NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1104 Multivariable Calculus
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Question 1

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\\ &= 2f_x e^{2u}+f_y v\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right) &= 2e^{2u}\left(\frac{\partial}{\partial v}f_x\right)+\left[\frac{\partial v}{\partial v}\cdot f_y+v\cdot\frac{\partial}{\partial v}f_y\right]\\ &= 2e^{2u}\left(f_{xx}\frac{\partial x}{\partial v}+f_{xy}\frac{\partial y}{\partial v}\right)+f_y+v\left(f_{yx}\frac{\partial x}{\partial v}+f_{yy}\frac{\partial y}{\partial v}\right)\\ &= 2e^{2u}\left(f_{xx}0+f_{xy}u\right)+f_y+v\left(f_{yx}0+f_{yy}u\right)\\ &= 2uf_{xy}e^{2u}+f_{yy}uv+f_y\end{aligned}$$

Question 2

(a)

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0}{h}=0$$
$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

When $(x,y)\neq(0,0)$:

$$f_x=\frac{y^5\left(x^4+y^4\right)-\mathrm{xy}^5 4x^3}{\left(x^4+y^4\right)^2}=\frac{y^9-3x^4y^5}{\left(x^4+y^4\right)^2}$$
$$f_y=\frac{5\mathrm{xy}^4\left(x^4+y^4\right)-\mathrm{xy}^5 4y^3}{\left(x^4+y^4\right)^2}=\frac{5x^5y^4+\mathrm{xy}^8}{\left(x^4+y^4\right)^2}$$

Hence:

$$f_{xy}(0,0)=\lim_{h\to 0}\frac{f_x(0,h)-f_x(0,0)}{h}=\lim_{h\to 0}\frac{h}{h}=1$$
$$f_{yx}(0,0)=\lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

(b) Yes, it is differentiable.

$$\Delta\mathrm{z}=f(\Delta\mathrm{x},\Delta\mathrm{y})-f(0,0)=\frac{\Delta\mathrm{x}\Delta\mathrm{y}^5}{\Delta\mathrm{x}^4+\Delta\mathrm{y}^4}=f_x(0,0)\Delta\mathrm{x}+f_y(0,0)\Delta\mathrm{y}+\epsilon_1\Delta\mathrm{x}+\epsilon_2\Delta\mathrm{y}$$

where $f_x(0,0)=0$, $f_y(0,0)=0$, $\epsilon_1=\frac{\Delta \mathrm{y}^5}{\Delta \mathrm{x}^4+\Delta \mathrm{y}^4}$ and $\epsilon_2=0$ and as $(\Delta \mathrm{x},\Delta \mathrm{y})\to(0,0)$, ϵ_1, $\epsilon_2\to 0$.
Hence by the definition of differentiability, $f(x,y)$ is differentiable at $(0,0)$.

Note: To show that $\epsilon_1\to 0$ as $(\Delta x,\Delta y)\to(0,0)$, notice that $\Delta y^4\leq \Delta x^4+\Delta y^4$,
So $0\leq|\epsilon_1|=\left|\frac{\Delta y^5}{\Delta x^4+\Delta y^4}\right|\leq\left|\frac{\Delta y^5}{\Delta y^4}\right|=|\Delta y|$
Hence, by squeeze theorem, $\epsilon_1\to 0$ as $(\Delta x,\Delta y)\to(0,0)$

(c) When $(x,y)\neq(0,0)$:

$$f_{xy}(x,y)=\frac{-15x^4y^4+9y^8}{(x^4+y^4)^2}-\frac{8y^3\left(-3x^4y^5+y^9\right)}{(x^4+y^4)^3}$$

Then we consider the path $x=t, y=t$, then:

$$\lim_{t\to 0} f_{xy}(t,t)=\frac{1}{2}$$

However, $f_{xy}(0,0)=1\neq\lim_{t\to 0} f_{xy}(t,t)=0.5$.
Hence, $f_{xy}(x,y)$ is not continuous at $(0,0)$.

Question 3

First, we need to use Lagrange Multiplier to find the extreme value, i.e. $\nabla f=\lambda\nabla g$, thus solve:

$$\begin{aligned} xy^2+xz^2&=2\lambda x\\ yx^2+yz^2&=2\lambda y\\ zy^2+zx^2&=2\lambda z\\ x^2+y^2+z^2&=4 \end{aligned}$$

Since $x,y,z\geq 1$,

$$\begin{aligned} y^2+z^2&=2\lambda && (1)\\ x^2+z^2&=2\lambda && (2)\\ y^2+x^2&=2\lambda && (3)\\ x^2+y^2+z^2&=4 && (4) \end{aligned}$$

Then substitute (1), (2), (3) to (4), we can obtain $\lambda=\frac{4}{3}$ and by subtract (1) from (2), we can deduce $x=y$ and reject the case $x=-y$. Similarly, we can deduce $y=z$ by subtract (2) from (3). Hence, when $x=y=z=\frac{2}{\sqrt{3}}$, we obtain a critical value $\frac{8}{3}$, but it may be a local minimum, local maximum, or neither.

If we have the extra constraint $x,y,z\geq 1$, then we have to take special care about this boundary value. By substitute $x=1$, we have to find the minimum value of the function $f(y,z)=\frac{y^2+z^2+y^2z^2}{2}$ subject to the condition that $y^2+z^2=3$ and $x,y,z\geq 1$. Then by substituting $y^2=3-z^2$ and $1\leq z^2\leq 2$, our job is to minimize:

$$f(z)=\frac{-z^4+3z^2+3}{2}=\frac{-(z^2-\frac{3}{2})^2+\frac{9}{4}+3}{2}\geq\frac{-\frac{1}{4}+\frac{9}{4}+3}{2}=\frac{5}{2}$$

Since $\frac{5}{2}\leq\frac{8}{3}$, then $(x,y,z)=(1,1,\sqrt{2})$ or $(x,y,z)=(1,\sqrt{2},1)$ are the global minimum points.

Hence, by symmetry, $(x,y,z)=(1,1,\sqrt{2})$ or $(1,\sqrt{2},1)$ or $(\sqrt{2},1,1)$ are the points that minimize the function whose value is 2.5.

Question 4

(a) First, we need to compute the directional vector, $\mathbf{u}$. Let $f(x,y) = 2(x-2)^2 + (y-1)^2$:

$$\nabla f(3, \sqrt{2}+1) = < 4(x-2),\ 2(y-1) > = < 4,\ 2\sqrt{2} >$$

Hence, the perpendicular directional vector which has the correct orientation as given from the question should be:

$$\mathbf{u} \cdot \nabla f = \mathbf{0} \rightarrow \mathbf{u} = < -\frac{\sqrt{3}}{3},\ \frac{\sqrt{6}}{3} >$$

Since it is easy to compute:

$$\nabla T(3, \sqrt{2}+1) = < 2x,\ -2y > = < 6,\ -2\sqrt{2}-2 >$$

Then:

$$D_{\mathbf{u}}T = \nabla T \cdot \mathbf{u} = \frac{1}{\sqrt{3}}(-10 - 2\sqrt{2})$$

(b)

$$\begin{aligned}
\int_0^1 \sqrt{\ln\left(\frac{1}{x}\right)}\, dx &= \int_0^1 \int_0^{\sqrt{\ln(\frac{1}{x})}} dy\, dx \\
&= \int_0^{\infty} \int_0^{e^{-y^2}} dx\, dy \\
&= \int_0^{\infty} e^{-y^2} dy \\
&= \frac{\sqrt{\pi}}{2}
\end{aligned}$$

Question 5

(a)

$$\int_0^2 \int_0^{4-y^2} \int_0^{y/2} f(x,y,z) dx\, dz\, dy = \int_0^1 \int_0^{4-4x^2} \int_{2x}^{\sqrt{4-z}} f(x,y,z) dy\, dz\, dx$$

(b) Firstly, we apply the Green's Theorem:

$$\begin{aligned}
\int_C (e^x + 6\text{xy})\, dx + \left(8x^2 + \sin\left(y^2\right)\right) dy &= -\iint_D (16x - 6x)\, dA \\
&= -\int_{-\sqrt{\frac{1}{5}}}^{\sqrt{\frac{1}{5}}} \int_0^{\sqrt{1-5x^2}} 10x\, dy\, dx \\
&= \int_{-\sqrt{\frac{1}{5}}}^{\sqrt{\frac{1}{5}}} \sqrt{1-5x^2}\, d(1-5x^2) \\
&= \left[\frac{\left(1-5x^2\right)^{1.5}}{1.5}\right]_{-\sqrt{\frac{1}{5}}}^{\sqrt{\frac{1}{5}}} \\
&= 0
\end{aligned}$$

Question 6

(a) Let A_1 denote the surface $z = 1, x \in [-1, 1], y \in [-1, 1]$, with outward pointing normal vector $< 0, 0, 1 >$.
So, $\mathbf{F} \cdot \hat{n} = -1$ on this surface, hence $\iint_{A_1} \mathbf{F} \cdot \hat{n} dS = -4$.

Let A_2 denote the surface $z = -1$ (with suitable range for x and y), with outward pointing normal vector $< 0, 0, -1 >$.
So, $\mathbf{F} \cdot \hat{n} = -1$ on this surface, hence $\iint_{A_2} \mathbf{F} \cdot \hat{n} dS = -4$.
Similarly,

Area	Equation	normal vector	$\mathbf{F} \cdot \hat{n}$	$\iint_{A_i} \mathbf{F} \cdot \hat{n} dS$
A_3	$x = 1$	$< 1, 0, 0 >$	$\dfrac{y}{1 + y^2 + z^2}$	$\displaystyle\int_{-1}^{1}\int_{-1}^{1} \frac{y}{1 + y^2 + z^2} dydz$
A_4	$x = -1$	$< -1, 0, 0 >$	$\dfrac{-y}{1 + y^2 + z^2}$	$\displaystyle\int_{-1}^{1}\int_{-1}^{1} \frac{-y}{1 + y^2 + z^2} dydz$
A_5	$y = 1$	$< 0, 1, 0 >$	$\dfrac{-x}{x^2 + 1 + z^2}$	$\displaystyle\int_{-1}^{1}\int_{-1}^{1} \frac{-x}{x^2 + 1 + z^2} dxdz$
A_6	$y = -1$	$< 0, -1, 0 >$	$\dfrac{x}{x^2 + 1 + z^2}$	$\displaystyle\int_{-1}^{1}\int_{-1}^{1} \frac{x}{x^2 + 1 + z^2} dxdz$

So, $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{A_1} \mathbf{F} \cdot d\mathbf{S} + \ldots + \iint_{A_6} \mathbf{F} \cdot d\mathbf{S} = -4 + (-4) = -8$
(Note that $\iint_{A_3+A_4} \mathbf{F} \cdot d\mathbf{S} = 0, \iint_{A_5+A_6} \mathbf{F} \cdot d\mathbf{S} = 0$)

(b) Let E_1 represents the cube in part (a), E_2 represents the sphere $x^2 + y^2 + z^2 = 6^2$ minus the cube in part (a) (i.e. a sphere with a cube hole), and E_3 represents $\mathbb{R}^3 - (E_1 \cup E_2)$.
Let S_1 denotes the cube surface (with normal vector pointing from the region E_1 to E_2), S_2 denotes the surface on the sphere(with normal vector pointing from the region E_3 to E_2)
When using the Divergence Theorem, one should be aware of the orientation of the normal vector.(Note that a minus sign is multiplied to the volume integral, since we have chosen normal vectors such that they point into the solid E_2)

$$\begin{aligned}
-\iiint_{E_2} div\mathbf{F}\, dV &= \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} \\
-\iiint_{E_2} -1 dV &= -8 + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} \\
\frac{4}{3}\pi \cdot 6^3 - 2^3 &= -8 + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} \\
288\pi &= \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}
\end{aligned}$$

(Note: One can easily show that $div\mathbf{F} = -1$ for all $(x, y, z) \neq (0, 0, 0)$)

Question 7

(a) Parameterize the surface by $f(x, y) = (x, y, 7 - y)$, so $\dfrac{\partial f}{\partial x} \times \dfrac{\partial f}{\partial y} =< 0, 1, 1 >$.

Apply Stokes' Theorem:

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S curl\mathbf{F}\cdot d\mathbf{S} \\
&= \iint_S < \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} > \cdot \mathbf{n}\, dA \\
&= \iint_{2x^2+y^2\le 1} < 1-2x^2-y^2,\ 1-2x^2-y^2,\ 4xz+2yz > \cdot < 0,\ -1,\ -1 > \ dydx \\
&= \iint_{2x^2+y^2\le 1} -\big(1-2x^2-y^2+4xz+2yz\big)\ dydx \\
&= -\iint_{2x^2+y^2\le 1} (1-2x^2-3y^2+28x-4xy+14y)\, dydx \\
&= -\iint_{2x^2+y^2\le 1} (1-2x^2-3y^2)\, dydx \quad \text{(We will justify this step later)} \\
&= -\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} 1-2x^2-3y^2 dydx \\
&= -\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (1-2x^2)\left[\sqrt{1-2x^2} - \left(-\sqrt{1-2x^2}\right)\right] - 3\cdot\frac{1}{3}\left[\sqrt{1-2x^2}^3 - \left(-\sqrt{1-2x^2}\right)^3\right] dx \\
&= -\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 0dx = 0
\end{aligned}$$

Note:

$$\begin{aligned}
\iint_{2x^2+y^2\le 1} (28-4y)x\, dxdy &= \int_{-1}^{1} \int_{-\frac{1}{2}\sqrt{1-y^2}}^{\frac{1}{2}\sqrt{1-y^2}} (28-4y)x\, dxdy \\
&= \int_{-1}^{1} (28-4y)\left[\frac{1}{2}x^2\right]_{x=-\frac{1}{2}\sqrt{1-y^2}}^{x=\frac{1}{2}\sqrt{1-y^2}} dy \\
&= \int_{-1}^{1} 0\, dy = 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\iint_{2x^2+y^2\le 1} 14y\, dydx &= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} 14y\, dydx \\
&= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 14\left[\frac{1}{2}y^2\right]_{y=-\sqrt{1-2x^2}}^{y=\sqrt{1-2x^2}} dy \\
&= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 0\, dx = 0
\end{aligned}$$

(b) First, we have to parameterize the surface. In order to do that, note that the circle in the yz-plane of radius 1 centered at $(0, 2, 0)$ can be parameterized by $(0, 2+\cos\phi, \sin\phi)$, $\phi \in [0, 2\pi]$. Then any point on the surface is a rotation of angle θ (about the z-axis) from some point on this circle. ($\theta \in [0, 2\pi]$)

Thus, the parametrization of the surface can be expressed as:

$$f(r,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2+\cos\phi \\ \sin\phi \end{bmatrix}$$
$$=< (2+\cos\phi)(-\sin\theta), (2+\cos\phi)(\cos\theta), \sin\phi >$$

One can show that:

$$\frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \phi} =< (2+\cos\phi)(-\sin\theta)(\cos\phi), (2+\cos\phi)(-\cos\theta)(\cos\phi), (2+\cos\phi)(\sin\phi) >$$

$$\left\|\frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \phi}\right\| = \sqrt{(2+\cos\phi)^2} = (2+\cos\phi)$$

(Since $(2+\cos\phi) \geq 1 > 0$)

Hence, the surface area is:

$$A_{\text{area}} = \int_0^{2\pi}\int_0^{2\pi} (2+\cos\phi)d\phi d\theta = 8\pi^2$$