NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

**MA1104 Multivariable Calculus**
AY 2011/12 Sem 2
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## Question 1

(i) The plane contains the point $(11,-3,16),(2,3,-1)$, and $\langle 2,3,-1\rangle + 1\cdot\langle 3,-4,3\rangle = (5,-1,2)$.
So, by solving the system

$$\begin{cases} 11a-3b+16c &= 95 \\ 2a+3b-c &= 95 \\ 5a-b+2c &= 95 \end{cases}$$

we get $a=25, b=12, c=-9$.

**Auditor's Note:** Alternatively, consider the vectors $\langle 3,-4,3\rangle$ and $\langle 11,-3,16\rangle - \langle 2,3,-1\rangle = \langle 9,-6,17\rangle$ that are parallel to the desired plane. Then the cross product $\langle 3,-4,3\rangle \times \langle 9,-6,17\rangle = \langle -50,-24,18\rangle$ must be normal to the plane: $\langle a,b,c\rangle = k\langle -50,-24,18\rangle$ for some $k\in\mathbb{R}$. Plugging in $x=2, y=3, z=-1$ into the given equation, we conclude that $k=-\frac{1}{2}, a=25, b=12, c=-9$.

(ii) Since $\langle 25,12,-9\rangle$ is a normal vector to the plane, so it must be perpendicular to $\langle\alpha,\beta,\gamma\rangle$
So, consider $\langle 3,-4,3\rangle \times \langle 25,12,-9\rangle = \langle 0,102,136\rangle$, note that $\sqrt{0^2+102^2+136^2}=170$.
After normalizing, one can check that the vector $\langle\alpha,\beta,\gamma\rangle = \langle 0,\frac{102}{170},\frac{136}{170}\rangle = \langle 0,\frac{3}{5},\frac{4}{5}\rangle$ works.

(iii) We try to solve for the point of intersection $Q$ of the line in part (ii) and the line $L$.
i.e. to solve $\langle 11,-3,16\rangle + t\langle 0,\frac{3}{5},\frac{4}{5}\rangle = \langle 2,3,-1\rangle + l\langle 3,-4,3\rangle$

By comparing the first coordinate, we get $l=3$
Substituting $l=3$ and simplifying, we get $t\langle 0,\frac{3}{5},\frac{4}{5}\rangle = \langle 0,-6,-8\rangle$, and hence $t=-10$.

Since $\langle 0,\frac{3}{5},\frac{4}{5}\rangle$ is a unit vector, the distance $d$ is 10.

(iv) $Q = \langle 11,-3,16\rangle + (-10)\langle 0,\frac{3}{5},\frac{4}{5}\rangle = \langle 11,-9,8\rangle$

## Question 2

(i) (By abuse of notation, let $\mathbf{r}$ denote $\mathbf{r}(t)$, $\mathbf{T}$ denote $\mathbf{T}(t)$, etc.)

Lemma: Let $\mathbf{r}'(t) = \langle\alpha'(t),\beta'(t),\gamma'(t)\rangle$, then $\dfrac{d|\mathbf{r}'(t)|}{dt} = \dfrac{d\sqrt{(\alpha'(t))^2+(\beta'(t))^2+(\gamma'(t))^2}}{dt}$
$= \dfrac{1}{\sqrt{(\alpha')^2+(\beta')^2+(\gamma')^2}}\cdot\dfrac{1}{2}\cdot(2\alpha'\alpha''+2\beta'\beta''+2\gamma'\gamma'') = \dfrac{\mathbf{r}'\cdot\mathbf{r}''}{|\mathbf{r}'|}$

So, with the formula $\dfrac{d|\mathbf{r}'|}{dt} = \dfrac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|}$, we get:

$$\begin{aligned}
\mathbf{T}' \cdot \mathbf{T} &= \left(\frac{\mathbf{r}''}{v} - \frac{\mathbf{r}'}{v^2}\frac{d|\mathbf{r}'|}{dt}\right) \cdot \frac{\mathbf{r}'}{v} \\
&= \left(\frac{\mathbf{r}''}{v} - \frac{\mathbf{r}'}{v^2}\frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|}\right) \cdot \frac{\mathbf{r}'}{v} \\
&= \frac{\mathbf{r}'' \cdot \mathbf{r}'}{v^2} - \frac{(\mathbf{r}' \cdot \mathbf{r}')(\mathbf{r}'' \cdot \mathbf{r}')}{v^4} \\
&= \frac{\mathbf{r}'' \cdot \mathbf{r}' - \mathbf{r}'' \cdot \mathbf{r}'}{v^2} = 0
\end{aligned}$$

**Auditor's Note:** Alternatively, we can reason that $\mathbf{T}$ has fixed magnitude 1. (note that we are guaranteed $\mathbf{r}' \neq \mathbf{0}$) If the derivative $\mathbf{T}'$ has any component along $\mathbf{T}$, the magnitude of $\mathbf{T}$ would change. It follows that $\mathbf{T}'$ must be orthogonal to $\mathbf{T}$.

(ii) $|\mathbf{T}' \times \mathbf{T}| = \left|\left(\frac{\mathbf{r}''}{v} - \frac{\mathbf{r}'\cdot\mathbf{r}''}{v^3}\mathbf{r}'\right) \times \frac{\mathbf{r}'}{v}\right| = \left|\frac{\mathbf{r}''\times\mathbf{r}'}{v^2}\right|$ (Since $\mathbf{r}' \times \mathbf{r}' = \mathbf{0}$)

On the other hand,

$$\begin{aligned}
|\mathbf{T}'| = \left|\frac{\mathbf{r}''}{v} - \frac{\mathbf{r}' \cdot \mathbf{r}''}{v^3}\mathbf{r}'\right| &= \frac{|(\mathbf{r}' \cdot \mathbf{r}')\mathbf{r}'' - (\mathbf{r}' \cdot \mathbf{r}'')\mathbf{r}'|}{v^3} \\
&= \frac{|\mathbf{r}' \times (\mathbf{r}' \times \mathbf{r}''))|}{v^3} \\
&= \frac{|\mathbf{r}'|\,|(\mathbf{r}' \times \mathbf{r}'')|}{v^3} \text{ (Since } \mathbf{r}' \text{ perpendicular to } (\mathbf{r}' \times \mathbf{r}'')) \\
&= \frac{|(\mathbf{r}' \times \mathbf{r}''))|}{v^2}
\end{aligned}$$

**Auditor's Note:** Alternatively, since $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal and $|\mathbf{T}| = 1$, $|\mathbf{T} \times \mathbf{T}'| = |\mathbf{T}|\,|\mathbf{T}'| = |\mathbf{T}'|$.

(iii) By the above calculation, $|\mathbf{T}'| = \frac{|\mathbf{r}'\times\mathbf{r}''|}{v^2}$

So, $|\mathbf{r}' \times \mathbf{r}''| = |\mathbf{T}'|v^2 = \kappa v^3$

(iv) $\mathbf{r}(t) = \langle t, t^6, 0\rangle$,
$\mathbf{r}'(t) = \langle 1, 6t^5, 0\rangle, \mathbf{r}'(1) = \langle 1, 6, 0\rangle$
$\mathbf{r}''(t) = \langle 0, 30t^4, 0\rangle, \mathbf{r}''(1) = \langle 0, 30, 0\rangle$
$v(1) = \sqrt{37}$
$\mathbf{r}' \times \mathbf{r}''(1) = \langle 0, 0, 30\rangle, |\mathbf{r}' \times \mathbf{r}''(1)| = 30$
$\kappa(1) = \frac{30}{\sqrt{37}^3}$

**Question 3**

(a)

$$0 \leq \left|\frac{x^3(12y^5 + 4xy^4)}{x^6 + 4y^8}\right| \leq \frac{|x|^3(12|y|^5 + 4|x||y|^4)}{2|x|^3(2|y|^4)} = 3|y| + 4|x|$$

So, by squeeze theorem, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{12x^3y^5 + 4x^4y^4}{x^6 + 4y^8} = 0$

(b) Consider the path $(x,y)=(t^4,t^3)$, we get

$$\lim_{t\to 0}\frac{12(t^4)^3(t^3)^4+4(t^4)^4(t^3)^4}{(t^4)^6+4(t^3)^8}=\lim_{t\to 0}\frac{12t^{24}+4t^{26}}{5t^{24}}=\lim_{t\to 0}\frac{12+4t^2}{5}=\frac{12}{5}$$

But going along the path $(x,y)=(0,t)$, we get the limit 0.

So, the limit $\displaystyle\lim_{(x,y)\to(0,0)}\frac{12x^3y^4+4x^4y^4}{x^6+4y^8}=0$ does not exist.

**Question 4**

(a)

$$0\le\left|\frac{xy}{\sqrt{x^2+y^2}}-0\right|\le\frac{|xy|}{\sqrt{2|x||y|}}=\frac{\sqrt{2|x||y|}}{2}$$

So, by Squeeze Theorem, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}=0$, and hence $f(x,y)$ is continuous at $(0,0)$

(b) $\displaystyle f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0$
Note: Similarly, $f_y(0,0)=0$. We will need it for part (d).

(c) Note: One can check that if $(x,y)\neq(0,0)$, then $\displaystyle\left.\frac{\partial f}{\partial x}\right|_{x,y}=\frac{y^3}{\sqrt{x^2+y^2}^3}$. So:

$$\begin{aligned}\frac{\partial^2 f}{\partial y\partial x}(0,0)&=\lim_{\Delta y\to 0}\frac{f_x(0,\Delta y)-f_x(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{\frac{\Delta y^3}{\sqrt{0^2+\Delta y^2}^3}-0}{\Delta y}\\&=\lim_{\Delta y\to 0}\frac{(\Delta y)^3}{\sqrt{|\Delta y|}^5}=\lim_{\Delta y\to 0}\sqrt{|\Delta y|}=0\end{aligned}$$

(d) Assume (for a contradiction) $f(x,y)$ is differentiable at $(0,0)$, then there exists functions $\epsilon_1(x,y),\epsilon_2(x,y)$ that satisfy $\displaystyle\lim_{(x,y)\to(0,0)}\epsilon_1(x,y)=\lim_{(x,y)\to(0,0)}\epsilon_2(x,y)=0$, and:

$$f(x,y)-f(0,0)=xf_x(0,0)+yf_y(0,0)+x\epsilon_1(x,y)+y\epsilon_2(x,y)$$

Notice that $\displaystyle f(x,y)-f(0,0)=\frac{xy}{\sqrt{x^2+y^2}}$ and $f_x(0,0)=f_y(0,0)=0$.
Consider the path $(x,y)=(t,t),t>0$.

$$\begin{aligned}f(t,t)-f(0,0)&=0+0+t\epsilon_1(t,t)+t\epsilon_2(t,t)\\\frac{t^2}{\sqrt{t^2+t^2}}&=t(\epsilon_1(t,t)+\epsilon_2(t,t))\\\frac{t}{\sqrt{2t^2}}&=\epsilon_1(t,t)+\epsilon_2(t,t)\\\frac{1}{\sqrt{2}}=\lim_{t\to 0^+}\frac{t}{\sqrt{2t^2}}&=\lim_{t\to 0^+}\epsilon_1(t,t)+\epsilon_2(t,t)=0\end{aligned}$$

So we have a contradiction. Hence $f(x,y)$ is not differentiable at $(0,0)$

**Question 5**

(i) Let $\phi(x,y,z) = x^2z + 2y^2 + 2z^5 - 5$. $\nabla\phi = \langle 2xz, 4y, x^2 + 10z^4\rangle$.
$\nabla\phi|_{(1,-1,1)} = \langle 2, -4, 11\rangle$ is a normal vector for the tangent plane.
So, the formula for the plane is $\nabla\phi|_{(1,-1,1)} \cdot \langle x-1, y+1, z-1\rangle = 0$ or $2x - 4y + 11z = 17$

(ii) On the tangent plane, if $(x,y) = (1.05, -0.99)$, then $z = \frac{17-2x+4y}{11} = \frac{10.94}{11} \approx 0.9945$

**Question 6**

(i) Let $\phi(x,y,z) = \frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{4^2} - 1$.
$\nabla\phi = \langle \frac{x}{2}, \frac{2y}{9}, \frac{z}{8}\rangle$ represents a normal vector for the tangent plane at $\langle x,y,z\rangle$
We want this vector to be parallel to $\langle 12, 8, 9\rangle$, a normal vector for the plane $12x + 8y + 9z = 51$
i.e. if we solve for $\nabla\phi \times \langle 12, 8, 9\rangle = \mathbf{0}$, we get $\begin{cases} 2y - z = 0 \\ \frac{3}{2}z - \frac{9}{2}x = 0 \\ 4x - \frac{8}{3}y = 0 \end{cases} \Rightarrow \begin{cases} z = 3x \\ y = \frac{3}{2}x \end{cases}$ Substitute back to the equation of the ellipsoid, we get $\frac{x^2}{4} + \frac{(\frac{3}{2}x)^2}{9} + \frac{(3x)^2}{16} = 1$
Hence, we get $x = \pm\frac{4}{\sqrt{17}}, y = \pm\frac{6}{\sqrt{17}}, z = \pm\frac{12}{\sqrt{17}}$ (Where all the $\pm$ are either all $+$ or all $-$)
Let $A$ be the point $\langle \frac{4}{\sqrt{17}}, \frac{6}{\sqrt{17}}, \frac{12}{\sqrt{17}}\rangle$, $B$ denote the point $\langle -\frac{4}{\sqrt{17}}, -\frac{6}{\sqrt{17}}, -\frac{12}{\sqrt{17}}\rangle$
The tangent planes at $A, B$ are parallel to the plane $L : 12x + 8y + 9z = 51$, hence must take the form $12x + 8y + 9z = M_A, 12x + 8y + 9z = M_B$ for some real numbers $M_A, M_B$.
By substituting, we get $M_A = 12\sqrt{17}, M_B = -12\sqrt{17}$.
The distance from $A$ to $L$ is $\frac{|51-12\sqrt{17}|}{\sqrt{12^2+8^2+9^2}} = 3 - \frac{12}{17}\sqrt{17}$

(ii) The distance from $B$ to $L$ is $\frac{|51-(-12\sqrt{17})|}{\sqrt{12^2+8^2+9^2}} = 3 + \frac{12}{17}\sqrt{17}$

**Question 7**

(i) One can check that by the change of coordinates $\begin{cases} x = \frac{1}{3} - \frac{2}{3}v + \frac{2}{3}w \\ y = \frac{2}{3} - \frac{1}{3}v - \frac{2}{3}w \\ z = \frac{2}{3} + \frac{2}{3}v + \frac{1}{3}w \end{cases}$,
the condition $x^2 + y^2 + z^2 \leq \pi^2$ becomes $u^2 + v^2 + w^2 \leq \pi^2$

and the Jacobian matrix $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = A, \det A = 1$. So,

$$\begin{aligned}
\iiint_B \cos\left(\frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z\right) dxdydz &= \iiint_B \cos u \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw \\
&= \iiint_B \cos u dudvdw \\
\left(\text{Letting } \begin{array}{l} v = r\cos\theta, r \in [0, \sqrt{\pi^2 - u^2}] \\ w = r\sin\theta, \theta \in [0, 2\pi] \end{array}\right) &= \int_{-\pi}^{\pi} \left[\cos u \int_0^{\sqrt{\pi^2-u^2}} rdr \int_0^{2\pi} d\theta\right] du \\
&= \int_{-\pi}^{\pi} (\cos u)(\pi^2 - u^2)\pi du \\
&= \pi^3 \left[\sin u\right]_{-\pi}^{\pi} - \pi \int_{-\pi}^{\pi} u^2 \cos u \, du \\
&= 0 - \pi(-4\pi) = 4\pi^2
\end{aligned}$$

Since:

$$\begin{aligned}
\int_{-\pi}^{\pi} u^2 \cos u \, du &= \left[u^2 \sin u\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2u \sin u \, du \\
&= 0 - \left(\left[-2u\cos u\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -2\cos u \, du\right) \\
&= 2\pi\cos\pi - 2(-\pi)\cos(-\pi) - 0 = -4\pi
\end{aligned}$$

(ii) Let $\hat{\mathbf{a}} = \langle a_1, a_2, a_3\rangle$. Intuitively, our answer must still be $4\pi^2$. This is because we can express the integral as $\iiint_B \cos(\mathbf{F} \cdot \hat{\mathbf{a}})\, dV$ where $\mathbf{F} = \langle x, y, z\rangle$. It is not difficult to see that the value of this expression does not depend on the direction of $\hat{\mathbf{a}}$.

An algebraic approach would be as follows: since $\hat{\mathbf{a}}$ is a unit vector we can find unit vectors $\hat{\mathbf{b}} = \langle b_1, b_2, b_3\rangle$ and $\hat{\mathbf{c}} = \langle c_1, c_2, c_3\rangle$ so that $\left\{\hat{a}, \hat{b}, \hat{c}\right\}$ is an orthonomal basis for $\mathbb{R}^3$. It follows that the matrix

$$X = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

is orthogonal, that is, $X^{-1} = X^T$. In particular, we have

$$(a_1\, a_2\, a_3) \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = (1\,0\,0)$$

Then if we take $u$, $v$, $w$ so that

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = X^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = X \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

we will have

$$\begin{aligned} a_1x + a_2y + a_3z &= (a_1\, a_2\, a_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= (a_1\, a_2\, a_3)\, X \begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ &= (1\,0\,0) \begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ &= u \end{aligned}$$

Since $X$ is orthogonal $\det X = \pm 1$. Without loss of generality let $\det X = 1$ (if otherwise replace $\hat{\mathbf{c}}$ with $-\hat{\mathbf{c}}$). Now our Jacobian determinant is 1. Since the transformation associated with $X$ only rotates vectors $|\langle x, y, z\rangle| = |\langle u, v, w\rangle|$, and $x^2 + y^2 + z^2 \le \pi^2 \iff u^2 + v^2 + w^2 \le \pi^2$. Now we proceed as in (i) to obtain $4\pi^2$.

**Question 8**

(i) $\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^3 - 2y & 3x^2 - 4y & z + 3y \end{vmatrix} = \langle 3, 3z^2, 6x + 2\rangle$

(ii) To prevent confusion, you can reparametrize the curve $C$ as $\langle 3\cos\theta, -3\sin\theta, 2\rangle, 0 \le \theta \le 4\pi$.
Let $D$ be the curve $\mathbf{r}(t) = \langle 3\cos\theta, -3\sin\theta, 2\rangle, 0 \le \theta \le 2\pi$,
let $f(\theta, r) = \langle r\cos\theta, -r\sin\theta, 2\rangle$ parametrize the disk $A$ bounded by the curve $D$, $0 \le r \le 3, 0 \le \theta \le 2\pi$.
So, $\frac{\partial f}{\partial r} \times \frac{\partial f}{\partial \theta} = \langle 0, 0, -r\rangle$. By Stokes' Theorem,

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} = 2\oint_D \mathbf{F} \cdot d\mathbf{r} = 2\iint_A \nabla \times \mathbf{F} \cdot dA \\ = 2\int_0^3 \int_0^{2\pi} \langle 3, 3z^2, 6x + 2\rangle \cdot \langle 0, 0, -r\rangle dr d\theta \\ = 2\int_0^3 \int_0^{2\pi} (6r\cos\theta + 2)(-r) dr d\theta \\ = -12\int_0^3 r^2 dr \int_0^{2\pi} \cos\theta d\theta - 4\int_0^3 r dr \int_0^{2\pi} d\theta = -36\pi \end{aligned}$$

**Question 9**

(i) By divergence theorem,

$$\oiint_S D_{\mathbf{n}}\phi d\sigma = \iint_S \nabla\phi \cdot \hat{\mathbf{n}} d\sigma = \iiint_B \nabla \cdot \nabla\phi dV = \iiint_B \Delta\phi dV$$

(ii) By divergence theorem,

$$\begin{aligned}
\oiint_S \phi(D_{\mathbf{n}}\phi)d\sigma = \iint_S \phi(\nabla\phi\cdot\hat{\mathbf{n}})d\sigma &= \iiint_B \nabla\cdot(\phi\nabla\phi)dV \\
&= \iiint_B \nabla\phi\cdot\nabla\phi + \phi(\nabla\cdot(\nabla\phi))dV \\
&= \iiint_B |\nabla\phi|^2 + \phi\Delta\phi dV \\
&= \iiint_B |\nabla\phi|^2 dV
\end{aligned}$$

**END OF SOLUTIONS**

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