NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA2101 Linear Algebra II
AY 2011/2012 Sem 2
Version 1: July 1, 2014

Written by	**Audited by**	**Contributors**
Sherman Yuen	Le Hoang Van	–

Question 1

(a) Since W_1 and W_2 are vector subspaces of vector space V, we have $W_1 + W_2 \subseteq V$.

Let $\boldsymbol{u}, \boldsymbol{v} \in W_1 + W_2$. Then $\exists \boldsymbol{u}_1, \boldsymbol{v}_1 \in W_1, \exists \boldsymbol{u}_2, \boldsymbol{v}_2 \in W_2$ such that $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2, \boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$. Let $c \in F$.

- $\mathbf{0} \in W_1, W_2$, so $\mathbf{0} = \mathbf{0} + \mathbf{0} \in W_1 + W_2$.
- $\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{u}_1 + \boldsymbol{u}_2) + (\boldsymbol{v}_1 + \boldsymbol{v}_2) = (\boldsymbol{u}_1 + \boldsymbol{v}_1) + (\boldsymbol{u}_2 + \boldsymbol{v}_2) \in W_1 + W_2$.
- $c\boldsymbol{u} = c(\boldsymbol{u}_1 + \boldsymbol{u}_2) = c\boldsymbol{u}_1 + c\boldsymbol{u}_2 \in W_1 + W_2$.

This shows that $W_1 + W_2$ is a vector subspace of V.

(b) (i) $\Rightarrow$ (ii): Let $W_1 + W_2$ be a direct sum. By the Second Isomorphism Theorem,

$$\begin{aligned} (W_1 + W_2)/W_2 &\cong W_1/(W_1 \cap W_2) = W_1/\{\mathbf{0}\} \cong W_1 \\ \dim((W_1 + W_2)/W_2) &= \dim(W_1) \\ \dim(W_1 + W_2) - \dim(W_2) &= \dim(W_1) \\ \dim(W_1 + W_2) &= \dim(W_1) + \dim(W_2) \end{aligned}$$

(ii) $\Rightarrow$ (i): Let $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$. Then

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \Rightarrow \dim(W_1 \cap W_2) = 0$$

This shows that $W_1 + W_2$ is a direct sum.

(c) No. Let $V = F^{\mathbb{N}}$ be an infinite dimensional vector space, and let $W_1 = W_2 = V$. Then $\dim(W_1 + W_2) = \infty = \infty + \infty = \dim(W_1) + \dim(W_2)$, so condition (ii) is satisfied. But $W_1 \cap W_2 = V \neq \{\mathbf{0}\}$, so $W_1 + W_2$ is not a direct sum and so condition (i) is not satisfied.

Question 2

(a) $\text{rank}(A) = \dim T_A(\mathbb{R}^n_c)$ and $\text{rank}(AB) = \dim T_A(X)$, where $X = T_B(\mathbb{R}^r_c) \subseteq \mathbb{R}^n_c$. Therefore we must have $T_A(T_B(\mathbb{R}^r_c)) = T_A(X) \subseteq T_A(\mathbb{R}^n_c) \Rightarrow \text{rank}(AB) = \dim T_A(T_B(\mathbb{R}^r_c)) \leq \dim T_A(\mathbb{R}^n_c) = \text{rank}(A)$. By transposing the matrix AB, we get a similar result, that is, $\text{rank}(AB) = \text{rank}((AB)^T) = \text{rank}(B^T A^T) \leq \text{rank}(B^T) = \text{rank}(B)$. Therefore, $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

For any matrix X, we must have $\text{rank}(X) \leq \min\{\text{number of rows of } X, \text{number of columns of } X\}$. Therefore, $\text{rank}(A) \leq n$, $\text{rank}(A) \leq m$, $\text{rank}(B) \leq n$, and $\text{rank}(A) \leq r$. Putting together, we get $\min\{\text{rank}(A), \text{rank}(B)\} \leq \min\{m, n, r\}$.

(b) (i) $\Rightarrow$ (ii): Let $T : V \to W$ be an isomorphism. Then there exists an isomorphism $T^{-1} : W \to V$. This gives $T^{-1} \circ T = I_V$ and $T \circ T^{-1} = I_W$.

(ii) $\Rightarrow$ (iii): Let $S : W \to V$ be a linear transformation such that $S \circ T = I_V$ and $T \circ S = I_W$. By putting the ordered basis B_V into the first equation, we get $[I_V]_{B_V} = [S \circ T]_{B_V} = [S]_{B_W,B_V}[T]_{B_V,B_W}$. Since the rank of the identity matrix is its size, we must have

$$\begin{aligned} n &= \operatorname{rank}([I_V]_{B_V}) \\ &= \operatorname{rank}([S]_{B_W,B_V}[T]_{B_V,B_W}) \\ &\leq \min\{\operatorname{rank}([S]_{B_W,B_V}), \operatorname{rank}([T]_{B_V,B_W})\} \\ &\leq \min\{n, m\} \end{aligned}$$

Similarly, by putting the ordered basis B_W into the second equation, we get $[I_W]_{B_W} = [T]_{B_V,B_W}[S]_{B_W,B_V}$, and by the same reasoning, we will have $m \leq \min\{n, m\}$. Therefore, we get $n = m$. This means that $[T]_{B_V,B_W}$ is a square matrix. Since its inverse matrix is $[S]_{B_W,B_V}$, we conclude that $[T]_{B_V,B_W}$ is invertible.

(iii) $\Rightarrow$ (i): Let $[T]_{B_V,B_W}$ be an invertible square matrix. Let $\boldsymbol{w} \in W$. Then for $\boldsymbol{v} \in V$, $T(\boldsymbol{v}) = \boldsymbol{w}$ if and only if $\boldsymbol{v}$ is a solution to the matrix equation

$$[T]_{B_V,B_W}[\boldsymbol{v}]_{B_V} = [\boldsymbol{w}]_{B_W}$$

Since $[T]_{B_V,B_W}$ is invertible, there exists one and only one solution for $[\boldsymbol{v}]_{B_V}$. Therefore the vector $\boldsymbol{v}$ exists and is unique. This shows that T is bijective. As T is a bijective linear transformation between 2 vector spaces, it is an isomorphism.

Remark: The notation used in this question (as well as in Question 4) is such that if $T : V \to W$ is a linear transformation and B and C are bases for V and W respectively, then the matrix for T relative to the ordered bases B and C is denoted as $[T]_{B,C}$. This may be different from other lecture notes, for example, in Professor Ma Siu Lun's notes, it is denoted as $[T]_{C,B}$.

Question 3

(a) $p_2(x)$ cannot be the characteristic polynomial of A because if $p_2(3) = 0$, then $m(3) = 0$. But $m(3) \neq 0$.

$p_3(x)$ cannot be the characteristic polynomial of A because the order of the polynomial is 5, whereas A is a 6-by-6 matrix.

Therefore, only $p_1(x)$ can be the characteristic polynomial of A. It satisfies the conditions of the characteristic polynomial for the minimal polynomial $m(x)$.

(b) Along the diagonals of the Jordan canonical forms of A, there should be 2 '2's and 4 '1's. For the eigenvalue 2, the Jordan block associated with 2 must be of order 1, since the order of $(x - 2)$ in the minimal polynomial is 1. For the eigenvalue 1, there must exist a Jordan block associated with 1 with order 2. Therefore, the possible Jordan canonical forms of A are

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Question 4

(a) (i) We have $q_1 = (1,1,0)^T = [\boldsymbol{w}_1 + \boldsymbol{w}_2]_{B_1}$. Therefore,

$$\begin{aligned}[\boldsymbol{w}_1 + \boldsymbol{w}_2]_{B_1} &= q_1 \\ &= Kq_1 \\ &= [T]_{B_1}[\boldsymbol{w}_1 + \boldsymbol{w}_2]_{B_1} \\ &= [T(\boldsymbol{w}_1 + \boldsymbol{w}_2)]_{B_1}\end{aligned}$$

This shows that $T(\boldsymbol{w}_1+\boldsymbol{w}_2) = \boldsymbol{w}_1+\boldsymbol{w}_2$. Therefore, $\boldsymbol{w}_1+\boldsymbol{w}_2$ is an eigenvector of T. Similarly, we get $T(\boldsymbol{w}_2+\boldsymbol{w}_3) = 2(\boldsymbol{w}_2+\boldsymbol{w}_3)$ and $T(\boldsymbol{w}_1+\boldsymbol{w}_3) = 3(\boldsymbol{w}_1+\boldsymbol{w}_3)$. So, $\boldsymbol{w}_2+\boldsymbol{w}_3, \boldsymbol{w}_1+\boldsymbol{w}_3$ are also eigenvectors of T. As the eigenvectors are associated to different eigenvalues, they are linearly independent.

(ii) $|B_1| = |B_2| = 3$. Furthermore, the vectors in B_2 are linearly independent and are linear combinations of vectors in the basis B_1 of W. Therefore B_2 is also a basis of W.

(b) (i) As each vector in B_3 is a linear combination of the vectors in B_2, we have $\text{span}(B_3) \subseteq \text{span}(B_2)$. $\det P = 3$, so P is invertible. Therefore, we have $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)P^{-1}$. Now, each vector in B_2 is a linear combination of the vectors in B_3, so $\text{span}(B_2) \subseteq \text{span}(B_3)$.

Since $\text{span}(B_3) = \text{span}(B_2)$ and $|B_3| = |B_2| = 3 = \dim(W)$, B_3 is a basis of W.

(ii) We have $[T]_{B_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ from part (a). Therefore, we see that $[T]_{B_2} = [I_W]_{B_3,B_2}[T]_{B_3}[I_W]_{B_2,B_3}$ is a diagonal matrix, where

$$\begin{aligned}Q &= [I_W]_{B_3,B_2} \\ &= ([\boldsymbol{u}_1]_{B_2} \quad [\boldsymbol{u}_2]_{B_2} \quad [\boldsymbol{u}_3]_{B_2}) \\ &= P\end{aligned}$$

Question 5

(a) Let $g(x) = (x+1)(x-1)(x+2)$. Then by the question, $g(H) = 0$.

If $m(x)$ is a minimal polynomial of H, then $m(x)$ must divide $g(x)$. Therefore there are 7 possibilities for the minimal polynomial of H. Furthermore, for each of the minimal polynomial, it must divide the characteristic polynomial.

(i) $m(x) = (x+1)$, $p(x) = (x+1)^4$

(ii) $m(x) = (x-1)$, $p(x) = (x-1)^4$

(iii) $m(x) = (x+2)$, $p(x) = (x+2)^4$

(iv) $m(x) = (x+1)(x-1)$, $p(x) = (x+1)^3(x-1)$, $(x+1)^2(x-1)^2$, $(x+1)(x-1)^3$

(v) $m(x) = (x-1)(x+2)$, $p(x) = (x-1)^3(x+2)$, $(x-1)^2(x+2)^2$, $(x-1)(x+2)^3$

(vi) $m(x) = (x+1)(x+2)$, $p(x) = (x+1)^3(x+2)$, $(x+1)^2(x+2)^2$, $(x+1)(x+2)^3$

(vii) $m(x) = (x+1)(x-1)(x+2)$, $p(x) = (x+1)^2(x-1)(x+2)$, $(x+1)(x-1)^2(x+2)$, $(x+1)(x-1)(x+2)^2$

(b) H is a diagonalizable matrix as the powers of the linear factors in the minimal polynomial are all 1.

(c) H is invertible as 0 is not an eigenvalue of H. By the definition of the minimal polynomial, $m(H) = 0$.

(i) $m(x) = (x + 1)$. Then $H + I = 0 \Rightarrow -IH = I \Rightarrow H^{-1} = -I \Rightarrow f(x) = -1$ satisfies the condition.

(ii) $m(x) = (x - 1)$. Then $H - I = 0 \Rightarrow H = I \Rightarrow H^{-1} = I \Rightarrow f(x) = 1$ satisfies the condition.

(iii) $m(x) = (x + 2)$. Then $H + 2I = 0 \Rightarrow \left(-\frac{1}{2}I\right) H = I \Rightarrow H^{-1} = -\frac{1}{2}I \Rightarrow f(x) = -\frac{1}{2}$ satisfies the condition.

(iv) $m(x) = (x + 1)(x - 1)$. Then $H^2 - I = 0 \Rightarrow H^2 = I \Rightarrow H^{-1} = H \Rightarrow f(x) = x$ satisfies the condition.

(v) $m(x) = (x - 1)(x + 2)$. Then $H^2 + H - 2I = 0 \Rightarrow (H + I)H = 2I \Rightarrow \frac{1}{2}(H + I)H = I \Rightarrow H^{-1} = \frac{1}{2}(H + I) \Rightarrow f(x) = \frac{1}{2}(x + 1)$ satisfies the condition.

(vi) $m(x) = (x + 1)(x + 2)$. Then $H^2 + 3H + 2I = 0 \Rightarrow -(H + 3I)H = 2I \Rightarrow -\frac{1}{2}(H + 3I)H = I \Rightarrow H^{-1} = -\frac{1}{2}(H + 3I) \Rightarrow f(x) = -\frac{1}{2}(x + 3)$ satisfies the condition.

(vii) $m(x) = (x + 1)(x - 1)(x + 2)$. Then $H^3 + 2H^2 - H - 2I = 0 \Rightarrow (H^2 + 2H - I)H = 2I \Rightarrow \frac{1}{2}(H^2 + 2H - I)H = I \Rightarrow H^{-1} = \frac{1}{2}(H^2 + 2H - I) \Rightarrow f(x) = \frac{1}{2}(x^2 + 2x - 1)$ satisfies the condition.

Question 6

(a)

$$\begin{aligned}
p_A(x) &= \det(x\boldsymbol{I} - A) \\
&= \begin{vmatrix} x-3 & 2 & 0 \\ 2 & x-3 & 0 \\ 0 & 0 & x-5 \end{vmatrix} \\
&= (x-3)^2(x-5) + 0 + 0 - 0 - 0 - (2)(2)(x-5) \\
&= (x-5)((x-3)^2 - 4) \\
&= (x-5)^2(x-1)
\end{aligned}$$

Therefore the eigenvalues of A are 1 and 5.

(b)

$$\begin{aligned}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in V_1(A) \quad &\Leftrightarrow \quad (1\boldsymbol{I} - A)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow \quad \begin{pmatrix} 1-3 & 2 & 0 \\ 2 & 1-3 & 0 \\ 0 & 0 & 1-5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow \quad \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}$$

Therefore, an orthonormal basis for $V_1(A)$ is $\left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right\}$.

$$\begin{aligned}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in V_5(A) \quad &\Leftrightarrow \quad (5\boldsymbol{I} - A) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow \quad \begin{pmatrix} 5-3 & 2 & 0 \\ 2 & 5-3 & 0 \\ 0 & 0 & 5-5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}$$

Therefore, an orthonormal basis for $V_1(A)$ is $\left\{ \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(c) The orthogonal matrix is $P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Question 7

(a) A is self-adjoint, so it is diagonalizable. So there exists an unitary matrix P such that $A = PDP^*$, where

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

is a diagonal matrix, and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A (may be repeated). As all the eigenvalues of A are positive, we have $\lambda_1, \lambda_2, \cdots, \lambda_n > 0$. Therefore we can define

$$D_0 = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix}$$

Note that $D_0^* = D_0$ and $D = D_0^2 = D_0 D_0^*$. Therefore if we set $G = PD_0$, we get

$$A = PDP^* = P(D_0 D_0^*)P^* = (PD_0)(D_0^* P^*) = (PD_0)(PD_0)^* = GG^*$$

Since P is orthogonal and D_0 has determinant $\left(\sqrt{\lambda_1}\sqrt{\lambda_2}\cdots\sqrt{\lambda_n}\right)$, G is invertible.

(b) Following the argument in (a), if we set $E = PD_0P^*$, then

$$\begin{aligned} E^2 &= PD_0P^*PD_0P^* \\ &= PD_0^2P^* \quad (\because P \text{ is unitary}) \\ &= A. \qquad\quad (\because D_0^2 = D) \end{aligned}$$

Now E is clearly invertible since P is unitary and D_0 is a diagonal matrix with non-zero diagonal entries. Moreover, E is self-adjoint because

$$\begin{aligned} E^* &= (PD_0P^*)^* \\ &= (P^*)^*D_0^*P^* \\ &= PD_0P^* \quad (\because (P^*)^* = P \text{ and } D_0 \text{ is diagonal}) \\ &= E. \end{aligned}$$

(c) Yes. Let $\boldsymbol{z} \in \mathbb{C}^n$ be a non-zero column vector. Then $L\boldsymbol{z}$ is a non-zero column vector since L is invertible. Then

$$\boldsymbol{z}^*L^2\boldsymbol{z} = \boldsymbol{z}^*L^*L\boldsymbol{z} = (L\boldsymbol{z})^*(L\boldsymbol{z}) > 0$$

since $L^* = L$. This shows that L^2 is positive definite.

Question 8

(a) Let $x \in W^\perp$ and $w \in W$. Since W is a T^*-invariant subspace of V, $\exists w_0 \in W$ such that $T^*(w) = w_0$. Therefore,

$$\langle T(x), w\rangle = \langle x, T^*(w)\rangle = \langle x, w_0\rangle = 0$$

since $x \in W^\perp$, $w_0 \in W$. But this shows that $\langle T(x), w\rangle = 0$ for all $w \in W$, implying that $T(x) \in W^\perp$. This shows that $W^\perp$ is a T-invariant subspace of V.

(b) No. Let $V = \mathbb{C}^2$, $U = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \middle| x \in \mathbb{C} \right\}$ and $T\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} b \\ 0 \end{pmatrix}$. Then $U^\perp = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \middle| y \in \mathbb{C} \right\}$.

But $U^\perp$ is not a T-invariant subspace of V because, for example, $T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin U^\perp$.

END OF SOLUTIONS

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