MA2101S - Linear Algebra II(S) Suggested Solutions

(Semester 2 : AY2015/16)

Written by : Pan Jing Bin
Audited by : Chong Jing Quan

Question 1

(a) First note that since $A^{Re},\ A^{Im},\ B$ and C are matrices with real coefficients, $A^{Re}B,\ CA^{Re},\ A^{Im}B,\ CA^{Im} \in M_{n\times n}(\mathbb{R})$.

$$\begin{aligned} AB = CA &\iff (A^{Re} + iA^{Im})B = C(A^{Re} + iA^{Im}) \\ &\iff A^{Re}B + iA^{Im}B = CA^{Re} + iCA^{Im} \\ &\iff A^{Re}B = CA^{Re} \ \wedge\ A^{Im}B = CA^{Im} - \text{(By comparing real and imaginary parts)} \end{aligned}$$

(b) Since A is invertible, $\det(A^{Re} + iA^{Im}) \neq 0$. Thus $p(i) \neq 0$ so p is not the zero polynomial.

$p(x)$ is a non-zero polynomial of degree n so it can have at most n roots. $\exists c \in \mathbb{R}$ such that $p(c) \neq 0$. Then $\det(A^{Re} + cA^{Im}) \neq 0$ so $A^{Re} + cA^{Im}$ is invertible.

(c) To prove 'only if' :

Since $P \in M_{n\times n}(\mathbb{R}),\ P \in M_{n\times n}(\mathbb{C})$. Simply choose $Q = P$ and we have $QBQ^{-1} = C$.

To prove 'if' :

$QBQ^{-1} = C \to QB = CQ$. By (a), $Q^{Re}B = CQ^{Re} \ \wedge\ Q^{Im}B = CQ^{Im}$.

By (b), $\exists d \in \mathbb{R}$ such that $Q^{Re} + dQ^{Im}$ is an invertible matrix with real coefficients. Then we have

$$Q^{Re}B + dQ^{Im}B = CQ^{Re} + dCQ^{Im} \to (Q^{Re} + dQ^{Im})B = C(Q^{Re} + dQ^{Im}).$$

Thus $(Q^{Re} + dQ^{Im})B(Q^{Re} + dQ^{Im})^{-1} = C$. Choose $P = Q^{Re} + dQ^{Im}$. Then $P \in M_n(\mathbb{R})$ and the proof is complete.

Question 2

Let K be a basis for V and let A and B be the standard matrix of α and β with respect to basis K. (Since V is finite dimensional)

(a)(i) False. Counterexample:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}, A + B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $c_A(x) = x^2 - x - 1 = (x - \frac{1+\sqrt{5}}{2})(x - \frac{1-\sqrt{5}}{2}),\ c_B(x) = x(x-1)$. Since $c_A(x)$ and $c_B(x)$ have no repeated factors, A and B are diagonalisable. Notice that $(A+B)^T$ is a $J_2(0)$ Jordan block, which is not diagonalisable. Thus $A + B$ is not diagonalisable.

(ii) False. Counterexample:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$c_A(x) = x^2 - x - 1 = (x - \frac{1+\sqrt{5}}{2})(x - \frac{1-\sqrt{5}}{2})$. Since $c_A(x)$ have no repeated factors, A is diagonalisable. B is obviously diagonalisable since it is a diagonal matrix. AB is obviously not diagonalisable since it is a $J_2(0)$ Jordan block.

(iii) True. Consider the polynomial $f(x) = x^2 - x = x(x-1)$. Since $f(\alpha) = 0_V$, $m_\alpha(x) \mid f(x)$ by definition of minimal polynomial. But $f(x)$ has no repeated factors so $m_\alpha(x)$ has no repeated factors as well. Thus α is diagonalisable.

(b) Let $\lambda_1, \lambda_2, ...\lambda_n$ be the eigenvalues of α^2.

$$m_{\alpha^2}(x) = (x - \lambda_1)(x - \lambda_2)...(x - \lambda_n).$$

Then $(\alpha^2 - \lambda_1)(\alpha^2 - \lambda_2)...(\alpha^2 - \lambda_n) = 0_V$. Since F is algebraically closed, $\sqrt{\lambda_i}$ exist $\forall\ 1 \leq i \leq n$.

$(\alpha + \sqrt{\lambda_1})(\alpha - \sqrt{\lambda_1})(\alpha + \sqrt{\lambda_2})(\alpha - \sqrt{\lambda_2})...(\alpha - \sqrt{\lambda_n}) = 0_V$. By definition of minimal polynomial, $m_\alpha(x) \mid (x+\sqrt{\lambda_1})(x-\sqrt{\lambda_1})(x+\sqrt{\lambda_2})(x-\sqrt{\lambda_2})...(x-\sqrt{\lambda_n})$. Since α is bijective, $\lambda_i \neq 0\ \forall\ 1 \leq i \leq n$. Thus $m_\alpha(x)$ has no repeated factors so it is diagonalisable.

Question 3

(a) Remark: The statement is trivial when V is a one-dimensional vector space. Thus we only consider the case where $\dim(V) > 1$. Assume that $\exists z \in V$ such that $\beta(z) \neq \lambda z\ \forall\ \lambda \in \mathbb{R}$. Write $\beta(z) = p + \lambda z$ for some $\lambda \in \mathbb{R}$ and p is orthogonal to z. Note that since z is not an eigenvalue of β, $p \neq 0_V$. Then

$$\begin{aligned}\phi(p, z) = 0 &\to \phi(\alpha(p), \alpha(z)) = 0 \\ &\to \phi(p, \beta(z)) = 0 \\ &\to \phi(p, p + \lambda z) = 0.\end{aligned}$$

But $\phi(p, p + \lambda z) = \phi(p, p) + \phi(p, \lambda z) = \phi(p, p) > 0$ since ϕ is positive definite. Thus there is a contradiction so the assumption is false and every nonzero vector $v \in V$ is an eigenvector of β.

(a)(ii) Claim 1: β only has one eigenvalue, λ.

Assume that β has more than 1 eigenvalue. Let λ_1 and λ_2 be 2 distinct eigenvalues of β. Then $\exists v_1, v_2 \in V$ such that $\beta(v_1) = \lambda_1 v_1\ \wedge\ \beta(v_2) = \lambda_2 v_2$.

But then $\beta(v_1 + v_2) = \lambda_1 v_1 + \lambda_2 v_2$. Since v_1 and v_2 are linearly independent and $\lambda_1 \neq \lambda_2, v_1 + v_2$ and $\lambda_1 v_1 + \lambda_2 v_2$ are linearly independent vectors. In other words, $v_1 + v_2$ is not an eigenvector of β, which contradicts (a)(i). Thus the assumption is false and β only has one eigenvalue.

Claim 2: $\lambda \geq 0$.

Assume $\lambda < 0$. First note that if $\lambda < 0$, the β is not the zero operator and so α cannot be the zero operator. $\exists w \in V$ such that $\alpha(w) \neq 0_V$.

$\phi(\alpha(w), \alpha(w)) > 0$ since ϕ is positive definite.

Thus $\phi(\beta(w), w) > 0 \to \lambda\phi(w, w) > 0 \to \phi(w, w) < 0$ since λ is negative. This is a contradiction as ϕ is positive definite so the assumption is false and $\lambda \geq 0$.

Obviously β is diagonalisable. Thus $\ker(\lambda I_V - \beta) = V$ (Since β only has one eigenvalue) so $\lambda I_V - \beta = 0_V$. We thus conclude that $\beta = \lambda I_V$ for some $\lambda \geq 0$.

(b) Consider 2 cases:

Case 1: $\beta = 0_V$.

Claim: $\ker(\beta) = \ker(\alpha)$.

Obviously $\ker \alpha \subseteq \ker \beta$. To prove $\ker \beta \subseteq \ker \alpha$:

Let $k \in \ker(\beta)$. Then $\phi(\beta(k), k) = 0 \to \phi(\alpha(k), \alpha(k)) = 0 \to \alpha(k) = 0_V$. Thus $k \in \ker(\alpha)$ so $\ker(\beta) \subseteq \ker(\alpha)$.

Thus $\beta = 0_V \to \alpha = 0_V$ so simply choose $k = 0$ and any orthogonal matrix P will suffice.

Case 2: $\beta \neq 0$.

Let B be an orthonormal basis for V with respect to ϕ. (Since ϕ is positive definite, such a basis exists) Let $A = [\alpha]_B$. Then $[\alpha^\star]_B = A^T$. Note that

$$\begin{aligned}[\beta]_B = \lambda I &\to [\alpha^\star]_B[\alpha]_B = \lambda I \\ &\to A^T A = \lambda I.\end{aligned}$$

Since $\lambda > 0$, $\sqrt{\lambda}$ exists. We have

$$(\frac{1}{\sqrt{\lambda}}A^T)(\frac{1}{\sqrt{\lambda}}A) = I \to (\frac{1}{\sqrt{\lambda}}A)^T(\frac{1}{\sqrt{\lambda}}A) = I.$$

Thus $\frac{1}{\sqrt{\lambda}}A$ is our orthogonal matrix. Choose B to be our basis and $[\alpha]_B = \sqrt{\lambda}(\frac{1}{\sqrt{\lambda}}A)$.

Question 4

(a) To prove existence:

Since λ is not an eigenvalue of T, $\gcd(x-\lambda, m_T(x)) = 1$. $\exists a(x), b(x) \in F[x]$ such that

$$(x-\lambda)a(x) + b(x)m_T(x) = 1.$$

Then $(T - \lambda I_V) \circ a(T) + b(T) \circ m_T(T) = I_V$.

Since $m_T(T) = 0_V$, $(T - \lambda I_V) \circ a(T) = I_V$.

If $\deg(a(x)) < k$, then choose $p_\lambda(x) = a(x)$ and the proof is complete. If $\deg(a(x)) \geq k$, perform the euclidean algorithm:

$$\begin{gathered}\exists q(x), r(x) \in F[x], \text{with } \deg(r(x)) < k, \text{ such that: } a(x) - q(x)m_T(x) = r(x). \\ a(T) - q(T)m_T(T) = r(T) \to a(T) = r(T) \to (T - \lambda I_V) \circ r(T) = I_V.\end{gathered}$$

Choose $p_\lambda(x) = r(x)$ and the proof is complete.

To prove uniqueness:

Let $p_\lambda(x), q_\lambda(x) \in F[x]$ be 2 polynomials of degree less than k satisfying:

$$(T - \lambda I_V) \circ p_\lambda(T) = I_V = (T - \lambda I_V) \circ q_\lambda(T).$$

Since λ is not an eigenvalue of $T, T - \lambda I_V$ is invertible. Thus:

$$p_\lambda(T) = q_\lambda(T) \to p_\lambda(T) - q_\lambda(T) = 0_V.$$

Recall that $p_\lambda(x)$ and $q_\lambda(x)$ are polynomials with degree less than k. Thus $\deg(p_\lambda(x) - q_\lambda(x)) < k$. By definition of minimial polynomial, $p_\lambda(x) - q_\lambda(x) = 0$ (Since $p_\lambda(T) - q_\lambda(T) = 0_V$) so $p_\lambda(x) = q_\lambda(x)$.

To prove $\deg(p_\lambda(x)) = k - 1$:

Assume $\deg(p_\lambda(x)) < k - 1$. Then $\deg((x-\lambda)p_\lambda(x) - 1) < k$. Note that $(x-\lambda)p_\lambda(x) - 1 \neq 0$.

But $(T - \lambda I_V)p_\lambda(T) - I_V = 0_V$, which contradicts the fact that the minimal polynomial has degree k. Thus the assumption is false and $\deg(p_\lambda(x)) \geq k-1$. Since $\deg(p_\lambda(x)) < k$, $\deg(p_\lambda(x)) = k-1$.

(b)(i) Let $f_i(x) = \frac{\prod_{j=1}^{k}(x-\lambda_j)}{x-\lambda_i}$ $\forall 1 \leq i \leq k$. Note that each $f_i(x)$ is a polynomial of degree $k-1$.

Claim: The set $\{f_1(x), f_2(x), ..., f_k(x)\}$ is linearly independent.

Proof: Consider the homogeneous equation

$$\begin{aligned} b_1 f_1(x) + b_2 f_2(x) + ... + b_k f_k(x) &= 0 \\ b_1 f_1(\lambda_1) + b_2 f_2(\lambda_1) + ... + b_k f_k(\lambda_1) &= 0 \\ b_1 f_1(\lambda_1) + 0 + ... + 0 &= 0 \\ b_1 f_1(\lambda_1) &= 0. \end{aligned}$$

Since each λ_i is distinct, $f_1(\lambda_1) \neq 0$ so $b_1 = 0$.

Repeat the algorithm for $\lambda_2, \lambda_3, ...\lambda_k$ and we get: $b_1 = b_2 = ... = b_k = 0$. Hence the homogeneous equation only has the trivial solution so the set $\{f_1(x), f_2(2), ..., f_k(x)\}$ is linearly independent.

Observe that

$$\sum_{i=1}^{k} c_i p_{\lambda_i}(x) = 0$$

$$(\prod_{j=1}^{k}(x-\lambda_j))(\sum_{i=1}^{k} c_i p_{\lambda_i}(x)) = 0$$

$$(\prod_{j=1}^{k}(T-\lambda_j I_V))(\sum_{i=1}^{k} c_i p_{\lambda_i}(T)) = 0_V$$

$$\sum_{i=1}^{k} c_i f_i(T) = 0_V.$$

Since each $f_i(x)$ is a polynomial of degree $k-1$, $\sum_{i=1}^{k} c_i f_i(x)$ is also a polynomial of degree $k-1$. Thus by definition of minimial polynomial

$$\sum_{i=1}^{k} c_i f_i(T) = 0 \to \sum_{i=1}^{k} c_i f_i(x) = 0.$$

Since $\{f_1(x), f_2(x), ..., f_k(x)\}$ is a linearly independent set, $c_i = 0 \ \forall 1 \leq i \leq k$.

Thus only the trivial solution exists to the homogeneous equation $\sum_{i=1}^{k} c_i p_{\lambda_i}(x) = 0$ hence each $p_{\lambda_i}(x)$ is linearly independent in $F[x]$.

(ii) To prove existence: Let P_{k-1} denote the vector space consisting of the zero polynomial and the polynomials of degree at most $k-1$. It is easy to check that $\{1, x, x^2, ..., x^{k-1}\}$ is a basis for P_{k-1}. Thus $\dim(P_{k-1}) = k$.

Let S $=$ span$\{p_{\lambda_1}(x), p_{\lambda_2}(x), ..., p_{\lambda_k}(x)\}$. Then obviously $S \subseteq P_{k-1}$.
Since $\dim(S) = k = \dim(P_{k-1})$, $S = P_{k-1}$. Thus $\exists c_1, c_2, ..., c_k \in \mathbb{F}$ such that:

$$\textstyle\sum_{i=1}^{k} c_i p_{\lambda_i}(x) = 1.$$

Then $\sum_{i=1}^{k} c_i p_{\lambda_i}(T) = I_V$ and the proof is complete.

To prove uniqueness: Let $\sum_{i=1}^{k} c_i p_{\lambda_i}(x)$ and $\sum_{i=1}^{k} d_i p_{\lambda_i}(x)$ be two polynomials satisfying:

$$\textstyle\sum_{i=1}^{k} c_i p_{\lambda_i}(T) = I_V = \sum_{i=1}^{k} d_i p_{\lambda_i}(T).$$

Then $\sum_{i=1}^{k} c_i p_{\lambda_i}(T) - \sum_{i=1}^{k} d_i p_{\lambda_i}(T) = 0_V$. Keep in mind that $\sum_{i=1}^{k} c_i p_{\lambda_i}(x) - \sum_{i=1}^{k} d_i p_{\lambda_i}(x)$ is a polynomial of degree less than k. Thus by definition of minimal polynomial:

$$\sum_{i=1}^{k} c_i p_{\lambda_i}(x) - \sum_{i=1}^{k} d_i p_{\lambda_i}(x) = 0.$$

$$\sum_{i=1}^{k}(c_i - d_i) p_{\lambda_i}(x) = 0.$$

By linear independence of $p_{\lambda_1}(x), p_{\lambda_2}(x), ..., p_{\lambda_k}(x)$, $c_i - d_i = 0 \ \forall 1 \leq i \leq k$.

Thus $\forall 1 \leq i \leq k$, $c_i = d_i$ so we conclude that $\sum_{i=1}^{k} c_i p_{\lambda_i}(x) = \sum_{i=1}^{k} d_i p_{\lambda_i}(x)$.