NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1104 Multivariable Calculus
AY 2007/2008 Sem 2

Question 1

(a) (i) Note that

$$\begin{aligned} PQ &= \sqrt{(1-0)^2+(0-5)^2+(0-0)^2} = \sqrt{26}, \\ QR &= \sqrt{(0-0)^2+(5-2)^2+(0-(-3))^2} = 3\sqrt{2}, \\ PR &= \sqrt{(1-0)^2+(0-2)^2+(0-(-3))^2} = \sqrt{14}. \end{aligned}$$

Thus, one has

$$\cos\angle PQR = \frac{PQ^2+QR^2-PR^2}{2\cdot PQ\cdot QR} = \frac{26+18-14}{2\cdot\sqrt{26}\cdot 3\sqrt{2}} = \frac{5\sqrt{13}}{26}.$$

(ii) We have

$$\begin{aligned} \overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} = \langle 0,5,0\rangle - \langle 1,0,0\rangle = \langle -1,5,0\rangle, \\ \overrightarrow{PR} &= \overrightarrow{OR} - \overrightarrow{OP} = \langle 0,2,-3\rangle - \langle 1,0,0\rangle = \langle -1,2,-3\rangle, \\ \Rightarrow \text{Area of triangle } PQR &= \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| \\ &= \frac{1}{2}\left|\langle -1,5,0\rangle \times \langle -1,2,-3\rangle\right| \\ &= \frac{1}{2}\left|\langle -15,-3,3\rangle\right| \\ &= \frac{1}{2}\sqrt{(-15)^2+(-3)^2+3^2} = \frac{9\sqrt{3}}{2}. \end{aligned}$$

(iii) From part (ii), one has a normal vector of the plane to be $\overrightarrow{PQ}\times\overrightarrow{PR} = \langle -15,-3,3\rangle$. Then the equation of the plane must satisfy the following equation:

$$\langle -15,-3,3\rangle\cdot\langle x-1,y,z\rangle = 0.$$

This gives us the equation of the plane containing the triangle PQR to be $5x+y-z-5=0$.

(b) From the conditions given, we deduce that the volume of the parallelpiped is equal to $|\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)| = \mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3) = 7$. This would then imply that $\mathbf{v}_2\cdot(\mathbf{v}_3\times\mathbf{v}_1) = 7$. Thus, one has

$$\begin{aligned} \mathbf{k}_2\times\mathbf{k}_3 &= \frac{\mathbf{v}_3\times\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)} \times \frac{\mathbf{v}_1\times\mathbf{v}_2}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)} \\ &= \frac{\mathbf{v}_3\times\mathbf{v}_1}{7}\times\frac{\mathbf{v}_1\times\mathbf{v}_2}{7} \\ &= \frac{1}{49}(((\mathbf{v}_3\times\mathbf{v}_1)\cdot\mathbf{v}_2)\mathbf{v}_1 - ((\mathbf{v}_3\times\mathbf{v}_1)\cdot\mathbf{v}_1)\mathbf{v}_2) \\ &= \frac{1}{49}(7\mathbf{v}_1) = \frac{1}{7}\mathbf{v}_1. \\ \Rightarrow \mathbf{k}_1\cdot(\mathbf{k}_2\times\mathbf{k}_3) &= \frac{1}{7}\mathbf{v}_1\cdot\frac{\mathbf{v}_2\times\mathbf{v}_3}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)} = \frac{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}{7\cdot 7} = \frac{1}{7}. \end{aligned}$$

Question 2

(i) In order to find the point(s) of intersection between the surface S and the plane Π_k, we need to solve the following pair of simultaneous equations:

$$\begin{aligned} x+y+z &= k, \qquad (1)\\ x^2+x+2y^2+3y &= z. \qquad (2)\end{aligned}$$

From equation (1), we get $z = k-x-y$. By substituting this into equation (2), we get $k-x-y = x^2+x+2y^2+3y$, or equivalently, $(x+1)^2+2(y+1)^2 = k+3$.
Since $(x+1)^2$ and $(y+1)^2$ are both non-negative, it follows that for the surface S to intersect the plane Π_k in at least one point, one must have $k+3 \geq 0$, or equivalently, $k \geq -3$.

In particular, when $k=-3$, the equation becomes $(x+1)^2+2(y+1)^2 = 0$, so there would be only 1 point of intersection, namely at $(x,y,z) = (x,y,k-x-y) = (-1,-1,-1)$.
Hence, the value of k for which the surface S is tangent to the plane Π_k is $k=-3$, and the coordinates of the point of tangency P is $(-1,-1,-1)$.

(ii) From part (i), we know that the surface S intersects the plane Π_k in at least one point if and only if $k \geq -3$. Hence, the values of k for which the surface S intersects the plane Π_k in more than one point is $k > -3$.
As the curve must satisfy the equations $(x+1)^2+2(y+1)^2 = k+3$ and $z = k-x-y$, it follows that a smooth parametrization of the curve of intersection C is

$$\mathbf{r}(t) = \left\langle \sqrt{k+3}\sin t - 1, \sqrt{\frac{k+3}{2}}\cos t - 1, k+2-\sqrt{k+3}\sin t - \sqrt{\frac{k+3}{2}}\cos t \right\rangle, \ 0 \leq t < 2\pi.$$

(iii) At $(1,-1,1)$, we have $k = 1-1+1 = 1$, and thus $\sin t = \frac{1}{\sqrt{k+3}}(x+1) = \frac{1}{\sqrt{1+3}}(1+1) = 1$, which implies that $t = \frac{\pi}{2}$. Hence, at $(1,-1,1)$, we have

$$\begin{aligned}\mathbf{r}'(t) &= \left\langle \sqrt{k+3}\cos t, -\sqrt{\frac{k+3}{2}}\sin t, -\sqrt{k+3}\cos t + \sqrt{\frac{k+3}{2}}\sin t \right\rangle\\ &= \left\langle \sqrt{1+3}\cos\frac{\pi}{2}, -\sqrt{\frac{1+3}{2}}\sin\frac{\pi}{2}, -\sqrt{1+3}\cos\frac{\pi}{2} + \sqrt{\frac{1+3}{2}}\sin\frac{\pi}{2} \right\rangle = \left\langle 0, -\sqrt{2}, \sqrt{2}\right\rangle.\end{aligned}$$

Thus the parametric equations for the tangent line to the curve of intersection C at $(1,-1,1)$ are $x=1$, $y=-1-\sqrt{2}t$, $z = 1+\sqrt{2}t$, $t \in \mathbb{R}$.

Question 3

(a) From the equation $f(x,y) = x^3-2xy-y^3$, we get $f_x(x,y) = 3x^2-2y$ and $f_y(x,y) = -2x-3y^2$.
This would further imply that $f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = -2$ and $f_{yy}(x,y) = -6y$.
In order to find the critical points of f, we need to solve the following set of simultaneous equations:

$$\begin{aligned} 3x^2-2y &= 0, \qquad (3)\\ -2x-3y^2 &= 0. \qquad (4)\end{aligned}$$

From equation (3), we get $y = \frac{3x^2}{2}$. By substituting this into equation (4), we get $27x^4+8x = 0$ (after simplification), or equivalently, $x(3x+2)(9x^2-6x+4) = 0$. As $9x^2-6x+4 = (3x+1)^2+3 > 0$, it follows that the only roots of the equation $x(3x+2)(9x^2-6x+4) = 0$ are $x=0$ or $x = -\frac{2}{3}$.
From there, we get $y=0$ and $y = \frac{2}{3}$ respectively. So the critical points are $(0,0)$ and $\left(-\frac{2}{3}, \frac{2}{3}\right)$.

When $(x, y) = (0, 0)$, we have $f_{xx} = 0$, $f_{xy} = -2$ and $f_{yy} = 0$. Thus $f_{xx}f_{yy} - f_{xy}^2 = -4 < 0$, so this implies that the critical point $(0, 0)$ is a saddle point.
When $(x, y) = \left(-\frac{2}{3}, \frac{2}{3}\right)$, we have $f_{xx} = -4 < 0$, $f_{xy} = -2$ and $f_{yy} = -4$. Thus $f_{xx}f_{yy} - f_{xy}^2 = 12 > 0$, so this implies that the critical point $\left(-\frac{2}{3}, \frac{2}{3}\right)$ is a maximum point.

(b) Let $V(x, y, z) = xyz$. In order to find the maximum volume of the box, we need to find the maximum value of $V(x, y, z)$, subject to the constraint $4 - x^2 - y^2 - z = 0$.
Let $R(x, y, z) = 4 - x^2 - y^2 - z$. Then it follows that $\nabla V(x, y, z) = \langle yz, xz, xy\rangle$ and $\nabla R(x, y, z) = \langle -2x, -2y, -1\rangle$. By the Method of Lagrange Multipliers, we have

$$\begin{aligned} & \nabla V(x,y,z) = \lambda \nabla R(x,y,z) \\ \Rightarrow\ & \langle yz, xz, xy\rangle = \lambda\langle -2x, -2y, -1\rangle \\ \Rightarrow\ & yz = -2\lambda x,\ xz = -2\lambda y,\ xy = -\lambda \\ \Rightarrow\ & yz = 2x^2y,\ xz = 2xy^2. \end{aligned}$$

Since x and y are assumed to be positive, it follows that $z = 2x^2 = 2y^2$, which would also imply that $x^2 = y^2$, or equivalently $x = y$.
So one has $4 - x^2 - y^2 - z = 4 - x^2 - x^2 - 2x^2 = 4(1 - x^2) = 0$, which would imply that $x^2 = 1$, or equivalently, $x = 1$.
Hence, $y = 1$ and $z = 2$, so the maximum volume of the box is equal to $V(1, 1, 2) = 1(1)(2) = 2$, and the corresponding coordinates of Q when this occurs is $(1, 1, 2)$.

Question 4

(a) We have

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = e^{x+2y}\cdot\frac{1}{t} + 2e^{x+2y}\cdot\left(-\frac{t}{s^2}\right) = e^{x+2y}\left(\frac{1}{t} - \frac{2t}{s^2}\right), \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = e^{x+2y}\cdot\left(-\frac{s}{t^2}\right) + 2e^{x+2y}\cdot\frac{1}{s} = e^{x+2y}\left(\frac{2}{s} - \frac{s}{t^2}\right). \end{aligned}$$

When $s = 2$, $t = -3$, we have $x = -\frac{2}{3}$ and $y = -\frac{3}{2}$, so one has $x + 2y = -\frac{11}{3}$. Thus

$$\begin{aligned} \frac{\partial z}{\partial s} &= e^{x+2y}\left(\frac{1}{t} - \frac{2t}{s^2}\right) = e^{-\frac{11}{3}}\left(\frac{1}{(-3)} - \frac{2(-3)}{2^2}\right) = \frac{7}{6}e^{-\frac{11}{3}}, \\ \frac{\partial z}{\partial t} &= e^{x+2y}\left(\frac{2}{s} - \frac{s}{t^2}\right) = e^{-\frac{11}{3}}\left(\frac{2}{2} - \frac{2}{(-3)^2}\right) = \frac{7}{9}e^{-\frac{11}{3}}. \end{aligned}$$

(b) Let $F(a) = \int_0^a \cos\left(t^2\right) dt$. Then it follows that $f(x, y) = F(x) - F(y^3)$. Thus, by the Fundamental Theorem of Calculus Part I, one has

$$f_x(x,y) = F'(x) = \cos(x^2),\ f_y(x,y) = -\tfrac{\partial}{\partial y}F(y^3) = -\left(\cos((y^3)^2)\cdot\tfrac{d}{dy}(y^3)\right) = -3y^2\cos(y^6).$$

(c) Note that a smooth parametrization of C is $\mathbf{r}(t) = \langle t, 2t, 3t\rangle$, $0 \le t \le 1$. This implies that $\mathbf{r}'(t) = \langle 1, 2, 3\rangle$, so one has $|\mathbf{r}'(t)| = |\langle 1, 2, 3\rangle| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$. Thus one has

$$\int_C xe^{yz}\, ds = \int_0^1 te^{(2t)(3t)}\left|\mathbf{r}'(t)\right| dt = \int_0^1 \sqrt{14}te^{6t^2}\, dt = \left[\frac{\sqrt{14}}{12}e^{6t^2}\right]_0^1 = \frac{\sqrt{14}}{12}\left(e^6 - 1\right).$$

Question 5

(a) It is easy to see that D is bounded by the curves $x = 0$, $y = 1$ and $x = y$. So the limits of integration are $0 \leq y \leq 1$ and $0 \leq x \leq y$. Thus

$$\begin{aligned}
\iint_D x\sqrt{y^2 - x^2}\,dA &= \int_0^1 \int_0^y x\sqrt{y^2 - x^2}\,dx\,dy \\
&= \int_0^1 \left[-\frac{1}{3}\left(y^2 - x^2\right)^{\frac{3}{2}}\right]_0^y dy \\
&= \int_0^1 \frac{y^3}{3}\,dy \\
&= \left[\frac{y^4}{12}\right]_0^1 = \frac{1}{12}.
\end{aligned}$$

(b) From the given question, we see that the given domain D of the surface is the disk $x^2 + y^2 \leq 3$. By converting to polar coordinates, i.e. $x = r\cos\theta$ and $y = r\sin\theta$, we see that one must have $0 \leq r \leq \sqrt{3}$ and $0 \leq \theta \leq 2\pi$. Thus

$$\begin{aligned}
\text{Required Area} &= \iint_D \sqrt{1 + z_x^2 + z_y^2}\,dA \\
&= \iint_D \sqrt{1 + y^2 + x^2}\,dA \\
&= \int_0^{2\pi} \int_0^{\sqrt{3}} r\sqrt{1 + r^2}\,dr\,d\theta \\
&= 2\pi\left[\frac{1}{3}(1 + r^2)^{\frac{3}{2}}\right]_0^{\sqrt{3}} = \frac{14\pi}{3}.
\end{aligned}$$

Question 6

(i) The domain of f is $\mathbb{R}^2$, and the range of f is $\mathbb{R}$.

(ii) It is easy to see that f is continuous on $\mathbb{R}^2\backslash\{(0,0)\}$. Hence, it remains to check if f is continuous at $(0,0)$. Let $x = r\cos\theta$ and $y = r\sin\theta$. Then one has

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2 + y^2} = \lim_{r\to0} \frac{(r\cos\theta)^3}{r^2} = \lim_{r\to0} r\cos^3\theta = 0 = f(0,0).$$

This shows that f is continuous at $(0,0)$. Hence f is continuous at all points $(x,y) \in \mathbb{R}^2$.

(iii) For $(x,y) \neq (0,0)$, we have $f_x(x,y) = -\frac{y^3}{(x^2+y^2)^2} \cdot 2x = -\frac{2xy^3}{(x^2+y^2)^2}$.
For $(x,y) = (0,0)$, by definition one has

$$f_x(0,0) = \lim_{h\to0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to0} \frac{0-0}{h} = 0.$$

(iv) We shall show that the limit $\lim\limits_{(x,y)\to(0,0)} f_x(x,y)$ does not exist.
Along the path $y = 0$, we see that $f_x(x,y) = f_x(x,0) = 0$ for all $x \neq 0$. So as (x,y) approaches $(0,0)$ along the path $y = 0$, we have $f_x(x,y) \to 0$.
Along the path $y = x$, we see that $f_x(x,y) = f_x(x,x) = -\frac{2x(x)^3}{(x^2+x^2)^2} = -\frac{1}{2}$ for all $x \neq 0$. So as (x,y) approaches $(0,0)$ along the path $y = x$, we have $f_x(x,y) \to -\frac{1}{2}$.

Thus, by the two-path test, we see that the limit $\lim_{(x,y)\to(0,0)} f_x(x,y)$ does not exist. Consequently, $f_x(x,y)$ is not continuous at $(0,0)$.

(v) By definition, one has

$$f_{xx}(0,0) = \lim_{h\to 0} \frac{f_x(0+h,0) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Question 7

(a) It is easy to see that D is bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$. Let $u = x + y$ and $v = x - y$. Then it follows that $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$, so the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{1}{2}\cdot\left(-\frac{1}{2}\right) - \frac{1}{2}\cdot\frac{1}{2} = -\frac{1}{2}.$$

By letting the image of D under the change of variables to be R, we easily see that R is bounded by the lines $v = -u$, $v = u$ and $u = 1$. So the limits of integration under the variables u and v are $0 \le u \le 1$ and $-u \le v \le u$. So one has

$$\iint_D f(x+y)\,dx\,dy = \iint_R f(u)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \int_0^1 \int_{-u}^{u} \frac{1}{2} f(u)\,dv\,du = \int_0^1 uf(u)\,du$$

as desired.

(b) Let the shell be denoted S. By converting into spherical coordinates, i.e. $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$, we see that the limits of integration must be $\sqrt{a} \le \rho \le \sqrt{b}$, $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. Thus, one has

$$\begin{aligned}
\text{Mass of the Shell} &= \iiint_S \frac{1}{x^2+y^2+z^2}\,dV \\
&= \int_0^{2\pi}\int_0^{\pi}\int_{\sqrt{a}}^{\sqrt{b}} \frac{1}{\rho^2}\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\int_{\sqrt{a}}^{\sqrt{b}} \sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{\pi}\sin\phi\,d\phi \int_{\sqrt{a}}^{\sqrt{b}} d\rho \\
&= 2\pi\left(\sqrt{b}-\sqrt{a}\right)\left[-\cos\phi\right]_0^{\pi} = 4\pi\left(\sqrt{b}-\sqrt{a}\right).
\end{aligned}$$

Question 8

(a) (i) As $\mathbf{F}$ is a vector field with continuous second order partial derivatives, one has $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$. Thus one has

$$\operatorname{div}(\operatorname{curl}\mathbf{F}) = \frac{\partial}{\partial x}(Ax-y^2)+\frac{\partial}{\partial y}(2xy-3y)+\frac{\partial}{\partial z}(Bxz) = A+2x-3+Bx = A-3+(2+B)x = 0.$$

By comparing coefficients, we must have $A = 3$ and $B = -2$.

(ii) Let S_1 and S_2 be the surfaces $z = \frac{\sqrt{13}}{13}\sqrt{1-x^2-5y^2}$ and $z = -\frac{\sqrt{13}}{13}\sqrt{1-x^2-5y^2}$ respectively. Then it follows that $S = S_1 + S_2$.
As S_1 has an outward pointing normal (and hence positive orientation), we see that the normal to the surface S_1 is pointing in the positive $z-$ axis. Hence, by the right hand rule, it follows that the boundary curve C for S_1 is the ellipse $x^2+5y^2 = 1$ in the $xy-$ plane with the counter-clockwise orientation.
By a similar reasoning as above, we also see that the boundary curve for S_2 is the curve $-C$, i.e. the ellipse $x^2+5y^2 = 1$ in the $xy-$ plane with the clockwise orientation.
Thus, by Stokes' Theorem, one has

$$\begin{aligned}
\iint_{S_1} (\text{curl } \mathbf{F}) \cdot d\mathbf{S} &= \int_C \mathbf{F}\cdot d\mathbf{r}, \iint_{S_2} (\text{curl } \mathbf{F}) \cdot d\mathbf{S} = \int_{-C} \mathbf{F}\cdot d\mathbf{r} \\
\Rightarrow \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S} &= \iint_{S_1} (\text{curl } \mathbf{F}) \cdot d\mathbf{S} + \iint_{S_2} (\text{curl } \mathbf{F}) \cdot d\mathbf{S} \\
&= \int_C \mathbf{F}\cdot d\mathbf{r} + \int_{-C} \mathbf{F}\cdot d\mathbf{r} \\
&= \int_C \mathbf{F}\cdot d\mathbf{r} - \int_C \mathbf{F}\cdot d\mathbf{r} = 0.
\end{aligned}$$

(iii) Let S denote the surface $x^2+4z^2 \le 1$ in the $xz-$ plane (i.e. $y=0$). Then it follows that the boundary curve of S is the curve C.
Since C has a -clockwise orientation, we see that an outward pointing normal $\mathbf{n}$ to the surface S is the vector $\langle 0,-1,0\rangle$. This would imply that on the surface S, one has

$$(\text{curl } \mathbf{F}) \cdot \mathbf{n} = \langle 2x-y^2, 2xy-3y, -2xz\rangle \cdot \langle 0,-1,0\rangle = y(2x-3) = 0(2x-3) = 0.$$

Therefore, by Stokes' Theorem, one has

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S 0\, dS = 0.$$

(b) By Green's Theorem, one has

$$\int_C (y^3-y)\,dx - 2x^3\,dy = \iint_R \frac{\partial}{\partial x}(-2x^3) - \frac{\partial}{\partial y}(y^3-y)\,dA = \iint_R 1-(6x^2+3y^2)\,dA, \qquad (5)$$

where R denotes the region enclosed by C. Denote the region enclosed by the ellipse $6x^2+3y^2 = 1$ in the $xy-$ plane by S. We shall show that the above integral is maximised when $R = S$.

Note that the surfaces $z = 0$ and $z = 1-(6x^2+3y^2)$ intersect at the ellipse $6x^2+3y^2 = 1$ in the $xy-$ plane, and observe that on the region S, the surface $z = 1-(6x^2+3y^2)$ is above the plane $z = 0$. Thus, for all points (x,y) in S, one has $1-(6x^2+3y^2) \ge 0$. Likewise, for all points (x,y) not in S, i.e. $(x,y) \in \mathbb{R}^2 \backslash S$, one has $1-(6x^2+3y^2) < 0$. Hence, one has

$$\begin{aligned}
\iint_R 1-(6x^2+3y^2)\,dA &= \iint_{(R\cap S)} 1-(6x^2+3y^2)\,dA + \iint_{(R\cap(\mathbb{R}^2\backslash S))} 1-(6x^2+3y^2)\,dA \\
&\le \iint_{(R\cap S)} 1-(6x^2+3y^2)\,dA \quad (\because \forall\,(x,y)\in\mathbb{R}^2\backslash S,\ 1-(6x^2+3y^2)<0) \\
&\le \iint_S 1-(6x^2+3y^2)\,dA \quad (\because R\cap S \subseteq S).
\end{aligned}$$

As equality holds if and only if $R\cap(\mathbb{R}^2\backslash S) = \phi$ and $R \cap S = S$, or equivalently, $R = S$, we see that the integral in equation (5) is maximised when $R = S$. So the positively oriented simple closed C is the ellipse $6x^2+3y^2 = 1$ in the $xy-$ plane, with orientation $x = \frac{\sqrt{6}}{6}\cos t$, $y = \frac{\sqrt{3}}{3}\sin t$, $0 \le t \le 2\pi$.