NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1102R Calculus
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Question 1

(a)

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{1+x}+\sqrt{1-x}-2}{x^2} &= \lim_{x\to 0}\frac{\sqrt{1+x}+\sqrt{1-x}-2}{x^2}.\frac{\sqrt{1+x}+\sqrt{1-x}+2}{\sqrt{1+x}+\sqrt{1-x}+2}\\ &= \lim_{x\to 0}\frac{(1+x)+(1-x)+2\sqrt{1-x^2}-4}{x^2(\sqrt{1+x}+\sqrt{1-x}+2)}\\ &= \lim_{x\to 0}\frac{2(\sqrt{1-x^2}-1)}{x^2(\sqrt{1+x}+\sqrt{1-x}+2)}.\frac{\sqrt{1-x^2}+1}{\sqrt{1-x^2}+1}\\ &= \lim_{x\to 0}\frac{-2x^2}{x^2(\sqrt{1+x}+\sqrt{1-x}+2)(\sqrt{1-x^2}+1)}\\ &= \lim_{x\to 0}\frac{-2}{(\sqrt{1+x}+\sqrt{1-x}+2)(\sqrt{1-x^2}+1)}\\ &= -\frac{1}{4}\end{aligned}$$

(b) Let $y=\frac{x}{x+1}$. This implies $x=\frac{y}{1-y}$. Note that as $x\to 0$, $y\to 0$. Note also that $\frac{x^2+1}{x}=x+\frac{1}{x}=\frac{y}{1-y}+\frac{1-y}{y}=\frac{1-2y+2y^2}{y-y^2}$. Thus,

$$\begin{aligned}\lim_{x\to 0}(2e^{\frac{x}{x+1}}-1)^{\frac{x^2+1}{x}} &= \lim_{y\to 0}(2e^y-1)^{\frac{1-2y+2y^2}{y-y^2}}\\ &= exp\left(\lim_{y\to 0}\frac{(1-2y+2y^2)\ln(2e^y-1)}{y-y^2}\right)\\ &= exp\left(\lim_{y\to 0}\frac{(4y-2)\ln(2e^y-1)+(1-2y+2y^2)\cdot\frac{2e^y}{2e^y-1}}{1-2y}\right)(UsingI'Hopital'srule)\\ &= e^2\end{aligned}$$

Question 2

(a) Let $u=\sqrt[3]{x}$. It implies that $\frac{dx}{du}=3u^2$. The wanted expression is therefore equal to

$$\int 3u^2\cdot e^u\ du$$

Using usual tabular integration, this is equal to

$$3u^2e^u-6u\cdot e^u+6e^u+C=e^{\sqrt[3]{x}}(3\sqrt[3]{x^2}-6\sqrt[3]{x}+6)+C$$

(b) Let
$$\frac{2x^3+5x^2+2x+2}{(x^2+2x+2)(x^2+2x-2)} = \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{x^2+2x-2}$$
From this, we get four equations: $A+C=2$, $2A+B+2C+D=5$, $-2A+2B+2C+2D=2$, and $-2B+2D=2$. Solving these yields $A=1$, $B=0$, and $C=D=1$. Hence,
$$\begin{aligned}\int \frac{2x^3+5x^2+2x+2}{(x^2+2x+2)(x^2+2x-2)}dx &= \int \frac{x}{x^2+2x+2}dx + \int \frac{x+1}{x^2+2x-2}dx \\ &= \frac{1}{2}\int \frac{d(x^2+2x+2)}{x^2+2x+2} - \int \frac{1}{(x+1)^2+1}dx + \frac{1}{2}\int \frac{d(x^2+2x-2)}{x^2+2x-2} \\ &= \frac{1}{2}\ln|x^2+2x+2| - \tan^{-1}(x+1) + \frac{1}{2}\ln|x^2+2x-2| + C.\end{aligned}$$

Question 3

(a) Note that $\ln y = \ln x + \cos x . \ln(\sin x)$. Then,
$$\frac{y'}{y} = \frac{1}{x} + (-\sin x)\ln(\sin x) + \cos x . \frac{1}{\sin x} . \cos x$$
Thus, the answer is $\frac{\pi}{2} \cdot (\frac{2}{\pi} - \ln 1 + 0) = 1$

(b) Let $f(x) = e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{6}$. Then, $f'(x) = e^x - 1 - \frac{x^2}{2}$, $f''(x) = e^x - 1 - x$, $f'''(x) = e^x$. Since $f'''(x) > 0$ for $x > 0$, then $f''(x)$ is increasing on $(0,\infty)$, and because $f''(0) = 0$, then $f''(x) > 0$ for $x > 0$. Thus, $f'(x)$ is increasing on $(0,\infty)$, and because $f'(0) = 0$, then $f'(x) > 0$ for $x > 0$. This means that $f(x)$ is increasing for $x > 0$, and because $f(0) = 0$, then $f(x) > 0$ for $x > 0$.

(c) $\frac{d}{dx}(\tanh x) = \frac{d}{dx}(\frac{\sinh x}{\cosh x}) = \frac{\cosh^2 x - sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$.
Using part(i), then
$$\frac{d}{dx}(\tanh^{-1} x) = \cosh^2(\tanh^{-1} x) = \frac{1}{sech^2(\tanh^{-1} x)} = \frac{1}{1-(\tanh(\tanh^{-1} x))^2} = \frac{1}{1-x^2}.$$

(d) Let $f(x) = x^5 + \sqrt{x}$ and consider the Riemann Sum on interval $(0,1)$, by dividing that interval into n equal subintervals and using the right endpoints. Then, $\int_0^1 f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} \frac{1}{n} \cdot ((\frac{i}{n})^5 + \sqrt{\frac{i}{n}}) = \lim_{n\to\infty}\sum_{i=1}^{n}(\frac{i^5}{n^6} + \frac{\sqrt{i}}{n\sqrt{n}})$. Thus, the answer is just $\int_0^1 f(x)\ dx = \frac{5}{6}$

Question 4

For part (i), we only need to show that the area of each of L-shape is $\sin\theta(2\cos\theta - \sin\theta)$. Note that the area of one L-shape can be seen as the area of big square subtracted by the area of small square. Here, the length of side of big square is $x = \cos\theta$ and that of small square is $x - y = \cos\theta - \sin\theta$. Thus, the area of each L-shape is $x^2 - (x-y)^2 = y(2x-y) = \sin\theta(2\cos\theta - \sin\theta)$.

For part (ii), let $f(\theta) = 4\sin\theta(2\cos\theta - \sin\theta)$ for $\theta \in (0, \pi/4)$. Then, $f'(\theta) = 8\cos^2\theta - 8\sin^2\theta - 8\sin\theta\cos\theta = 4(\cos(2\theta) - sin(2\theta))$. So, $f'(\theta) = 0$ only when $\sin(2\theta) = \cos(2\theta)$, which is $\theta = \pi/8$. It is also easy to check that $f'(\theta) < 0$ for $\theta > \pi/8$ and $f'(\theta) > 0$ for $\theta < \pi/8$. Then, the maximum are appears at $\theta = \pi/8$. The area is thus 2.243

Question 5

(a) It is easy to show that P is $(1, 1)$. So, the length of curve is $\int_0^1 \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_0^1 \sqrt{1 + \frac{9}{4}x}\, dx$. Now by letting $x = \frac{4}{9}\tan^2\theta$, and hence $\frac{dx}{d\theta} = \frac{8}{9}\tan\theta\sec^2\theta$, the last expression is equal to $\int_0^{\theta'} \frac{8}{9}\sec^3\theta\tan\theta d\theta = \int_0^{\theta'} \frac{8}{9}\sec^2\theta d(\sec\theta) = \left[\frac{8}{27}\sec^3\theta\right]_0^{\theta'}$. Here, θ' is the angle such that $\frac{4}{9}\tan^2\theta' = 1$, which means $\tan\theta' = \frac{3}{2}$ and $\sec\theta' = \frac{\sqrt{13}}{2}$. Hence, the length of curve is $\frac{8}{27} \cdot \frac{13\sqrt{13}}{8} = \frac{13}{27}\sqrt{13}$

(b) Note that the volume can be obtained by revolving the line $y = r + \frac{R-r}{h}.x$ about the x-axis with boundary $x = 0$ and $x = h$. Hence, the volume is

$$
\begin{aligned}
\pi. \int_0^h \left(r + \frac{R-r}{h}.x\right)^2 dx &= \pi. \int_0^h \left(r^2 + \frac{2r(R-r)x}{h} + \frac{(R-r)^2}{h^2}x^2\right) dx \\
&= \pi. \left[r^2 x + \frac{r(R-r)x^2}{h} + \frac{(R-r)^2 x^3}{3h^2}\right]_0^h \\
&= \pi. \left(r^2 h + r(R-r)h + \frac{(R-r)^2 h}{3}\right) \\
&= \frac{\pi}{3}(r^2 + rR + R^2)h
\end{aligned}
$$

Question 6

(a) Note that

$$3y^2\frac{dy}{dx} + \frac{3y^3}{x} = \frac{3\cos x}{x}$$

Now, let $z = y^3$, so $\frac{dz}{dx} = 3y^2\frac{dy}{dx}$ and

$$\frac{dz}{dx} + \frac{3z}{x} = \frac{3\cos x}{x}$$

So,

$$x^3\frac{dz}{dx} + 3x^2 x = 3x^2\ \cos x$$

Then,

$$\frac{d(zx^3)}{dx} = 3x^2\ \cos x$$

Hence,

$$y^3 x^3 = \int 3x^2\ \cos x\, dx = 3x^2\ \sin x + 6x\ \cos x - 6\sin x + C$$

With the initial value condition, we get $C = 6\pi$. In other words,

$$y = \frac{\sqrt[3]{3x^2\ \sin x + 6x\ \cos x - 6\sin x + 6\pi}}{x}$$

(b) Let V be the volume of water at time t. Then, $V = \frac{\pi r^2 h}{3}$ and because $r = \frac{Rh}{H}$, then $V = \frac{\pi R^2}{3H^2} \cdot h^3$. Thus, $\frac{dV}{dh} = b.h^2$ for some constant $b > 0$. Using Toricelli's Law, we know that $\frac{dV}{dt} = -a\sqrt{h}$ for some constant $a > 0$. Now, using two last equations,

$$\frac{dh}{dt} = -\frac{c}{\sqrt{h^3}}$$

for some $c > 0$.

Next, from last differential equation,

$$\int h^{3/2} dh = \int -c\, dt$$

and hence

$$h^{5/2} = -\frac{5}{2}ct + d$$

for some constant d. We know that at $t = 0$, $h = H$, so $d = H^{5/2}$. At $t = 1$, $h = \frac{H}{2}$, so $(\frac{H}{2})^{5/2} = -\frac{5}{2}c + H^{5/2}$. Hence, $\frac{5}{2}c = H^{5/2}\left(1 - \frac{1}{2^{5/2}}\right)$. In other words,

$$h^{5/2} = -H^{5/2}(1 - \frac{1}{2^{5/2}})t + H^{5/2}$$

Therefore, to drain completely ($h = 0$), it takes $\frac{1}{1-\frac{1}{2^{5/2}}} = 1.215$ hours.

Question 7

(a) Assume othereise that for some reals a, b, with $a < b$, we have $f'(a) < 0$ and $f'(b) > 0$. We want to proof that there exists $c \in (a, b)$ such that $f'(c) = 0$. Note that on the close interval $[a, b]$, absolute maximum and minimum exists. If the absolute maximum or minimum appears somewhere at $c \in (a, b)$, then the point $(c, f(c))$ is also a local maximum or minimum, which means, by Fermat's Theorem, $f'(c) = 0$ and we are done. We left the case when the absolute maximum and minimum only appears at both endpoints. The first case is when the absolute minimum appears at a and the absolute maximum appears at b. Thus, $f(a) < f(x)$ for all $x \in (a, b)$. Then, $f'(a) = \lim_{x\to a^+} \dfrac{f(x) - f(a)}{x - a} > 0$. A contradiction. The last case, when the absolute maximum appears at a and the absolute minimum appears at b also leads to contradiction since $f'(b) = \lim_{x\to b^-} \dfrac{f(x) - f(b)}{x - b} < 0$.

(b) Note that, since the graph of $f(x) = \sin x$ is concave down on $(0, \pi/6)$, then it is easy to show that for $t \in (0, \pi/6)$, $\frac{3t}{\pi} \leq \sin t \leq \frac{1}{2}$. Thus,

$$\left(\int_0^{\pi/6} \left(\frac{3t}{\pi}\right)^x dt\right)^{\frac{1}{x}} \leq \left(\int_0^{\pi/6} (\sin t)^x dt\right)^{\frac{1}{x}} \leq \left(\int_0^{\pi/6} (1/2)^x dt\right)^{\frac{1}{x}}$$

The left hand side, by simple computation is equal to $(\frac{\pi}{6} \cdot \frac{1}{x+1} \cdot (\frac{1}{2})^x)^{\frac{1}{x}} = \frac{1}{2} \cdot (\frac{\pi}{6(x+1)})^{\frac{1}{x}}$ and the right hand side is $(\frac{\pi}{6} \cdot (\frac{1}{2})^x)^{\frac{1}{x}} = \frac{1}{2} \cdot (\frac{\pi}{6})^{\frac{1}{x}}$. Then, by taking limit for x going to infinity on both sides, the limit for LHS is $\frac{1}{2}$ and for the RHS is $\frac{1}{2}$ also. Thus by squeeze theorem,

$$\lim_{x\to\infty} \left(\int_0^{\pi/6} (\sin t)^x dt\right)^{\frac{1}{x}} = \frac{1}{2}$$