NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Teo Wei Hao

MA4211 Functional Analysis
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Question 1

(a) Notice that $T : X \to Y$ is a well-defined function, with $T(x) = y$ iff $Ax = By$ for all $x \in X$ and $y \in Y$. Also, for all $x \in X$, we have $Ax = (B \circ T)(x)$.

Now let $x_1, x_2 \in X$, $\alpha \in \mathbb{C}$, and $y_1, y_2 \in Y$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$.
This implies that $Ax_1 = By_1$ and $Ax_2 = By_2$.
Thus $A(\alpha x_1 + x_2) = \alpha Ax_1 + Ax_2 = \alpha By_1 + By_2 = B(\alpha y_1 + y_2)$.
This give us $T(\alpha x_1 + x_2) = \alpha y_1 + y_2 = \alpha T(x_1) + T(x_2)$, i.e. T is linear.
We can now simplify the notation of $T(x)$ to Tx now that it is linear.

Let $(x_n)_{n\in\mathbb{Z}^+}$ be a sequence in X such that $(x_n)_{n\in\mathbb{Z}^+}$ and $(Tx_n)_{n\in\mathbb{Z}^+}$ converges to $x \in X$ and $y \in Y$ respectively.
Since A is bounded linear, it is continuous, and so $(Ax_n)_{n\in\mathbb{Z}^+}$ converges to Ax.
Similarly, B is continuous, and so $(Ax_n)_{n\in\mathbb{Z}^+} = (BTx_n)_{n\in\mathbb{Z}^+}$ converges to By.
By uniqueness of limit, we have $Ax = By$, i.e. $Tx = y$, and thus the graph of T is closed in $X \oplus Y$.
Since X and Y are Banach spaces, by Closed Graph Theorem, we have T to be bounded.

(b) Let $(m_k)_{k\in\mathbb{Z}^+}$ be a sequence in M that converges to some $h \in H$.
We are given that there exists $m \in M$ such that m is the projection of h in M, i.e. $h - m \perp M$.

Let $\varepsilon \in \mathbb{R}^+$. Then there exists $K \in \mathbb{Z}^+$ such that $\|m_K - h\|_H < \varepsilon$.
Since M is a subspace of H, we have $m - m_K \in M$, and so $\langle h - m, m - m_K\rangle = 0$. Thus,

$$\begin{aligned}\|h-m\|_H^2 &\leq \|h-m\|_H^2 + 2\mathrm{Re}(\langle h-m, m-m_K\rangle) + \|m-m_K\|_H^2 \\ &= \|(h-m)+(m-m_K)\|_H^2 \\ &= \|h-m_K\|_H^2 < \varepsilon^2,\end{aligned}$$

i.e. $\|h - m\|_H < \varepsilon$. Thus we conclude that $\|h - m\|_H = 0$, i.e. $h = m \in M$, and so M is closed.

Question 2

(a) Since $z \in X - M$, we have M to be a proper closed subspace of X, and $d(z, M) > 0$.
Let $N = \mathrm{span}(M \cup \{z\})$, and $\lambda : N \to \mathbb{C}$ be a linear functional such that $\lambda(x) = 0$ for all $x \in M$, and $\lambda(z) = d(z, M)$ (this can be constructed by extension from the zero functional on M).

For all $m \in M$, we have $|\lambda(m)| = 0$, and for all $\alpha \in \mathbb{C} - \{0\}$, we have,

$$|\lambda(m+\alpha z)| = |\alpha| d(z,M) \leq |\alpha| \left\| z - \frac{-m}{\alpha} \right\|_X = \|m + \alpha z\|_X.$$

And so λ is a bounded linear functional with $\|\lambda\| \leq 1$.

Let $(m_k)_{k\in\mathbb{Z}^+}$ be a sequence in M such that $(\|z - m_k\|_X)_{k\in\mathbb{Z}^+}$ converges to $d(z, M)$ from above.
Then for all $\varepsilon \in \mathbb{R}^+$, there exists $K \in \mathbb{Z}^+$ such that $\dfrac{d(z, M)}{\|z - m_K\|_X} > 1 - \varepsilon$.

This give us $|\lambda(z - m_K)| = d(z, M) = \dfrac{d(z, M)}{\|z - m_K\|_X}\|z - m_K\|_X > (1 - \varepsilon)\|z - m_K\|_X$.
Thus $\|\lambda\| > 1 - \varepsilon$ for all $\varepsilon \in \mathbb{R}^+$, i.e. $\|\lambda\| = 1$.
Therefore by Hahn-Banach Theorem, we can extend λ to $f \in X^*$ with $\|f\| = \|\lambda\| = 1$.

Now we would like to deduce that $\|x\|_X = \sup\{|g(x)| \mid g \in X^*, \|g\| = 1\}$ for all $x \in X$ from the above.
Firstly, it is trivial to see that $\sup\{|g(0_X)| \mid g \in X^*, \|g\| = 1\} = 0 = \|0_X\|_X$.
Next, we consider $z \in X - \{0_X\}$. For all $g \in X^*$, $\|g\| = 1$, we have $|g(z)| \leq \|g\|\|z\|_X = \|z\|_X$.
Thus $\sup\{|g(z)| \mid g \in X^*, \|g\| = 1\} \leq \|z\|_X$.
Now, let $M = \{0_X\}$, which is a proper closed subspace of X.
Then by the above, there exists $f \in X^*$ such that $\|f\| = 1$ and $|f(z)| = |d(z, M)| = \|z\|_X$.
Thus, we conclude that $\sup\{|g(z)| \mid g \in X^*, \|g\| = 1\} = \|z\|_X$.

(b) For all $n \in \mathbb{Z}^+$, let $y_n = x_n - x_{n-1}$.
Then $(y_n)_{n\in\mathbb{Z}^+}$ is a sequence in X such that $y_1 = b - a$, and for all $n \in \mathbb{Z}^+$, we have $y_{n+1} = \dfrac{-y_n}{2}$.
This give us $y_{n+1} = (-1)^n \dfrac{b-a}{2^n}$, and so $\displaystyle\sum_{k=1}^{\infty} \|y_k\|_X = \|b - a\|_X \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 2\|b - a\|_X < \infty$.
Since X is Banach, we have $\displaystyle\sum_{k=1}^{\infty} y_k$ to exists.
Using the fact that $\displaystyle\sum_{k=1}^{\infty}\left(\frac{-1}{2}\right)^{k-1} = \frac{2}{3}$, we get $\displaystyle\sum_{k=1}^{\infty} y_k = \frac{2}{3}(b - a)$.
For all $n \in \mathbb{Z}^+$, since $\displaystyle\sum_{k=1}^{n} y_k = x_n - x_0$, we have,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(x_0 + \sum_{k=1}^{n} y_k\right) = x_0 + \sum_{k=1}^{\infty} y_k = a + \frac{2}{3}(b - a) = \frac{2b + a}{3}.$$

Question 3

(a) We notice that the fact that f is a well-defined function has been established in lecture.

For all $x \in H$, let $(x_n)_{n\in\mathbb{Z}^+}$ be a sequence in H that converges to x.
Let $\varepsilon \in \mathbb{R}^+$. Then there exists $N \in \mathbb{Z}^+$ such that for all $n \in \mathbb{Z}_{\geq N}$, we have $\|x_n - x\| < 1$ and $\|x_n - x\| < \dfrac{1}{8(\|f(x) - x\| + 1)}$.
For all $n \in \mathbb{Z}_{\geq N}$, since $f(x), f(x_n) \in S$, we have,

$$\|f(x_n) - x\| \leq \|f(x_n) - x_n\| + \|x_n - x\| \leq \|f(x) - x_n\| + \|x_n - x\| \leq \|f(x) - x\| + 2\|x_n - x\|.$$

Also, since S is convex, we have $\dfrac{f(x) + f(x_n)}{2} \in S$, and so $\|x - f(x)\| \leq \left\|x - \dfrac{f(x) + f(x_n)}{2}\right\|$.
Together with Parallelogram Law, we have,

$$\begin{aligned}
\|f(x_n) - f(x)\|^2 &= \|(x - f(x)) - (x - f(x_n))\|^2 \\
&= 2(\|x - f(x)\|^2 + \|x - f(x_n)\|^2) - \|(x - f(x)) + (x - f(x_n))\|^2 \\
&= 2(\|x - f(x)\|^2 + \|x - f(x_n)\|^2) - 4\left\|x - \frac{f(x) + f(x_n)}{2}\right\|^2 \\
&\leq 2(\|f(x_n) - x\|^2 + \|f(x) - x\|^2) - 4\|f(x) - x\|^2 \\
&= 2(\|f(x_n) - x\| + \|f(x) - x\|)(\|f(x_n) - x\| - \|f(x) - x\|) \\
&\leq 8(\|f(x) - x\| + \|x_n - x\|)\|x_n - x\| < \varepsilon,
\end{aligned}$$

i.e. $(f(x_n))_{n\in\mathbb{Z}^+}$ converges to $f(x)$. This implies that f is continuous.

(b) Let us have $\langle x, y\rangle = 0$.
Then for all $\alpha \in \mathbb{C}$, we have,

$$\|x+\alpha y\|^2 = \|x\|^2 + 2\mathrm{Re}(\overline{\alpha}\langle x, y\rangle) + |\alpha|^2\|y\|^2 \quad = \quad \|x\|^2 + |\alpha|^2\|y\|^2 \geq \|x\|^2,$$

and so $\|x+\alpha y\| \geq \|x\|$.

Next, instead let us have $\|x+\alpha y\| \geq \|x\|$ for all $\alpha \in \mathbb{C}$, i.e. $0 \leq \|x+\alpha y\|^2 - \|x\|^2$.
Then by letting $\alpha = \dfrac{-\langle x, y\rangle}{\|y\|^2}$, we can get $\|x+\alpha y\|^2 - \|x\|^2 = -\dfrac{|\langle x, y\rangle|^2}{\|y\|^2}$.
This implies that $\dfrac{|\langle x, y\rangle|^2}{\|y\|^2} \leq 0$, and so $|\langle x, y\rangle| = 0$, i.e. $\langle x, y\rangle = 0$.

Question 4

(a) Pythagoras Theorem tells us that in any inner product space X, if for $x, y \in X$ we have $\langle x, y\rangle = 0$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

In a real inner product space X, let us have $x, y \in X$ such that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.
We can simplify to $\langle x, y\rangle + \langle y, x\rangle = 0$, i.e. $2\langle x, y\rangle = 0$. Thus $\langle x, y\rangle = 0$.

The same is not necessarily true if X is instead a complex inner product space.
Let $X = \mathbb{C}$, with the usual inner product $\langle x, y\rangle = x\overline{y}$ for all $x, y \in X$.
Now $\|i+1\|^2 = 2 = \|i\|^2 + \|1\|^2$, however $\langle i, 1\rangle = i \neq 0$.

(b) As a by-product of Riesz Representation Theorem, we have that for all $z \in X$, $\|f_z\|_{X^*} = \|z\|_X$.
Also, for all $x, z_1, z_2 \in X$, since $\langle x, z_1 - z_2\rangle = \langle x, z_1\rangle - \langle x, z_2\rangle$, we have $f_{z_1 - z_2} = f_{z_1} - f_{z_2}$.
This give us $\|f_{z_1} - f_{z_2}\|_{X^*} = \|z_1 - z_2\|_X$.

Let $(z_n)_{n\in\mathbb{Z}^+}$ be a Cauchy sequence in X.
Then the above give us $(f_{z_n})_{n\in\mathbb{Z}^+}$ to be a Cauchy sequence in X^*.
Since X^* is complete, and T is bijective, there exists a unique $z \in X$ such that $(f_{z_n})_{n\in\mathbb{Z}^+}$ converges to f_z. The above norm relation again give us $(z_n)_{n\in\mathbb{Z}^+}$ to converges to z, and so X is complete, i.e. X is a Hilbert space.

Question 5

(a) It is not necessarily true that $T = 0_{\mathcal{B}(X)}$.
Let $X = \mathbb{R}^2$ be an inner product space with $\langle\cdot,\cdot\rangle$ being the usual real vector dot product.
Let $T \in \mathcal{B}(X)$ be such that for all $x_1, x_2 \in \mathbb{R}$, we have $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$.
For all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}$, we have $\langle Tx, x\rangle = -x_1x_2 + x_2x_1 = 0$, however $T \neq 0_{\mathcal{B}(X)}$.

(b) We shall assume the notation $T^0 = I$. The below is briefly what was established in tutorial:-
If $\|T\| < 1$, then since $\mathcal{B}(H)$ is complete, and $\displaystyle\sum_{k=0}^{\infty}\|T^k\| = \frac{1}{1-\|T\|} < \infty$, we have $\displaystyle\sum_{k=0}^{\infty} T^k \in \mathcal{B}(H)$.
Then we can verify that $(I-T)\left(\displaystyle\sum_{k=0}^{\infty} T^k\right) = I$, and so $I - T$ is invertible.
This implies that if $x \in H$ is such that $(I-T)(x) = 0_H$, then $x = 0_H$.

Similarly since $\|T^*\| = \|T\|$, we can also conclude that $(I - T^*)(x) = 0_H$ implies that $x = 0_H$. Thus $\{x \in H \mid Tx = x\} = \{0_H\} = \{x \in H \mid T^*x = x\}$.

Else, we have $\|T\| = 1$. This also give us $\|T^*\| = 1$.
Notice that we have $0_H \in \{x \in H \mid Tx = x\}$ and $0_H \in \{x \in H \mid T^*x = x\}$.
Let $z \in \{x \in H \mid Tx = x\} - \{0_H\}$. Then we have $\|z\|^2 = \langle z, Tz\rangle = \langle T^*z, z\rangle \leq \|z\|\|T^*z\|$.
This give us $\|z\| \leq \|T^*z\|$.
Since $\|T^*\| = 1$, we have $\|T^*z\| \leq \|z\|$, and thus we conclude that $\|T^*z\| = \|z\|$.
This give us equality to the Cauchy-Schwarz Inequality $|\langle T^*z, z\rangle| = \|z\|\|T^*z\|$, which give us T^*z to be a scalar multiple of z. Since $\|T^*z\| = \|z\|$ and $\langle T^*z, z\rangle = \|z\|\|T^*z\|$, we get $T^*z = z$.
This implies that $\{x \in H \mid Tx = x\} \subseteq \{x \in H \mid T^*x = x\}$.
Using the fact that $(T^*)^* = T$, similarly we can get $\{x \in H \mid T^*x = x\} \subseteq \{x \in H \mid Tx = x\}$.
Therefore $\{x \in H \mid Tx = x\} = \{x \in H \mid T^*x = x\}$.

(c) Notice that the Fourier expansion and Bessel's equality with respect to the orthonormal basis $\{x_1, x_2, \ldots\}$ give us that for all $h \in H$, we have $h = \sum_{n=1}^{\infty} \langle h, x_n\rangle x_n$ and $\|h\|^2 = \sum_{n=1}^{\infty} |\langle h, x_n\rangle|^2$.
Let $T : H \to H$ be such that for all $h \in H$, $T(h) = \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) \langle h, x_n\rangle x_n$.
Then for all $h_1, h_2 \in H$, $\alpha \in \mathbb{C}$, we have,

$$\begin{aligned}
T(\alpha h_1 + h_2) &= \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) \langle \alpha h_1 + h_2, x_n\rangle x_n \\
&= \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) (\alpha\langle h_1, x_n\rangle + \langle h_2, x_n\rangle) x_n \\
&= \alpha \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) \langle h_1, x_n\rangle x_n + \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) \langle h_2, x_n\rangle x_n \\
&= \alpha T(h_1) + T(h_2),
\end{aligned}$$

and so T is linear (we shall hereon write $T(h)$ as Th).
Also for all $h \in H$, we have,

$$\begin{aligned}
\|Th\|^2 &= \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^2 |\langle h, x_n\rangle|^2 \\
&\leq \sum_{n=1}^{\infty} 4|\langle h, x_n\rangle|^2 \\
&= 4\|h\|^2,
\end{aligned}$$

and so T is bounded, i.e. $T \in \mathcal{B}(H)$.

This give us $Y = \left\{ y \in H \mid \sum_{n=1}^{\infty} \left| \left(1 + \frac{1}{n}\right) \langle y, x_n\rangle \right|^2 \leq 1 \right\} = \{y \in H \mid \|Ty\| \leq 1\} = T^{-1}[S]$, where $S = \{h \in H \mid \|h\| \leq 1\}$. Since S is closed and T is continuous, we have $Y = T^{-1}[S]$ to be closed.
Also for all $y \in Y$, we have,

$$\|y\|^2 = \sum_{n=1}^{\infty} |\langle y, x_n\rangle|^2 \quad < \quad \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^2 |\langle y, x_n\rangle|^2 \leq 1,$$

and so Y is bounded.
Let $y_1, y_2 \in Y$. Then for all $\alpha \in [0, 1]$, we have,

$$\|T((1 - \alpha)y_1 + \alpha y_2)\| \quad = \quad \|(1 - \alpha)Ty_1 + Ty_2\| \leq (1 - \alpha)\|Ty_1\| + \alpha\|Ty_2\| = 1,$$

and so $(1-\alpha)y_1 + \alpha y_2 \in Y$, i.e. Y is convex.

Let $\varepsilon \in \mathbb{R}^+$. By Archimedean's property, there exists $N \in \mathbb{Z}^+$ such that $N+1 > \dfrac{1}{\varepsilon}$.
We note that $T\left(\dfrac{N}{N+1}x_N\right) = x_n$ and $\|x_n\| = 1$, and so $\dfrac{N}{N+1}x_N \in Y$.
As $\left\|\dfrac{N}{N+1}x_N\right\| = \dfrac{N}{N+1} = 1 - \dfrac{1}{N+1} > 1-\varepsilon$, we conclude that $\sup\{\|y\| \mid y \in Y\} \geq 1$.
Since we have $\|y\| < 1$ for all $y \in Y$ as a by-product from establishing boundedness above, we conclude that there does not exists $y_0 \in Y$ such that $\|y_0\| = \sup\{\|y\| \mid y \in Y\}$.

Question 6

(a) It has been established in lecture that if Y is a Banach space, then $\mathcal{B}(X,Y)$ is a Banach space.

Instead, let us have $\mathcal{B}(X,Y)$ to be Banach.
Since $X \neq \{0_X\}$, let us fix $x \in X - \{0_X\}$, such that $\|x\|_X = 1$.
By consequence of Hahn-Banach Theorem, there exists a bounded linear functional λ on X with $\|\lambda\|_{X^*} = 1$, such that $\lambda(x) = \|x\|_X = 1$.
Let $y \in Y$. Then we have $f_y \in \mathcal{B}(\mathbb{C}, Y)$ such that $f_y(\alpha) = \alpha y$.
We also note that $\|f_y\|_{\mathcal{B}(\mathbb{C},Y)} = \|y\|_Y$ and $\|f_{y_1} - f_{y_2}\|_{\mathcal{B}(\mathbb{C},Y)} = \|y_1 - y_2\|_Y$ for $y_1, y_2 \in Y$.
As the composition of 2 bounded linear operators, we have $T_y = f_y\lambda \in \mathcal{B}(X,Y)$. Also, $T_y(x) = y$.

Now, let $(y_n)_{n\in\mathbb{Z}^+}$ be a Cauchy sequence in Y.
For all $m_1, m_2 \in \mathbb{Z}^+$, $x' \in X$, since

$$\begin{aligned} \|(T_{y_{m_1}} - T_{y_{m_2}})x'\|_Y &= \|(f_{y_{m_1}} - f_{y_{m_2}})(\lambda x')\|_Y \\ &\leq \|f_{y_{m_1}} - f_{y_{m_2}}\|_{\mathcal{B}(\mathbb{C},Y)}|\lambda x'| \\ &\leq \|f_{y_{m_1}} - f_{y_{m_2}}\|_{\mathcal{B}(\mathbb{C},Y)}\|\lambda\|_{X^*}\|x'\|_X, \end{aligned}$$

we have $\|T_{y_{m_1}} - T_{y_{m_2}}\|_{\mathcal{B}(X,Y)} \leq \|f_{y_{m_1}} - f_{y_{m_2}}\|_{\mathcal{B}(\mathbb{C},Y)}\|\lambda\|_{X^*} = \|y_{m_1} - y_{m_1}\|_Y\|\lambda\|_{X^*}$.
Thus we conclude that $(T_{y_n})_{n\in\mathbb{Z}^+}$ is Cauchy , and so it converges to some $T \in \mathcal{B}(X,Y)$.
Since for all $n \in \mathbb{Z}^+$, $\|y_n - T(x)\|_Y = \|(T_{y_n} - T)(x)\|_Y \leq \|T_{y_n} - T\|_{\mathcal{B}(X,Y)}\|x\|_X$, we have $(y_n)_{n\in\mathbb{Z}^+}$ to converges to $T(x)$ in Y, and so Y is Banach.

(b) The statement may not be true.
Let us consider the complex Hilbert space ℓ^2.
Let $A \in \mathcal{B}(\ell^2)$ be such that $A(x_k)_{k\in\mathbb{Z}^+} = (2x_{k+1})_{k\in\mathbb{Z}^+}$ for all $(x_k)_{k\in\mathbb{Z}^+} \in \ell^2$, i.e. a left shift followed by a scalar multiple of 2.
Notice that $\left(\dfrac{1}{2^k}\right)_{k\in\mathbb{Z}^+} \in \ell^2$, and $A\left(\dfrac{1}{2^k}\right)_{k\in\mathbb{Z}^+} = \left(\dfrac{1}{2^k}\right)_{k\in\mathbb{Z}^+}$. Thus 1 is an eigenvalue of A.
Now, A^* is the right shift follow by scalar multiple by 2, i.e. for all $x = (x_k)_{k\in\mathbb{Z}^+} \in \ell^2$, we have,

$$A^*(x_1, x_2, x_3, \ldots) = (0, 2x_1, 2x_2, 2x_3, \ldots).$$

This give us $\|A^*x\|^2 = \displaystyle\sum_{k=1}^{\infty}(2x_k)^2 = 4\|x\|^2$, and so $A^*x = x$ must give us $x = 0_{\ell^2}$ (alternatively, comparing coefficient will give us the same conclusion), and so 1 is not eigenvalue of A^*.