MA1100 - Discrete Mathematics Suggested Solutions

(Semester 1, AY2022/2023)

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Question 1

Claim: $\bigcup_{n=1}^{\infty} A_n = \mathbb{Z}^+$.
($\subseteq$) Let $l \in \bigcup_{n=1}^{\infty} A_n$. Then $\exists k \in \mathbb{Z}^+$ such that $l \in A_k$. Since $A_k = \{j \in \mathbb{Z}^+ | k \leq j \leq 2k\} \subseteq \mathbb{Z}^+$, so $l \in \mathbb{Z}^+$ and $\bigcup_{n=1}^{\infty} A_n \subseteq \mathbb{Z}^+$.
($\supseteq$) Let $n \in \mathbb{Z}^+$, then $n \in A_n$ and $n \in \bigcup_{n=1}^{\infty} A_n$. So $\mathbb{Z}^+ \subseteq \bigcup_{n=1}^{\infty} A_n$.
Hence, $\bigcup_{n=1}^{\infty} A_n = \mathbb{Z}^+$ as desired.

Claim: $\bigcap_{n=1}^{\infty} A_n = \emptyset$.
Since $\emptyset \subseteq \bigcap_{n=1}^{\infty} A_n$, we only need to show $\bigcap_{n=1}^{\infty} A_n \subseteq \emptyset$. By definition, $\bigcap_{n=1}^{\infty} A_n \subseteq A_1$ and $\bigcap_{n=1}^{\infty} A_n \subseteq A_3$. Then $\bigcap_{n=1}^{\infty} A_n \subseteq A_1 \cap A_3$. So $\bigcap_{n=1}^{\infty} A_n = \emptyset$ as desired.

Question 2

(i) Let $x_1, x_2 \in \mathbb{R} - \{2\}$ such that $f(x_1) = f(x_2)$. Then

$$\begin{aligned} f(x_1) &= f(x_2) \\ 1 + \frac{1}{x_1 - 2} &= 1 + \frac{1}{x_2 - 2} \\ x_1 - 2 &= x_2 - 2 \quad \text{since } x_1, x_2 \neq 2 \\ x_1 &= x_2. \end{aligned}$$

So f is injective as desired.

(ii) Claim: $R(f) = \mathbb{R} - \{1\}$.
Let $y \in R(f)$. Then $\exists a \in \mathbb{R} - \{2\}$ such that $f(x) = y$. Since $y = 1 + \frac{1}{a-2} \neq 1$, so $y \in \mathbb{R} - \{1\}$.
Let $y \in \mathbb{R} - \{1\}$. Take $a = 2 + \frac{1}{y-1}$, where $a \neq 2$. Then

$$\begin{aligned} f(a) &= 1 + \frac{1}{a - 2} \\ &= 1 + \frac{1}{2 + \frac{1}{y-1} - 2} \\ &= 1 + y - 1 \\ &= y. \end{aligned}$$

So $y \in R(f)$. Hence, $R(f) = \mathbb{R} - \{1\}$ as desired.

(iii) Claim: f is not invertible.
To show f is not invertible is equivalent to show f is not bijective. Since $R(f) = \mathbb{R} - \{1\} \neq \mathbb{R}$, f is not surjective and hence not bijective. So f is not invertible.

Question 3

(a)

$$\begin{aligned} f[X] &= \{f(x) | x \in X\} \\ &= \{f(-1), f(0), f(1)\} \\ &= \{0, 1\} \end{aligned}$$

$$\begin{aligned} f^{-1}[Y] &= \{x \in \mathbb{R} | f(x) \in Y\} \\ &= \{\pm 1, \pm 2\} \end{aligned}$$

(b) ($\subseteq$) Let $y \in g[\bigcup_{i \in I} C_i]$. Then $\exists x \in \bigcup_{i \in I} C_i$ such that $g(x) = y$. So $\exists i \in I$ such that $x \in C_i$ and $g(x) = y \in g[C_i]$. Hence, $y \in \bigcup_{i \in I} g[C_i]$.

($\supseteq$) Let $y \in \bigcup_{i \in I} g[C_i]$. Then $\exists i \in I$ such that $y \in g[C_i]$ and $\exists x \in C_i$ such that $g(x) = y$. So $x \in C_i$, $x \in \bigcup_{i \in I} C_i$ and $g(x) = y \in g[\bigcup_{i \in I} C_i]$.
Hence, $g[\bigcup_{i \in I} C_i] = \bigcup_{i \in I} g[C_i]$ as desired.

Question 4

Let $n \in \mathbb{Z}$, we want to show that $n(7n^2 + 5) = 6k$ for some $k \in \mathbb{Z}$. We know $n \equiv r \mod 6$ for some integer r with $0 \leq r < 6$. Consider all 6 cases, then

n	$n(7n^2 + 5)$
$0 \mod 6$	$0(7(0)^2 + 5) \equiv 0 \mod 6$
$1 \mod 6$	$1(7(1)^2 + 5) \equiv 12 \mod 6 \equiv 0 \mod 6$
$2 \mod 6$	$2(7(2)^2 + 5) \equiv 66 \mod 6 \equiv 0 \mod 6$
$3 \mod 6$	$3(7(3)^2 + 5) \equiv 6(34) \mod 6 \equiv 0 \mod 6$
$4 \mod 6$	$4(7(4)^2 + 5) \equiv 6(78) \mod 6 \equiv 0 \mod 6$
$5 \mod 6$	$5(7(5)^2 + 5) \equiv 36(25) \mod 6 \equiv 0 \mod 6.$

So $n(7n^2 + 5)$ is divisible by 6 as desired.

Question 5

(i)

$$\begin{aligned} 12378 &= 4 \times 3054 + 162 \\ 3054 &= 18 \times 162 + 138 \\ 162 &= 1 \times 138 + 24 \\ 138 &= 5 \times 24 + 18 \\ 24 &= 1 \times 18 + 6 \\ 18 &= 3 \times 6 + 0 \end{aligned}$$

So $\gcd(12378, 3054) = 6$ as desired.

(ii)

$$\begin{aligned}
6 &= 24 - 18 \\
&= 24 - (138 - 5 \times 24) \\
&= 6 \times 24 - 138 \\
&= 6 \times (162 - 138) - 138 \\
&= 6 \times 162 - 7 \times 138 \\
&= 6 \times 162 - 7 \times (3054 - 18 \times 162) \\
&= 132 \times 162 - 7 \times 3054 \\
&= 132 \times (12378 - 4 \times 3054) - 7 \times 3054 \\
&= 132 \times 12378 - 535 \times 3054
\end{aligned}$$

So $x = 132,\ y = -535$ as desired.

Question 6

(i) Reflexive: Let $(a, b) \in \mathbb{R}^2$. $b - a^3 = b - a^3 \Leftrightarrow (a, b) \sim (a, b)$.
Symmetric: Let $(a, b), (c, d) \in \mathbb{R}^2$ such that $(a, b) \sim (c, d)$, then

$$\begin{aligned}
b - a^3 &= d - c^3 \\
d - c^3 &= b - a^3 \\
(c, d) &\sim (a, b).
\end{aligned}$$

Transitive: Let $(a, b), (c, d), (e, f) \in \mathbb{R}^2$ such that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. We know $b - a^3 = d - c^3$ and $d - c^3 = f - e^3$, then $b - a^3 = f - e^3$. So $(a, b) \sim (e, f)$.
Hence, $\sim$ is an equivalence relation as desired.

(ii) By definition of the graph of a function, consider the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^3 + b - a^3$.

(iii) Each partition in the quotient set represents a solution curve $y = x^3 + b - a^3$ for some a, b. The quotient set consists of all these curves. Consider the function $g([(x, y)]) = b - a^3$, where $[(a, b)]$ is the partition in which (x, y) lies. By definition of a partition, g is injective and surjective, hence a bijection as desired.

Question 7

(i) To show two sets are equinumerous, it suffices to show there exists a bijection f between the 2 sets.
Consider the function $f : A \to A \times \{b_0\}$ such that $f(a) = (a, b_0)$.
Injective: Let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. Then $(a_1, b_0) = (a_2, b_0)$ and $a_1 = a_2$.
Surjective: Let $(a, b_0) \in A \times \{b_0\}$. Then by definition, for $a \in A$, $f(a) = (a, b)$.
So f is bijective. Hence $A \times \{b_0\}$ is equinumerous with A as desired.

(ii) Consider the set $\{b_0\} \subseteq B$, then $A \times \{b_0\} \subseteq A \times B$. Since A is uncountable, and from (i), $A \times \{b_0\}$ is uncountable, hence $A \times B$ is also uncountable as desired.