NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA2214 Combinatorial Analysis
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Question 1

(a) (i) For each natural number i, let a_i be the number of ways to place i identical balls into $n+1$ distinct boxes. The particular value a_n shall be counted in two different ways.

The first way involves placing the $n+1$ boxes beside each other and representing each of the n boundaries between boxes as 1s, while balls are represented as 0s. a_n is then the number of ways to arrange these n 0s with the n 1s in a straight line. The number of ways to do so is

$$\binom{2n}{n}.$$

The second way involves finding a suitable ordinary generating function for a_i and taking the coefficient of x^n as a_n. Since there are $n+1$ distinct boxes and each box can hold any number of identical balls (including none), the suitable ordinary generating function for a_i is

$$\begin{aligned}
& (1+x+x^2+x^3+\dots)^{n+1} \\
= & \frac{1}{(1-x)^{n+1}} \\
= & \frac{1}{(1-x)^n}\cdot\frac{1}{1-x} \\
= & \sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k\cdot\sum_{j=0}^{\infty}x^j.
\end{aligned}$$

a_n, which is the coefficient of x_n in the above ordinary generating function, is

$$\begin{aligned}
& \sum_{k=0}^{n}\left(\binom{n+k-1}{k}\cdot 1\right) \\
= & \sum_{k=0}^{n}\binom{n-1+k}{k}.
\end{aligned}$$

Hence,

$$\sum_{k=0}^{n}\binom{n-1+k}{k}=\binom{2n}{n}.$$

(ii) Observe that

$$\begin{aligned}
\frac{\binom{n}{k}}{\binom{2n-1}{k}} &= \frac{\frac{n!}{(n-k)!k!}}{\frac{(2n-1)!}{(2n-1-k)!k!}} \\
&= \frac{\frac{n!}{(n-k)!}}{\frac{(2n-1)!}{(2n-1-k)!}} \\
&= \frac{\frac{(2n-1-k)!}{(n-k)!}}{\frac{(2n-1)!}{n!}} \\
&= \frac{\frac{(2n-1-k)!}{(n-k)!(n-1)!}}{\frac{(2n-1)!}{n!(n-1)!}} \\
&= \frac{\binom{2n-1-k}{n-k}}{\binom{2n-1}{n}} \\
&= \frac{\binom{n-1+(n-k)}{(n-k)}}{\binom{2n-1}{n}}.
\end{aligned}$$

(Note: The above can actually be done with an application of a well known binomial identity $\binom{n}{m}\binom{m}{r} \equiv \binom{n}{r}\binom{n-r}{m-r}$.)

Therefore,

$$\begin{aligned}
\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{2n-1}{k}} &= \sum_{k=0}^{n} \frac{\binom{n-1+(n-k)}{(n-k)}}{\binom{2n-1}{n}} \\
&= \frac{1}{\binom{2n-1}{n}} \sum_{k=0}^{n} \binom{n-1+k}{k} \\
&= \frac{\binom{2n}{n}}{\binom{2n-1}{n}} \\
&= \frac{\frac{(2n)!}{n!n!}}{\frac{(2n-1)!}{n!(n-1)!}} \\
&= \frac{2n}{n} \\
&= 2.
\end{aligned}$$

(b) It is known that D_n, the derangement of n distinct items (more specifically, the number of ways to arrange integers 1 to n in a straight row such that each integer i is not at the ith position), is given by the formula

$$D_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!.$$

The following values of D_n will be useful for our subsequent solution:

- $D_2 = 1$;
- $D_3 = 2$;
- $D_4 = 9$;
- $D_5 = 44$;

- $D_6 = 265$;
- $D_8 = 14833$.

The problem at the carnival can now be broken into five different cases (note that it is not possible to have exactly 1 father and 7 mothers, or exactly 1 mother and 7 fathers, with their respective child, since the remaining father or mother respectively will have to be paired with his or her own child as well):

Case 1: (2 children have their father with them and the other 6 children have their mother with them). The number of ways to choose the 2 children having their fathers (and hence the other 6 children having their mothers), and then arranging the remaining unpaired 6 fathers and 2 mothers with the children such that these remaining parents do not end up with their child, is $\binom{8}{2}D_6D_2 = 7420$.

Case 2: (3 children have their father with them and the other 5 children have their mother with them). By a similar method of counting, the number of ways is $\binom{8}{3}D_5D_3 = 4928$.

Case 3: (4 children have their father with them and the other 4 children have their mother with them). By a similar method of counting, the number of ways is $\binom{8}{4}D_4D_4 = 5670$.

Case 4: (5 children have their father with them and the other 3 children have their mother with them). By a similar method of counting, the number of ways is $\binom{8}{5}D_3D_5 = 4928$.

Case 5: (6 children have their father with them and the other 2 children have their mother with them). By a similar method of counting, the number of ways is $\binom{8}{6}D_2D_6 = 7420$.

Case 6: (8 children have their father with them and the other 0 children have their mother with them). By a similar method of counting, the number of ways is $D_8 = 14833$.

Case 7: (0 children have their father with them and the other 8 children have their mother with them). By a similar method of counting, the number of ways is $D_8 = 14833$.

Hence the total number of ways is

$$7420 + 4928 + 5670 + 4928 + 7420 + 14833 + 14833 = 60032.$$

Question 2

(a) If the 6 boxes were distinct, then the number of ways to arrange the 30 distinct objects into them is simply 6^{30}. Since the boxes are identical, every bunch of 6! ways to arrange the items are repeated among themselves. Hence, the desired number of ways is

$$\frac{6^{30}}{6!}.$$

(b) The first method to count $|T|$ is to consider the number of ways to assign values to (a, b, c, d, e) for each particular value of f, from 2 to $n + 1$ (there is no way to assign values to them when $f = 1$). For example, when f is 2, a, b, c, d, e can each hold only the value of 1, hence there is 1^5 way to assign values to (a, b, c, d, e). When f is 3, a, b, c, d, e can each hold either 1 or 2, hence there are 2^5 ways to assign values to (a, b, c, d, e). In general, when f is $r + 1$ where $1 \le r + 1 \le n + 1$, a, b, c, d, e can each hold integers from 1 to r, and hence there are r^5 ways to assign values to (a, b, c, d, e).

Therefore, by considering each value of f from 1 to $n+1$,

$$\begin{aligned} |T| &= 1^5 + 2^5 + \ldots + n^5 \\ &= \sum_{r=1}^{n} r^5. \end{aligned}$$

The second method to count $|T|$ is much more involved. When $n \geq 6$, there can be 2, 3, 4, 5 or 6 distinct integers amongst a, b, c, d, e and f. Considering the 5 different cases involved:

Case 1: There are 2 distinct integers amongst a, b, c, d, e and f. Let them be x_1 and x_2, where $x_1 < x_2$. There are $\binom{n+1}{2}$ ways to select them. By the definition of T, $f = x_2$. a, b, c, d and e are all x_1. Hence, the number of ways for this case is $\binom{n+1}{2}$.

Case 2: There are 3 distinct integers amongst a, b, c, d, e and f. Let them be x_1, x_2 and x_3, where $x_1 < x_2 < x_3$. There are $\binom{n+1}{3}$ ways to select them. By the definition of T, $f = x_3$. a, b, c, d and e can each be x_1 or x_2, of which there needs to be at least 1 x_1 and 1 x_2. Hence, the number of ways for this case is $\binom{n+1}{3}(2^5 - 2) = 30\binom{n+1}{3}$.

Case 3: There are 4 distinct integers amongst a, b, c, d, e and f. Let them be x_1, x_2, x_3 and x_4, where $x_1 < x_2 < x_3 < x_4$. There are $\binom{n+1}{4}$ ways to select them. By the definition of T, $f = x_4$. a, b, c, d and e can each be x_1, x_2 or x_3, of which there must be at least 1 of x_1 , 1 of x_2 and 1 of x_3. Hence the number of ways for this case is $\binom{n+1}{4}(\binom{3}{1} \times \frac{5!}{3!} + \binom{3}{2} \times \frac{5!}{2!2!}) = 150\binom{n+1}{4}$.

Case 4: There are 5 distinct integers amongst a, b, c, d, e and f. Let them be x_1, x_2, x_3, x_4 and x_5, where $x_1 < x_2 < x_3 < x_4 < x_5$. There are $\binom{n+1}{5}$ ways to select them. By the definition of T, $f = x_5$. a, b, c, d and e can each be x_1, x_2, x_3 or x_4, of which there must be at least 1 of x_1 , 1 of x_2, 1 of x_3 and 1 of x_4 (the remaining one being any one of x_1, x_2, x_3 or x_4). Hence the number of ways for this case is $\binom{n+1}{5}(\binom{4}{1} \times \frac{5!}{2!}) = 240\binom{n+1}{5}$.

Case 5: There are 6 distinct integers amongst a, b, c, d, e and f. Let them be x_1, x_2, x_3, x_4, x_5 and x_6, where $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$. There are $\binom{n+1}{6}$ ways to select them. By the definition of T, $f = x_6$. There is exactly 1 of x_1, x_2, x_3, x_4 and x_5 amongst a, b, c, d and e. Hence the number of ways for this case is $5!\binom{n+1}{6} = 120\binom{n+1}{6}$.

By considering all 5 cases above and by using the representation that $\binom{n}{k} = 0$ if $k > n$,

$$|T| = \binom{n+1}{2} + 30\binom{n+1}{3} + 150\binom{n+1}{4} + 240\binom{n+1}{5} + 120\binom{n+1}{6}, \quad n \geq 1.$$

And therefore,

$$\sum_{r=1}^{n} r^5 = \binom{n+1}{2} + 30\binom{n+1}{3} + 150\binom{n+1}{4} + 240\binom{n+1}{5} + 120\binom{n+1}{6}, \quad n \geq 1.$$

Question 3

(a) Arranged in alphabetical order, the letters in concern are ACCEEIILMMNNOOTTU. The Principle of Inclusion and Exclusion shall be used to solve this question.

Let

- A be the set of all possible 17-letter permutations of the 17 letters above;
- P_1, P_2, P_3, P_4 be the properties that the permutation contains the blocks CAT, TEL, MUM and CNN respectively;
- $E(m)$ be the number of elements of S possessing exactly m of the 4 properties for $0 \leq m \leq 4$;

- $\omega(P_{i_1}P_{i_2}\dots P_{i_m})$ be the number of elements of A possessing the properties P_{i_1}, P_{i_2}, $\dots$, P_{i_m}, where $1 \leq m \leq 4$;
- $\omega(m) = \sum(\omega(P_{i_1}P_{i_2}\dots P_{i_m}))$, $\omega(0) = |A|$.

By arranging the respective block/s with the remaining letters left not in the block/s, taking note of repetition,

- $\omega(P_1) = \dfrac{15!}{(2!)^5}$;
- $\omega(P_2) = \dfrac{15!}{(2!)^5}$;
- $\omega(P_3) = \dfrac{15!}{(2!)^6}$;
- $\omega(P_4) = \dfrac{15!}{(2!)^5}$;
- $\omega(P_1P_2) = \dfrac{13!}{(2!)^4} + \dfrac{13!}{(2!)^4}$ (noting CAT and TEL blocks, as well as CATEL blocks);
- $\omega(P_1P_3) = \dfrac{13!}{(2!)^4}$;
- $\omega(P_1P_4) = \dfrac{13!}{(2!)^4}$;
- $\omega(P_2P_3) = \dfrac{13!}{(2!)^4}$;
- $\omega(P_2P_4) = \dfrac{13!}{(2!)^3}$;
- $\omega(P_3P_4) = \dfrac{13!}{(2!)^4}$;
- $\omega(P_1P_2P_3) = \dfrac{11!}{(2!)^3} + \dfrac{11!}{(2!)^3}$ (noting CAT and TEL blocks, as well as CATEL blocks);
- $\omega(P_1P_2P_4) = \dfrac{11!}{(2!)^3} + \dfrac{11!}{(2!)^3}$ (noting CAT and TEL blocks, as well as CATEL blocks);
- $\omega(P_1P_3P_4) = \dfrac{11!}{(2!)^3}$;
- $\omega(P_2P_3P_4) = \dfrac{11!}{(2!)^2}$;
- $\omega(P_1P_2P_3P_4) = \dfrac{9!}{(2!)^2} + \dfrac{9!}{(2!)^2}$ (noting CAT and TEL blocks, as well as CATEL blocks).

Hence (correct to the number of significant figures the calculator can display),

- $\omega(0) = |S| = \dfrac{(17)!}{(2!)^7} = 2.778808032 \times 10^{12}$;
- $\omega(1) = \omega(P_1) + \omega(P_2) + \omega(P_3) + \omega(P_4) = 1.43026884 \times 10^{11}$;
- $\omega(2) = \omega(P_1P_2) + \omega(P_1P_3) + \omega(P_1P_4) + \omega(P_2P_3) + \omega(P_2P_4) + \omega(P_3P_4) = 3113510400$;
- $\omega(3) = \omega(P_1P_2P_3) + \omega(P_1P_2P_4) + \omega(P_1P_3P_4) + \omega(P_2P_3P_4) = 34927200$;
- $\omega(4) = \omega(P_1P_2P_3P_4) = 181440$.

And therefore, by the Principle of Inclusion and Exclusion,

(i) Number of elements in A which do not contain any of the four blocks is

$$\begin{aligned} & E(0) \\ = \; & \omega(0) - \omega(1) + \omega(2) - \omega(3) + \omega(4) \\ = \; & 2.638859913 \times 10^{12}. \end{aligned}$$

(ii) Number of elements in A which contains exactly one of the four blocks is

$$\begin{aligned} & E(1) \\ = \; & \omega(1) - 2\omega(2) + 3\omega(3) - 4\omega(4) \\ = \; & 1.36903919 \times 10^{11}. \end{aligned}$$

(iii) Number of elements in A which contains exactly two of the four blocks is

$$\begin{aligned} & E(2) \\ = \; & \omega(2) - \binom{3}{2}\omega(3) + \binom{4}{2}\omega(4) \\ = \; & 3009817440. \end{aligned}$$

(iv) Number of elements in A which contains exactly three of the four blocks is

$$\begin{aligned} & E(3) \\ = \; & \omega(3) - \binom{4}{3}\omega(4) \\ = \; & 34201440. \end{aligned}$$

(v) Number of elements in A which contains all the four blocks is

$$\begin{aligned} & E(4) \\ = \; & \omega(4) \\ = \; & 181440. \end{aligned}$$

(b) Let $A = \{1, 2, 3, \ldots, n-9\}$, where $n \geq 10$ and let

$$B = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in A^6 \mid y_6 \geq y_5 \geq y_4 \geq y_3 \geq y_2 \geq y_1\}.$$

By letting $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3 - 2$, $y_4 = x_4 - 5$, $y_5 = x_5 - 9$ and $y_6 = x_6 - 9$, it can easily be observed that B and T have a bijective relationship, since an inverse relationship is easily constructed by making x_1 to x_6 be the subject of the six equations above. As such, $|T| = |B|$ and hence it suffices to find $|B|$.

$|B|$ is the number of ways to arrange 6 identical objects (representing y_1 to y_6) into $n-9$ distinct boxes labeled 1 to $n-9$ (representing the $n-9$ numbers in A). The object in the smallest numbered box will be y_1, the second smallest y_2 and so on, and identical objects in the same box can be interchangeably labeled with no difference whatsoever. Hence, $|B|$ is the number of ways to arrange the 6 identical objects with the $n-10$ barriers between the boxes, and therefore,

$$|T| = |B| = \binom{n-10+6}{6} = \binom{n-4}{6}.$$

Question 4

(a) (i) A suitable exponential generating function for a_n, based on the stated criterion, is

$$\begin{aligned}
& (e^x)\left(\frac{e^{4x}+e^{-4x}}{2}\right)\left(e^{4x}-1-4x-\frac{(4x)^2}{2!}\right) \\
= \;& \frac{1}{2}(e^{5x}+e^{-3x})(e^{4x}-1-4x-8x^2) \\
= \;& \frac{1}{2}(e^{9x}+e^{x}-e^{5x}-e^{-3x})-2x(e^{5x}+e^{-3x})-4x^2(e^{5x}+e^{-3x}).
\end{aligned}$$

(ii) Note that for any positive integer a, the x^n term of the expansion of e^{ax} is $\frac{a^n}{n!}$. Since a^n is $n!$ times the coefficient of x^n in the above exponential generating function,

$$a_n = \frac{1}{2}(9^n+1-5^n-(-3)^n)-2n(5^{n-1}+(-3)^{n-1})-4n(n-1)(5^{n-2}+(-3)^{n-2}).$$

(iii) By using the above expression,

$$\begin{aligned}
a_8 &= \frac{1}{2}(9^8+1-5^8-(-3)^8)-16(5^7+(-3)^7)-224(5^6+(-3)^6) \\
&= 16446464.
\end{aligned}$$

(b) (i) A suitable ordinary generating function for b_n, based on the stated criterion, is

$$\begin{aligned}
& (1-x)^{-1}(1-x^2)^{-4}\left(\frac{(1-x)^{-4}-(1+x)^{-4}}{2}\right) \\
= \;& \frac{1}{2}(1-x)^{-5}(1+x)^{-4}((1-x)^{-4}-(1+x)^{-4}) \\
= \;& \frac{1}{2}(1-x)^{-9}(1+x)^{-4}-\frac{1}{2}(1-x)^{-5}(1+x)^{-8} \\
= \;& \frac{1}{2}\sum_{k=0}^{\infty}\binom{k+8}{8}x^k\sum_{r=0}^{\infty}\binom{r+3}{3}(-1)^r x^r-\frac{1}{2}\sum_{k=0}^{\infty}\binom{k+4}{4}x^k\sum_{r=0}^{\infty}\binom{r+7}{7}(-1)^r x^r.
\end{aligned}$$

(ii) b_n is the coefficient of x^n in the above generating function. Hence,

$$\begin{aligned}
b_n &= \frac{1}{2}\sum_{r=0}^{n}\binom{r+3}{3}(-1)^r\binom{n-r+8}{8}-\frac{1}{2}\sum_{r=0}^{n}\binom{r+7}{7}(-1)^r\binom{n-r+4}{4} \\
&= \frac{1}{2}\sum_{r=0}^{n}(-1)^r\left(\binom{r+3}{3}\binom{n-r+8}{8}-\binom{r+7}{7}\binom{n-r+4}{4}\right).
\end{aligned}$$

(iii) By using the above generating function,

$$\begin{aligned}
b_6 &= \frac{1}{2}\sum_{r=0}^{6}(-1)^r\left(\binom{r+3}{3}\binom{14-r}{8}-\binom{r+7}{7}\binom{10-r}{4}\right) \\
&= \frac{1}{2}\Bigg(\binom{14}{8}-\binom{10}{4}-\binom{4}{3}\binom{13}{8}+\binom{8}{7}\binom{9}{4}+\binom{5}{3}\binom{12}{8}-\binom{9}{7}\binom{8}{4}-\binom{6}{3}\binom{11}{8} \\
&\quad +\binom{10}{7}\binom{7}{4}+\binom{7}{3}\binom{10}{8}-\binom{11}{7}\binom{6}{4}-\binom{8}{3}\binom{9}{8}+\binom{12}{7}\binom{5}{4}+\binom{9}{3}-\binom{13}{7}\Bigg) \\
&= 216.
\end{aligned}$$

Question 5

(a) (i) Among the n-digit integers counted in a_n, let b_n be the number of those that end with 2, and let c_n be the number of those that end with 3. Note that in general $b_n \neq c_n$.

Constructing 3 recurrence relations between a_n, b_n and c_n,

$$\begin{aligned} a_n &= 4a_{n-1} + b_n + c_n; & (1)\\ b_n &= 3a_{n-2} + b_{n-1} + c_{n-1}; & (2)\\ c_n &= 4a_{n-2} + c_{n-1}. & (3) \end{aligned}$$

The RHS of (1) represents the quantity of relevant numbers that ends with 0, 1, 4 or 5 ($4a_{n-1}$), that ends with 2 (b_n), and that ends with 3 (c_n). The RHS of (2) represents the quantity of relevant numbers that ends with 02, 42 or 52 ($3a_{n-2}$), that ends with 22 (b_{n-1}) and that ends with 32 (c_{n-1}). The RHS of (3) represents the quantity of relevant numbers that ends with 03, 13, 43 or 53 ($4a_{n-2}$) and that ends with 33 (c_{n-1}).

Note that the first digit may not be 0. $a_1 = 5$ since a single digit number can be 1 to 5. Similarly, $b_2 = c_2 = 4$. By (1), $a_2 = 28$. By manual counting, $b_3 = 23$ and $c_3 = 24$. By (1) again $a_3 = 159$.

From (1),

$$b_n = a_n - 4a_{n-1} - c_n. \qquad (4)$$

Substitute (4) into (2),

$$\begin{aligned} a_n - 4a_{n-1} - c_n &= 3a_{n-2} + a_{n-1} - 4a_{n-2} - c_{n-1} + c_{n-1}\\ c_n &= a_n - 5a_{n-1} + a_{n-2}. & (5) \end{aligned}$$

Substitute (5) into (3),

$$\begin{aligned} a_n - 5a_{n-1} + a_{n-2} &= 4a_{n-2} + a_{n-1} - 5a_{n-2} + a_{n-3}\\ a_n &= 6a_{n-1} - 2a_{n-2} + a_{n-3}. \end{aligned}$$

where $a_1 = 5$, $a_2 = 28$ and $a_3 = 159$.

(ii) Using the above recurrence relation,

$$\begin{aligned} a_4 &= 6a_3 - 2a_2 + a_1\\ &= 903.\\ a_5 &= 6a_4 - 2a_3 + a_2\\ &= 5128.\\ a_6 &= 6a_5 - 2a_4 + a_3\\ &= 29121. \end{aligned}$$

(b) Observe that among the relevant numbers, the first digit can be 4, 5, 6, 7, 8, or 9, the last digit can be 1, 3, 5, 7, or 9, and the first and last digit must be different. If the first digit is odd, there are 4 ways to choose the last digit. If the first digit is even, there are 5 ways to choose the last digit. Hence $b_2 = 3 \times 4 + 3 \times 5 = 27$.

For a 3 digit number satisfying the conditions, there are 27 ways to choose the first and last digit. Since the first and last digit will be different, there are 8 ways to choose the 2nd digit so that the 2nd digit will be different from the first and last digit. Hence $b_3 = 27 \times 8 = 216$.

For an n digit number where $n > 3$, consider the 3rd last digit with the last digit. If these two digits are different, then the number of ways to choose the 2nd last digit is 8, and the number of ways to choose the rest of the digits is b_{n-1}. If these two digits are the same, then the number of ways to choose the 2nd last digit is 9, and the number of ways to choose the rest of the digits is b_{n-2}. Hence a recurrence relation for b_n is

$$b_n = 8b_{n-1} + 9b_{n-2}.$$

where $b_2 = 27$ and $b_3 = 216$.

By using the above recurrence relation,

$$\begin{aligned}
b_4 &= 8b_3 + 9b_2 \\
&= 1971. \\
b_5 &= 8b_4 + 9b_3 \\
&= 17712. \\
b_6 &= 8b_5 + 9b_4 \\
&= 159435.
\end{aligned}$$