MA2108S - Mathematical Analysis I(S) Suggested Solutions

(Semester 2 : AY2014/15)

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Question 1

(a) We want to show that for all $\epsilon > 0$, there exists N such that $\left|\frac{a_n^n}{n!}\right| < \epsilon\ \forall n \geq N$. Since $\{a_n\}_{n=1}^{\infty}$ is a positive sequence, it suffices to prove that $\frac{a_n^n}{n!} < \epsilon$.

Let $\epsilon > 0$. Since $\{a_n\}_{n=1}^{\infty}$ converges to α, it is bounded , ie, there exists $B \geq |a_n|\ \forall n \in \mathbb{N}$.

$\exists N' \in \mathbb{N}$ such that $N' \geq B$ and $\exists N > N'$ such that $N > \frac{B^{N'+1}}{N'!\epsilon}$. Then $\frac{1}{N} < \frac{N'!\epsilon}{B^{N'+1}}$.

$$\begin{aligned}
\forall n \geq N,\ \frac{a_n^n}{n!} &\leq \frac{B^n}{n!} \\
&= (\frac{B}{n})(\frac{B}{n-1})...(\frac{B}{N'+1})(\frac{B^{N'}}{N'!}) \\
&\leq (\frac{B}{n})(\frac{B^{N'}}{N'!}) \\
&= (\frac{1}{n})(\frac{B^{N'+1}}{N'!}) \\
&< \epsilon.
\end{aligned}$$

(b) Claim : $1 \leq x_n \leq 2\ \forall n \in \mathbb{N}$.

The case for $n = 1$ is clear. Suppose that $1 \leq x_n \leq 2$ for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
x_{n+1} = \frac{3}{1+x_n} &\implies \frac{3}{1+2} \leq x_{n+1} \leq \frac{3}{1+1} \\
&\implies 1 \leq x_{n+1} \leq \frac{3}{2} \\
&\implies 1 \leq x_{n+1} \leq 2.
\end{aligned}$$

The rest follows from mathematical induction. Now observe that

$$\begin{aligned}
|x_{n+2} - x_{n+1}| &= \left| \frac{3}{1 + x_{n+1}} - \frac{3}{1 + x_n} \right| \\
&= \frac{3|x_{n+1} - x_n|}{(1 + x_{n+1})(1 + x_n)} \\
&\leq \frac{3}{4}|x_{n+1} - x_n|.
\end{aligned}$$

Since the sequence is a contraction, it is Cauchy and thus has a limit.

Question 2

Method 1 : The idea is that if a sequence is increasing and starts off with $f(a) > a$ and ends with $f(b) < b$, it must 'cross' the $f(x) = x$ line at some point.

If $f(a) = a$ or $f(b) = b$, then we are done, so assume $f(a) > a$ and $f(b) < b$. Let the first interval $I_1 = [a, b]$ and pick the midpoint of I_1, call it m. If $f(m) = m$, we are done, so for the sake of argument we will assume $f(m) < m$ or $f(m) > m$. From here, we recursively pick I_2 as one of two intervals $[a, m]$ or $[m, b]$ such that the following condition $(\star)$ is fulfilled:

Let s, t be the starting and ending points of the chosen interval I_2 respectively. Then:

$$f(s) \geq s \quad \text{and} \quad f(t) \leq t \tag{$\star$}$$

Because the function is monotone increasing, we will not encounter a case whereby $f(m) > f(a), f(b)$ or $f(m) < f(a), f(b)$. It must be that $f(a) \leq f(m) \leq f(b)$ and this method output a well-defined interval. Now, keep applying the above method to get a nested sequence of intervals such that

$$I_1 \supset I_2 \supset I_3 \cdots$$

and that for each interval, if s, t are the starting and ending points respectively, then $(\star)$ is fulfilled. If at any poitn $f(s) = s$ or $f(t) = t$, then we are done.

Now, by Nested Interval Property, there exists some point which is in the intersection of all the intervals, x_0. x_0 is in the intersection of all the intervals, so it fulfills $(\star)$, implying that $f(x_0) \leq x_0$ and $f(x_0) \geq x_0$.

Method 2 : Consider the sequence $\{a, f(a), f^2(a), f^3(a), ...\}$.

Claim 1 : $\forall n \in \mathbb{Z}_{\geq 0},\ f^n(a) \in [a, b]$.

Proof : We will prove by mathematical induction. The base case is trivial since $a \in [a, b]$.

Induction step : Assume $f^n(a) \in [a, b]$. Since f is monotonically increasing, we have

$$\begin{gathered} a \leq f^n(a) \leq b \\ f(a) \leq f(f^n(a)) \leq f(b) \\ a \leq f^{n+1}(a) \leq b \end{gathered}$$

so $f^{n+1}(a) \in [a, b]$.

Claim 2 : $f^{n+1}(a) \geq f^n(a)\ \forall n \in \mathbb{Z}_{\geq 0}$.

Proof : Again by mathematical induction. The base case is trivial since it is given in the question that $f(a) \geq a$.

Induction step : Assume $f^{n+1}(a) \geq f^n(a)$. Since f is monotonically increasing, we have
$f(f^{n+1}(a)) \geq f(f^n(a)) \implies f^{n+2}(a) \geq f^{n+1}(a)$.

Thus the sequence $\{a, f(a), f^2(a), f^3(a), ...\}$ is monotonically increasing and bounded above. By the monotone convergence theorem, the sequence converges to some $L \in [a, b]$. It is easy to check that $f(L) = L$.

(b) Suppose for contradiction that $\lim x_n \neq b$. That means that there exist some $\epsilon > 0$ such that $\forall N \in \mathbb{N},\ \exists n > N$ such that $|x_n - b| \geq \epsilon$. This implies there are infinitely many points $|x_n - b| \geq \epsilon$. Since $x_n \leq b$, $|x_n - b| \geq \epsilon \implies b - x_n \geq \epsilon$. Construct a subsequence $\{y_n\}$ of $\{x_n\}$ consisting of all terms x_n that fulfill the inequality $b - x_n \geq \epsilon$. Then $y_n \leq b - \epsilon$ so $\gamma(y_n) \leq \gamma(b - \epsilon) < \gamma(b)$. This means that

$$|\gamma(y_n) - \gamma(b)| \geq |\gamma(b - \epsilon) - \gamma(b)|\ \forall n \in \mathbb{N}$$

so $\{\gamma(y_n)\}$ cannot converge to $\gamma(b)$. Yet, this is a contradiction since $\gamma(y_n)$ is a subsequence of $\gamma(x_n)$.

Question 3

We want to show the convergence of

$$\sum_{n=0}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\frac{\sin(\frac{n\pi}{2})}{n}.$$

First, $\sin(\frac{n\pi}{2}) = 1$ if $n \bmod 4 = 1$, $\sin(\frac{n\pi}{2}) = -1$ if $n \bmod 4 = 3$, $\sin(\frac{n\pi}{2}) = 0$ if $n \bmod 4 = 0,2$. The sum simplifies to:

$$\sum_{n\text{ odd}}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\frac{\phi(n)}{n}$$

where $\phi(n) = 1$ if $n \bmod 4 = 1$, $\phi(n) = -1$ if $n \bmod 4 = 3$ and $\phi(n) = 0$ if $n \bmod 4 = 0,2$. This sum will converge by the Alternating Series Test if we can show $\frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$ is monotone decreasing. We want to show:

$$\frac{1}{n}+\frac{1}{2n}+\cdots+\frac{1}{n^2} > \frac{1}{n+1}+\frac{1}{2(n+1)}+\cdots+\frac{1}{n(n+1)}+\frac{1}{(n+1)^2}$$

This can be done by moving terms to the left,

$$\begin{aligned}\frac{1}{n}-\frac{1}{n+1}+\frac{1}{2n}-\frac{1}{2(n+1)}+\cdots+\frac{1}{n^2}-\frac{1}{(n+1)n} &> \frac{1}{(n+1)^2}\\ \iff \frac{1}{n(n+1)}+\frac{1}{2(n)(n+1)}+\cdots\frac{1}{n\times n(n+1)} &> \frac{1}{(n+1)^2}\\ \iff \frac{1}{n(n+1)}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) &> \frac{1}{(n+1)^2}\end{aligned}$$

The last statement is obviously true, since $\frac{1}{n(n+1)} > \frac{1}{(n+1)^2}$ and $\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) > 1$. Applying the Alternating Series Test shows convergence.

(b) First note that $a_n \geq 0$ for all n. Now suppose a_n is some non-zero positive value, then there exists some $k \in \mathbb{Z}$ such that $\frac{1}{k} < a_1$. Now we show by induction that if $a_n > \frac{1}{k}$, then $a_{n+1} > \frac{1}{k+1}$.

Assume $a_n > \frac{1}{k}$, then,

$$a_n - a_{n+1} = a_{n+1}^2 \implies a_{n+1} = \sqrt{a_n+\frac{1}{4}}-\frac{1}{2} = \frac{-1+\sqrt{1+4a_n}}{2}.$$

Note that the negative root is rejected as a_n is positive. Now,

$$a_{n+1} = \frac{-1+\sqrt{1+4a_n}}{2} > \frac{-1+\sqrt{1+\frac{4}{k}}}{2} \tag{1}$$

Further,

$$\begin{aligned}
&\sqrt{1+\frac{4}{k}} > 1+\frac{2}{k+1} \\
\iff\ & 1+\frac{4}{k} > 1+\frac{4}{k+1}+\frac{4}{(k+1)^2} \\
\iff\ & \frac{1}{k} > \frac{1}{k+1}+\frac{1}{(k+1)^2} \\
\iff\ & \frac{1}{k}-\frac{1}{k+1} > \frac{1}{(k+1)^2} \\
\iff\ & \frac{1}{k\,(k+1)} > \frac{1}{(k+1)^2}
\end{aligned}$$

The last inequality is obviously true, means that we can replace (1) with:

$$\begin{aligned}
a_{n+1} &> \frac{-1+\sqrt{1+\frac{4}{k}}}{2} \\
&> \frac{-1+1+\frac{2}{k+1}}{2} \\
&> \frac{1}{k+1}
\end{aligned}$$

If $a_1 = 0$, then $\{a_n\}_{n=1}^{\infty}$ is the zero sequence. If $a_1 > 0$, then $\exists K \in \mathbb{R}^+$ such that $a_1 > K$. Then $\forall n \in \mathbb{N}$, $a_n \geq \frac{K}{n}$. This means that $\{a_n\}_{n=1}^{\infty}$ cannot converge by comparison to p-series. As a result, we conclude that $\{a_n\}_{n=1}^{\infty}$ must be the zero sequence.

Question 4

(a) Note that

$$\min\{f(x_1),\cdots f(x_n)\} \leq \sum_{k=1}^{n} \lambda_k f(x_k) \leq \max\{f(x_1),\cdots f(x_n)\}.$$

Since $[a,b]$ is connected, the range of f is connected as f is continuous. By the intermediate value theorem, f takes on every value between $\min\{f(x_1),\cdots f(x_n)\}$ and $\max\{f(x_1),\cdots f(x_n)\}$. So there exists an $x_0 \in [a,b]$ such that $f(x_0) = \sum_{k=1}^{n} \lambda_k f(x_k)$.

(b) Put

$$f(x) = \frac{2^{x^2}+3^{x^2}}{2^x+3^x}.$$

$$\begin{aligned}
\lim_{x\to 0} f(x)^{1/x} &= \exp\left\{\lim_{x\to 0}\frac{\ln f(x)}{x}\right\} \\
&= \exp\left\{\lim_{x\to 0}\frac{f'(x)}{f(x)}\right\} \qquad (2)
\end{aligned}$$

$$\frac{d}{dx}\left(\frac{2^{x^2}+3^{x^2}}{2^x+3^x}\right) = \frac{\left(\ln(2)\cdot\ 2^{x^2+1}x + 2\ln(3)\cdot\ 3^{x^2}x\right)(2^x+3^x) - (2^x\ln(2)+3^x\ln(3))\left(2^{x^2}+3^{x^2}\right)}{(2^x+3^x)^2}$$

$$\frac{f'(x)}{f(x)} = \frac{\left(\ln(2)\cdot\ 2^{x^2+1}x + 2\ln(3)\cdot\ 3^{x^2}x\right)(2^x+3^x) - (\ln(2)\cdot\ 2^x+\ln(3)\cdot\ 3^x)\left(2^{x^2}+3^{x^2}\right)}{(2^x+3^x)\left(2^{x^2}+3^{x^2}\right)}$$

Take the limit as $x \to 0$ gives:

$$\frac{f'(x)}{f(x)} = \frac{0-2(\ln 2+\ln 3)}{2^2}$$

Subbing in back into (2), we have

$$\begin{aligned}\lim_{x\to 0} f(x)^{\frac{1}{x}} &= \exp\left\{\frac{-(\ln 2+\ln 3)}{2}\right\}\\ &= \exp\left\{\frac{-\ln 2}{2}\right\}\cdot\exp\left\{\frac{-\ln 3}{2}\right\}\\ &= \frac{1}{\sqrt{6}}.\end{aligned}$$

Question 5

(a) We want to prove that $f(x) = \cos(rx)$ for some $r \in \mathbb{R}$ or $f(x) = 0$, and these are the only functions that satisfies the properties:

- $f(x+y) + f(x-y) = 2f(x)f(y)$ for $x, y \in \mathbb{R}$.
- $|f| \leq 1$.

Obviously the zero function satisfies both properties. We will now prove that if f is not the zero function, then $f(x) = \cos(rx)$ for some $r \in \mathbb{R}$.

Claim 1 : If f is not the zero function, then $f(0) = 1$.

Proof : Set $x = 0, y = 0$. One has:

$$f(0) + f(0) = 2[f(0)]^2 \implies f(0) = 0 \text{ or } f(0) = 1.$$

If $f(0) = 0$, for $y \in \mathbb{R}$, by setting $x = 0$, one has $f(y) + f(y) = 0 \implies f(y) = -f(y) = 0$ for all $y \in \mathbb{R}$, which means that f is the zero function. Thus if f is not the zero function, then $f(0) = 1$.

Claim 2 : If f is not the zero function then there exists a point a such that $0 < f(a) \leq 1$.

Proof : Assume, for the sake of contradiction, that $f(a) \leq 0\ \forall a \in \mathbb{R}$. Since f is not the zero function, $\exists a' \in \mathbb{R}$ such that $f(a') < 0$. By choosing $x = y = a'$, we have

$$f(2a') + f(0) = 2f(a')^2.$$

Since $f(a') \neq 0,\ 2f(a') > 0$ so $f(2a') > 0$ or $f(0) > 0$. This is a contradiction and the proof is complete.

There will be a point $0 \leq \theta < \pi$ such that $f(a) = \cos(\theta)$. We can use strong induction to show that $f(ma) = \cos(m\theta)$ and $f(\frac{a}{2^n}) = \cos(\frac{\theta}{2^n})$. Using the base cases $f(0) = 1$ and $f(a) = \cos(\theta)$, suppose $f(ka) = \cos(k\theta)$ for $0 \leq k \leq m-1$. To show $f(ma) = \cos(m\theta)$, set $x = (m-1)a,\ y = a$.

$$\begin{aligned} f(ma) + f((m-2)a) &= 2f((m-1)a)f(a) \\ f(ma) + \cos((m-2)\theta) &= 2\cos((m-1)\theta)\cos(\theta) \\ f(ma) &= 2\cos((m-1)\theta)\cos(\theta) - \cos((m-2)\theta) \end{aligned}$$

However, factor formula tells us that $\cos(m\theta) + \cos((m-2)\theta) = 2\cos((m-1)\theta)\cos(\theta)$, which gives us $f(ma) = \cos(m\theta)$. One may replace m with $-k$ to see that $f(ma) = \cos(m\theta)$ holds for all $m \in \mathbb{Z}$, not just for positive m.

Further, we want to show $f(\frac{a}{2^n}) = \cos(\frac{\theta}{2^n})$. Using the base case $f(a) = \cos(\theta)$, assume true for $k = m-1$, set $x = y = \frac{a}{2^n}$. One has:

$$f\left(\frac{a}{2^n} + \frac{a}{2^n}\right) = 2f\left(\frac{a}{2^n}\right)f\left(\frac{a}{2^n}\right) - f(0)$$

Since f is not the zero function, $f(0) = 1$. Thus

$$\cos\left(\frac{\theta}{2^{n-1}}\right) = 2f\left(\frac{a}{2^n}\right)^2 - 1.$$

However, the double angle formula gives us $\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1$. Replacing $x = \frac{\theta}{2^{n-1}}$ gives $f(\frac{a}{2^n}) = \cos(\frac{\theta}{2^n})$.

Consider the set $\mathscr{G} = \{\frac{m}{2^n}, m \in \mathbb{Z}, n \in \mathbb{N}\}$. $\mathscr{G}$ is a dense set in $\mathbb{R}$. This is because $\mathbb{Q}$ is dense in $\mathbb{R}$, so given some real number r, there exists a rational number $\frac{p}{q}$ such that $|r - \frac{p}{q}| < \epsilon/2$. This means that $r \in (\frac{p}{q} - \frac{\epsilon}{2}, \frac{p}{q} + \frac{\epsilon}{2})$. The width of this interval is ϵ. One may choose n sufficiently big such that $\frac{1}{2^n} < \epsilon$. So the sequence $\frac{1}{2^n}, \frac{2}{2^n}, \frac{3}{2^n} \cdots$ will not 'skip' over the $(\frac{p}{q} - \frac{\epsilon}{2}, \frac{p}{q} + \frac{\epsilon}{2})$ interval, meaning there is some element of $\mathscr{G}$ also within $(\frac{p}{q} - \frac{\epsilon}{2}, \frac{p}{q} + \frac{\epsilon}{2})$, which is within an ϵ distance from r, so $\mathscr{G}$ is dense in $\mathbb{R}$.

Write $r = \frac{\theta}{a}$, for $r \in \mathbb{R}$. We know that $f(x) = \cos(rx)$ for all $x \in \mathscr{G}$. Since $f(x) = \cos(rx)$ on a dense set, $f(x) = \cos(rx)$ for all $x \in \mathbb{R}$. This comes from the continuity of f. Take a point $b \in \mathbb{R}$. Since $\mathscr{G}$ is dense, there exists a sequence in $(g_n) \to b$, where each $g_n \in \mathscr{G}$. Continuity of f gives us $(f(g_n)) \to f(b)$, which is $\lim_{x \to b} \cos(rx)$. Since $\cos(rx)$ is continuous, $\lim_{x \to b} \cos(rx) = \cos(rb) = f(b)$, and we are done.

(b) Set the limit of $\frac{\sum_{i=1}^{n} a_i}{n}$ to be L. Then note

$$\frac{a_1 + \cdots + a_n}{n} = \frac{a_1 + \cdots + a_{n-1}}{n-1} \cdot \left(1 - \frac{1}{n}\right) + \frac{a_n}{n}$$

For any sequence, if (x_n) converges to L, then so does x_{n-1}. This means that the sequence $\frac{a_1+\cdots+a_{n-1}}{n-1}$ converges to L. Taking limits on both sides:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = \lim_{n\to\infty} \frac{a_1 + \cdots + a_{n-1}}{n-1} \cdot \left(1 - \frac{1}{n}\right) + \frac{a_n}{n} \\
&L = L(1) + \lim_{n\to\infty} \frac{a_n}{n} \\
&\lim_{n\to\infty} \frac{a_n}{n} = 0.
\end{aligned}$$