NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA3205 Set Theory
AY 2014/2015 Sem 1
Version 1: December 16, 2014

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(This is the original solution provided by the lecturer himself)

Question 1

(i) True.

Let $\beta_0 = \min\{\beta \in \mathbf{ORD} : X \subseteq \beta\}$. For any $\alpha \in X, \alpha \in \beta_0 \Rightarrow \alpha < \beta_0 \Rightarrow \alpha + 1 \leq \beta_0$. So β_0 is an upper bound for $\{\alpha + 1 : \alpha \in X\}$. On the other hand, if $\beta < \beta_0$, then $X \not\subseteq \beta$. So for some $\alpha \in X, \beta \leq \alpha < \alpha + 1$. Hence, β is not an upper bound for $\{\alpha + 1 : \alpha \in X\}$. This completes the prove.

(ii) False.

Let $\alpha = \omega > 0, \beta = 2, \gamma = 3$, then $\beta < \gamma$ but $2 \cdot \omega = \omega = 3 \cdot \omega$.

(iii) True.

$\aleph_1 \leq 2^{\aleph_0}$ by Cantor's Theorem, so $\aleph_1^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \otimes \aleph_0} = 2^{\aleph_0}$. On the other hand, $2 \leq \aleph_1$, so $2^{\aleph_0} \leq \aleph_1^{\aleph_0}$. Combining these two gives the desired result.

(iv) True.

If $\beta \neq \gamma$, then either $\beta < \gamma$ or $\beta > \gamma$. Suppose WLOG that $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$, which is a contradiction.

(v) False.

Let $\alpha = \omega_1 + \omega$, where $+$ denotes ordinal addition, then $\omega_1 \leq \alpha < \omega_2$ is a limit ordinal. Suppose $X \subseteq \alpha$ is cofinal in α and $\mathrm{otp}(X) = \omega_1$. Let $f : \omega_1 \to X$ be an isomorphism. Since X is cofinal in α, there exists $\xi < \omega_1$ such that $f(\xi) \geq \omega_1$. Since f is order-preserving, $\mathrm{Im}_f(\{\zeta < \omega_1 : \zeta \geq \xi\}) \subseteq \{\omega_1 + n :\in \omega\}$. However, $\{\omega_1 + n :\in \omega$ is countable while $\{\zeta < \omega_1 : \zeta \geq \xi\}$ is uncountable. This is a contradiction.

Question 2

Since ω is a limit ordinal, $\omega \cdot \omega = \sup\{\omega \cdot n : n \in \omega\}$. We first check that $\omega \cdot \omega$ is a limit ordinal. Note that $0 < \omega = \omega \cdot 1 \leq \omega \cdot \omega$, so $\omega \cdot \omega \neq 0$. Next, for any $n \in \omega, n < n+1$ and so $\omega \cdot n < \omega \cdot (n+1)$. If $\beta < \omega \cdot \omega$, then $\beta < \omega \cdot n$ for some $n \in \omega$. So $\beta \leq \omega \cdot n < \omega \cdot (n+1) \leq \omega \cdot \omega$. Thus, $\beta < \omega \cdot \omega \Rightarrow \beta + 1 < \omega \cdot \omega$, showing that $\omega \cdot \omega$ is indeed a limit ordinal.

Now, fix any $\alpha < \omega \cdot \omega$. By the paragraph above, $\alpha + (\omega \cdot \omega) = \sup\{\alpha + \delta : \delta < \omega \cdot \omega\}$. Note that $0 \leq \alpha \Rightarrow \omega \cdot \omega \leq \alpha + (\omega \cdot \omega)$. So it suffice to show that $\omega \cdot \omega$ is not an upper bound for $\{\alpha + \delta : \delta < \omega \cdot \omega\}$. Fix any $\delta < \omega \cdot \omega$, there exists $n, m \in \omega$ such that $\alpha < \omega \cdot n$ and $\delta < \omega \cdot m$. So $\alpha + \delta < \alpha + (\omega \cdot m) \leq (\omega \cdot n) + (\omega \cdot m) = \omega + (n+m) \leq \omega \cdot \omega$ as needed.

Question 3

Note that for each $n \in \omega, |X_n| = \kappa_n > 0$. So X is non-empty. Since $|X|$ and λ are both cardinals, if $|X| \not> \lambda$, then $|X| \leq \lambda$, which implies that $X \lessapprox \lambda$. Since X is non-empty, there is an onto funciton $G : \lambda \to X$.

Fix $n \in \omega$. Note that $\kappa_n < \kappa_{n+1} \leq \lambda$. So if $\beta < \kappa_n$, then $\beta \in \lambda$ and so $G(\beta) \in X$. This means that $G(\beta)$ is a function with domain ω and $G(\beta)(n+1) \in X_{n+1}$. Consider $Z_{n+1} = \{G(\beta)(n+1) : \beta < \kappa_n\} \subseteq X_{n+1}$ and $|Z_{n+1}| \leq \kappa$. Hence, $X_{n+1} \setminus Z_{n+1} \neq \phi$ because $|X_{n+1}| = \kappa_{n+1} > \kappa_n$. Define $Y_{n+1} = X_{n+1} \setminus Z_{n+1}$ and $Y_0 = X_0$. Now $\langle Y_n : n \in \omega \rangle$ is a sequence of non-empty sets such that $\forall n \in \omega[Y_n \subseteq X_n]$. Therefore, $\prod_{n\in\omega} Y_n \neq \phi$ and $\prod_{n\in\omega} Y_n \subseteq \prod_{n\in\omega} X_n = X$. Choose $f \in \prod_{n\in\omega} Y_n$. Since G is onto, there exists $\beta \in \lambda$ such that $G(\beta) = f$. There is some $n \in \omega$ such that $\beta < \kappa_n$. Thus, $G(\beta)(n+1) \in Z_{n+1}$. However, $G(\beta)(n+1) = f(n+1) \in Y_{n+1} = X_{n+1} \setminus Z_{n+1}$. This is a contradiction.

Question 4

(a) Suppose $I = (r, s)$ with $r < s$. If $I \prec I$, then $s \leq r < s$, which is absurd. So $I \not\prec I$.

Next, suppose that $I = (r, s), J = (t, u)$ and $K = (v, w)$ with $r < s, t < u$ and $v < w$. If $I \prec J$ and $J \prec K$, then $s \leq t$ and $u \leq v$. Thus $s \leq t < u \leq v$, which impleis that $I \prec K$.

(b) We prove by induction. To make the prove smoother, we prove a slightly stronger statement. We induction on n that if $A \subseteq \mathcal{F}$ is an antichain of size $n+1$, then there exists $I = (r, s)$ and $J = (t, u)$ both members of A such that $r \leq t < s \leq u$ and $\bigcap A = (t, s)$.

When $n = 0$, $A = \{I\}$ for some $I = (r, s)$ with $r < s$. Letting $t = r, u = s$ and $J = I$, we have $r = t < s = u$ and $\bigcap A = (r, s) = (t, u)$ as needed.

Now, suppose that the claim holds for some n. We prove it for $n+1$. Let $A \subseteq \mathcal{F}$ be an antichain of size $n+2$. Fix $K = (v, w) \in A$. Let $B = A \setminus \{K\}$. By the inductive hypothesis, there exists $I = (r, s)$ and $J = (t, u)$ both members of B such that $r \leq t < s \leq u$ and $\bigcap B = (t, s)$. Note that $\bigcap A = (t, s) \cap (v, w)$. Also K is incomparable with both I and J. Therefore $v < s$ and $t < w$. We divide into four cases:

Case 1 : $t \leq v$ and $w \leq s$: $\bigcap A = (v, w)$, so the intervals K and K will work.
Case 2 : $t \leq v$ and $s < w$: $\bigcap A = (v, s)$, so the interval I and K will work.
Case 3 : $v < t$ and $w \leq s$: $\bigcap A = (t, w)$, so the interval K and J will work.
Case 4 : $v < t$ and $s < w$: $\bigcap A = (t, s)$, so the interval I and J will work.

Question 5

Let C denotes the set of all continuous function from $\mathbb{R}$ to $\mathbb{R}$. Define

$$X = \{\langle f, r, s, y\rangle \in C \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} : r < s\}$$

X has cardinality $\mathfrak{c}$. Let $\langle\langle f_\alpha, r_\alpha, s_\alpha, y_\alpha\rangle : \alpha < \mathfrak{c}\rangle$ enumerates X. We construct a sequence $\langle g_\alpha : \alpha < \mathfrak{c}\rangle$ with the following properties holding at each $\alpha < \mathfrak{c}$.

(1) g_α is a function, $\text{dom}(g_\alpha) \subseteq \mathbb{R}$ and $\text{ran}(g_\alpha) \subseteq \mathbb{R}$.
(2) $\forall \xi < \alpha[g_\xi \subseteq g_\alpha]$.
(3) $z_\alpha \in \text{dom}(g_\alpha)$.
(4) $\exists x \in \text{dom}(g_\alpha) \cap (r_\alpha, s_\alpha)[g_\alpha(x) = y_\alpha - f_\alpha(x)]$.
(5) $|\text{dom}(g_\alpha)| \leq \max\{|\alpha|, \omega\}$.

Suppose for a moment that we have constructed such a sequence. Let $g = \bigcup_{\alpha<\mathfrak{c}} g_\alpha$. Conditions (1) and (2) guarantee that g is a function with $\text{dom}(g) = \bigcup_{\alpha<\mathfrak{c}}\text{dom}(g_\alpha)$ and $\text{ran}(g_\alpha) \subseteq \mathbb{R}$. Condition (3) ensures that $\bigcup_{\alpha<\mathfrak{c}}\text{dom}(g_\alpha) = \mathbb{R}$. Thus $g : \mathbb{R} \to \mathbb{R}$. Suppose $\langle f, r, s, y\rangle \in X$, then $\langle f, r, s, y\rangle = \langle f_\alpha, r_\alpha, s_\alpha, y_\alpha\rangle$for some $\alpha < \mathfrak{c}$. By clause (4), there is $x \in (r, s)$ such that $g(x) = g_\alpha(x) = y - f(x)$ and hence $(f + g)(x) = y$ as needed.

To construct such a sequence, we proceed by induction. Fix $\alpha < \mathfrak{c}$ and suume that $\langle g_\xi : \xi < \alpha\rangle$ has already been constructed in such a way that conditions (1) to (5) are satified. Put $h = \bigcup_{\xi<\alpha} g_\xi$. Again by (1) and (2), h is a function with $\text{dom}(h) = \bigcup_{\xi<\alpha}\text{dom}(g_\xi)$ and $\text{ran}(h) \subseteq \mathbb{R}$. For convinience, denote $\kappa = \max\{|\alpha|, \omega\}$. By (5), $|\text{dom}(g_\xi)| \leq \max\{|\xi|, \omega\} \leq \kappa$. Thus $\text{dom}(h)$ is the union of at most $|\alpha| \leq \kappa$ many sets each having size less than κ. Since κ is an infinite cardinal, $|\text{dom}(h)| \leq \kappa < \mathfrak{c}$. Since (r_α, s_α) has size $\mathfrak{c}$, it is possible to choose $x \in (r_\alpha, s_\alpha)\setminus \text{dom}(h)$. If $z_\alpha \in \text{dom } (h) \cup \{x\}$, then let $g_\alpha = h \cup \{\langle x_\alpha, y_\alpha - f_\alpha(x)\rangle\}$. If $z_\alpha \notin \text{dom } (h) \cup \{x\}$, then let $g_\alpha = h \cup \{\langle x_\alpha, y_\alpha - f_\alpha(x)\rangle, \langle z_\alpha, 0\rangle\}$. In either case, it is easy to see that conditions (1) to (4) are satisfied by g_α. For (5), note that $\text{dom}(g_\alpha)$ is the unioon of $\text{dom}(h)$ and a finite set, so $\text{dom}(g_\alpha)$ is the union of at most κ many sets (in fact, is just two sets) each having size at most κ. Hence, $|\text{dom}(g_\alpha)| \leq \kappa = \max\{|\alpha|, \omega\}$ as needed. This completes the construction.

END OF SOLUTIONS

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