MA2108 - Mathematical Analysis I Suggested Solutions

AY18/19 Semester 2

Author: Chong Jing Quan Reviewer: Pan Jing Bin

Question 1

(a) (i) Note that $(n^2 8^n + n^3 4^n)^{\frac{1}{3n}} = n^{\frac{2}{3n}}(8^n + n\times 4^n)^{\frac{1}{3n}}$ and $\lim_{n\to\infty} n^{\frac{2}{3n}} = 1$. We now wish to compute the limit $\lim_{n\to\infty}(8^n + n\times 4^n)^{\frac{1}{3n}}$.

We first show that $2\times 8^n > 8^n + n\times 4^n > 8^n$ for all positive integers n. The inequality $8^n + n\times 4^n > 8^n$ is obvious. To show that $2\times 8^n > 8^n + n\times 4^n$, it is equivalent to prove that $8^n > n\times 4^n$. Indeed, the inequality is obvious for $n = 1$. Suppose the inequality holds for $n = k$. Then

$$8^{k+1} = 8\times 8^k > 8(k\times 4^k) > k\times 4^{k+1}$$

and we are done by induction.

Hence, we have

$$(2\times 8^n)^{\frac{1}{3n}} = 2^{\frac{1}{3n}}\times 2 > (8^n + n\times 4^n)^{\frac{1}{3n}} > (8^n)^{\frac{1}{3n}}.$$

Applying limit on both sides, we get

$$\lim_{n\to\infty} 2^{\frac{1}{3n}+1} \geq \lim_{n\to\infty}(8^n + n\times 4^n)^{\frac{1}{3n}} \geq \lim_{n\to\infty}(8^n)^{\frac{1}{3n}}.$$

Since $\lim_{n\to\infty} 2^{\frac{1}{3n}+1} = \lim_{n\to\infty}(8^n)^{\frac{1}{3n}} = 2$, it follows from squeeze theorem that $\lim_{n\to\infty}(8^n + n\times 4^n)^{\frac{1}{3n}} = 2$.

In conclusion, $\lim_{n\to\infty}(n^2 8^n + n^3 4^n)^{\frac{1}{3n}} = 1\times 2 = 2$.

(ii) We have

$$\frac{(n+1)^{2n^2}(n-1)^{2n^2}}{(n^2+1)^{2n^2}} = \frac{(n^2-1)^{2n^2}}{(n^2+1)^{2n^2}} = \left(\frac{n^2-1}{n^2+1}\right)^{2n^2} = \left(1-\frac{2}{n^2+1}\right)^{2n^2} = \frac{\left(1-\frac{2}{n^2+1}\right)^{2n^2+2}}{\left(1-\frac{2}{n^2+1}\right)^2}.$$

Since

$$\lim_{n\to\infty}\left(1-\frac{2}{n^2+1}\right)^{2n^2+2} = \lim_{n\to\infty}\left(\left(1-\frac{2}{n^2+1}\right)^{n^2+1}\right)^2 = \left(\lim_{n\to\infty}\left(1-\frac{2}{n^2+1}\right)^{n^2+1}\right)^2 = e^{-4}$$

and

$$\lim_{n\to\infty}\left(1-\frac{2}{n^2+1}\right)^2 = 1,$$

it is now easy to see that the required limit is e^{-4}.

(iii) We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^n}{\sqrt{9^n+6^{n+2}}-\sqrt{9^n-n}} &= \lim_{n\to\infty} \frac{2^n\left(\sqrt{9^n+6^{n+2}}+\sqrt{9^n-n}\right)}{9^n+6^{n+2}-(9^n-n)} \\
&= \lim_{n\to\infty} \frac{2^n\left(\sqrt{9^n+6^{n+2}}+\sqrt{9^n-n}\right)}{6^{n+2}+n} \\
&= \lim_{n\to\infty} \frac{\sqrt{1+\frac{6^{n+2}}{9^n}}+\sqrt{1-\frac{n}{9^n}}}{36+\frac{n}{6^n}} \\
&= \frac{\sqrt{1+0}+\sqrt{1-0}}{36+0} \\
&= \frac{1}{18}.
\end{aligned}$$

(b) The function is only continuous at $x=3$. Let $\varepsilon>0$ be given. Take $\delta=\min\left\{1,\frac{\varepsilon}{4}\right\}$ so that $0<|x-3|<\delta \implies |f(x)-4|<\varepsilon$. Indeed, we have

$$|f(x)-4| \le \sup\left\{\frac{4}{|x-1|}|x-3|, |x-3|\right\} < 4|x-3| < 4\times\frac{\varepsilon}{4} = \varepsilon.$$

Thus, the function is continuous at $x=3$.

For $x\neq 3$, consider two cases. If x is rational, then $f(x)=\frac{8}{x-1}$. Consider a sequence of irrational numbers $(x_n)_{n=1}^{\infty}$ that converges to x. Then, $f(x_k)=x_k+1$ for each positive integer k.

Since $x\neq 3$, the limit $\lim_{k\to\infty}(x_k+1)=x+1$ does not equal to $f(x)=\frac{8}{x-1}$. Thus, the function is not continuous at rational values other than 3. The case for x is irrational can be handled similarly.

Question 2

(a) (i) We have

$$0<\sqrt{4^n+n^2+1}-2^n = \frac{4^n+n^2+1-4^n}{\sqrt{4^n+n^2+1}+2^n} = \frac{n^2+1}{\sqrt{4^n+n^2+1}+2^n} < \frac{n^2+1}{2^{n+1}}.$$

Since $\lim_{n\to\infty}\sqrt[n]{\frac{n^2+1}{2^{n+1}}}=\frac{1}{2}$, the series $\sum_{n=1}^{\infty}\frac{n^2+1}{2^{n+1}}$ converges by root test. Hence, the original series converges by comparison test.

(ii) Using ratio test, we see that

$$L := \lim_{n\to\infty}\left|\frac{\frac{(2(n+1))!}{(n+1)!(n+1)^{n+1}}}{\frac{(2n)!}{n!n^n}}\right| = \lim_{n\to\infty} 2\times\frac{2n+1}{n+1}\times\frac{n^n}{(n+1)^n}.$$

Since $\lim_{n\to\infty}\frac{2n+1}{n+1}=2$ and $\lim_{n\to\infty}\frac{n^n}{(n+1)^n}=\lim_{n\to\infty}(1-\frac{1}{n+1})^n=\frac{1}{e}$, we have $L=\frac{4}{e}>1$. Thus, the series diverges.

(b) We note that the series is an alternating series since n^2-n-3 is increasing for $n\geq 3$ and $\sin\left(n\pi+\frac{\pi}{2}\right)=1$ for even n and $\sin\left(n\pi+\frac{\pi}{2}\right)=-1$ for odd n. Hence, the series converges by alternating series test.

However, since for $n>3$, we have $n^2-n-3<n^2 \implies \frac{1}{\sqrt{n^2-n-3}}>\frac{1}{n}$, the series does not converge absolutely by comparison to p-series. Hence, the series converge conditionally.

Question 3

(a) The sequence converges. We first show that $x_n > 5$ by induction. The case for $n = 1$ is clear. Suppose $x_k > 5$ for some positive integer k. We want show that $x_{k+1} > 5$. We have

$$x_k > 5 \implies \frac{3}{x_k} < \frac{3}{5} \implies \frac{3}{x_k} + 1 < \frac{8}{5} \implies \frac{8}{\frac{3}{x_k} + 1} = x_{k+1} > 5,$$

which completes the induction step. We now want to show that the sequence converges to 5. We have

$$|x_{n+1} - 5| = \left| \frac{8x_n}{x_n + 3} - 5 \right| = \frac{3}{3 + x_n} |x_n - 5| < \frac{3}{8} |x_n - 5|.$$

Thus, the sequence contracts and converges to 5.

(b) Yes, such a c always exists. Suppose such a c does not exist.

Claim: $(f(2c))^2 > f(c) \cdot f(4c)\ \forall c \in [1,2] \ \vee\ (f(2c))^2 < f(c) \cdot f(4c)\ \forall c \in [1,2]$.

Proof: Assume that $\exists c_1 \in [1,2]$ such that $(f(2c_1))^2 > f(c_1) \cdot f(4c_1)$ and $\exists c_2 \in [1,2]$ such that $(f(2c_2))^2 < f(c_2) \cdot f(4c_2)$.

Then the continuous function $g(x) = (f(2x))^2 - f(x) \cdot f(4x)$ will have the property:

$$g(c_1) > 0 \wedge g(c_2) < 0.$$

By the intermediate value theorem, $\exists c' \in (\inf\{c_1, c_2\}, \sup\{c_1, c_2\})$ such that $g(c') = 0$. Then $(f(2c'))^2 = f(c') \cdot f(4c')$ which is a contradiction.

Without loss of generality, suppose that $(f(2c))^2 > f(c) \cdot f(4c)\ \forall c \in [1,2]$. Then, we have

$$(f(2))^2 > f(1) \cdot f(4) \text{ and } (f(4))^2 > f(2) \cdot f(8).$$

Multiplying both inequalities yield

$$(f(2)f(4))^2 > f(1)f(2)f(4)f(8) \implies f(2)f(4) > f(1)f(8)$$

which is a contradiction. The proof is similar for the latter case.

Question 4

(a) The statement is true. Suppose $X := \{x_{g(n)}\}_{n=1}^{\infty}$ and $Y := \{y_{g(n)}\}_{n=1}^{\infty}$ are two subsequences so that $z = \lim_{n\to\infty} \frac{x_{g(n)}}{y_{g(n)}}$.

Since X and Y are both bounded sequences, by Bolzano-Weierstrass theorem, there exists subsequences $\{x_{f(n)}\}_{n=1}^{\infty}$ and $\{y_{h(n)}\}_{n=1}^{\infty}$ of X and Y respectively so that $\lim_{n\to\infty} x_{f(n)} = x$ and $\lim_{n\to\infty} y_{h(n)} = y$. In particular, there is no subsequence of Y that converges to 0. Suppose there is one such sequence $\{y_{k(n)}\}_{n=1}^{\infty}$. Then, for a given $\varepsilon_1 > 0$, there exists a positive integer N so that $|y_{k(n)}| < \varepsilon_1$ for all $n \geq N$. Hence, $\left| \frac{x_{k(n)}}{y_{k(n)}} \right| > \frac{1}{\varepsilon_1}$, which contradicts the fact that $\left| \frac{x_{k(n)}}{y_{k(n)}} \right|$ is bounded. Thus $y \neq 0$ so $\frac{x}{y}$ is well-defined. Now, it is easy to check that $\frac{x}{y} = z$.

(b) The statement is true. Let $\varepsilon > 0$ be given. Since g is continuous on $[0,1]$, it is uniformly continuous. Thus there exists $\delta_1 > 0$ so that $|x - y| < \delta_1 \implies |g(x) - g(y)| < \varepsilon$. On the other hand, by uniform continuity of f, there exists $\delta_2 > 0$ so that $|x - y| < \delta_2 \implies |f(x) - f(y)| < \delta_1$.

Consequently, $|x - y| < \delta_2 \implies |f(x) - f(y)| < \delta_1 \implies |h(x) - h(y)| = |g \circ f(x) - g \circ f(y)| < \varepsilon$.

Question 5

(a) (i) The limit is $12\cos 5$. Clearly,
$$\lim_{x\to 5}\cos x = \cos 5.$$
Now, we examine the limit $L := \lim_{x\to 5}\left(\lfloor 2x\rfloor + \left\lfloor \frac{75}{x^2}\right\rfloor\right)$. Observe that
$$\lim_{x\to 5^-}\left(\lfloor 2x\rfloor + \left\lfloor \frac{75}{x^2}\right\rfloor\right) = \lim_{x\to 5^-}\lfloor 2x\rfloor + \lim_{x\to 5^-}\left\lfloor \frac{75}{x^2}\right\rfloor = 9+3 = 12$$
and
$$\lim_{x\to 5^+}\left(\lfloor 2x\rfloor + \left\lfloor \frac{75}{x^2}\right\rfloor\right) = \lim_{x\to 5^+}\lfloor 2x\rfloor + \lim_{x\to 5^+}\left\lfloor \frac{75}{x^2}\right\rfloor = 10+2 = 12.$$
Hence, $L = 12$. Thus, the final limit is $12\cos 5$.

(ii) We show that the limit $\lim_{x\to 3}\sin\left(\frac{x}{3-x}\right)$ does not exist. Let L be the limit and let $\varepsilon = 1$. Suppose $L \geq 0$. Then, for each positive integer n, we see that if $x = \dfrac{\frac{9\pi}{2}+6n\pi}{1+\frac{3\pi}{2}+2n\pi}$, then $\sin\left(\frac{x}{3-x}\right) = -1$. In fact, for any $\delta > 0$, there exists a positive integer N so that $|x-3| = \dfrac{3}{1+\frac{3\pi}{2}+2n\pi} < \delta$ for all $n \geq N$. Thus, we get $\left|\sin\left(\frac{x}{3-x}\right) - L\right| \geq 1 = \varepsilon$. The proof is similar for $L < 0$.

Suppose the limit $\lim_{x\to 3}(x+1)\sin\left(\frac{x}{3-x}\right)$ exists. Then, we see that $\lim_{x\to 3}(x+1)\sin\left(\frac{x}{3-x}\right)\cdot\frac{1}{x+1} = \lim_{x\to 3}\sin\left(\frac{x}{3-x}\right)$ also exists since $\lim_{x\to 3}\frac{1}{x+1}$ exists. This is a contradiction and we are done.

(b) Yes, such a number always exists. We first prove that $\lim_{x\to\infty} f(x) = \sup\{f(x) : x\in[1,\infty)\}$.

Let $L := \sup\{f(x) : x \in [1,\infty)\}$. Then, for each $\varepsilon > 0$, there exists $x_0 \geq 1$ so that $f(x_0) > L - \frac{\varepsilon}{2}$. Since f is continuous, there exists $\delta > 0$ (and less than x_0) such that
$$0 < |x - x_0| < \delta \implies f(x_0) - \frac{\varepsilon}{2} < f(x) < f(x_0) + \frac{\varepsilon}{2},$$
which implies that
$$L - \varepsilon < f(x) \leq L < L + \varepsilon.$$

Note that there exists a positive integer n such that $x_0 < n\delta$. Let $M = n(x_0+\delta)$ and $x > M$. Also, let $k = \left\lfloor \frac{x}{x_0}\right\rfloor$. Then, we have $k \leq \frac{x}{x_0} < k+1$. But since $x > M$, it follows that $\frac{x}{x_0} > n + n\frac{\delta}{x_0} > n+1$ and so $k > n$.

Since
$$-k\delta < 0 \leq x - kx_0 < x_0 < n\delta < k\delta,$$
we get that $|x - kx_0| < k\delta \implies \left|\frac{x}{k} - x_0\right| < \delta$. As such, we have
$$L + \varepsilon > L \geq f(x) = f\left(\frac{kx}{k}\right) \geq f\left(\frac{x}{k}\right) > L - \varepsilon$$
and so $\lim_{n\to\infty} f(x) = L$. Since $0 \leq f(x) \leq 1$, we conclude that $0 \leq L \leq 1$.

Question 6

(a) Let $\varepsilon > 0$ be given. Then choose $\delta = \min\left\{\frac{1}{4}, 2\varepsilon\right\}$ so that $0 < |x-2| < \delta \implies \left|\frac{x^2-6}{2x-5} - 2\right| < \varepsilon$. We have

$$\left|\frac{x^2-6}{2x-5} - 2\right| = \left|\frac{x^2-4x+4}{2x-5}\right| = \left|\frac{(x-2)^2}{2x-5}\right| = |x-2|\left|\frac{x-2}{2x-5}\right| < \frac{1}{2}|x-2| < \frac{1}{2} \times 2\varepsilon = \varepsilon.$$

The result follows.

(b) Let $M = f(2) - f(1) > 0$. Split the interval $[1,2]$ into two equal intervals, i.e. $[1,1.5]$ and $[1.5,2]$. Note that we either have

$$f(2) - f(1.5) \geq f(1.5) - f(1) \iff f(2) - f(1.5) \geq \frac{M}{2}$$

or

$$f(1.5) - f(1) \geq f(2) - f(1.5) \iff f(1.5) - f(1) \geq \frac{M}{2}.$$

Pick the subinterval $I_1 := [x_1, y_1]$ so that $f(y_1) - f(x_1) \geq \frac{M}{2}$ and $y_1 - x_1 = \frac{1}{2}$.

Suppose that we have picked one such interval $I_n := [x_n, y_n]$ where $f(y_n) - f(x_n) \geq \frac{M}{2^n}$ and $y_n - x_n = \frac{1}{2^n}$ for some positive integer n. Split I_n into two subintervals of equal length, $I_{n_1} := [x_{n_1}, y_{n_1}]$ and $I_{n_2} := [x_{n_2}, y_{n_2}]$. Set I_{n+1} to be the interval I_{n_k} that satisfy the inequality $f(y_{n_k}) - f(x_{n_k}) \geq \frac{M}{2^{n+1}}$.

Since $[1,2] \supset I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ are closed intervals whose length tends to 0, by Nested Interval Theorem, there exists a value a so that $a \in \bigcap_{n=1}^{\infty} I_n$.

Now, for each $t \in [0,1]$, choose the smallest positive integer n so that $I_n \subseteq (a-t, a+t)$. Then, we must have $I_{n-1} \not\subseteq (a-t, a+t)$. Observe that at least one of $a-t$ or $a+t$ is in I_{n-1}. If $a-t \in I_{n-1}$, we have

$$a - (a-t) = t \leq y_{n-1} - x_{n-1} = \frac{1}{2^{n-1}},$$

and if $a+t \in I_{n-1}$, we have

$$(a+t) - a = t \leq y_{n-1} - x_{n-1} = \frac{1}{2^{n-1}}.$$

Thus, in either case we have $t \leq \frac{1}{2^{n-1}}$.

Finally, since f is increasing, we get

$$f(a+t) - f(a-t) \geq f(y_n) - f(x_n) \geq \frac{M}{2^n} = \frac{M}{2} \times \frac{1}{2^{n-1}} \geq \frac{M}{2}t.$$

Pick $c = \frac{M}{2}$ to complete the proof.