MA2101S - Linear Algebra II(S) Suggested Solutions

(Semester 2 : AY2019/2020)

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Question 1

(i) Claim : $\forall\ a \cdot 1 \in \mathbb{F}_p,\ (a \cdot 1)^p = a \cdot 1$.

Proof : If $a = 0$, then trivially $(a \cdot 1)^p = (0 \cdot 1)^p = 0 \cdot 1 = a \cdot 1$. Thus we only consider the case where $a \neq 0$.

$$\begin{aligned} \text{For } a \neq 0,\ (a \cdot 1)^p &= \underbrace{(1 + 1 + ... + 1)^p}_{\text{a times}} \\ &= 1^p + 1^p + ... + 1^p - \text{By the hint in Question 1.} \\ &= 1 + 1 + ... + 1 \\ &= a \cdot 1. \end{aligned}$$

To prove that F is an $\mathbb{F}_p$-linear operator:

Let $u, v \in \mathbb{F}_q$ and $x, y \in \mathbb{F}_p$.

$$\begin{aligned} F(xu + yv) &= (xu + yv)^p \\ &= x^p u^p + y^p v^p - \text{By the hint in Question 1.} \\ &= xu^p + yv^p - \text{By our claim.} \\ &= xF(u) + yF(v). \end{aligned}$$

Thus F is an $\mathbb{F}_p$-linear operator.

To prove that F is an isomorphism:

Let $w \in \ker(F)$. Then $F(w) = 0_V \to w^p = 0$. Since $\mathbb{F}_q$ is a field,
$w^p = 0 \to w = 0$. Thus $\ker(F) = \{0_V\}$ so F is injective.

Since F is a linear operator, F is injective $\to$ F is surjective. (By the pigeonhole principle) Thus we conclude that F is an isomorphism.

(ii) By our claim in part(i), $\forall\ a \cdot 1 \in \mathbb{F}_p,\ F(a \cdot 1) = a \cdot 1$. Thus $\mathbb{F}_p \subseteq E_1$.

Claim: $E_1 = \mathbb{F}_p$.

Let $k \in E_1$. Then $F(k) = k \to k^p - k = 0$ so k is a root of the p-degree polynomial $x^p - x = 0$. Since a p-degree polynomial have at most p roots, $|E_1| \leq p$. But $|\mathbb{F}_p| = p\ \wedge\ \mathbb{F}_p \subseteq E_1$. Thus we must have: $E_1 = \mathbb{F}_p$.

Our desired basis is simply: $\{1\}$.

Question 2

(i) We first make 2 observations:

Observation 1: $\forall p(x, y) \in P_1, \Delta(p(x, y)) = 0_V$.

Observation 2: $\forall p(x, y) \in P_n, \Delta(p(x, y)) \in P_{n-2}$.

It is now easy to see that $\forall p(x,y) \in P_n,\ \Delta^{\lfloor \frac{n}{2}+1 \rfloor}(p(x,y)) = 0_V$. Thus $m_\Delta(x) \mid x^{\lfloor \frac{n}{2}+1 \rfloor}$ so $m_\Delta(x) = x^j$ for some $1 \le j \le \lfloor \frac{n}{2}+1 \rfloor$.

On the other hand, $\Delta^{\lfloor \frac{n}{2} \rfloor}(x^n) = \begin{cases} n! & \text{if n is even.} \\ n!x & \text{if n is odd.} \end{cases}$

In both cases, $\Delta^{\lfloor \frac{n}{2} \rfloor}(x^n) \neq 0_V$ so $j > \lfloor \frac{n}{2} \rfloor$. Thus we must have $m_\Delta(x) = x^{\lfloor \frac{n}{2}+1 \rfloor}$.

(ii) We start with the canonical basis for P_3, which is $B = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$.

By (i), $m_\Delta(x) = x^2$. Thus we separate each vector in the basis into 2(mutually exclusive) groups: $\ker(\Delta)$ & $\ker(\Delta^2) \setminus \ker(\Delta)$.

$$\begin{aligned} \Delta(1) = 0\ ,\ \Delta(x) = 0 \\ \Delta(y) = 0\ ,\ \Delta(x^2) = 2 \\ \Delta(xy) = 0\ ,\ \Delta(y^2) = 2 \\ \Delta(x^3) = 6x\ ,\ \Delta(x^2y) = 2y \\ \Delta(xy^2) = 2x\ ,\ \Delta(y^3) = 6y. \end{aligned}$$

Obviously $1, x, y, xy \in \ker(\Delta)$.

$\Delta(x^2) = 2 \to \Delta(\frac{1}{2}x^2) = 1$. Thus $x^2 \in \ker(\Delta^2) \setminus \ker(\Delta)$ and the vector pair $\{1, \frac{1}{2}x^2\}$ form the first $J_2(0)$ Jordan block. Then note that $\Delta(x^2 - y^2) = 0$. Thus $x^2 - y^2 \in \ker(\Delta)$ so we replace y^2 in B with $x^2 - y^2$. Such a replacement will not affect the linear independence of our set so B is still a basis.

$\Delta(x^3) = 6x \to \Delta(\frac{1}{6}x^3) = x$. Thus $x^3 \in \ker(\Delta^2) \setminus \ker(\Delta)$ and the vector pair $\{x, \frac{1}{6}x^3\}$ form the second $J_2(0)$ Jordan block. Similarly, $y^3 \in \ker(\Delta^2) \setminus \ker(\Delta)$ and the vector pair $\{y, \frac{1}{6}y^3\}$ form the third $J_2(0)$ Jordan block.

For the last 2 vectors, x^2y and xy^2, it is easy to see that:

$$y^3 - 3x^2y,\ x^3 - 3xy^2 \in \ker(\Delta).$$

Thus we replace x^2y with $y^3 - 3x^2y$ and xy^2 with $x^3 - 3xy^2$.

After reordering, our resultant basis B is:

$$\{1, \tfrac{1}{2}x^2, x, \tfrac{1}{6}x^3, y, \tfrac{1}{6}y^3, xy, x^2 - y^2, x^3 - 3xy^2, y^3 - 3x^2y\}$$

Which will result in the following standard matrix:

$$[\Delta]_{B,B} = \begin{pmatrix} J_2(0) & & & \\ & J_2(0) & & \\ & & J_2(0) & \\ & & & 0_{4\times4} \end{pmatrix}.$$

Question 3

(i) Recall that:

$$P \text{ is invertible} \iff (P^T)^{-1} \text{ is invertible } \& \ D \text{ is diagonal} \to D = D^T.$$

$$\begin{aligned} A \text{ is diagonalizable} &\iff A = PDP^{-1} \text{ for some invertible matrix } P, \text{ diagonal matrix } D \\ &\iff A^T = (P^{-1})^T D^T P^T \\ &\iff A^T = (P^T)^{-1} D P^T \\ &\iff A^T = QDQ^{-1} \text{ where } Q = (P^T)^{-1} \\ &\iff A^T = QDQ^{-1} \text{ for some invertible matrix Q, diagonal matrix D} \\ &\iff A^T \text{ is diagonalisable.} \end{aligned}$$

(ii) Let $Au = \lambda u, A^T v = \mu v$ for some $\lambda, \mu \in \mathbb{C}$.

$$\begin{aligned}
\mathrm{ad}_A(uv^T) &= A(uv^T) - (uv^T)A \\
&= (Au)v^T - u(v^T A) \\
&= \lambda uv^T - u(A^T v)^T \\
&= \lambda uv^T - u(\mu v)^T \\
&= \lambda uv^T - \mu uv^T \\
&= (\lambda - \mu)uv^T.
\end{aligned}$$

(iii) A is diagonalizable $\rightarrow A^T$ is diagonalizable.

$\exists$ basis of eigenvectors $B_1 = \{u_1, u_2, ..., u_n\}$ for $\mathbb{C}^n$ with respect to A.
Similarly, $\exists$ a basis of eigenvectors $B_2 = \{v_1, v_2, ..., v_n\}$ for $\mathbb{C}^n$ with respect to A^T.

Claim: The set $B = \{u_i v_j^T \mid 1 \leq i \leq n, 1 \leq j \leq n\}$ is a basis for $M_{n\times n}(\mathbb{C})$.

Proof: Since $|B| = n^2 = \dim(M_{n\times n}(\mathbb{C}))$ and $B \subseteq M_{n\times n}(\mathbb{C})$, it suffice to prove that B is a linearly independent set.

Consider the homogeneous equation:

$$\begin{aligned}
&a_{1,1}(u_1v_1^T) + a_{1,2}(u_1v_2^T) + ... + a_{1,n}(u_1v_n^T) + \\
&a_{2,1}(u_2v_1^T) + a_{2,2}(u_2v_2^T) + ... + a_{2,n}(u_2v_n^T) + \\
&\qquad\vdots \\
&a_{n,1}(u_nv_1^T) + a_{n,2}(u_nv_2^T) + ... + a_{n,n}(u_nv_n^T) = 0_{n\times n}.
\end{aligned}$$

Collecting the terms:

$$u_1(\textstyle\sum_{i=1}^n a_{1,i}v_i^T) + u_2(\sum_{i=1}^n a_{2,i}v_i^T) + ... + u_n(\sum_{i=1}^n a_{n,i}v_i^T) = 0_{n\times n}$$

Let $\sum_{i=1}^n a_{k,i}v_i^T = \begin{pmatrix} e_{k,1} & e_{k,2} & ... & e_{k,n}\end{pmatrix}$ for each $k \in \mathbb{N}, 1 \leq k \leq n$.

Rewriting the homogeneous equation:

$$\begin{aligned}
u_1 \begin{pmatrix} e_{1,1} & ... & e_{1,n}\end{pmatrix} + u_2 \begin{pmatrix} e_{2,1} & ... & e_{2,n}\end{pmatrix} + ... + u_n \begin{pmatrix} e_{n,1} & ... & e_{n,n}\end{pmatrix} &= 0_{n\times n}. \\
\begin{pmatrix} e_{1,1}u_1 & ... & e_{1,n}u_1\end{pmatrix} + \begin{pmatrix} e_{2,1}u_2 & ... & e_{2,n}u_2\end{pmatrix} + ... + \begin{pmatrix} e_{n,1}u_n & ... & e_{n,n}u_n\end{pmatrix} &= 0_{n\times n}. \\
\begin{pmatrix} \sum_{i=1}^n e_{i,1}u_i & \sum_{i=1}^n e_{i,2}u_i & ... & \sum_{i=1}^n e_{i,n}u_i\end{pmatrix} &= 0_{n\times n}.
\end{aligned}$$

By linear independence of $B_1 = \{u_1, u_2, ..., u_n\}$, each $e_{i,j} = 0$.

Thus we reduce the homogeneous equation

$$u_1(\textstyle\sum_{i=1}^n a_{1,i}v_i^T) + u_2(\sum_{i=1}^n a_{2,i}v_i^T) + ... + u_n(\sum_{i=1}^n a_{n,i}v_i^T) = 0_{n\times n}$$

to :

$$(\textstyle\sum_{i=1}^n a_{1,i}v_i^T) = (\sum_{i=1}^n a_{2,i}v_i^T) = ... = (\sum_{i=1}^n a_{n,i}v_i^T) = 0_V.$$

By linear independence of $B_2 = \{v_1, v_2, ..., v_n\}$, we conclude that:

$$\forall\ 1 \leq i \leq n, 1 \leq j \leq n, a_{i,j} = 0.$$

Since only the trivial solution exists to the homogeneous equation, the set B is linearly independent.

By (ii), each vector in B is also an eigenvector of ad_A. Thus B is a basis of eigenvectors for $M_{n\times n}(\mathbb{C})$ with respect to ad_A so it is diagonalisable.

Question 4

Let $A = \begin{pmatrix} u_1 & u_2 & ... & u_n \end{pmatrix}$ for column vectors $u_1, u_2, ..., u_n$.
Since $\det(A) \neq 0, u_1, u_2, ..., u_n$ form a basis for $\mathbb{R}^n$.

First apply the Gram-Schmidt process on the vectors $u_1, u_2, ..., u_n$ to obtain an orthogonal basis $\{v_1, v_2, ..., v_n\}$:

$$\begin{aligned}
v_1 &= u_1, \\
v_2 &= u_2 - \frac{\langle u_2, u_1 \rangle}{\langle u_1, u_1 \rangle} v_1, \\
v_3 &= u_3 - \frac{\langle u_3, u_1 \rangle}{\langle u_1, u_1 \rangle} v_1 - \frac{\langle u_3, u_2 \rangle}{\langle u_2, u_2 \rangle} v_2, \\
&\vdots \\
v_n &= u_n - \frac{\langle u_n, u_1 \rangle}{\langle u_1, u_1 \rangle} v_1 - ... - \frac{\langle u_n, u_{n-1} \rangle}{\langle u_{n-1}, u_{n-1} \rangle} v_{n-1}.
\end{aligned}$$

Construct matrix P' as follows:

$$P' = \begin{pmatrix} \frac{v_1}{||v_1||} & \frac{v_2}{||v_2||} & \cdots & \frac{v_n}{||v_n||} \end{pmatrix}.$$

$\{v_1, v_2, ..., v_n\}$ is an orthogonal basis for $\mathbb{R}^n$ so $\{\frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}, ..., \frac{v_n}{||v_n||}\}$ is an orthonormal basis for $\mathbb{R}^n$. Since the columns of P' form an orthonormal basis, P' is an orthogonal matrix. Consider 2 cases:

Case 1 : $\det(P') = 1$.

Then P' is a special orthogonal matrix. Choose $P = P'$ and construct matrix B as follows:

$$B = \begin{pmatrix}
||v_1|| & \frac{\langle u_2, u_1 \rangle}{\langle u_1, u_1 \rangle}||v_1|| & \frac{\langle u_3, u_1 \rangle}{\langle u_1, u_1 \rangle}||v_1|| & ... & \frac{\langle u_n, u_1 \rangle}{\langle u_1, u_1 \rangle}||v_1|| \\
0 & ||v_2|| & \frac{\langle u_3, u_2 \rangle}{\langle u_2, u_2 \rangle}||v_2|| & ... & \frac{\langle u_n, u_2 \rangle}{\langle u_2, u_2 \rangle}||v_2|| \\
0 & 0 & ||v_3|| & ... & \frac{\langle u_n, u_3 \rangle}{\langle u_3, u_3 \rangle}||v_3|| \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & ... & ||v_n||
\end{pmatrix}$$

Case 2 : $\det(P') = -1$

Then construct matrix P by:

$$P = \begin{pmatrix} -\frac{v_1}{||v_1||} & \frac{v_2}{||v_2||} & \cdots & \frac{v_n}{||v_n||} \end{pmatrix}.$$

Then $\det(P) = 1$ and $\{-\frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}, ..., \frac{v_n}{||v_n||}\}$ is still an orthonormal basis so P is special orthogonal. Construct matrix B as follows:

$$B = \begin{pmatrix} -||v_1|| & \frac{\langle u_2,u_1\rangle}{\langle u_1,u_1\rangle}||v_1|| & \frac{\langle u_3,u_1\rangle}{\langle u_1,u_1\rangle}||v_1|| & \cdots & \frac{\langle u_n,u_1\rangle}{\langle u_1,u_1\rangle}||v_1|| \\ 0 & ||v_2|| & \frac{\langle u_3,u_2\rangle}{\langle u_2,u_2\rangle}||v_2|| & \cdots & \frac{\langle u_n,u_2\rangle}{\langle u_2,u_2\rangle}||v_2|| \\ 0 & 0 & ||v_3|| & \cdots & \frac{\langle u_n,u_3\rangle}{\langle u_3,u_3\rangle}||v_3|| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & ||v_n|| \end{pmatrix}$$

It is easy to check that in both cases, B is upper triangular, P is special orthogonal and $A = PB$.

Question 5

(i)

$$\begin{aligned} A = -A^T &\to iA = -iA^T \\ &\to iA = \overline{iA^T} \text{ - Recall that } A = \overline{A} \text{ since } A \in M_{n\times n}(\mathbb{R}) \\ &\to iA = (iA)^*. \end{aligned}$$

Thus iA is a Hermitian matrix. Let u be an eigenvector of A associated with eigenvalue λ.

$$\begin{aligned} Au = \lambda u &\to iAu = i\lambda u \\ u^*(iA)u &= u^*(iA)^*u \\ u^*(iAu) &= (iAu)^*u \\ u^*(i\lambda u) &= (i\lambda u)^*u \\ i\lambda(u^*u) &= -i\overline{\lambda}(u^*u) \end{aligned}$$

Since u is an eigenvector, $u \neq 0_V$ so $u^*u \neq 0$. Thus we have:

$$\begin{aligned} i\lambda &= -i\overline{\lambda} \\ \lambda &= -\overline{\lambda} \end{aligned}$$

Thus λ is purely imaginary.

(ii) Let λ be the eigenvalue associated with eigenvector u and write $\lambda = ix$ for $x \in \mathbb{R}$. Then:

$$\begin{aligned} A(u) &= A(\mathrm{Re}(u) + i\mathrm{Im}(u)) \\ ix\mathrm{Re}(u) - x\mathrm{Im}(u) &= A(\mathrm{Re}(u)) + iA(\mathrm{Im}(u)) \end{aligned}$$

Note that $x \in \mathbb{R}, \mathrm{Re(u)},\ \mathrm{Im(u)} \in \mathbb{R}^n,\ A \in M_{n\times n}(\mathbb{R})$. Thus we can safely compare real and imaginary parts:

$$A(\mathrm{Re}(u)) = -x\mathrm{Im}(u)\ ,\ A(\mathrm{Im}(u)) = x\mathrm{Re}(u)$$

Since $A(\mathrm{Re}(u)), A(\mathrm{Im}(u)) \in \mathrm{span}_{\mathbb{R}}\{\mathrm{Re}(u), \mathrm{Im}(u)\}$, $\mathrm{span}_{\mathbb{R}}\{\mathrm{Re}(u), \mathrm{Im}(u)\}$ is L_A-invariant.

(iii) It suffices to prove that $\exists$ an orthonormal basis C for $\mathbb{R}^n$ such that $[L_A]_C = D$. We will prove via mathematical induction.

Base Case: If A is a 1×1 matrix, then $A = -A^T \to A = 0_{1\times 1}$. Then choose C to be any orthonormal basis and $[L_A]_C = 0_{1\times 1} = J$.

Induction Step: Consider A as a matrix in $M_{n\times n}(\mathbb{C})$, for $n \geq 2$. Consider 2 cases:

Case 1: $A = 0_{n\times n}$.

Similarly, choose C to be any orthonormal basis and $[L_A]_C = J$.

Case 2: $A \neq 0_{n\times n}$.

$A = -A^T \to A = -A^*$ so obviously A is normal. Thus A is unitarily diagonalisable. Since A is diagonalisable and $A \neq 0_{n\times n}$, A has a non-zero eigenvalue, λ.

Let u be an eigenvector associated with eigenvalue λ. Then

$$\begin{aligned}\overline{Au} = \overline{\lambda u} &\to \overline{A}\overline{u} = \overline{\lambda}\overline{u}\\ &\to A\overline{u} = \overline{\lambda}\overline{u}\end{aligned}$$

so $\overline{u}$ is also an eigenvector of A associated with eigenvalue $\overline{\lambda}$. By (i), $\lambda = -\overline{\lambda}$. Since $\lambda \neq 0, \lambda \neq \overline{\lambda}$. In other words, u and $\overline{u}$ are eigenvectors associated with different eigenvalues so $u \neq \overline{u}$.

Claim 1: $C_1 = \{\frac{\mathrm{Re}(u)}{||\mathrm{Re}(u)||}, \frac{\mathrm{Im}(u)}{||\mathrm{Re}(u)||}\}$ is an orthonormal set.

Proof: It suffices to prove the following 2 statements:

$$||\mathrm{Re}(u)|| = ||\mathrm{Im}(u)||. \tag{1}$$
$$\langle \mathrm{Im}(u), \mathrm{Re}(u)\rangle = 0. \tag{2}$$

Recall that since A is unitarily diagonalisable, and u and $\overline{u}$ are eigenvectors associated with different eigenvalues, u and $\overline{u}$ are orthogonal vectors so
$\langle u, \overline{u}\rangle = 0$. Then:

$$\langle \mathrm{Re}(u) + i\mathrm{Im}(u), \mathrm{Re}(u) - i\mathrm{Im}(u)\rangle = 0.$$
$$\langle \mathrm{Re}(u), \mathrm{Re}(u)\rangle - \langle \mathrm{Im}(u), \mathrm{Im}(u)\rangle + 2i\langle \mathrm{Im}(u), \mathrm{Re}(u)\rangle = 0.$$

Comparing real parts: $\langle \mathrm{Re}(u), \mathrm{Re}(u)\rangle = \langle \mathrm{Im}(u), \mathrm{Im}(u)\rangle \to ||\mathrm{Re}(u)|| = ||\mathrm{Im}(u)||$.

Comparing imaginary parts: $\langle \mathrm{Im}(u), \mathrm{Re}(u)\rangle = 0$.

Let $W = \mathrm{span}\{\frac{\mathrm{Re}(u)}{||\mathrm{Re}(u)||}, \frac{\mathrm{Im}(u)}{||\mathrm{Re}(u)||}\}$ over $\mathbb{R}$. By (ii), W is L_A-invariant and
$A(\frac{\mathrm{Re}(u)}{||\mathrm{Re(u)}||}) = -\lambda\frac{\mathrm{Im}(u)}{||\mathrm{Re}(u)||}$ and $A(\frac{\mathrm{Im}(u)}{||\mathrm{Re}(u)||}) = \lambda\frac{\mathrm{Re}(u)}{||\mathrm{Re}(u)||}$. Thus:

$$[L_A|_W]_{C_1} = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0\end{pmatrix}.$$

Claim 2: $W^\perp$ is L_A-invariant.

Proof: First note that since the canonical basis for $\mathbb{R}^n$ is also orthonormal, $L_A^*(v) = A^*v \;\; \forall v \in \mathbb{R}^n$.

Let $w \in W^\perp$. Then $\forall v \in W$:

$$\begin{aligned}\langle v, w\rangle &= 0.\\ \langle L_A(v), w\rangle &= 0. - \text{(Recall that } W \text{ is } L_A\text{-invariant)}\\ \langle v, L_A^*(w)\rangle &= 0.\\ \langle v, A^*w\rangle &= 0.\\ \langle v, -Aw\rangle &= 0.\\ \langle v, Aw\rangle &= 0.\end{aligned}$$

Since $\dim(W) = 2$, $\dim(W^\perp) = n-2$ so by our induction hypothesis,
$\exists$ orthonormal basis C_2 for $W^\perp$ such that

$$[L_A|_{W^\perp}]_{C_2} = \begin{pmatrix} J' & 0 \\ 0 & 0_{(n-2-2m)\times(n-2-2m)}\end{pmatrix}$$

Where J' is of the same form as J.

Choose $C = C_1 \cup C_2$. Then C is still orthonormal and

$$[L_A]_C = \begin{pmatrix} [L_A|_W]_{C_1} & 0_{2\times(n-2)} \\ 0_{(n-2)\times 2} & [L_A|_{W^\perp}]_{C_2}\end{pmatrix} = \begin{pmatrix} J & 0 \\ 0 & 0_{(n-2-2m)\times(n-2-2m)}\end{pmatrix}.$$

Question 6

(i) Let $u, v \in U$ and $x, y \in \mathbb{F}$.

$$\begin{aligned}\mathfrak{Lim}(xu + yv) &= \mathcal{B}(xu + yv, u_1)u_1 + \mathcal{B}(xu + yv, u_2)u_2 + ... + \mathcal{B}(xu + yv, u_n)u_n \\ &= x\mathcal{B}(u, u_1)u_1 + y\mathcal{B}(v, u_1)u_1 + x\mathcal{B}(u, u_2)u_2 + ... + y\mathcal{B}(v, u_n)u_n \\ &= x\mathfrak{Lim}(u) + y\mathfrak{Lim}(v).\end{aligned}$$

Thus $\mathfrak{Lim}$ is a linear transformation.

$$\begin{aligned}w \in \ker(\mathfrak{Lim}) &\iff \mathcal{B}(w, u_1)u_1 + \mathcal{B}(w, u_2)u_2 + ... + \mathcal{B}(w, u_n)u_n = 0_V \\ &\iff \mathcal{B}(w, u_1) = \mathcal{B}(w, u_2) = ... = \mathcal{B}(w, u_n) = 0 \\ &\iff w \in U^{\perp}.\end{aligned}$$

Thus $\ker(\mathfrak{Lim}) = U^{\perp}$.

(ii) Yes. Obviously $R(\mathfrak{Lim}) \subseteq U$. Thus to prove $R(\mathfrak{Lim}) = U$, it suffices to prove that $\dim(R(\mathfrak{Lim})) = \dim(U) = n$.

Claim: $\mathfrak{Lim}(u_1), \mathfrak{Lim}(u_2), ..., \mathfrak{Lim}(u_n)$ are linearly independent vectors.

Proof: Assume, for the sake of contradiction, that $\exists\ a_1, a_2, ..., a_n$, not all zero, such that:

$$a_1\mathfrak{Lim}(u_1) + a_2\mathfrak{Lim}(u_2) + ... + a_n\mathfrak{Lim}(u_n) = 0_V.$$

$\mathfrak{Lim}(a_1u_1 + a_2u_2 + ... + a_nu_n) = 0_V \to a_1u_1 + a_2u_2 + ... + a_nu_n \in \ker(\mathfrak{Lim})$. By (i), $a_1u_1 + a_2u_2 + ... + a_nu_n \in U^{\perp}$. But by definition of subspace, $a_1u_1 + a_2u_2 + ... + a_nu_n \in U$. This is a contradiction as $U^{\perp} \cap U = \{0_V\}$. (Since $\mathcal{B}$ is non-degenerate)

Thus the assumption is false and $\mathfrak{Lim}(u_1), \mathfrak{Lim}(u_2), ..., \mathfrak{Lim}(u_n)$ are linearly independent vectors.

Since $\mathfrak{Lim}(u_1), \mathfrak{Lim}(u_2), ..., \mathfrak{Lim}(u_n) \in R(\mathfrak{Lim})$, $\dim(R(\mathfrak{Lim})) = n$ and our proof is complete.

Clearly $\mathfrak{Lim}$ has domain V, kernel $U^{\perp}$ and range U. Thus by the first isomorphism theorem, $V/U^{\perp} \cong U$.

(iii) Disprove by counterexample: Choose $V = \mathbb{R}^{\infty}$, the vector space consisting of all infinite sequences of real numbers. Define the vector e_n by:

$$e_n = \underbrace{(0\ ,\ 0\ , ...\ ,\ 0\ ,\ 1\ ,\ 0\ ,\ ...\ )}_{\text{1 at the Nth entry, 0 otherwise}}.$$

Define $U = \operatorname{span}\{e_1, e_2, ...\}$. Then $U^{\perp} = \{0_V\}$ so $V/U^{\perp} = V$. U has a countable basis while V has an uncountable basis so $V \ncong U$. Thus $V/U^{\perp} \ncong U$.

Remark: One way of showing that $\mathbb{R}^{\infty}$ has an uncountable basis is by considering the linearly independent set: $\{(t, t^2, t^3, ...) \mid t \in \mathbb{R}\}$ which has the same cardinality as $\mathbb{R}$.