NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA1104 Multivariable Calculus
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Question 1

(i) Let $f(x,y,z) = x^2 + 3y^2 - 3z^2 - 1$
$\bigtriangledown f(x,y,z) = \langle 2x, 6y, -6z \rangle$
$\bigtriangledown f(1,1,1) = \langle 2, 6, -6 \rangle = 2\langle 1, 3, -3 \rangle$
Hence a vertor normal to the tangent plane to the surface S at the point $(1,1,1)$ is $\langle 1,3,-3\rangle$. Therefore the equation of the tangent plane at $(1,1,1)$ is;

$$\begin{aligned}\langle 1,3,-3\rangle \cdot \langle x,y,z\rangle &= \langle 1,3,-3\rangle \cdot \langle 1,1,1\rangle \\ x + 3y - 3z &= 1.\end{aligned}$$

(ii) Let $g(x,y,z) = x^2 + 5y^2 - z^2 - 5$
$\bigtriangledown g(x,y,z) = \langle 2x, 10y, -2z \rangle$
$\bigtriangledown g(1,1,1) = \langle 2, 10, -2 \rangle = 2\langle 1, 5, -1 \rangle$
Hence a vertor parallell to the tangent line to the curve of intersection of the 2 surfaces at $(1,1,1)$ is;

$$\begin{aligned}\mathbf{u} &= \langle 1,3,-3\rangle \times \langle 1,5,-1\rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -3 \\ 1 & 5 & -1 \end{vmatrix} \\ &= \langle 12, -2, 2\rangle \\ &= 2\langle 6, -1, 1\rangle\end{aligned}$$

Whence a parametrization of the tangent line is
$\mathbf{r}(t) = \langle 1,1,1\rangle + t\langle 6,-1,1\rangle$ for $t \in \mathbb{R}$.
Hence a parametric equation of the tangent line is;

$$\begin{aligned}x &= 1 + 6t, \\ y &= 1 - t, \\ z &= 1 + t, \ t \in \mathbb{R}\end{aligned}$$

Question 2

(a) (i) Using Chain Rule;

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \\ &= f_x \cdot (2v) + f_y \cdot (3) \\ &= 2vf_x + 3f_y\end{aligned}$$

(ii) Using Chain Rule;

$$\begin{aligned}
\frac{\partial^2 z}{\partial v^2} &= \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) \\
&= \frac{\partial}{\partial v}\left(2vf_x + 3f_y\right) \\
&= 2\cdot f_x + 2v\cdot\frac{\partial}{\partial v}\left(f_x\right) + 3\cdot\frac{\partial}{\partial v}\left(f_y\right) \\
&= 2f_x + 2v\left(f_{xx}\cdot\frac{\partial x}{\partial v} + f_{xy}\cdot\frac{\partial y}{\partial v}\right) + 3\left(f_{yx}\cdot\frac{\partial x}{\partial v} + f_{yy}\cdot\frac{\partial y}{\partial v}\right) \\
&= 2f_x + 2v\left(f_x x\cdot(2v) + f_{xy}\cdot(3)\right) + 3\left(f_{yx}\cdot(2v) + f_{yy}\cdot(3)\right) \\
&= 2f_x + 4v^2 f_{xx} + 12v f_{xy} + 9f_{yy}
\end{aligned}$$

(b) (i)

$$\begin{aligned}
T(x,y) &= (x+3y)e^{y-x^2} \\
\nabla T(x,y) &= \left\langle \frac{\partial}{\partial x}\left((x+3y)e^{y-x^2}\right), \frac{\partial}{\partial y}\left((x+3y)e^{y-x^2}\right)\right\rangle \\
&= \left\langle e^{y-x^2}\left(1-2x^2-6xy\right), e^{y-x^2}\left(x+3y+3\right)\right\rangle \\
\Rightarrow \nabla T(1,1) &= \langle -7,7\rangle = 7\langle -1,1\rangle
\end{aligned}$$

Therefore, the direction one should go to get the maximum rate of decrease in T is given by $-\nabla T(1,1) = \langle 7,-7\rangle$.

(ii) Let $g(x,y) = 2x^2 - xy^3$

$$\begin{aligned}
\bigtriangledown g(x,y) &= \left\langle 4xy - y^3, 2x^2 - 3xy^2\right\rangle \\
\Rightarrow \bigtriangledown g(1,1) &= (3,1)
\end{aligned}$$

Hence a vector normal to the tangent line to the curve C at (1,1) is $\langle 3,-1\rangle$. Thus, a vector parallel to the tangent line to the curve C at (1,1) is $\langle 1,3\rangle$. Thus,$\hat{\mathbf{u}} = \frac{\langle 1,3\rangle}{\sqrt{1^2+3^2}} = \frac{1}{\sqrt{10}}\langle 1,3\rangle$

(iii) Hence, at (1,1);

$$\begin{aligned}
D_u T &= \nabla T(1,1)\cdot\hat{\mathbf{u}} \\
&= \langle -7,7\rangle\cdot\left(\frac{1}{\sqrt{10}}\langle 1,3\rangle\right) \\
&= \frac{14}{\sqrt{10}}
\end{aligned}$$

Question 3

(a)

$$\begin{aligned}
\int_0^4 \int_{\sqrt{x}}^2 \frac{3}{5+y^3} dydx &= \int_0^2 \int_0^{y^2} \frac{3}{5+y^3} dxdy \\
&= \int_0^2 \left[\frac{3x}{5+y^3}\right]_0^{y^2} dy \\
&= \int_0^2 \frac{3y^2}{5+y^3} dy \\
&= \left[\ln(5+y^3)\right]_0^2 \\
&= \ln 13 - \ln 5 \\
&= \ln \frac{13}{5}
\end{aligned}$$

(b) Observe that $z = 1 \pm \sqrt{1-x^2-y^2}$ is a sphere of radius 1 centered at (0,0,1).
In sphereical coordinates,

$$\begin{aligned}
z = 1 \pm \sqrt{1-x^2-y^2} &\Rightarrow x^2+y^2+(z-1)^2 = 1 \\
&\Rightarrow p^2 = 2p\cos\phi \\
&\Rightarrow p = 2\cos\phi
\end{aligned}$$

Hence;

$$\begin{aligned}
\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{1-\sqrt{1-x^2-y^2}}^{1+\sqrt{1-x^2-y^2}} (x^2+y^2+z^2)^{\frac{3}{2}} dzdydx &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\cos\phi} (p^3 \cdot p^2 \sin\phi) dpd\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\cos\phi} (p^5 \sin\phi) dpd\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left[\frac{p^6}{6}\sin\phi\right]_0^{2\cos\phi} d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left(\frac{32}{3}\cos^6\phi \sin\phi\right) d\phi d\theta \\
&= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{2}} \frac{32}{3}\cos^6\phi\sin\phi \; d\phi\right) \\
&= 2\pi \cdot \frac{32}{3}\left[-\frac{\cos^7\phi}{7}\right]_0^{\frac{\pi}{2}} \\
&= \frac{64}{21}\pi
\end{aligned}$$

Question 4

(a) Let $D = \{(x,y) \in \mathbb{R}^2 : x+y \leq 6, x \geq 0, y \geq 0\}$.
We shall consider the boundary of D.

1. $x+y=6, x,y \geq 0$
Let $g(x,y) = x+y \Rightarrow \bigtriangledown g(x,y) = \langle 1,1 \rangle$.

Consider $f(x, y) = x^3 + y^3 - 5xy \Rightarrow \triangledown f(x, y) = \langle 3x^2 - 5y, 3y^3 - 5x \rangle$.
By Lagrange Multipliers, consider $\triangledown f = \lambda \cdot \triangledown g$, i.e.

$$3x^2 - 5y = \lambda \tag{1}$$
$$3y^2 - 5x = \lambda \tag{2}$$
$$x + y = 6\lambda \tag{3}$$

By (1) and (2), $3x^2 - 5y = 3y^2 - 5x$.
by (3), $x = 6 - y$, whence

$$\begin{aligned} 3(6-y)^2 - 5y &= 3y^2 - 5(6-y) \\ 108 - 36y + 3y^2 - 5y &= 3y^2 - 30 + 5y \\ 46y &= 138 \\ y &= 3 \\ \Rightarrow x &= 3 \end{aligned}$$

Hence a candidate point is (3,3), where $f(3, 3) = 9$

2. $x = 0, 0 \leq y \leq 6$.
Consider $f(0, y) = y^3$ Which is monotone increasing on the close interval [0,6].
Hence on $x = 0, 0 \leq y \leq 6, f$ achieves absolute minimum at (0,0) and maximum at (0,6), i.e.
$f(0, 0) = 0, f(0, 6) = 216$

3. $y = 0, 0 \leq x \leq 6$.
By symmetry, on $y = 0, 0 \leq x \leq 6, f$ achieves absolute minimum at (0,0) and maximum at (6,0), i.e.
$f(0, 0) = 0, f(6, 0) = 216$

We shall now consider the interior region of D.
$f(x, y) = x^3 + y^3 - 5xy \Rightarrow f_x = 3x^2 - 5y, f_y = 3y^2 - 5x$.
Consider $f_x = f_y = 0$, and $x, y \geq 0$ i.e.
$3x^2 - 5y = 0 \Rightarrow x = \sqrt{\frac{5}{3}y}$ and $3y^2 - 5x = 0 \Rightarrow y = \sqrt{\frac{5}{3}x}$.
Thus we get $x = \sqrt{\frac{5}{3}\sqrt{\frac{5}{3}x}} \Rightarrow x = \frac{5}{3}$.(We reject $x = 0$ as it is in the interior region.)
Hence we also have $y = \frac{5}{3}$, making $(\frac{5}{3}, \frac{5}{3})$ another candidate point; with $f(\frac{5}{3}, \frac{5}{3}) = -\frac{125}{27}$.

By Closed Interval Method,

Point(x, y)	$f(x, y)$
$(\frac{5}{3}, \frac{5}{3})$	$-\frac{125}{27}$
(0,0)	0
(3,3)	9
(0,6)	216
(6,0)	216

Whence the maximum value attained is 216 at (0,6) and (6,0) and minimum value attained is $-\frac{125}{27}$ at $(\frac{5}{3}, \frac{5}{3})$.

(b) Consider a change of variables, i.e.

$$\begin{aligned}
u = xy, v = \frac{y}{x} &\Rightarrow x = \sqrt{\frac{u}{v}}, y = \sqrt{uv} \\
\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\
&= \begin{vmatrix} \frac{1}{2v}\left(\frac{u}{v}\right)^{-\frac{1}{2}} & \frac{1}{2}\left(\frac{u}{v}\right)^{\frac{1}{2}}\left(-\frac{u}{v^2}\right) \\ \frac{1}{2}(uv)^{-\frac{1}{2}}(v) & \frac{1}{2}(uv)^{-\frac{1}{2}}(u) \end{vmatrix} \\
&= \frac{1}{2v}.
\end{aligned}$$

Thus;

$$\begin{aligned}
\iint_R \frac{y}{x}\cos\left(\frac{y}{x}\pi\right) dA &= \int_1^4 \int_2^5 v\cos(v\pi) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \\
&= \int_1^4 \int_2^5 v\cos(v\pi)\frac{1}{2v} dudv \\
&= \frac{1}{2}\int_1^4 \int_2^5 v\cos(v\pi) dudv \\
&= \frac{1}{2}\left(\int_2^5 du\right)\left(\int_1^4 \cos(v\pi) dv\right) \\
&= \frac{3}{2}\left[\frac{\sin(v\pi)}{\pi}\right]_1^4 \\
&= \frac{3}{2}(0) \\
&= 0
\end{aligned}$$

Question 5

(a) (i) $\mathbf{F} = \left\langle 2xe^y + 1, x^2e^y + 2y + x \right\rangle$ It is not a conversative vector field since;

$$\begin{aligned}
\frac{\partial}{\partial x}(x^2e^y + 2y + x) &= 2xe^y + 1 \\
&\neq 2xe^y \\
&= \frac{\partial}{\partial y}(2xe^y + 1).
\end{aligned}$$

(ii) Let $\mathbf{G} = \left\langle 2xe^y + 1, x^2e^y + 2y \right\rangle$. Then $\mathbf{F} = \mathbf{G} + \langle 0, x \rangle$.

Obeserve that $\mathbf{G}$ is conservative since;
$\frac{\partial}{\partial x}(x^2e^y + 2y + x) = 2xe^y = \frac{\partial}{\partial y}(2xe^y + 1)$.
Hence there exist a potential function f such that $\mathbf{G} = \bigtriangledown f$
whence; $f_x = 2xe^y + 1, f_y = x^2e^y + 2y$.
Consider;

$$\begin{aligned}
f(x,y) &= \int f_x dx \\
&= x^2e^y + x + \phi(y) \\
\Rightarrow f_y &= x^2e^y + \phi'(y)
\end{aligned}$$

By comparison,

$$\begin{aligned}\phi'(y) &= 2y \\ \phi(y) &= y^2 + C\end{aligned}$$

Choose $C = 0$.
Hence a potential function f is;

$$f(x, y) = x^2 e^y + x + y^2.$$

(iii) Parametrize C:

$$\begin{aligned}\mathbf{r}(t) &= \langle t, \sin t\rangle\,,\ 0 \le t \le \pi \\ \mathbf{r}'(t) &= \langle 1, \cos t\rangle\end{aligned}$$

Hence;

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (\mathbf{G} + \langle 0, x\rangle)\cdot d\mathbf{r} \\ &= \int_C \mathbf{G}\cdot d\mathbf{r} + \int_C \langle 0, x\rangle \cdot d\mathbf{r} \\ &= [f(\pi, 0) - f(0, 0)] + \int_C \langle 0, t\rangle \cdot \langle 1, \cos t\rangle\, dt \\ &= (\pi^2 + \pi + 0 - 0) + \int_0^\pi t\cos t dt \\ &= \pi^2 + \pi + [t\sin t]_0^\pi - \int_0^\pi \sin t dt \\ &= \pi^2 + \pi + 0 - [-\cos t]_0^\pi \\ &= \pi^2 + \pi - 2\end{aligned}$$

(b)

$$\mathbf{F} = \left\langle \tfrac{x^3}{x^4+y^4}, \tfrac{y^3}{x^4+y^4}\right\rangle$$

Observe that we cannot apply Green's Theorem directly due to $\mathbf{F}$ not being defined at (0,0). So the region D enclosed by C is not simply connected.

Instead we consider curve $C' = C_1 \cup C_2 \cup C_3 \cup C_4$ in negative orientation, where;

$$\begin{aligned}C_1 &: \mathbf{r}(t) = \langle t, 1\rangle\,,\ -1 \le t \le 1 \\ C_2 &: \mathbf{r}(t) = \langle 1, t\rangle\,,\ -1 \le t \le 1 \\ C_3 &: \mathbf{r}(t) = \langle t, -1\rangle\,,\ -1 \le t \le 1 \\ C_4 &: \mathbf{r}(t) = \langle -1, t\rangle\,,\ -1 \le t \le 1\end{aligned}$$

Now for $C_1, \mathbf{r}'(t) = \langle 1, 0 \rangle$; thus,

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_{-1}^{1} \left\langle \frac{t^3}{1+t^4}, \frac{1}{1+t^4} \right\rangle \cdot \langle 1, 0 \rangle \, dt \\ &= \int_{-1}^{1} \frac{t^3}{1^4+t^4} dt \\ &= \left[\frac{1}{4} \ln\left(1+t^4\right) \right]_{-1}^{1} \\ &= \frac{1}{4}(\ln 2 - \ln 2) \\ &= 0\end{aligned}$$

In a similar fashion;

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_{1}^{-1} \left\langle \frac{1}{1+t^4}, \frac{t^3}{1+t^4} \right\rangle \cdot \langle 0, 1 \rangle \, dt \\ &= \int_{1}^{-1} \frac{t^3}{1^4+t^4} dt \\ &= 0\end{aligned}$$

$$\begin{aligned}\int_{C_3} \mathbf{F} \cdot d\mathbf{r} &= \int_{1}^{-1} \left\langle \frac{t^3}{1+t^4}, -\frac{1}{1+t^4} \right\rangle \cdot \langle 1, 0 \rangle \, dt \\ &= \int_{1}^{-1} \frac{t^3}{1^4+t^4} dt \\ &= 0\end{aligned}$$

$$\begin{aligned}\int_{C_4} \mathbf{F} \cdot d\mathbf{r} &= \int_{-1}^{1} \left\langle -\frac{1}{1+t^4}, \frac{t^3}{1+t^4} \right\rangle \cdot \langle 0, 1 \rangle \, dt \\ &= \int_{-1}^{1} \frac{t^3}{1^4+t^4} dt \\ &= 0\end{aligned}$$

Whence;

$$\begin{aligned}\int_{C'} \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r} \\ &= 0\end{aligned}$$

Now let R be the region enclosed by C and C'. Then, we apply the extanded Green's Theorem from tutorial (where there is a hole in the region), we have;

$$\begin{aligned}\int_{C \cup C'} \mathbf{F} \cdot d\mathbf{r} &= \iint_{R'} \frac{\partial}{\partial x} \left(\frac{y^3}{x^3+y^3} \right) - \frac{\partial}{\partial y} \left(\frac{x^3}{x^3+y^3} \right) dA \\ &= \iint_{R} 0 \; dA \\ &= 0\end{aligned}$$

Thus we have;

$$\begin{aligned}
&\Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} + \int_{C'} \mathbf{F}\cdot d\mathbf{r} = 0\\
&\Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} + 0 = 0\\
&\Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} = 0
\end{aligned}$$

Question 6

(a) Let S be the portion of the cylinder $y^2+z^2=4$ that lie within the cylinder $x^2+z^2=4$ and $y\geq 0$. Hence by symmetry, the surface required is;

$$A = 2\times \iint_S 1dS$$

Parametrizing S;

$$\mathbf{r}(u,v) = \left\langle u, \sqrt{4-v^2}, v\right\rangle,\ u^2+v^2\leq 4.$$

Hence;

$$\begin{aligned}
A &= 2\iint_S 1dS\\
&= 2\int_{-2}^{2}\int_{-\sqrt{4-v^2}}^{\sqrt{4-v^2}} \|\mathbf{r}_u\times\mathbf{r}_v\| dudv\\
&= 2\int_{-2}^{2}\int_{-\sqrt{4-v^2}}^{\sqrt{4-v^2}} \left\|\langle 1,0,0\rangle \times \left\langle 0, -\frac{v}{\sqrt{4-v^2}}, 1\right\rangle\right\| dudv\\
&= 2\int_{-2}^{2}\int_{-\sqrt{4-v^2}}^{\sqrt{4-v^2}} \frac{2}{\sqrt{4-v^2}} dudv\\
&= 2\int_{-2}^{2} 4\ dv\\
&= 32.
\end{aligned}$$

(b) (i) $\mathbf{r}(u,v) = \langle v\cos u, v\sin u, v\rangle$, where $0\leq u\leq 2\pi,\ 0\leq v\leq 1$.

(ii) Consider; $\mathbf{r}_u = \langle -v\sin u, v\cos u, 0\rangle,\ \mathbf{r}_v = \langle \cos u, \sin u, 1\rangle$
$\Rightarrow \mathbf{r}_u\times\mathbf{r}_v = \langle v\cos u, v\sin u, -v\rangle$

Therefore,

$$\begin{aligned}
\mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v) &= \left\langle 2v\sin u\cos(2v^2), (2v-2v\cos u)\cos(2v^2), 1-2v\sin u\cos(2v^2)\right\rangle \cdot \langle v\cos u, v\sin u, -v\rangle\\
&= 4v^2\sin u\cos(2v^2) - v
\end{aligned}$$

Hence;

$$\begin{aligned}
\iint_H \mathbf{F}\ d\mathbf{S} &= \int_0^1 \int_0^{2\pi} \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v) du dv \\
&= \int_0^1 \int_0^{2\pi} 4v^2 \sin u \cos(2v^2) - v\ du dv \\
&= \int_0^1 \int_0^{2\pi} 4v^2 \sin u \cos(2v^2)\ du dv - \int_0^1 v\ du dv \\
&= \left(\int_0^1 4v^2 \cos(2v^2) dv\right)\left(\int_0^{2\pi} \sin u\ du\right) - (2\pi)\left(\int_0^1 v dv\right) \\
&= \left(\int_0^1 4v^2 \cos(2v^2) dv\right)(0) - (2\pi)\left[\frac{v^2}{2}\right]_0^1 \\
&= 0 - \pi \\
&= -\pi
\end{aligned}$$

(iii) Observe that if $\mathbf{G} = \left\langle \sin p^2, x, \sin p^2 \right\rangle$, then we have curl $\mathbf{G} = \mathbf{F}$.
Therefore by Strokes' Theorem,

$$\textstyle\iint_T \mathbf{F}\cdot d\mathbf{S} = \iint_T \text{curl } \mathbf{G}\cdot d\mathbf{S} = \int_C \mathbf{G}\cdot d\mathbf{r}$$

where C consists of three lines, $x+y=1$ and $z=0, y+z=1$ and $x=0, x+z=1$ and $y=0$, oriented in the positive orientation.

Consider $C_1 : x+y=1$ and $z=0, 0 \le y \le 1$.
It can be parametrized as $\mathbf{r}(t) = \langle 1-t, t, 0\rangle,\ 0 \le t \le 1 \Rightarrow \mathbf{r}'(t) = \langle -1, 1, 0\rangle$.
Thus we get;

$$\begin{aligned}
\int_{C_1} \mathbf{G}\cdot d\mathbf{r} &= \int_0^1 \left\langle \sin p^2, 1-t, \sin p^2 \right\rangle \cdot \langle -1, 1, 0\rangle\, dt \\
&= \int_0^1 1 - t - \sin p^2 dt
\end{aligned}$$

Consider $C_2 : y+z=1$ and $x=0, 0 \le z \le 1$.
It can be parametrized as $\mathbf{r}(t) = \langle 0, 1-t, t\rangle,\ 0 \le t \le 1 \Rightarrow \mathbf{r}'(t) = \langle 0, -1, 1\rangle$.
Thus we get;

$$\begin{aligned}
\int_{C_2} \mathbf{G}\cdot d\mathbf{r} &= \int_0^1 \left\langle \sin p^2, 0, \sin p^2 \right\rangle \cdot \langle 0, -1, 1\rangle\, dt \\
&= \int_0^1 \sin p^2 dt
\end{aligned}$$

Consider $C_3 : z+x=1$ and $y=0, 0 \le x \le 1$.
It can be parametrized as $\mathbf{r}(t) = \langle t, 0, 1-t\rangle,\ 0 \le t \le 1 \Rightarrow \mathbf{r}'(t) = \langle 1, 0, -1\rangle$.
Thus we get;

$$\begin{aligned}
\int_{C_3} \mathbf{G}\cdot d\mathbf{r} &= \int_0^1 \left\langle \sin p^2, t, \sin p^2 \right\rangle \cdot \langle 1, 0, -1\rangle\, dt \\
&= \int_0^1 0\ dt \\
&= 0
\end{aligned}$$

Thus;

$$\begin{aligned}
\int_C \mathbf{G}\cdot d\mathbf{r} &= \int_{C_1} \mathbf{G}\cdot d\mathbf{r} + \int_{C_2} \mathbf{G}\cdot d\mathbf{r} + \int_{C_3} \mathbf{G}\cdot d\mathbf{r} \\
&= \int_0^1 1 - t - \sin p^2 dt + \int_0^1 \sin p^2 dt + 0 \\
&= \int_0^1 1 - t \ dt \\
&= \left[t - \frac{t^2}{2} \right]_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

Whence by Strokes' Theorem,

$$\iint_T \mathbf{F}\cdot d\mathbf{S} = \tfrac{1}{2}$$

END OF SOLUTIONS

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