MA1102R – Calculus
AY2019/20 SEM 2 Solutions

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Question 1

a. i. Note that $f(-1) < 0$ and $f(2) > 0$, hence by IVT, there is a root between -1 and 2.

ii. First, note that $f'(x) = 3x^2 - 2x + 1 = (3x+1)(x-1)$. Also, $f(\frac{-1}{3})$ and $f(1)$ is larger than 0. This means the interval $[-\frac{1}{3}, 1]$ and $[1, \infty)$ have no zeroes. Since we have at most one zeroes in $(-\infty, -\frac{1}{3}]$, f have at most one zeroes.

b. i. Suppose there exists two different real numbers x, y so that $g(x) = g(y)$. Then,

$$\begin{aligned} & \frac{\sqrt{x}}{\sqrt{x}-3} = \frac{\sqrt{y}}{\sqrt{y}-3} \\ \iff & \sqrt{xy} - 3\sqrt{x} = \sqrt{xy} - 3\sqrt{y} \\ \iff & x = y \end{aligned}$$

a contradiction. Hence, g is one to one.

ii. Let $y = \frac{\sqrt{x}}{\sqrt{x}-3}$. Then, $y - 1 = \frac{3}{\sqrt{x}-3}$. Hence, $\sqrt{x} - 3 = \frac{3}{y-1}$ and $x = \left(3 + \frac{3}{y-1}\right)^2$. We conclude that $g^{-1}(x) = \left(3 + \frac{3}{x-1}\right)^2$

iii. The domain of g^{-1} is $\mathbb{R}\backslash\{1\}$. The range is $\mathbb{R}_{\geq 0}$

c. We use chain rule. Let $u = x$, $du = dx$, $dv = \sec^2 x dx$ and $v = \tan x$. Then,

$$\int x \sec^2 x dx = x(\tan x) - \int \tan x dx = x \tan x - \ln|\cos x| + C.$$

Question 2

a. First, recall that the $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$. By L-Hopital's rule,

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) &= \lim_{x\to 0}\left(\frac{x^2 - \sin^2 x}{x^2\sin^2 x}\right) \\
&= \lim_{x\to 0}\left(\frac{x^2 - \sin^2 x}{x^4\frac{\sin^2 x}{x^2}}\right) \\
&= \lim_{x\to 0}\left(\frac{x^2 - \sin^2 x}{x^4}\right) \\
&= \lim_{x\to 0}\left(\frac{x + \sin x}{x}\right)\lim_{x\to 0}\left(\frac{x - \sin x}{x^3}\right) \\
&= 2\lim_{x\to 0}\left(\frac{1 - cosx}{3x^2}\right) \\
&= \frac{2}{3}\lim_{x\to 0}\left(\frac{\sin x}{2x}\right) \\
&= \frac{1}{3}.
\end{aligned}$$

b. Let ϵ be given. Pick $\delta = \min\{1, \frac{2\epsilon}{7}\}$. Then, since $|x-1| < 1$, $0 < x < 2$. Hence, $1 + x^2 > 1$ and $2x^2 - x + 1 < 7$ (This can be verified by graphing). Now,

$$\begin{aligned}
\left|x + \frac{1}{x^2+1} - \frac{3}{2}\right| &= \left|\frac{2x^3 - 3x^2 + 2x - 1}{2(x^2+1)}\right| \\
&= |\frac{(x-1)(2x^2 - x + 1)}{2(x^2+1)}| \\
&< \frac{4\epsilon}{7}\frac{7}{2\times 1} = \epsilon.
\end{aligned}$$

Hence, the limit is $\frac{3}{2}$.

Problem 3

a. Note that $\sin x = \sin(\pi - x)$. Hence, $f(0) = f(\pi)$ by symmetry. We will find $f(0)$.

$$\begin{aligned}
\lim_{x\to 0^+}\sin(x)^{\sin(x)} &= \exp\left[\lim_{x\to 0^+}\sin(x)(\ln\sin(x))\right] \\
&= \exp\left[\lim_{x\to 0^+}\frac{\ln(\sin(x))}{\frac{1}{\sin(x)}}\right]
\end{aligned}$$

The top goes to $-\infty$ while the bottom goes to ∞. We may use L'Hopital.

$$\begin{aligned}
&= \exp\left[\lim_{x\to 0^+} -\frac{\frac{\cos(x)}{\sin(x)}}{\frac{1}{\sin^2(x)}}\right] \\
&= \exp\left[\lim_{x\to 0^+} -\frac{\sin(x)\cos(x)}{1}\right] \\
&= 1
\end{aligned}$$

Hence, $f(0) = f(\pi) = 1$.

b. We find the derivative of $y = \sin x^{\sin x}$. Note that

$$\begin{aligned}
y &= \sin x^{\sin x} \\
\ln y &= \sin x(\ln \sin x) \\
\frac{1}{y}dy &= (\cos x \ln \sin x + \sin x \left(\frac{1}{\sin x}\right) \cos x)dx \\
\frac{dy}{dx} &= y(\cos x \ln \sin x + \cos x) \\
&= \sin x^{\sin x}(\cos x \ln \sin x + \cos x) \\
&= \sin x^{\sin x} \cos x(\ln \sin x + 1)
\end{aligned}$$

Note that f is increasing if $\sin x^{\sin x} \cos x(\ln \sin x + 1) > 0$. Since $\sin x > 0$, $\sin x^{\sin x} > 0$. Also, $\ln \sin x + 1 > 0$ if and only if $\ln \sin x > -1$, which means that $\sin x > \frac{1}{e}$. Hence, $x > \arcsin \frac{1}{e}$. Finally, $\cos x > 0$ if $x < \frac{\pi}{2}$. Combining, we get that f is increasing in the interval $(\arcsin \frac{1}{e}, \frac{\pi}{2}) \cup (\pi - \arcsin \frac{1}{e}, \pi)$.

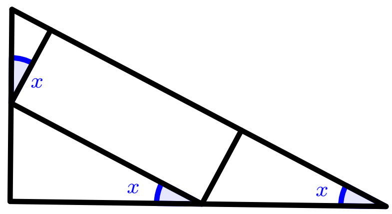
c. By similar reasoning, f is decreasing at the interval $(0, \arcsin \frac{1}{e}) \cup (\frac{\pi}{2}, \pi - \arcsin \frac{1}{e})$

d. The maximum and minimum occurs when $f' = 0$ or the endpoints. We note that the zeroes are located in $x = 0, \frac{\pi}{2}, \pi, \arcsin \frac{1}{e}, \pi - \arcsin \frac{1}{e}$. We note that $f(0) = f(\pi) = f(\frac{\pi}{2}) = 1$ and $f(\arcsin \frac{1}{e}) = f(\pi - \arcsin \frac{1}{e}) < 1$. Hence, the absolute maximum points are $(0, 1), (\frac{\pi}{2}, 1), (\pi, 1)$. and the absolute minimum points are

$$\left(\arcsin \frac{1}{e}, \arcsin \frac{1}{e}^{\arcsin \frac{1}{e}}\right), \left(\pi - \arcsin \frac{1}{e} \& \left(\pi - \arcsin \frac{1}{e}\right)^{\pi - \arcsin \frac{1}{e}}\right)$$

.

Problem 4



First, note by AAA criteria that all three similar triangles are similar to the large right triangle. Let the length of the box as l and the width as w. (l denotes the segment that coincides with the long hypotanuse). Then, $5 = l + \frac{3w}{4} + \frac{4w}{3} = l + \frac{25w}{12}$ by similarity. Hence, $l = 5 - \frac{25w}{12}$. We want to maximize lw, which is equal to $w(5 - \frac{25w}{12}) = -\frac{25}{12}w^2 + 5w = -\frac{1}{12}(5w - 6)^2 + 3$. Hence, the maximum area is 3, which is achieved by $w = \frac{6}{5}$ and $l = \frac{3}{\frac{6}{5}} = \frac{5}{2}$.

Problem 5

a. By arc length formula, the length is

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{d}{dx} \int_0^x \sqrt{\cos 2t} dt\right)^2} = \int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 2x} dx = \int_0^{\frac{\pi}{4}} \sqrt{2} \cos x dx = 1.$$

b. i. We will prove that $\lim_{x\to 0^+} x\ln x = f(0)$. By L-hopital's rule,

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0 = f(0).$$

Hence, f is continuous at 0 from the right.

ii. Since f is continuous from the right, we just add like normal. By chain rule, one can verify that $\int x^2\ln x^2 = \frac{x^3}{27}(9\ln x^2 - 6\ln x + 2) + C$. The volume is

$$\begin{aligned} V &= \pi\left|\int_0^1 (x\ln x)^2\right| + \pi\left|\int_1^2 (x\ln x)^2\right| \\ &= \pi\left|\frac{x^3}{27}(9\ln x^2 - 6\ln x + 2)\right|_0^1 + \pi\left|\frac{x^3}{27}(9\ln x^2 - 6\ln x + 2)\right|_1^2 \\ &= \frac{2}{27}\pi + \frac{2}{27}(7 + 36\ln 2^2 - 24\ln 2). \end{aligned}$$

Problem 6

a. Note that

$$\begin{aligned} \lim_{n\to\infty}\sum_{k=1}^{n} \ln\sqrt[n]{1+\frac{k}{n}} &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right) \\ &= \int_0^1 \ln(1+x)dx \\ &= |(x+1)\ln(x+1) - (x+1)|_0^1 \\ &= 2\ln 2 - 1. \end{aligned}$$

(Recall that $\int \ln x = x\ln x - x + C$ by chain rule).

b. We prove a lemma.

Lemma 1 *For all positive integer m,*

$$\frac{1}{x(x+1)\cdots(x+m)} = \frac{1}{m!}\left(\sum_{i=0}^{m}\binom{m}{i}\frac{1}{x+i}(-1)^i\right).$$

Proof 1 *We use induction on m. For $m = 1$, the identity is obvious. Assume for $m = k-1$, the identity is correct. Then,*

$$
\begin{aligned}
\frac{1}{x\cdots(x+k)} &= \frac{1}{k}\left(\frac{1}{x\cdots(x+k-1)} - \frac{1}{(x+1)\cdots(x+k)}\right)\\
&= \frac{1}{k!}\left(\left(\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{1}{x+i}(-1)^i\right) - \left(\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{1}{x+i+1}(-1)^i\right)\right)\\
&= \frac{1}{k!}\left(\left(\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{1}{x+i}(-1)^i\right) - \left(\sum_{i=1}^{k}\binom{k-1}{i}\frac{1}{x+i}(-1)^i\right)\right)\\
&= \frac{1}{k!}\left(\binom{k-1}{0}\frac{1}{x} + \sum_{i=1}^{k-1}\left(\binom{k-1}{i} + \binom{k-1}{i-1}\right)\frac{1}{x+i}(-1)^i + \binom{k-1}{k}\frac{1}{x+k}(-1)^k\right)\\
&= \frac{1}{k!}\left(\sum_{i=0}^{k}\frac{1}{x+i}\binom{k}{i}\frac{1}{x+i}(-1)^i\right).
\end{aligned}
$$

(Note that here we define $\binom{k-1}{k} = 1$). Also, we use Pascal's identity i.e. $\binom{n}{k} = \binom{n}{k-1} + \binom{n-1}{k-1}$). Hence, our lemma is proven.

Now, the answer of our problem is simply

$$
\int \frac{1}{m!}\left(\sum_{i=0}^{m}\binom{m}{i}\frac{1}{x+i}(-1)^i\right) = \frac{1}{m!}\left(\sum_{i=0}^{m}\binom{m}{i}\ln|x+i|(-1)^i\right) + C.
$$

Proof 2 *We want to split the fraction $\frac{1}{x(x+1)...(x+m)}$ into partial fractions. To do this, put,*

$$
\frac{1}{x(x+1)...(x+m)} = \frac{A_0}{x} + \frac{A_1}{x+1} + \frac{A_2}{x+2} + \cdots + \frac{A_m}{x+m}.
$$

Multiplying the denominator out, we get,

$$
[A_0(x+1)...(x+m)] + [A_1(x)...(x+m)] + \cdots + [A_m(x)...(x+m-1)] = 1
$$

For each A_i, it is multiplied by $(x)(x+1)...(x+i-1)(x+i+1)...(x+m)$. Subbing in $x=-i$ removes all terms and leaves

$$
A_i(-i)(-i+1)...(-i+i-1)(-i+i+1)...(-i+m) = 1 \implies A_i(-1)^i i!(m-i)! = 1.
$$

Hence, the term A_i in the partial fraction expansion must be,

$$
A_i = (-1)^i\frac{1}{i!(m-i)!} = (-1)^i\frac{1}{m!}\binom{m}{i}
$$

When the integral acts on each $\frac{A_i}{x+i}$ term, we have,

$$
\int \frac{A_i}{x+i} = (-1)^i\frac{1}{m!}\binom{m}{i}\ln|x+i|
$$

Hence,

$$
\int \frac{A_0}{x} + \frac{A_1}{x+1} + \frac{A_2}{x+2} + \cdots + \frac{A_m}{x+m} = \sum_{i=0}^{m}\left[(-1)^i\frac{1}{m!}\binom{m}{i}\ln|x+i|\right] + C
$$

Problem 7

a.

$$\begin{aligned}
\frac{dy}{dx} + \frac{y}{e^y + x} &= 0 \\
e^y dy + xdy + ydx &= 0 \\
xdy + ydx &= -e^y dy \\
(xy)' &= -e^y dy \\
xy &= \int -e^y dy \\
xy &= -e^y + C
\end{aligned}$$

Since $y(0) = 1$, we see that $C = e$. Hence, the solution is $xy + e^y = e$.

b. i. Let k be the height of the "grey cone" and v be the volume at a given time. So $h = 16 - k$. Then, $\frac{dk}{dt} = \frac{\sqrt{k}}{2}$, which means $\frac{2}{\sqrt{k}} dk = dt$. Hence, $t = 4\sqrt{k} + C$. At $t = 0$, $k = 0$. Hence, $C = 0$. Thus, $t = 4\sqrt{k}$. Finally, the height of the water is $h = 16 - 4\sqrt{t}$.

ii. When $h = 0$, $4\sqrt{t} = 16$, hence $t = 16$.

Problem 8

Proof 1

Let $F(x) = \int_0^x f(x)dx$. Since it converges, it has a limit L as $x \to \infty$. Assume by contradiction that $xf(x)$ does not converge to 0. Then there exists $c > 0$ so that $xf(x) > 0$ for arbitiary large x. Suppose $x_0 f(x_0) > c$ for x_0 large enough and $\int_{x_0}^{\infty} f(t)dt < d$. Pick $x_1 > 2x_0$ such that $x_1 f(x_1) > c$. Since f is monotone decreasing, $\int_{x_0}^{x_1} f(x)dx > (x_1 - x_0)f(x_1) > (x_1 - x_0)\frac{c}{x_1} = c\left(1 - \frac{x_0}{x_1}\right) > \frac{c}{2}$. Hence, $d > \int_{x_0}^{\infty} f(t)dt > \frac{c}{2}$. Since $\lim_{x \to \infty} \int_x^{\infty} f(t)dt = 0$, we can pick x_0 large enough so that d is smaller than $\frac{c}{2}$, contradiction. Hence, the limit is 0.

Proof 2

Set $F(x) = \int_0^x f(x)dx$. Since $F(x) \to L$, for $\epsilon/4$ there exists some $N > 0$ such that whenever $x > N$, $F(x) \in (L + \epsilon/4, L - \epsilon/4)$. Pick any $n, m > N$, and we have $F(n) \in (L + \epsilon/4, L - \epsilon/4)$ and $F(m) \in (L + \epsilon/4, L - \epsilon/4)$. This means that whenever $n, m > N$[1],

$$-\epsilon/2 < \int_m^n f(x)dx < \epsilon/2. \tag{1}$$

Now, either there is some point x_0 which f touches the x-axis or not. If there isn't, since f is decreasing, f lies entirely above the x-axis. Fix $k \in \mathbb{R}$ such that $k > N$. Then $2k > N$ and from (1),

$$-\epsilon/2 < \int_k^{2k} f(x)dx < \epsilon/2 \implies 0 < \int_k^{2k} f(x)dx < \epsilon/2.$$

However, since f is decreasing,

$$0 < kf(2k) \leq \int_k^{2k} f(x)dx < \epsilon/2 \implies 2kf(2k) < \epsilon$$

[1] This is a theorem in analysis saying that convergent sequences are Cauchy.

(Here, $kf(2k)$ is derived from $\int_k^{2k} f(2k)dx \leq \int_k^{2k} f(x)dx$). So for any $x > 2k$, $0 < xf(x) < \epsilon$, implying that $xf(x)$ converges to 0.

Else, suppose there is some point x_0 where the f touches the x-axis. Put $M = \max\{x_0, N\}$. Since f is decreasing, whenever $x > M$, $f(x) \leq 0$. Again, fix $k \in \mathbb{R}$ such that $k > M$, then $2k > M$. From (1),

$$-\epsilon/2 < \int_k^{2k} f(x) < \epsilon/2 \implies -\epsilon/2 < \int_k^{2k} f(x) \leq 0.$$

Again, since f is decreasing,

$$-\epsilon/2 < \int_k^{2k} f(x) \leq kf(2k) \leq 0 \implies -\epsilon < 2kf(2k) \leq 0$$

Whenever $x > 2k$, $-\epsilon < xf(x) < 0$. For all ϵ, there exists some $N \in \mathbb{R}$ such that $n > N$ implies $|nf(n)| < \epsilon$, $xf(x)$ must converge to 0 as $x \to \infty$.