Quadratic Functions — Assignment

Instructions. Answer all questions. Where a sketch is requested, sketch the graph clearly and label all key features (intercepts, turning point, and line of symmetry where appropriate). No sketches are provided in the solutions.

Section 1: Completing the Square (Part 1a–1b)

Q1. Express $x^2 - 14x + 10$ in the form $(x + a)^2 + b$.

Q2. Express $x^2 + 3x + 9$ in the form $(x + a)^2 + b$.

Q3. Express $x^2 + \frac{5}{2}x - 7$ in the form $(x + a)^2 + b$.

Q4. Express $x^2 + 8x + 1$ in the form $(x + a)^2 + b$.

Q5. Express $x^2 - 9x + 2$ in the form $(x + a)^2 + b$.

Q6. Express $5x^2 - 20x + 7$ in the form $a(x + b)^2 + c$.

Q7. Express $-3x^2 + 6x + 1$ in the form $a(x + b)^2 + c$.

Q8. Express $\frac{1}{2}x^2 - 3x + 4$ in the form $a(x + b)^2 + c$.

Q9. Express $6x^2 + 5x - 1$ in the form $a(x + b)^2 + c$.

Q10. Express $-\frac{2}{3}x^2 + 4x - 5$ in the form $a(x + b)^2 + c$.

Section 2: Sketching Quadratic Graphs (Part 2a–2b)

Q11. (i) Express $y = 3x^2 - 12x + 7$ in the form $a(x - h)^2 + k$.

(ii) Hence, sketch the graph and state the turning point, the x-intercepts (if any), the y-intercept, and the line of symmetry.

Q12. (i) Express $y = -2x^2 + 8x + 1$ in the form $a(x - h)^2 + k$.

(ii) Hence, sketch the graph and state the turning point, the x-intercepts (if any), the y-intercept, and the line of symmetry.

Q13. (i) Express $y = x^2 + 4x + 6$ in the form $a(x - h)^2 + k$.

(ii) Hence, sketch the graph and state the number of x-intercepts.

Q14. (i) Express $f(x) = -3x^2 - 6x + 12$ in the form $a(x - h)^2 + k$.

(ii) Hence, state the maximum value of $f(x)$ and the value of x at which it occurs.

(iii) Sketch the graph and label the key features.

Q15. (i) Express $y = 2x^2 - 4x - 6$ in the form $a(x-h)^2 + k$.

(ii) Hence, solve $2x^2 - 4x - 6 \geq 0$.

(iii) Sketch the graph and label the key features.

Q16. Sketch the graph of $y = (x-5)(x+1)$ and state the x-intercepts, the y-intercept, the turning point, and the line of symmetry.

Q17. Sketch the graph of $y = -2(x-3)(x+4)$ and state the x-intercepts, the y-intercept, the turning point, and the line of symmetry.

Q18. Sketch the graph of $y = 3(2x-1)(x+2)$ and state the x-intercepts, the y-intercept, the turning point, and the line of symmetry.

Q19. Sketch the graph of $y = (x-2)(x-2)$ and state the x-intercepts (including multiplicity), the y-intercept, the turning point, and the line of symmetry.

Q20. Sketch the graph of $y = (x+3)(5-2x)$ and state the x-intercepts, the y-intercept, the turning point, and the line of symmetry.

Section 3: Maximum/Minimum Values and Inequalities (Part 3)

Q21. Given $f(x) = 2x^2 - 8x + 11$, find the smallest possible value of $f(x)$ and state the value of x at which it occurs.

Q22. Given $y = -3x^2 + 12x - 7$, find the range of possible values of y.

Q23. Show that for all real x,

$$5x^2 - 20x + 18 \geq -2,$$

and state when equality holds.

Q24. Show that for all real x,

$$-2x^2 + 4x - 9 \leq -7,$$

and state when equality holds.

Q25. Explain why $x^2 + 6x + 10$ is always positive for all real x, and state its minimum value.

Q26. Explain why $-4x^2 - 4x - 2$ is always negative for all real x, and state its maximum value.

Q27. Find the maximum value of $y = -2x^2 + 7x - 3$ and the value of x at which it occurs.

Q28. Given $y = 3x^2 - 12x + k$, the minimum value of y is 5. Find k, and hence state the range of y.

Q29. Find the least value of $x^2 - 4x + 7$. Hence find the range of

$$\frac{1}{x^2 - 4x + 7}.$$

Q30. Let $g(x) = 2x^2 - 5x + 4$.

(i) Find the range of $g(x)$.

(ii) Hence solve $2x^2 - 5x + 4 \leq 1$.

Section 4: Applications of Quadratic Functions (A Maths Chap 1 Section 4)

Q31. A cable between two towers has height above the roadway

$$y = \frac{1}{900}(x - 150)^2 + 9, \qquad 0 \leq x \leq 300,$$

where x (m) is the horizontal distance from the left tower.

(i) Find the height of each tower above the roadway.

(ii) Find the distance between the towers.

(iii) Find the possible distances from the left tower where the cable is 20 m above the roadway.

Q32. Two towers are 18 m above the roadway and 90 m apart. The cable between them just touches the roadway halfway between the towers. Let x be the horizontal distance (m) from the left tower and y the height (m) above the roadway.

(i) Find a quadratic function for y in terms of x.

(ii) Find the cable's height above the roadway at a point 15 m from the left tower.

Q33. A bridge cable is supported by vertical wires. The two end wires are 35 m long and are 180 m apart. The shortest wire, at the midpoint, is 8 m long. Let x be the horizontal distance (m) from the left end wire and y the wire length (m).

(i) Find y in the form $y = a(x - h)^2 + k$.

(ii) Find the wire length when $x = 50$.

Q34. A skateboard ramp is modelled by

$$y = 0.5x^2 - 4x + 9,$$

where x is horizontal distance (m) and y is height above the ground (m).

(i) Find the y-intercept and the turning point.

(ii) Find the minimum height of the ramp above the ground.

(iii) For safety, the ramp height is limited to 3 m. Find the maximum possible width of the curved part (the distance between the two points where $y = 3$).

Q35. A flare is launched from a platform 2 m above ground. Its height (m) is modelled by

$$y = -\frac{1}{25}x^2 + \frac{8}{5}x + 2,$$

where x is the horizontal distance (m).

(i) Find the maximum height and where it occurs.

(ii) Find the horizontal distance from the launch point when the flare hits the ground.

(iii) A mast at $x = 30$ m is 15 m tall. Determine whether the flare clears the mast.

Q36. Two shells are fired from ground level at the origin. Their paths are modelled (in metres) by

$$\text{Shell 1: } y = -\frac{1}{200}x^2 + 3x, \qquad \text{Shell 2: } y = -\frac{1}{500}(x - 350)^2 + 245.$$

(i) Assuming Shell 1 hits an enemy frigate at sea level, how far from the origin is the frigate?

(ii) Find the maximum height of Shell 1.

(iii) Determine whether Shell 2 will hit the same frigate. Justify your answer.

(iv) What is the horizontal distance from the origin to where Shell 2 lands?

Q37. A wire of length 60 cm is bent into a rectangle. The width is x cm.

(i) Express the area A in terms of x.

(ii) Find the maximum possible area.

(iii) Show that this maximum occurs only when the rectangle is a square.

Q38. The height (m) of a ball at time t seconds is

$$y = -3t^2 + 18t + c,$$

where c is a constant.

(i) Use the discriminant to find the range of c if the ball does not reach 40 m.

(ii) If $c = 10$, write y in the form $a(t - h)^2 + k$ and state the maximum point.

(iii) Explain how your result in (ii) is consistent with the condition found in (i).

Q39. Three water jets are modelled by

$$A: \; y = -0.25x^2 + 4, \quad B: \; y = -0.16x^2 + 3.2, \quad C: \; y = -0.10x^2 + 3.6,$$

where x is horizontal distance (m) and y is height (m).

(i) Which jet sends water the farthest horizontally (largest positive x-intercept)?

(ii) Which jet sends water the highest?

(iii) Which jet has the steepest curve (largest $|a|$ value)? Give a brief reason.

(iv) A fourth jet is modelled by $D: \ y = -0.2x^2 + 2x + k$. Find the range of k such that the maximum height is at least 10 m.

Q40. A soccer ball is kicked from ground level. Its height is

$$h(t) = 22t - 5t^2,$$

where t is in seconds and h in metres.

(i) Express $h(t)$ in the form $a(t - p)(t - q)$.

(ii) Find the maximum height and the time when it occurs.

(iii) Find the time when the ball returns to the ground.

(iv) Find the time when the ball is at height 18 m on the way down.

Solutions

Q1. $(x-7)^2-39$.

Q2. $\left(x+\frac{3}{2}\right)^2+\frac{27}{4}$.

Q3. $\left(x+\frac{5}{4}\right)^2-\frac{137}{16}$.

Q4. $(x+4)^2-15$.

Q5. $\left(x-\frac{9}{2}\right)^2-\frac{73}{4}$.

Q6. $5(x-2)^2-13$.

Q7. $-3(x-1)^2+4$.

Q8. $\frac{1}{2}(x-3)^2-\frac{1}{2}$.

Q9. $6\left(x+\frac{5}{12}\right)^2-\frac{49}{24}$.

Q10. $-\frac{2}{3}(x-3)^2+1$.

Q11. $y=3(x-2)^2-5$. Turning point $(2,-5)$, line of symmetry $x=2$, y-intercept $(0,7)$, x-intercepts $\left(2\pm\frac{\sqrt{15}}{3},0\right)$.

Q12. $y=-2(x-2)^2+9$. Turning point $(2,9)$, line of symmetry $x=2$, y-intercept $(0,1)$, x-intercepts $\left(2\pm\frac{3\sqrt{2}}{2},0\right)$.

Q13. $y=(x+2)^2+2$. Turning point $(-2,2)$, line of symmetry $x=-2$, y-intercept $(0,6)$, no x-intercepts.

Q14. $f(x)=-3(x+1)^2+15$. Maximum 15 at $x=-1$. Turning point $(-1,15)$, line of symmetry $x=-1$, y-intercept $(0,12)$, x-intercepts $\left(-1\pm\sqrt{5},0\right)$.

Q15. $y=2(x-1)^2-8$. x-intercepts $(-1,0),(3,0)$; y-intercept $(0,-6)$. Hence $2x^2-4x-6\geq 0$ for $x\leq -1$ or $x\geq 3$. Turning point $(1,-8)$, line of symmetry $x=1$.

Q16. x-intercepts $(-1,0),(5,0)$; y-intercept $(0,-5)$; line of symmetry $x=2$; turning point $(2,-9)$ (minimum).

Q17. x-intercepts $(-4,0),(3,0)$; y-intercept $(0,24)$; line of symmetry $x=-\frac{1}{2}$; turning point $\left(-\frac{1}{2},\frac{49}{2}\right)$ (maximum).

Q18. x-intercepts $(-2,0),\left(\frac{1}{2},0\right)$; y-intercept $(0,-6)$; line of symmetry $x=-\frac{3}{4}$; turning point $\left(-\frac{3}{4},-\frac{75}{8}\right)$ (minimum).

Q19. $y=(x-2)^2$. x-intercept $(2,0)$ (double root); y-intercept $(0,4)$; line of symmetry $x=2$; turning point $(2,0)$ (minimum).

Q20. x-intercepts $(-3,0),\left(\frac{5}{2},0\right)$; y-intercept $(0,15)$; line of symmetry $x=-\frac{1}{4}$; turning point $\left(-\frac{1}{4},\frac{121}{8}\right)$ (maximum).

Q21. $f(x)=2(x-2)^2+3$. Minimum value 3 at $x=2$.

Q22. $y=-3(x-2)^2+5$. Range: $y\leq 5$ (maximum 5 at $x=2$).

Q23. $5x^2-20x+18=5(x-2)^2-2\geq -2$; equality at $x=2$.

Q24. $-2x^2+4x-9=-2(x-1)^2-7\leq -7$; equality at $x=1$.

Q25. $x^2+6x+10=(x+3)^2+1\geq 1>0$; minimum value 1.

Q26. $-4x^2-4x-2=-4\left(x+\frac{1}{2}\right)^2-1<0$; maximum value -1 at $x=-\frac{1}{2}$.

Q27. $y=-2\left(x-\frac{7}{4}\right)^2+\frac{25}{8}$. Maximum $\frac{25}{8}$ at $x=\frac{7}{4}$.

Q28. $y=3(x-2)^2+(k-12)$. Minimum $k-12=5\Rightarrow k=17$. Range: $y\geq 5$.

Q29. $x^2-4x+7=(x-2)^2+3\geq 3$. Least value 3. So $0<\frac{1}{x^2-4x+7}\leq\frac{1}{3}$.

Q30. $g(x)=2\left(x-\frac{5}{4}\right)^2+\frac{7}{8}$.

(i) Range: $g(x)\geq\frac{7}{8}$.

(ii) $2\left(x-\frac{5}{4}\right)^2+\frac{7}{8}\leq 1\Rightarrow\left(x-\frac{5}{4}\right)^2\leq\frac{1}{16}\Rightarrow 1\leq x\leq\frac{3}{2}$.

Q31. (i) $y(0) = 34$ and $y(300) = 34$, so each tower is 34 m high.

(ii) Distance between towers: 300 m.

(iii) $20 = \frac{1}{900}(x-150)^2 + 9 \Rightarrow (x-150)^2 = 9900 \Rightarrow x = 150 \pm 30\sqrt{11} \approx 50.5$ or 249.5.

Q32. (i) Vertex $(45, 0)$: $y = a(x-45)^2$. Using $(0, 18)$ gives $18 = a(45)^2 \Rightarrow a = \frac{2}{225}$, so $y = \frac{2}{225}(x-45)^2$.

(ii) $y(15) = \frac{2}{225}(15-45)^2 = 8$ m.

Q33. (i) Vertex $(90, 8)$: $y = a(x-90)^2+8$. Using $(0, 35)$ gives $35 = 8100a+8 \Rightarrow a = \frac{1}{300}$, so $y = \frac{1}{300}(x-90)^2 + 8$.

(ii) $y(50) = \frac{1}{300}(50-90)^2 + 8 = \frac{40}{3}$ m.

Q34. (i) y-intercept $(0, 9)$; turning point at $x = 4$, $y(4) = 1$, so $(4, 1)$.

(ii) Minimum height: 1 m.

(iii) $0.5x^2 - 4x + 9 = 3 \Rightarrow x^2 - 8x + 12 = 0 \Rightarrow x = 2, 6$. Width $= 6 - 2 = 4$ m.

Q35. (i) Vertex at $x = 20$, maximum height $y(20) = 18$ m.

(ii) $0 = -\frac{1}{25}x^2 + \frac{8}{5}x + 2 \Rightarrow x = 20 \pm 15\sqrt{2}$; physical root $x = 20 + 15\sqrt{2} \approx 41.2$ m.

(iii) $y(30) = 14 < 15$, so it does not clear the mast.

Q36. (i) $0 = -\frac{1}{200}x^2 + 3x \Rightarrow x = 600$ m (non-zero root).

(ii) Maximum at $x = 300$: $y(300) = 450$ m.

(iii) At $x = 600$, Shell 2 has $y = -\frac{1}{500}(250)^2 + 245 = 120 > 0$, so it does not hit.

(iv) Shell 2 lands when $0 = -\frac{1}{500}(x-350)^2 + 245 \Rightarrow (x-350)^2 = 350^2 \Rightarrow x = 700$ m.

Q37. (i) Perimeter 60: length $= 30 - x$. So $A = x(30-x) = -x^2 + 30x$.

(ii) Maximum at $x = 15$: $A_{\max} = 225$ cm^2.

(iii) When $x = 15$, length $= 15$, so the rectangle is a square; maximum is unique at the vertex.

Q38. (i) Not reach 40 means $-3t^2 + 18t + (c-40) = 0$ has no real root: $\Delta = 12c - 156 < 0 \Rightarrow c < 13$.

(ii) If $c = 10$: $y = -3(t-3)^2+37$, maximum point $(3, 37)$.

(iii) Maximum height is $27 + c$; requiring $27 + c < 40$ gives $c < 13$ (consistent with (i)).

Q39. (i) Positive x-intercepts: $A : 4$, $B : \sqrt{20}$, $C : 6$; farthest is C.

(ii) Highest is A (height 4 at $x = 0$).

(iii) Narrowest is A (largest $|a|$).

(iv) $-0.2x^2+2x+k = -0.2(x-5)^2+(5+k)$, so max height $5 + k \geq 10 \Rightarrow k \geq 5$.

Q40. (i) $h(t) = 22t - 5t^2 = -5t\left(t - \frac{22}{5}\right)$.

(ii) Vertex at $t = \frac{11}{5}$; maximum height $h\left(\frac{11}{5}\right) = \frac{121}{5}$ m.

(iii) Returns to ground at $t = \frac{22}{5}$ s.

(iv) $22t - 5t^2 = 18 \Rightarrow t = \frac{11 \pm \sqrt{31}}{5}$; on the way down: $t = \frac{11 + \sqrt{31}}{5} \approx 3.31$ s.