NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Zheng Shaoxuan

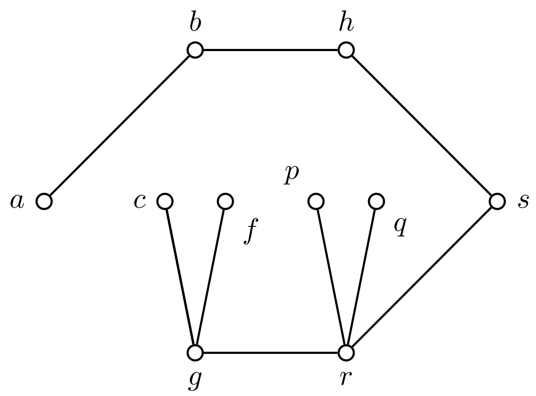
MA3233 Algorithmic Graph Theory
AY 2005/2006 Sem 2

Question 1

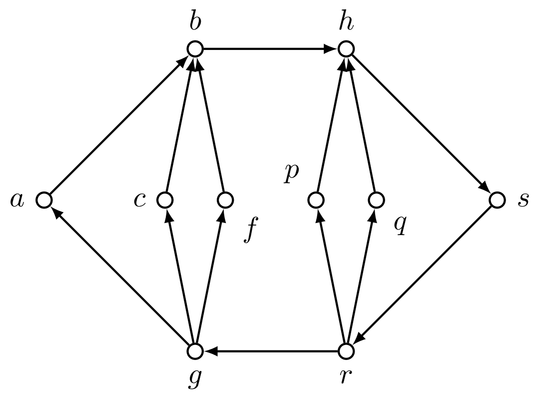
(i) By simple inspection, $(4, 4, 4, 4, 2, 2, 2, 2, 2, 2)$.

(ii) The 6 vertices of degree 2 needs to have their degrees each increase by 2 for the graph to become 4-regular. Hence the total degree needs to increase by 12 and hence 6 new edges needs to be included.

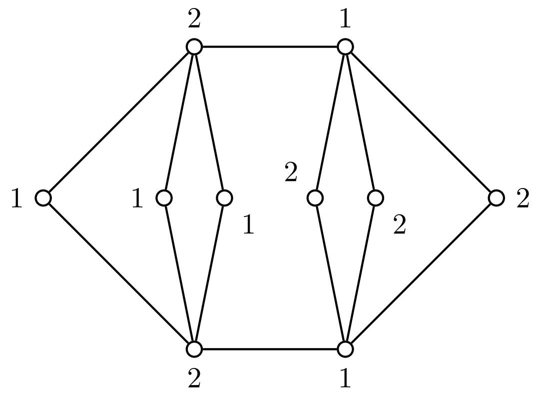
(iii) There are many many possible answers, below is one of them:



(iv) Using (iii), we label all edges in the DFST as 'forward' arrows and all edges not in the DFST as 'backward' arrows. This gives us the following one-way system.

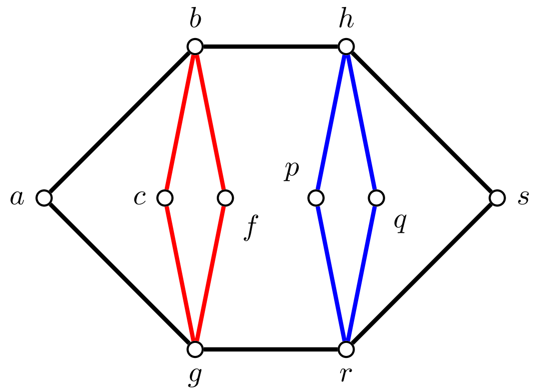


(v) A 2-colouring of G exists, as shown.



Hence $\chi(G) = 2$ and therefore G is bipartite. (You may also argue from the fact that G contains no odd cycles.

(vi) G is eulerian since all vertices have even degree. Below shows the edges of G in three different colours, edges of each colour forming a disjoint cycle.

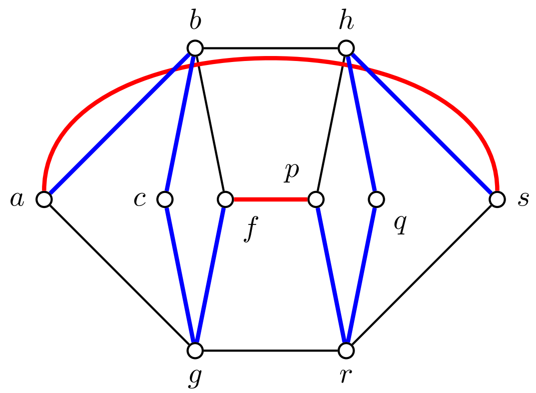


(vii) Suppose G is hamiltonian. Since $d(c) = d(f) = 2$, the edges bc, gc, bf and gf have to be within the hamiltonian cycle. But these edges form a C_4 by themselves, a contradiction! Hence G is not hamiltonian (you can use the same argument with regards to the other vertices with degree 2 as well).

Another proof would be to consider the set $S = \{b, h, g, r\}$. Since $|S| = 4 < 6 = c(G - S)$, G is not hamiltonian.

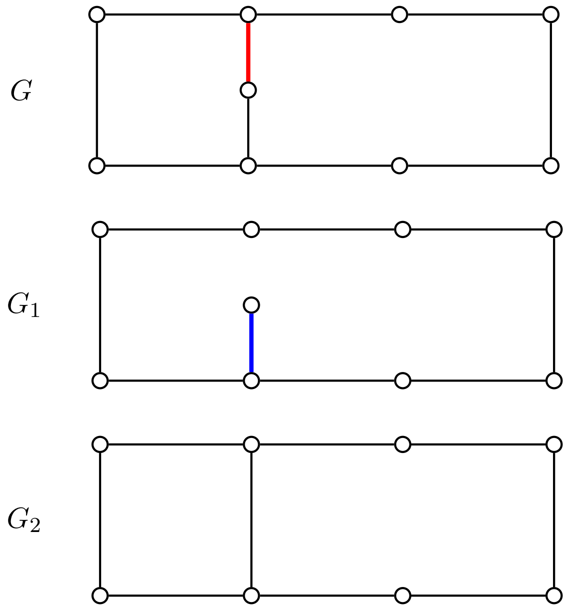
(viii) One additional edge is not enough to make a resultant hamiltonian graph. By taking the same $S = \{b, h, g, r\}$, adding any edge between the vertices a, c, f, p, q and s to form G' will still lead to $|S| = 4 < 5 = c(G' - S)$, and hence G' is still not hamiltonian.

Two additional edges is possible and hence 2 is the least number of new edges to be added to G so that the resulting graph is hamiltonian. Below is the graph with the two new edges as and fp (coloured red) (there are other possible choices of the two edges) as well as the resultant hamiltonian graph (coloured blue and red):



Question 2

Define the following graphs as such:



We aim to find $\tau(G)$, the number of spanning trees of G. By considering the removal of the bolded edge in G, $\tau(G) = \tau(G_1) + \tau(G_2)$.

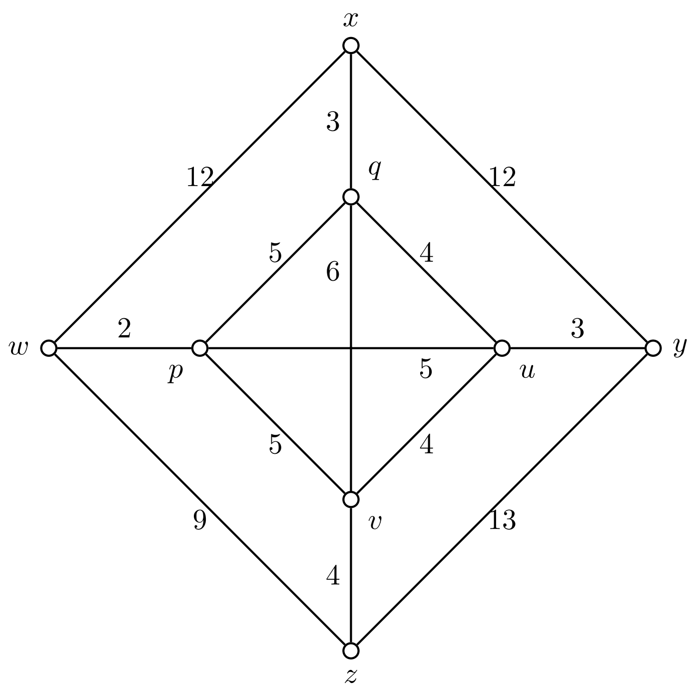
The bolded edge in $\tau(G_1)$ does not play a part in the computation of the number of spanning trees of G and hence it can be removed from G_1 without affecting $\tau(G_1)$. Hence $\tau(G_1) = \tau(C_8) = 8$.

G_2 is simply a C_4 and a C_6 sharing a common edge. Hence $\tau(G_2) = 4 \times 6 - 1 = 23$.

Hence the number of spanning trees of G, $\tau(G) = 8 + 23 = 31$.

Question 3

Apply Edmond's Algorithm to the following graph:



There exists 4 odd vertices in the graph, w, x, y and z. The least weight and path of least weight between each pair of these vertices are:

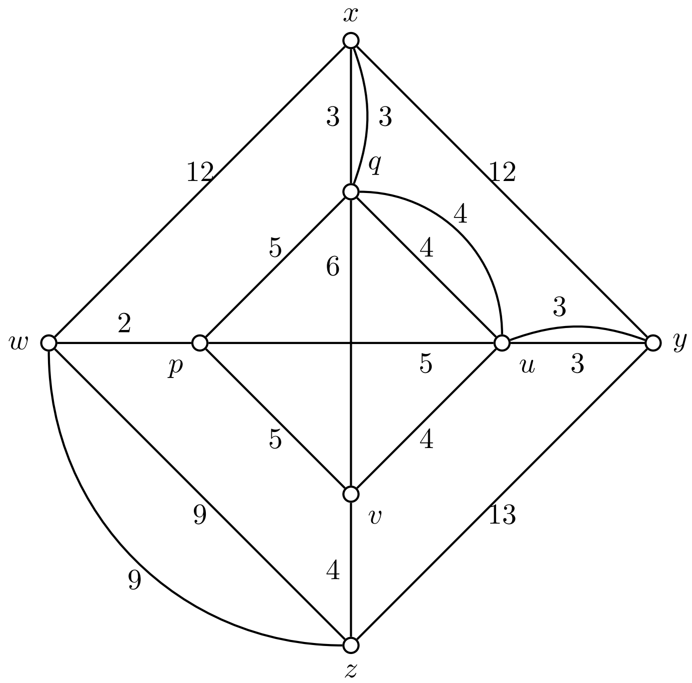
- $w - x$: 10 (via $xqpw$)
- $w - y$: 10 (via $wpuy$),
- $w - z$: 9 (via wz),
- $x - y$: 10 (via $xquy$),
- $x - z$: 13 (via $xqvz$),
- $y - z$: 11 (via $zvuy$).

The weights of the 3 possible pairings between these 4 vertices are:

- $w - x$ and $y - z$: $10 + 11 = 21$,
- $w - y$ and $x - z$: $10 + 13 = 23$,
- $w - z$ and $x - y$: $9 + 10 = 19$.

The minimum weight pairing is $w - z$ and $x - y$.

We append the paths of least weights of the two paths within the minimum weight pairing into the original graph. We obtain:

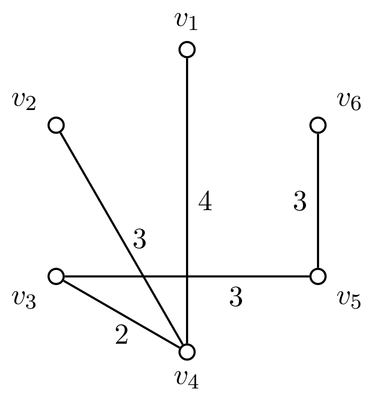


Using Fluerry's algorithm, we construct an eulerian trail of this new multigraph. One such trail can be $xqxwzwpquqvupvzyuyx$, and this is also the closed walk with minimum weight which contains all the edges in the original graph. (there are many other possible answers).

The weight of our closed walk is $(19) + (12+3+12+2+5+4+3+5+4+9+4+13+6+5) = 106$.

Question 4

(a) Using Kruskal's algorithm, the following edges are chosen from the table because they individually have the least weight and do not result in any cycles, hence forming the minimum weight spanning tree of the weighted K_6 (the vertices are labelled v_1 to v_6 rather than 1 to 6 to avoid confusion with the weighted edges):



The weight of this minimum weight spanning tree is $4+3+3+3+2 = 15$.

(b) Apply Christofide's algorithm to the above minimum weight spanning tree (this can be done since by observation of the weight matrix, triangular inequality is obeyed):

The odd vertices in the above MWST are v_1, v_2, v_4 and v_6.

The weights of the edges between each possible pair of the 4 vertices are:

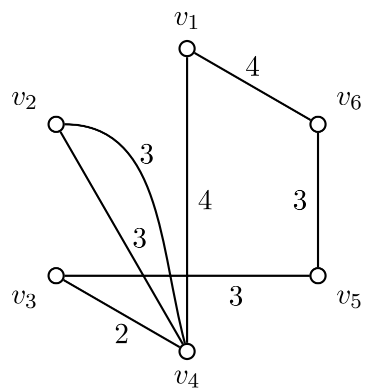
- v_1v_2: 5;
- v_1v_4: 4;
- v_1v_6: 4;
- v_2v_4: 3;
- v_2v_6: 7;
- v_4v_6: 8.

The weights of the 3 possible pairings of edges are:

- v_1v_2 and v_4v_6: $5 + 8 = 13$;
- v_1v_4 and v_2v_6: $4 + 7 = 11$;
- v_1v_6 and v_2v_4: $4 + 3 = 7$.

The minimum weight pairing of edges is v_1v_6 and v_2v_4.

We include these two edges into the above minimum weight spanning tree to obtain the following multigraph:

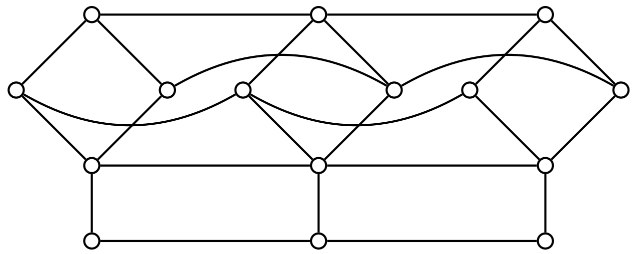


Using Fluerry's Algorithm, an eulerian cycle of this graph is $v_1v_6v_5v_3v_4v_2v_4v_1$. By eliminating repeated visits to vertices, an approximately minimum weight hamiltonian cycle of the weighted K_6 is (according to the original labelling) $1 - 6 - 5 - 3 - 4 - 2 - 1$.

Hence, an approximate solution for the TSP is $4 + 3 + 3 + 2 + 3 + 5 = 20$.

Question 5

(a) $H \times P_3$ looks like this:



(b) (i) Yes. Consider G as C_3. Since $\forall v \in C_3$, $d(v) = 2$ and $\forall u \in C_5$, $d(u) = 2$, then $C_3 \times C_5$ is a 4-regular graph and hence is eulerian.

(ii) Semi-eulerian graphs require exactly 2 odd vertices in the graph. Suppose $\exists v_{x,y} \in G \times C_5$ such that $v_x \in G$, $v_y \in C_5$ and $d(v_{x,y})$ is odd, $d(v_x)$ has to be odd since $d(v_y) = 2$.

For this v_x, $\forall v_{y'} \in C_5$, $v_{x,y'} \in G \times C_5$ is such that $d(v_{x,y'})$ is odd too, since $d(v_{y'}) = 2$ and $d(v_x)$ is odd.

Hence, if there exists 1 odd vertex in $G \times C_5$, there exists at least 5 odd vertices in $G \times C_5$. Therefore $G \times C_5$ is not semi-eulerian for all graphs G.

Question 6

Grinberg's theorem states that if a planar graph G has hamiltonian cycle C, with α_i denoting the number of i-gonal faces of G interior to C and β_i denoting the number of i-gonal faces of G exterior to C, then $\sum_{i\geq 3}(i-2)(\alpha_i - \beta_i) = 0$ (Note: this theorem is not taught in some semesters of the MA3233 course). We can use the contrapositive of this theorem to prove that this hamiltonian cycle C that contains the edge xy does not exists for this planar graph G.

Observing the graph G, suppose there is such a cycle C that contains the edge xy. G has one 3-gonal face, two 4-gonal faces, five 5-gonal faces and one 8-gonal face (remember to count the infinite face!). Given the graph and the edge xy, we know for sure that $\alpha_3 = 1$, $\beta_3 = 0$, $\alpha_8 = 0$ and $\beta_8 = 1$, as the 3-gonal face has to be within C and the infinite face has to be exterior of C.

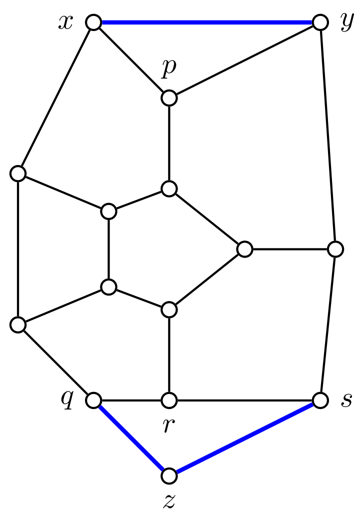
Furthermore, the bottom-most 4-gonal face has to be within C since the two edges incident to the bottom-most vertex (of degree 2) has to be included inside C. This leaves one more 4-gonal face. If this face is within C, then $\alpha_4 = 2$ and $\beta_4 = 0$. If this face is exterior of C, then $\alpha_4 = 1$ and $\beta_4 = 1$.

We evaluate $\sum_{i\geq 3}(i-2)(\alpha_i - \beta_i)$, which turns out to be $(1) + (-6) + 2(\alpha_4 - \beta_4) + 3(\alpha_5 - \beta_5)$. If $\alpha_4 = 2$ and $\beta_4 = 0$, then $\sum_{i\geq 3}(i-2)(\alpha_i - \beta_i) = -1 + 3(\alpha_5 - \beta_5)$, which is never 0 since $\alpha_5 - \beta_5$ is an integer. If $\alpha_4 = 1$ and $\beta_4 = 1$, then $\sum_{i\geq 3}(i-2)(\alpha_i - \beta_i) = -5 + 3(\alpha_5 - \beta_5)$, which is never 0 since $\alpha_5 - \beta_5$ is an integer.

Hence the hamiltonian cycle of G containing xy cannot exist!

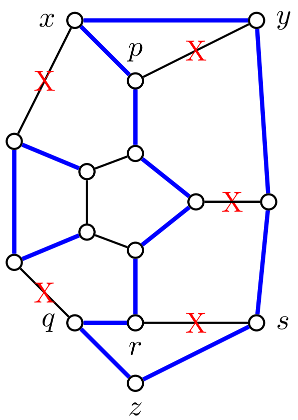
Alternatively, without using Grinberg theorem (if it is not in the syllabus), then we would use a more primitive method to prove the required statement:

Suppose such a hamiltonian cycle containing xy does exist. Consider the same graph with the following labels, where for this and all subsequent graphs in the question, the blue edges indicate edges that have to be included due to vertices having only two possible adjacent edges left that could be in the hamiltonian cycle and red crosses represent edges that are not possible to be in the hamiltonian cycle due to similar logical deductions:



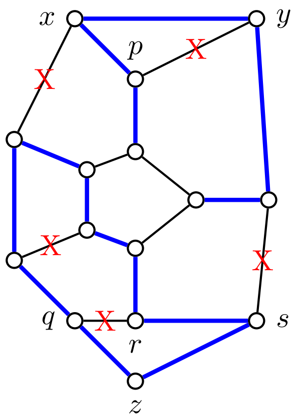
We consider the following 4 cases:

Case 1: If xp and qr are both within the hamiltonian cycle, through logical deduction we obtain:



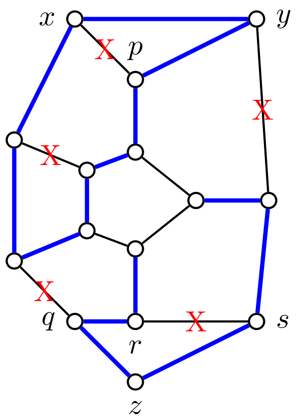
The blue edges which have to be within the hamiltonian cycle already form a smaller cycle by themselves, a contradiction!

Case 2: If xp is within the hamiltonian cycle and qr is not, through logical deduction we obtain



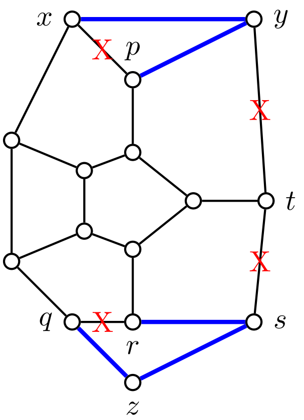
The blue edges which have to be within the hamiltonian cycle already form a smaller cycle by themselves, a contradiction!

Case 3: If xp is not within the hamiltonian cycle and qr is, through logical deduction we obtain



The blue edges which have to be within the hamiltonian cycle already form a smaller cycle by themselves, a contradiction!

Case 4: If xp and qr are both not within the hamiltonian cycle, through logical deduction we obtain:



Only one possible edge in the hamiltonian cycle is incident to vertex t, a contradiction!

These 4 cases effectively shows that no matter how we select edges to be in the hamiltonian cycle, we quickly end with a contradiction. Therefore, there is no hamiltonian cycle of G containing the edge xy.

Question 7

(a) (i) Perform the greedy colouring algorithm on G. Let this colouring be represented by $\theta(v_x)$ for x from 1 to 12.

- $\theta(v_1) = 1$ and $\theta(v_2) = 1$ because they are not adjacent to each other;
- $\theta(v_3) = 2$ because v_3 is adjacent to v_2;
- $\theta(v_4) = 1$ because v_4 is not adjacent to v_1 or v_2;
- $\theta(v_5) = 2$ because v_5 is adjacent to v_4, but not adjacent to v_3;
- $\theta(v_6) = 2$ because v_6 is adjacent to v_2, but not adjacent to v_3 or v_5;
- $\theta(v_7) = 1$ because v_7 is not adjacent to v_1, v_2 or v_4;
- $\theta(v_8) = 3$ because v_8 is adjacent to v_1 as well as v_3;
- $\theta(v_9) = 3$ because v_9 is adjacent to v_7 as well as v_6, but not adjacent to v_8;
- $\theta(v_{10}) = 4$ because v_{10} is adjacent to v_1, v_3, as well as v_9;
- $\theta(v_{11}) = 3$ because v_{11} is adjacent to v_1 as well as v_5, but not adjacent to v_8 or v_9;
- $\theta(v_{12}) = 5$ because v_{12} is adjacent to v_2, v_5, v_8 as well as v_{10}.

Hence, 5 colours are produced by applying the greedy colouring algorithm on G

(ii) Consider a colouring θ such that $\theta(v_3) = \theta(v_7) = \theta(v_{11}) = \theta(v_{12}) = 1$, $\theta(v_1) = \theta(v_2) = \theta(v_5) = \theta(v_9) = 2$, and $\theta(v_4) = \theta(v_6) = \theta(v_8) = \theta(v_{10}) = 3$. We may routinely verify that θ is a 3-colouring for G by checking that for each vertex v in G, v is not adjacent to any vertex of the same colour. Hence, $\chi(G) \leq 3$.

Also, $\chi(G) \geq 3$ since $v_1v_5v_4v_1$ is a C_3, meaning that G is not bipartite.

Therefore, $\chi(G) = 3$.

(b) Perform a depth-first search on H starting on a vertex w such that $d(w) < \Delta(H)$. This is possible since H is not regular. We obtain a sequence of vertices $w = v_1, v_2, v_3, \ldots, v_n$, where n is the order of H. Relabel these vertices as $u_1, u_2, \ldots, u_n$ such that $\forall i = 1, \ldots, n$, $u_i = v_{n-i+1}$ i.e $u_1 = v_n$, $u_2 = v_{n-1}$ etc.

We can see that $\forall i = 1, \ldots, n-1$, u_i is adjacent to u_j for some $j > i$. This is true since by DFS, each vertex is adjacent to a vertex which was encountered earlier while performing the algorithm, i.e. $\forall i = 2, \ldots, n$, v_i is adjacent to v_k for some $k < i$.

Perform greedy colouring algorithm on H based on u_1, u_2, ..., u_n. By the above, $\forall i = 1, \ldots, n-1$, each vertex u_i is adjacent to at most $\Delta(H) - 1$ coloured vertices as each u_i is adjacent to at least one uncoloured vertex at u_i's point of colouring.

For the last vertex u_n to be coloured, u_n itself is adjacent to at most $\Delta(H) - 1$ coloured vertices since $u_n = v_1$, and $d(v_1) < \Delta(H)$.

Therefore, the number of colours required to colour H using greedy colouring algorithm is at most $\Delta(H)$.

Question 8

(i) We first claim that for a graph G satisfying the stated condition, at least $n-1$ vertices of G each have a degree of at least $n-2$.

Suppose not. There exists two vertices, v_1 and v_2, such that $d(v_1) < n-2$ and $d(v_2) < n-2$, i.e., v_1 and v_2 are each not adjacent to at least 2 vertices in G.

If v_1 is not adjacent to v_2, and both v_1 and v_2 are not adjacent to another vertex u_1, then for any other vertex u_2, consider the subgraph G' induced by v_1, v_2, u_1 and u_2. For any 3 vertices chosen out of the 4 vertices as stated, there contains at least a pair of vertices which is not adjacent. Hence, G' does not contain a C_3.

If the above case is not true, then v_1 must be not adjacent to a vertex w_1, and v_2 must be not adjacent to a vertex w_2, for v_1, v_2, w_1 and w_2 being distinct vertices. Consider the subgraph G' induced by v_1, v_2, w_1 and w_2. For any 3 vertices chosen out of the 4 vertices as stated, there contains either the pair of vertices v_1, w_1, or the pair of vertices v_2, w_2, which in either case the pair of vertices are not adjacent to each other. Hence, G' does not contain a C_3, and we have arrived at a contradiction!

Hence, at least $n-1$ vertices of G each have a degree of at least $n-2$. Therefore,

$$\begin{aligned}
\sum_{v \in V(G)} d(v) &\geq (n-1)(n-2) \\
\therefore e(G) &\geq \frac{(n-1)(n-2)}{2} \\
&= \binom{n-1}{2}.
\end{aligned}$$

(ii) For equality to hold, $\sum_{v \in V(G)} d(v) = (n-1)(n-2)$. This means that in this case, the sharpness of our claim holds, i.e., exactly $n-1$ vertices in G have exactly degrees of $n-2$ each, and the remaining vertex has degree 0.

Such graphs of order n can only be characterised by an isolated vertex (the vertex of degree 0) together with a K_{n-1} (containing the $n-1$ vertices of degree $n-2$ each). From such graphs, the induced subgraph of any four vertices containing the isolated vertex contains exactly one C_3, and the induced subgraph of any four vertices not containing the isolated vertex is a K_4, which contains a C_3. Hence these graphs of order n are the only ones satisfying the stated condition as well as holding the equality in (i).