NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Chua Hongshen

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Question 1

(a) (i) $\mathbf{A} = \begin{pmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -3 & 3 \\ 3 & -6 & -6 & 6 & 2 \end{pmatrix} \overset{R_2+2R_1}{\rightarrow} \begin{pmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 3 & 3 \\ 3 & -6 & -6 & 6 & 2 \end{pmatrix} \overset{R_3-3R_1}{\rightarrow}$

$$\begin{pmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & -3 & -3 & 2 \end{pmatrix} \overset{R_3+R_2}{\rightarrow} \begin{pmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \overset{\frac{1}{3}R_2}{\rightarrow} \begin{pmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

$$\overset{\frac{1}{5}R_3}{\rightarrow} \begin{pmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \overset{R_2-R_3}{\rightarrow} \begin{pmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{B}$$

$\therefore$ **A** and **B** are row equivalent.

(ii) $\{(1, -2, 0, 4, 0), (0, 0, 1, 1, 0), (0, 0, 0, 0, 1)\}$

(iii) No, row operations do not perseve column space.

(iv) $\{(-2, 0, 0), (4, 1, 0)\}$
$Nullity(\mathbf{A}) = 2$

(v) Since $Rank(\mathbf{A}^T) = Rank(\mathbf{A}) = 3$, $\therefore A^T$ has full rank.
Hence, $Nullity(\mathbf{A}^T) = 3 - 3 = 0$
$\therefore$ Nullspace of $\mathbf{A}^T$ has only trival element, $\{0\}$,
i.e. $\mathbf{A}^T\mathbf{x} = 0$ has trivial solution only.

(vi) Since **A** has full rank, $\therefore$ the column space of A is the whole $\Re^3$.
$\therefore$ Every $\mathbf{b} \in \Re^3$ is also in the column space of **A** and so is a solution of $\mathbf{A}x = \mathbf{b}$,
i.e. $\mathbf{A}x = \mathbf{b}$ is consistent for every $\mathbf{b} \in \Re^3$.

(b) (i) $\mathbf{P} = \mathbf{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ 5 & -1 & 7 \end{pmatrix}$

$\therefore$ Transition matrix from **T** to **S** is $\mathbf{P}^{-1} = \frac{1}{21} \begin{pmatrix} 5 & 2 & 2 \\ -31 & 17 & -4 \\ -8 & 1 & 1 \end{pmatrix}$

(ii) $(\mathbf{w})_S = \frac{1}{21}\begin{pmatrix} 5 & 2 & 2 \\ -31 & 17 & -4 \\ -8 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{21}\begin{pmatrix} 11 \\ -1 \\ -5 \end{pmatrix}$

Question 2

(a) (i) Since $det\begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} = 5 \neq 0$

$\therefore$ **S** forms a basis for $\Re^3$.

(ii) Notice that $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\therefore (\mathbf{b})_S = (2, -3, 2)$.

(iii) Let $\mathbf{n} = (x, y, z)$ such that $n \cdot u_1 = n \cdot u_3 = 0$,
Then,

$$2x - z = 0 \quad (1)$$
$$y + z = 0 \quad (2)$$

Solving the equations, we have $x = s, y = -2s, z = 2s$, where $s \in \Re$.
$\therefore$ The equation of P is $x - 2y + 2z = 0$.

(iv) Let the orthogonal vectors be v_1 and v_2,
By Gram-Schmidt Process,
$v_1 = u_1 = (2, 0, -1)$
$v_2 = u_2 - \frac{u_2 \cdot v_1}{\|v_1\|^2} v_1$
$= (3, 0, 1) - \frac{(3,0,1)\cdot(2,0,-1)}{\|(2,0,-1)\|^2}(2, 0, -1)$
$= (3, 0, 1) - \frac{5}{5}(2, 0, -1)$
$= (1, 0, 2)$

(v) Projection of **b** onto V is
$\frac{\mathbf{b}\cdot v_1}{\|v_1\|^2} v_1 + \frac{\mathbf{b}\cdot v_2}{\|v_2\|^2} v_2$
$= \frac{(1,2,-1)\cdot(2,0,-1)}{\|(2,0,-1)\|^2}(2, 0, -1) + \frac{(1,2,-1)\cdot(1,0,2)}{\|(1,0,2)\|^2}(1, 0, 2)$
$= \frac{3}{5}(2, 0, -1) - \frac{1}{5}(1, 0, 2)$
$= (1, 0, -1)$
$\therefore$ The distance from **b** to V is $\|(1, 0, -1)\| = \sqrt{2}$.

(b) (i) Observe that, $V - W = \text{span}\{(1, 0, 0), (0, -1, 0), (0, 0, -1)\}$

And $det\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -1 \neq 0$

$\therefore V - W = \Re^3$

(ii) To prove that $V - W \subseteq V + W$,
For any arbitrary $\mathbf{u} \in \Re^3$, if $\mathbf{u} \in V - W$, then $\mathbf{u} = \mathbf{v} - \mathbf{w}$ for some $\mathbf{v} \in V$ and $\mathbf{w} \in W$
$\therefore \mathbf{u} = \mathbf{v} + (\textbf{-w})$, where $\mathbf{v} \in V$ and $\textbf{-w} \in W$
i.e. $\mathbf{u} \in V + W$

To prove that $V + W \subseteq V - W$,
For any arbitrary $\mathbf{u} \in \Re^3$, if $\mathbf{u} \in V + W$, then $\mathbf{u} = \mathbf{v} + \mathbf{w}$ for some $\mathbf{v} \in V$ and $\mathbf{w} \in W$
$\therefore \mathbf{u} = \mathbf{v} - (\textbf{-w})$, where $\mathbf{v} \in V$ and $\textbf{-w} \in W$
i.e. $\mathbf{u} \in V - W$

Combining both, we have $V - W = V + W$

Question 3

(a) (i) $\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \overset{R_2-2R_1}{\rightarrow} \begin{pmatrix} 1 & 2 & 2 \\ 0 & -3 & -2 \\ 2 & 2 & 1 \end{pmatrix} \overset{R_3-2R_1}{\rightarrow} \begin{pmatrix} 1 & 2 & 2 \\ 0 & -3 & -2 \\ 0 & -2 & -3 \end{pmatrix} \overset{R_3-\frac{2}{3}R_2}{\rightarrow} \begin{pmatrix} 1 & 2 & 2 \\ 0 & -3 & -2 \\ 0 & 0 & -\frac{5}{3} \end{pmatrix}$

$\therefore det(\mathbf{A}) = (1)(-3)(-\frac{5}{3}) = 5$

(ii) $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix}$

(iii) To find the eignenvalue of $\mathbf{A}$, we solve the equation $det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$det\begin{pmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{pmatrix} = 0$$
$5 + 9x + 3x^2 - x^3 = 0$
$(x+1)^2(x-5) = 0$
$\therefore x = -1$ or 5

When $x = -1$, $\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}\mathbf{x} = 0$

$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{x} = 0$
$\Rightarrow \mathbf{x} = \text{span}\ \{(-1,1,0), (-1,0,1)\}$

When $x = 5$, $\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \mathbf{x} = 0$

$\Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} = 0$
$\Rightarrow \mathbf{x} = \text{span}\ \{(1, 1, 1)\}$

To convert the vectors into orthonormal matrix, by Gram-Schmidt Process, letting the orthogonal vectors for span $\{(-1, 1, 0), (-1, 0, 1)\}$ be v_1 and v_2,
$v_1 = u_1 = (-1, 1, 0)$
$v_2 = u_2 - \frac{u_2 \cdot v_1}{\|v_1\|^2} v_1$
$= (-1, 0, 1) - \frac{(-1,0,1)\cdot(-1,1,0)}{\|(-1,1,0)\|^2}\ (-1, 1, 0)$
$= (-1, 0, 1) - \frac{1}{2}(-1, 1, 0)$
$= (-\frac{1}{2}, -\frac{1}{2}, 1)$

$\therefore \mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{6}}{3} \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(b) (i) When $a = -1$, $det \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix} = det \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} - det \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 0$
$\therefore$ The system is inconsistent.

To find the least square solution, we solve $A^T Ax = A^T b$,

$i.e.\ \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & -2 \\ 1 & -2 & 3 \end{pmatrix} x = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

Consider the matrix, $\begin{pmatrix} 3 & -2 & 1 & 0 \\ -2 & 2 & -2 & -1 \\ 1 & -2 & 3 & 2 \end{pmatrix} \overset{Gaussian-JordanElimination}{\rightarrow} \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore$ A solution for x is $(-1, -\frac{3}{2}, 0)$.

(ii) To have unique solution, we must have $det \begin{pmatrix} 1 & a & 1 \\ -1 & 0 & 1 \\ a & 1 & a \end{pmatrix} = det \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} - det \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} =$

$2a^2 - 2 \neq 0$.
This can only be true when $a \neq \pm 1$.

Question 4

(a) (i) $T_1 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\therefore$ The standard matrix for T_1, $\mathbf{A}_{T_1} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 4 \end{pmatrix}$

(ii) Observe that

$$T_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = T_2(3u_1 - u_2 + u_3) = 3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$$

$$T_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = T_2(-5u_1 + 3u_2 - 2u_3) = -5 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ 1 \\ 1 \end{pmatrix}$$

$$T_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = T_2(2u_1 - u_2 + u_3) = 2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore T_2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4 & -8 & 3 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

So the standard matrix for T_2, $\mathbf{A}_{T_2} = \begin{pmatrix} 4 & -8 & 3 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$

(iii) Notice that $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = T_2(u_1 + 2u_3) = T_2 \circ T_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$
Also, $det(\mathbf{A}_{T_1}) = 1 \neq 0$ and $det(\mathbf{A}_{T_2}) = 3 \neq 0$
$\therefore det(\mathbf{A}_{T_2}\mathbf{A}_{T_1}) = 3 \neq 0$
i.e. The transformation $T_2 \circ T_1$ is one-to-one.

$\therefore w = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

(iv) Since $det(\mathbf{A}_{T_2}\mathbf{A}_{T_1}) = 3 \neq 0$,
$\therefore$ The transformation $T_1 \circ T_2$ is one-to-one.

$\therefore Ker(T_1 \circ T_2) = \{\mathbf{0}\}$.

(b) (i) $ab^T = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -2 & -2 \\ 1 & -1 & -1 \end{pmatrix} \overset{Gaussian-Jordan Elimination}{\rightarrow} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

So, $Rank(ab^T) = 1$

(ii) To prove the 'if' part,
If $\mathbf{A} = ab^T$, then $1 \leq Rank(A) = Rank(ab^T) \leq Rank(a)Rank(b) \leq 1$
The first inequality is due to the fact that $\mathbf{A}$ is a non-zero matrix.
$\therefore Rank(A) = 1$

To prove the 'only if' part,
If $\mathbf{A}$ has rank 1, then the $RREF$ of $\mathbf{A}$ has only 1 non-zero row,

$i.e.$ $\mathbf{A}$ is of the form $\begin{pmatrix} a_1 & k_1 a_1 & \cdots & k_{n-1} a_1 \\ a_2 & k_1 a_2 & \cdots & k_{n-1} a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_m & k_1 a_m & \cdots & k_{n-1} a_m \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \begin{pmatrix} 1 & k_1 & \cdots & k_{n-1} \end{pmatrix} = ab^T$

Here, $a_i, k_j \in \Re$ where $i = 1, 2 \ldots n; j = 1, 2 \ldots n-1$ and $a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}, b = \begin{pmatrix} 1 \\ k_1 \\ \vdots \\ k_{n-1} \end{pmatrix}$ are non-zero matices.

Combining both, we have $\mathbf{A}$ has rank 1 if and only if $\mathbf{A} = ab^T$ for some non-zero matrices a and b.