NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Wang Xingyin

MA3227 Numerical Analysis 2
AY 2009/2010 Sem 1

Question 1

(a) Let $A = L + D + U$

D: a diagonal matrix with dagonal entries equal to those of A's
L: a lower triangular matrix with 0 as diagonal entries and the same entries below its diagonal as those of A's
U: an upper triangular matrix with 0 as diagonal entries and the same entries above its diagonal as those of A's

For Jacobi iteration: $P = D$, and
for Gauss-Seidel iteration: $P = L + D$

(b) $Px^{(k)} = (P - A)x^{(k-1)} + b$
$Px = (P - A)x + b$

Taking the difference, we have

$$\begin{aligned} P(x^{(k)} - x) &= (P - A)(x^{(k-1)} - x) \\ x^{(k)} - x &= (I - P^{-1}A)(x^{(k-1)} - x) \end{aligned}$$

Inductively,

$$\begin{aligned} x^{(k)} - x &= (I - P^{-1}A)^k (x^{(0)} - x) \\ \left\| x^{(k)} - x \right\| &= \left\| (I - P^{-1}A)^k (x^{(0)} - x) \right\| \\ &\leq \left\| (I - P^{-1}A)^k \right\| \left\| (x^{(0)} - x) \right\| \end{aligned}$$

Given $\rho(I - P^{-1}A) < 1$, $\lim_{k\to\infty} \left\| (I - P^{-1}A)^k \right\| = 0$, and
$\left\| x^{(0)} - x \right\|$ is a constant independent of k
so $\lim_{k\to\infty} \left\| x^{(k)} - x \right\| = 0$.

Question 2

(a) Consider backward Euler method to solve $y'(t) = \lambda y(t)$
We have,

$$\begin{aligned} \frac{y^{(n+1)} - y^{(n)}}{\Delta t} &= \lambda y^{(n+1)} \\ y^{(n+1)} &= \frac{y^{(n)}}{1 - \lambda \Delta t} \end{aligned}$$

Inductively,

$$y^{(n)} = (\frac{1}{1-\lambda\Delta t})^n y^{(0)}$$
$$\left|y^{(n)}\right| = \left|\frac{1}{1-\lambda\Delta t}\right|^n \left|y^{(0)}\right|$$

Given $\lambda < 0$, $\Delta t > 0$ $\forall n \in N$, $\left|\frac{1}{1-\lambda\Delta t}\right|^n < 1$ so $\left|y^{(n)}\right| \leq \left|y^{(n)}\right|$.

(b) Consider forward Euler method to solve $y'(t) = \lambda y(t)$
We have,

$$\frac{y^{(n+1)} - y^{(n)}}{\Delta t} = \lambda y^{(n)}$$
$$y^{(n+1)} = (1+\lambda\Delta t)y^{(n)}$$

Inductively,

$$y^{(n)} = (1+\lambda\Delta t)^n y^{(0)}$$
$$\left|y^{(n)}\right| = \left|(1+\lambda\Delta t)\right|^n \left|y^{(0)}\right|$$

Given $0 \leq \Delta t \leq \frac{2}{-\lambda}$ and $\lambda < 0$, so $|1+\lambda\Delta t| \leq 1$ and $\left|y^{(n)}\right| \leq \left|y^{(n)}\right|$.

Question 3

(a) Assuming $y(t) \in C^2[0,T]$,

$$\begin{aligned}\tau_n &= y(t_{n+1}) - y(t_n) - \Delta t f(t_n, y(t_n)) \\ &= [y(t_n) + y'(t_n)\Delta t + O(\Delta t^2)] - y(t_n) - \Delta t y'(t_n) \\ &= O(\Delta t^2)\end{aligned}$$

So $|\tau_n| \leq C\Delta t^2$ for some constant C.

(b)

$$\begin{aligned}|e_{n+1}| &= |y(t_{n+1}) - y_{n+1}| \\ &= |y(t_n) + \Delta t f(t_n, y(t_n)) + \tau_n - (y_n + \Delta t f(t_n, y_n)| \\ &\leq |y(t_n - y_n)| + \Delta t\,|f(t_n, y(t_n) - f(t_n, y_n)| + |\tau_n| \\ &\leq |e_n| + \Delta t L\,|y(t_n) - y_n| + C\Delta t^2 \\ &= (1+\Delta t L)\,|e_n| + C\Delta t^2\end{aligned}$$

Inductively, for $0 \leq n \leq \frac{T}{\Delta t}$,

$$\begin{aligned}|e_n| &\leq (1+\Delta t L)^n\,|e_0| + C\Delta t^2 \sum_{i=0}^{n-1}(1+\Delta t L)^i \\ &= 0 + C\Delta t^2 \frac{(1+\Delta t L)^n - 1}{(1+\Delta t L) - 1} \\ &\leq C\Delta t^2 \frac{e^{n\Delta t L} - 1}{\Delta t L} \\ &= C\frac{e^{LT} - 1}{L}\Delta t\end{aligned}$$

Question 4

(a) $\|H\vec{w}\|_2^2 = (H\vec{w})^T(H\vec{w}) = \vec{w}^T H^T H\vec{w} = \vec{w}^T\vec{w} = \|\vec{w}\|_2^2$
So $\|H\vec{w}\|_2 = \|\vec{w}\|_2$

(b) Let $\vec{v} = \text{sign}(w_1)\,\|\vec{w}\|_2\,\vec{e}_1 + \vec{w}$
Then $H = I - 2\dfrac{\vec{v}\vec{v}^*}{\vec{v}^*\vec{v}}$.

Question 5

(a) Consider $y' = i\lambda y$, $y(0) = y_0$,
then $f(t_n, y_n) = i\lambda y_n$.

Applying (A), we have,

$$\begin{aligned} y_{n+1} &= y_n + \frac{\Delta t}{2}(i\lambda y_n + i\lambda[y_n + \Delta t(i\lambda y_n)]) \\ &= y_n + \frac{\Delta t}{2}[i\lambda y_n + i\lambda y_n - \Delta t\lambda^2 y_n] \\ &= y_n + i\lambda\Delta t y_n - \frac{1}{2}\lambda^2\Delta t^2 y_n \\ &= (1 + i\lambda\Delta t - \frac{1}{2}\lambda^2\Delta t^2)y_n \end{aligned}$$

Applying (B), we have,

$$\begin{aligned} y_{n+1} &= y_n + \Delta t i\lambda[y_n + \frac{\Delta t}{2}i\lambda y_n] \\ &= y_n + i\lambda\Delta t y_n - \frac{1}{2}\lambda^2\Delta t^2 y_n \\ &= (1 + i\lambda\Delta t - \frac{1}{2}\lambda^2\Delta t^2)y_n \end{aligned}$$

We obtain the same relation between y_n and y_{n+1} when (A) and (B) are applied.

(b)

$$y_{n+1} = (1 + i\lambda\Delta t - \frac{1}{2}\lambda^2\Delta t^2)y_n$$

Inductively,

$$|y_n| = \left|1 + i\lambda\Delta t - \frac{1}{2}\lambda^2\Delta t^2\right|^n |y_0|$$

$$\begin{aligned} \left|1 + i\lambda\Delta t - \frac{1}{2}\lambda^2\Delta t^2\right|^2 &= (1 - \frac{1}{2}\lambda^2\Delta t^2)^2 + (\lambda\Delta t)^2 \\ &= 1 - \lambda^2\Delta t^2 + \frac{1}{4}\lambda^4\Delta t^4 + \lambda^2\Delta t^2 \\ &= 1 + \frac{1}{4}\lambda^4\Delta t^4 \\ &> 1 \text{ for any fixed } \Delta t \end{aligned}$$

Hence $|y_n| \to \infty$ as $n \to \infty$.

(c) Applying (C), we have,

$$\begin{aligned} y_{n+1} &= y_n + \frac{\Delta t}{2}[i\lambda y_n + i\lambda y_{n+1}] \\ (1 - \frac{i\lambda\Delta t}{2})y_{n+1} &= (1 + \frac{i\lambda\Delta t}{2})y_n \end{aligned}$$

Since $\left|1 - \frac{i\lambda\Delta t}{2}\right| = \left|1 + \frac{i\lambda\Delta t}{2}\right| \neq 0$, $|y_{n+1}| = |y_n|$.
Inductively, $|y_n| = |y_0|$.

Question 6

(a) Suppose λ is an arbitrary eigenvalue of $A^{(k-1)}$, then $\left|\lambda I - A^{(k-1)}\right| = 0$

$$\begin{aligned} \left|\lambda I - A^{(k)}\right| &= \left|\lambda I - R^{(k)}Q^{(k)}\right| \\ &= \left|\lambda Q^{(k),T}Q^{(k)} - Q^{(k),T}Q^{(k)}R^{(k)}Q^{(k)}\right| \\ &= \left|Q^{(k),T}\right|\left|\lambda I - A^{(k-1)}\right|\left|Q^{(k)}\right| \\ &= 0 \end{aligned}$$

So λ is an eigenvalue of $A^{(k)}$
Similarly, we can prove any eigenvalue of $A^{(k)}$ is also eigenvalue of $A^{(k-1)}$, so $A^{(k)}$ and $A^{(k-1)}$ have the same eigenvalues.

(b)

- $A = \vec{u}\vec{v}^T = [v_1\vec{u} \vdots v_2\vec{u} \vdots \cdots \vdots v_n\vec{u}]$
  Suppose $\vec{q_1} = \hat{u} \neq \vec{0}$ and $\text{span}\{\vec{q_1}, \vec{q_2}, \ldots \vec{q_n}\} = \Re^n$
  $\vec{q_i}^T\vec{q_j} = 0$ if $i \neq j$ and $\vec{q_i}^T\vec{q_i} = 1$
  Choose $\vec{r_{jk}} = v_k \left\|\vec{u}\right\|$ if $j = 1$ and $r_{jk} = 0$ if $j \neq 1$
  Then $A = QR$.

- $QR = \vec{u}\vec{v}^T$
  Comparing the first column $r_{11}\vec{q_1} = v_1\vec{u}$
  So $\vec{u} = k\vec{q_1}$ for some $k \in \Re$ and $\vec{q_i}^T\vec{u} = 0$ for $i = 2, 3, \ldots, n$

  $RQ = Q^TQRQ = Q^T\vec{u}\vec{v}^TQ = (Q^T\vec{u})(Q^T\vec{v})^T$, where
  $Q^T\vec{v} = [\vec{q_1}^T\vec{v} \vdots \vec{q_2}^T\vec{v} \vdots \cdots \vdots \vec{q_n}^T\vec{v}]^T$
  $Q^T\vec{u} = [\vec{q_1}^T\vec{u} \vdots \vec{0} \vdots \cdots \vdots \vec{0}]^T$So RQ is an upper triangular matrix with 0s below its first row, and the $(1,1)$ entry is
  $(\vec{q_1}^T\vec{u})(\vec{q_1}^T\vec{v}) = k(\vec{q_1}^T\vec{q_1})(\vec{q_1}^T\vec{v}) = \vec{u}^T\vec{v}$
  Eigenvalues of RQ are $\vec{u}^T\vec{v}$ and 0
  Eigenvalues of $A = QR$ are $\vec{u}^T\vec{v}$ and 0

Question 7

(a)

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(t, y(t))dt$$

Using first order polynomial approximation,

$$\begin{aligned}
&\int_{t_n}^{t_{n+1}} [\frac{t - t_{n-1}}{t_n - t_{n-1}} f_n + \frac{t - t_n}{t_{n-1} - t_n} f_{n-1}]dt \\
=&\int_{t_n}^{t_{n+1}} [\frac{f_n}{\Delta t}(t - t_{n-1}) - \frac{f_{n-1}}{\Delta t}(t - t_n)]dt \\
=&\frac{\Delta t}{2}(3f_n - f_{n-1})
\end{aligned}$$

A second order $AB2$ scheme is $y_{n+1} = y_n + \frac{\Delta t}{2}(3f_n - f_{n-1})$ or $\frac{y_{n+1} - y_n}{\Delta t} = \frac{3}{2}f_n - \frac{1}{2}f_{n-1}$.

(b) Consider applying $AB-2$ to $y'(t) = \lambda y(t)$

$$\frac{y_{n+1} - y_n}{\Delta t} = \lambda(\tfrac{3}{2}y_n - \tfrac{1}{2}y_{n-1})$$

Replacing y_n by z^n

$$\frac{z_{n+1} - z_n}{\Delta t} = \lambda(\tfrac{3}{2}z_n - \tfrac{1}{2}z_{n-1})$$

$$\lambda\Delta t = \frac{z^{n+1} - z^n}{\frac{3}{2}z^n - \frac{1}{2}z^{n-1}} = \frac{z - 1}{\frac{1}{2}(3 - \frac{1}{z})}$$

At the boundary of the stability region, $|z| = 1$ or $z = e^{i\theta}, \theta \in \Re$. In the programme, $r = z - 1$ and $s = \frac{1}{2}(3 - \frac{1}{z})$. The programme varies θ and plots r/s, which is the boundary of the stability region.