NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Zheng Shaoxuan

MA1104 Multivariable Calculus
AY 2005/2006 Sem 1

Question 1

(a) Let P be the point of the given ellipsoid, closest to the given plane than any other point on the ellipsoid. Furthermore, let $f(x,y,z) = x^2 + y^2 + 2x^2 - 1$ and $g(x,y,z) = x + y + z - 100$. Hence, $\nabla f = \langle 2x, 2y, 4z \rangle$ and $\nabla g = \langle 1, 1, 1 \rangle$.

By the given condition, the normal vector of the ellipsoid at the point P must be parallel to the normal vector of the plane. Hence, $\nabla f = \lambda \nabla g$ and $f(x,y,z) = 0$. This yields the system of equations:

$$\begin{cases} 2x &= \lambda; \\ 2y &= \lambda; \\ 4z &= \lambda; \\ x^2 + y^2 + 2z^2 &= 1. \end{cases}$$

By substituting λ from the first three equations to the fourth,

$$\begin{aligned} \frac{1}{4}\lambda^2 + \frac{1}{4}\lambda^2 + \frac{1}{8}\lambda^2 &= 1 \\ \lambda &= \pm\sqrt{\frac{8}{5}}. \end{aligned}$$

Hence, by substituting λ into the above three equations, the point P has coordinates

$$\left(\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}, \sqrt{\frac{1}{10}}\right) \quad \text{or} \quad \left(-\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}, -\sqrt{\frac{1}{10}}\right)$$

A point on the given plane is $\langle 100, 0, 0 \rangle$. Hence, a vector pointing from the plane to P is

$$\left\langle \sqrt{\frac{2}{5}} - 100, \sqrt{\frac{2}{5}}, \sqrt{\frac{1}{10}} \right\rangle \quad \text{or} \quad \left\langle -\sqrt{\frac{2}{5}} - 100, -\sqrt{\frac{2}{5}}, -\sqrt{\frac{1}{10}} \right\rangle$$

The distance of the former vector to the plane is given by

$$\begin{aligned} &\left| \frac{\left\langle \sqrt{\frac{2}{5}} - 100, \sqrt{\frac{2}{5}}, \sqrt{\frac{1}{10}} \right\rangle \cdot \langle 1,1,1 \rangle}{\sqrt{1^2+1^2+1^2}} \right| \\ = &\ \frac{100 - \sqrt{\frac{2}{5}} - \sqrt{\frac{2}{5}} - \sqrt{\frac{1}{10}}}{\sqrt{3}}. \end{aligned}$$

The distance of the latter vector to the plane is given by

$$\begin{aligned} &\left| \frac{\left\langle -\sqrt{\frac{2}{5}} - 100, -\sqrt{\frac{2}{5}}, -\sqrt{\frac{1}{10}} \right\rangle \cdot \langle 1,1,1 \rangle}{\sqrt{1^2+1^2+1^2}} \right| \\ = &\ \frac{\sqrt{\frac{2}{5}} + \sqrt{\frac{2}{5}} + \sqrt{\frac{1}{10}} + 100}{\sqrt{3}}. \end{aligned}$$

Since the former distance value is smaller, that is the smallest distance to be considered between the ellipsoid and the plane. Hence, after simplification, the distance between the ellipsoid and the plane is

$$\frac{200\sqrt{3}-\sqrt{30}}{6}.$$

(b) (i) They are the cone and the hyperbolic paraboloid.

(ii) Choosing the cone with equation of $z^2 = x^2 + y^2$ (the reason for this choice is that it is the easiest one to explain), all the lines that lie on the surface can be described as $r(t)$ for $0 \leq \theta \leq 2\pi$, $t \in \mathbb{R}$, where

$$\boldsymbol{r}(t) = \langle t\cos\theta, t\sin\theta, t\rangle.$$

This is because $\boldsymbol{r}(t)$ is a line for each particular value of θ since the x, y and z coordinates varies linearly with the variable t. Also, by substituting $x = t\cos\theta$ and $y = t\sin\theta$ into the equation of the cone, we find that, indeed, $z^2 = t^2\cos^2\theta + t^2\sin^2\theta = t^2$, and hence, $z = t$ or $-t$ which coincides with the particular z value that the line holds.

Question 2

(a) We have,

$$\begin{aligned}
& \lim_{(x,y)\to(0,0)} \frac{\sin(x^4+y^4)}{\sin(x^2+y^2)} \\
= {} & \lim_{r\to 0} \frac{\sin(r^4(\cos^4\theta+\sin^4\theta))}{\sin(r^2(\cos^2\theta+\sin^2\theta))} \\
= {} & \lim_{r\to 0} \frac{\sin(r^4(\cos^4\theta+\sin^4\theta))}{r^4(\cos^4\theta+\sin^4\theta)} \cdot \frac{r^2}{\sin r^2} \cdot \frac{r^4(\cos^4\theta+\sin^4\theta)}{r^2} \\
= {} & \lim_{r\to 0} \frac{\sin(r^4(\cos^4\theta+\sin^4\theta))}{r^4(\cos^4\theta+\sin^4\theta)} \cdot \lim_{r\to 0} \frac{r^2}{\sin r^2} \cdot \lim_{r\to 0} r^2(\cos^4\theta+\sin^4\theta) \\
= {} & 1\cdot 1\cdot 0 \\
= {} & 0.
\end{aligned}$$

(b) (i) We have,

$$\begin{aligned}
& \lim_{(x,y)\to(0,0)} f(x,y) \\
= {} & \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} \\
= {} & \lim_{r\to 0} \frac{r^4\cos^2\theta\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \\
= {} & \lim_{r\to 0} r^2\cos^2\theta\sin^2\theta \\
= {} & 0 \\
= {} & f(0,0).
\end{aligned}$$

Therefore, f is continuous at $(0,0)$.

(ii) By the definition of partial differentiation,

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0-0}{h} \\ &= 0. \end{aligned}$$

By the definition of partial differentiation,

$$\begin{aligned} f_y(0,0) &= \lim_{h\to 0} \frac{(0,h)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0-0}{h} \\ &= 0. \end{aligned}$$

(iii) Note that for all real h,

$$\begin{aligned} f_x(h,0) &= \lim_{t\to 0} \frac{f(h+t,0)-f(h,0)}{t} \\ &= \lim_{h\to 0} \frac{0-0}{t} \\ &= 0. \end{aligned}$$

$$\begin{aligned} f_y(0,h) &= \lim_{t\to 0} \frac{f(0,h+t)-f(0,h)}{t} \\ &= \lim_{h\to 0} \frac{0-0}{t} \\ &= 0. \end{aligned}$$

$$\begin{aligned} f_x(0,h) &= \lim_{t\to 0} \frac{f(t,h)-f(0,h)}{t} \\ &= \lim_{t\to 0} \frac{\frac{t^2h^2}{t^2+h^2}}{t} \\ &= \lim_{t\to 0} \frac{th^2}{t^2+h^2} \\ &= \frac{0}{0+h^2} \\ &= 0. \end{aligned}$$

$$\begin{aligned} f_y(h,0) &= \lim_{t\to 0} \frac{f(h,t)-f(h,0)}{t} \\ &= \lim_{t\to 0} \frac{\frac{t^2h^2}{t^2+h^2}}{t} \\ &= \lim_{t\to 0} \frac{th^2}{t^2+h^2} \\ &= \frac{0}{0+h^2} \\ &= 0. \end{aligned}$$

Hence,

$$\begin{aligned} f_{xx}(0,0) &= \lim_{h\to 0} \frac{f_x(h,0)-f_x(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0-0}{h} \\ &= 0. \end{aligned}$$

$$\begin{aligned} f_{yy}(0,0) &= \lim_{h\to 0}\frac{f_y(0,h)-f_y(0,0)}{h} \\ &= \lim_{h\to 0}\frac{0-0}{h} \\ &= 0. \end{aligned}$$

$$\begin{aligned} f_{yx}(0,0) &= \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} \\ &= \lim_{h\to 0}\frac{0-0}{h} \\ &= 0. \end{aligned}$$

$$\begin{aligned} f_{xy}(0,0) &= \lim_{h\to 0}\frac{f_x(0,h)-f_x(0,0)}{h} \\ &= \lim_{h\to 0}\frac{0-0}{h} \\ &= 0. \end{aligned}$$

(iv) For all real h, let the proposition $P(n)$ be the following:

$$P(n):\ \frac{\partial^n f}{\partial x^n}(h,0)=0,\ n\in\mathbb{N}.$$

$P(1)$ is true as illustrated in (iii).
Assume that $P(k)$ is true for $k\in\mathbb{N}$. Consider $P(k+1)$:

$$\begin{aligned} \frac{\partial^{k+1} f}{\partial x^{k+1}}(h,0) &= \lim_{t\to 0}\frac{\frac{\partial^k f}{\partial x^k}(h+t,0)-\frac{\partial^k f}{\partial x^k}(h,0)}{t} \\ &= \lim_{t\to 0}\frac{0-0}{t} \\ &= 0. \end{aligned}$$

Since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$, by Mathematical Induction, $P(n)$ is true for all natural numbers n. Therefore, $\frac{\partial^n f}{\partial x^n}(0,0)=0$.

(v) No. It suffices to show that f_{xxy} does not exist at $(0,0)$.
Working out the expression of f_{xx},

$$\begin{aligned} f_x(x,y) &= \frac{(x^2+y^2)(2xy^2)-(x^2y^2)(2x)}{(x^2+y^2)^2} \\ &= \frac{2xy^4}{(x^2+y^2)^2} \\ f_{xx}(x,y) &= \frac{(x^2+y^2)^2(2y^4)-(2xy^4)(2)(x^2+y^2)(2x)}{(x^2+y^2)^4} \\ &= \frac{2x^2y^4+2y^6-8x^2y^4}{(x^2+y^2)^3} \\ &= \frac{2y^4(y^2-3x^2)}{(x^2+y^2)^3}. \end{aligned}$$

For any $h\neq 0$

$$\begin{aligned} f_{xx}(0,h) &= \frac{2h^4(h^2)}{(h^2)^3} \\ &= 2. \end{aligned}$$

Therefore,

$$\begin{aligned} f_{xxy}(0,0) &= \lim_{h\to 0} \frac{f_{xx}(0,h) - f_{xx}(0,0)}{h} \\ &= \lim_{h\to 0} \frac{2}{h}. \end{aligned}$$

And hence f_{xxy} does not exist at $(0,0)$.

Question 3

(i) Let $g(x,y,z) = x^2 + y^2 + z^2 - h$, where $0 \le h \le 1$. $\nabla f = \langle 2x + y + z, 2y + x + z, 2z + x + y \rangle$, $\lambda \nabla g = \langle 2x\lambda, 2y\lambda, 2z\lambda \rangle$. By the Method of Lagrange Multiplier, $\nabla f = \lambda \nabla g$ and $g(x,y,z) = 0$, resulting in the following system of equations:

$$\left\{ \begin{aligned} 2x + y + z &= 2x\lambda; \\ x + 2y + z &= 2y\lambda; \\ x + y + 2z &= 2z\lambda; \\ x^2 + y^2 + z^2 &= h. \end{aligned} \right.$$

By adding the first three equations, we obtain $\lambda = 2$. Hence, by substituting λ back into the first three equations, it is easy to find out that $x = y = z$. Hence, for $0 \le h \le 1$, the critical points hold the following x, y and z values:

$$x^2 = y^2 = z^2 = \frac{h}{3}.$$

By substituting this back into $f(x,y,z)$:

$$f(x,y,z) = 2h.$$

And hence, the maximum value of f is 2 and the minimum value of f is 0.

(ii) We have,

$$\begin{aligned} \int_C f(x,y,z)\, ds &= \int_0^1 f(\cos t, \sin t, t)\sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt \\ &= \sqrt{2}\int_0^1 \cos^2 t + \sin^2 t + t^2 + \sin t\cos t + t\sin t + t\cos t\, dt \\ &= \sqrt{2}\int_0^1 1 + t^2 + \frac{1}{2}\sin 2t + t(\sin t + \cos t)\, dt \\ &= \sqrt{2}\left[t + \frac{1}{3}t^3 - \frac{1}{4}\cos 2t + t(\sin t - \cos t) - \int \sin t - \cos t\, dt\right]_0^1 \\ &= \sqrt{2}\left[t + \frac{1}{3}t^3 - \frac{1}{4}\cos 2t + t(\sin t - \cos t) + \sin t + \cos t\right]_0^1 \\ &= \sqrt{2}\left(\frac{7}{12} - \frac{1}{4}\cos 2 + 2\sin 1\right). \end{aligned}$$

Question 4

(a) Let D be the domain of the unit circle around the origin. By Green's Theorem,

$$\begin{aligned}
\int_C F\cdot\, d\boldsymbol{r} &= \int_C (x^2+y^2)\ dx + (y^3+x)\ dy \\
&= \iint_D (1-2y)\ dA \\
&= \int_0^{2\pi}\int_0^1 r(1-2r\sin\theta)\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^1 r-2r^2\sin\theta\ dr\ d\theta \\
&= \int_0^{2\pi}\left[\frac{1}{2}r^2-\frac{2}{3}r^3\sin\theta\right]_0^1\ d\theta \\
&= \int_0^{2\pi}\frac{1}{2}-\frac{2}{3}\sin\theta\ d\theta \\
&= \left[\frac{1}{2}\theta+\frac{2}{3}\cos\theta\right]_0^{2\pi} \\
&= \pi+\frac{2}{3}-\frac{2}{3} \\
&= \pi.
\end{aligned}$$

(b) Let $G=\langle P,Q,R\rangle$. Hence,

$$\text{curl } G=\left\langle \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right\rangle.$$

If $F=\text{curl } G$, then by observation with the given F, a suitable G would be the following:

$$G(x,y,z)=\left\langle xz^2,\frac{3}{2}x^2y,\frac{1}{2}y^2z\right\rangle.$$

Let C be the unit circle with the anticlockwise orientation, which is also the bounding curve of surface S. C has equation $r(t)=\langle\cos t,\sin t,0\rangle$, where $0\le t\le 2\pi$. Since S is negatively oriented, by Stoke's Theorem,

$$\begin{aligned}
\iint_S F\cdot\, d\boldsymbol{S} &= \iint_S \text{curl } G\cdot\, d\boldsymbol{S} \\
&= -\int_C G\cdot\, dr \\
&= -\int_0^{2\pi} G(\cos t,\sin t,0)\cdot r'(t)\ dt \\
&= -\int_0^{2\pi}\left\langle 0,\frac{3}{2}\cos^2 t\sin t,0\right\rangle\cdot\langle -\sin t,\cos t,0\rangle\ dt \\
&= -\frac{3}{2}\int_0^{2\pi}\cos^3 t\sin t\ dt \\
&= =\frac{3}{2}\left[-\frac{1}{4}\cos^4 t\right]_0^{2\pi} \\
&= 0.
\end{aligned}$$

(c) This writer believes there is a typographic error in this question and hence will not provide a model answer for it.

Question 5

(a) (1. $\Rightarrow$ 2.) $\int_C \boldsymbol{F}\cdot\ dr$ is independent of path $\Rightarrow \oint_C \boldsymbol{F}\cdot\ dr = 0$, and hence for any positive number B, $|\int_C \boldsymbol{F}\cdot\ dr| < B$ whenever C is closed.

(2. $\Rightarrow$ 1.) The equivalent contrapositive, i.e. (NOT 1.) $\Rightarrow$ (NOT 2.) shall be shown instead. $\int_C \boldsymbol{F}\cdot\ dr$ is dependent on path $\Rightarrow \oint_C \boldsymbol{F}\cdot\ dr \neq 0$. Suppose $\left|\oint_C \boldsymbol{F}\cdot\ dr\right| = a$, where $a > 0$. Let the closed curve C_M, where M is a positive integer, be C looped upon itself exactly M times. For all $B > 0$, let $M = \lceil \frac{B}{a} \rceil$. Therefore, $|\int_{C_M} \boldsymbol{F}\cdot\ dr| = M|\int_C \boldsymbol{F}\cdot\ dr| = aM = a\lceil \frac{B}{a} \rceil \geq B$. This represents the negation of (2.)

Hence, statement 1. and statement 2. are equivalent.

(b) We have,

$$\begin{aligned}\text{curl } \boldsymbol{F}(x,y,z) &= \langle \cos z - \cos z, \cos x - \cos x, \cos y - \cos y\rangle \\ &= \langle 0,0,0\rangle.\end{aligned}$$

Therefore, $\boldsymbol{F}$ is conservative.

Question 6

(a) By spherical coordinates, $B = \{(\rho,\theta,\phi) \mid 0 \leq \rho \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}$. Hence,

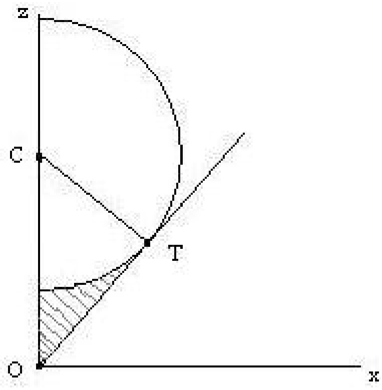
$$\begin{aligned}\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\ dV &= \int_0^\pi \int_0^{2\pi} \int_0^1 \rho^2 \sin\phi\ e^{(\rho^2)^{\frac{3}{2}}}\ d\rho\ d\theta\ d\phi \\ &= \left[\int_0^\pi \sin\phi\ d\phi\right]\left[\int_0^{2\pi} d\theta\right]\left[\int_0^1 \rho^2\ e^{\rho^3}\ d\rho\right] \\ &= [-\cos\phi]_0^\pi\ [2\pi] \left[\frac{1}{3}e^{\rho^3}\right]_0^1 \\ &= (2)(2\pi)\left(\frac{1}{3}e - \frac{1}{3}\right) \\ &= \frac{4\pi}{3}(e-1).\end{aligned}$$

(b) Figure 1 below shows one symmetric half of a slice of the visualization in the xz-plane for positive x and positive z, with the sphere appearing as a semicircle and the cone appearing as a slanted line. O is the origin, C is the center of the sphere and T is the touching point, also the tangent, of the sphere with the cone on this slice.

It can be seen that $\angle TOC = \dfrac{\pi}{4}$ since the cone has slant of magnitude 1, $\angle CTO = \dfrac{\pi}{2}$ since T is a tangent point of the sphere and the cone and $CT = 1$ as the sphere has unit radius. Hence, $OT = 1$, and hence T has xz-coordinates $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$. By Pythagoras' Theorem, C has xz-coordinates $(0, \sqrt{2})$.

The line has equations $z = x$, while the circle has equation $x^2 + (z-\sqrt{2})^2 = 1$. In particular, the lower half of the circle has equation $z = \sqrt{2} - \sqrt{1-x^2}$.

AY05-06 Sem 1 Qn 6b Diagram.jpg



The revolution of the shaded area about the z-axis is the region X. Since region X has unit density, the mass of solid X is simply the volume of region X. By the volume of revolution using cylindrical shells,

$$\begin{aligned}
\text{Mass of X} &= \int_0^{\frac{1}{\sqrt{2}}} 2\pi x(\sqrt{2} - \sqrt{1-x^2} - x)\ dx \\
&= 2\pi \int_0^{\frac{1}{\sqrt{2}}} \sqrt{2}x - x^2 - x\sqrt{1-x^2})\ dx \\
&= 2\pi \left[\frac{\sqrt{2}}{2}x^2 - \frac{x^3}{3} + \frac{1}{3}(1-x^2)^{\frac{3}{2}} \right]_0^{\frac{1}{\sqrt{2}}} \\
&= 2\pi \left[\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{12} + \frac{1}{3}\left(\frac{\sqrt{2}}{4} - 1 \right) \right] \\
&= 2\pi \left[\frac{\sqrt{2}}{4} - \frac{1}{3} \right] \\
&= \frac{\pi}{6}(3\sqrt{2} - 4).
\end{aligned}$$

By rotational symmetry of region X about the z-axis, the x-coordinate and y-coordinate of the center of mass of X are 0. The z-coordinate of the center of mass is simply the weighted average of the middle of the heights of each cylindrical shell considered above. Hence,

$$\text{z-coordinate of center of mass of X} = \frac{\int_0^{\frac{1}{\sqrt{2}}} 2\pi x(\sqrt{2} - \sqrt{1-x^2} - x)\frac{\sqrt{2}-\sqrt{1-x^2}+x}{2}\ dx}{\frac{\pi}{6}(3\sqrt{2} - 4)}.$$

Computing the integral in the numerator of the fraction,

$$\begin{aligned}
& \int_0^{\frac{1}{\sqrt{2}}} 2\pi x(\sqrt{2}-\sqrt{1-x^2}-x)\frac{\sqrt{2}-\sqrt{1-x^2}+x}{2}\ dx \\
= & \ \pi \int_0^{\frac{1}{\sqrt{2}}} x((\sqrt{2}-\sqrt{1-x^2})^2-x^2)\ dx \\
= & \ \pi \int_0^{\frac{1}{\sqrt{2}}} x(2-2\sqrt{2}\sqrt{1-x^2}+1-x^2-x^2)\ dx \\
= & \ \pi \int_0^{\frac{1}{\sqrt{2}}} 3x-2x^3-2\sqrt{2}x\sqrt{1-x^2}\ dx \\
= & \ \pi \left[\frac{3}{2}x^2-\frac{1}{2}x^4+\frac{2\sqrt{2}}{3}(1-x^2)^{\frac{3}{2}}\right]_0^{\frac{1}{\sqrt{2}}} \\
= & \ \pi \left[\frac{3}{4}-\frac{1}{8}+\frac{1}{3}-\frac{2\sqrt{2}}{3}\right] \\
= & \ \pi \left[\frac{23}{24}-\frac{2\sqrt{2}}{3}\right] \\
= & \ \frac{\pi}{24}(23-16\sqrt{2}).
\end{aligned}$$

Hence, the z-coordinate of the center of mass of X is

$$\begin{aligned}
& \frac{\frac{\pi}{24}(23-16\sqrt{2})}{\frac{\pi}{6}(3\sqrt{2}-4)} \\
= & \ \frac{23-16\sqrt{2}}{4(3\sqrt{2}-4)}.
\end{aligned}$$

And therefore, the center of mass of X has coordinates $\left(0,0,\dfrac{23-16\sqrt{2}}{4(3\sqrt{2}-4)}\right)$.