NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Chan Yu Ming, Yang Cheng

MA3220 Ordinary Differential Equations
AY 2008/2009 Sem 2

SECTION A

Question 1

$$\frac{dy}{dt} + y = e^t, \quad t > 0, \quad y(0) = 0$$

Integrating factor $= e^{\int dt} = e^t$

$$\begin{aligned} e^t \left(\frac{dy}{dt} + y \right) &= e^{2t} \\ \frac{d}{dt}\left(ye^t\right) &= e^{2t} \\ ye^t &= \frac{1}{2}e^{2t} + c \end{aligned}$$

Using the initial condition $y(0) = 0$, we obtain $c = -\frac{1}{2}$.

So the solution is given by

$$ye^t = \frac{1}{2}\left(e^{2t} - 1\right)$$

Question 2

$$\frac{dy}{dt} + y = e^{-t}, \quad t > 0, \quad y(0) = 0$$

Integrating factor $= e^{\int dt} = e^t$

$$\begin{aligned} e^t \left(\frac{dy}{dt} + y \right) &= 1 \\ \frac{d}{dt}\left(ye^t\right) &= 1 \\ ye^t &= t + c \end{aligned}$$

Using the initial condition $y(0) = 0$, we obtain $c = 0$.

So the solution is given by

$$y = te^{-t}$$

Question 3

To find all the equilibrium solutions of $\dfrac{dy}{dt} = y(8 - y^2)$:

$$y(8 - y^2) = 0$$

$\therefore$ The equilibrium solutions are $y \equiv 0$, $y \equiv 2\sqrt{2}$ and $y \equiv -2\sqrt{2}$.

Let $f(y) = y(8 - y^2)$. Thus, $f'(y) = -3y^2 + 8$. We have:

$$\begin{array}{rcll} f'(2\sqrt{2}) & = & -3(8) + 8 = -16 < 0 & \textbf{(Asymptotically Stable)} \\ f'(-2\sqrt{2}) & = & -3(8) + 8 = -16 < 0 & \textbf{(Asymptotically Stable)} \\ f'(0) & = & -3(0) + 8 = 8 > 0 & \textbf{(Unstable)} \end{array}$$

$\therefore$ $y \equiv 2\sqrt{2}$ and $y \equiv -2\sqrt{2}$ are asymptotically stable equilibrium solutions, while $y \equiv 0$ is an unstable equilibrium solution.

Question 4

$$\begin{aligned} \frac{dy}{dt} &= t(1 + y^2), \quad y(0) = 0 \\ \int \frac{1}{1 + y^2}\, dy &= \int t\, dt \\ \tan^{-1} y &= \frac{t^2}{2} + c \end{aligned}$$

Using the initial condition $y(0) = 0$, we have $c = 0$.

$\therefore y = \tan\left(\dfrac{t^2}{2}\right)$.

Question 5

$$y'' - 2y' - 3y = x(e^x + e^{-x})$$

First, let us find two linearly independent solutions $y_1(x),\ y_2(x)$ of the homogenous differential equation $y'' - 2y' - 3y = 0$:

$$\begin{aligned} m^2 - 2m - 3 &= 0 \\ (m + 1)(m - 3) &= 0 \\ m &= -1 \text{ or } m = 3 \end{aligned}$$

$\therefore$ Let $y_1(x) = e^{-x}$ and $y_2(x) = e^{3x}$.

We shall now find a particular solution of the original differential equation using variation of parameters.
The Wronskian W is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{vmatrix} = 4e^{2x}$$

$$\begin{aligned} \text{Let } u_1(x) &= -\int \frac{y_2(x)f(x)}{W(x)}\,dx \\ &= -\int \frac{xe^{3x}(e^x+e^{-x})}{4e^{2x}}\,dx \\ &= -\frac{1}{4}\int xe^{2x}+x\,dx \\ &= -\frac{x^2}{8}-\frac{1}{8}xe^{2x}+\frac{1}{16}e^{2x} \end{aligned}$$

$$\begin{aligned} \text{Let } u_2(x) &= \int \frac{y_1(x)f(x)}{W(x)}\,dx \\ &= \int \frac{xe^{-x}(e^x+e^{-x})}{4e^{2x}}\,dx \\ &= \frac{1}{4}\int xe^{-2x}+xe^{-4x}\,dx \\ &= -\frac{1}{8}xe^{-2x}-\frac{1}{16}xe^{-4x}-\frac{1}{16}e^{-2x}-\frac{1}{64}e^{-4x} \end{aligned}$$

Thus, a particular solution y_p is given by:

$$\begin{aligned} y_p(x) &= u_1(x)y_1(x)+u_2(x)y_2(x) \\ &= -\frac{1}{4}xe^x-\frac{1}{8}x^2e^{-x}-\frac{1}{16}xe^{-x}-\frac{1}{64}e^{-x} \end{aligned}$$

Therefore, the general solution is given by:

$$y = Ae^{-x}+Be^{3x}-\frac{1}{4}xe^x-\frac{1}{8}x^2e^{-x}-\frac{1}{16}xe^{-x}-\frac{1}{64}e^{-x}$$

, which can be written more simply as:

$$y = Ce^{-x}+De^{3x}-\frac{1}{4}xe^x-\frac{1}{8}x^2e^{-x}-\frac{1}{16}xe^{-x}$$

Question 6

(1) Since both the eigenvalues are positive, so (0,0) is an **Unstable Proper Nodal Source**. Thus, the phase portrait is either d or f. The solution is given by $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$.

Observe that if $c_2 \neq 0$, we have $\frac{y(t)}{x(t)} = \frac{c_1e^{2t}-c_2e^t}{c_1e^{2t}+c_2e^t} = \frac{c_1e^t-c_2}{c_1e^t+c_2} \to -\frac{c_2}{c_2} = -1$ as $t \to -\infty$. If $c_2 = 0$, then $\frac{y(t)}{x(t)} \equiv 1$, i.e. $y = x$. This means the phase portrait is f, because $c_2 = 0$ corresponds to the line $y = x$, and all other solution curves (i.e. $c_2 \neq 0$) are tangential to the line $y = -x$.

(2) Since both the eigenvalues are positive, so (0,0) is an **Unstable Proper Nodal Source**. Thus, the phase portrait is either d or f. But the answer to the previous part is f, so let us conjecture that the correct phase portrait for (2) is d.

To see this, note that the solution is given by $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$.

Observe that if $c_1 \neq 0$, we have $\frac{y(t)}{x(t)} = \frac{c_1e^{t}-c_2e^{2t}}{c_1e^{t}+c_2e^{2t}} = \frac{c_1-c_2e^t}{c_1+c_2e^t} \to \frac{c_1}{c_1} = 1$ as $t \to -\infty$. If $c_1 = 0$, then $\frac{y(t)}{x(t)} \equiv -1$, i.e. $y = -x$. This means the phase portrait is d, because $c_1 = 0$ corresponds to the line $y = -x$, and all other solution curves (i.e. $c_1 \neq 0$) are tangential to the line $y = x$.

(3) Since both the eigenvalues are negative, so (0,0) is an **Asymptotically Stable Proper Nodal Sink**. Thus, the phase portrait is either c or e.

Note that the solution is given by $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$.

Observe that if $c_1 \neq 0$, we have $\frac{y(t)}{x(t)} = \frac{c_1e^{-t}-c_2e^{-2t}}{c_1e^{-t}+c_2e^{-2t}} = \frac{c_1-c_2e^{-t}}{c_1+c_2e^{-t}} \to \frac{c_1}{c_1} = 1$ as $t \to \infty$. If $c_1 = 0$, then $\frac{y(t)}{x(t)} \equiv -1$, i.e. $y = -x$. This means the phase portrait is c, because $c_1 = 0$ corresponds to the line $y = -x$, and all other solution curves (i.e. $c_1 \neq 0$) are tangential to the line $y = x$.

(4) Since both the eigenvalues are negative, so (0,0) is an **Asymptotically Stable Proper Nodal Sink**. Thus, the phase portrait is either c or e.

Note that the solution is given by $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$.

Observe that if $c_2 \neq 0$, we have $\frac{y(t)}{x(t)} = \frac{c_1e^{-2t}-c_2e^{-t}}{c_1e^{-2t}+c_2e^{-t}} = \frac{c_1e^{-t}-c_2}{c_1e^{-t}+c_2} \to -\frac{c_2}{c_2} = -1$ as $t \to \infty$. If $c_2 = 0$, then $\frac{y(t)}{x(t)} \equiv 1$, i.e. $y = x$. This means the phase portrait is e, because $c_2 = 0$ corresponds to the line $y = x$, and all other solution curves (i.e. $c_2 \neq 0$) are tangential to the line $y = -x$.

(5) Since one of the eigenvalues is positive and the other is negative, so (0,0) is an **Unstable Saddle Point**. Thus, the phase portrait is either a or b.

Note that the solution is given by $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$.

Observe that if $c_1 \neq 0$, we have $\frac{y(t)}{x(t)} = \frac{c_1e^{t}-c_2e^{-2t}}{c_1e^{t}+c_2e^{-2t}} = \frac{c_1-c_2e^{-3t}}{c_1+c_2e^{-3t}} \to \frac{c_1}{c_1} = 1$ as $t \to \infty$. This means the phase portrait is a. This is because $c_1 = 0$ corresponds to the line $y = -x$, and all other solution curves (i.e. $c_1 \neq 0$) are approaching the oblique asymptote $y = x$.

(6) Since one of the eigenvalues is positive and the other is negative, so (0,0) is an **Unstable Saddle Point**. Thus, the phase portrait is either a or b.

Note that the solution is given by $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$.

Observe that if $c_2 \neq 0$, we have $\frac{y(t)}{x(t)} = \frac{c_1e^{-t}-c_2e^{2t}}{c_1e^{-t}+c_2e^{2t}} = \frac{c_1e^{-3t}-c_2}{c_1e^{-3t}+c_2} \to -\frac{c_2}{c_2} = -1$ as $t \to \infty$. This means the phase portrait is b. This is because $c_2 = 0$ corresponds to the line $y = x$, and all other solution curves (i.e. $c_2 \neq 0$) are approaching the oblique asymptote $y = -x$.

SECTION B

Question 7

Since the matrix $\mathbf{A}$ has no eigenvalues equal to 1, so we can try a particular solution of the form:

$$\mathbf{v}(t) = \mathbf{c}e^t$$

for some unknown vector $\mathbf{c}$. Thus, $\mathbf{v}'(t) = \mathbf{c}e^t$. Substituting into the original differential equation,

$$\begin{aligned}
\mathbf{c}e^t - \mathbf{A}\mathbf{c}e^t &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \\
(\mathbf{I} - \mathbf{A})\mathbf{c} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\therefore \mathbf{c} &= (\mathbf{I} - \mathbf{A})^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}$$

So a particular solution is given by $\mathbf{v}(t) = (\mathbf{I} - \mathbf{A})^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$

Question 8

$$t^2 y'' + ty' - 4y = t^{11} + t^{12}, \quad y(1) = y'(1) = 0$$

Let us first solve the homogenous equation $t^2y'' + ty' - 4y = 0$. Try a solution of the form $y_c(t) = t^r$. Then $y_c'(t) = rt^{r-1}$, $y_c''(t) = r(r-1)t^{r-2}$. Thus,

$$\begin{aligned}
t^2 r(r-1)t^{r-2} + trt^{r-1} - 4t^r &= 0 \\
r(r-1) + r - 4 &= 0 \\
r^2 - 4 &= 0 \\
r &= \pm 2
\end{aligned}$$

So the general solution to the homogenous equation is $y_c(t) = At^2 + \frac{B}{t^2}$

Now, let us find a particular solution of the form $y_p(t) = Ct^{11} + Dt^{12}$. Then

$$\begin{aligned}
y_p'(t) &= 11Ct^{10} + 12Dt^{11} \\
y_p''(t) &= 110Ct^9 + 132Dt^{10}
\end{aligned}$$

Substituting into the original differential equation:

$$\begin{aligned}
t^2(110Ct^9 + 132Dt^{10}) + t(11Ct^{10} + 12Dt^{11}) - 4(Ct^{11} + Dt^{12}) &= t^{11} + t^{12} \\
117Ct^{11} + 140Dt^{12} &= t^{11} + t^{12}
\end{aligned}$$

Comparing coefficients of t^{11} and t^{12}, we have $C = \frac{1}{117}$, $D = \frac{1}{140}$. Therefore, the general solution of the original differential equation is:

$$y(t) = At^2 + \frac{B}{t^2} + \frac{1}{117}t^{11} + \frac{1}{140}t^{12}$$

Finally, let us use the initial conditions to determine A and B. We have:

$$y(1) = A + B + \frac{257}{16380} = 0$$
$$y'(t) = 2At - \frac{2B}{t^3} + \frac{11}{117}t^{10} + \frac{3}{35}t^{11}$$
$$y'(1) = 2A - 2B + \frac{736}{4095} = 0$$

, which gives $A = -\frac{19}{360}$, $B = \frac{27}{728}$.

$\therefore$ The final solution is $y(t) = -\frac{19}{360}t^2 + \frac{27}{728t^2} + \frac{1}{117}t^{11} + \frac{1}{140}t^{12}$.

Question 9

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-5t}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ y_0 \end{pmatrix}$$

Let us first solve the homogenous equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \text{where } A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

To find the eigenvalues of A,

$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$
$$\lambda^2 - 1 = 0$$
$$\lambda = \pm 1$$

Note that $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, i.e. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of A corresponding to eigenvalues 1 and -1 respectively. So the general solution to the homogenous equation can be written as

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}$$

Since the non-homogenous term is $-\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-5t}$, let us try a particular solution of the form $\mathbf{v}(t) = \mathbf{c}e^{-5t}$ for some vector $\mathbf{c}$.

Now, $\mathbf{v}'(t) = -5\mathbf{c}e^{-5t}$. Substituting into the original differential equation:

$$\begin{aligned}
-5\mathbf{c}e^{-5t} &= A\mathbf{c}e^{-5t} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-5t} \\
(A + 5\mathbf{I})\mathbf{c} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\mathbf{c} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{c} &= \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{c} &= \frac{1}{6}\begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}$$

The general solution to the original equation is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \frac{1}{6} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-5t}$$

Substituting the initial condition $\mathbf{x}(0) = \begin{pmatrix} 0 \\ y_0 \end{pmatrix}$, we obtain:

$$\begin{pmatrix} 0 \\ y_0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which gives $c_1 = \frac{y_0}{2} - \frac{1}{6}$ and $c_2 = -\frac{y_0}{2}$.
The final solution is therefore given by:

$$\mathbf{x}(t) = \left(\frac{y_0}{2} - \frac{1}{6}\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t - \frac{y_0}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \frac{1}{6} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-5t}$$

Finally, note that as $t \to \infty$, the latter two terms will always vanish, but the term $\left(\frac{y_0}{2} - \frac{1}{6}\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$ will vanish if and only if $\frac{y_0}{2} - \frac{1}{6} = 0$. To satisfy the condition $\lim_{t \to \infty} \mathbf{x}(t) = \mathbf{0}$, we must have $y_0 = \dfrac{1}{3}$.