MA2108S - Mathematical Analysis I(S) Suggested Solutions

(Semester 2 : AY2016/17)

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Question 1

We use the AM-GM inequality here, which is the statement that for any $a, b > 0$, one has:

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

This can be devired from expanding $(a-b)^2 \geq 0$. Then,

$$a_{n+1} = \frac{a_n + b_n}{2} \geq \sqrt{a_n b_n}$$

and

$$b_{n+1} = \frac{2a_n b_n}{a_n + b_n} \leq \frac{a_n b_n}{\sqrt{a_n b_n}} \leq \sqrt{a_n b_n}.$$

From this, we instantly see that $a_i \geq b_i$ for all i. We also note that a_n, b_n satisfies the recurrence,

$$a_n b_n = a_{n+1} b_{n+1} \implies \frac{a_{n+1}}{a_n} = \frac{b_n}{b_{n+1}} \tag{1}$$

Observe that

$$a_{n+1} = \frac{a_n + b_n}{2} \leq \frac{a_n + a_n}{2} = a_n$$

so (a_n) is monotone decreasing. From (1), and the fact that $\{a_n\}$ is monotone decreasing, we deduce that $\{b_n\}$ is monotonically increasing.

Furthermore, $a_{n+1} = \frac{a_n+b_n}{2} \geq 0$ so $\{a_n\}$ is bounded below by 0. From the monotone convergence theorem, (a_n) converges to some $L \geq 0$. Since $\{b_n\}$ is monotone increasing and bounded above by a, again, $\{b_n\}$ converges to some $M \geq 0$. Finally

$$\begin{aligned}
a_{n+1} = \frac{a_n + b_n}{2} &\implies \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n + b_n}{2} \\
&\implies L = \frac{L+M}{2} \\
&\implies L = M
\end{aligned}$$

so we conclude that both $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and that they are equal.

Question 2

We need the identity:

$$a^n - 1 = (a-1)(1 + a + a^2 + \cdots a^{n-1})$$

Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n + a_n^2 + \cdots a_n^k - k}{a_n - 1} &= \lim_{n\to\infty} \frac{(a_n - 1) + (a_n^2 - 1) + \cdots (a_n^k - 1)}{a_n - 1} \\
&= \lim_{n\to\infty} 1 + (a_n + 1) + (a_n^2 + a_n + 1) + \cdots + (a_n^{k-1} + \cdots + a_n^2 + a_n + 1)
\end{aligned}$$

Since $\lim_{n\to\infty} a_n = 1$, we apply the limit laws on each term

$$\begin{aligned}
&= 1 + (1+1) + (1+1+1) + \cdots \overbrace{(1 + 1 + \cdots + 1)}^{k \text{ times}} \\
&= \frac{k(k+1)}{2}.
\end{aligned}$$

Question 3

$\sum_{k=1}^{\infty} |\frac{(-1)^{k+1}}{k}|$ is just the p-series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ which obviously diverges. Thus the alternating harmonic series is not absolutely convergent in $\mathbb{R}$.

There are numerous ways to show that the alternating harmonic series is convergent.

Way 1: Expansion of $\ln(2)$

Taking the series expansion of $\ln(2)$ immediately gives the alternating harmonic series.

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Way 2: Alternating Series Test

Set $a_n = \frac{1}{n}$. $\{a_n\} \to 0$ and the sequence is monotone decreasing, by alternating series test, $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Question 4

Since f is continuous on a compact interval, it is uniformly continuous. That is, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. We want to show that there exists an N sufficiently large for all ϵ such that for all $n > N$,

$$\left| \frac{1}{n} \sum_{k=1}^{n} (-1)^k f\left(\frac{k}{n}\right) \right| < \epsilon \tag{2}$$

We know there exists a δ such that $|f(x + \delta) - f(x)| < \frac{\epsilon}{2}$ for all x. Choose N_1 sufficiently big such that $\frac{1}{N_1} < \delta$. Further, set $f(1) = b$ and choose N_2 such that $\frac{b}{N_2} < \frac{\epsilon}{2}$. Put $N = \max\{N_1, N_2\}$.

If $n > N$ is even, the left hand side of (2) can be expanded to:

$$\begin{aligned}
\left| \frac{1}{n} \sum_{k=1}^{n} (-1)^k f\left(\frac{k}{n}\right) \right| &= \left| \frac{f(\frac{2}{n}) - f(\frac{1}{n}) + f(\frac{4}{n}) - f(\frac{3}{n}) + \cdots + f(\frac{n}{n}) - f(\frac{n-1}{n})}{n} \right| \\
&\leq \frac{|f(\frac{2}{n}) - f(\frac{1}{n})| + |f(\frac{4}{n}) - f(\frac{3}{n})| + \cdots + |f(\frac{n}{n}) - f(\frac{n-1}{n})|}{n} \\
&< \frac{\frac{n\epsilon}{2}}{n} \\
&= \epsilon.
\end{aligned}$$

If n is odd, we have

$$\begin{aligned}
\left| \sum_{k=1}^{n} (-1)^k f\left(\frac{k}{n}\right) \right| &= \left| \frac{f(\frac{2}{n}) - f(\frac{1}{n}) + f(\frac{4}{n}) - f(\frac{3}{n}) + \cdots + f(\frac{n-1}{n}) - f(\frac{n-2}{n}) + f(\frac{n}{n})}{n} \right| \\
&\leq \frac{|f(\frac{2}{n}) - f(\frac{1}{n})| + |f(\frac{4}{n}) - f(\frac{3}{n})| + \cdots + |f(\frac{n-1}{n}) - f(\frac{n-2}{n})| + |f(\frac{n}{n})|}{n} \\
&< \frac{\frac{n\epsilon}{2} + b}{n} \\
&< \epsilon.
\end{aligned}$$

Thus proving our desired statement (2).

Question 5

Note that $\frac{1}{n}(f(x_1) + f(x_2) + \cdots f(x_n))$ is simply the mean of n values. Set $L = \min\{f(x_1), f(x_2) \cdots f(x_n)\}$, $M = \max\{f(x_1), f(x_2) \cdots f(x_n)\}$. Since f is continuous, by the intermediate value theorem, f takes on every value between L and M. Since $L \leq \frac{1}{n}(f(x_1) + f(x_2) + \cdots f(x_n)) \leq M$, it will surely take on the value $\frac{1}{n}(f(x_1) + f(x_2) + \cdots f(x_n))$, meaning there exists some $x_0 \in \mathbb{R}$ such that $f(x_0) = \frac{1}{n}(f(x_1) + f(x_2) + \cdots f(x_n))$.

Question 6

We have $g(x) = \sup\{f(y) \in \mathbb{R} : y \in [a, x]\}$.

Showing Well-definiteness

Since the sup is unique if it exists, it suffices to show that $\forall x \in [a, b],\ \sup\{f(y) \in \mathbb{R} : y \in [a, x]\}$ exists. Fix $x \in [a, b]$

Then $[a, x]$ is compact. Since f is continuous, the range of f is bounded and thus $\{f(y) \in \mathbb{R} : y \in [a, x]\}$ is bounded. By the least upper bound property of $\mathbb{R}$, $\sup\{f(y) \in \mathbb{R} : y \in [a, x]\}$ exists and the proof is complete.

Showing continuous

Assume, for the sake of contradiction, that g is not continuous at some $x \in [a, b]$. Then $\exists \epsilon > 0$ such that $\forall \delta > 0, \exists z \in [a, b]$ such that

$$|x - z| < \delta \ \wedge \ |g(x) - g(z)| \geq \epsilon.$$

On the other hand, by continuity of f at x

$$\exists \delta' > 0 \text{ such that } \forall y \in [a, b],\ |x - y| < \delta' \to |f(x) - f(y)| < \epsilon.$$

Using $\delta = \delta',\ \exists z' \in [a, b]$ such that

$$|x - z'| < \delta' \ \wedge \ |g(x) - g(z')| \geq \epsilon.$$

Without loss of generality, assume $z' > x$. Then

$$\sup\{f(y) \in \mathbb{R} \ : \ y \in [a, x]\} \leq \sup\{f(y) \in \mathbb{R} \ : \ y \in [a, z']\}$$

so we have $g(z') - g(x) \geq \epsilon$.

Since $[a, z']$ is a compact interval, by the extreme value theorem, $\exists k \in [a, z']$ such that

$$f(k) = \sup\{f(y) \in \mathbb{R} \ : \ y \in [a, z']\} = g(z')$$

Note that since $f(k) > \sup\{f(y) \in \mathbb{R} \ : \ y \in [a, x]\},\ k \in (x, z']$. Then

$$\begin{aligned} f(k) - f(x) &\geq g(z') - g(x) \\ &\geq \epsilon. \end{aligned}$$

But $|x - k| \leq |x - z'| < \delta'$ so $|f(k) - f(x)| < \epsilon$ by continuity of f. There is a contradiction so we conclude that g must be continuous on $[a, b]$.