NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1101R Linear Algebra I
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Question 1

(a) (i)Since the rows are linearly independent, $\{(1,0,1,0,1),(0,0,1,0,1),(0,0,0,1,1)\}$ is a basis for the rowspace of $\boldsymbol{A}$.

(ii) From a(i), we know that the rank of $\boldsymbol{A}$ is 3, so the dimension for the column space is 3. But the column space of m x n matrix is a subspace of m-dimensional Euclidean space, and a subspace of $\mathbb{R}^3$ with dimension 3 is the whole space, hence column space of $\boldsymbol{A} = \mathbb{R}^3$, with basis vectors $\{\boldsymbol{e1}, \boldsymbol{e2}, \boldsymbol{e3}\}$.

(iii)$\{(0,1,0,0,0),(0,0,0,0,1)\}$.

(iv)$\{(0,1,0,0,0),(0,0,1,1,-1)\}$. By the rank-nullity theorem, nullspace($\boldsymbol{R}$) has dimension 2, thus there are two basis vectors. The two vectors (0,1,0,0,0),(0,0,1,1,-1) are linearly independent. Since $\boldsymbol{R}$(0,1,0,0,0)^T $= \mathbf{0}$ and $\boldsymbol{R}$(0,0,1,1,-1)^T $= \mathbf{0}$, (0,1,0,0,0) and (0,0,1,1,-1) are basis vectors of the nullspace of $\boldsymbol{R}$.

(v) $\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} + \lambda \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + \mu \begin{pmatrix}0\\0\\1\\1\\-1\end{pmatrix}, \lambda, \mu \in \mathbb{R}.$

(b) $\boldsymbol{B} = \begin{pmatrix}1&1&1\\2&2&2\\3&3&3\end{pmatrix},$

$\boldsymbol{C} = \begin{pmatrix}1&1&1\\2&2&2\\3&3&4\end{pmatrix}.$

Using Gaussian elimination, $\{(1,1,1)\}$ is a basis for the rowspace of $\boldsymbol{B}$. Thus, rank($\boldsymbol{B}$)=1. Similarly, $\{(1,1,1),(0,0,1)\}$ is a basis for the rowspace of $\boldsymbol{C}$. Thus, rank($\boldsymbol{C}$)=2.

(c) We show that rank($\boldsymbol{PA}$) $\leq$ rank($\boldsymbol{A}$), and rank($\boldsymbol{PA}$) $\geq$ rank($\boldsymbol{A}$) .

1. rank($\boldsymbol{PA}$) $\leq$ rank($\boldsymbol{A}$).
Proof: Given any vector $\boldsymbol{x}$ such that $\boldsymbol{x} \neq \mathbf{0}$ and $\boldsymbol{Ax} = \mathbf{0}$, $\mathbf{PAx} = \mathbf{P}(\mathbf{Ax}) = \mathbf{P}(\mathbf{0}) = \mathbf{0}$. Thus, nullspace($\mathbf{A}$) $\subseteq$ nullspace($\mathbf{PA}$) and nullity($\mathbf{A}$) $\leq$ nullity($\mathbf{PA}$). Since both $\mathbf{PA}$ and $\mathbf{A}$ have p columns, by the rank-nullity theorem (Theorem 4.3.3), rank($\boldsymbol{PA}$) $\leq$ rank($\boldsymbol{A}$).

2. rank($\boldsymbol{PA}$) $\geq$ rank($\boldsymbol{A}$).
Proof: Assume to the contrary that rank($\boldsymbol{PA}$) $<$ rank($\boldsymbol{A}$). Then by the rank-nullity theorem, nullity($\mathbf{A}$) $<$ nullity($\mathbf{PA}$). Thus, there exists a vector $\mathbf{v}$ such that $\mathbf{Av} = \mathbf{b}, \mathbf{b} \neq \mathbf{0}$ but $\mathbf{PAv} = \mathbf{Pb} = 0$. This means that some linear combination of the columns in $\mathbf{P}$ gives $\mathbf{0}$, and that the columns are linearly dependent. However, rank($\mathbf{P}$) $= n$, implying that all of the n columns in $\mathbf{P}$ are linearly independent, which is a contradiction. Thus, rank($\boldsymbol{PA}$) $\geq$ rank($\boldsymbol{A}$).

Since rank($\boldsymbol{PA}$) $\leq$ rank($\boldsymbol{A}$), and rank($\boldsymbol{PA}$) $\geq$ rank($\boldsymbol{A}$),rank($\boldsymbol{PA}$) $=$ rank($\boldsymbol{A}$).

Question 2

(a) (i) $\det(\mathbf{A}-\lambda\mathbf{I}) = (1-\lambda)^3 - (1-\lambda) = 1+3\lambda^2-3\lambda-\lambda^3+\lambda-1 = 3\lambda^2-2\lambda-\lambda^3$.
$3\lambda^2-2\lambda-\lambda^3 = 0 \Rightarrow$the eigenvalues $\lambda = 0, 1, 2$.

(ii) For $\lambda = 0$, $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x} = 0$

$$\Leftrightarrow \begin{pmatrix} 0-1 & 0 & 1 \\ 0 & 0-1 & 0 \\ 1 & 0 & 0-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, t \in \mathbb{R}.$$

Hence the basis for the eigenspace of $\mathbf{A}$ associated with the eigenvalue 0 is $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

For $\lambda = 1$, $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x} = 0$

$$\Leftrightarrow \begin{pmatrix} 1-1 & 0 & 1 \\ 0 & 1-1 & 0 \\ 1 & 0 & 1-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, t \in \mathbb{R}.$$

Hence the basis for the eigenspace of $\mathbf{A}$ associated with the eigenvalue 1 is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

For $\lambda = 2$, $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x} = 0$

$$\Leftrightarrow \begin{pmatrix} 2-1 & 0 & 1 \\ 0 & 2-1 & 0 \\ 1 & 0 & 2-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, t \in \mathbb{R}.$$

Hence the basis for the eigenspace of $\mathbf{A}$ associated with the eigenvalue 2 is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

(iii)$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

$\mathbf{P}$ can be obtained by noticing that $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are orthogonal. To make $\mathbf{P}$ an orthogonal matrix, the columns must be orthonormal, thus we need to make all the columns have norm 1 by using a scaling factor.

(b) Since $\mathbf{B}$ is a triangular matrix, its eigenvalues are a, a, b.

For $\lambda = a$, $(\lambda\mathbf{I}-\mathbf{B})\mathbf{x} = 0$

$$\Leftrightarrow \begin{pmatrix} a-a & 1 & 0 \\ 0 & a-b & 1 \\ 0 & 0 & a-a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, t \in \mathbb{R}.$$

Hence the basis for the eigenspace of $\mathbf{B}$ associated with the eigenvalue a is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

For $\lambda = b$, $(\lambda\mathbf{I} - \mathbf{B})\mathbf{x} = 0$

$$\Leftrightarrow \begin{pmatrix} b-a & 1 & 0 \\ 0 & b-b & 1 \\ 0 & 0 & b-a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ a-b \\ 0 \end{pmatrix}, t \in \mathbb{R}.$$

Hence the basis for the eigenspace of $\mathbf{B}$ associated with the eigenvalue b is $\begin{pmatrix} 1 \\ a-b \\ 0 \end{pmatrix}$.

There are at most 2 linearly independent eigenvectors, but $\mathbf{B}$ is of order 3, hence $\mathbf{B}$ is not diagonalizable.

(c) Consider an orthogonal matrix $\mathbf{Q}$ with eigenvalue(s) λ and corresponding eigenvector(s) $\mathbf{x}$. Then $\mathbf{Qx} = \lambda\mathbf{x}$.
Since $\mathbf{Q}$ is orthogonal,
$\mathbf{Ix} = \mathbf{Q^TQx} = \mathbf{Q^T}(\lambda\mathbf{x}) = \lambda(\mathbf{Q^Tx}) = \lambda^2\mathbf{x}$.
$\Rightarrow \lambda^2 = 1$
$\Rightarrow \lambda = \pm 1$.

Question 3

(a) (i) The reduced row echelon form of $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, whose rows are the vectors in $\mathbf{U}$, is $\mathbf{I}$, thus the vectors in $\mathbf{U}$ are linearly independent.

(ii)$\mathbf{V}$ is the set of all possible vectors of the form $\begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $a, b, c \in \mathbb{R}$ or the column space of $\begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, which is span($\mathbf{S}$), where $\mathbf{S} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

Since $\mathbf{V}$ =span($\mathbf{S}$) and every vector in $\mathbf{S} \in \mathbb{R}^4$, $\mathbf{V}$ is a subspace of $\mathbb{R}^4$.

(iii) $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \in \mathbf{W}$ but $\begin{pmatrix} 2 \\ 2 \\ 4 \\ 2 \end{pmatrix}$ is not. If $\begin{pmatrix} 2 \\ 2 \\ 4 \\ 2 \end{pmatrix}$ were in $\mathbf{W}$, we would obtain an inconsistent system of linear equations, which is a contradiction.

(b) (i) $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

(ii)$\frac{1}{5}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}$.

(iii)$(\mathbf{w})_S = \begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}3\\ -1\end{pmatrix}$.

(c) 2, 3, or 4.

As in Exercise 3.10, we define the subspace $U + V = \{\mathbf{u} + \mathbf{v} | \mathbf{u} \in U \text{ and } \mathbf{v} \in V.\}$

If $W = U + V$, it is the smallest subspace of $\mathbb{R}^5$ that contains U and V.

The subspace W contains the subspaces U and V.

Proof: We show that W contains U. The proof that W contains V is similar. Let $\mathbf{u}$ be a vector in U. Then $\mathbf{u} = \mathbf{u} + \mathbf{0}$, where $\mathbf{u} \in U$, $\mathbf{0} \in V$, so $\mathbf{u} \in W$. Thus, every vector in U is contained in W.

The subspace W is the smallest subspace of $\mathbb{R}^5$ that contains U and V because if W^* is another subspace which contains U and V, then $W \subseteq W^*$. Let $\mathbf{w} \in W$. Then $\mathbf{w} = \mathbf{u}+\mathbf{v}, \mathbf{u} \in U \subseteq W^*, \mathbf{v} \in V \subseteq W^*$. Since W^* is a subspace, $\mathbf{w} \in W^*$. Thus, $W \subseteq W^*$.

From Exercise 3.36, $\dim(\mathbf{U}+\mathbf{V}) = \dim(\mathbf{U}) + \dim(\mathbf{V}) - \dim(\mathbf{U}\cap\mathbf{V}) \leq 3+3-2 = 4$.
$\dim(\mathbf{U}+\mathbf{V}) = \dim(\mathbf{U}) + \dim(\mathbf{V}) - \dim(\mathbf{U}\cap\mathbf{V}) \geq 2+2-2 = 2$.

Example where $\dim(\mathbf{W})=2$: $\mathbf{u_1} = (1,0,0,0,0)^T$, $\mathbf{u_2} = (0,1,0,0,0)^T$, $\mathbf{u_3} = (2,0,0,0,0)^T$,$\mathbf{v_1} = (1,0,0,0,0)^T$, $\mathbf{v_2} = (0,1,0,0,0)^T$, $\mathbf{v_3} = (2,0,0,0,0)^T$.

Example where $\dim(\mathbf{W})=3$: $\mathbf{u_1} = (1,0,0,0,0)^T$, $\mathbf{u_2} = (0,1,0,0,0)^T$, $\mathbf{u_3} = (0,0,1,0,0)^T$, $\mathbf{v_1} = (1,0,0,0,0)^T$, $\mathbf{v_2} = (0,1,0,0,0)^T$, $\mathbf{v_3} = (0,0,1,0,0)^T$.

Example where $\dim(\mathbf{W})=4$: $\mathbf{u_1} = (1,0,0,0,0)^T$, $\mathbf{u_2} = (0,1,0,0,0)^T$, $\mathbf{u_3} = (0,0,1,0,0)^T$, $\mathbf{v_1} = (1,0,0,0,0)^T$, $\mathbf{v_2} = (0,1,0,0,0)^T$, $\mathbf{v_3} = (0,0,0,1,0)^T$.

Question 4

(a) (i)A plane.

(ii)Denote the basis vectors of $\mathbf{V}$ by $\mathbf{v_1}$ and $\mathbf{v_2}$. Using the Gram-Schmidt process,

$$\mathbf{v_1} = \begin{pmatrix}1\\ 1\\ 0\end{pmatrix}.$$

$$\mathbf{v_2} = \begin{pmatrix}1\\ 3\\ 2\end{pmatrix} - 2\begin{pmatrix}1\\ 1\\ 0\end{pmatrix} = \begin{pmatrix}-1\\ 1\\ 2\end{pmatrix}.$$

(iii)The projection of $\mathbf{w}$ onto $\mathbf{V}$ is $1\begin{pmatrix}1\\ 1\\ 0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}-1\\ 1\\ 2\end{pmatrix} = \begin{pmatrix}2/3\\ 4/3\\ 2/3\end{pmatrix}$.

(iv)The answer is $\begin{pmatrix}1/3\\ 5/3\\ 1/3\end{pmatrix}$. To obtain it, we find the reflection of $\mathbf{w}$ about the space spanned by the basis vectors of $\mathbf{v}$.

$$\begin{pmatrix}1\\ 1\\ 1\end{pmatrix} - \begin{pmatrix}2/3\\ 4/3\\ 2/3\end{pmatrix} = \begin{pmatrix}1/3\\ -1/3\\ 1/3\end{pmatrix},$$

$$\begin{pmatrix}2/3\\ 4/3\\ 2/3\end{pmatrix} - \begin{pmatrix}1/3\\ -1/3\\ 1/3\end{pmatrix} = \begin{pmatrix}1/3\\ 5/3\\ 1/3\end{pmatrix}.$$

(b) A least squares solution $\mathbf{x}$ satisfies the equation $\mathbf{A^T A x} = \mathbf{A^T b}$.
$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.
Simplifying,
$\begin{pmatrix} 3 & 4 \\ 4 & 8 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$.

One value of $\mathbf{x}$ that satisfies this equation is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

(c) False.
Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 0 & 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1/3 \\ 5/3 \\ 1/3 \end{pmatrix}$.
We can verify that $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ax} = \mathbf{c}$ are inconsistent systems. Alternatively, it follows for Q4(a) that both $\mathbf{b}$ and $\mathbf{c}$ do not lie in the column space of $\mathbf{A}$ and hence are inconsistent. Note that $\mathbf{A^T b} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $\mathbf{A^T c} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1/3 \\ 5/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$.
Thus, both the least squares soutions to $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ax} = \mathbf{c}$ are the same and work out to be $\begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$.

Question 5

(a) (i) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(ii)Since $\begin{pmatrix} x+y \\ x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$,

$R(T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(iii) The kernel of T corresponds to all vectors $\mathbf{v}$ that satisfy $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent, $\mathbf{v}$ can only be the zero vector. Hence Ker(T)$= \{\mathbf{0}\}$.

(iv) $(T \circ S) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ corresponds to $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
$= \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
$= \begin{pmatrix} x+2y+z \\ x+y+z \\ y \end{pmatrix}$.

(v) Yes.
Since $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$, we can let the transformation T' correspond to $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(vi) No.
Let $\mathbf{A}$ be the matrix that corresponds to T and $\mathbf{B}$ be the matrix corresponding to Q. The range of $T \circ Q$ is the column space of the matrix $\mathbf{AB}$. Since $\mathbf{AB} = (\mathbf{Ab_1 Ab_2 Ab_3})$, each column of $\mathbf{AB}$ is a combination of the columns of $\mathbf{A}$. Thus, column space $\mathbf{AB} \subseteq$ column space $\mathbf{A}$.
$\text{rref}\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right)$, implying that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is not in the column space of $\mathbf{A}$. Thus, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is not in the column space of $\mathbf{AB}$, so no such linear transformation exists.

(b) Let $\mathbf{A}$ be the matrix that corresponds to the transformation P.
Using the reasoning in part (a), the column space of $\mathbf{A^{m+1}} \subseteq$ the column space of $\mathbf{A^m} \subseteq$ the column space of $\mathbf{A} \subseteq \mathbb{R}^n$. Thus, $\dim(\mathbf{A^{m+1}}) \leq \dim(\mathbf{A^m}) \leq n$.

Let us define a sequence of matrices starting with $\mathbf{A^{m_1}} = \mathbf{A}$.If we assume to the contrary that there is no such integer k, given any $i \in \mathbb{N}, i \geq 2$, there exists m_i such that $\dim(\ \mathbf{A^{m_i}}) < \dim(\mathbf{A^{m_{i-1}}})$. In particular, $\dim(\ \mathbf{A^{m_{n+2}}}) \leq n + 1 - (n+2) < 0$, which is a contradiction since all matrices have a nonnegative dimension.