NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Xiong Xi, Lin Mingyan Simon

MA1104 Multivariable Calculus
AY 2010/2011 Sem 2

Question 1

(a) By the Chain Rule, one has

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\
&= \frac{1}{x+y}\cdot e^t + \frac{1}{x+y}\cdot(-e^{-t}) \\
&= \frac{x-y}{x+y}, \\
\frac{d^2w}{dt^2} &= \frac{d}{dt}\left(\frac{dw}{dt}\right) \\
&= \left(\frac{\partial}{\partial x}\left(\frac{dw}{dt}\right)\right)\frac{dx}{dt} + \left(\frac{\partial}{\partial y}\left(\frac{dw}{dt}\right)\right)\frac{dy}{dt} \\
&= \frac{x+y-(x-y)}{(x+y)^2}\cdot e^t + \frac{-(x+y)-(x-y)}{(x+y)^2}\cdot(-e^{-t}) \\
&= \frac{2y}{(x+y)^2}\cdot e^t + \frac{2x}{(x+y)^2}\cdot e^{-t} \\
&= \frac{4}{(e^t+e^{-t})^2}.
\end{aligned}$$

At $t=0$, we have $\frac{d^2w}{dt^2} = \frac{4}{(e^0+e^{-0})^2} = 1$.

(b) (i) By definition, one has

$$\begin{aligned}
f_x(0,0) &= \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} \\
&= \lim_{h\to 0}\frac{h^2\sin\left(\frac{1}{\sqrt{h^2}}\right)}{h} \\
&= \lim_{h\to 0} h\sin\left(\frac{1}{\sqrt{h^2}}\right).
\end{aligned}$$

Since $\left|\sin(\frac{1}{\sqrt{h^2}})\right| \le 1$, one has $\left|h\sin(\frac{1}{\sqrt{h^2}})\right| \le |h|$. As $\lim_{h\to 0}|h| = 0$, it follows from the Squeeze Theorem that $\lim_{h\to 0} h\sin\left(\frac{1}{\sqrt{h^2}}\right) = 0$. Hence, $f_x(0,0)=0$.
Similarly, by symmetry one has $f_y(0,0)=0$.

(ii) Suppose on the contrary that the function $f_x(x,y)$ is continuous at $(0,0)$. We have

$$\begin{aligned} f_x(x,y) &= 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) + (x^2+y^2)\left(-\frac{1}{2}\cdot\frac{2x}{(x^2+y^2)^{3/2}}\right)\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) \\ &= 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right). \end{aligned}$$

By letting $x = r\cos\theta$, $y = r\sin\theta$, the above expression becomes

$$f_x(x,y) = 2r\cos\theta\sin\frac{1}{r} - \cos\theta\cos\frac{1}{r} = \cos\theta\left(2r\sin\frac{1}{r} - \cos\frac{1}{r}\right). \tag{1}$$

Since $f_x(x,y)$ is continuous at $(0,0)$, it follows that the limit $\lim\limits_{(x,y)\to(0,0)} f_x(x,y)$ exists, and by equation (1), one has

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{r\to 0}\cos\theta\left(2r\sin\frac{1}{r} - \cos\frac{1}{r}\right). \tag{2}$$

By a similar argument in Question 1b(i), we see that the limit $\lim\limits_{r\to 0} 2r\cos\theta\sin\frac{1}{r}$ exists. However, the limit $\lim\limits_{r\to 0}\cos\theta\cos\frac{1}{r}$ does not exist. Hence, the limit on the RHS (and hence LHS) of equation (2) does not exist, which is a contradiction. So $f_x(x,y)$ is not continuous at $(0,0)$.

(iii) Note that f is differentiable at (a,b) if and only if $f_x(a,b)$ and $f_y(a,b)$ exist, and $\triangle f(x,y)$ satisfies some equation $\triangle f(x,y) = f_x(a,b)\triangle x + f_y(a,b)\triangle y + \epsilon_1\triangle x + \epsilon_2\triangle y$, in which each of ϵ_1, ϵ_2 tends to 0 as both $\triangle x, \triangle y$ tends to 0. Clearly, we note that $f_x(0,0) = f_y(0,0) = 0$, and

$$\begin{aligned} \triangle f(x,y) &= f(\triangle x, \triangle y) - f(0,0) \\ &= \left((\triangle x)^2 + (\triangle y)^2\right)\sin\left(\frac{1}{\sqrt{(\triangle x)^2 + (\triangle y)^2}}\right) \\ &= (\triangle x)^2\sin\left(\frac{1}{\sqrt{(\triangle x)^2 + (\triangle y)^2}}\right) + (\triangle y)^2\sin\left(\frac{1}{\sqrt{(\triangle x)^2 + (\triangle y)^2}}\right) \\ &= f_x(0,0)\cdot\triangle x + f_y(0,0)\cdot\triangle y \\ &\quad + \left[\triangle x\sin\left(\frac{1}{\sqrt{(\triangle x)^2 + (\triangle y)^2}}\right)\right]\cdot\triangle x + \left[\triangle y\sin\left(\frac{1}{\sqrt{(\triangle x)^2 + (\triangle y)^2}}\right)\right]\cdot\triangle y \end{aligned}$$

Let $\epsilon_1 = \triangle x\sin\left(\frac{1}{\sqrt{(\triangle x)^2+(\triangle y)^2}}\right)$ and $\epsilon_2 = \triangle y\sin\left(\frac{1}{\sqrt{(\triangle x)^2+(\triangle y)^2}}\right)$. Then by a similar argument in Question 1b(i), we note that each of ϵ_1 and ϵ_2 tends to 0 as both $\triangle x, \triangle y$ tend to 0. So f is differentiable at $(0,0)$.

Question 2

(a) A parametrization of the ellipse is $\mathbf{r}(\theta) = \left\langle\sqrt{6}\cos\theta, \sqrt{3}\sin\theta\right\rangle, 0 \le \theta \le 2\pi$.

At $(2,1)$, we have $\sqrt{6}\cos\theta = 2$ and $\sqrt{3}\sin\theta = 1$, which gives us $\cos\theta = \frac{2}{\sqrt{6}}$ and $\sin\theta = \frac{1}{\sqrt{3}}$. Thus

$$\begin{aligned}
\mathbf{r}'(\theta) &= \left\langle -\sqrt{6}\sin\theta, \sqrt{3}\cos\theta \right\rangle \\
&= \left\langle -\sqrt{6}\cdot\frac{1}{\sqrt{3}}, \sqrt{3}\cdot\frac{2}{\sqrt{6}} \right\rangle \\
&= \left\langle -\sqrt{2}, \sqrt{2} \right\rangle.
\end{aligned}$$

Thus, a unit direction vector is $\mathbf{u} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$.
Next, from the equation $T(x,y) = 100 - 6xy - 5y^2$, we get $\nabla T(x,y) = \langle -6y, -6x - 10y\rangle$. Thus $\nabla T(2,1) = \langle -6, -22\rangle$. Hence

$$\begin{aligned}
D_{\mathbf{u}}T(x,y) &= \nabla T(x,y)\cdot\mathbf{u} \\
&= \langle -6, -22\rangle \cdot \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle = -8\sqrt{2}.
\end{aligned}$$

Therefore, the rate of change of the temperature is $-8\sqrt{2}$ ℃/m.

(b) Let $P(x_0, y_0, z_0)$ be a point on the paraboloid $z = \frac{x^2}{4} + \frac{y^2}{4}$. Then the distance from P to $(3,0,0)$ is equal to $\sqrt{(x_0-3)^2 + y_0^2 + z_0^2}$.
Let $f(x,y,z) = (x-3)^2 + y^2 + z^2$. In order to find the shortest possible distance between P and $(3,0,0)$, we need to find the smallest value of $f(x,y,z)$, subject to the constraint $\frac{x^2}{4} + \frac{y^2}{25} - z = 0$. Let $g(x,y,z) = \frac{x^2}{4} + \frac{y^2}{25} - z$. Then $\nabla f(x,y,z) = \langle 2x-6, 2y, 2z\rangle$ and $\nabla g(x,y,z) = \left\langle \frac{x}{2}, \frac{2y}{25}, -1\right\rangle$. By the Method of Lagrange Multipliers, one has

$$\begin{aligned}
& \nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) \\
\Rightarrow\quad & \langle 2x_0 - 6, 2y_0, 2z_0\rangle = \lambda\left\langle \frac{x_0}{2}, \frac{2y_0}{25}, -1\right\rangle \\
\Rightarrow\quad & 2x_0 - 6 = \frac{\lambda x_0}{2}, \quad y_0 = \frac{\lambda y_0}{25}, \quad 2z_0 = -\lambda.
\end{aligned}$$

From the equation $y_0 = \frac{\lambda y_0}{25}$, one has $y_0 = 0$ or $\lambda = 25$.
If $\lambda = 25$, then this forces $z_0 = -\frac{25}{2}$. However, by the equation $z_0 = \frac{x_0^2}{4} + \frac{y_0^2}{25}$, we must have $z_0 \geq 0$, which is a contradiction. So there are no solutions for this case.
If $y_0 = 0$, then we must have $2x_0 - 6 = \frac{\lambda x_0}{2}$ and $z_0 = \frac{x_0^2}{4} + \frac{y_0^2}{25} = \frac{x_0^2}{4} = -\frac{\lambda}{2}$. By solving the simultaneous equations, one has $\lambda = -2$, $x_0 = 2$ and $z_0 = 1$. Thus the point closest to $(3,0,0)$ is $P(2,0,1)$, and the distance is equal to $\sqrt{(3-2)^2 + 0^2 + 1^2} = \sqrt{2}$.

Question 3

(a) First Solution:
Let the solid that we are integrating over be denoted V. Note that all the points (x,y,z) in V must satisfy the following set of inequalities:

$$0 \leq x \leq 4, \quad 0 \leq y \leq \frac{4-x}{2}, \quad 0 \leq z \leq \frac{12 - 3x - 6y}{4},$$

which is equivalent to the following set of inequalities:

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad 0 \leq 3x + 6y + 4z \leq 12.$$

From the above, if we integrate with respect to y, followed by x and finally z, then we see that the limits of integration must be the following:

$$\begin{aligned} & 0 \leq 6y \leq 12 - 3x - 4z \\ \Rightarrow \quad & 0 \leq y \leq \frac{12 - 3x - 4z}{6}, \\ & 0 \leq 3x \leq 12 - 6y - 4z \leq 12 - 4z \\ \Rightarrow \quad & 0 \leq x \leq \frac{12 - 4z}{3}, \\ & 0 \leq 4z \leq 12 - 3x - 6y \leq 12 \\ \Rightarrow \quad & 0 \leq z \leq 3. \end{aligned}$$

Hence, by Fubini's Theorem, one has

$$\int_0^4 \int_0^{(4-x)/2} \int_0^{(12-3x-6y)/4} dz\, dy\, dx = \int_0^3 \int_0^{(12-4z)/3} \int_0^{(12-3x-4z)/6} dy\, dx\, dz.$$

Second Solution:
Note that

$$\int_0^4 \int_0^{(4-x)/2} \int_0^{(12-3x-6y)/4} dz\, dy\, dx = \iiint_E dV,$$

where E is the solid bounded by the planes $z = 0$,$x = 0$, $y = 0$ and $z = (12 - 3x - 6y)/4$. We see that the projection D of solid E onto the xz-plane is the triangle formed by the x-axis, z-axis and the line $3x + 4z = 12$. For a fixed point $(x, z) \in D$, we shall integrate $f(x, y, z)$ from the left boundary curve $y = 0$ to the right boundary curve $z = (12 - 3x - 6y)/4$, which can be rewritten as $y = (12 - 3x - 4z)/6$. Therefore, by Fubini's Theorem,

$$\begin{aligned} \iiint_E dV &= \iint_D \left[\int_0^{(12-3x-4z)/6} dy \right] dA \\ &= \int_0^3 \int_0^{(12-4z)/3} \int_0^{(12-3x-4z)/6} dy\, dx\, dz \end{aligned}$$

(b) First Solution:
Let the solid that we are integrating over be denoted E. Since E is bounded by $y = 1$, $y = 7$ and $y^2 + 2 = x^2 + z^2$, it follows that all points (x, y, z) in E must satisfy the following set of inequalities:

$$1 \leq y \leq 7, \quad 0 \leq x^2 + z^2 \leq y^2 + 2.$$

By converting to polar coordinates in the $xz-$plane, i.e. $x = r\cos\theta$, $z = r\sin\theta$, where $r \geq 0$ and $0 \leq \theta \leq 2\pi$, the above set of inequalities is equivalent to:

$$1 \leq y \leq 7, \quad 0 \leq r \leq \sqrt{y^2 + 2}.$$

Hence,

$$\begin{aligned}
\text{Volume of } E &= \iiint_E dV \\
&= \int_1^7 \int_{-\sqrt{y^2+2}}^{\sqrt{y^2+2}} \int_{-\sqrt{y^2-x^2+2}}^{\sqrt{y^2-x^2+2}} dz\,dx\,dy \\
&= \int_1^7 \int_0^{2\pi} \int_0^{\sqrt{y^2+2}} r\,dr\,d\theta\,dy \\
&= 2\pi \int_1^7 \left[\frac{r^2}{2}\right]_0^{\sqrt{y^2+2}} dy \\
&= \pi \int_1^7 y^2 + 2\,dy \\
&= \pi \left[\frac{y^3}{3} + 2y\right]_1^7 \\
&= 126\pi.
\end{aligned}$$

Second Solution (By the method suggested in the textbook):

Let the solid that we are integrating over be denoted E. Let us break E into two parts E_1 and E_2, where E_1 is bounded by the curves $x^2 + z^2 = 3$, $y = 1$ and $y = 7$, and $E_2 = E - E_1$. Note that the volume of E_1 is equal to $3 \cdot \pi \cdot (7 - 1) = 18\pi$.

For E_2, let the projection of E_2 on $xz-$plane be D. Note that at $y = 1$, we have $x^2+z^2 = 1^2+2 = 3$, and at $y = 7$, we have $x^2 + z^2 = 7^2 + 2 = 51$.
Hence, we see that the equation of D is the annulus $3 \le x^2 + z^2 \le 51$. Moreover, the left boundary curve of E_2 is the curve $y = \sqrt{x^2 + z^2 - 2}$ (since $y > 0$) and the right boundary curve of E_2 is the curve $y = 7$.

By converting to polar coordinates in the $xz -$ plane, i.e. $x = r\cos\theta$, $z = r\sin\theta$, where $r \ge 0$ and $0 \le \theta \le 2\pi$, we see that one must have $\sqrt{3} \le r \le \sqrt{51}$. Hence,

$$\begin{aligned}
\text{Volume of } E_2 &= \iiint_{E_2} dV \\
&= \iint_D \left[\int_{\sqrt{x^2+z^2-2}}^7 dy\right] dA \\
&= \iint_D 7 - \sqrt{x^2 + z^2 - 2}\,dA \\
&= \int_0^{2\pi} \int_{\sqrt{3}}^{\sqrt{51}} \left(7 - \sqrt{r^2 - 2}\right) r\,dr\,d\theta \\
&= 2\pi \left[\frac{7r^2}{2} - \frac{\sqrt{(r^2-2)^3}}{3}\right]_{\sqrt{3}}^{\sqrt{51}} \\
&= 108\pi.
\end{aligned}$$

Thus, Volume of E = Volume of E_1 + Volume of $E_2 = 18\pi + 108\pi = 126\pi$.

Question 4

From $x = u^{1/3}v^{2/3}$ and $y = u^{2/3}v^{1/3}$, we get $u = \frac{x^2}{y}$ and $v = \frac{y^2}{x}$.
Note that R is bounded by the curves $y = \sqrt{x}$, $y = \sqrt{2x}$, $y = \frac{x^2}{3}$ and $y = \frac{x^2}{4}$.
By letting the image of R under the change of variables to be S, we get that the boundaries of S to be $u = 1$, $u = 2$, $v = 3$ and $v = 4$.
The Jacobian is

$$\begin{aligned}\frac{\partial(x,y)}{\partial(u,v)} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= \left(\frac{1}{3}u^{-\frac{2}{3}}v^{\frac{2}{3}}\right)\left(\frac{1}{3}u^{\frac{2}{3}}v^{-\frac{2}{3}}\right) - \left(\frac{2}{3}u^{\frac{1}{3}}v^{-\frac{1}{3}}\right)\left(\frac{2}{3}u^{-\frac{1}{3}}v^{\frac{1}{3}}\right) \\ &= \frac{1}{9} - \frac{4}{9} = -\frac{1}{3}.\end{aligned}$$

Thus, one has

$$\begin{aligned}\text{Area of Region } R &= \iint_R dA \\ &= \iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA \\ &= \int_1^2 \int_3^4 \frac{1}{3}\, du\, dv = \frac{1}{3}.\end{aligned}$$

Question 5

(a) (i) Let $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ where $P(x,y,z) = y^2\cos x + z^3$, $Q(x,y,z) = 2y\sin x - 4$ and $R(x,y,z) = 3xz^2 + 2$.
Then one sees that $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} = 0$, $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} = 3z^2$ and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2y\cos x$.
As P, Q and R all have continuous partial derivatives on $\mathbb{R}$, by the Component Test for Conservative Fields, we have that $\mathbf{F}$ is a conservative vector field on $\mathbb{R}$.

(ii) Let a potential function of $\mathbf{F}$ be f. Then $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$ so one has $f_x(x,y,z) = y^2\cos x + z^3$, $f_y(x,y,z) = 2y\sin x - 4$ and $f_z(x,y,z) = 3xz^2 + 2$. By integrating f_x with respect to x, we get $f(x,y,z) = y^2\sin x + z^3x + g(y,z)$, where g is some function of y and z with continuous first partial derivatives.

By differentiating the above with respect to y, we get $f_y(x,y,z) = 2y\sin x + g_y(y,z)$, so one has $g_y(y,z) = -4$.
By integrating g_y with respect to y, we get $g(y,z) = -4y + h(z)$, where h is some continuously differentiable function of z. This implies that $f(x,y,z) = y^2\sin x + z^3x - 4y + h(z)$.

By differentiating the above with respect to z, we get $f_z(x,y,z) = 3xz^2 + h'(z)$, so one has $h'(z) = 2$.
Hence one has $h(z) = 2z + C$ for some constant C so this implies that $f(x,y,z) = y^2\sin x + z^3x - 4y + 2z + C$.

Therefore, by the Fundamental Theorem for Line Integrals, one has

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= f(\pi,\pi,\pi) - f(0,1,-1)\\ &= (\pi^2\sin\pi + \pi^3\pi - 4\pi + 2\pi + C) - (1^2\sin 0 + (-1)^3(0) - 4(1) + 2(-1) + C)\\ &= \pi^4 - 2\pi + 6.\end{aligned}$$

(b) To find the boundary curve C of S, we need to solve the simultaneous equations $x^2+y^2+z^2=5^2$ and $z=5\sqrt{2}$. Then the equation of C is given by $z=\frac{5}{\sqrt{2}}$, and $x^2+y^2=25/2$. Hence, a parametrization of the curve C is $\mathbf{r}(t)=\left\langle \frac{5}{\sqrt{2}}\cos t, \frac{5}{\sqrt{2}}\sin t, \frac{5}{\sqrt{2}}\right\rangle$, $0\le t\le 2\pi$.
Also we have

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) &= \left\langle \frac{5}{\sqrt{2}}\sin t, \left(\frac{5}{\sqrt{2}}\cos t - 2\cdot\frac{5}{\sqrt{2}}\cos t\cdot\frac{5}{\sqrt{2}}\right), \left(\frac{5}{\sqrt{2}}\sin t\cdot\frac{5}{\sqrt{2}}\cos t\right)\right\rangle\\ &= \left\langle \frac{5}{\sqrt{2}}\sin t, \left(\frac{5}{\sqrt{2}} - 25\right)\cos t, \frac{25}{2}\sin t\cos t\right\rangle,\\ \mathbf{r}'(t) &= \left\langle -\frac{5}{\sqrt{2}}\sin t, \frac{5}{\sqrt{2}}\cos t, 0\right\rangle.\end{aligned}$$

Therefore, by Stokes' Theorem,

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \int_C \mathbf{F}\cdot d\mathbf{r}\\ &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt\\ &= \int_0^{2\pi} \left\langle \frac{5}{\sqrt{2}}\sin t, \left(\frac{5}{\sqrt{2}} - 25\right)\cos t, \frac{25}{2}\sin t\cos t\right\rangle\cdot\left\langle -\frac{5}{\sqrt{2}}\sin t, \frac{5}{\sqrt{2}}\cos t, 0\right\rangle dt\\ &= \int_0^{2\pi} -\frac{25}{2}\sin^2 t + \left(\frac{25}{2} - \frac{125}{\sqrt{2}}\right)\cos^2 t\,dt\\ &= \int_0^{2\pi} -\frac{125}{2\sqrt{2}} + \left(\frac{25}{2} - \frac{125}{2\sqrt{2}}\right)\cos 2t\,dt\\ &= \left[-\frac{125t}{2\sqrt{2}} + \left(\frac{25}{4} - \frac{125}{4\sqrt{2}}\right)\sin 2t\right]_0^{2\pi} = -278\,(3.\text{s.f.}).\end{aligned}$$

Question 6

Let $\mathbf{F}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$ where $P(x,y,z)=-\frac{z}{y}$, $Q(x,y,z)=y\sin y$ and $R(x,y,z)=z^2$. Then

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle\\ &= \left\langle 0-0, -\frac{1}{y} - 0, 0 - \frac{z}{y^2}\right\rangle = \left\langle 0, -\frac{1}{y}, -\frac{z}{y^2}\right\rangle.\end{aligned}$$

By Stokes' Theorem, we have

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \iint_{S_1} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S},\ \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S},$$

where S_1 and S_2 denote the surfaces with the boundary curves C_1 and C_2 respectively. Note that the projection D of both S_1 and S_2 on the $xz-$plane is the disk $x^2+z^2 \le 1$. For S_1, let $y = g(x,z) = 10 + x^2 + 3z^2$. Then one has

$$\begin{aligned}
\mathbf{r}_x \times \mathbf{r}_z &= \left\langle -\frac{\partial g}{\partial x}, 1, -\frac{\partial g}{\partial z}\right\rangle = \langle -2x, 1, -6z\rangle \\
\Rightarrow \operatorname{curl}\mathbf{F}\cdot(\mathbf{r}_x\times\mathbf{r}_z) &= \left\langle 0, -\frac{1}{y}, -\frac{z}{y^2}\right\rangle \cdot \langle -2x, 1, -6z\rangle \\
&= \frac{6z^2 - y}{y^2} \\
&= \frac{6z^2 - (10+x^2+3z^2)}{(10+x^2+3z^2)^2} = \frac{3z^2-10-x^2}{(10+x^2+3z^2)^2}, \\
\Rightarrow \iint_{S_1} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \iint_D \operatorname{curl}\mathbf{F}\cdot(\mathbf{r}_x\times\mathbf{r}_z)\,dA = \iint_D \frac{3z^2-10-x^2}{(10+x^2+3z^2)^2}\,dA.
\end{aligned}$$

For S_2, let $y = h(x,z) = 2 - x$. Then one has

$$\begin{aligned}
\mathbf{r}'_x \times \mathbf{r}'_z &= \left\langle -\frac{\partial h}{\partial x}, 1, -\frac{\partial h}{\partial z}\right\rangle = \langle 1, 1, 0\rangle \\
\Rightarrow \operatorname{curl}\mathbf{F}\cdot(\mathbf{r}'_x\times\mathbf{r}'_z) &= \left\langle 0, -\frac{1}{y}, -\frac{z}{y^2}\right\rangle \cdot \langle 1, 1, 0\rangle \\
&= -\frac{1}{y} = -\frac{1}{2-x}, \\
\Rightarrow \iint_{S_2} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \iint_D \operatorname{curl}\mathbf{F}\cdot(\mathbf{r}'_x\times\mathbf{r}'_z)\,dA = \iint_D -\frac{1}{2-x}\,dA.
\end{aligned}$$

Note that for all points (x,z) on D, we have

$$\frac{3z^2-10-x^2}{(10+x^2+3z^2)^2} \ge \frac{3(0)^2-10-1^2}{(10+0^2+3(0)^2)^2} = -\frac{11}{100} > -\frac{1}{3} = -\frac{1}{2-(-1)} \ge -\frac{1}{2-x}.$$

Thus,

$$\begin{aligned}
\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \iint_{S_1} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} \\
&= \iint_D \frac{3z^2-10-x^2}{(10+x^2+3z^2)^2}\,dA \\
&> \iint_D -\frac{1}{2-x}\,dA \\
&= \iint_{S_2} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r},
\end{aligned}$$

so the given assertion is not true.