A Math Quadratic - Consolidated Questions - Chp 1

Instructions. Answer all questions. Where a sketch is requested, sketch the graph clearly and label all key features (turning point, intercepts, and line of symmetry where appropriate).

1. Express $2x^2 - 7x + 1$ in the form $a(x+b)^2 + c$.

2. Express $-4x^2 + 12x - 5$ in the form $a(x+b)^2 + c$.

3. Express $\frac{3}{2}x^2 - 5x + \frac{7}{3}$ in the form $a(x+b)^2 + c$. Give exact values.

4. Express $x^2 - \frac{7}{3}x + \frac{5}{9}$ in the form $(x+a)^2 + b$.

5. A quadratic curve has turning point $(-3, 5)$ and passes through the point $(1, -3)$.

(i) Find the equation of the curve in the form $y = a(x-h)^2 + k$.

(ii) Hence write the equation in the form $y = ax^2 + bx + c$.

6. For $y = 4x^2 - 12x + 5$:

(i) Express y in the form $a(x-h)^2 + k$.

(ii) Sketch the graph and state the turning point, the x-intercepts (if any), the y-intercept, and the line of symmetry.

7. For $y = -2x^2 - 4x + 3$:

(i) Express y in the form $a(x-h)^2 + k$.

(ii) Sketch the graph and state the turning point, the x-intercepts (if any), the y-intercept, and the line of symmetry.

8. Sketch the graph of $y = (x-4)(x+2)$ and state the x-intercepts, the y-intercept, the turning point, and the line of symmetry.

9. A quadratic curve is U-shaped, has x-intercepts -1 and 5, and passes through $(0, -10)$.

(i) Find the equation of the curve in expanded form.

(ii) State the turning point and the line of symmetry.

(iii) Sketch the graph.

10. A quadratic curve opens downward, has line of symmetry $x = 1$, and passes through $(1, 4)$ and $(3, 0)$.

(i) Find the equation of the curve.

(ii) Find the intercepts and state the turning point.

(iii) Sketch the graph.

11. Let $f(x) = -2x^2 + 10x - 7$.

(i) Express $f(x)$ in the form $a(x-h)^2 + k$.

(ii) Hence state the maximum value of $f(x)$ and the value of x at which it occurs.

(iii) Solve the equation $f(x) = 1$.

12. For $y = x^2 - 6x + 13$:

(i) Find the minimum value of y and the value of x at which it occurs.

(ii) Find the values of x for which $y = 17$.

13. The quadratic function is $y = -x^2 + ax + 4$, where a is a constant. The maximum value of y is 9.

(i) Find the possible values of a.

(ii) For each value of a, state the turning point.

14. The graph of $y = 2x^2 + px + q$ has line of symmetry $x = 3$ and passes through the point $(0, 6)$.

(i) Find p and q.

(ii) Find the turning point and the x-intercepts.

(iii) Sketch the graph.

15. The equation $x^2 - (p+1)x + p = 0$ has two real roots that differ by 3. Find p.

16. A firework leaves the ground at $x = 1$ m and lands at $x = 9$ m. Its maximum height is 8 m at $x = 5$ m.

(i) Find a quadratic model for its height y (m) in terms of x (m).

(ii) Find y at $x = 7$.

(iii) A wall at $x = 7$ is 6 m tall. Determine whether the firework clears the wall.

(iv) Find the values of x for which $y = 6$.

(v) Sketch the graph.

17. A suspension cable is modelled by

$$y = \frac{1}{500}(x-100)^2 + 6, \qquad 0 \le x \le 200,$$

where x (m) is the horizontal distance from the left tower and y (m) is the height above the road.

(i) Find the height of each tower above the road.

(ii) Find the height of the cable at $x = 40$.

(iii) A truck is 12 m tall. Find the values of x at which the cable height equals the truck height (i.e. the boundary points where the truck can just pass).

(iv) Sketch the graph.

18. A farmer has 200 m of fencing to enclose three identical rectangular pens side-by-side against a straight river (no fencing along the river). Let x m be the dimension perpendicular to the river and L m be the total length along the river.

(i) Express the total area A in terms of x.

(ii) Find the maximum total area and the corresponding values of x and L.

19. Monthly profit (in dollars) from producing x hundreds of units is

$$P(x) = -2x^2 + 180x - 2000.$$

(i) Find the value of x that gives the maximum profit and the maximum profit.

(ii) Find the break-even values of x.

(iii) Find the values of x for which $P(x) = 1600$.

20. A performer's height h (m) after travelling x m horizontally is quadratic. He starts at height 2 m when $x = 0$, passes through $(10, 12)$, and lands on the ground (i.e. $h = 0$) at $x = 30$.

(i) Find $h(x) = ax^2 + bx + c$.

(ii) Find the maximum height and the value of x where it occurs.

(iii) A barrier at $x = 12$ is 13 m high. Determine whether he clears the barrier.

(iv) Sketch the graph.

Solutions

1. $2x^2 - 7x + 1 = 2\left(x - \frac{7}{4}\right)^2 - \frac{41}{8}$.

2. $-4x^2 + 12x - 5 = -4\left(x - \frac{3}{2}\right)^2 + 4$.

3. $\frac{3}{2}x^2 - 5x + \frac{7}{3} = \frac{3}{2}\left(x - \frac{5}{3}\right)^2 - \frac{11}{6}$.

4. $x^2 - \frac{7}{3}x + \frac{5}{9} = \left(x - \frac{7}{6}\right)^2 - \frac{29}{36}$.

5. Turning point $(-3, 5)$ gives $y = a(x+3)^2 + 5$. Using $(1, -3)$: $-3 = 16a + 5 \Rightarrow a = -\frac{1}{2}$. So $y = -\frac{1}{2}(x+3)^2 + 5$. Expanding: $y = -\frac{1}{2}x^2 - 3x + \frac{1}{2}$.

6. $y = 4\left(x - \frac{3}{2}\right)^2 - 4$. Turning point $\left(\frac{3}{2}, -4\right)$, axis $x = \frac{3}{2}$, y-intercept $(0, 5)$, x-intercepts $x = \frac{1}{2}, \frac{5}{2}$.

7. $y = -2(x+1)^2 + 5$. Turning point $(-1, 5)$, axis $x = -1$, y-intercept $(0, 3)$, x-intercepts $x = -1 \pm \frac{\sqrt{10}}{2}$.

8. x-intercepts $(-2, 0), (4, 0)$; y-intercept $(0, -8)$; axis $x = 1$; turning point $(1, -9)$.

9. $y = a(x+1)(x-5)$ and $y(0) = -10$ gives $-5a = -10 \Rightarrow a = 2$. So $y = 2(x+1)(x-5) = 2x^2 - 8x - 10$. Axis $x = 2$, turning point $(2, -18)$.

10. Vertex at $(1, 4)$ gives $y = a(x-1)^2 + 4$ and $y(3) = 0$ gives $4a + 4 = 0 \Rightarrow a = -1$. So $y = -(x-1)^2 + 4 = -x^2 + 2x + 3$. x-intercepts $x = -1, 3$; y-intercept $(0, 3)$; turning point $(1, 4)$.

11. $f(x) = -2\left(x - \frac{5}{2}\right)^2 + \frac{11}{2}$. Maximum value $\frac{11}{2}$ at $x = \frac{5}{2}$. Solve $f(x) = 1$: $-2\left(x - \frac{5}{2}\right)^2 + \frac{11}{2} = 1 \Rightarrow \left(x - \frac{5}{2}\right)^2 = \frac{9}{4} \Rightarrow x = 1$ or 4.

12. $y = (x-3)^2 + 4$. Minimum value 4 at $x = 3$. For $y = 17$: $(x-3)^2 = 13 \Rightarrow x = 3 \pm \sqrt{13}$.

13. $y = -\left(x - \frac{a}{2}\right)^2 + \frac{a^2}{4} + 4$. Maximum value is $\frac{a^2}{4} + 4 = 9 \Rightarrow a^2 = 20$. So $a = \pm 2\sqrt{5}$. Turning point is $\left(\frac{a}{2}, 9\right)$, i.e. $(\sqrt{5}, 9)$ or $(-\sqrt{5}, 9)$.

14. Axis $x = 3$ gives $-\frac{p}{4} = 3 \Rightarrow p = -12$. Using $(0, 6)$ gives $q = 6$. Turning point at $x = 3$: $y = 2(3)^2 - 12(3) + 6 = -12$, so $(3, -12)$. x-intercepts: $2x^2 - 12x + 6 = 0 \Rightarrow x^2 - 6x + 3 = 0 \Rightarrow x = 3 \pm \sqrt{6}$.

15. If roots are r, s, then $(r-s)^2 = (r+s)^2 - 4rs = (p+1)^2 - 4p = (p-1)^2 = 9$. So $p = 4$ or $p = -2$.

16. Vertex at $(5, 8)$ gives $y = a(x-5)^2 + 8$. Using $y(1) = 0$: $16a + 8 = 0 \Rightarrow a = -\frac{1}{2}$. So $y = -\frac{1}{2}(x-5)^2 + 8$. $y(7) = 6$; the firework does not clear the wall (it meets it exactly at 6 m). For $y = 6$: $-\frac{1}{2}(x-5)^2 + 8 = 6 \Rightarrow (x-5)^2 = 4 \Rightarrow x = 3$ or 7.

17. $y(0) = \frac{10000}{500} + 6 = 26$, so each tower is 26 m high. $y(40) = \frac{3600}{500} + 6 = \frac{66}{5}$. For a 12 m truck: $12 = \frac{1}{500}(x-100)^2 + 6 \Rightarrow (x-100)^2 = 3000$, so $x = 100 \pm 10\sqrt{30}$ (approximately 45.23 or 154.77).

18. Fencing gives $L + 4x = 200 \Rightarrow L = 200 - 4x$. So $A = xL = x(200 - 4x) = 200x - 4x^2$. Maximum at $x = \frac{200}{8} = 25$ and then $L = 100$. Maximum area $= 25 \times 100 = 2500$ m^2.

19. Vertex at $x = \frac{-180}{2(-2)} = 45$. Maximum profit $P(45) = -2(45)^2 + 180(45) - 2000 = 2050$. Break-even: $-2x^2 + 180x - 2000 = 0 \Rightarrow x = 45 \pm 5\sqrt{41}$. For $P(x) = 1600$: $-2x^2 + 180x - 2000 = 1600 \Rightarrow x^2 - 90x + 1800 = 0 \Rightarrow x = 30$ or 60.

20. $c = 2$. From $(10, 12)$: $100a + 10b + 2 = 12 \Rightarrow 10a + b = 1$. From $(30, 0)$: $900a + 30b + 2 = 0 \Rightarrow 900a + 30b = -2$. Solving gives $a = -\frac{4}{75}$ and $b = \frac{23}{15}$. So $h(x) = -\frac{4}{75}x^2 + \frac{23}{15}x + 2$. Maximum at $x = -\frac{b}{2a} = \frac{115}{8}$ with height $h\left(\frac{115}{8}\right) = \frac{625}{48}$. At $x = 12$, $h(12) = \frac{318}{25} < 13$, so he does not clear the barrier.