NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA 3110 Mathematical Analysis II
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Question 1

(a) Since $g \in C^{101}$, $\therefore$ By Taylor Series expansion, we can write (for $x \in (1,2)$),

$$g(x) = g(1) + \frac{g'(1)}{1!}(x-1) + \frac{g''(1)}{2!}(x-1)^2 + ... + \frac{g^{(100)}(1)}{100!}(x-1)^{100} + \frac{g^{(101)}(c)}{101!}(x-1)^{101}$$

$$g(x) - g(1) = \frac{g^{(101)}(c)}{101!}(x-1)^{101}$$

for some $c \in (1, x)$. Similarly, for $x \in (0,1)$,

$$g(x) - g(1) = \frac{g^{(101)}(d)}{101!}(x-1)^{101}$$

for some $d \in (x, 1)$.

Since $g^{(101)}(1) = 2 > 0$ and $g^{(101)}$ is continuous, there exists $\delta > 0$ such that $g^{(101)}(c) > 0$ for all $c \in (1-\delta, 1+\delta)$.

Now, if $x \in (1-\delta, 1)$, then $(x-1)^{101} < 0 \Rightarrow g(x) < g(1)$ and if $x \in (1, 1+\delta)$, then $(x-1)^{101} > 0 \Rightarrow g(x) > g(1)$. Since $g(x) < g(1)$ and $g(x) < g(1)$ occurs in the neighbourhood of 1, hence $g(1)$ is neither a minimum nor a maximum point, in fact it is a point of inflection.

(b) Note that, $f \in C^1$. By letting $k = \frac{a+b}{2}$, then by Talylor's Series expansion at $x = a, b$ we have

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(c)}{2!}(x-a)^2 \\ \Rightarrow f(k) &= f(a) + f'(a)\Big(\frac{b-a}{2}\Big) + \frac{f''(c)}{2}\Big(\frac{b-a}{2}\Big)^2 \text{ and} \qquad (1) \\ f(x) &= f(a) + \frac{f'(b)}{1!}(x-b) + \frac{f''(c)}{2!}(x-b)^2 \\ \Rightarrow f(k) &= f(b) - f'(b)\Big(\frac{b-a}{2}\Big) + \frac{f''(c')}{2}\Big(\frac{b-a}{2}\Big)^2 \qquad (2) \end{aligned}$$

for some $c \in (a, k), c' \in (k, b)$.

$(1) = (2)$:

$$\begin{aligned} f(a) + \frac{f''(c)}{2}\Big(\frac{b-a}{2}\Big)^2 &= f(b) + \frac{f''(c')}{2}\Big(\frac{b-a}{2}\Big)^2 \\ \frac{4}{(b-a)^2}|f(b) - f(a)| &= \Big|\frac{f''(c)}{2} - \frac{f''(c')}{2}\Big| \\ &\le \Big|\frac{f''(c)}{2}\Big| + \Big|\frac{f''(c')}{2}\Big| \\ &\le |f''(C)| \end{aligned}$$

where

$$C = \begin{cases} c, & \text{if } |f''(c)| \geq |f''(c')| \\ c', & \text{if } |f''(c)| < |f''(c')| \end{cases}$$

Question 2

(a) Let $P = \{x_0 = a, x_1, x_2, ..., x_{n-1}, x_n = b\}$ be a partition of $[a, b]$, and define

$$M_i(f, P) = \sup\{f(x) | x \in [x_{i-1}, x_i]\}$$
$$m_i(f, P) = \inf\{f(x) | x \in [x_{i-1}, x_i]\}$$

as per the usual definition. If $u_i \in [x_{i-1}, x_i]$ such that $|h(u_i)| = M_i(|h|, P)$, then $h^2(u_i) = M_i(h^2, P)$. Similarly, if $v_i \in [x_{i-1}, x_i]$ such that $|h(v_i)| = m_i(|h|, P)$, then $h^2(v_i) = m_i(h^2, P)$. Therefore, for any partition P, $(M_i(|h|, P))^2 = M_i(h^2, P)$ and $(m_i(|h|, P))^2 = m_i(h^2, P)$.

Let $\epsilon > 0$ be given. Then there exists a partition P such that

$$\sum_{i=1}^{n} \Big(M_i(h^2, P) - m_i(h^2, P) \Big) \Delta x_i < \frac{\epsilon^2}{4(b-a)}$$

Define a set $A \subseteq \{1, 2, ..., n\}$ such that $i \in A \Leftrightarrow M_i(|h|, P) < \frac{\epsilon}{2(b-a)}$ (that is, A contains precisely the indices i such that $|h(x)| < \frac{\epsilon}{2(b-a)}$ for all $x \in [x_{i-1}, x_i]$). Then,

$$\begin{aligned}
&\sum_{i=1}^{n} \Big(M_i(|h|, P) - m_i(|h|, P) \Big) \Delta x_i \\
&= \sum_{\substack{i=1 \\ i \in A}}^{n} \Big(M_i(|h|, P) - m_i(|h|, P) \Big) \Delta x_i \quad + \quad \sum_{\substack{i=1 \\ i \notin A}}^{n} \Big(M_i(|h|, P) - m_i(|h|, P) \Big) \Delta x_i \\
&= \sum_{\substack{i=1 \\ i \in A}}^{n} \Big(M_i(|h|, P) - m_i(|h|, P) \Big) \Delta x_i \quad + \quad \sum_{\substack{i=1 \\ i \notin A}}^{n} \left(\frac{(M_i(|h|, P))^2 - (m_i(|h|, P))^2}{M_i(|h|, P) + m_i(|h|, P)} \right) \Delta x_i \\
&= \sum_{\substack{i=1 \\ i \in A}}^{n} \Big(M_i(|h|, P) - m_i(|h|, P) \Big) \Delta x_i \quad + \quad \sum_{\substack{i=1 \\ i \notin A}}^{n} \left(\frac{M_i(h^2, P)^2 - m_i(h^2, P)}{M_i(|h|, P) + m_i(|h|, P)} \right) \Delta x_i \\
&< \sum_{\substack{i=1 \\ i \in A}}^{n} \left(\frac{\epsilon}{2(b-a)} - 0 \right) \Delta x_i \quad + \quad \sum_{\substack{i=1 \\ i \notin A}}^{n} \left(\frac{M_i(h^2, P)^2 - m_i(h^2, P)}{\frac{\epsilon}{2(b-a)} + 0} \right) \Delta x_i \\
&= \frac{\epsilon}{2(b-a)} \sum_{\substack{i=1 \\ i \in A}}^{n} \Delta x_i \quad + \quad \frac{2(b-a)}{\epsilon} \sum_{\substack{i=1 \\ i \notin A}}^{n} \Big(M_i(h^2, P)^2 - m_i(h^2, P) \Big) \Delta x_i \\
&< \frac{\epsilon}{2(b-a)} \Big(b - a \Big) \quad + \quad \frac{2(b-a)}{\epsilon} \left(\frac{\epsilon^2}{4(b-a)} \right) \\
&= \frac{\epsilon}{2} + \frac{2\epsilon^2}{4\epsilon} \\
&= \epsilon
\end{aligned}$$

This shows that $|h|$ is Riemann integrable on $[a, b]$.

(b) (i) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{f(x)}{x} \cdot x = \lim_{x\to 0} \frac{f(x)}{x} \cdot \lim_{x\to 0} x = L \cdot 0 = 0$. Since $f(x)$ is continuous at 0, we have $f(0) = 0$.

Now, $g(0) = \int_0^1 f(0 \cdot t)dt = \int_0^1 0dt = 0$.

(ii) Let $s = xt$, then $ds = xdt$, therefore

$$\begin{aligned} g'(x) &= \frac{d}{dx}\int_0^1 f(xt)dt \\ &= \frac{d}{dx}\left[\frac{1}{x}\int_0^x f(s)ds\right] \\ &= \frac{1}{x}f(x) - \frac{1}{x^2}\int_0^x f(s)ds \end{aligned}$$

On the other hand,

$$\begin{aligned} g'(0) &= \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} \\ &= \lim_{x\to 0} \frac{g(x)}{x} \\ &= \lim_{x\to 0} \frac{\int_0^1 f(xt)\mathrm{d}t}{x} \\ &= \lim_{x\to 0} \int_0^1 \frac{f(xt)}{x}\mathrm{d}t \\ &= \lim_{x\to 0} \int_0^1 \frac{f(xt)}{xt} \times t\mathrm{d}t \\ &= \int_0^1 L \times t\mathrm{d}t \\ &= \frac{1}{2}Lt^2|_0^1 \\ &= \frac{L}{2} \end{aligned}$$

(iii)

$$\begin{aligned} \lim_{x\to 0} g'(x) &= \lim_{x\to 0}\left[\frac{1}{x}f(x) - \frac{1}{x^2}\int_0^x f(s)ds\right] \\ &= \lim_{x\to 0}\frac{f(x)}{x} - \lim_{x\to 0}\frac{\int_0^x f(s)ds}{x^2} \\ &= L - \lim_{x\to 0}\frac{f(x)}{2x} \\ &= L - \frac{1}{2}\lim_{x\to 0}\frac{f(x)}{x} \\ &= L - \frac{L}{2} \\ &= \frac{1}{2}L \\ &= g'(0) \end{aligned}$$

where we have used the L'Hopital's Rule. Hence, g' is continuous at 0.

Question 3

(a) Let $\epsilon > 0$ be given. Let $a < \alpha < \beta < b$ and let γ and η be selected such that $a < \gamma < \alpha$ and $\beta < \eta < b$ (to be precise, we can let $\gamma = \frac{a+\alpha}{2}$ and $\eta = \frac{\beta+b}{2}$). Since g' is continuous on $[\gamma, \eta]$, it is uniformly continuous on $[\gamma, \eta]$, that is, $\exists \delta_1 > 0$ such that

$$x, y \in [\gamma, \eta], |x - y| < \delta_1 \quad \Rightarrow \quad \left|g'(x) - g'(y)\right| < \epsilon$$

Let $\delta = \min\{\delta_1, \eta - \beta\}$ and choose $K \in \mathbb{N}$ such that $K > 1/\delta$. Let $x \in [\alpha, \beta]$. Then whenever $n \geq K$, there exists a $c \in \left(x, x + \frac{1}{n}\right)$ such that

$$g'(c) = \frac{g\left(x + \frac{1}{n}\right) - g(x)}{\frac{1}{n}} = g_n(x)$$

by the Mean Value Theorem (c is a function of x and n). We can see that $x < c < x + \frac{1}{n}$, so $|x - c| < \frac{1}{n} \leq \frac{1}{K} < \delta \leq \delta_1$. Also, $\gamma < x < c < x + \frac{1}{n} \leq \beta + \frac{1}{n} < \beta + \delta \leq \beta + (\eta - \beta) = \eta$, so $x, c \in [\gamma, \eta]$. Therefore,

$$x, c \in [\gamma, \eta], |x - c| < \delta_1 \quad \Rightarrow \quad |g_n(x) - g'(x)| = \left|\frac{g\left(x + \frac{1}{n}\right) - g(x)}{\frac{1}{n}} - g'(x)\right| = |g'(c) - g'(x)| < \epsilon$$

Therefore, g_n converges uniformly to g' on $[\alpha, \beta]$. We will get

$$\lim_{n\to\infty} \int_\alpha^\beta g_n(x)\mathrm{d}x = \int_\alpha^\beta \lim_{n\to\infty} g_n(x)\mathrm{d}x = \int_\alpha^\beta g'(x)\mathrm{d}x = g(\beta) - g(\alpha)$$

by the Fundamental Theorem of Calculus.

(b) (i) For all $n \in \mathbb{N}, x \in \mathbb{R}, \left|\frac{1}{n}\sin\left(\frac{x}{n+1}\right)\right| \leq \left|\frac{1}{n} \cdot \frac{x}{n+1}\right| = |x| \cdot \frac{1}{n(n+1)}$ and

$$\sum_{n=1}^{\infty} |x| \cdot \frac{1}{n(n+1)} = |x| \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = |x| \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1} = |x| \lim_{N\to\infty} 1 - \frac{1}{N} = |x|$$

Hence, by Comparison Test, $\sum\limits_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{x}{n+1}\right)$ converges absolutely on $\mathbb{R}$.

For all $n \in \mathbb{N}, x \in [-r, r], r > 0, \left|\frac{1}{n}\sin\left(\frac{x}{n+1}\right)\right| \leq |x| \cdot \frac{1}{n(n+1)} \leq r \cdot \frac{1}{n(n+1)}$ and $\sum\limits_{n=1}^{\infty} r \cdot \frac{1}{n(n+1)} = r$.

Hence, by Weierstrass M-test, $\sum\limits_{n=1}^{\infty} \frac{1}{n}\sin(\frac{x}{n+1})$ converges uniformly on $[-r, r]$.

$$|f(x)| = \left|\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{x}{n+1}\right)\right| \leq \sum_{n=1}^{\infty} \left|\frac{1}{n}\sin\left(\frac{x}{n+1}\right)\right| \leq \sum_{n=1}^{\infty} |x| \cdot \frac{1}{n(n+1)} = |x|$$

(ii) $\sum\limits_{n=1}^{\infty} f_n(0)$ converges to the number 0. For all $n \in \mathbb{N}, x \in \mathbb{R}, |f_n'(x)| = \left|\frac{1}{n(n+1)}\cos\left(\frac{x}{n+1}\right)\right| \leq \frac{1}{n(n+1)}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$, so $\sum\limits_{n=1}^{\infty} f_n'(x)$ converges uniformly on $\mathbb{R}$ by the Weierstrass M-test.

Therefore, f is differentiable on $\mathbb{R}$ and

$$\begin{aligned}
|f'(x)| &= \left|\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=1}^{\infty} f_n(x)\right| \\
&= \left|\sum_{n=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x} f_n(x)\right| \\
&= \left|\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\cos\left(\frac{x}{n+1}\right)\right| \\
&\leq \sum_{n=1}^{\infty}\left|\frac{1}{n(n+1)}\cos\left(\frac{x}{n+1}\right)\right| \\
&\leq \sum_{n=1}^{\infty}\left|\frac{1}{n(n+1)}\right| \\
&= 1
\end{aligned}$$

(c) (i) Choose x_n such that

$$x_n = \begin{cases} \frac{\pi}{2}, & \text{when } n \text{ is even} \\ \frac{-\pi}{2}, & \text{when } n \text{ is odd} \end{cases}$$

Then

$$\begin{aligned}
\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}+\sin x_n} &= \sum_{m=1}^{\infty}\frac{(-1)^{2m+1}}{\sqrt{2m}+\sin x_{2m}} + \frac{(-1)^{(2m+1)+1}}{\sqrt{2m+1}+\sin x_{2m+1}} \\
&= \sum_{m=1}^{\infty}\frac{-1}{\sqrt{2m}+\sin\frac{\pi}{2}} + \frac{1}{\sqrt{2m+1}+\sin\frac{-\pi}{2}} \\
&= \sum_{m=1}^{\infty}\frac{-1}{\sqrt{2m}+1} + \frac{1}{\sqrt{2m+1}-1} \\
&= \sum_{m=1}^{\infty}\frac{2+\sqrt{2m}-\sqrt{2m+1}}{\sqrt{4m^2+2m}+\sqrt{2m+1}-\sqrt{2m}-1}
\end{aligned}$$

Note that for any m, $0 < \sqrt{2m+1}-\sqrt{2m} < 1$. The numerator is therefore between 1 and 2, and the denominator is greater than 0. Therefore each summand is positive, and we are ready to use the Comparison Test.

$$\begin{aligned}
\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}+\sin x_n} &> \sum_{m=1}^{\infty}\frac{1}{\sqrt{4m^2+2m}+\sqrt{2m+1}-\sqrt{2m}-1} \\
&> \sum_{m=1}^{\infty}\frac{1}{\sqrt{4m^2+2m}} \\
&\geq \sum_{m=1}^{\infty}\frac{1}{\sqrt{4m^2+2m^2}} \\
&= \sum_{m=1}^{\infty}\frac{1}{\sqrt{6m^2}} \\
&= \frac{1}{\sqrt{6}}\sum_{m=1}^{\infty}\frac{1}{m}
\end{aligned}$$

which diverges. Therefore the series does not converge uniformly on $\mathbb{R}$.

(ii) We use the same argument as part (i) to show that the series diverges. This time, we choose x_n such that

$$x_n = \begin{cases} \min\{\frac{\pi}{4}, R\}, & \text{when } n \text{ is even} \\ \max\{\frac{-\pi}{4}, -R\}, & \text{when } n \text{ is odd} \end{cases}$$

Then, let $r = \sin x_{2m} = -\sin x_{2m+1} > 0, s = x_{2m} = -x_{2m+1} > 0$ and we can be sure that $\cos\frac{x_n}{n} \geq \frac{\sqrt{2}}{2} > \frac{1}{2}$ for all n. Therefore,

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\cos(x_n/n)}{\sqrt{n}+\sin x_n} &= \sum_{m=1}^{\infty} \frac{(-1)^{2m+1}\cos(x_{2m}/2m)}{\sqrt{2m}+\sin x_{2m}} + \frac{(-1)^{(2m+1)+1}\cos(x_{2m+1}/(2m+1))}{\sqrt{2m+1}+\sin x_{2m+1}} \\
&= \sum_{m=1}^{\infty} \frac{-\cos(s/2m)}{\sqrt{2m}+r} + \frac{\cos(s/(2m+1))}{\sqrt{2m+1}-r} \\
&= \sum_{m=1}^{\infty} \frac{\sqrt{2m}\cos\left(\frac{s}{2m+1}\right) + r\cos\left(\frac{s}{2m+1}\right) - \sqrt{2m+1}\cos\left(\frac{s}{2m}\right) + r\cos\left(\frac{s}{2m}\right)}{\sqrt{4m^2+2m} + r\left(\sqrt{2m+1}+\sqrt{2m}-1\right)} \\
&= \sum_{m=1}^{\infty} \frac{\sqrt{2m}\cos\left(\frac{s}{2m+1}\right) - \sqrt{2m+1}\cos\left(\frac{s}{2m}\right) + r\cos\left(\frac{s}{2m+1}\right) + r\cos\left(\frac{s}{2m}\right)}{\sqrt{4m^2+2m} + r\left(\sqrt{2m+1}+\sqrt{2m}-1\right)}
\end{aligned}$$

From the numerator, the first 2 terms give a small negative number, whereas the last 2 terms give a value close to $2r$ for large m. So using the same concept as the previous part,

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\cos(x_n/n)}{\sqrt{n}+\sin x_n} &> \sum_{m=1}^{\infty} \frac{r}{\sqrt{4m^2+2m} + r\left(\sqrt{2m+1}+\sqrt{2m}-1\right)} \\
&> \sum_{m=1}^{\infty} \frac{r}{\sqrt{4m^2+2m}} \\
&\geq \sum_{m=1}^{\infty} \frac{r}{\sqrt{6m^2}} \\
&= \frac{r}{\sqrt{6}} \sum_{m=1}^{\infty} \frac{1}{m}
\end{aligned}$$

which diverges. Hence the convergence is not uniform.

Question 4

(a) Using the Ratio Test, the series $\sum_{n=0}^{\infty} \frac{x^{3n}}{n+1}$ converges if

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{x^{3(n+1)}}{n+2}}{\frac{x^{3n}}{n+1}} \right| = \lim_{n\to\infty} \left| x^3 \frac{n+2}{n+1} \right| = \left| x^3 \right|$$

is less than 1, and diverges if $\rho > 1$. Therefore the series converges for $|x| < 1$ and diverges for $|x| > 1$.

When $x = 1$, the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges as it is a p-series with $p = 1$. When $x = -1$, the series $\sum\limits_{n=0}^{\infty} \frac{(-1)^{3n}}{n+1} = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges by the Alternating Series Test.

Therefore the series converges for $x \in [-1, 1)$ and diverges otherwise.

For $y \in (-1,1)\backslash\{0\}$,

$$\begin{aligned} \sum_{n=0}^{\infty} y^n &= \frac{1}{1-y} \\ \sum_{n=0}^{\infty} \int y^n \mathrm{d}y = \int \sum_{n=0}^{\infty} y^n \mathrm{d}y &= \int \frac{1}{1-y} \mathrm{d}y \\ \sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1} &= -\ln(1-y) \\ \sum_{n=0}^{\infty} \frac{y^n}{n+1} &= -\frac{\ln(1-y)}{y} \end{aligned}$$

If we let $y = x^3$, note that $y \in (-1,1)\backslash\{0\} \Leftrightarrow x \in (-1,1)\backslash\{0\}$, so substitute this into the equation above to get the closed form for the series. By Abel's Theorem, the series is uniformly continuous in the interval $[-1, 0]$. Therefore, by L'Hopital's Rule,

$$\sum_{n=0}^{\infty} \frac{0^n}{n+1} = \lim_{y \to 0} -\frac{\ln(1-y)}{y} = \lim_{y \to 0} -\frac{-1}{1-y} = 1$$

Therefore

$$\sum_{n=0}^{\infty} \frac{x^{3n}}{n+1} = \begin{cases} 1, & x = 0 \\ -\frac{\ln(1-x^3)}{x^3}, & x \neq 0 \end{cases}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \lim_{x \to -1^+} \Big(\sum_{n=0}^{\infty} \frac{x^n}{n+1} \Big) = \lim_{x \to -1^+} \Big(-\frac{\ln(1-x^3)}{x^3} \Big) = \ln 2$$

(b)

$$\begin{aligned} g(x) &= (1-3x^2)\cos(x^2) \\ &= (1-3x^2) \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} - 3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n+2} \end{aligned}$$

On the other hand,

$$g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n$$

Hence,

$$\frac{g^{(2010)}(0)}{2010!} = -3\frac{(-1)^{502}}{(2(502))!} \Rightarrow g^{(2010)}(0) = -3 \cdot \frac{2010!}{1004!}$$

$$g^{(2011)}(0) = 0$$

$$\frac{g^{(2012)}(0)}{2012!} = \frac{(-1)^{503}}{(2(503))!} \Rightarrow g^{(2012)}(0) = -\frac{2012!}{1006!}$$

END OF SOLUTIONS

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