NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA1101R Linear Algebra I
AY 2014/2015 Sem 1
Version 1: May 28, 2015

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Question 1

(a) (i)

$$\text{Basis}=\{(1,1,0,1,0),(0,0,1,1,0),(0,0,0,0,1)\}$$

(ii) Additional vectors would be;

$$(0,1,0,0,0),(0,0,0,1,0)$$

(iii) Since $\boldsymbol{R}$ is already in rref form we can deduce the solutions for $\boldsymbol{Rx}=\boldsymbol{0}$ directly. We will get that;

$$x=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}-a-b\\b\\-a\\a\\0\end{pmatrix}=a\begin{pmatrix}-1\\0\\-1\\1\\0\end{pmatrix}+b\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix}$$

Hence the basis of the nullspace of $\boldsymbol{A}$ will be;

$$\left\{\begin{pmatrix}-1\\0\\-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix}\right\}$$

(b)

$$\boldsymbol{B}\xrightarrow{R_2-(x-1)R_3}\begin{pmatrix}x & x(x-1) & 0\\0 & x-1 & 0\\0 & 0 & x+1\end{pmatrix}\xrightarrow{R_1-(x)R_2}\begin{pmatrix}x & 0 & 0\\0 & x-1 & 0\\0 & 0 & x+1\end{pmatrix}=\boldsymbol{B'}$$

(i) Clearly $\forall\ x\in\mathbb{R}$ there exist two $a,b\in\{x,x+1,x-1\}$ such that $a,b\neq 0$ Moreover for any two $a,b\in\{x,x+1,x-1\}, a\neq b$. Therefore this means that there will two non zero rows in $\boldsymbol{B'}$, and thus there will be at least two pivot points in $\boldsymbol{B'}$, thus rank($\boldsymbol{B}$)=rank($\boldsymbol{B'}$)=1 is not possible.

(ii) From matrix $\boldsymbol{B'}$, if $x=-1,0,1$, exactly one element in $\{x,x+1,x-1\}$ will be 0. Hence only two non zero rows in $\boldsymbol{B'}$ and thus only two pivot points in $\boldsymbol{B'}$. Thus we will have rank($\boldsymbol{B}$)=rank($\boldsymbol{B'}$)=2.

(iii) From (i) and (ii). Considering $\boldsymbol{B'}$, if $x\neq -1,0,1$ then all elements in $\{x,x+1,x-1\}$ will be non zero. Therefore, $\boldsymbol{B'}$ will have 3 pivot points and thus rank($\boldsymbol{B}$)=rank($\boldsymbol{B'}$)=3.

(c)

$$\boldsymbol{C} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Clearly rank($\boldsymbol{C}$)=1. By dimension theorem; nullity($\boldsymbol{C}$)$= 4 - 1 = 3$.

Question 2

(a) (i)

$$V = \{(a+b, a, b, 2a) | a, b \in \mathbb{R}\}$$

(ii)

$$V = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} = a \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ a, b \in \mathbb{R}$$

(iii)

$$\text{Basis} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

So dim(V)$= 2$.

(b) (i) Clearly the transition matrix from T to S is $\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$.

(ii) the transition matrix from S to T is $\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$.

(ii) $[\boldsymbol{w}]_S = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} [\boldsymbol{w}]_T = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

(c) Clearly False. Now as $V = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$, $\dim(\mathbb{R}^4) = 4$, and $\text{span}(V) = \mathbb{R}^4$. Thus the vectors in V are linear independent and non zero. Thus there exist no $a, b, c, d \in \mathbb{R}$ such that $a\boldsymbol{u}_1 + b\boldsymbol{u}_2 = \boldsymbol{u}_3 + \boldsymbol{u}_4$ and $c\boldsymbol{u}_3 + d\boldsymbol{u}_4 = \boldsymbol{u}_1 + \boldsymbol{u}_2$. So clearly the vector $\boldsymbol{v} = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3 + \boldsymbol{u}_4$ is such that $\boldsymbol{v} \notin U_1, U_2 \Rightarrow \boldsymbol{v} \notin U_1 \cup U_2$ but $\boldsymbol{v} \in \mathbb{R}^4$. Therefore $U_1 \cap U_2 \neq \mathbb{R}^4$

Question 3

(a)

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\boldsymbol{A}^T\boldsymbol{b} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\begin{aligned}
\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} &= \boldsymbol{A}^T\boldsymbol{b} \\
\Rightarrow \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}\boldsymbol{x} &= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\
\Rightarrow \boldsymbol{x} &= \begin{pmatrix} -2 \\ 0 \end{pmatrix}
\end{aligned}$$

Thus $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ is the least squares solution. Hence we have;

$$\begin{aligned}
\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \\
\Rightarrow \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_{\min} &= \sqrt{(-3)^2 + 1^2 + 1^2} \\
&= \sqrt{11}
\end{aligned}$$

(b) (i)

$$\begin{aligned}
\boldsymbol{u'} = \boldsymbol{u} &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
\boldsymbol{v'} = \boldsymbol{v} &- \frac{\boldsymbol{v} \cdot \boldsymbol{u'}}{\|\boldsymbol{u'}\|^2}\boldsymbol{u'} \\
&= \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{1^2 + 1^2 + 1^2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{4}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}
\end{aligned}$$

Thus an orthogonal basis is $\{(1,1,1)^T, (\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3})^T\}$

(ii)

$$\begin{aligned}
\boldsymbol{w}_{\text{proj}} &= \frac{\boldsymbol{w}\cdot\boldsymbol{u'}}{\|\boldsymbol{u'}\|^2}\boldsymbol{u'} + \frac{\boldsymbol{w}\cdot\boldsymbol{v'}}{\|\boldsymbol{v'}\|^2}\boldsymbol{v'} \\
&= \frac{\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}\frac{4}{3}\\-\frac{2}{3}\\-\frac{2}{3}\end{pmatrix}}{(\frac{4}{3})^2+(-\frac{2}{3})^2+(-\frac{2}{3})^2}\begin{pmatrix}\frac{4}{3}\\-\frac{2}{3}\\-\frac{2}{3}\end{pmatrix} \\
&= \frac{2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{\frac{2}{3}}{\frac{24}{9}}\begin{pmatrix}\frac{4}{3}\\-\frac{2}{3}\\-\frac{2}{3}\end{pmatrix} \\
&= \frac{2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}\frac{1}{2}\\-\frac{1}{4}\\-\frac{1}{4}\end{pmatrix} \\
&= \frac{2}{3}\left(\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\-\frac{1}{4}\\-\frac{1}{4}\end{pmatrix}\right) \\
&= \frac{2}{3}\begin{pmatrix}\frac{3}{2}\\\frac{3}{4}\\\frac{3}{4}\end{pmatrix} \\
&= \begin{pmatrix}1\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}
\end{aligned}$$

(c) Let $\boldsymbol{u}_1 = \frac{1}{\sqrt{2}}\boldsymbol{v}_1 - \frac{1}{\sqrt{2}}\boldsymbol{v}_2, \boldsymbol{u}_2 = \frac{1}{\sqrt{2}}\boldsymbol{v}_1 + \frac{1}{\sqrt{2}}\boldsymbol{v}_2, \boldsymbol{u}_3 = \boldsymbol{v}_3$. Clearly $\boldsymbol{u}_1\cdot\boldsymbol{u}_3 = \boldsymbol{u}_2\cdot\boldsymbol{u}_3 = 0$. Now;

$$\begin{aligned}
\boldsymbol{u}_1\cdot\boldsymbol{u}_2 &= (\frac{1}{\sqrt{2}}\boldsymbol{v}_1 - \frac{1}{\sqrt{2}}\boldsymbol{v}_2)\cdot(\frac{1}{\sqrt{2}}\boldsymbol{v}_1 + \frac{1}{\sqrt{2}}\boldsymbol{v}_2) \\
&= \frac{1}{2}\cdot(\boldsymbol{v}_1 - \boldsymbol{v}_2)\cdot(\boldsymbol{v}_1 + \boldsymbol{v}_2) \\
&= \frac{1}{2}\cdot(\|(\boldsymbol{v}_1\|^2 + \boldsymbol{v}_1\cdot\boldsymbol{v}_2 - \boldsymbol{v}_1\cdot\boldsymbol{v}_2 - \|\boldsymbol{v}_2\|^2) \\
&= \frac{1}{2}(0) \\
&= 0
\end{aligned}$$

So $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ are orthogonal. Now we want to show that their length are all 1.

$$\begin{aligned}
\|\boldsymbol{u}_1\|^2 &= (\frac{1}{\sqrt{2}}\boldsymbol{v}_1 - \frac{1}{\sqrt{2}}\boldsymbol{v}_2)^2 \\
&= \frac{1}{2}\cdot(\|\boldsymbol{v}_1\|^2 - \boldsymbol{v}_1\cdot\boldsymbol{v}_2 - \boldsymbol{v}_2\cdot\boldsymbol{v}_1 + \|\boldsymbol{v}_2\|^2) \\
&= \frac{1}{2}(1 - 0 - 0 + 1) \\
&= 1
\end{aligned}$$

Thus we have $\|\boldsymbol{u}_1\| = 1$ and similarly, we have $\|\boldsymbol{u}_2\| = 1$. And $\|\boldsymbol{u}_3\| = \|\boldsymbol{v}_3\| = 1$. Thus $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthonormal basis.

Question 4

(a) (i) As $\boldsymbol{A}$ is a triangular matrix, the diagonal entries are the eigenvalues, i.e. 1 and 2.

(ii) For $\lambda = 2$

$$\left(\begin{pmatrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix} - \begin{pmatrix}1&1&0&0\\0&2&0&0\\0&0&2&1\\0&0&0&1\end{pmatrix}\right)\boldsymbol{x} = \boldsymbol{0}$$

$$\Rightarrow \begin{pmatrix}1&-1&0&0\\0&0&0&0\\0&0&0&-1\\0&0&0&1\end{pmatrix}\boldsymbol{x} = \boldsymbol{0}$$

$$\Rightarrow \boldsymbol{x} = a\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\ a, b \in \mathbb{R}$$

Hence basis for $\boldsymbol{E}_2 = \left\{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right\}$.

For $\lambda = 1$

$$\left(\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} - \begin{pmatrix}1&1&0&0\\0&2&0&0\\0&0&2&1\\0&0&0&1\end{pmatrix}\right)\boldsymbol{x} = \boldsymbol{0}$$

$$\Rightarrow \begin{pmatrix}0&-1&0&0\\0&-1&0&0\\0&0&-1&-1\\0&0&0&0\end{pmatrix}\boldsymbol{x} = \boldsymbol{0}$$

$$\Rightarrow \boldsymbol{x} = a\begin{pmatrix}0\\0\\-1\\1\end{pmatrix} + b\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ a, b \in \mathbb{R}$$

Hence basis for $\boldsymbol{E}_1 = \left\{\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\}$.

(b) We claim that a possibility is $\boldsymbol{C} = \begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}2&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}^{-1} = \begin{pmatrix}2&0\\1&1\end{pmatrix}$.

Now $\boldsymbol{B} = \begin{pmatrix}1&0\\1&1\end{pmatrix}$ We use the fact that if $\boldsymbol{D}$ is a diagonal matrix such that $\boldsymbol{D} = \begin{pmatrix}d_1&&0\\&\ddots&\\0&&d_n\end{pmatrix}$ then $\boldsymbol{D}^m = \begin{pmatrix}d_1^m&&0\\&\ddots&\\0&&d_n^m\end{pmatrix}$. Since $\boldsymbol{B} = \begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}4&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}^{-1}$, it is appriopate to let $\boldsymbol{C} = \begin{pmatrix}1&0\\1&1\end{pmatrix}\boldsymbol{D}\begin{pmatrix}1&0\\1&1\end{pmatrix}^{-1}$ for some 2 by 2 matrix $\boldsymbol{D}$. Hence $\boldsymbol{C}^2 = \begin{pmatrix}1&0\\1&1\end{pmatrix}\boldsymbol{D}^2\begin{pmatrix}1&0\\1&1\end{pmatrix}^{-1} =$

$\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1}$. Therefore we have $\boldsymbol{D}^2 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$. Thus a possible solution for $\boldsymbol{D} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) Since $\boldsymbol{M}$ is non-invertible, 0 is an eigenvalue.
Moreover as;

$$\boldsymbol{M} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{M} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\boldsymbol{M}$ has eigenvalues $-1, 0, 2$.
Now we note $\boldsymbol{M}$ is 3×3 symmetric matrix. This means $\boldsymbol{M}$ is orthogonally diagonalizable. Hence, three orthogonally, linearly independent vectors are the eigenvectors of $\boldsymbol{M}$. Now, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ are linear independent eigenvectors of $\boldsymbol{M}$. Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the eigenvector of $\boldsymbol{M}$ that is not in $\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$. Therefore;

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0 \text{ and } \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow a + b = 0, \quad a - b + c = 0$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = c \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}, \quad c \in \mathbb{R}$$

Thus $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ is also an eigenvector of $\boldsymbol{M}$. Now to reaffirm their linear independency;

$$\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -6 \neq 0.$$

Thus a basis of $\mathbb{R}^3$ which consist entirely of eigenvectors of $\boldsymbol{M}$ would be $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$.

Question 5

(a) Let the standard matrix of T be $\boldsymbol{A}$. $\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{R_3+R_2} \left(\begin{array}{cc|c} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{array}\right)$$

Thus for $\boldsymbol{Ax} = \mathbf{0}$. The only solution is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore, we have kernel(T)= 0.

(ii)

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

Thus, $(S \circ T) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + y \\ 2y \end{pmatrix}$.

(b) Now $P = \{x + y + z = 0 | x, y, z \in \mathbb{R}\} = \{(a - b, b, a) | a, b \in \mathbb{R}\} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \ a, b \in \mathbb{R}$.

Moreover;

$$\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -1 \neq 0$$

So let $V = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$. Now $\dim(V) = 3$, so span$\{V\} = \mathbb{R}^3$. Thus;

$$\forall\ \boldsymbol{u}\ \in \mathbb{R}^3, \exists\ a, b, c \in \mathbb{R} \text{ such that } \boldsymbol{u} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Therefore we will get;

$$\begin{aligned}
\forall \boldsymbol{u} \in \mathbb{R}^3, (F \circ F)(\boldsymbol{u}) &= F(F(\boldsymbol{u})) \\
&= F\left(F\left(a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \right) \\
&= F\left(F\left((a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right) + F\left(c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \right) \\
&= F\left(F\left(a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right) + \mathbf{0} \right) \\
&= F\left(\begin{pmatrix} k \\ k \\ k \end{pmatrix} \right) \quad \text{for some fixed } k \in \mathbb{R} \\
&= \mathbf{0}
\end{aligned}$$

(c) Let $\boldsymbol{A}$ be the standard matrix of T.
Clearly we have column space of $\boldsymbol{A} = \text{span}\{T(\boldsymbol{u}_1, T(\boldsymbol{u}_2, T(\boldsymbol{u}_3)\} = \mathbb{R}^2$.
Hence we have $\text{rank}(\boldsymbol{A}) = 2$. Moreover, the standard matrix of T is a 3×2 matrix. Hence we have $\min\{2, 3\} = 2 = \text{rank}(\boldsymbol{A})$. Thus $\boldsymbol{A}$ is full rank.

Question 6

(a) False. Consider $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

(b) True. Now, $(1, 0, 1, 0), (1, 1, 1, 1) \in W$. However. $(1, 0, 1, 0) + (1, 1, 1, 1) = (2, 1, 2, 1) \notin W$ Under closure property, W is not a subspace.

(c) True. We shall prove by contradiction. Suppose not, i.e. $\text{span}\{\boldsymbol{u}\} \neq \text{span}\{\boldsymbol{v}\}$.
WLOG, $\exists\, \boldsymbol{w} \in \text{span}\{\boldsymbol{u}\}$ s.t. $\boldsymbol{w} \notin \text{span}\{\boldsymbol{v}\}$. So we have $\boldsymbol{w} \notin \text{span}\{\boldsymbol{u}\} \cap \text{span}\{\boldsymbol{v}\}$ but $\boldsymbol{w} \in \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$.
A contradiction.

(d) False. Consider $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

(e) True. We shall prove by contradiction. Lets say it is consistent. So $\boldsymbol{Ax} = \boldsymbol{b}$.
Now we know that $\boldsymbol{A} = (\boldsymbol{c}_1 \;\; \boldsymbol{c}_2)$ and $\boldsymbol{x}$ has to be nonzero, or else $\boldsymbol{b}$ is zero. Lets say $\boldsymbol{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ for some real a, b, where both are not zero simultaneously. Hence;

$$\begin{aligned} &\boldsymbol{Ax} = \boldsymbol{b} \\ \Rightarrow &(\boldsymbol{c}_1 \;\; \boldsymbol{c}_2) \begin{pmatrix} a \\ b \end{pmatrix} = \boldsymbol{b} \\ \Rightarrow &a\ \boldsymbol{c}_1 + b\ \boldsymbol{c}_2 = \boldsymbol{b} \\ \Rightarrow &a\ (\boldsymbol{b} \cdot \boldsymbol{c}_1) + b\ (\boldsymbol{b} \cdot \boldsymbol{c}_2) = \|\boldsymbol{b}\|^2 \\ \Rightarrow &0 + 0 = \|\boldsymbol{b}\|^2 \\ \Rightarrow &0 = \|\boldsymbol{b}\|^2 \end{aligned}$$

However, $\|\boldsymbol{b}\|^2 > 0$, a contradiction.

END OF SOLUTIONS

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