NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Teo Wei Hao

MA1104 Multivariable Calculus
AY 2006/2007 Sem 1

Question 1

(i) To get a unit vector perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$, we use

$$\begin{aligned}\frac{\boldsymbol{u}\times\boldsymbol{v}}{|\boldsymbol{u}\times\boldsymbol{v}|} &= \frac{\langle(1)(-7)-(5)(4),(4)(3)-(-7)(-2),(-2)(5)-(3)(1)\rangle}{|\langle(1)(-7)-(5)(4),(4)(3)-(-7)(-2),(-2)(5)-(3)(1)\rangle|}\\ &= \frac{\langle -27,-2,-13\rangle}{\sqrt{27^2+2^2+13^2}}\\ &= \frac{1}{\sqrt{902}}\langle -27,-2,-13\rangle.\end{aligned}$$

(ii) Volume of the parallelepiped is

$$\begin{aligned}|\boldsymbol{w}\cdot(\boldsymbol{u}\times\boldsymbol{v})| &= |\langle 1,6,2\rangle\cdot\langle 27,2,13\rangle|\\ &= |27+12+26| = 65.\end{aligned}$$

Question 2

(i) We have velocity of $\boldsymbol{r}(t)$ to be

$$\begin{aligned}\boldsymbol{r}'(t) &= \left\langle \frac{d}{dt}(5t), \frac{d}{dt}(12\sin t), \frac{d}{dt}(12\cos t)\right\rangle\\ &= \langle 5, 12\cos t, -12\sin t\rangle.\end{aligned}$$

(ii) We have speed of $\boldsymbol{r}(t)$ to be

$$\begin{aligned}|\boldsymbol{r}'(t)| &= \sqrt{5^2+(12\cos t)^2+(-12\sin t)^2}\\ &= 13.\end{aligned}$$

(iii) Let the length in question be L. Since $\boldsymbol{r}(2\pi) = (10\pi, 0, 12)$ and $\boldsymbol{r}(3\pi) = (15\pi, 0, -12)$, we have,

$$\begin{aligned}L &= \int_{2\pi}^{3\pi} |\boldsymbol{r}'(t)|\; dt\\ &= [13t]_{2\pi}^{3\pi}\\ &= 13\pi.\end{aligned}$$

(iv) Since $|\boldsymbol{r}'(t)| = 13$ and $|\boldsymbol{s}'(t)| = 1$ for all $t \in \mathbb{R}$, we have $\boldsymbol{s}(t) = \frac{1}{13}\boldsymbol{r}(t) = \frac{1}{13}\langle 5t, 12\sin t, 12\cos t\rangle$

Question 3

(i) For $f(x, y)$ to be well-defined, $x^4 + 3y^8 \neq 0$. Since $x^4 \geq 0$ and $y^8 \geq 0$, with equality only when $x = 0$ and $y = 0$ respectively, the above only give us $(x, y) \neq (0, 0)$.
Thus the domain of $f(x, y)$ is $\mathbb{R}^2\backslash\{(0, 0)\}$.

(ii) The limit does not exists.
Assume on the contrary that the limit does exists. Then we must have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x, y) &= \lim_{x\to 0} f(x, 0) \\ &= \lim_{x\to 0} 0 \\ &= 0.\end{aligned}$$

However at the same time, we also have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x, y) &= \lim_{y\to 0} f(y^2, y) \\ &= \lim_{y\to 0} \frac{(y^2)^2y^4}{(y^2)^4 + 3y^8} \\ &= \lim_{y\to 0} \frac{1}{4} \\ &= \frac{1}{4}.\end{aligned}$$

This is a contradiction, and thus the limit does not exists.

Question 4

(i) Let $f(x, y, z) = x^2z - xy^2 + yz^2 + 2$. Thus we have $f_x(x, y, z) = 2xz - y^2$, $f_y(x, y, z) = -2xy + z^2$, $f_z(x, y, z) = x^2 + 2yz$. Therefore the tangent plane of $f(x, y, z) = 0$ at $(0, -2, 1)$ is

$$\begin{aligned}f_x(0, -2, 1)(x - 0) + f_y(0, -2, 1)(y - (-2)) + f_z(0, -2, 1)(z - 1) &= 0 \\ (-4)(x) + (1)(y + 2) + (-4)(z - 1) &= 0 \\ -4x + y - 4z + 6 &= 0.\end{aligned}$$

(ii) Since $(0, -2, 1)$ lies on the line, and $\nabla f(0, -2, 1) = \langle -4, 1, -4\rangle$ is parallel to the line, for equation of the line, we have

$$\boldsymbol{r}(x, y, z) = \langle 0, -2, 1\rangle + \lambda\langle -4, 1, -4\rangle$$

and thus

$$\begin{cases} x = -4\lambda \\ y = -2 + \lambda \\ z = 1 - 4\lambda. \end{cases}$$

By eliminating λ, we get the symmetric equation of the line to be $\dfrac{-x}{4} = y + 2 = \dfrac{1 - z}{4}$.

Question 5

By substituting $2z + y = 0$ into $\dfrac{x^2}{2} + \dfrac{y^2}{4} + z^2 = 1$, we get $\dfrac{x^2}{2} + \dfrac{y^2}{4} + \left(\dfrac{-y}{2}\right)^2 = 1$, i.e. $x^2 + y^2 = 2$. Thus the curve $\boldsymbol{r}(t)$ can also be described as the intersection of $2z + y = 0$ and $x^2 + y^2 = 2$. This implies that we can let $x = \sqrt{2}\sin t$, and get $y = \sqrt{2}\cos t$ (from $x^2 + y^2 = 2$), $z = \dfrac{-\cos t}{\sqrt{2}}$ (from $2z + y = 0$), $t \in \mathbb{R}$ to trace out the intersection.
Therefore $\boldsymbol{r}(t) = \left\langle \sqrt{2}\sin t, \sqrt{2}\cos t, \dfrac{-\cos t}{\sqrt{2}} \right\rangle$, $t \in \mathbb{R}$ is the curve we wanted.

Question 6

(i) We have $f_x(x, y) = 6x - 6y$ and $f_y(x, y) = 2y - 6x$.

When $f_x(x, y) = 0$, we get $x = y$.
When $f_y(x, y) = 0$, we get $y = 3x$.
Combining the above, we have $\nabla f(x, y) = \langle 0, 0\rangle$ only when $(x, y) = (0, 0)$.

(ii) We have $f_{xx}(x, y) = 6$, $f_{yy}(x, y) = 2$, $f_{xy}(x, y) = -6$, and so $D = f_{xx}f_{yy} - (f_{xy})^2 = -24$. Thus $D|_{(0,0)} < 0$, i.e. $(0, 0)$ is a saddle point.

Question 7

(i) Yes.
Let $f(x, y)$ be a function such that $f_x(x, y) = 2xy + 2y^2$, $f_y(x, y) = x^2 + 4xy + 3y^2$.

By integrating f_x with respect to x, we get $f(x, y) = x^2y + 2xy^2 + g(y)$ for some scalar function $g(y)$. Differentiating this result with respect to y, we get $f_y(x, y) = x^2 + 4xy + g'(y)$, i.e. $g'(y) = 3y^2$. Now by integrating $g'(y)$ with respect to y, we have $g(y) = y^3 + c$ for some arbitrary value c. Therefore we can let $c = 0$, and establish that $f(x, y) = x^2y + 2xy^2 + y^3$ is a function that satisfy the condition $\boldsymbol{F}(x, y) = \nabla f(x, y)$.

(ii) We have $\boldsymbol{r}(0) = \langle 0, 0\rangle$ and $\boldsymbol{r}(1/2) = \langle \sin(\pi/2), 8(1/2)^2\rangle = \langle 1, 2\rangle$. Since $\boldsymbol{F}$ is conservative, we get

$$\begin{aligned} \int_C \boldsymbol{F}(x, y) \cdot d\boldsymbol{r} &= f(\boldsymbol{r}(1/2)) - f(\boldsymbol{r}(0)) \\ &= 18. \end{aligned}$$

Question 8

(i) By substituting $x = u - y$ into $v = 2x - 3y$, we get $y = \dfrac{2u - v}{5}$, and so $x = \dfrac{3u + v}{5}$.

(ii) Let the image of R be S. By substituting the results of (8i.) into all the boundaries, we get S to be the region bounded by $u = 0$, $u = 2$, $v = 2$, $v = 5$.

(iii) We have $\frac{\partial x}{\partial u} = \frac{3}{5}$, $\frac{\partial x}{\partial v} = \frac{1}{5}$, $\frac{\partial y}{\partial u} = \frac{2}{5}$, $\frac{\partial y}{\partial v} = \frac{-1}{5}$. Thus the Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = (\frac{3}{5})(\frac{-1}{5}) - (\frac{1}{5})(\frac{2}{5}) = \frac{-1}{5}$.

(iv) Using (8ii.), we have

$$\begin{aligned}
\iint_R (x+y)(2x-3y)\ dx\ dy &= \iint_S (uv)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv \\
&= \int_2^5 \int_0^2 \frac{1}{5}uv\ du\ dv \\
&= \frac{1}{5}\int_2^5 v\ dv \int_0^2 u\ du \\
&= \frac{1}{5}\left[\frac{1}{2}v^2\right]_2^5 \left[\frac{1}{2}u^2\right]_0^2 \\
&= \frac{1}{5}\left(\frac{21}{2}\right)(2) = \frac{21}{5}.
\end{aligned}$$

Question 9

(i) We have

$$\begin{aligned}
\mathrm{div}\boldsymbol{F} &= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(x^2 z) \\
&= y^2 + z^2 + x^2.
\end{aligned}$$

(ii) Let E be the region bounded by S_1 and S_2. We see that E is given by $\rho \in [0,2]$, $\theta \in [0,2\pi]$, $\phi \in [0,\frac{\pi}{4}]$. By the Divergence Theorem, we have,

$$\begin{aligned}
\iint_S \boldsymbol{F}(x,y,z)\cdot d\boldsymbol{S} &= \iiint_E \mathrm{div}\boldsymbol{F}\ dV \\
&= \iiint_E x^2+y^2+z^2\ dV \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^2 \rho^2(\rho^2\sin\phi)\ d\rho\ d\phi\ d\theta \\
&= \int_0^{2\pi} 1\ d\theta \int_0^{\frac{\pi}{4}} \sin\phi\ d\phi \int_0^2 \rho^4\ d\rho \\
&= [\theta]_0^{2\pi}[-\cos\phi]_0^{\frac{\pi}{4}}\left[\frac{1}{5}\rho^5\right]_0^2 \\
&= (2\pi)\left(1-\frac{\sqrt{2}}{2}\right)\left(\frac{32}{5}\right) = \frac{32\pi}{5}(2-\sqrt{2}).
\end{aligned}$$

Question 10

Let $F(k) = \int_0^k e^{-s^2}\ ds$, i.e. $u(x,t) = \dfrac{2}{\sqrt{\pi}}F\left(\dfrac{x}{\sqrt{t}}\right)$.

By Fundamental Theorem of Calculus, we have $F'(k) = e^{-k^2}$, and so $F'\left(\dfrac{x}{\sqrt{t}}\right) = e^{-(\frac{x}{\sqrt{t}})^2} = e^{\frac{-x^2}{t}}$.

Thus, we have

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{2}{\sqrt{\pi}} F'\left(\frac{x}{\sqrt{t}}\right)\left[\frac{\partial}{\partial t}\left(\frac{x}{\sqrt{t}}\right)\right] \\
&= \frac{2}{\sqrt{\pi}} e^{\frac{-x^2}{t}}\left(\frac{-x}{2t\sqrt{t}}\right) \\
&= -\frac{x}{t\sqrt{t\pi}} e^{\frac{-x^2}{t}}, \\
\frac{\partial u}{\partial x} &= \frac{2}{\sqrt{\pi}} F'\left(\frac{x}{\sqrt{t}}\right)\left[\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{t}}\right)\right] \\
&= \frac{2}{\sqrt{\pi}} e^{\frac{-x^2}{t}}\left(\frac{1}{\sqrt{t}}\right) = \frac{2}{\sqrt{t\pi}} e^{\frac{-x^2}{t}} \\
\frac{\partial^2 u}{\partial x^2} &= \left(\frac{2}{\sqrt{t\pi}} e^{\frac{-x^2}{t}}\right)\left(\frac{-2x}{t}\right) \\
&= 4\left(-\frac{x}{t\sqrt{t\pi}} e^{\frac{-x^2}{t}}\right) \\
&= 4\frac{\partial u}{\partial t}.
\end{aligned}
$$

Thus we see that $K = 1/4$.

Question 11

Let $\boldsymbol{F} = f\nabla g - g\nabla f = \langle fg_x - gf_x, fg_y - gf_y, fg_z - gf_z\rangle$. This give us

$$
\begin{aligned}
\text{div}\boldsymbol{F} &= \frac{\partial}{\partial x}(fg_x - gf_x) + \frac{\partial}{\partial y}(fg_y - gf_y) + \frac{\partial}{\partial z}(fg_z - gf_z) \\
&= (f_x g_x + fg_{xx} - g_x f_x - gf_{xx}) + (f_y g_y + fg_{yy} - g_y f_y - gf_{yy}) + (f_z g_z + fg_{zz} - g_z f_z - gf_{zz}) \\
&= (fg_{xx} - gf_{xx}) + (fg_{yy} - gf_{yy}) + (fg_{zz} - gf_{zz}) \\
&= f(g_{xx} + g_{yy} + g_{zz}) - g(f_{xx} + f_{yy} + f_{zz}) \\
&= f\nabla^2 g - g\nabla^2 f.
\end{aligned}
$$

Thus by Divergence Theorem, we have

$$
\begin{aligned}
\iiint_E \text{div}\boldsymbol{F}\ dV &= \iint_S \boldsymbol{F}\cdot d\boldsymbol{S} \\
\iiint_E (f\nabla^2 g - g\nabla^2 f)\ dV &= \iint_S (f\nabla g - g\nabla f)\cdot d\boldsymbol{S}
\end{aligned}
$$