NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Agus Leonardi, Chang Hai Bin, Dr. Ku Cheng Yeaw

MA1104 Multivariable Calculus
AY 2009/2010 Sem 2

Question 1

(a) The equation of tangent plane to the surface $f(x, y, z) = k$ at the point (x_0, y_0, z_0) is given by $\nabla f(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$.

In this case, $f(x, y, z) = z^2 - 2x^2 - 2y^2$, hence $\nabla f(x, y, z) = (-4x, -4y, 2z), \nabla f(1, -1, 4) = (-4, 4, 8)$.

$\therefore$ After simplifying, the equation of tangent plane is $(x - 1) - (y + 1) - 2(z - 4) = 0$.

(b) We have $f_x = 6y^2 - 6x^2$ and $f_y = 12xy - 12y^3$.
For (x, y) to be a critical point, we need $f_x = f_y = 0$, or:

$$\begin{cases} y^2 = x^2 \\ y = 0 \text{ or } x = y^2 \end{cases} \Rightarrow (x, y) = (0, 0) \text{ or } (1, 1) \text{ or } (1, -1)$$

Define the discriminant:

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

We also have

$$f_{xx} = -12x, \quad f_{yy} = -36y^2, \quad f_{xy} = 12y$$

Hence, $D(0, 0) = 0$, $D(1, 1) = 288 > 0$ and $D(1, -1) = 288 > 0$. We also have $f_{xx}(1, 1) = f_{xx}(1, -1) = -12 < 0$.

$\therefore$ By the second derivative test, $(1, -1)$ and $(1, 1)$ are local maxima.

For the point $(0, 0)$, consider $f(x, y)$ at the plane $y = 0$. We have $f(x, 0) = -2x^3$, which does not have local extremum at $x = 0$.

$\therefore$ $(0, 0)$ is a saddle point.
Hence, there are 3 critical points for the function $f(x, y)$:
the 2 local maxima at $(x, y) = (1, -1)$ and $(1, 1)$,
and the the saddle point at $(x, y) = (0, 0)$

Question 2

(a) (i) $f_x(0, 0) = \lim\limits_{h\to 0} \dfrac{f(h, 0) - f(0, 0)}{h} = \lim\limits_{h\to 0} h^2 \cos\left(\dfrac{1}{h^3}\right)$

For all $h \neq 0$, we have $-h^2 \leq h^2 \cos\left(\dfrac{1}{h^3}\right) \leq h^2$.

As $\lim\limits_{h\to 0} -h^2 = \lim\limits_{h\to 0} h^2 = 0$, we have $\lim\limits_{h\to 0} h^2 \cos\left(\dfrac{1}{h^3}\right) = 0$ by Squeeze Theorem.

Similarly, $f_y(0, 0) = 0$ by symmetry.

(ii) The function is differentiable on (a,b) if there exists functions ϵ_1, ϵ_2, such that:
$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b) = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$
with $\epsilon_1, \epsilon_2 \to 0$ when $\Delta x, \Delta y \to 0$.

So, when $(a,b) = (0,0) : f(a,b) = 0, f_x(a,b) = 0, f_y(a,b) = 0$,
If we can find the functions ϵ_1, ϵ_2, such that $f(\Delta x, \Delta y) = \epsilon_1 \Delta x + \epsilon_2 \Delta y$, and $\lim\limits_{(\Delta x, \Delta y) \to (0,0)} (\epsilon_1, \epsilon_2) = (0,0)$, then the function is differentiable at (0,0).

$$\begin{aligned} f(\Delta x, \Delta y) &= \left((\Delta x)^3 + (\Delta y)^3\right) \cdot \cos\left(\frac{1}{(\Delta x)^3 + (\Delta y)^3}\right) \\ &= \left[(\Delta x)^2 \cdot \cos\left(\frac{1}{(\Delta x)^3 + (\Delta y)^3}\right)\right] \Delta x + \left[(\Delta y)^2 \cdot \cos\left(\frac{1}{(\Delta x)^3 + (\Delta y)^3}\right)\right] \Delta y \end{aligned}$$

Since $0 \leq (\Delta x)^2 \cos\left(\frac{1}{(\Delta x)^3 + (\Delta y)^3}\right) \leq (\Delta x)^2$,
by Squeeze theorem, $\lim\limits_{(\Delta x, \Delta y) \to (0,0)} \left[(\Delta x)^2 \cos\left(\frac{1}{(\Delta x)^3 + (\Delta y)^3}\right)\right] = 0$
Similarly, $\lim\limits_{(\Delta x, \Delta y) \to (0,0)} \left[(\Delta y)^2 \cos\left(\frac{1}{(\Delta x)^3 + (\Delta y)^3}\right)\right] = 0$

By letting $\begin{cases} \epsilon_1 = \left[(\Delta x)^2 \cos\left(\frac{1}{(\Delta x)^3 + (\Delta y)^3}\right)\right] \\ \epsilon_2 = \left[(\Delta y)^2 \cos\left(\frac{1}{(\Delta x)^3 + (\Delta y)^3}\right)\right] \end{cases}$, we can conclude that the function is differentiable on $(0,0)$.

(iii) Note that for all $(x,y) \neq (0,0)$,

$$f_x(x,y) = (3x^2)\cos\left(\frac{1}{x^3+y^3}\right) + \left(x^3+y^3\right)\left(-\sin\frac{1}{x^3+y^3}\right)\left(\frac{-3x^2}{(x^3+y^3)^2}\right)$$

If we approach $(0,0)$ along the line $x = y$, we have:

$$\begin{aligned} \lim_{t\to 0} f_x(x,y)|_{(t,t)} &= \lim_{t\to 0}(3t^2)\cos\left(\frac{1}{2t^3}\right) + \frac{6t^5}{4t^6}\sin\left(\frac{1}{2t^3}\right) \\ &= \lim_{t\to 0}(3t^2)\cos\left(\frac{1}{2t^3}\right) + \frac{3}{2t}\sin\left(\frac{1}{2t^3}\right) \end{aligned}$$

The limit does not exists, since $\lim\limits_{t\to 0} \frac{3}{2t}\sin\left(\frac{1}{2t^3}\right)$ does not exists, and
$\lim\limits_{t\to 0}(3t^2)\cos\left(\frac{1}{2t^3}\right) = 0$ (easily shown with Squeeze Theorem).

Hence, $\lim\limits_{t\to 0} \lim\limits_{(x,y)\to(0,0)} f_x(x,y)$ does not exists, and hence $f_x(x,y)$ is not continuous at $(0,0)$.

(b)

$$\begin{aligned} \int_0^2 \int_{x/2}^1 \sin(y^2)\, dy\, dx &= \int_0^1 \int_0^{2y} \sin(y^2)\, dx\, dy \\ &= \int_0^1 2y\sin(y^2)\, dy \\ &= [-\cos(y^2)]_0^1 \\ &= 1 - \cos 1 \end{aligned}$$

Question 3

(a) All the points in the solid V we are integrating over satisfy:

$$\begin{cases} 0 \le x \le z, \\ 0 \le z \le y, \\ 0 \le y \le a, \end{cases} \iff 0 \le x \le z \le y \le a$$

This solid is a tetrahedron (or triangular pyramid), with vertices $(x, y, z) = (0, 0, 0), (0, a, 0), (0, a, a)$, and (a, a, a).

From the inequality $0 \le x \le z \le y \le a$, if we integrate dy first, then dz, then dx, the integration limits will be $z \le y \le a, x \le z \le a, 0 \le x \le a$.

$$\begin{aligned}
\int_0^a \int_0^y \int_0^z f(x)\, dx\, dz\, dy &= \int_0^a \int_x^a \int_z^a f(x)\, dy\, dz\, dx \\
&= \int_0^a \int_x^a (a-z) f(x)\, dz\, dx \\
&= \int_0^a \left[az - \frac{z^2}{2} \right]_x^a f(x)\, dx \\
&= \int_0^a \left[\frac{a^2}{2} - ax + \frac{x^2}{2} \right] f(x)\, dx \\
&= \frac{1}{2} \int_0^a (a-x)^2 f(x)\, dx
\end{aligned}$$

(b) The solid can be described in cylindrical coordinates as:

$$E = \{(r, \theta, z) : 0 \le \theta \le 2\pi, 0 \le r \le \sqrt{2}, \sqrt{6 - r^2} \le z \le 4 - r^2\}$$

$$\begin{aligned}
\text{Volume } &= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{\sqrt{6-r^2}}^{4-r^2} r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} 4r - r^3 - r\sqrt{6 - r^2}\, dr\, d\theta \\
&= \int_0^{2\pi} \left[2r^2 - \frac{r^4}{4} + \frac{1}{3}(6 - r^2)^{\frac{3}{2}} \right]_0^{\sqrt{2}} d\theta \\
&= \int_0^{2\pi} \left(\frac{17}{3} - 2\sqrt{6} \right) d\theta \\
&= 2\pi \left(\frac{17}{3} - 2\sqrt{6} \right)
\end{aligned}$$

(c) Using the substitution $u = xy, v = x^2 y$, we have $x = u^{-1}v, y = u^2 v^{-1}$. The Jacobian of the transformation is:

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = (-u^{-2}v)(-u^2 v^{-2} - (u^{-1})(2uv^{-1}) = -v^{-1}$$

$$\begin{aligned}\iint_D e^{xy}\,dA &= \int_{20}^{40}\int_{10}^{20} e^u\,|-v^{-1}|\,du\,dv\\ &= \int_{10}^{20} e^u\,du \int_{20}^{40}\frac{1}{v}\,dv\\ &= (e^{20}-e^{10})\ln 2\end{aligned}$$

Question 4

(a) Let $D = \{(L,K) : L \geq 0, K \geq 0, 2L + 5K \leq 150\}$.
We use the closed interval method to find its absolute maximum.

First, $\dfrac{\partial P}{\partial L} = \dfrac{400}{3}L^{-1/3}K^{1/3}$ and $\dfrac{\partial P}{\partial K} = \dfrac{200}{3}L^{2/3}K^{-2/3}$. Note that there is no value of K and L which makes $\dfrac{\partial P}{\partial L} = 0$ and $\dfrac{\partial P}{\partial K} = 0$.

Next, we find the extreme values along the boundary of D.
Along $L = 0$, we have $P(L,K) = 0$, hence no critical point there.
Along $K = 0$, we have $P(L,K) = 0$, hence no critical point there.
Along $2L + 5K = 150$, we have $P(L,K) = 200\left(\dfrac{150-5K}{2}\right)^{2/3}K^{1/3}$. To find critical point in this case, we use logarithmic differentiation.

$$\begin{aligned} & \ln P = \ln 200 + \frac{2}{3}\ln\left(\frac{150-5K}{2}\right) + \frac{1}{3}\ln K\\ \Leftrightarrow \quad & \frac{dP}{dK} = P\left(\frac{1}{3K} - \frac{2}{3(30-K)}\right)\end{aligned}$$

Solving $\dfrac{dP}{dK} = 0$, we have $P = 0$ or $\dfrac{1}{3K} = \dfrac{2}{3(30-K)}$.
For the first case, we have $K = 0$ or $150 - 5K = 0$ giving two points $(K,L) = (30,0)$ and $(K,L) = (0,75)$.
For the second case, solving the equation we get another point $(K,L) = (10,50)$.
Comparing all the values obtained above, we have $(K,L) = (10,50)$ maximises the production.

(b) Let $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$. We have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -6xy^2$$

Therefore $\mathbf{F}$ is conservative.

Moreover, we have $\nabla f(x,y) = \mathbf{F}(x,y)$ when $f(x,y) = 2x^5 - x^2y^3$.
By fundamental theorem of line integrals, we have $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = f(2,1) - f(0,0) = 60$.

(c) Let C be the curve as given in the question, but with parameter domain reduced to $0 \leq t \leq \pi$.

The required area, A, is:

$$\begin{aligned} A &= 2\oint_C x\ dy \\ &= 2\int_0^{\pi} \sin t\cos^2 t\ dt \\ &= 2\left[-\frac{\cos^3 t}{3}\right]_0^{\pi} \\ &= \frac{4}{3} \end{aligned}$$

Question 5

(a) We have $\dfrac{\partial}{\partial x}\left(\dfrac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \dfrac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$.

Proceeding similarly and factoring, we have:

$$\begin{aligned} \text{div } \mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= \frac{(x^2+y^2+z^2)^{1/2}\left[3(x^2+y^2+z^2) - 3x^2 - 3y^2 - 3z^2\right]}{(x^2+y^2+z^2)^3} \\ &= 0 \end{aligned}$$

(b) We use the parametric representation

$$\mathbf{r}(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi$$

We have

$$\begin{aligned} \mathbf{r}_\phi \times \mathbf{r}_\theta &= \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\rangle \\ \mathbf{F} &= \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle \end{aligned}$$

Therefore,

$$\mathbf{F}\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta) = \sin\phi$$

The flux is

$$\begin{aligned} \iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta)\ dA \\ &= \int_0^{2\pi}\int_0^{\pi} \sin\phi\ d\phi\ d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\ d\phi \\ &= 4\pi \end{aligned}$$

(c) Consider the region $S_1 = S' - S$. By divergence theorem, the flux of $\mathbf{F}$ across S_1 is equal to

$$\iiint_{S_1} \text{div } \mathbf{F}\ dV = 0$$

The flux of $\mathbf{F}$ across S' is equal to the sum of that across S and that across S_1, i.e. $4\pi + 0 = 4\pi$.

Question 6

(a) Assume for a contradiction there exist such a vector field $\mathbb{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$. Since condition (iii) says $\mathbb{F}$ and $\mathbb{G}$ agree on $z = 0$, it is necessary that $\mathbb{F}$ is not defined at $(0, 0, 0)$. Moreover, condition (i) - the existence of partial derivatives - implies that $\mathbb{F}$ is defined for all $(x, y, z) \neq (0, 0, 0)$.

Let S be a surface whose boundary is described by $x^2 + 4y^2 = 1$ on the xy-plane with positive orientation. Moreover, we shall choose S so that it does not contain $(0, 0, 0)$ for the argument below to work (e.g. we can choose $S = \{(x, y, z) : x^2 + 4y^2 + z^2 = 1,\ x, y, z \geq 0\}$ with upward pointing normal, but not $S = \{(x, y, 0) : x^2 + 4y^2 \leq 1\}$ since curl $\mathbf{F}$ is not defined at $(0, 0, 0)$).

By condition (ii),

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0 \tag{1}$$

On the other hand, we can parametrise the boundary of S by

$$\mathbf{r}(t) = \langle \cos t, \frac{1}{2}\sin t, 0\rangle, \quad 0 \leq t \leq 2\pi$$

Then

$$\mathbf{r}'(t) = \langle -\sin t, \frac{1}{2}\cos t, 0\rangle$$

By Stoke's Theorem,

$$\begin{aligned}
\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \mathbf{G} \cdot \mathbf{r}'(t)\, dt \quad \text{since } \mathbf{F} \text{ and } \mathbf{G} \text{ agrees on } z = 0 \\
&= \int_0^{2\pi} \langle -\frac{1}{2}\sin t, \cos t, 0\rangle \cdot \langle -\sin t, \frac{1}{2}\cos t, 0\rangle\, dt \\
&= \int_0^{2\pi} \frac{1}{2}\, dt \\
&= \pi,
\end{aligned}$$

contradicting (1). Therefore, no such $\mathbf{F}$ exists.