NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA3252 Linear and Network Optimization
AY 2008/2009 Sem 2

Question 1

(a) (i) Referring to the graph given, we can deduce the constraints of the linear program by determining the region of intersection.

$$L_1 : x_1 \geq 0$$
$$L_2 : x_2 \geq 5$$
$$L_3 : x_1 + x_2 \geq 30$$
$$L_4 : x_1 \leq 20$$
$$L_5 : x_1 - x_2 \geq 0$$

(ii) We want to maximize the following

$$\max \quad c_1x_1 + c_2x_2$$

Note that $c_i > 0$ is the return for each unit of ith investment made. Let $c_1x_1 + c_2x_2 = K$ where $K \in \mathbb{R}$. We push this line along the gradient of $\frac{-c_1}{c_2}$ and we will reach the corner point B. So the best investment option is $(x_1, x_2) = (20, 20)$.

(b) Let

$$z = \max\{3|x_1| - 5x_2, |x_2 - 4x_3|\}$$
$$b = \max\{2\max\{x_2, 0\}, |-4x_1 + 2x_2| - 5x_3\}$$
$$c = \max\{x_2, 0\}$$

then we want to

$$\min \qquad z$$
$$st.\ z \geq 3|x_1| - 5x_2 \Rightarrow \begin{cases} x_1 + \frac{5}{3}x_2 \geq -\frac{1}{3}z \\ -x_1 + \frac{5}{3}x_2 \geq -\frac{1}{3}z \end{cases}$$
$$z \geq |x_2 - 4x_3| \Rightarrow \begin{cases} 4x_3 - x_2 \geq -z \\ x_2 - 4x_3 \geq -z \end{cases}$$
$$b \leq 10 \Rightarrow \begin{cases} 2\max\{x_2, 0\} \leq 10, \\ |-4x_1 + 2x_2| - 5x_3 \leq 10 \end{cases} \Rightarrow \begin{cases} x_2 \leq 5 \\ -10 - 5x_3 \leq -4x_1 + 2x_2 \leq 10 + 5x_3 \end{cases}$$
$$b \geq 2c$$
$$b \geq |-4x_1 + 2x_2| - 5x_3 \Rightarrow \begin{cases} b + 5x_3 + 4x_1 - 2x_2 \geq 0 \\ b + 5x_3 - 4x_1 + 2x_2 \geq 0 \end{cases}$$
$$c \geq x_2 \quad \Rightarrow b \geq 2x_2$$
$$c \geq 0 \quad \Rightarrow b \geq 0$$

Question 2

(a) (i) We introduce a slack variable s_1 and an artificial variable y into the standard form of the linear programming.

$$\begin{array}{rccccccccccc} \min & x_1 & + & 3x_2 & + & x_3 & & & + & My & & \\ st. & x_1 & + & x_2 & + & 2x_3 & + & s_1 & & & = & 5 \\ & & & 3x_2 & + & 2x_3 & & & + & y & = & 6 \\ & x_1 & , & x_2 & , & x_3 & , & s_1 & , & y & \geq & 0 \end{array}$$

Choose $x_B = (s_1, y)' = (0, 0)$. Note that $c_B = (0, M)'$, we obtain the starting $\bar{c}$-row as follow:

$$\bar{c} - \text{row} = (c - \text{row}) - 0 \times (s_1 - \text{row}) - M \times (y - \text{row})$$

Basic	x_1	x_2	x_3	s_1	y	Solutions
c	1	3	1	0	M	0
$\bar{c}$	1	$3-3M$	$1-2M$	0	0	$-6M$
s_1	1	1	2	1	0	5
y	0	3	2	0	1	6
$\bar{c}$	1	0	-1	0	$M-1$	-6
s_1	1	0	$\frac{4}{3}$	1	$\frac{-1}{3}$	3
x_2	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	2
$\bar{c}$	$\frac{7}{4}$	0	0	$\frac{3}{4}$	$M-\frac{5}{4}$	$\frac{-15}{4}$
x_3	$\frac{3}{4}$	0	1	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{9}{4}$
x_2	$\frac{-1}{2}$	1	0	$\frac{-1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

In the first iteration, we choose x_2 as the entering variable, as $3 - 3M < 1 - 2M < 0$. Let i be any variable of the LP, we compute the ratio, $r_{x_2 i}$ where $r_{x_2 i} = |\frac{B^{-1}b_i}{B^{-1}A_2}|$ and we select the min of $r_{x_2 i}$ and variable i as leaving variable. We compute and find out that $r_{x_2 s_1} = 5$ and $r_{x_2 y} = 2$. Hence, y will be the leaving variable.
In the second iteration, we choose x_3 as the entering variable, because $\overline{c_3}$ is the only component with has nonpositive value. We compute the ration, $r_{x_3 i}$ where $r_{x_3 i} = |\frac{B^{-1}b_i}{B^{-1}A_3}|$ and we select the min of $r_{x_3 i}$ and variable i as leaving variable. We compute and find out that $r_{x_3 s_1} = \frac{9}{4}$ and $r_{x_3 x_2} = 3$. Hence s_1 will be the leaving variable.
In the third iteration, notice that $\bar{\mathbf{c}} > 0$. So we have an optimal solution

$$\mathbf{x} = (x_1\ x_2\ x_3\ s_1) = (0\ \frac{1}{2}\ \frac{9}{4}\ 0)$$

.

(ii)

$$\begin{aligned} \max \quad & 5p_1 + 6p_2 \\ st. \quad & p_1 \leq 1 \\ & p_1 + 3p_2 \leq 3 \\ & 2p_1 + 2p_2 \leq 1 \\ & p_1 \leq 0, p_2 \text{ free} \end{aligned}$$

(iii) Complementary Slackness theorem states that let $\boldsymbol{x}$ and $\boldsymbol{p}$ be feasible solutions to the primal problem and dual problem respectively. The vector $\boldsymbol{x}$ and $\boldsymbol{p}$ are optimal solutions for the two respective problems if and only if

$$\begin{aligned} p_i(\boldsymbol{a_i'x} - b_i) &= 0 \quad \forall i \\ (c_j - \boldsymbol{p'A_j})x_j &= 0 \quad \forall j \end{aligned}$$

By applying the above theorem, we substitute the value of $\mathbf{x}$ into the following equations,

$$x_1(p_1 - 1) = 0$$

$$x_2(p_1 + 3p_2 - 3) = 0 \Rightarrow 2p_1 + 6p_2 = 6$$
$$x_3(2p_1 + 2p_2 - 1) = 0 \Rightarrow 2p_1 + 2p_2 = 1$$
$$p_2 = \frac{5}{4} \quad p_1 = \frac{-3}{4}$$

(b) ($\Longrightarrow$) Assume (1) is true, then $\exists x \in \mathbb{R}^n$ such that $\boldsymbol{Ax} = \boldsymbol{b}$. For any $\boldsymbol{y} \in \mathbf{R}^n$ st. $\boldsymbol{A}'\boldsymbol{y} = \mathbf{0}$. we have

$$\begin{aligned}\boldsymbol{y}'\boldsymbol{b} &= \boldsymbol{y}'\boldsymbol{Ax}\\ &= (\boldsymbol{y}'\boldsymbol{A})(\boldsymbol{x})\\ &= \mathbf{0}'\boldsymbol{x}\\ &= 0 \neq 1\end{aligned}$$

which $\boldsymbol{y}'\boldsymbol{b} = 0 \neq 1$, a contradiction.
($\Longleftarrow$) Let $\mathbf{z}$ be a vector in $\mathbb{R}^m$ which satisfies $\mathbf{A'y} = \mathbf{0}$ and $\mathbf{z'b} = 1$. So we have $\mathbf{z'A} = \mathbf{0}$ as (1) $\mathbf{z'b} = 1$ as (2). $\forall\ \mathbf{x} \in \mathbb{R}^n$, we consider the following equation

$$\mathbf{z'Ax} = \mathbf{0} \quad (3)$$

Suppose $\exists \mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax} = \mathbf{b}$, then from (3), $\mathbf{z'Ax} = \mathbf{0} \Rightarrow \mathbf{z'b} = \mathbf{0}$, which contradicts (2). Thus when *Alternative 2* holds, *Alternative 1* cannot hold.

Question 3

(i) We complete the tableau From the above tableau, we can deduce $\mathbf{B}^-1$. Note that x_3, x_4, x_5

Basic	x_1	x_2	x_3	x_4	x_5	Solution
$\overline{c}$	0	0	0	δ	1	-7
x_1	1	0	0	-1	0	3
x_3	0	0	1	γ	3	β
x_2	0	1	0	α	-4	1

is the initial basic variables. So we know that

$$B^{-1} = \begin{pmatrix} A_3 & A_4 & A_5 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & \gamma & 3 \\ 0 & \alpha & -4 \end{pmatrix}$$

(ii)

$$\underline{c}' - \underline{c}_B' B^{-1} A \geq 0$$

$$c_4 - (c_1 \quad c_2 + \epsilon \quad c_3) B^{-1} A_4 \geq 0 \Leftrightarrow \overline{c_4} - (0 \quad \epsilon \quad 0) B^{-1} A_4 \geq 0$$

$$c_5 - (c_1 \quad c_2 + \epsilon \quad c_3) B^{-1} A_5 \geq 0 \Leftrightarrow \overline{c_5} - (0 \quad \epsilon \quad 0) B^{-1} A_5 \geq 0$$

$$\delta - (0 \quad \epsilon \quad 0) = \begin{pmatrix} -1 \\ \gamma \\ \alpha \end{pmatrix} = \delta - \epsilon\gamma \geq 0 \Rightarrow \epsilon \geq \frac{\delta}{\gamma}$$

$$1 - (0 \quad \epsilon \quad 0) = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = 1 - 3\epsilon \geq 0 \Rightarrow \epsilon \leq \frac{1}{3}$$

(iii) Given $\beta = 0$, we have the following table

Basic	x_1	x_2	x_3	x_4	x_5	Solution
$\overline{c}$	0	0	0	δ	1	-7
x_1	1	0	0	-1	0	3
x_3	0	0	1	γ	3	0
x_2	0	1	0	α	-4	1

to make sure the current basic feasible solution is optimal. We have $\delta \geq 0$ so that $\underline{c} > \underline{0}$. Note that also this optimal solution might be a degenerate optimal solution.

(iv) If at any iteration, the constraint coefficients $B^{-1}A_j$ of a nonbasic variable x_j are all non-positive, the solution space is unbounded in that direction. If, the reduced cost $\overline{c_j}$ of that nonbasic variable is negative (respectively positive) in the minimization (respectively maximization) problem, then the objective value is also unbounded.
To have a feasible solution, $\beta \geq 0$. To make sure x_4 is chosen as entering variable, $\delta < 0$, $\gamma \geq 0$ and $\alpha \leq 0$. In particular, if $\beta > 0$, then $\gamma \leq 0$. If $\beta = 0$, then γ is free.

(v) To have the primal problem infeasible, $\beta < 0$. Then to ensure dual problem is feasible, $\delta > 0$ and $\gamma < 0$.

Question 4

(a) (i)

$$\begin{aligned} \max \quad & 100P + 75T - 20L \\ st. \quad & P \geq 20 \\ & T \geq 5 \\ & 0.3P + 0.5T \leq 100 \qquad \Rightarrow 3P + 5T \leq 100 \\ & L \leq 250 \\ & 0.5P + T \leq L \\ & 0.6P + 0.8T \leq 40 \end{aligned}$$

(ii) Yes, the company should accept the offer. Referring to the sensitivity report, note that the Shadow Price for raw material is \$ 30. Hence, the net profit will be \$ 10.

$$2 \times 30 - 50 = 10$$

(iii) Note that the constraints of T is tight. Considering the Shadow Price for T, the increase in shadow price is $2 \times$ -\$ 95 = -\$ 190. Hence they will lose extra \$ 190 of profit.

(b) Let $S = \{\mathbf{Ax} = b\}$ be the feasible region for a standard LP problem. We try to study the property of Ax.

$$Ax = \begin{cases} a_{11}x_1 + \cdots + a_{1i}x_i^+ + a_{1i}x_i^- + \cdots + a_{1n}x_n = b_1 \\ \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mi}x_i^+ + a_{mi}x_i^- + \cdots + a_{mn}x_n = b_m \end{cases}$$

By observations, the coefficients of x_i^+ and x_i^- are not equal for each row.

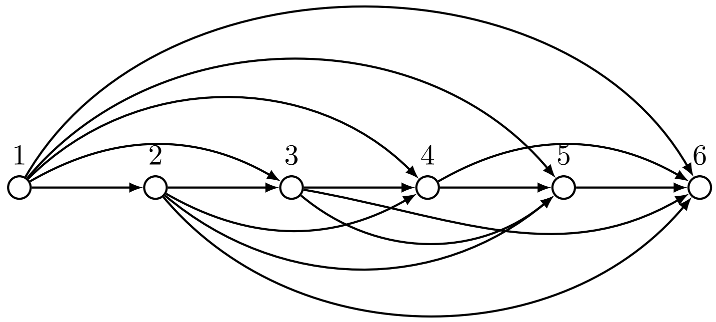
To prove the statement, we assume to the contrary that there exists a step of the simplex method that 2 of the variables x_i^+, x_i^- is equal to zero.

Basic	x_1	$\ldots$	x_i^+	$\ldots$	x_i^-	$\ldots$	x_n	Solution
$\overline{c}$				$\underline{c}' - \underline{c}'_B B^{-1}A$				$\underline{c}'_B B^{-1}b$
x_{i_1}								
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_i^+								
$\vdots$	$B^{-1}A_1$	$\cdots$	$B^{-1}A_i$	$\cdots$	$B^{-1}A_i$	$\cdots$	$B^{-1}A_n$	$B^{-1}b$
x_i^-								
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_{i_m}								

Note that $B^{-1}b > \mathbf{0}$ for a feasible solution. So we will have x_i^+, x_i^- both not equal to zero. So, x_i^+ and x_i^- must be both basic variables. However, notice from the above tableau that the columns for both x_i^+ and x_i^- are identical.
Considering A, the matrix consisting only the columns of basic variables, we find out that A is a singular matrix, since there are 2 columns which are identical. Since A is singular, we obtain a contradiction. Therefore, in each step of the simplex method, at most one of the variables x_i^+, x_i^- is not equal to zero.

Question 5

(a)



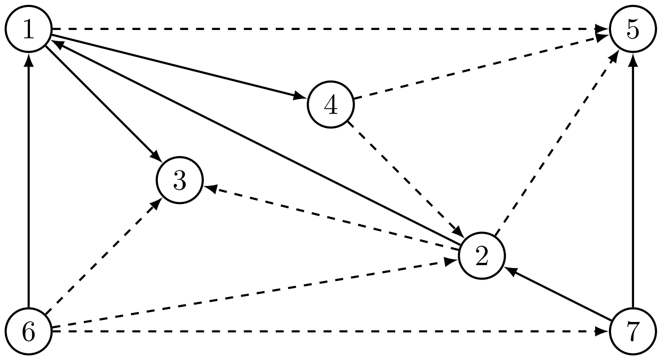
Referring to the graph above, we set the vertex i be the starting of ith year. Let $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{(i, j) : 1 \leq i < j \leq 6\}$. We define the cost, c_{ij} as

$$c_{ij} = \begin{cases} 40 + 20 + 40 & = 60 \quad \text{if } j - i = 1 \\ 40 + 20 + 40 + 30 & = 90 \quad \text{if } j - i = 2 \\ 40 + 20 + 40 + 30 + 40 & = 130 \quad \text{if } j - i = 3 \\ 40 + 20 + 40 + 30 + 40 + 60 & = 190 \quad \text{if } j - i = 4 \\ 40 + 20 + 40 + 30 + 40 + 60 + 70 & = 260 \quad \text{if } j - i = 5 \end{cases}$$

We can then model this problem as a minimum cost problem, where

$$\begin{aligned} \min \quad & \sum_{(i,j) \in E} c_{ij} x_{ij} \\ st. \quad & \sum_{j \in O(i)} x_{ij} - \sum_{j \in I(i)} x_{ji} = \begin{cases} 0 & \text{if } i \neq 1, 6 \\ 1 & \text{if } i \neq 1 \\ -1 & \text{if } i = 6 \end{cases} \\ & x_{ij} \geq 0 \qquad \forall (i, j) \in E \end{aligned}$$

(b)

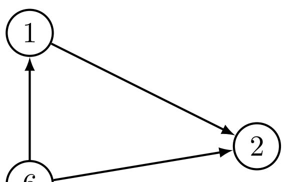


(i)

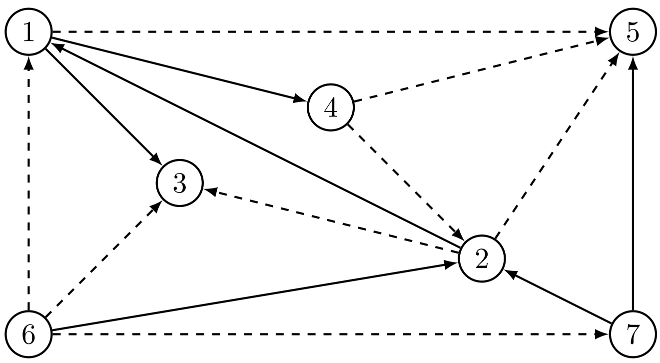
We try to compute $p_i = c_{ij} - p_j$. Let $p_6 = 0$, then

$$\begin{aligned}
p_6 - p_1 = 56 &\Rightarrow p_1 = -56 \\
p_1 - p_3 = 48 &\Rightarrow p_3 = -104 \\
p_1 - p_4 = 28 &\Rightarrow p_4 = -84 \\
p_2 - p_1 = 7 &\Rightarrow p_2 = -49 \\
p_7 - p_2 = 33 &\Rightarrow p_7 = 82 \\
p_7 - p_5 = 19 &\Rightarrow p_5 = 63
\end{aligned}$$

Then we compute $\overline{c_{ij}} = c_{ij} - (p_i - p_j)$, and we find out that $\overline{c_{62}} = 48 - (0 - (-49)) = -1 < 0$ while other $\overline{c_{ij}} > 0$, then we choose arc $(6, 2)$ to enter. By considering the following graph,



we find the value of $\theta^* = 9$ and decides that arc $(6, 1)$ leaves.
After this iteration, we obtain a new graph,



We try to use the similar method as above to compute $p_i = c_{ij} - p_j$. Let $p_6 = 0$, then

$$
\begin{aligned}
p_6 - p_2 = 48 &\Rightarrow p_2 = -48 \\
p_7 - p_2 = 33 &\Rightarrow p_7 = -15 \\
p_7 - p_5 = 19 &\Rightarrow p_5 = -34 \\
p_2 - p_1 = 7 &\Rightarrow p_1 = -55 \\
p_1 - p_3 = 48 &\Rightarrow p_3 = -103 \\
p_1 - p_4 = 28 &\Rightarrow p_4 = -83
\end{aligned}
$$

Then we compute $\overline{c_{ij}} = c_{ij} - (p_i - p_j)$, and we find out that $\forall (i, j)$ arc in this graph, $\overline{c_{ij}} > 0$, then we conclude that this dual solution is optimal. By Complementary Slackness Theorem, the prime solution is optimal as well. Hence we have

$$
\boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{pmatrix} = \begin{pmatrix} -55 \\ -48 \\ -103 \\ -83 \\ -34 \\ 0 \\ -15 \end{pmatrix} \quad \& \quad \boldsymbol{x} = \begin{pmatrix} x_{13} \\ x_{14} \\ x_{15} \\ x_{21} \\ x_{23} \\ x_{25} \\ x_{42} \\ x_{45} \\ x_{61} \\ x_{62} \\ x_{63} \\ x_{67} \\ x_{72} \\ x_{75} \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 0 \\ 12 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 9 \\ 0 \\ 0 \\ 3 \\ 2 \end{pmatrix}
$$

(ii) Note that ϵ is small enough so that it will not affect the optimal solutions. Now, considering the flow of the optimal solution's graph, an extra flow of ϵ flow passes through $(6, 2)$, incurring an extra cost of $c_{62} \times \epsilon$. Then the extra flow of ϵ passes through $(2, 1)$, incurring an extra cost of $c_{21} \times \epsilon$. Since ϵ flows out at node 1, there is no further cost incurred. Hence the change in value of the optimal cost is

$$
\delta \quad \text{in optimal cost} = \epsilon \times (c_{62} + c_{21}) = 55\epsilon
$$