NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lau Tze Siong

MA3201 Algebra II
AY 2005/2006 Sem 1

Question 1

(a) Since $f(\sqrt{-1}) = \sqrt{-1} - \sqrt{-1} - (-1) - 1 = 0$.Hence $t^2 + 1$ is a factor of f. Also since $f(1) = (1) + (1) - (1) - 1 = 0$, $t - 1$ is a factor of f. Hence $t^3 - t^2 + t - 1$ is a factor of f. By comparing coefficients we have $f = (t^2+t+1)(t^2+1)(t-1)$. Since $(t^2+t+1), (t^2+1)$ has no roots in $\mathbb{R}$, they are irreducible in $\mathbb{R}[t]$. Since $\mathbb{R}$ is a field, $\mathbb{R}[t]$ is a Euclidean Domain. Therefore $(t^2+t+1), (t^2+1)$ is prime in $\mathbb{R}[t]$.
Hence the prime factorization of f in $\mathbb{R}[t]$ is $(t^2+1)(t^2+t+1)(t-1)$.

(b) The prime factorization of f in $\mathbb{C}[t]$ is $(t-i)(t+i)(t+\frac{1-i\sqrt{3}}{2})(t+\frac{1+i\sqrt{3}}{2})(t-1)$.

Question 2

(a) Since $2 \nmid 1$, $2 \mid 2$,$2 \mid 10$ and $2^2 \nmid 10$, by Eisenstein's Criterion, $t^9 + 2t + 10$ is irreducible in $\mathbb{Q}$.

(b) Claim: $t^5 + t^2 + 1$ is irreducible in $\mathbb{Q}[t]$
Proof:
By Gauss's Lemma, it suffices to show that $t^5 + t^2 + 1$ is irreducible in $\mathbb{Z}[t]$.
Suppose that $t^5 + t^2 + 1$ is reducible in $\mathbb{Z}[t]$.
Case 1) $t^5 + t^2 + 1$ has a linear factor.
If $t^5 + t^2 + 1$ has a linear factor then it must be either $(t+1)$ or $(t-1)$. We can easily check that $1, -1$ are not roots of $t^5 + t^2 + 1$.
Hence $t^5 + t^2 + 1$ has no linear factors.

Case 2)$t^5 + t^2 + 1$ has a quadratic factor.
Hence $t^5 + t^2 + 1 = (t^3 + at^2 + bt + c)(t^2 + dt + e)$. By comparing coefficient of t^0.
Case2.1)$c = e = 1$
Comparing coefficients for t^4, t^3, t^2, t we have

$$\begin{aligned} a + d &= 0 \\ ad + 1 + b &= 0 \\ a + 1 + bd &= 1 \\ b + d &= 0 \end{aligned}$$

Solving we have

$$a^2 - a - 1 = 0$$

which has no solutions in $\mathbb{Z}$.

Case 2.2)$c = e = -1$

$$\begin{aligned} a + d &= 0 \\ ad - 1 + b &= 0 \\ -a - 1 + bd &= 1 \\ -b - d &= 0 \end{aligned}$$

Solving we have

$$a^2 - a + 1 = 0$$

which has no solutions in $\mathbb{Z}$.

Hence $t^5 + t^2 + 1$ is irreducible in $\mathbb{Z}[t]$. Therefore $t^5 + t^2 + 1$ is irreducible in $\mathbb{Q}[t]$.

Question 3

$t_1^3 + t_2^3 + t_3^3 = (t_1 + t_2 + t_3)^3 - 3(t_1t_2^2 + t_1t_3^2 + t_2t_3^2 + t_2t_1^2 + t_3t_1^2 + t_3t_2^2) - 6(t_1t_2t_3)$

Question 4

Since A satisfies $A^2 - A - 6I = 0$. A satisfies the polynomial $x^2 - x - 6 = (x + 2)(x - 3)$. Let $m(x)$ be the minimal polynomial for A. Hence we have $m(x) \mid (x + 2)(x - 3)$ Therefore $m(x)$ can be expressed as a product of distinct linear factors. Therefore A is diagonalize.
The eigenvalues of A are either (3 and -2)or 3 or -2.

Question 5

Let $\phi : K[x, y]/(xy - 1) \to S$ be a ring isomorphism where $S = Im(\phi)$ and U be the set of units in S.
Since ϕ is a isomorphism, $\phi(1+\langle xy-1\rangle) = 1_S$. We can express $1+\langle xy-1\rangle$ as $xy-1+1+\langle xy-1\rangle = xy + \langle xy - 1\rangle$. Hence $\phi(1 + \langle xy - 1\rangle) = \phi(x + \langle xy - 1\rangle)\phi(y + \langle xy - 1\rangle) = 1_S$. Hence $\phi(x + \langle xy - 1\rangle)$ and $\phi(y + \langle xy - 1\rangle)$ are units in S. Hence $\phi(x + \langle xy - 1\rangle), \phi(y + \langle xy - 1\rangle) \in U$.

For any $\sum_{i=1}^{n} a_i x^{p_i} y^{q_i} + \langle xy - 1\rangle \in K[x, y]/(xy - 1)$ such that $a_i \in K$ and $p_i, q_i \in \mathbb{N} \cup \{0\}$,

$$\phi(\sum_{i=1}^{n} a_i x^{p_i} y^{q_i} + \langle xy - 1\rangle) = \sum_{i=1}^{n} \phi(a_i + \langle xy - 1\rangle)\phi(x^{p_i} + \langle xy - 1\rangle)\phi(y^{q_i} + \langle xy - 1\rangle) \in U$$

since $\phi(a_i + \langle xy - 1\rangle), \phi(x^{p_i} + \langle xy - 1\rangle), \phi(y^{q_i} + \langle xy - 1\rangle) \in S$.
Hence every element of S is a unit.
Now suppose S is a polynomial ring of one variable over K.
Hence $t + 1 \in S$ but $t + 1$ is not a unit.
Therefore $K[x, y]/(xy - 1)$ is not isomorphic to a polynomial ring in one variable over a field K.