NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Associate Professor Victor Tan

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MA1100 Fundamental Concepts of Mathematics
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Question 1

(a) Case 1: Let $n = 3k$ for certain $k \in \mathbb{N}$. So,

$$\begin{aligned} n^3 + 2n &= (3k)^3 + 2(3k) \\ &= 3(9k^3 + 2k) \end{aligned}$$

By closure properties, since $k \in \mathbb{N}$, $9k^3 + 2k$ is an integer, and hence, $n^3 + 2n$ is divisible by 3.
Case 2: Let $n = 3k + 1$ for certain $k \in \mathbb{N}$. So,

$$\begin{aligned} n^3 + 2n &= (3k+1)^3 + 2(3k+1) \\ &= (27k^3 + 27k^2 + 9k + 1) + (6k + 2) \\ &= 27k^3 + 27k^2 + 15k + 3 \\ &= 3(9k^3 + 9k^2 + 5k + 1) \end{aligned}$$

So, $n^3 + 2n$ is divisible by 3.
Case 3: Let $n = 3k + 2$ for certain $k \in \mathbb{N}$. So,

$$\begin{aligned} n^3 + 2n &= (3k+2)^3 + 2(3k+2) \\ &= (27k^3 + 3(3k)^2(2) + 3(3k)2^2 + 2^3) + (6k + 4) \\ &= 27k^3 + 54k^2 + 42k + 12 \\ &= 3(9k^3 + 18k^2 + 14k + 4) \end{aligned}$$

So, $n^3 + 2n$ is divisible by 3.

Combining the 3 cases, $n^3 + 2n$ is divisible by 3 for all natural number n.

(b) For base case $n = 1$,$n^3 + 2n = 3$, which is divisible by 3.

So, the statement S is true for $n = 1$.

Assume that the statement S is true for $n = k$, and $k \in \mathbb{N}$. ie. $k^3 + 2k = 3M$ for some $M \in \mathbb{Z}$. Then,

$$\begin{aligned} (k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= k^3 + 2k + (3k^2 + 3k + 3) \\ &= 3(M + k^2 + k + 1) \end{aligned}$$

So, $n^3 + 2n$ is divisible by 3 for $n = k + 1$. Hence, by the Principle of Mathematical Induction, $n^3 + 2n$ is divisible by 3 for all natural number n.

Question 2

(a) The relation R is not reflexive. Counter-example: $0 \nsim 0$, since $|0-0| = 0 \leq 3$
R is symmetric.
If $a \sim b$, ie $|a-b| > 3$, then $|b-a| = |a-b| > 3$,
hence, $b \sim a$, R is symmetric.
R is not transitive.
Counter-example: $a = 0, b = 10, c = 0$. $|a-b| = |b-c| = 10 > 3$, but $|a-c| = 0 \leq 3$.
So, $0 \sim 10, 10 \sim 0$, but $0 \nsim 0$

(b) If we use $0, 1, ..., 11$ to represent the equivalent classes in $\mathbb{Z}_{12}$, then
$[a]_{12} \cdot [b]_{12} = [0]_{12} \leftrightarrow ab$ is certain integer multiple of 12.
When a=0, any value of b (from 0 to 11) will satisfy the above property.
When a=1, only when b=0 (when chosen from 0 to 11) will satisfy the above property.
The rest are similar. The combinations satisfying the property is shown in the table below:

$a \cdot b$	0	1	2	3	4	5	6	7	8	9	10	11
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	**0**											
2	**0**						**0**					
3	**0**				**0**				**0**			
4	**0**			**0**			**0**			**0**		
5	**0**											
6	**0**		**0**		**0**		**0**		**0**		**0**	
7	**0**											
8	**0**			**0**			**0**			**0**		
9	**0**				**0**				**0**			
10	**0**						**0**					
11	**0**											

Note: In exam, you're only needed to LIST the possible pairs. No justification required.

Question 3

(i) Counter-example: f(2)=f(−2)=0, but 2 $\neq$-2

(ii) Choose $A = [0, \infty)$, $B = [-4, \infty)$.
To show $\hat{f} : [0, \infty) \to [-4, \infty)$ is a injection:
If $\hat{f}(a) = \hat{f}(b), a^2 - 4 = b^2 - 4, a^2 = b^2$
So $a = b$ ($a = -b$ is not possible if $a, b \in [0, \infty)$), hence injective.

To show the range of $\hat{f}$ with domain $[0, \infty)$ is $[-4, \infty)$,
$\forall y \in [-4, \infty)$, we can find $x = \sqrt{y+4} \geq 0$,, such that $\hat{f}(x) = y$
And for $y < -4$, there are no $x \in \mathbb{R}$, such that $f(x) = y$ (or else $x^2 < 0$)

Reason for choosing $A = [0, \infty)$:
From the graph of f, we notice that f is symmetric about $x = 0$. For $\hat{f} : A \to \mathbb{R}$ to be injective, 0 must not be an interior point of A. thus we can choose either $[0, \infty)$ or $(-\infty, 0]$.
i.e. A is not the form $(-\alpha, \beta), [-\alpha, \beta), (-\alpha, \beta], [-\alpha, \beta]$ for positive (or infinite) α, β, or else $\hat{f}(-\gamma) = \hat{f}(\gamma), \gamma = min\{\frac{\alpha}{2}, \frac{\beta}{2}, 1\}$
Note: We can choose A=$(-\infty, 0]$instead.

Reason for choosing $B = [-4, \infty)$:
It is not possible to have $x \in B$, with $x < -4$ (or else $\hat{f}$ is not surjective)
Opting out any element in $[-4, \infty)$ out of B would mean that some of the elements in $[0, \infty)$ would not have an image to map onto, and hence $\hat{f}$ would not be a function.

(iii) $\hat{f}(\hat{f}^{-1}(y)) = y$ for all y∈ B.
$(\hat{f}^{-1}(y))^2 - 4 = y,$
$\hat{f}^{-1}(y) = \sqrt{y+4}$ (Take positive square root)

(iv) Counter-example: $g \circ f(2) = g(f(2)) = g(0) = g(f(-2)) = g \circ f(-2)$

(v) Let h(x)= $2^{1/4}$ for all real x.
So, $f \circ h(x) = f(h(x)) = f(2^{1/4}) = \sqrt{2} - 4$ for all real x.
So, the range $= \{\sqrt{2} - 4\}$ contains irrational points only.

Question 4

(a) (i)

$$\begin{aligned}
262 &= 2 \cdot 124 + 14 \\
124 &= 8 \cdot 14 + 12 \\
14 &= 1 \cdot 12 + 2 \\
12 &= 6 \cdot 2 + 0
\end{aligned}$$

So, $gcd(124, 262) = 2$.

(ii)

$$\begin{aligned}
2 &= 14 \ - \ 12 \\
&= 14 - (124 - 8 \cdot 14) \\
&= 9 \cdot 14 - 124 \\
&= 9 \cdot (262 - 2 \cdot 124) - 124 \\
&= 9 \cdot 262 - 19 \cdot 124
\end{aligned}$$

So, let x=-19, y=9 (Other choices of x,y are also possible)

(b) (i) $a_1 = 2, b_1 = 4, c_1 = 6$
$gcd(2, 4) = gcd(2, 6) = gcd(2, 24) = 2$

(ii) $a_2 = 4, b_2 = 2, c_2 = 6$
$gcd(4, 2) = gcd(4, 6) = 2$, and $gcd(4, 12) = 4$

(iii) Given $gcd(a, b) = gcd(a, c) = 2$,
So a, b, c are divisible by 2. a, bc are both divisible by 2.
Hence, $gcd(a, bc)$ must be certain multiple of 2 (even number).

As $gcd(a, b) = gcd(a, c) = 2$,
We can write 2 in terms of linear combination of (a, b) and (a, c).
ie. for suitable $K_1, K_2, L_1, L_2 \in \mathbb{Z}$

$$\begin{cases} K_1a + K_2b & = 2 \\ L_1a + L_2c & = 2 \end{cases}$$

By multiplying the 2 equations, we get

$$\begin{aligned}
4 &= K_1L_1a^2 + K_1L_2ac + K_2L_1ab + K_2L_2bc \\
&= (K_1L_1a + K_1L_2c + K_2L_1b)a + (K_2L_2)bc
\end{aligned}$$

So, 4 is a linear combination of (a, bc). Hence, $gcd(a, bc) \leq 4$.
Since we know that $gcd(a, bc)$ is even(in above), $gcd(a, bc)$ must be either 2 or 4.

Question 5

(a) 1, 3, 4, 7, 11, 18, 29, 47, 76, 123

(b) Fibonacci Sequence: 1, 1, 2, 3, 5, 8, 13,......
$L_2 = 3 = 1 + 2 = F_1 + F_3, L_3 = 4 = 1 + 3 = F_2 + F_4$. Base cases, n=2 and 3, are true.
Assume that $L_n = F_{n-1} + F_{n+1}$ is true for $n = k, k \in \mathbb{N}, k \geq 3$, then:

$$\begin{aligned} L_{k+1} &= L_k + L_{k-1} \\ &= (F_{k+1} + F_{k-1}) + (F_k + F_{k-2}) \\ &= F_{k+2} + F_k \\ &= F_{(k+1)+1} + F_{(k+1)-1} \end{aligned}$$

So, by Strong Principle of Mathematical Induction, $L_n = F_{n+1} + F_{n-1}$ for all natural n>1

(c) The problem comes from the inductive step used on case $n = 2$ and $n = 3$.
Note that L_2 and F_2 are not defined in terms of $L_0, L_1, F_0, or F_1$ (in fact, L_0 and F_0 is not defined at all)
So, the inductive step CANNOT be used to prove the case $n = 2$ (and in fact, by checking the definition of L_2, F_2, we know $L_2 > F_2$, and the statement is false)
For us to use the inductive step to prove case $n = 3$, we need the statement be true for $n = 1$, and $n = 2$.
But since the statement is false for case $n = 2$, the inductive step fails to provide us the truth of the statement for case $n = 3$.
And since the truth of the statement for case $n = 3$ is unknown, the inductive fails to prove the case $n = 4$, and $n = 5$, and so on.

Question 6

(a) (i) $[1]_4 = \{1, 5, 9\}, [2]_4 = \{2, 6, 10\}$,
$[3]_4 = \{3, 7\}, [4]_4 = \{4, 8\}$.

(ii)

$$\begin{aligned} card(R) &= card([1] \times [1]) + card([2] \times [2]) + card([3] \times [3]) + card([4] \times [4]) \\ &= 3^2 + 3^2 + 2^2 + 2^2 = 26 \end{aligned}$$

(b) (i) We can count the number of partitions using the table below, which is 52. So, the number of equivalence relations is 52.

partition	type	number of ways
{{a},{b},{c},{d},{e}}	11111	1
{{a,b},{c},{d},{e}}	2111	10
{{a,b},{c,d},{e}}	221	15
{{a,b,c},{d},{e}}	311	10
{{a,b,c},{d,e}}	32	10
{{a,b,c,d},{e}}	41	5
{{a,b,c,d,e}}	5	1
sum		52

Note: We can also use the property of the Bell numbers, with $B_5 = 52$

(ii) S= $\{(a,a),(a,d),(d,a),(d,d),(b,b),(c,c),(c,e),(e,c),(e,e)\}$
(Since d is in the same equivalence class with a,
and e is neither in the equivalence class including a and b. By referring to the table above, the only possible "case" will be the "221" case)

Question 7

(a) Prove by contradiction.
Suppose (for a contradiction) that n is a positive odd integer of the form $4k+3$, and n does not have prime factor of the form $4k+3$.

Case 1: n has at least one even prime factor,
ie. of the form $4k+2$ or $4k$, then n is an even number (product of even number to any natural number is even), and so n can take the form $4k$ or $4k+2$(even), but not $4k+3$ (odd). Hence, a contradiction.

Case 2: n has no prime factors of the form $4k$, or $4k+2$,
And by the assumption, no prime factors of n can take the form $4k+3$
Hence, all of the prime factors of n (assume to be $a_1, a_2, ...a_j$) take the form 4k+1.

$$\begin{aligned} n = a_1 \cdot a_2 \cdot ... \cdot a_j &\equiv 1 \cdot 1 \cdot ... \cdot 1 \mod 4 \\ &\equiv 1 \mod 4 \end{aligned}$$

So, n is not of the form 4k+3.

Conclusion: Either case, there is a contradiction.

Hence, if n is a positive odd integer of the form $4k+3$, then n does not have prime factor of this form as well.

(b)

$$\begin{aligned} 2^{p-1} + 2^p + ... + 2^{2p-2} &= 2^{p-1} \cdot (1 + 2 + 2^2 + ... + 2^{p-1}) \\ &= 2^{p-1} \cdot (2^p - 1) \text{ (Geometric Sum)} \end{aligned}$$

Given $2^p - 1$ is prime, so the proper divisor of $2^{p-1} + 2^p + ... + 2^{2p-2}$ include:

Those without $2^p - 1$, ie. $1, 2, 4, ..., 2^{p-1}$,
and those with $(2^p - 1)$, ie. $2^p - 1$,$(2^p - 1) \cdot 2$, ..., $(2^p - 1) \cdot 2^{p-2}$

$$\begin{aligned} &(1 + 2 + 4 + ... + 2^{p-1}) + ((2^p - 1) + (2^p - 1) \cdot 2 + ... + (2^p - 1) \cdot 2^{p-2}) \\ &= (2^p - 1) + (2^p - 1)(2^{p-1} - 1) \\ &= (2^p - 1)\left(1 + 2^{p-1} - 1\right) \\ &= (2^p - 1)(2^{p-1}) \end{aligned}$$

So, $2^{p-1} + 2^p + ... + 2^{2p-2}$ is a perfect number.

Question 8

(a) Define: $S_2 = \{2^n : n \in \mathbb{N}\}$
$S_3 = \{3^n : n \in \mathbb{N}\}$
....
$S_p = \{p^n : n \in \mathbb{N}\}$ for prime numbers p.
$S_0 = \{x : x = 1 \text{ or } x \neq k^n \ \forall k, n \in \mathbb{N}\}$

Note that these sets form a partition of $\mathbb{N}$.
(All other elements not in the form k^n will be assigned to S_0, and $S_2, S_3, S_5, ...$ are disjoint due to properties of prime numbers.)

As there are infinitely many prime numbers, there are infinitely many sets S_p, where p is prime.
And each of $S_2, S_3, S_5, ...$ contains infinitely many elements (exists a bijection from $\mathbb{N}$ to each of them)

We can try to prove S_0 also contains infinitely many elements. (See Notes)
Alternatively, let $R_2 = S_2 \cup S_0$, and let C=$\{R_2, S_3, S_5, S_7, S_{11}, ...\}$
This partition C would satisfy the required condition.

Note:Consider $K_0 \subset S_0, K_0 = \{2 \times 3, 2 \times 3^2, 2 \times 3^3, ...\}, K_0$ is infinite and hence S_0 is infinite.

(b) Functions maps each value in the domain to a specific value in the codomain.
Let $f_{(m,n)}$ be the function which $f_{(m,n)}(0) = m, f_{(m,n)}(1) = n$,
then A=$\{f_{(m,n)} : (m,n) \in \mathbb{N} \times \mathbb{N}\}$

There exists a bijection ϕ between A and $\mathbb{N} \times \mathbb{N}$.
e.g. $\phi : A \to \mathbb{N} \times \mathbb{N}, \phi(f_{(m,n)}) = (m,n)$
and $\mathbb{N} \times \mathbb{N}$ is countable.

So, A is countable.

Note: We only concern ourselves with the value of the function on$\{0,1\}$.
The value of the function outside these 2 points are ignored.
For example, $f(x) = 0$ and $g(x) = x(x-1)$ are considered the same function under domain $\{0,1\}$

To show ϕ is bijective,
Note that for every $(x,y) \in \mathbb{N} \times \mathbb{N}$, consider $f_{(x,y)}(\alpha) = x + (y-x)(\alpha)$, so that $f_{(x,y)}(0) = m, f_{(x,y)}(1) = n.\phi(f_{(x,y)}) = (x,y)$So, ϕ is surjective.
If $(x_1, y_1) = \phi(f_{(x_1,y_1)}) = \phi(f_{(x_2,y_2)}) = (x_2, y_2)$, then

$$f_{(x_1,y_1)}(0) = x_1 = x_2 = f_{(x_2,y_2)}(0),$$
$$f_{(x_1,y_1)}(1) = y_1 = y_2 = f_{(x_2,y_2)}(1)$$

Hence $f_{(x_1,y_1)} = f_{(x_2,y_2)}$ (Under the domain $\{0,1\}$), ϕ is injective

Question 9

(a) Define $f : C \to \mathbb{Q}^+$ by
$f(\,[(a,b)]\,) = \dfrac{a}{b}$,
To show that the function is well defined, suppose $(x_1, y_1) \sim (x_2, y_2)$,
$x_1y_2 = x_2y_1$. $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2}$.
So $f(\,[(x_1, y_1)]\,) = f(\,[(x_2, y_2)]\,)$,

To show that it is injective, if $f(\,[(a,b)]\,) = f(\,[(c,d)]\,)$
then $\dfrac{a}{b} = \dfrac{c}{d}, \to ad = bc$ (for $b, d \in \mathbb{N}$)
hence $[(a,b)] = [(c,d)]$

To show that it is surjective, for all positive rational number q, we can write
$q = \dfrac{m}{n}$ with $gcd(m,n) = 1$,
then $f(\,[(m,n)]\,) = \dfrac{m}{n} = q$.

Hence, this function f is bijective.

(b) f is a bijection from C to $\mathbb{N} \times \mathbb{N}$, a countable set.
So, C is a countably infinite set. (From result of (a))

Let S=$[(x,y)] \in C$ If $gcd(x,y) = u > 1$, let

$$x = u \times x_1, \quad y = u \times y_1, \quad x \times y_1 = x_1 \times u \times y_1 = x_1 \times y$$

So $[(x,y)] = [(x_1,y_1)]$, with $gcd(x_1,y_1) = 1$. As $(x_1,y_1), (2 \times x_1, 2 \times y_1), (3 \times x_1, 3 \times y_1), ..., \in S$ define

$$\phi : \mathbb{N} \to [x_1, y_1], \quad n \mapsto (nx, ny)$$

ϕ is a bijection from $\mathbb{N} \to [x_1, y_1]$. So, S is countably infinite.

Conclusion: S and C are both countably infinite. $|S| = |C|$