NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lee Yung Hei, Joseph Nah

MA1102R Calculus
AY 2007/2008 Sem 1

Question 1

(a) Since $\lim_{x\to 2} x^2 - 4 = 0$ and $\lim_{x\to 2} x^3 - 8 = 0$, we apply L'Hôpital's rule to get $\lim_{x\to 2} \frac{2x}{3x^2} = \frac{2\cdot 2}{3\cdot 2^2} = \frac{1}{3}$.

(b) Using L'Hôpital's rule repeatedly, we get,

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin(x^2)} - \frac{1}{x^2}\right) = \lim_{x\to 0}\frac{x^2-\sin(x^2)}{x^2\sin(x^2)} &= \lim_{x\to 0}\frac{x^2}{\sin(x^2)}\lim_{x\to 0}\frac{x^2-\sin(x^2)}{x^4} \\
&= 1\cdot\lim_{x\to 0}\frac{2x-2x\cos(x^2)}{4x^3} \\
&= \lim_{x\to 0}\frac{1-\cos(x^2)}{2x^2} \\
&= \lim_{x\to 0}\left(\frac{1-\cos^2(x^2)}{x^4}\right)\left(\frac{x^2}{2(1+\cos(x^2))}\right) \\
&= \lim_{x\to 0}\left(\frac{\sin(x^2)}{x^2}\right)^2\lim_{x\to 0}\frac{x^2}{2(1+\cos(x^2))} \\
&= 1^2\cdot 0 = 0.
\end{aligned}$$

(c) Since $\lim_{x\to 0^+}\sin x = \lim_{m\to\infty}\frac{1}{m}$, we have $\lim_{x\to 0^+}(\sin x)^{\sin x} = \lim_{m\to\infty}\left(\frac{1}{m}\right)^{\frac{1}{m}} = \lim_{m\to\infty} m^{(-\frac{1}{m})}$.
Using L'Hôpital's rule, $\lim_{m\to\infty}\ln\left(m^{(-\frac{1}{m})}\right) = \lim_{m\to\infty}\frac{-\ln m}{m} = \lim_{m\to\infty}\frac{-1}{m} = 0$.
Since $f:\mathbb{R}^+\to\mathbb{R}$ such that $f(x)=e^x$ is continuous on $\mathbb{R}$, we have

$$\lim_{x\to 0^+}(\sin x)^{\sin x} = \lim_{m\to\infty} m^{(-\frac{1}{m})} = \lim_{m\to\infty} f\left(\ln\left(m^{(-\frac{1}{m})}\right)\right) = f\left(\lim_{m\to\infty} m^{(-\frac{1}{m})}\right) = f(0) = 1.$$

Question 2

(a) We have,

$$\begin{aligned}
\int_0^1 \frac{x^3+2}{4-x^2}\,dx &= \int_0^1\left(-x+\frac{2.5}{2-x}-\frac{1.5}{2+x}\right)dx \\
&= \left[-\frac{x^2}{2} - 2.5\ln(2-x) - 1.5\ln(2+x)\right]_0^1 \\
&= -\frac{1}{2} + 4\ln 2 - 1.5\ln 3.
\end{aligned}$$

(b) We have,

$$\begin{aligned}\int_0^1 x^3 e^{x^2}\,dx = \int_0^1 \frac{x^3}{2x}\cdot 2xe^{x^2}\,dx &= \int_0^1 \frac{x^2}{2}\cdot 2xe^{x^2}\,dx \\ &= \left[\frac{x^2}{2}\cdot e^{x^2}\right]_0^1 - \int_0^1 xe^{x^2} \\ &= \frac{e}{2} - \left[\frac{e^{x^2}}{2}\right]_0^1 = \frac{1}{2}.\end{aligned}$$

Question 3

(a) Let $b_n = \dfrac{1}{\sqrt{\ln\ln n}}$, $n \in \mathbb{Z}_{\geq 3}$.
Since $3 > e$ and ln is an increasing function, for all $n \geq 3$, we have $\ln\ln n > 0$, and so $b_n > 0$.
Also, $\ln\ln(n+1) > \ln\ln(n)$, and so $b_{n+1} = \dfrac{1}{\sqrt{\ln\ln(n+1)}} < \dfrac{1}{\sqrt{\ln\ln n}} = b_n$, and $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{\ln\ln n}} = 0$.
Therefore by Alternating Series test, the series is convergent.

(b) We notice that $(\ln\ln n)^{\ln n} = e^{\ln\left((\ln\ln n)^{\ln n}\right)} = e^{(\ln n)(\ln\ln\ln n)} = e^{\ln\left(n^{(\ln\ln\ln n)}\right)} = n^{\ln\ln\ln n}$.
When $n \geq e^{e^{e^2}}$, we have $\ln\ln\ln n \geq 2$, and so $\dfrac{1}{(\ln\ln n)^{\ln n}} = \dfrac{1}{n^{\ln\ln\ln n}} \leq \dfrac{1}{n^2}$.
Since $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2}$ is convergent, by Comparison Test, $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(\ln\ln n)^{\ln n}}$ is convergent.

(c) $\displaystyle\sum_{n=1}^{\infty}\left(\sin\frac{1}{2n} - \sin\frac{1}{2n+1}\right) = \sum_{n=2}^{\infty}\left((-1)^n \sin\frac{1}{n}\right)$.
Since $\lim\limits_{n\to\infty} \sin\dfrac{1}{n} = 0$ and $0 < \sin\frac{1}{n+1} < \sin\frac{1}{n}$ for all $n > 2$.
By Alternating Series Test, the series is convergent.

Question 4

Let the length of the track be a and the radius of the semicircles be $\dfrac{b}{2}$.
From the length of the track, we have $2a + \pi b = 5$.
Therefore, the shaded area, $A = ab = \left(\dfrac{5}{2} - \dfrac{\pi}{2}b\right)b = \dfrac{5}{2}b - \dfrac{\pi}{2}b^2$.
This give us $\dfrac{dA}{db} = \dfrac{5}{2} - \pi b$ and $\dfrac{d^2A}{db^2} = -\pi$. When $\dfrac{dA}{db} = 0$, we have $\dfrac{5}{2} - \pi b = 0$, i.e. $b = \dfrac{5}{2\pi}$.
Since $\dfrac{d^2A}{db^2} = -\pi$, A attain maximum when $b = \dfrac{5}{2\pi}$.
This give us $A = \dfrac{5}{2}b - \dfrac{\pi}{2}b^2 = \dfrac{5}{2}\left(\dfrac{5}{2\pi}\right) - \dfrac{\pi}{2}\left(\dfrac{5}{2\pi}\right)^2 = \dfrac{25}{8\pi}$.

Question 5

It suffice to consider only the part with $y \geq 0$.
Thus we have $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1 \;\Rightarrow\; y^{\frac{2}{3}} = 1 - x^{\frac{2}{3}} \;\Rightarrow\; y = \left(\sqrt{1 - x^{\frac{2}{3}}}\right)^3$.

This give us $\dfrac{dy}{dx} = \left(\dfrac{3}{2}\right)\sqrt{1-x^{\frac{2}{3}}}\left(-\frac{2}{3}x^{-\frac{1}{3}}\right) = -x^{-\frac{1}{3}}\sqrt{1-x^{\frac{2}{3}}}$.

$$\begin{aligned}
\text{Surface area of revolution} &= 2\int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= 4\pi\int_0^1 \left(\sqrt{1-x^{\frac{2}{3}}}\right)^3\sqrt{1+x^{-\frac{2}{3}}\left(1-x^{\frac{2}{3}}\right)}\,dx \\
&= -\frac{12\pi}{5}\int_0^1 \left(\sqrt{1-x^{\frac{2}{3}}}\right)^3\left(-\frac{2}{3}x^{-\frac{1}{3}}\right)\left(\frac{5}{2}\right)\,dx \\
&= -\frac{12\pi}{5}\left[\left(\sqrt{1-x^{\frac{2}{3}}}\right)^5\right]_0^1 = \frac{12\pi}{5}.
\end{aligned}$$

Question 6

Notice that $\displaystyle\lim_{n\to\infty}\sqrt[n]{\left|\left(1+\frac{1}{n}\right)^n x^n\right|} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\cdot|x| = |x|$.

Thus radius of convergence of $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n x^n$, $R=1$.

When $x=1$, we have $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n x^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 0$, and so by the Test of Divergence, $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n x^n$ is divergent when $x=1$.

When $x=-1$, we have $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n x^n = \lim_{n\to\infty}(-1)^n\left(1+\frac{1}{n}\right)^n$, which does not exists.

Thus by the Test of Divergence, $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n x^n$ is divergent when $x=-1$.

Therefore, the interval of convergence of $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n x^n$ is $x\in(-1,1)$.

Question 7

(a) By L'Hôpital's rule, we have $\displaystyle\lim_{y\to0} y\ln y = \lim_{y\to0}\frac{\ln y}{\frac{1}{y}} = \lim_{y\to0}\frac{\frac{1}{y}}{\frac{-1}{y^2}} = \lim_{y\to0} -y = 0$.

Since $\ln 0$ is undefined, we have,

$$\begin{aligned}
\int_0^1 \ln x\,dx &= \lim_{y\to0}\int_y^1 \ln x\,dx \\
&= \lim_{y\to0}\left[x\ln x - x\right]_y^1 \\
&= (0-1) - \lim_{y\to0}(y\ln y - y) \\
&= -1 - \lim_{y\to0} y\ln y - \lim_{y\to0} y \\
&= -1+0+0 = -1.
\end{aligned}$$

(b) For all $x \in \mathbb{R}^+$, we have $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x} > 0$, and so $\ln x$ is increasing on $\mathbb{R}^+$.

Thus by considering Riemann sum for $\displaystyle\int_0^{1-\frac{1}{n}} \ln x \, dx$ with $n-1$ intervals of width $\frac{1}{n}$, we have

$$\int_0^{1-\frac{1}{n}} \ln x \, dx \leq \sum_{i=1}^{n-1} \frac{1}{n} \ln\left(\frac{i}{n}\right) \quad = \quad \frac{1}{n}\left(\ln \frac{(n-1)!}{n^{n-1}}\right) = \ln\left(\frac{(n-1)!}{n^{n-1}}\right)^{\frac{1}{n}}.$$

Also by considering Riemann sum for $\displaystyle\int_{\frac{1}{n}}^{1} \ln x \, dx$ with $n-1$ intervals of width $\frac{1}{n}$, we have

$$\int_{\frac{1}{n}}^{1} \ln x \, dx \geq \sum_{i=1}^{n-1} \frac{1}{n} \ln\left(\frac{i}{n}\right) \quad = \quad \frac{1}{n}\left(\ln \frac{(n-1)!}{n^{n-1}}\right) = \ln\left(\frac{(n-1)!}{n^{n-1}}\right)^{\frac{1}{n}}.$$

(c) From (7b.), we have

$$\begin{aligned}
\lim_{n\to\infty} \int_0^{1-\frac{1}{n}} \ln x \, dx \leq &\quad \lim_{n\to\infty} \ln\left(\frac{(n-1)!}{n^{n-1}}\right)^{\frac{1}{n}} &&\leq \lim_{n\to\infty} \int_{\frac{1}{n}}^{1} \ln x \, dx \\
\int_0^1 \ln x \, dx \leq &\quad \lim_{n\to\infty} \ln\left(\frac{(n-1)!}{n^{n-1}}\right)^{\frac{1}{n}} &&\leq \int_0^1 \ln x \, dx \qquad \left(\text{Since } \lim_{n\to\infty} \frac{1}{n} = 0\right) \\
-1 \leq &\quad \lim_{n\to\infty} \ln\left(\frac{(n-1)!}{n^{n-1}}\right)^{\frac{1}{n}} &&\leq -1 \qquad (\text{From Q7a}).
\end{aligned}$$

Thus by Squeeze theorem, we have $\displaystyle\lim_{n\to\infty} \ln\left(\frac{(n-1)!}{n^{n-1}}\right)^{\frac{1}{n}} = -1$.

Since $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = e^x$ is a continuous function on $\mathbb{R}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n \cdot (n-1)!}{n \cdot n^{n-1}}\right)^{\frac{1}{n}} &= \lim_{n\to\infty} \left(\frac{(n-1)!}{n^{n-1}}\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} f\left(\ln\left(\frac{(n-1)!}{n^{n-1}}\right)^{\frac{1}{n}}\right) \\
&= f\left(\lim_{n\to\infty} \ln\left(\frac{(n-1)!}{n^{n-1}}\right)^{\frac{1}{n}}\right) \\
&= f(-1) = e^{-1}.
\end{aligned}$$

Question 8

(a) Since $f'(x)$ is continuous on $[a, b]$, by Extreme Value Theorem, there exists $c_1, c_2 \in [a, b]$ such that for all $x \in [a, b]$, we have $f'(c_1) \leq f'(x) \leq f'(c_2)$. Let $c = c_1$ if $|f'(c_1)| \geq |f'(c_2)|$, and $c = c_2$ otherwise. Then for all $x \in [a, b]$, we have $|f'(x)| \leq |f'(c)|$.

By Mean Value Theorem, for all $s \in (a, u)$, there exists $m \in (a, s)$ such that $\dfrac{f(s) - f(a)}{s - a} = f'(m)$. Thus, $f(s) = |f(s) - f(a)| = |f'(m)||s - a| = |f'(m)|(s - a) \leq |f'(c)|(s - a)$.

Similarly for all $t \in (u, b)$, there exists $n \in (t, b)$ such that $\dfrac{f(b) - f(t)}{b - t} = f'(n)$.
Thus, $f(t) = |f(b) - f(t)| = |f'(n)||b - t| = |f'(n)|(b - t) \leq |f'(c)|(b - t)$.
Therefore $c \in [a, b]$ is what we wanted.

(b) Let $u = \dfrac{a+b}{2}$.

Then there exists $r \in [a, b]$ such that for every $x \in \left(a, \dfrac{a+b}{2}\right)$, we have $f(x) \leq |f'(r)|(x - a)$.

Therefore,

$$\begin{aligned}
\int_a^{\frac{a+b}{2}} f(x)\ dx &\leq \int_a^{\frac{a+b}{2}} |f'(r)|(x-a)\ dx \\
&= |f'(r)| \int_a^{\frac{a+b}{2}} (x-a)\ dx \\
&= |f'(r)| \left[\frac{(x-a)^2}{2}\right]_a^{\frac{a+b}{2}} \\
&= |f'(r)| \frac{(b-a)^2}{8}.
\end{aligned}$$

Similarly, there exists $r \in [a, b]$ such that for every $x \in \left(\dfrac{a+b}{2}, b\right)$, we have $f(x) \leq |f'(r)|(b - x)$.

So,

$$\begin{aligned}
\int_{\frac{a+b}{2}}^{b} f(x)\ dx &\leq \int_{\frac{a+b}{2}}^{b} |f'(r)|(b-x)\ dx \\
&= |f'(r)| \int_{\frac{a+b}{2}}^{b} (b-x)\ dx \\
&= |f'(r)| \left[\frac{-(b-x)^2}{2}\right]_{\frac{a+b}{2}}^{b} \\
&= |f'(r)| \frac{(b-a)^2}{8}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_a^b f(x)\ dx &= \int_a^{\frac{a+b}{2}} f(x)\ dx + \int_{\frac{a+b}{2}}^{b} f(x)\ dx \\
&\leq |f'(r)| \frac{(b-a)^2}{8} + |f'(r)| \frac{(b-a)^2}{8} \\
&= |f'(r)| \frac{(b-a)^2}{4} \\
|f'(r)| &\geq \frac{4}{(b-a)^2} \int_a^b f(x)\ dx.
\end{aligned}$$