cube roots using prime factorization:

$$\begin{aligned} 54756 &= 2^2 \times 3^4 \times 13^2 \\ \sqrt{54756} &= 2 \times 3^2 \times 13 = 234 \\ 1728 &= 2^6 \times 3^3 \\ \sqrt[3]{1728} &= 2^2 \times 3 = 12 \end{aligned}$$

Approximation

Significant Figures

Rules of identifying number of significant digits:

1. All non-zero digits are significant.
2. Zeros between non-zero digits are significant.

   Eg. 302 (3 sf)

   Eg. 10.2301 (6 sf)
3. In a whole number, zeros after the last nonzero digit may or may not be significant. It depends on the estimation being made.

Eg.

$7436000 = 7000000$ (1 sf)

$7436000 = 7400000$ (2 sf)

$7436000 = 7440000$ (3 sf)

$7436000 = 7436000$ (4 sf)

$7436000 = 7436000$ (5 sf)

$7436000 = 7436000$ (6 sf)

4. In a decimal number, zeros before the 1st non-zero digit are not significant.

   Eg. 0.004 (1 sf)

   Eg. 0.07008 (4 sf)

5. In a decimal number, zeros after the last non-zero digit are significant.

   Eg. 6.40 (3 sf)

   Eg. 12.000 (5 sf)

   Eg. 20300.000 (8 sf)

   Eg. 0.0700800 (6 sf)

Decimal Place Rounding

Examples:

$0.7374 = 0.74$ (2 dp)

$58.301 = 58.30$ (2 dp)

$207.6296 = 207.630$ (3 dp)

$207.6296 = 207.63$ (2 dp)

$207.977 = 208.0$ (1 dp)

$207.977 = 207.98$ (2 dp)

$18.997 = 19.00$ (2 dp)

Standard Form

$\pm A \times 10^n$, where $1 \leq A < 10$ and n is an integer.

Examples:

$1350000 = 1.35 \times 10^6$

$0.000375 = 3.75 \times 10^{-4}$

Common Prefixes

10^{12}	trillion	tera	T
10^9	billion	giga	G
10^6	million	mega	M
10^3	thousand	kilo	k
10^{-3}	thousandth	milli	m
10^{-6}	millionth	micro	μ
10^{-9}	billionth	nano	n
10^{-12}	trillionth	pico	p

Indices

Rules of Indices:

Assume that a, b, m, n are non-zero.

$$a^0 = 1$$

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$(a^n)^m = a^{nm}$$

$$a^{-n} = \frac{1}{a^n}$$

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Caution:
1. For indices (powers) that are not integers, the above rules of indices hold only for $a, b > 0$.
2. Likewise, if some of the indices are negative or there is division by either a or b, then the above rules hold only for $a, b > 0$.
3. $\sqrt{(-8)^2} = \sqrt{64} = 8$, NOT -8.
4. $\sqrt[3]{-27} = -3$.

Equalities of Indices:
1. If $a^m = a''$, then $m = n$.
2. If $a^m = b^m$, then $a = b$.

Percentage, Ratio, Rate

- To express a percentage as a fraction or decimal, divide by 100:

  $x\% = \frac{x}{100}$

  Eg, $23.5\% = \frac{23.5}{100} = \frac{235}{1000} = \frac{47}{200}$

  Eg, $401\% = \frac{401}{100} = 4.01$

- To express any number as a percentage, multiply it by 100%.

  Eg, $0.165 = 0.165 \times 100\% = 16.5\%$.

- Expressing a quantity A as a percentage of a quantity B:

  $\frac{A}{B} \times 100\%$

  Eg, Express 63.7 as a percentage of 98.

  Answer: $\frac{63.7}{98} \times 100\% = 65\%$

  In words, we say that 63.7 is 65% of 98.

- Increase or decrease a quantity by a given percentage:

  Eg, Increase 45 by 2.4%:

  Answer: $45 \times \left(1 + \frac{2.4}{100}\right) = 45 \times 1.024 = 46.08$

  Eg, Decrease 45 by 90%:

  Answer: $45 \times \left(1 - \frac{90}{100}\right) = 45 \times 0.1 = 4.5$

- Percentage Increase and Percentage Decrease:

  When a quantity increases, the percentage increase is

  $\frac{\text{final value} - \text{initial value}}{\text{initial value}} \times 100\%$

  When a quantity dereases, the percentage decrease is

  $\frac{\text{initial (bigger) value} - \text{final (smaller) value}}{\text{initial value}} \times 100\%$

  Percentage increase will always be > 0 if the quantity has increased.

  Percentage decrease will always be > 0 if the quantity has decreased.

  Percentage change is

  $\frac{\text{final value} - \text{initial value}}{\text{initial value}} \times 100\%$

  regardless of whether the quantity has increased or decreased. Percentage change can be either positive or negative depending on whether the quantity has increased or decreased.

- When writing ratios such as $a : b$, a, b are positive integers. Always reduce ratios to the simplest form, eg, $10 : 6$ is to be reduced to $5 : 3$. The ratio $a : b$ expressed in fraction form is $\frac{a}{b}$.

  Eg, If 7 times of x is equal to 5 times of y, then $x : y = 5 : 7$ (note the switching of the order)

- We can use ratios to increase and decrease quantities. For example, if we increase a quantity x in the ratio $6 : 5$, the new quantity is $\frac{6}{5}x$; if we decrease a quantity x in the ratio $5 : 6$, the new quantity is $\frac{5}{6}x$.

- Various units of measurement:

  Mass:

  1 kg = 1000 g

  1 g = 1000 mg

  Length:

  1 km = 1000 m

  1 m = 100 cm

  1 cm = 10 mm

  Area:

  1 km^2 = 10^6 m^2

  1 m^2 = 10000 cm^2

  Volume:

  1 l = 1000 ml

  1 cm^3 = 1 ml

  1 m^3 = 10^6 cm^3 = 1000 l

  Time:

  1 hr = 60 min

  1 min = 60 sec

- Distance = Speed $\times$ Time

- Average speed = (Total Distance) / (Total Time Taken)

- Conversion of units for speed:

  26km/h = 26000m/h = $\frac{26000}{3600}$m/s = $\frac{65}{9}$m/s

  35m/s = 0.035km/s = (0.035 × 3600)km/h = 126km/h

- Density = Mass / Volume.

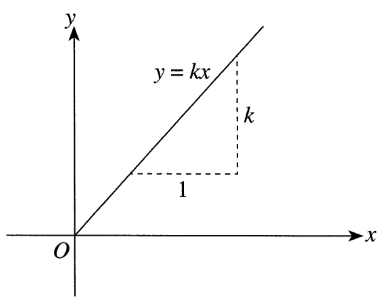
  Units are usually g/cm^3 or kg/m^3.

  1g/cm^3 = 1000kg/m^3.

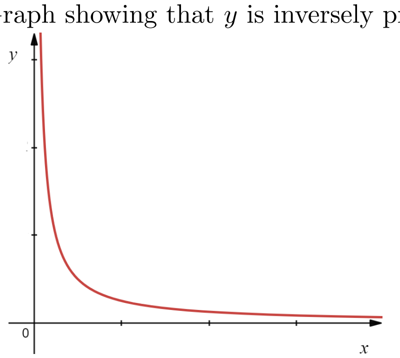
  Eg, 0.235g/cm^3 = 235kg/m^3.

Direct and Inverse Proportion

If y is directly proportional to x, then $y = kx$, where k is a constant and $k \neq 0$. The ratios $\frac{x}{y}$ and $\frac{y}{x}$ are constant. Furthermore, the graph on y against x (or of x against y) is a straight line through the origin.
Graph showing that y is directly proportional to x:



If y is inversely proportional to x, then $y = \frac{k}{x}$, where k is a constant and $k \neq 0$. The product xy is constant.
Graph showing that y is inversely proportional to x:



Map Scales

- Linear scale:

  1 : n means 1 unit length on map represents n units length on ground.

  Eg. 1 : 5000 means

  1 cm represents 5000 cm

  which implies 1 cm represents 50 m

  which implies 1 cm represents 0.05 km

- Representative Fraction (RF):

  If the linear scale is 1 : n, the RF is expressed as $\frac{1}{n}$.

  Eg, if 3 cm represents 6 m, then RF is $\frac{1}{200}$.

- Area Scale:

  If linear scale is 1 : 20000, then it means

  1 cm represents 20000 cm

  which implies 1 cm represents 0.2 km

  which implies 1^2 cm^2 represents (0.2)^2 km^2

  which implies 1 cm^2 represents 0.04 km^2

Number Patterns

Common number patterns:

- Constant difference

  Eg, $-5, -2, 1, 4, 7, 10, \ldots$

  The n^{th} term, denoted T_n, is given by

  $T_n = a + d(n-1)$

  where a is the first term and d is the common difference.

  For the sequence $-5, -2, 1, 4, 7, 10, \ldots$,

  $T_n = -5 + (n-1)(3) = 3n - 8$.

  Alternatively,

  $T_n = b + dn$

  where b is the term that would have come before the first term (ie, the "zeroth" term).

- Constant multiple (or common ratio)

  Eg, $3, 15, 75, 375, \ldots$

  $T_n = a \times r^{n-1}$

  where a is the first term, r is the common ratio, that is, r is the number that when multiplied a term gives the next term.

  For the sequence $3, 15, 75, 375, \ldots$

  $T_n = 3 \times 5^{n-1}$.

- Perfect squares and perfect cubes

  $1, 4, 9, 16, 25, \ldots\ :\ T_n = n^2$

  $1, 8, 27, 64, 125, \ldots\ :\ T_n = n^3$

  $2, 8, 18, 32, 50, \ldots\ :\ T_n = 2n^2$

  $3, 10, 29, 66, 127, \ldots\ :\ T_n = n^3 + 2$

Simultaneous Linear Equations

Method 1: Elimination

$5x - 2y = 21$ —(1)

$2x - y = 8$ —(2)

$(1) \times 2 : 10x - 4y = 42$ —(3)

$(2) \times 5 : 10x - 5y = 40$ —(4)

(3) - (4) : $y = 2$

Sub into (1): $5x - 2(2) = 21$

$5x - 4 = 21$

$5x = 25$

$x = 5$

Method 2: Substitution

$5x - 2y = 21$ —(1)

$2x - y = 8$ —(2)

From (2):

$y = 2x - 8$ —(3)

Sub (3) into (1): $5x - 2(2x - 8) = 21$

$5x - 4x + 16 = 21$

$x - 16 = 21$

$x = 5$

Sub into (2):

$2(5) - y = 8$

$10 - y = 8$

$y = 2$

Inequalities

Inequality sign is reversed when both sides are multiplied or divided by a negative number.

Eg. $-3x + 4 \geq 12$

$-3x \geq 12 - 4$

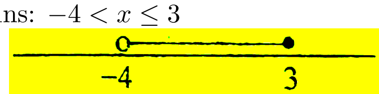
$-3x \geq 8$

$x \leq -\frac{8}{3}$

Eg. $3(x - 1) < 4x + 1 \leq 7 + 2x$

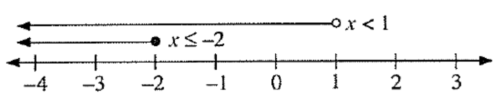
$3(x-1) < 4x + 1$	$4x + 1 \leq 7 + 2x$
$3x - 3 < 4x + 1$	$4x - 2x \leq 7 - 1$
$3x - 4x < 1 + 3$	$2x \leq 6$
$-x < 4$	$x \leq 3$
$x > -4$	

Ans: $-4 < x \leq 3$



Eg. $5x + 4 \leq 3x < 6 - 3x$

$5x + 4 \leq 3x$	$3x < 6 - 3x$
$2x \leq -4$	$6x < 6$
$x \leq -2$	$x < 1$



Ans: $x \leq -2$.

Expansion

Eg.

$2p - 3(p + 1)$

$= 2p - 3p - 3$

$= -p - 3$

Eg.

$5x - (x + 1)(2x - 3)$

$= 5x - \left(2x^2 - 3x + 2x - 3\right)$

$= 5x - 2x^2 + 3x - 2x + 3$

$= -2x^2 + 6x + 3$

Factorization and Identities

Factorization of Quadratic Expressions:
$5x^2 + 9x - 2$

$\times$	$5x$	-1
x	$5x^2$	$-x$
2	$10x$	-2

$\therefore \quad 5x^2 + 9x - 2 = (5x - 1)(x + 2)$

Identities:

1. $(a + b)^2 = a^2 + 2ab + b^2$
2. $(a - b)^2 = a^2 - 2ab + b^2$
3. $(a + b)(a - b) = a^2 - b^2$

Common Factorisation Techniques:

- Common Factors
  Eg. $6a^3b - 2a^2b = 2a^2b(3a - 1)$
- Grouping
  Eg.
  $6p^2 - 3pq - 10ap + 5aq$
  $= 3p(2p - q) - 5a(2p - q)$
  $= (3p - 5a)(2p - q)$
- Using Difference of Two Squares
  Eg. $9a^2 - 1$
  $= (3a)^2 - (1)^2$
  $= (3a + 1)(3a - 1)$
  Eg. $16a^4 - 81$
  $= (4a^2 + 9)(4a^2 - 9)$
  $= (4a^2 + 9)(2a + 3)(2a - 3)$
- Combination of methods:
  Eg. $3x^3 - 12xy^2 = 3x\left(x^2 - 4y^2\right)$ common factor
  $= 3x(x + 2y)(x - 2y)$ then diff. of 2 squares
  Always try common factor first
  Eg.
  $4 - p^2 + 6pq - 9q^2$
  $= 4 - (p^2 - 6pq + 9q^2)$
  $= (2)^2 - (p - 3q)^2$
  $= (2 + (p - 3q))(2 - (p - 3q))$
  $= (2 + p - 3q)(2 - p + 3q)$

Algebraic Fractions

Eg.
$$\frac{x+2}{3} - \frac{x-5}{2} = \frac{2(x+2)}{6} - \frac{3(x-5)}{6}$$
$$= \frac{2(x+2) - 3(x-5)}{6}$$
$$= \frac{2x + 4 - 3x + 15}{6}$$
$$= \frac{19 - x}{6}$$

Eg.
$$\frac{5}{x+1} - \frac{2}{x-3}$$
$$= \frac{5(x-3)}{(x+1)(x-3)} - \frac{2(x+1)}{(x-3)(x+1)}$$
$$= \frac{5(x-3) - 2(x+1)}{(x+1)(x-3)}$$
$$= \frac{5x - 15 - 2x - 2}{(x+1)(x-3)}$$
$$= \frac{3x - 17}{(x+2)(x-5)}$$

Eg.
$$\frac{5}{3x} + \frac{2}{x}$$
$$= \frac{5}{3x} + \frac{6}{3x}$$
$$= \frac{5+6}{3x}$$
$$= \frac{11}{3x}$$

Eg.
$$\frac{3}{(x+2)^2} - \frac{4}{x+2}$$
$$= \frac{3}{(x+2)^2} - \frac{4(x+2)}{(x+2)^2}$$
$$= \frac{3 - 4(x+2)}{(x+2)^2}$$
$$= \frac{3 - 4x - 8}{(x+2)^2}$$
$$= \frac{-4x - 5}{(x+2)^2}$$

Eg.

$$\frac{7}{x^2-9} - \frac{1}{x-3}$$
$$= \frac{7}{(x+3)(x-3)} - \frac{1}{x-3}$$
$$= \frac{7}{(x+3)(x-3)} - \frac{x-3}{(x-3)^2}$$
$$= \frac{7-(x+3)}{(x+3)(x-3)}$$
$$= \frac{7-x-3}{(x+3)(x-3)}$$
$$= \frac{4-x}{(x+3)(x-3)}$$

Eg.

$$\frac{3x+2}{x^2-4} + \frac{1}{x+2} - \frac{2}{x-2}$$
$$= \frac{3x+2}{(x+2)(x-2)} + \frac{1}{x+2} - \frac{2}{x-2}$$
$$= \frac{3x+2+(x-2)-2(x+2)}{(x+2)(x-2)}$$
$$= \frac{3x+2+x-2-2x-4}{(x+2)(x-2)}$$
$$= \frac{2x-4}{(x+2)(x-2)}$$
$$= \frac{2(x-2)}{(x+2)(x-2)}$$
$$= \frac{2}{x+2}$$

Eg.

$$\frac{9}{x-5} + \frac{3}{5-x}$$
$$= \frac{9}{x-5} - \frac{3}{x-5}$$
$$= \frac{9-3}{x-5}$$
$$= \frac{6}{x-5}$$

Eg.

$$\frac{2x}{3y-8x} + \frac{11x}{80x-30y}$$
$$= \frac{2x}{3y-8x} + \frac{11x}{-10(3y-8x)}$$
$$= \frac{2x}{3y-8x} - \frac{11x}{10(3y-8x)}$$
$$= \frac{20x-11x}{10(3y-8x)}$$
$$= \frac{9x}{10(3y-8x)}$$

Eg.

$$\frac{4}{x^2-4} + \frac{1}{2-x}$$
$$= \frac{4}{(x+2)(x-2)} - \frac{1}{x-2}$$
$$= \frac{4-(x+2)}{(x+2)(x-2)}$$
$$= \frac{4-x-2}{(x+2)(x-2)}$$
$$= \frac{2-x}{(x+2)(x-2)}$$
$$= \frac{-x-2)}{(x+2)(x-2)}$$
$$= -\frac{1}{x+2}$$

Eg.

$$\frac{6p^3}{7q} \div \frac{2p}{21q^2}$$
$$= \frac{6p^3}{7q} \times \frac{21q^2}{2p} \quad \text{[do cancelling]}$$
$$= 9p^2q$$

Eg.

$$\frac{4pq^2+4pqr}{9pqr^2+9pq^2r} = \frac{4pq(q+r)}{9pqr(r+q)}$$
$$= \frac{4}{9r}$$

Eg.
$$\frac{5k^2-17k-12}{5k^2-10k-40} = \frac{(5k+3)(k-4)}{5\left(k^2-2k-8\right)}$$
$$= \frac{(5k+3)(k-4)}{5(k-4)(k+2)}$$
$$= \frac{5k+3}{5(k+2)}$$

Eg.
$$\frac{xy-z^2-xz+yz}{y^2-2yz+z^2} \div \frac{11}{2xz+x^2+z^2}$$
$$= \frac{xy-xz+yz-z^2}{y^2-2yz+z^2} \times \frac{2xz+x^2+z^2}{11}$$
$$= \frac{x(y-z)+z(y-z)}{(y-z)^2} \times \frac{(x+z)^2}{11}$$
$$= \frac{(x+z)(y-z)}{(y-z)^2} \times \frac{(x+z)^2}{11}$$
$$= \frac{(x+z)^3}{11(y-z)}$$

Eg.
$$\frac{1}{x}+1$$
$$= \left(\frac{1}{x}+1\right) \div (x+1)$$
$$= \frac{1+x}{x} \times \frac{1}{x+1}$$
$$= \frac{1}{x}$$

Eg.
$$\frac{1-\frac{1}{x}}{1-\frac{1}{x^2}}$$
$$= \left(1-\frac{1}{x}\right) \div \left(1-\frac{1}{x^2}\right)$$
$$= \left(\frac{x-1}{x}\right) \div \left(\frac{x^2-1}{x^2}\right)$$
$$= \frac{x-1}{x} \times \frac{x^2}{x^2-1^2}$$
$$= \frac{x-1}{x} \times \frac{x^2}{(x+1)(x-1)}$$
$$= \frac{x}{x+1}$$

Making Subject of Formula

Eg: Make a the subject
$$y = m(x-a)+b$$
$$y-b = m(x-a)$$
$$m(x-a) = y-b$$
$$x-a = \frac{y-b}{m}$$
$$a = x-\frac{y-b}{m}$$

Eg: Make x the subject
$$ax-by = 3-2x$$
$$ax+2x = 3+by$$
$$x(a+2) = by+3$$
$$x = \frac{by+3}{a+2}$$

Eg: Make d the subject
$$T = 0.25\pi d^2$$
$$\pi d^2 = 4T$$
$$d^2 = \frac{4T}{\pi}$$
$$d = \pm\sqrt{\frac{4T}{\pi}}$$
Note the $\pm$ when taking square-roots in this type of question.

Eg: Make c the subject
$$d = \frac{8-c}{c+7}$$
$$d(c+7) = 8-c$$
$$cd+7d = 8-c$$
$$cd+c = 8-7d$$
$$c(d+1) = 8-7d$$
$$c = \frac{8-7d}{d+1}$$

Eg: Make q the subject

$$5a = \sqrt{\frac{b^2}{q} - \frac{3c}{4}}$$
$$25a^2 = \frac{b^2}{q} - \frac{3c}{4}$$
$$\frac{b^2}{q} = 25a^2 + \frac{3c}{4}$$
$$\frac{b^2}{q} = \frac{100a^2 + 3c}{4}$$
$$\frac{q}{b^2} = \frac{4}{100a^2 + 3c}$$
$$q = \frac{4b^2}{100a^2 + 3c}$$

Eg: Make y the subject

$$\frac{x\left(yz - w^2\right)}{2} - \frac{y}{3} = 6y$$
$$3x\left(yz - w^2\right) - 2y = 36y$$
$$3xyz - 3w^2x - 2y = 36y$$
$$3xyz - 38y = 3w^2x$$
$$y(3xz - 38) = 3w^2x$$
$$y = \frac{3w^2x}{3xz - 38}$$

Eg: Make x the subject

$$p\sqrt{x} + q = r + s\sqrt{x}$$
$$p\sqrt{x} - s\sqrt{x} = r - q$$
$$\sqrt{x}(p - s) = r - q$$
$$\sqrt{x} = \frac{r - q}{p - s}$$
$$x = \left(\frac{r - q}{p - s}\right)^2$$

Solving Quadratic Equations and Equations Involving Algebraic Fractions

Three methods of solving quadratic equations:

1. Factorisation

$$3x^2 - 2x - 8 = 0$$
$$(3x + 4)(x - 2) = 0$$
$$3x + 4 = 0 \text{ or } x - 2 = 0$$
$$x = -\frac{4}{3} \text{ or } x = 2$$

2. Using General Formula

$$ax^2 + bx + c = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Eg. $12x^2 - x - 25 = 0$

$$a = 2, b = -1, c = -25$$
$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-25)}}{2(2)}$$
$$x = \frac{1 \pm \sqrt{201}}{4}$$
$$x = \frac{1 + \sqrt{201}}{4} \text{ or } \frac{1 - \sqrt{201}}{4}$$
$$x = 3.79(2\text{dp}) \text{ or } -3.29(2\text{dp})$$

3. Complete the Square

$$x^2 - 8x + 13 = (x + p)^2 + q$$
$$x^2 - 8x + 13$$
$$= x^2 - 8x + \left(\frac{8}{2}\right)^2 - \left(\frac{8}{2}\right)^2 + 13$$
$$= x^2 - 8x + 16 - 16 + 13$$
$$= (x - 4)^2 - 3$$

To Solve $\quad x^2 - 8x + 13 = 0$

First Change To $(x - 4)^2 - 3 = 0$

$$(x - 4)^2 = 3$$
$$x - 4 = \pm\sqrt{3}$$
$$x - 4 = \sqrt{3} \text{ or } x - 4 = -\sqrt{3}$$
$$x = \sqrt{3} + 4 \text{ or } x = -\sqrt{3} + 4$$
$$x = 5.73(2\text{dp}) \text{ or } 2.27(2\text{dp})$$

Equations involving algebraic fractions:

Eg:

$$\begin{aligned}
3x &= \frac{1}{x} - 4 \\
3x^2 &= 1 - 4x \\
3x^2 + 4x - 1 &= 0 \\
x &= \frac{-4 \pm \sqrt{4^2 - 4(3)(-1)}}{2(3)} \\
&= \frac{-4 \pm \sqrt{28}}{6}
\end{aligned}$$

$$x = \frac{-4 + \sqrt{28}}{6} \quad \text{or} \quad x = \frac{-4 - \sqrt{28}}{6}$$

$$x = 0.22 \quad \text{or} \quad x = -1.55(2 \text{ d.}p.)$$

Eg:

$$\begin{aligned}
x + 1 &= \frac{20}{x+2} \\
(x+1)(x+2) &= 20 \\
x^2 + 3x + 2 &= 20 \\
x^2 + 3x - 18 &= 0 \\
(x+6)(x-3) &= 0
\end{aligned}$$

$$x + 6 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = -6 \quad \text{or} \quad x = 3$$

Eg:

$$\frac{2-x}{x+1} + \frac{1}{x-3} = \frac{3}{5}$$

$$5(2-x)(x-3) + 5(x+1) = 3(x+1)(x-3)$$

$$5\left(2x - 6 - x^2 + 3x\right) + 5x + 5 = 3\left(x^2 - 3x + x - 3\right)$$

$$5\left(-x^2 + 5x - 6\right) + 5x + 5 = 3\left(x^2 - 2x - 3\right)$$

$$-5x^2 + 25x - 30 + 5x + 5 = 3x^2 - 6x - 9$$

$$8x^2 - 36x + 16 = 0$$

$$2x^2 - 9x + 4 = 0$$

$$(2x-1)(x-4) = 0$$

$$2x - 1 = 0 \text{ or } x - 4 = 0$$

$$x = \frac{1}{2} \text{ or } x = 4$$

Eg:

$$\frac{2x+5}{x^2+4x+3} + \frac{2}{x+3} = 1$$

$$\frac{2x+5}{(x+1)(x+3)} + \frac{2}{x+3} = 1$$

$$\begin{aligned}
2x + 5 + 2(x+1) &= (x+1)(x+3) \\
2x + 5 + 2x + 2 &= x^2 + 4x + 3 \\
4x + 7 &= x^2 + 4x + 3 \\
x^2 - 4 &= 0 \\
x^2 &= 4 \\
x &= \pm\sqrt{4} \\
x &= 2 \text{ or } x = -2
\end{aligned}$$

Eg:

$$\frac{x^2+2}{(5x-4)(2x-1)} = \frac{1}{3}$$

$$\begin{aligned}
3\left(x^2+2\right) &= (5x-4)(2x-1) \\
3x^2 + 6 &= 10x^2 - 13x + 4
\end{aligned}$$

$$7x^2 - 13x - 2 = 0$$

$$(7x+1)(x-2) = 0$$

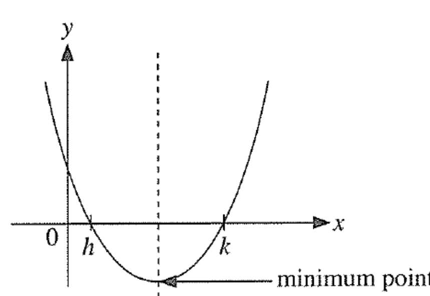
$$\therefore 7x + 1 = 0 \quad \text{or} \quad x - 2 = 0$$

$$x = -\frac{1}{7} \quad \text{or} \quad x = 2$$

Quadratic Graphs

1. Suppose that a quadratic function $y = ax^2 + bx + c$ $(a \neq 0)$ can be factorized into $a(x-h)(x-k)$.

   Consider the case where $a > 0$. The graph is concave upwards (smiley face) and there is a minimum point.



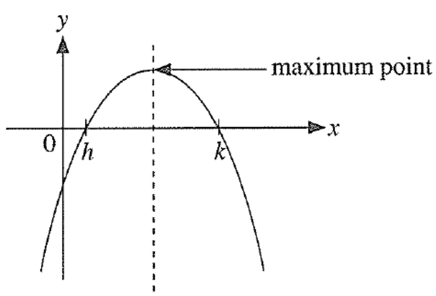
The graph intersects the x-axis at $(h, 0)$ and $(k, 0)$.

The line of symmetry is $x = \frac{h+k}{2}$.

The minimum point also occurs at $x = \frac{h+k}{2}$. To find the y-coordinate of the minimum point, substitute $x = \frac{h+k}{2}$ into the expression $y = ax^2 + bx + c$.

2. Suppose that a quadratic function $y = ax^2 + bx + c$ $(a \neq 0)$ can be factorized into $a(x-h)(x-k)$.

   Consider the case where $a < 0$. The graph is concave downwards (sad face) and there is a maximum point.



   The graph intersects the x-axis at $(h, 0)$ and $(k, 0)$.

   The line of symmetry is $x = \frac{h+k}{2}$.

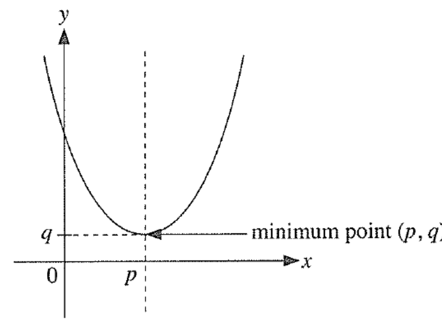
   The maximum point also occurs at $x = \frac{h+k}{2}$. To find the y-coordinate of the maximum point, substitute $x = \frac{h+k}{2}$ into the expression $y = ax^2 + bx + c$.

3. Suppose we complete the square quadratic function $y = ax^2 + bx + c$ $(a \neq 0)$ to obtain $y = a(x-p)^2 + q$.

   If $a > 0$, then the graph is smiley face, and has a minimum point at (p, q). The minimum value of y is q.

   The equation $ax^2 + bx + c = h$ has two solutions if $h > q$, one solution if $h = q$, and no solutions if $h < q$.

   The line of symmetry $x = p$.

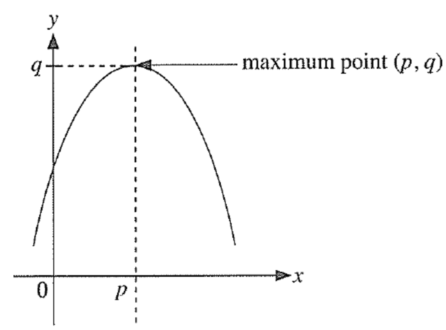


4. Suppose we complete the square quadratic function $y = ax^2 + bx + c$ $(a \neq 0)$ to obtain $y = a(x-p)^2 + q$.

   If $a < 0$, then the graph is sad face, and has a maximum point at (p, q). The maximum value of y is q.

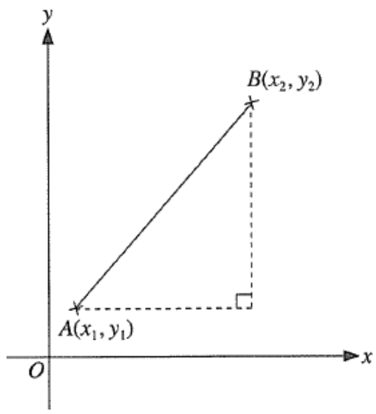
   The equation $ax^2 + bx + c = h$ has two solutions if $h < q$, one solution if $h = q$, and no solutions if $h > q$.

   The line of symmetry $x = p$.

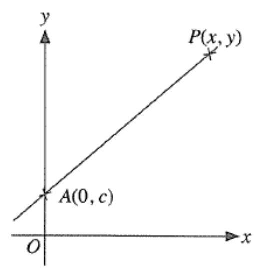


Coordinate Geometry

- The length of a line segment with end points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by
  $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
  or equivalently
  $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



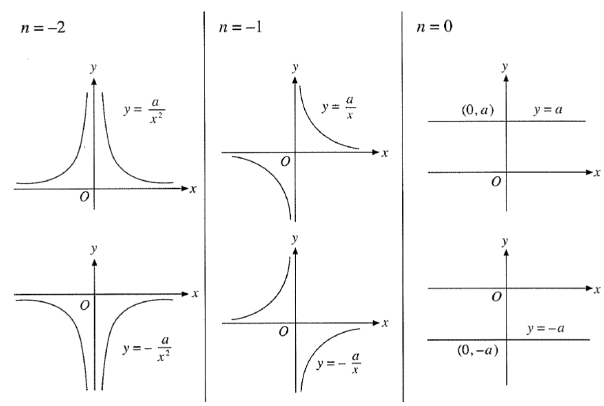
- The gradient of a line passing through $A\,(x_1, y_1)$ and $B\,(x_2, y_2)$ is given by
$$\frac{y_2 - y_1}{x_2 - x_1}$$
or equivalently
$$\frac{y_1 - y_2}{x_1 - x_2}.$$
- Collinear Points:
If gradient of AB = gradient of BC = gradient of AC, we say that A, B, C are collinear. This means that A, B, C lie on the same straight line (not necessarily in the order $A-B-C$; it could also be in any order such as $B-C-A$.)
Note: For collinear points, we only need to check the equality of one pair of gradients, for example, gradient of AB = gradient of BC. Then automatically, gradient of AB = gradient of AC and gradient of AC = gradient of BC as well.
When checking that gradient of AB = gradient of BC, there must be a common point (in this case, B) in order to conclude that the points are collinear.
If gradient of AB = gradient of CD (ie, no common points), then we cannot conclude any of the points are collinear.
- A horizontal line is parallel to the x-axis. If all the points on the line has y-coordinate equal to b, then the equation of the line is
$$y = b.$$
A vertical line is parallel to the y-axis. If all the points on the line has x-coordinate equal to a, then the equation of the line is
$$x = a.$$
If a line has a gradient m and y-intercept c, then its equation is
$$y = mx + c.$$
The equation $y = mx + c$ is known as the gradient-intercept form.

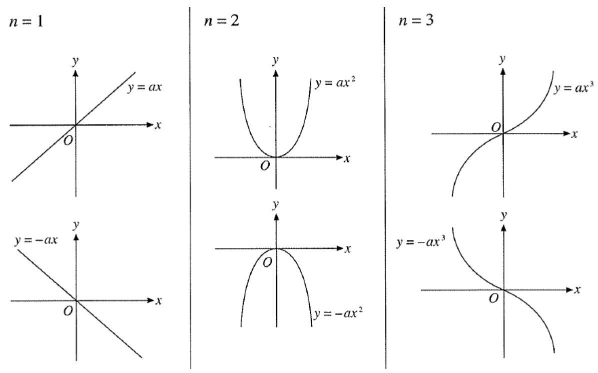


Graphs and Functions

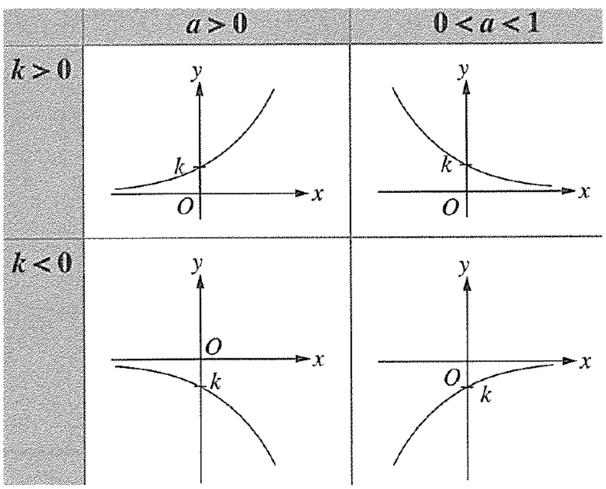
Below is a summary of the graphs of the form $y = ax^n$, where $n = -2, -1, 0, 1, 2$ and 3.

In the below sketches, we have assumed $a > 0$.

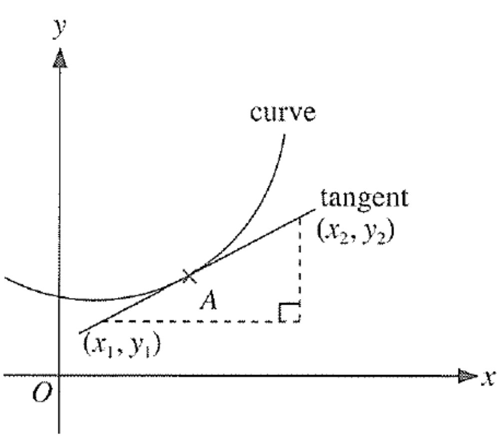




These are graphs of the form $y = ka^x$.



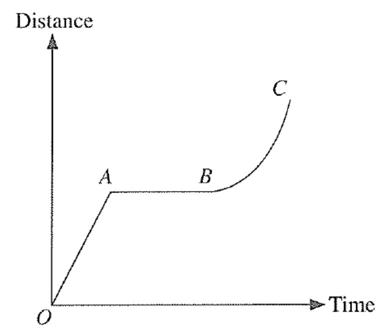
Tangent to graphs:



A straight line which touches a curve at a single point A is called a tangent to the curve at the point A.

The gradient of the curve at the point A is equal to the gradient of the tangent to the curve at A. In general, the gradient of the curve at any point is equal to the gradient of the tangent to the curve at that point.

Distance-Time Graphs:



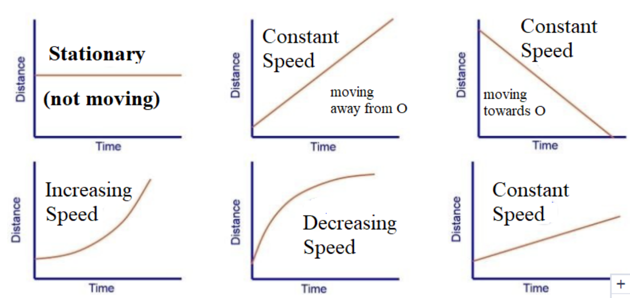
The gradient of a distance-time graph gives the speed of the object.

OA is a straight line, i.e. speed of object = gradient of OA.

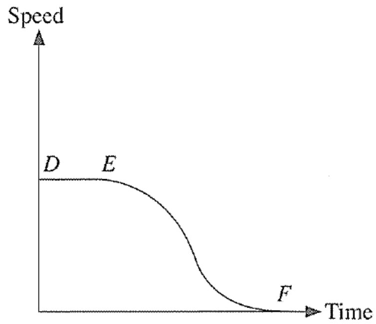
AB is a horizontal line, i.e. speed of object = zero.

BC is a curve, i.e. instantaneous speed of object = gradient of the tangent to the curve at a point.

Various types of distance-time graphs:



Speed-Time Graphs:



The gradient of a speed-time graph gives the acceleration of the object.

The area under a speed-time graph gives the distance travelled by the object.

DE is a horizontal line, i.e. acceleration of object = zero.

EF is a curve, i.e. instantaneous acceleration of object = gradient of tangent to the curve at a point.

Total distance travelled = area under graph from D to F

Various types of speed-time graphs: