NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to YU Shih-Hsien

MA3220 Ordinary Differential Equations
AY 2008/2009 Sem 1

Question 1

(a) $y' + y = e^t$, thus $e^t(y' + y = e^{2t})$.

$\frac{d}{dt}(e^t y) = d^{2t}$. Integrate both sides, we get $ye^t = (1/2)e^{2t} + c$, where c is an arbitrary constant.

$y = (1/2)e^t + ce^{-t}$

Now we apply the initial condition $y(0) = 0$. Then $y(0) = (1/2) + c = 0 \Rightarrow c = -1/2$. Thus $y = \frac{e^{2t}-1}{2e^t}$.

(b) $y' + y = e^{-t}$, thus $e^t(y' + y) = 1$, so $\frac{d}{dt}(y' + y) = 1$.

Integrating both sides we get $ye^t = t + c$, where c is an arbitrary constant.

$y = (t + c)/e^t$

Apply the initial condition to this equation, $y(0) = c = 0 \Rightarrow c = 0$, so $y = \frac{t}{e^t}$

(c) By the definition of equilibrium solutions, $y'(t) = 0 \Rightarrow y(8 - y^2) = 0$, so $y = 0,\ 2\sqrt{2},\ -2\sqrt{2}$.

Draw the graph of $f(y) = y(8 - y^2)$, then equilibrium solutions are zeros. If $f(y) > 0$ on the left side of a solution and $f(y) < 0$ on its right side, then the solution is stable. Else it is instable. By this, we see that 0 is instable, $2\sqrt{2}$,$-2\sqrt{2}$ are stable.

(d) $\frac{d}{dt}y(t) = t(1 + y^2)$, $dy/(1 + y^2) = t\ dt$. Integrating both sides we get $\arctan y = t^2 + c$, where c is an arbitrary constant, so $y = \tan(t^2/2 + c)$.

Apply the initial condition $y(0) = \tan c = 0 \Rightarrow c = k\pi$, where k belong to integers.

$y = \tan t^2 + k\pi$,k is integer.

(e) Solve $y'' - 2y' - 3y = xe^x$, $y'' - 2y' - 3y = xe^{-x}$ first, then add the two particular solutions up to get a particular solution of original equation. Solve $y'' - 2y' - 3y = 0$ then add this solution to the particular solution to obtain the general one.

We solve $y'' - 2y' - 3y = 0$ now.

Let $y = e^{rx}$. We have characteristic equation $r^2 - 2r - 3 = 0$ and it has two distinct real solutions $r = -1$, $r = 3$. Thus $y(x) = c_1e^{-x} + c_2e^{3x}$.

Now we solve $y'' - 2y' - 3y = xe^x$. Substitute $y = Ae^x$, we obtain the solution $-\frac{x}{4}e^x$. Similarly, a particular solution of the other equation is $(-\frac{x^2}{8} - \frac{x}{16})e^{-x}$, so the general solution is $y = c_1e^{3x} + c_2e^{-x} - \frac{x}{4}e^x + (-\frac{x^2}{8} - \frac{x}{16})e^{-x}$, c_1,c_2 are reals.

Question 2

f, d, c, e, a, b.

Question 3

Let $\mathbf{x} = e^t \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We get

$$(1-a)x - by = 1$$

$$-cx + (1-d)y = 1$$

$\mathbf{A}$ has no eigenvalue of 1 means that $ad - a - d + 1 - bc \neq 0$.

Case 1:

$a = 1$, $(bc \neq 0)$.

$$\mathbf{x} = \begin{pmatrix} -\frac{b-d+1}{bc} \\ -\frac{1}{b} \end{pmatrix}$$

Case 2:

$a \neq 1$

$$\mathbf{x} = \begin{pmatrix} \frac{a+d-b-ad+ab-1}{(1-a)(-ad+a+d-1+bc)} \\ \frac{a-c-1}{-ad+a+d-1+bc} \end{pmatrix}$$

Question 4

$(-\frac{23+\sqrt{17}}{256\sqrt{17}} - \frac{25+\sqrt{17}}{304\sqrt{17}})t^{\frac{-1+\sqrt{17}}{2}} + (\frac{23-\sqrt{17}}{256\sqrt{17}} + \frac{25-\sqrt{17}}{304\sqrt{17}})t^{\frac{-1-\sqrt{17}}{2}} + \frac{e^{11u}}{128} + \frac{e^{12u}}{152}$,

Solution:

Let $t = e^u$. Then $u = \ln t$.

Substitute this into the equation we get:

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 4y = e^{11u} + e^{12u}$$

Firstly, solve $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 4y = e^{11u}$.

Let $y = ve^{11u}$. Then we get $v''e^{11u} + 23v'e^{11u} + 128ve^{11u} = e^{11u}$, that is $v'' + 23v' + 128v = 1$, so $v = \frac{1}{128}$, $y = \frac{1}{128}e^{11u}$

Then solve $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 4y = e^{12u}$. Similarly we get $y = \frac{1}{152}e^{12u}$.

Moreover, the general solution of $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 4y = 0$ is $c_1 e^{\frac{-1+\sqrt{17}}{2}u} + c_2 e^{\frac{-1-\sqrt{17}}{2}u}$. So the solution should be

$$c_1 e^{\frac{-1+\sqrt{17}}{2}u} + c_2 e^{\frac{-1-\sqrt{17}}{2}u} + \frac{1}{128}e^{11u} + \frac{1}{152}e^{12u}$$

Substitute that $u = \ln t$.

$$c_1 t^{\frac{-1+\sqrt{17}}{2}} + c_2 t^{\frac{-1-\sqrt{17}}{2}} + \frac{1}{128}t^{11} + \frac{1}{152}t^{12}$$

By initial conditions, we get the final solution.

Question 5

The eigenvalue and the corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are: $1, -1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Suppose that $\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Let $\mathbf{x} = \mathbf{T}\mathbf{y}$.

$\mathbf{y}' = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{y} - \mathbf{T}^{-1}e^{-5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{y} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-5t}$

$\mathbf{y} = \begin{pmatrix} \frac{e^{-5t}}{6} + c_1e^t \\ c_2e^{-t} \end{pmatrix}$, $\mathbf{x} = \mathbf{T}\mathbf{y} = \begin{pmatrix} \frac{e^{-5t}}{6} + c_1e^t + c_2e^{-t} \\ \frac{e^{-5t}}{6} + c_1e^t - c_2e^{-t} \end{pmatrix}$.

By $\mathbf{x}(0) = \begin{pmatrix} 0 \\ y_0 \end{pmatrix}$, $c_1 = \frac{y_0}{2} - \frac{1}{6}$. And we need $c_1 = 0$. So $y_0 = \frac{1}{3}$.