NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

Written by Lin Mingyan, Simon
Audited by Chua Hongshen

MA2108 Mathematical Analysis I

AY 2012/2013 Sem 1

Question 1

(a) (i) We have

$$\lim_{n\to\infty}\frac{2n+1-n^2}{n^2-2n+6}=\lim_{n\to\infty}\frac{\frac{2}{n}+\frac{1}{n^2}-1}{1-\frac{2}{n}+\frac{6}{n^2}}=\frac{-1}{1}=-1.$$

(ii) Since $\left(\left(1+\frac{1}{n+2}\right)^{n+2}\right)$ is a subsequence of $\left(\left(1+\frac{1}{n}\right)^n\right)$, we have $\lim\limits_{n\to\infty}\left(1+\frac{1}{n+2}\right)^{n+2}=e$. This implies that

$$\lim_{n\to\infty}\left(1+\frac{1}{n+2}\right)^{2n+4}=\lim_{n\to\infty}\left(\left(1+\frac{1}{n+2}\right)^{n+2}\right)^2=e^2.$$

Furthermore, we have

$$\lim_{n\to\infty}\left(1+\frac{1}{n+2}\right)^4=\left(\lim_{n\to\infty}\left(1+\frac{1}{n+2}\right)\right)^4=1.$$

Hence, we have

$$\lim_{n\to\infty}\left(1+\frac{1}{n+2}\right)^{2n}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n+2}\right)^{2n+4}}{\left(1+\frac{1}{n+2}\right)^4}=e^2.$$

(iii) We have

$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt{(n+a)(n+b)}-n\right) &= \lim_{n\to\infty}\frac{(\sqrt{(n+a)(n+b)}-n)(\sqrt{(n+a)(n+b)}+n)}{\sqrt{(n+a)(n+b)}+n}\\ &= \lim_{n\to\infty}\frac{(n+a)(n+b)-n^2}{\sqrt{n^2+(a+b)n+ab}+n}\\ &= \lim_{n\to\infty}\frac{(a+b)n+ab}{\sqrt{n^2+(a+b)n+ab}+n}\\ &= \lim_{n\to\infty}\frac{(a+b)n+ab}{\sqrt{n^2+(a+b)n+ab}+n}\\ &= \lim_{n\to\infty}\frac{(a+b)+\frac{ab}{n}}{\sqrt{1+\frac{a+b}{n}+\frac{ab}{n^2}}+1}\\ &= \frac{a+b}{2}.\end{aligned}$$

(b) We note that $x_{2n-1}=1+\frac{1}{n}$ and $x_{2n}=-\frac{1}{n}$ for all $n\in\mathbb{N}$. Let us show that $-\frac{2}{n}\le x_n\le 1+\frac{2}{n}$ for all $n\in\mathbb{N}$. Indeed, if $n=2k-1$ is odd, then we have $x_n=1+\frac{1}{k}>0>-\frac{2}{2k-1}=-\frac{2}{n}$, and

$x_n = 1+\frac{1}{k} < 1+\frac{2}{2k-1} = 1+\frac{2}{n}$. On the other hand, if $n = 2k$ is even, then we have $x_n = -\frac{1}{k} = -\frac{2}{n}$, and $x_n = -\frac{1}{k} < 0 < 1+\frac{2}{n}$, which completes the claim. Hence, it follows that if (x_{n_k}) is a convergent subsequence of (x_n), then we must have

$$0 = \lim_{k\to\infty} -\frac{2}{n_k} \leq \lim_{k\to\infty} x_{n_k} \leq \lim_{k\to\infty} \left(1+\frac{2}{n_k}\right) = 1.$$

This implies that $\limsup x_n \leq 1$ and $\liminf x_n \geq 0$. Now, since $\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1$, and $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} -\frac{1}{n} = 0$, we must have $\limsup x_n = 1$ and $\liminf x_n = 0$.

Question 2

(a) (i) For all $n \in \mathbb{N}$, we have $\frac{2n^2+1}{3n^3+2n} \geq \frac{2n^2}{3n^3+2n} \geq \frac{2n^2}{3n^3+2n^3} = \frac{2}{5n}$. As the series $\sum_{n=1}^{\infty} \frac{2}{5n} = \frac{2}{5}\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, it follows from the Comparison Test that the series $\sum_{n=1}^{\infty} \frac{2n^2+1}{3n^3+2n}$ is divergent.

(ii) For each $n \in \mathbb{N}$, let us define $a_n := 4^n \left(\frac{n}{n+2}\right)^{n^2}$. Then we have $|a_n|^{\frac{1}{n}} = 4\left(\frac{n}{n+2}\right)^n$. Now, we note that

$$\left(\frac{n}{n+2}\right)^n = \left(\frac{n}{n+1}\cdot\frac{n+1}{n+2}\right)^n = \left(\frac{n}{n+1}\right)^n \left(\frac{n+1}{n+2}\right)^n.$$

Furthermore, we have

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}, \text{ and}$$

$$\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n+1}\right)^n} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n+1}\right)}{\left(1+\frac{1}{n+1}\right)^{n+1}} = \frac{1}{e}.$$

Hence, we have

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty} 4\left(\frac{n}{n+2}\right)^n = \lim_{n\to\infty} 4\left(\frac{n}{n+1}\right)^n\left(\frac{n+1}{n+2}\right)^n = 4\cdot\frac{1}{e}\cdot\frac{1}{e} = \frac{4}{e^2} < 1.$$

Therefore, we have the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 4^n\left(\frac{n}{n+2}\right)^{n^2}$ to converge absolutely by the Root Test.

(iii) For each $n \in \mathbb{N}$, let us define $b_n := \frac{(-1)^n n!(n+1)!}{(2n)!}$. Then we have

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{(n+1)!(n+2)!}{(2n+2)!}\cdot\frac{(2n)!}{n!(n+1)!} = \frac{(n+1)(n+2)}{(2n+1)(2n+2)} = \frac{n+2}{2(2n+1)}.$$

As

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\frac{n+2}{2(2n+1)} = \lim_{n\to\infty}\frac{1+\frac{2}{n}}{2\left(2+\frac{1}{n}\right)} = \frac{1}{4} < 1,$$

it follows that the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{(-1)^n n!(n+1)!}{(2n)!}$ is absolutely convergent by the Ratio Test.

(b) (i) For all $n \in \mathbb{N}$, we have

$$a_{4n-3} = \frac{1}{2n-1}, \quad a_{4n-2} = -\frac{1}{6n-4}, \quad a_{4n-1} = -\frac{1}{6n-2}, \quad a_{4n} = -\frac{1}{6n}.$$

(ii) For each $n \in \mathbb{N}$, let us define $b_{2n-1} := a_{4n-3} = \frac{1}{2n-1}$ and $b_{2n} := -(a_{4n-2} + a_{4n-1} + a_{4n}) = \frac{1}{6n-4} + \frac{1}{6n-2} + \frac{1}{6n}$. Let us show that the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ is convergent. Clearly, we have $b_n > 0$ for all $n \in \mathbb{N}$. Next, let us show that $b_n > b_{n+1}$ for all $n \in \mathbb{N}$. To this end, we first note that we have

$$b_{2n} = \frac{1}{6n-4} + \frac{1}{6n-2} + \frac{1}{6n} > \frac{3}{6n+3} = \frac{1}{2n+1} = b_{2n+1}$$

for all $n \in \mathbb{N}$. Furthermore, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{1}{3n-1} + \frac{1}{3n+1} + \frac{1}{3n+3} &= \frac{(3n-1)(3n+1) + (3n-1)(3n+3) + (3n+1)(3n+3)}{(3n-1)(3n+1)(3n+3)} \\
&= \frac{27n^2 + 18n - 1}{(9n^2-1)(3n+3)} \\
&< \frac{27n^2 + 18n}{(9n^2-4)(3n+3)} \\
&= \frac{9n}{(3n+2)(3n+3)} \\
&< \frac{9n}{3n(3n+3)} \\
&= \frac{1}{n+1} \\
&< \frac{1}{n}.
\end{aligned}$$

This implies that

$$b_{2n} = \frac{1}{6n-4} + \frac{1}{6n-2} + \frac{1}{6n} < \frac{1}{2n-1} = b_{2n-1}$$

for all $n \in \mathbb{N}$, and this completes the proof.

Finally, let us show that $\lim_{n\to\infty} b_n = 0$. Clearly, we have $\lim_{n\to\infty} b_{2n-1} = 0$. As $0 < b_{2n} < b_{2n-1}$ for all $n \in \mathbb{N}$, it follows from Squeeze Theorem that $\lim_{n\to\infty} b_{2n} = 0$. This implies that for all $\varepsilon > 0$, there exist $K_1, K_2 \in \mathbb{N}$, such that $|b_{2k-1}| < \varepsilon$ for all $k \geq K_1$, and $|b_{2k}| < \varepsilon$ for all $k \geq K_2$.

Let $K = \max\{2K_1, 2K_2\}$. Let us show that $|b_n| < \varepsilon$ for all $n \geq K$. If $n = 2k-1$ is odd, then we have $2k-1 \geq 2K \geq 2K_1$. This implies that $k \geq K_1$, and hence we have $|b_n| = |b_{2k-1}| < \varepsilon$. If $n = 2k$ is even, then we have $2k \geq 2K \geq 2K_2$. This implies that $k \geq K_2$, and hence we have $|b_n| = |b_{2k}| < \varepsilon$, and this completes the claim.

By the Alternating Series Test, we have the series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ to be convergent. By defining $T_m := \sum_{n=1}^{m} (-1)^{n+1} b_n$ for all $m \in \mathbb{N}$, it is easy to see that the sequence (T_n) of partial sums converge. As we have $S_{4n} = T_{2n}$ for all $n \in \mathbb{N}$, and the subsequence (T_{2n}) converges, we have the sequence (S_{4n}) to converge as desired.

(iii) The series $\sum_{n=1}^{\infty} a_n$ converges. Indeed, let s denote the limit of the sequence (S_{4n}). We note that $S_{4n+1} = S_{4n} + a_{4n+1}$, $S_{4n+2} = S_{4n} + a_{4n+1} + a_{4n+2}$ and $S_{4n+3} = S_{4n} + a_{4n+1} + a_{4n+2} + a_{4n+3}$ for all $n \in \mathbb{N}$. As $\lim_{n\to\infty} a_{4n+1} = \lim_{n\to\infty} \frac{1}{2n+1} = 0$, it follows that $\lim_{n\to\infty} S_{4n+1} = \lim_{n\to\infty} (S_{4n} + a_{4n+1}) = s$. Similarly, we have $\lim_{n\to\infty} S_{4n+2} = \lim_{n\to\infty} S_{4n+3} = s$. By a similar argument as in part (ii), we must have $\lim_{n\to\infty} S_n = s$. So the sequence (S_n) of partial sums of $\sum_{n=1}^{\infty} a_n$ converges, and hence $\sum_{n=1}^{\infty} a_n$ is a convergent series as desired.

Question 3

(a) (i) We have

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to 1}\frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)}=\lim_{x\to 1}\frac{1}{\sqrt{x}+1}=\frac{1}{1+1}=\frac{1}{2}.$$

(ii) We have

$$\lim_{x\to\infty}\frac{\sqrt{x}-x}{\sqrt{x}+x}=\lim_{x\to\infty}\frac{\frac{1}{\sqrt{x}}-1}{\frac{1}{\sqrt{x}}+1}=\frac{0-1}{0+1}=-1.$$

(iii) For $x\in(2,3)$, we have $[x]=2$, so that $\frac{[x]-x}{x-2}=\frac{2-x}{x-2}=-1$. Hence, we have $\lim\limits_{x\to 2^+}\frac{[x]-x}{x-2}=-1$.

(b) Let $\varepsilon>0$ be given. As f is uniformly continuous on $[a,\infty)$, it follows that there exists some $\delta_1>0$, such that for all $x,y\in[a,\infty)$ that satisfies $|x-y|<\delta_1$, we have $|f(x)-f(y)|<\varepsilon$. Next, since f is continuous on $[0,\infty)$, it is continuous (hence uniformly continuous) on $[0,a]$, so it follows that there exists some $\delta_2>0$, such that for all $x,y\in[0,a]$ that satisfies $|x-y|<\delta_2$, we have $|f(x)-f(y)|<\varepsilon$. Finally, since f is continuous at a, it follows that there exists $\delta_3>0$, such that for all $x\in[0,\infty)$ that satisfies $|x-a|<\delta_3$, we have $|f(x)-f(a)|<\frac{\varepsilon}{2}$.

Now, set $\delta:=\min\{\delta_1,\delta_2,\delta_3\}>0$. Let us show that for all $x,y\in[0,\infty)$ that satisfies $|x-y|<\delta$, we have $|f(x)-f(y)|<\varepsilon$. Without loss of generality, let us assume that $x<y$. If $x<y\le a$ or $a\le x<y$, then the conclusion is immediate. Henceforth, let us assume that $x<a<y$. Then we have $|x-a|=a-x<y-x=|x-y|<\delta<\delta_3$, and $|y-a|=y-a<y-x=|x-y|<\delta<\delta_3$, so we have $|f(x)-f(y)|\le|f(x)-f(a)|+|f(a)-f(y)|\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$, and this completes the proof. As $\varepsilon>0$ is arbitrary, this shows that f is uniformly continuous on $[0,\infty)$ as desired.

(c) (i) Yes.

(ii) Firstly, since $\frac{1}{2n-1}>\frac{1}{2n}$ for all $n\in\mathbb{N}$, and the series $\sum\limits_{n=1}^{\infty}\frac{1}{2n}=\frac{1}{2}\sum\limits_{n=1}^{\infty}\frac{1}{n}$ is divergent, it follows that the series $\sum\limits_{n=1}^{\infty}\frac{1}{2n-1}$ is divergent by the Comparison Test.

Now, let us define a_1 to be the smallest positive integer such that $\sum\limits_{n=1}^{a_1}\frac{1}{2n-1}>c$ (note that such an a_1 must exist since the sequence of partial sums of the series $\sum\limits_{n=1}^{\infty}\frac{1}{2n-1}$ tends to ∞), and let us define b_1 to be the smallest integer such that $\sum\limits_{n=1}^{a_1}\frac{1}{2n-1}-\sum\limits_{n=1}^{b_1}\frac{1}{2n}<c$ (note that such an b_1 must exist since the sequence of partial sums of the series $\sum\limits_{n=1}^{\infty}-\frac{1}{2n}$ tends to $-\infty$).

Based on this, let us recursively define a_k and b_k for all positive integers $k>1$ as follows: We define a_k to be the smallest integer such that

$$\sum_{n=1}^{a_k}\frac{1}{2n-1}-\sum_{n=1}^{b_{k-1}}\frac{1}{2n}=\sum_{n=1}^{a_{k-1}}\frac{1}{2n-1}-\sum_{n=1}^{b_{k-1}}\frac{1}{2n}+\sum_{n=a_{k-1}+1}^{a_k}\frac{1}{2n-1}>c,$$

and define b_k to be the smallest integer such that

$$\sum_{n=1}^{a_k}\frac{1}{2n-1}-\sum_{n=1}^{b_k}\frac{1}{2n}=\sum_{n=1}^{a_k}\frac{1}{2n-1}-\sum_{n=1}^{b_{k-1}}\frac{1}{2n}-\sum_{n=b_{k-1}+1}^{b_k}\frac{1}{2n}<c.$$

Then it is clear that $a_k<a_{k+1}$ and $b_k<b_{k+1}$ for all $k\in\mathbb{N}$.

Next, let us define $A_1 = \sum\limits_{n=1}^{a_1} \frac{1}{2n-1}$, $A_2 = -\sum\limits_{n=1}^{b_1} \frac{1}{2n}$, $A_{2k-1} = \sum\limits_{n=a_{k-1}+1}^{a_k} \frac{1}{2n-1}$ and $A_{2k} = -\sum\limits_{n=b_{k-1}+1}^{b_k} \frac{1}{2n}$ for all positive integers $k > 1$. Let us show that

$$\sum_{n=1}^{2k-1} A_n = \sum_{n=1}^{a_k} \frac{1}{2n-1} - \sum_{n=1}^{b_{k-1}} \frac{1}{2n} \leq c + \frac{1}{2a_k - 1}$$

for all $k \in \mathbb{N}$. Indeed, since a_1 is the smallest positive integer such that $\sum\limits_{n=1}^{a_1} \frac{1}{2n-1} > c$, we must have $\sum\limits_{n=1}^{a_1-1} \frac{1}{2n-1} \leq c$, and hence we must have $\sum\limits_{n=1}^{a_1} \frac{1}{2n-1} = \frac{1}{2a_1-1} + \sum\limits_{n=1}^{a_1-1} \frac{1}{2n-1} \leq c + \frac{1}{2a_1-1}$. By a similar reasoning, we can also show that $\sum\limits_{n=1}^{2k-1} A_n \leq c + \frac{1}{2a_k-1}$ for all $k > 1$. Similarly, we have

$$\sum_{n=1}^{2k} A_n = \sum_{n=1}^{a_k} \frac{1}{2n-1} - \sum_{n=1}^{b_k} \frac{1}{2n} \geq c - \frac{1}{2b_k}$$

for all $k \in \mathbb{N}$.

Now, for each $1 \leq i \leq a_1$ and $1 \leq j \leq b_1$, let us define $x_i = \frac{1}{2i-1}$ and $x_{a_0+j} = -\frac{1}{2j}$. Furthermore, for each positive integer $N \in \mathbb{N}$, and $1 \leq k \leq a_{N+1} - a_N$, $1 \leq \ell \leq b_{N+1} - b_N$, let us define $x_{a_N+b_N+k} = \frac{1}{2(a_N+k)-1}$ and $x_{a_{N+1}+b_N+\ell} = -\frac{1}{2(b_N+\ell)}$ (In other words, the sequence (x_n) is defined as follows:

$$(x_n) = (1, \frac{1}{3}, \cdots, \frac{1}{2a_1-1}, -\frac{1}{2}, -\frac{1}{4}, \cdots, -\frac{1}{2b_1}, \frac{1}{2a_1+1}, \cdots, \frac{1}{2a_2-1}, -\frac{1}{2b_1+2}, \cdots) \quad).$$

Then it is evidently clear that $\sum\limits_{n=1}^{\infty} x_n$ is a rearrangement of the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. We would like to show that the series $\sum\limits_{n=1}^{\infty} x_n$ converges to c. To this end, for each $n > a_1 + b_1$, we define $f(n)$ to be the unique positive integer K such that $a_K + b_K < n \leq a_{K+1} + b_{K+1}$. We first note that $\sum\limits_{n=1}^{a_{N+1}+b_N} x_n = \sum\limits_{n=1}^{2N+1} A_n$ and $\sum\limits_{n=1}^{a_N+b_N} x_n = \sum\limits_{n=1}^{2N} A_n$ for all $N \in \mathbb{N}$. This implies that $\sum\limits_{n=1}^{a_{N+1}+b_N} x_n \leq c + \frac{1}{2a_{N+1}-1}$ and $\sum\limits_{n=1}^{a_N+b_N} x_n \geq c - \frac{1}{2b_N}$ for all $N \in \mathbb{N}$. Furthermore, for each positive integer $N \in \mathbb{N}$, and $1 \leq k \leq a_{N+1} - a_N$, we have

$$\sum_{n=1}^{a_N+b_N+k} x_n = \sum_{n=1}^{a_N+b_N} x_n + \sum_{n=a_N+1}^{a_N+k} \frac{1}{2n-1} \geq \sum_{n=1}^{a_N+b_N} x_n \geq c - \frac{1}{2b_N}, \text{ and}$$

$$\sum_{n=1}^{a_N+b_N+k} x_n \leq \sum_{n=1}^{a_N+b_N} x_n + \sum_{n=a_N+1}^{a_{N+1}} \frac{1}{2n-1} = \sum_{n=1}^{a_{N+1}+b_N} x_n \leq c + \frac{1}{2a_{N+1}-1}.$$

Similarly, for each positive integer $N \in \mathbb{N}$, and $1 \leq \ell \leq b_{N+1} - b_N$, we have

$$c - \frac{1}{2b_{N+1}} \leq \sum_{n=1}^{a_{N+1}+b_N+\ell} x_n \leq c + \frac{1}{2a_{N+1}-1}.$$

Hence, for all $N \in \mathbb{N}$ and $a_N + b_N < M \leq a_{N+1} + b_{N+1}$, we have

$$c - \frac{1}{2b_{f(M)}} = c - \frac{1}{2b_N} \leq \sum_{n=1}^{M} x_n \leq c + \frac{1}{2a_N - 1} = c + \frac{1}{2a_{f(M)} - 1}.$$

As $\lim\limits_{M\to\infty} f(M) = \infty$, it follows that $\lim\limits_{M\to\infty} c - \frac{1}{2b_{f(M)}} = c = \lim\limits_{M\to\infty} c + \frac{1}{2a_{f(M)-1}}$. By Squeeze Theorem, we have $\sum\limits_{n=1}^{\infty} x_n = \lim\limits_{M\to\infty} \sum\limits_{n=1}^{M} x_n = c$ as desired, and we are done.

(iii) It is possible to obtain two rearrangements $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$ of $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ such that $\sum\limits_{n=1}^{\infty} x_n = c = \sum\limits_{n=1}^{\infty} y_n$, but $\sum\limits_{n=1}^{\infty} x_n$ cannot be obtained by rearranging finite terms from $\sum\limits_{n=1}^{\infty} y_n$. Indeed, for each $n \in \mathbb{N}$, let us define x_n as in part (ii), and define $y_{2n-1} = x_{2n}$ and $y_{2n} = x_{2n-1}$. Then it is clear that $\sum\limits_{n=1}^{\infty} y_n$ is a rearrangement of the series $\sum\limits_{n=1}^{\infty} x_n$, and hence $\sum\limits_{n=1}^{\infty} y_n$ is also a rearrangement of the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Furthermore, by the definition of y_n for all $n \in \mathbb{N}$, it is also clear that $\sum\limits_{n=1}^{\infty} x_n$ cannot be obtained by rearranging finite terms from $\sum\limits_{n=1}^{\infty} y_n$.

Finally, for each $k \in \mathbb{N}$, let us define $S_k := \sum\limits_{n=1}^{k} x_n$ and $T_k := \sum\limits_{n=1}^{k} y_n$. Then it is clear that $T_{2k} = S_{2k}$ and $T_{2k+1} = T_{2k} + y_{2k+1} = S_{2k} + x_{2k+2}$ for all $k \in \mathbb{N}$. As the series $\sum\limits_{n=1}^{\infty} x_n$ converges to c, it follows that $\lim\limits_{k\to\infty} S_{2k} = c$, and $\lim\limits_{k\to\infty} x_{2k+2} = 0$. This implies that

$$\lim_{k\to\infty} T_{2k} = c = \lim_{k\to\infty} T_{2k+1} = \lim_{k\to\infty} T_{2k-1}.$$

By a similar argument as in Question 2 part (b)(iii), we have $\lim\limits_{k\to\infty} T_k = c$, and hence we have $\sum\limits_{n=1}^{\infty} y_n = c$, and this completes the claim.

Question 4

(a) Let us first show that $f(x) > f(1)$ for all $x \in (0,1)$. Arguing by contradiction, suppose there exists some $y \in (0,1)$ such that $f(1) \geq f(y)$. As we have $f(0) > f(1) \geq f(y)$, it follows from Intermediate Value Theorem that there exists some $c \in [0,y]$, such that $f(c) = f(1)$, which is a contradiction.

Next, let us show that f is strictly decreasing on $[0,1]$. Arguing by contradiction, suppose f is not strictly decreasing on $[0,1]$. Then there exist $a, b \in [0,1]$, such that $a \leq b$ and $f(a) \leq f(b)$. As we have $f(1) < f(a) \leq f(b)$, it follows from Intermediate Value Theorem that there exists some $d \in [b,1]$, such that $f(d) = f(a)$, a contradiction. So f is strictly decreasing on $[0,1]$ as desired.

(b) Arguing by contradiction, suppose that f is unbounded on $[a,b]$. Then for each $n \in \mathbb{N}$, there exists some $x_n \in [a,b]$, such that $|f(x_n)| > n$. By Bolzano-Weierstrass Theorem, there exists a convergent subsequence (x_{n_k}) of (x_n). Let us denote the limit of the sequence (x_{n_k}) by x. By assumption, there exists some $\delta_x > 0$ and $M \in \mathbb{N}$, such that for all $y \in [a,b]$ with $|x-y| < \delta_x$, we have $|f(y)| \leq M$. On the other hand, since $\lim\limits_{k\to\infty} x_{n_k} = x$, it follows that there exists some $K \in \mathbb{N}$, such that $|x_{n_k} - x| < \delta_x$ for all $k \geq K$. By letting $N = \max\{K, M\}$, we see that $|x_{n_N} - x| < \delta_x$, and hence, by assumption, we have $|f(x_{n_N})| \leq M \leq N \leq n_N < |f(x_{n_N})|$, which is a contradiction. So f must be bounded on $[a,b]$ as desired.

(c) $(\Leftarrow)$: Firstly, we suppose that the series $\sum\limits_{n=1}^{\infty} 2^n a(2^n)$ is convergent. Then this would imply that the series $\sum\limits_{n=0}^{\infty} 2^n a(2^n)$ is convergent. Let $s := \sum\limits_{n=0}^{\infty} 2^n a(2^n)$. For each $k \in \mathbb{N}$, let us define $b(k) :=$

$a\left(2^{[\log_2 k]}\right)$ (in other words, if n is the unique non-negative integer such that $2^n \le k \le 2^{n+1}-1$, then $b(k) = a(2^n)$).

Let us show that the series $\sum_{n=1}^{\infty} b(n)$ is convergent. To this end, let us define $S_k := \sum_{n=0}^{k} 2^n a(2^n)$ and $T_k := \sum_{n=1}^{k} b(n)$ for all $k \in \mathbb{N}$. Then we have $\lim_{n\to\infty} S_n = s$. Moreover, it is easy to verify that $T_{2^{k+1}-1} = S_k$ for all $k \in \mathbb{N}$. Furthermore, for each positive integer $n \ge 4$, we have

$$S_{[\log_2 n]-1} = T_{2^{[\log_2 n]}-1} \le T_n \le T_{2^{[\log_2 n]+1}-1} = S_{[\log_2 n]}.$$

As $\lim_{n\to\infty} [\log_2 n] = \infty$, this would imply that $\lim_{n\to\infty} S_{[\log_2 n]-1} = s = \lim_{n\to\infty} S_{[\log_2 n]}$. By Squeeze Theorem, we have $\lim_{n\to\infty} T_n = s$, and this completes the claim. Now, as $(a(n))$ is a decreasing sequence, we have $a(n) \le a\left(2^{[\log_2 k]}\right) = b(n)$ for all $n \in \mathbb{N}$. By Comparison Test, we have the series $\sum_{n=1}^{\infty} a(n)$ to be convergent.

($\Rightarrow$): Conversely, we suppose that the series $\sum_{n=1}^{\infty} a(n)$ is convergent. Then this would imply that the series $\sum_{n=1}^{\infty} 2a(n)$ is convergent. Consequently, we have the series $\sum_{k=1}^{\infty} \sum_{n=2^{k-1}}^{2^k-1} 2a(n) = \sum_{n=1}^{\infty} 2a(n)$ to be convergent. Now, since $(a(n))$ is a decreasing sequence, we have $a(2^m) \le a(n)$ for all $2^{m-1} \le n \le 2^m - 1$. This implies that $2^{m-1}a(2^m) \le \sum_{n=2^{m-1}}^{2^m-1} a(n)$, or equivalently, $2^m a(2^m) \le \sum_{n=2^{m-1}}^{2^m-1} 2a(n)$ for all $m \in \mathbb{N}$. Consequently, we have the series $\sum_{m=1}^{\infty} 2^m a(2^m)$ to be convergent by the Comparison Test as desired.