NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1102R Calculus
AY 2006/2007 Sem 1

Question 1

(a) Since $\lim_{x\to 0} e^{4x} - 1 - 4x = 0$ and $\lim_{x\to 0} x^2 = 0$, we apply L'Hôpital's rule to get

$$\begin{aligned}\lim_{x\to 0}\frac{e^{4x}-1-4x}{x^2} &= \lim_{x\to 0}\frac{4e^{4x}-4}{2x}\\ &= \lim_{x\to 0}\frac{16e^{4x}}{2}\\ &= 8.\end{aligned}$$

(b) Since $-1 \leq \cos\frac{1}{x^2} \leq 1$ for all $x \in \mathbb{R} - \{0\}$, we have $-x^2 \leq x^2\cos\frac{1}{x^2} \leq x^2$ for all $x \in \mathbb{R} - \{0\}$. Since $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$, by Squeeze Theorem, $\lim_{x\to 0} x^2\cos\frac{1}{x^2} = 0$.

Question 2

(a) We have,

$$\begin{aligned}\int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx &= \int_0^1 \frac{1}{2(x+1)} - \frac{x-1}{2(x^2+1)}\,dx\\ &= \left[\frac{1}{2}\ln(x+1)\right]_0^1 - \int_0^1 \frac{2x}{4(x^2+1)} - \frac{1}{2(x^2+1)}\,dx\\ &= \frac{1}{2}\ln 2 - \left[\frac{1}{4}\ln(x^2+1) - \frac{1}{2}\tan^{-1}x\right]_0^1\\ &= \frac{1}{4}\ln 2 + \frac{\pi}{8}.\end{aligned}$$

(b) We have,

$$\begin{aligned}\int_1^e t(\ln t)^2\,dt &= \left[\frac{t^2}{2}(\ln t)^2\right]_1^e - \int_1^e t\ln t\,dt\\ &= \frac{e^2}{2} - \left(\left[\frac{t^2}{2}\ln t\right]_1^e - \int_1^e \frac{t}{2}\,dx\right)\\ &= \frac{e^2}{2} - \left(\frac{e^2}{2} - \left[\frac{t^2}{4}\right]_1^e\right)\\ &= \frac{e^2-1}{4}.\end{aligned}$$

Question 3

By equating the equations of the curve $y = 2 - x^2$ and $y = x^2$, we see that they intersect when $x = 1$. Using Cylindrical Shells method, we obtain the volume as the integral,

$$\begin{aligned}\int_0^1 2\pi x\left[(2-x^2)-(x^2)\right]\,dx &= \pi\left(\int_0^1 4x\,dx - \int_0^1 4x^3\,dx\right)\\ &= \pi\left(\left[2x^2\right]_0^1 - \left[x^4\right]_0^1\right)\\ &= \pi.\end{aligned}$$

Question 4

(a) We have $\left(\frac{1}{\ln(n+1)}\right)_{n\in\mathbb{Z}^+}$ to be positive, decreasing and $\lim_{n\to\infty}\frac{1}{\ln(n+1)} = 0$.

Hence by Alternating Series Test, $\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n+1)}$ is convergent.

(b) Applying Root test, we have $\lim_{n\to\infty}\frac{n^{\frac{1}{n}}}{(\ln n)^{\frac{1}{2}}} = \frac{1}{\left(\lim_{n\to\infty}\ln n\right)^{\frac{1}{2}}} = 0$.

Hence the series $\sum_{n=2}^{\infty}\frac{n}{(\ln n)^{\frac{n}{2}}}$ converges.

(c) Since $\sum_{i=1}^{n}\frac{1}{i} \le 1 + \sum_{i=2}^{n}\int_{i-1}^{i}\frac{1}{t}\,dt = 1 + \int_1^n \frac{1}{t}\,dt = 1 + \ln n$ for all $n \in \mathbb{Z}^+$, we have,

$$\begin{aligned}\ln\frac{1}{2\cdot\sqrt{2}\cdot\cdots\cdot\sqrt[n]{2}} = (-\ln 2)\sum_{i=1}^{n}\frac{1}{i} &\le (-\ln 2)(1+\ln n)\\ &= -\ln 2 - (\ln 2)(\ln n)\\ &= \ln\left(\frac{1}{2}\right) + \ln\frac{1}{n^{\ln 2}} = \ln\frac{1}{2n^{\ln 2}}.\end{aligned}$$

Since $\ln x$ is an increasing function in x on $\mathbb{R}^+$, we have $\frac{1}{2\cdot\sqrt{2}\cdot\cdots\cdot\sqrt[n]{2}} \le \frac{1}{2}\left(\frac{1}{n^{\ln 2}}\right)$.

Since $\sum_{n=1}^{\infty}\frac{1}{n^{\ln 2}}$ is a p-series with $p < 1$, it diverges, and so by comparison test, this series diverges.

Question 5

(a) Differentiating $f(x)$ with respect to x, we have,

$$f'(x) = x^2e^x + 2xe^x.$$

Let $f'(x) = 0$, then we have,

$$\begin{aligned}x^2e^x + 2xe^x &= 0\\ x(x+2)e^x &= 0.\end{aligned}$$

Since f' exists on $\mathbb{R}$, we have $x = 0$ and $x = -2$ to be the only critical points.
Differentiating $f'(x)$ with respect to x,

$$f^{(2)}(x) = x^2 e^x + 4xe^x + 2e^x.$$

Since $f^{(2)}(0) > 0$, $(0, 0)$ is a local minimum. Since $f^{(2)}(-2) < 0$, $(-2, 4e^{-2})$ is a local maximum.

(b) By Increasing/Decreasing test, f is increasing on $(-\infty, -2)$ and decreasing on $(-2, 0)$.
Thus for $x < 0$, since $2 < e$, we have $x^2 e^x = f(x) < f(-2) = \frac{4}{e^2} < \frac{4}{2^2} = 1$, i.e. $e^x < \frac{1}{x^2}$.

(c) If $e^x = \frac{1}{x^2}$ then we have $f(x) = x^2 e^x = 1$. Since $f(1) = e > 1$, $f(0) = 0$, and f is a continuous function, by Intermediate Value Theorem, there exists $a \in [0, 1]$ such that $f(a) = 1$.

By (5b.), $x \in (-\infty, 0)$ give us $f(x) \neq 1$.
Assume on the contrary that there exist $x_1, x_2 \in [0, \infty)$ such that $x_1 \neq x_2$, and $f(x_1) = 1 = f(x_2)$.
Then by Rolle's Theorem, there exist $x_3 \in (x_1, x_2)$ such that $f'(x_3) = 0$, a contradiction since $x_3 \neq 0$ and $x_3 \neq -2$. Therefore, there is exactly 1 real number satisfying $e^x = \frac{1}{x^2}$.

Question 6

Let the 2 equal angles of the isosceles triangle be α, and the 2 equal sides of the triangle be x, and the area be A. We note that $\alpha \in \left[0, \frac{\pi}{2}\right]$ and $x \in \left[\frac{P}{4}, \frac{P}{2}\right]$.
Then we have $P = x + x + 2x\cos\alpha = 2x(1 + \cos\alpha)$, and $A = \frac{1}{2}(P - 2x)x\sin\alpha$.
Since $\cos\alpha = \frac{P}{2x} - 1$, we have $\sin\alpha = \sqrt{1 - \cos^2\alpha} = \frac{\sqrt{4Px - P^2}}{2x}$.
Therefore $A = \frac{1}{2}(P - 2x)\frac{\sqrt{4Px - P^2}}{2}$. Differentiating A with respect to x, we have,

$$\frac{dA}{dx} = \frac{4P^2 - 12Px}{4\sqrt{4Px - P^2}}.$$

Setting $\frac{dA}{dx} = 0$, together with the fact that $\sqrt{4Px - P^2} > 0$ for all $x \in \left(\frac{P}{4}, \frac{P}{2}\right)$, we get,

$$\begin{aligned} 4P^2 - 12Px &= 0 \\ x &= \frac{P}{3}. \end{aligned}$$

Therefore $x = \frac{P}{4}$, $x = \frac{P}{2}$ and $x = \frac{P}{3}$ are the only critical points.
When $x = \frac{P}{4}$ or $x = \frac{P}{2}$, we have $A = 0$. When $x = \frac{P}{3}$, we have $A = \frac{P^2\sqrt{3}}{36}$.
Hence $x = \frac{P}{3}$, $A = \frac{P^2\sqrt{3}}{36}$ is a global maximum point, i.e. the isosceles triangle with the greatest area with a fixed perimeter P is equilateral.

Question 7

Differentiating $f(x)$ with respect to x, we have,

$$f'(x) = \frac{1 - \ln x}{x^2}.$$

Since f' exists on the domain of f, if $f(x)$ is decreasing, then $f'(x) < 0$. Hence we have,

$$\begin{aligned} \frac{1-\ln x}{x^2} &< 0 \\ 1-\ln x &< 0 \\ \ln x &> 1 \\ x &> e. \end{aligned}$$

Since $e < 3 < a < b$, we have

$$\begin{aligned} f(b) &< f(a) \\ \frac{\ln b}{b} &< \frac{\ln a}{a} \\ a\ln b &< b\ln a \\ \ln b^a &< \ln a^b \\ b^a &< a^b \quad \text{(since ln is an increasing function).} \end{aligned}$$

Question 8

(a) Suppose not. Then $\int_a^b f(x)\ dx \leq 0$.

By Mean Value Theorem for Integrals, there exists $c \in (a,b)$ such that $\int_a^b f(x)\ dx = (b-a)f(c)$. Since $b-a > 0$, we have $f(c) \leq 0$, a contradiction as all $x \in (a,b)$ have $f(x) > 0$.

(b) Consider the function $h : [a,b] \to \mathbb{R}$ such that

$$h(y) \quad = \quad \int_a^b f(x)(g(x)-g(y))\ dx = \int_a^b f(x)g(x)\ dx - g(y)\int_a^b f(x)\ dx.$$

Since g is continuous, h is continuous.

By Extreme Value Theorem, there exist $x_1, x_2 \in [a,b]$ such that $g(x_1) \geq g(x)$ and $g(x_2) \leq g(x)$ for all $x \in [a,b]$, i.e. $f(x)(g(x)-g(x_1)) \leq 0$ and $f(x)(g(x)-g(x_2)) \geq 0$.
This give us $h(x_1) = \int_a^b f(x)(g(x)-g(x_1))\ dx \leq 0$ and $h(x_2) = \int_a^b f(x)(g(x)-g(x_2))\ dx \geq 0$.

Hence by Intermediate Value Theorem, there exist $c \in [x_1, x_2]$ such that $h(c) = 0$.

Therefore $c \in [a,b]$ is such that, $\int_a^b f(x)g(x)\ dx - g(c)\int_a^b f(x)\ dx = h(c) = 0$, and so we have $\int_a^b f(x)g(x)\ dx = g(c)\int_a^b f(x)\ dx$.