NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lau Tze Siong

MA2108S Mathematical Analysis I (version S)
AY 2006/2007 Sem 2

Question 1

We will proceed to prove the forward direction first.
Suppose k is an ordered field and k is archimedean,
for any $x, y \in k$,
Case 1)
suppose that they are of opposite signs, i.e. $x < 0 < y$, be letting $r = 0$, we are done.
Case 2)
If $x < y \leq 0$ then $0 \leq -y < -x$, by letting $x' = -x$ and $y' = -y$, we are have, $0 \leq x' < y'$.
Hence it suffices to prove the case where $0 \leq x < y$.
Since $0 \leq x < y$, we have $y - x > 0$,
By the archimedean property of k, there exists a $n \in \mathbb{N}$ such that $n(y - x) > 1$. Hence

$$y - x > \frac{1}{n} \ . \tag{1}$$

Let $S = \left\{p \in \mathbb{N} | p. \left(\frac{1}{n}\right) > x\right\} \subseteq \mathbb{N}$. Since $\frac{1}{n}, x \in k$ and $\frac{1}{n}, x > 0$, by the archimedean property of k, there exist a $p \in \mathbb{N}$ such that $p.\frac{1}{n} > x$. Hence S is non-empty. By the well-ordering principle of $\mathbb{N}$, there exist a least element $m \in S$ of S. Hence $m - 1 \notin S$, i.e. $(m-1)\frac{1}{n} \leq x$.

Claim: $x < \frac{m}{n} < y$
Proof:
Since $m \in S$, we have $\frac{m}{n} > x$.
Since $m - 1 \notin S$, we have $\frac{m-1}{n} \leq x$. Hence $\frac{m}{n} \leq x + \frac{1}{n} < x + (y - x) = y$ from (1).
Hence we have $x < \frac{m}{n} < y$.

Therefore if the ordering $\leq$ on k is archimedean then $\mathbb{Q}$ is dense in k.

Now we prove the other direction.
Suppose $\mathbb{Q}$ is dense in k.(We are suppose to find n such that $n.a > b$)
For any $a, b \in k_{>0}$,
Case 1)
Suppose that $b < a$, then by letting $n = 1$, we are done.
Case 2)
Suppose that $a \leq b$,
Since $a > 0$, we have $0 < a \leq b < a + b$. Since $\mathbb{Q}$ is dense in k, there exists $p, q \in \mathbb{Q}$ such that $0 < p < a \leq b < q < a + b$. By the archimedean property of $\mathbb{Q}$, there exists $n \in \mathbb{N}$ such that $np > q$. Since $na > np$ and $b < q$, we have $b < q < np < na$.
Hence for any $a, b \in k_{>0}$, there exists $n \in \mathbb{N}$ such that $na > b$.

Hence, if k is an ordered field such that $\mathbb{Q}$ is dense in k, then the ordering $\leq$ on k is archimedean.

Question 2

Consider the double sum,

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j - a_jb_i)^2 &\geq 0 \\
\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i^2b_j^2 - 2a_ia_jb_ib_j + a_j^2b_i^2) &\geq 0 \\
\sum_{i=1}^{n}\sum_{j=1}^{n}a_i^2b_j^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}a_j^2b_i^2 &\geq 2\sum_{i=1}^{n}\sum_{j=1}^{n}a_ia_jb_ib_j \\
2\sum_{i=1}^{n}a_i^2\sum_{j=1}^{n}b_j^2 &\geq 2\sum_{i=1}^{n}a_ib_i\sum_{j=1}^{n}a_jb_j \\
\sum_{i=1}^{n}a_i^2\sum_{j=1}^{n}b_j^2 &\geq \left(\sum_{i=1}^{n}a_ib_i\right)^2
\end{aligned}$$

Question 3

Claim 1) $(x_n)_{n\geq 1}$ is a increasing sequence and is bounded above by 2.
Proof:
We will prove the above claim by induction.
For the base case, we have $x_1 = 1 < \sqrt{2} = x_2$ and $x_1 = 1 \leq 2$.
Suppose that for some $k \in \mathbb{N}$,$x_k < x_{k+1}$ and $x_k \leq 2 < 3$, then $\sqrt{1+x_k} < \sqrt{1+x_{k+1}}$ and $x_{k+1} = \sqrt{1+x_k} < \sqrt{1+3} = 2$. Hence we have $x_{k+1} < x_{k+2}$ and $x_{k+1} \leq 2$.
By induction, we have $x_n < x_{n+1}$ and $x_n \leq 2$ for all $n \in \mathbb{N}$.
Since $(x_n)_{n\geq 1}$ is a increasing sequence and is bounded above, by the Completeness Axiom, $\lim\limits_{n\to\infty} x_n = x$ exists.

Hence $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \sqrt{1+x_n}$. Since $\sqrt{\ } : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, we have $\lim\limits_{n\to\infty} x_n = \sqrt{1+\lim\limits_{n\to\infty} x_n}$. Hence x is the solution of the equation $x = \sqrt{1+x}$. Solving, we have $x = \lim\limits_{n\to\infty} x_n = \dfrac{\sqrt{5}+1}{2}$

Question 4

(a) We will show that $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$.

For any $x \in \mathbb{R}$, $\lim_{k\to\infty} \frac{\frac{x^{k+1}}{k+1!}}{\frac{x^k}{k!}} = \lim_{k\to\infty} x/k = 0$.

Hence, by the ratio test, the sum $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$

(b) It suffices to show that for any $M \in \mathbb{N}$, there exist a $N \in \mathbb{N}$ such that $\displaystyle\sum_{n=1}^{N}\frac{1}{n} > M$.

For any given $M \in \mathbb{N}$,

Let $N = 2^{2M} - 1$.
Then we have,

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots + \frac{1}{2^{2M-1}} \\
&> \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \ldots + \frac{1}{2^{2M-1}} \\
&= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} \\
&= \frac{2M}{2} = M
\end{aligned}$$

Hence the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge in $\mathbb{R}$.

Question 5

The statement is true.
Proof:
Consider the following $\mathbb{N} \times \mathbb{N}$ matrix,

$$C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \ldots & \ldots \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \ldots & \ldots \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \ldots & \ldots \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{1}{n} & \ldots & \ldots & \ldots & \frac{1}{n} & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}$$

Since
(i)$\forall j \in \mathbb{N}, \displaystyle\lim_{i\to\infty} c_{ij} = 0$; (ii)$\displaystyle\lim_{i\to\infty} \sum_{j=0}^{\infty} c_{ij} = 1$; (iii)$\displaystyle\sup_{i\in\mathbb{N}} \sum_{j=0}^{\infty} |c_{ij}| < \infty$. Hence C is a Toeplitz matrix.

By Toeplitz Theorem, $a_n = \displaystyle\sum_{j=0}^{\infty} c_{ij} x_j$ converges in $\mathbb{R}$ and the sequence $(a_n)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ with $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} x_n$

Toeplitz Theorem
Suppose $\mathcal{C} = (c_{ij})$ is a $\mathbb{N} \times \mathbb{N}$ matrix with entries in $\mathbb{R}$ such that:
(i) $\forall j \in \mathbb{N}$, $\lim_{n\to\infty} c_{ij} = 0$;
(ii) $\lim_{i\to\infty} \sum_{j=0}^{\infty} c_{ij} = 1$;
(iii) $\sup_{i\in\mathbb{N}} \sum_{i=1}^{\infty} |c_{ij}| < +\infty$;
Let $(a_j)_{j\in\mathbb{N}}$ be a convergent sequence in $\mathbb{R}$ then $b_i = \sum_{j=0}^{\infty} c_{ij} a_j$ converges in $\mathbb{R}$ and the series $(b_i)_{i\in\mathbb{N}}$ converges in $\mathbb{R}$ with $\lim_{i\to\infty} b_i = \lim_{j\to\infty} a_j$.
Proof:
Let $A := \{|a_i| \mid i \in \mathbb{N}\}$. Since $\sup_{i\in\mathbb{N}} \sum_{i=1}^{\infty} |c_{ij}| < +\infty$, for all $i \in \mathbb{N}$, $\sum_{i=1}^{\infty} |c_{ij}|$ converges to a real

number. Hence $\sum_{i=1}^{\infty} c_{ij}$ is absolutely convergent. Since for all $i \in \mathbb{N}$,

$$\begin{aligned}
\sum_{j=0}^{\infty} |c_{ij}a_j| &= \sum_{j=0}^{\infty} |c_{ij}||a_j| \\
&\leq \sum_{j=0}^{\infty} \sup(A)|c_{ij}| \\
&= \sup(A) \sum_{j=0}^{\infty} |c_{ij}|
\end{aligned}$$

and $(a_n)_{n\in\mathbb{N}}$ is convergent, A is bounded. $\sup(A) \in \mathbb{R}$. Hence $\sum_{j=0}^{\infty} a_j c_{ij}$ is absolutely convergent. Hence $b_i = \sum_{j=0}^{\infty} a_j c_{ij}$ converges in $\mathbb{R}$ for all $i \in \mathbb{N}$.

Now let $k = \sup_{i\in\mathbb{N}} \sum_{j=0}^{\infty} |c_{ij}|$ and $(x_j)_{j\in\mathbb{N}}$ to be a null sequence.
Given any $\epsilon \in \mathbb{R}_{>0}$,
there exists $(n_0 + 1) \in \mathbb{N} \in \mathbb{N}$ such that for all $j \in \mathbb{N}_{\geq n_0+1}$ one has $|x_j| < \frac{\epsilon}{2k}$.
Hence

$$\begin{aligned}
\left|\sum_{j=0}^{\infty} c_{ij}x_j\right| &\leq \left|\sum_{j=0}^{n_0} c_{ij}x_j\right| + \left|\sum_{j=n_0+1}^{\infty} c_{ij}x_j\right| \\
&\leq \left|\sum_{j=0}^{n_0} c_{ij}x_j\right| + \sum_{n_0+1}^{\infty} |c_{ij}||x_j| \\
&\leq \left|\sum_{j=0}^{n_0} c_{ij}x_j\right| + \frac{\epsilon}{2k} \sum_{n_0+1}^{\infty} |c_{ij}| \\
&\leq \left|\sum_{j=0}^{n_0} c_{ij}x_j\right| + \frac{\epsilon}{2k} \sum_{0}^{\infty} |c_{ij}| \\
&\leq \left|\sum_{j=0}^{n_0} c_{ij}x_j\right| + \frac{\epsilon}{2k} \sup_{i\in\mathbb{N}} \sum_{n_0+1}^{\infty} |c_{ij}| \\
&< \left|\sum_{j=0}^{n_0} c_{ij}x_j\right| + \frac{\epsilon}{2}
\end{aligned}$$

by (i), for each $0 \leq j \leq n_0$, we have $\lim_{i\to\infty} c_{ij} = 0$.
Hence for each $0 \leq j \leq n_0$, there exist $m_j \in \mathbb{N}$ such that for all $i \in \mathbb{N}_{\geq m_j}$, one has $|c_{ij}| < \frac{\epsilon}{2(n_0+1)x}$ where $x = \sup\{|x_i| \mid i \in \{0, 1, 2, ..., n_0\}\}$. Hence we for all $j \in \{0, ..., n_0\}$ and for all $i \in \mathbb{N}_{\geq m}$ where $m = \max(m_0, m_1, ...m_j)$, one has $|c_{ij}| < \frac{\epsilon}{2(n_0+1)x}$ where $x = \sup\{|x_j| \mid j \in \{0, 1, 2, ..., n_0\}\}$.

Hence,

$$\begin{aligned}\left|\sum_{j=0}^{\infty} c_{ij}x_j\right| &< \left|\sum_{j=0}^{n_0} c_{ij}x_j\right| + \frac{\epsilon}{2} \\ &\leq \sum_{j=0}^{n_0} |c_{ij}||x_j| + \frac{\epsilon}{2} \\ &\leq x\sum_{j=0}^{n_0} |c_{ij}| + \frac{\epsilon}{2} \\ &< x\frac{(n_0+1)\epsilon}{2(n_0+1)x} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

Hence, given any epsilon, we have found a $m \in \mathbb{N}$ such that for all $i \in \mathbb{N}_{\geq m}$, one has $\left|\sum_{j=0}^{\infty} c_{ij}x_j\right| < \epsilon$. Hence $\lim_{n\to\infty} \sum_{j=0}^{\infty} c_{ij}x_j = 0$. Since $(a_j)_{j\in\mathbb{N}}$ converges to a, $(a_j - a)_{j\in\mathbb{N}}$ is a null sequence. Hence

$$\begin{aligned}\lim_{i\to\infty}\sum_{j=0}^{\infty} c_{ij}(a_j - a) &= 0 \\ \lim_{i\to\infty}\sum_{j=0}^{\infty} c_{ij}a_j - a\lim_{i\to\infty}\sum_{j=0}^{\infty} c_{ij} &= 0 \\ \lim_{i\to\infty}\sum_{j=0}^{\infty} c_{ij}a_j - a &= 0 \\ \lim_{i\to\infty}\sum_{j=0}^{\infty} c_{ij}a_j &= a \\ \lim_{i\to\infty} b_i &= \lim_{j\to\infty} a_j\end{aligned}$$

□

Question 6

The statement is false.
Consider the sequence $(x_n)_{n\in\mathbb{N}_{\geq 1}} = (1, 0, 1, 0, 1, 0, 1, ...)$, then $\lim_{n\to\infty} a_n = 0$ but $(x_n)_{n\in\mathbb{N}_{\geq 1}}$ does not converge.

Question 7

We need to show that for any $\epsilon \in \mathbb{R}$, there exist a $M \in \mathbb{N}$ such that $\left|\sum_{n=1}^{M} x_{\sigma(n)} - \sum_{n=1}^{M} x_n\right| < \epsilon$.

For any given ϵ,
Since $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, we know that there exist a $M_1 \in \mathbb{N}$ such that for any $m, n \in \mathbb{N}_{\geq M_1}$ with $m \leq n$ we have $\sum_{i=m}^{n} |x_n| < \frac{\epsilon}{2}$.
Hence we can choose M so that the set

$S = \{\sigma(n) | n \in \mathbb{N} \text{ and } n \leq M\} \supseteq \{1, 2, 3, ..., M_1\}$. This is possible since σ is a bijection.
Let the set $P = S \backslash \{1, 2, 3, ..., M_1\}$ Hence

$$\begin{aligned}
\left| \sum_{n=1}^{M} x_{\sigma(n)} - \sum_{n=1}^{M} x_n \right| &= \left| \sum_{n \in P} x_n - \sum_{n=M_1+1}^{M} x_n \right| \\
&\leq \left| \sum_{n \in P} x_n \right| + \left| \sum_{n=M_1+1}^{M} x_n \right| \\
&\leq \sum_{n \in P} |x_n| + \sum_{n=M_1+1}^{M} |x_n| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Hence $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges and $\sum_{n=1}^{\infty} x_{\sigma(n)} = \sum_{n=1}^{\infty} x_n$

Question 8

We will first show that f is continuous.
For any $a \in \mathbb{R}$ Given any $\epsilon \in \mathbb{R}$, choose $\delta = \frac{\epsilon}{c}$, Hence we have for any $x \in \mathbb{R}$ such that $|x - a| < \delta$, we have $|f(x) - f(a)| \leq c|x - a| = c\left(\frac{\epsilon}{c}\right) = \epsilon$. Hence f is continuous.
Existence:
If $f(0) = 0$, then we are done.

Suppose $f(0) = k > 0$, for any $x \in \mathbb{R}_{>0}$ we have $k - cx \leq f(x) \leq k + cx$. Consider the function $h(x) = f(x) - x$.
$h(0) = k > 0$ and $h(\frac{k}{1-c} + 1) = f(\frac{k}{1-c} + 1) - \frac{k}{1-c} - 1 \leq k + c\left(\frac{k}{1-c} + 1\right) - \frac{k}{1-c} - 1 = c - 1 < 0$. By Intermediate Value Theorem, there exist a $b \in [0, \frac{k}{1-c} + 1]$ such that $h(b) = 0$ and $f(b) = b$.

Suppose $f(0) = k < 0$, for any $x \in mathdsR_{<0}$ we have $k - cx \leq f(x) \leq k + cx$. Consider the function $h(x) = f(x) - x$.
$h(0) = k < 0$ and $h(\frac{k}{1-c} - 1) = f(\frac{k}{1-c} - 1) - \frac{k}{1-c} + 1 \leq k + c\left(\frac{k}{1-c} - 1\right) - \frac{k}{1-c} + 1 = 1 - c > 0$. By Intermediate Value Theorem, there exist a $b \in [0, \frac{k}{1-c} + 1]$ such that $h(b) = 0$ and $f(b) = b$.

Uniqueness:
Suppose there exist $d_1, d_2 \in \mathbb{R}$ such that $f(d_1) = d_1$ and $f(d_2) = d_2$. Then we have $|f(d_1) - f(d_2)| = |d_1 - d_2| \leq c|d_1 - d_2|$. Since $0 < c < 1$, we must have $d_1 = d_2$.

Hence there exist a unique $p \in \mathbb{R}$ such that $f(p) = p$.

Question 9

By Heine-Borel Theorem , $[a, b]$ is compact in $\mathbb{R}$. Since the continuous image of a compact set is compact we have $f([a, b])$ is compact. Hence $f([a, b])$ is closed and bounded by Heine-Borel Theorem. Since the $\sup\{f(x) \in \mathbb{R} | x \in [a, b]\}$ is a limit point and all limit points of $f([a, b])$ is in $f([a, b])$. Hence, there exists a $p \in [a, b]$ such that $f(p) = \sup\{f(x) \in \mathbb{R} | x \in [a, b]\}$.

Question 10

Since $[a,b]$ is a connected subset of $\mathbb{R}$ and the continuous image of a connected set is connected. We have $f([a,b])$ as a connected subset of $\mathbb{R}$ and $[f(a), f(b)] \subseteq f([a,b])$, if not $f([a,b])$ is disconnected. Therefore for any $f(a) \leq t \leq f(b)$ there exist a $x \in [a,b]$ such that $f(x) = t$.