NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA2108 Mathematical Analysis I
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Question 1

(a) No. We will illustrate a counter-example. Let (x_n) be a sequence of positive real numbers with $x_n = \frac{1}{n^2}$ if n is odd and $x_n = \frac{1}{n^3}$ if n is even. Then

$$\sum_{n=1}^{\infty} x_n = \frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^2} + \cdots < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

By Comparison Test, $\sum x_n$ is convergent. However, $\frac{1}{5^2} > \frac{1}{4^3}$. Hence, (x_n) is not decreasing.

(b) First, we establish the inequality

$$n^{\frac{1}{n+1}} \leq (-2 + n\ln n)^{\frac{1}{1+n}} \leq (2\cos n + n\ln n)^{\frac{1}{\sin n + n}} \leq (2 + n\ln n)^{\frac{1}{-1+n}} \leq (n^2)^{\frac{1}{n-1}}$$

Since

$$\lim_{x\to\infty} n^{\frac{1}{n+1}} = \lim_{x\to\infty} (n^2)^{\frac{1}{n-1}} = 1$$

By Squeeze Theorem,

$$\lim_{x\to\infty} (2\cos n + n\ln n)^{\frac{1}{\sin n + n}} = 1$$

Question 2

(a) (i) By Root Test, since

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{n}\right)^{n^2} e^{-2n}} &= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n e^{-2} \\
&= e \times e^{-2} \\
&= e^{-1} \\
&< 1
\end{aligned}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2} e^{-2n}$ converges.

(ii) By Raabe's Test, since

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{2\cdot 4\cdots(2n+2)}{5\cdot 7\cdots(2n+5)}}{\frac{2\cdot 4\cdots(2n)}{5\cdot 7\cdots(2n+3)}}\right| \\
&= \left|\frac{2n+2}{2n+5}\right| \\
&= 1 - \frac{3}{2n+5}
\end{aligned}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdots (2n)}{5 \cdot 7 \cdots (2n+3)}$ converges.

(iii) By Comparison Test,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n+1-3\lfloor (n+1/3 \rfloor}}{n} &= 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots \\
&> \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \frac{1}{6} + \frac{1}{6} - \frac{1}{6} + \frac{1}{9} + \frac{1}{9} - \frac{1}{9} + \cdots \\
&= \frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \cdots \\
&= \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right)
\end{aligned}$$

Since the harmonic series diverges, therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1-3\lfloor (n+1/3 \rfloor}}{n}$ diverges.

(b) By using Limit Comparison Test with $\frac{1}{n^{2q}}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{1}{(n^2+n)^q}}{\frac{1}{n^{2q}}} &= \lim_{n\to\infty} \frac{n^{2q}}{(n^2+n)^q} \\
&= \lim_{n\to\infty} \frac{1}{(\frac{n^2+n}{n^2})^q} \\
&= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^q} \\
&= 1
\end{aligned}$$

This means that if the series $\sum_{n=1}^{\infty} \frac{1}{n^{2q}}$ converges or diverges, then $\sum_{n=1}^{\infty} \frac{1}{(n^2+n)^q}$ converges or diverges respectively.
By the p-series, we know that $\sum_{n=1}^{\infty} \frac{1}{n^{2q}}$ converges if $q > \frac{1}{2}$, and it diverges if $q \le \frac{1}{2}$. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{(n^2+n)^q}$ converges if $q > \frac{1}{2}$, and it diverges if $q \le \frac{1}{2}$.

(c) Since $\sum_{n=1}^{\infty} na_n^3$ is convergent, $lim_{n\to\infty} na_n^3 = 0$. Therefore, the sequence (na_n^3) is bounded, say $na_n^3 \le M$ for all $n \in \mathbb{N}$. Note that

$$0 < na_n^3 \le M$$

so

$$0 < a_n^3 \le \frac{M}{n}$$

Therefore, by Squeeze Theorem, $\lim_{n\to\infty} a_n^3 = 0$.

By $\epsilon - \delta$ definition, there exists $\epsilon^3 > 0$ such that $\forall \delta > 0$,

$$\begin{aligned}
|x^3 - 0| &< \epsilon^3 \\
|x - 0| &< \epsilon
\end{aligned}$$

Therefore, $\lim_{n\to\infty} a_n = 0$.

By Limit Comparison Test, since

$$\begin{aligned}\lim_{n\to\infty}\frac{n^2a_n^7}{na_n^3} &= \lim_{n\to\infty} na_n^4\\ &= \lim_{n\to\infty} n\cdot a_n\cdot a_n^3\\ &= 0\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} n^2a_n^7$ is convergent.

Question 3

(a) Let $\epsilon > 0$. We want to prove that $\forall\epsilon > 0, \exists\delta > 0$ s.t. $|\frac{x^2+1}{3x+2} - \frac{2}{5}| < \epsilon$ whenever $|x-1| < \delta$. Suppose $\delta = \frac{1}{5}$. Then,

$$\begin{gathered}-\frac{1}{5} < x-1 < \frac{1}{5}\\ 3 < 5x-1 < 5\\ \frac{22}{5} < 3x+2 < \frac{28}{5}\end{gathered}$$

Hence, $|5x+1| < 5$ and $|\frac{1}{3x+2}| < \frac{5}{22}$. Set $\delta = \inf(\frac{1}{5}, \frac{22}{5}\epsilon)$, then $\forall|x-1| < \delta$,

$$\begin{aligned}\left|\frac{x^2+1}{3x+2} - \frac{2}{5}\right| &= \left|\frac{5x^2+5-6x-4}{5(3x+2)}\right|\\ &= \left|\frac{(5x-1)(x-1)}{5(3x+2)}\right|\\ &\le \frac{|5x-1||x-1|}{5|3x+2|}\\ &< \frac{5|x-1|}{5}\times\frac{5}{22}\\ &= \frac{5}{22}|x-1|\\ &< \epsilon\end{aligned}$$

Therefore, $\lim_{x\to 1}\frac{x^2+1}{3x+2} = \frac{2}{5}$.

(b) Using l-Hopital's Rule,

$$\begin{aligned}\lim_{x\to 0^-}\frac{e^{\frac{1}{x}}}{x^2} &= \lim_{x\to 0^-}\frac{x^{-2}}{e^{-x^{-1}}}\\ &= \lim_{x\to 0^-}\frac{-2x^{-3}}{x^{-2}e^{-x^{-1}}}\\ &= \lim_{x\to 0^-}\frac{-2x^{-1}}{e^{-x^{-1}}}\\ &= \lim_{x\to 0^-}\frac{2x^{-2}}{x^{-2}e^{-x^{-1}}}\\ &= \lim_{x\to 0^-}\frac{2}{e^{-x^{-1}}}\\ &= \lim_{x\to 0^-} 2e^{x^{-1}}\\ &= 0\end{aligned}$$

(c) Let $\epsilon = 1$ and $\alpha = K(L-1)$. We observe that

$$\begin{aligned} L-1 < \frac{f(x)}{x} < L+1 &\Rightarrow x(L-1) < f(x) < x(L+1) \\ &\Rightarrow f(x) > x(L-1) \\ &\Rightarrow f(x) > K(L-1) \\ &\Rightarrow f(x) > \alpha \end{aligned}$$

Since for any $\alpha \in \mathbb{R}$, $\exists K > a$ for some $a \in \mathbb{R}$ such that for any $x > K$, $f(x) > \alpha$, we have $\lim_{x\to\infty} f(x) = \infty$.

Question 4

(a) Let $\epsilon > 0$. Set $\delta = \frac{\epsilon}{4}$. If $x, y \in \mathbb{R}$ and $|x-y| < \delta$, then

$$\begin{aligned} \left|\frac{1}{2x^2+1} - \frac{1}{2y^2+1}\right| &= \left|\frac{2x^2-2y^2}{(2x^2+1)(2y^2+1)}\right| \\ &= \left|\frac{2(x+y)(x-y)}{(2x^2+1)(2y^2+1)}\right| \\ &\le \frac{2|x-y||x+y|}{|2x^2+1||2y^2+1|} \\ &\le 2|x-y|\left(\frac{|x|}{|2x^2+1||2y^2+1|} + \frac{|y|}{|2x^2+1||2y^2+1|}\right) \\ &\le 2|x-y|(1+1) \\ &< \epsilon \end{aligned}$$

Therefore, $f(x)$ is uniformly continuous on $\mathbb{R}$.

(b) An example will be $f(x) = \sin(\frac{1}{x^2})$, on $A = (0, \infty)$. Consider the sequence (x_n) and (y_n) in A where for each n,

$$x_n = \frac{1}{\sqrt{\frac{\pi}{2} + 2n\pi}}, y_n = \frac{1}{\sqrt{\frac{3\pi}{2} + 2n\pi}}$$

Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$, we have $\lim_{n\to\infty}(x_n - y_n) = 0$. However,

$$|f(x_n) - f(y_n)| = |\sin(\frac{\pi}{2} + 2n\pi) - \sin(\frac{3\pi}{2} + 2n\pi)| = 2$$

Hence, by Nonuniform Continuity Criteria, $f(x)$ is continuous and bounded on A but not uniformly continuous.

(c) Consider f on $[c-1, c+1]$. By Uniform Continuity Theorem, f is uniformly continuous on $[c-1, c+1]$. So there exists $\delta_A > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in [c-1, c+1] \text{ and } |x-y| < \delta_A$$

Also, since f is uniformly continuous on $(-\infty, c)$, there exists $\delta_B > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in (-\infty, c) \text{ and } |x-y| < \delta_B$$

Now, set $\delta_1 = \inf\{\delta_A, \delta_B, 1\}$. Suppose $x, y \in \mathbb{R}$ such that $|x-y| < \delta_1$. Without loss of generality, we may assume that $x < y$. Then either $x, y \in [c-1, c+1]$ or $x, y \in (-\infty, c)$ (note that the case of $x \leq c-1$ and $y \geq c$ never occur since otherwise $|x-y| = y-x \geq 1$, contradicting the assumption that $|x-y| < \delta_1 \leq 1$.) Now, the result follows from the 2 equations, that is

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in (-\infty, c+1] \text{ and } |x-y| < \delta_1$$

Also, since f is uniformly continuous on (c, ∞), there exists $\delta_C > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in (c, \infty) \text{ and } |x-y| < \delta_C$$

Now, set $\delta_2 = \inf\{\delta_A, \delta_C, 1\}$. Suppose $x, y \in \mathbb{R}$ such that $|x-y| < \delta_2$. Without loss of generality, we may assume that $x < y$. Then either $x, y \in [c-1, c+1]$ or $x, y \in (c, \infty)$ (note that the case of $x \leq c$ and $y \geq c+1$ never occur since otherwise $|x-y| = y-x \geq 1$, contradicting the assumption that $|x-y| < \delta_2 \leq 1$.) Now, the result follows from the 2 equations, that is

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in [c-1, \infty) \text{ and } |x-y| < \delta_2$$

Now, set $\delta = \inf\{\delta_1, \delta_2, 2\}$. Suppose $x, y \in \mathbb{R}$ such that $|x-y| < \delta$. Without loss of generality, we may assume that $x < y$. Then either $x, y \in (-\infty, c+1]$ or $x, y \in [c-1, \infty)$ (note that the case of $x \leq c-1$ and $y \geq c+1$ never occur since otherwise $|x-y| = y-x \geq 2$, contradicting the assumption that $|x-y| < \delta \leq 2$.) Now, the result follows from the 2 equations, that is

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in (-\infty, \infty) \text{ and } |x-y| < \delta$$

We conclude that f is uniformly continuous on $\mathbb{R}$.

Question 5

(a) From (i), since (a_{2k-1}) is bounded and monotone decreasing, by Monotone Convergence Theorem, (a_{2k-1}) is convergent.
From (ii), since (a_{2k}) is bounded and monotone increasing, by Monotone Convergence Theorem, (a_{2k}) is convergent.
From (iii), since $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$, there exists $N \in \mathbb{R}$ such that for all $m, n > N$, $|a_{n+1} - a_n| < \frac{\epsilon}{m-n}$. Hence, for $m > n$,

$$\begin{aligned} |a_m - a_n| &= |(a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \cdots + (a_{n+1} - a_n)| \\ &\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \\ &< \frac{\epsilon}{m-n} + \frac{\epsilon}{m-n} + \cdots + \frac{\epsilon}{m-n} \\ &= \epsilon \end{aligned}$$

Therefore, a_n is a Cauchy sequence and is convergent. Since (a_{2k-1}) and (a_{2k}) are subsequences of (a_n), they all have the same limit, and so there exists $\alpha \in \mathbb{R}$ such that

$$\alpha = \lim_{n\to\infty} a_n = \lim_{k\to\infty} a_{2k-1} = \lim_{k\to\infty} a_{2k}$$

(b) We are given that $f(x) = f(\frac{x+a}{b}) = f(\frac{x}{b} + \frac{a}{b})$. Hence,

$$\begin{aligned} f(x) &= f(\frac{x}{b} + \frac{a}{b}) \\ &= f(\frac{x}{b^2} + \frac{a}{b^2} + \frac{a}{b}) \\ &= f(\frac{x}{b^n} + \frac{a}{b^n} + \cdots + \frac{a}{b}) \end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} f(x) &= \lim_{n\to\infty} f\left(\frac{x}{b^n}+\frac{a}{b^n}+\cdots+\frac{a}{b}\right)\\ &= f\left(\frac{\frac{a}{b}}{1-\frac{1}{b}}\right)\\ &= f\left(\frac{a}{b-1}\right)\\ &= c\end{aligned}$$

where c is a constant.

Hence, $f(x)$ is a constant function.

When $b=1$, $f(x)=f(x+a)$. This is a periodic function but not a constant function. One example of a function that satisfies this will be $f(x)=\sin x$, where $a=2\pi$.

Question 6

(a) We shall prove by induction.

$k=1$:

By Cauchy Condensation Test, $\sum_{n=N_1}^{\infty}\frac{1}{n\ln n}$ converges $\Rightarrow \sum_{n=N_1}^{\infty} 2^n\frac{1}{2^n\ln 2^n}$ converges $\Rightarrow \sum_{n=N_1}^{\infty}\frac{1}{n\ln 2}$ converges. However, by p-series, $\sum_{n=N_1}^{\infty}\frac{1}{n}$ diverges, so $\sum_{n=N_1}^{\infty}\frac{1}{n\ln n}$ diverges.

Suppose this is true for $k=m$. Then we have,

$$\sum_{n=N_m}^{\infty}\frac{1}{n(\ln n)(\ln_2 n)(\ln_3 n)\cdots(\ln_m n)} \text{ diverges}$$

$k=m+1$:

By Cauchy Condensation Test,

$$\begin{aligned}\sum_{n=N_{m+1}}^{\infty}\frac{1}{n(\ln n)(\ln_2 n)\cdots(\ln_{m+1} n)} \text{ converges} &\Rightarrow \sum_{n=N_{m+1}}^{\infty} 2^n\frac{1}{2^n(\ln 2^n)(\ln_2 2^n)\cdots(\ln_{m+1} 2^n)} \text{ converges}\\ &\Rightarrow \sum_{n=N_{m+1}}^{\infty}\frac{1}{(n\ln 2)(\ln n\ln 2)\cdots(\ln_m n\ln 2)} \text{ converges}\end{aligned}$$

However, by Comparison Test,

$$\sum_{n=N_{m+1}}^{\infty}\frac{1}{(n\ln 2)(\ln n\ln 2)\cdots(\ln_{k-1} n\ln 2)} \text{ converges} \Rightarrow \sum_{n=N_{m+1}}^{\infty}\frac{1}{n(\ln n)(\ln_2 n)\cdots(\ln_m n)} \text{ converges}$$

since $\ln 2<1$.
However, we arrive at a contrdiction, since $\sum_{n=N_{m+1}}^{\infty}\frac{1}{n(\ln n)(\ln_2 n)\cdots(\ln_m n)}$ diverges. Therefore,

$$\sum_{n=N_{m+1}}^{\infty}\frac{1}{n(\ln n)(\ln_2 n)(\ln_3 n)\cdots(\ln_{m+1} n)} \text{ diverges.}$$

By Mathematical Induction, since the case for $k=1$ is true and $k=m$ true implies $k=m+1$ true, $\sum_{n=N_k}^{\infty}\frac{1}{n(\ln n)(\ln_2 n)\cdots(\ln_k n)}$ diverges for all $k\in\mathbb{N}$.

(b) We take the logarithmic function on both sides to obtain

$$\ln(f(x) - \lfloor x \rfloor) = \lfloor x \rfloor \ln(x - \lfloor x \rfloor)$$

The function $\lfloor x \rfloor$ is continuous on $\mathbb{R} \setminus \mathbb{Z}$. Since $x > \frac{1}{2}$, $\lfloor x \rfloor$ is continuous on $[\frac{1}{2}, \infty) \setminus \mathbb{Z}$.
Since $\ln(f(x) - \lfloor x \rfloor)$ is made up of a composition of $\ln x$, x and $\lfloor x \rfloor$, it is continuous on $[\frac{1}{2}, \infty) \setminus \mathbb{Z}$.
Since e^x is continuous on $\mathbb{R}$, $f(x) - \lfloor x \rfloor$ is continuous on $[\frac{1}{2}, \infty) \setminus \mathbb{Z}$.
Therefore, f is continuous on $[\frac{1}{2}, \infty) \setminus \mathbb{Z}$.

It suffices to prove that f is continuous on $\mathbb{Z}^+$.
Let $a \in \mathbb{Z}^+$.

$$\begin{aligned}
\lim_{x \to a^+} f(x) &= \lim_{x \to a^+} \left(\lfloor x \rfloor + (x - \lfloor x \rfloor)^{\lfloor x \rfloor} \right) \\
&= a + (a - a)^a \\
&= a \\
\lim_{x \to a^-} f(x) &= \lim_{x \to a^-} \left(\lfloor x \rfloor + (x - \lfloor x \rfloor)^{\lfloor x \rfloor} \right) \\
&= (a-1) + (a - (a-1))^{a-1} \\
&= (a-1) + 1 \\
&= a
\end{aligned}$$

Since $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$, we conclude that f is continuous on $\mathbb{Z}^+$.
Hence, f is continuous on $[\frac{1}{2}, \infty)$.

Question 7

(a) Suppose c is rational and $c \neq 0$. Let (x_n) be a sequence of irrational numbers converging to c. If f is continuous at c, then $f(c) = \lim f(x_n) = 0$. On the other hand, by definition, $f(c) = \sin c$. So $\sin c = 0$ which gives us $c = 0$, contradicting our assumption. So f is NOT continuous at rational numbers $c \neq 0$.

Similarly, suppose c is irrational and $c \neq n\pi$ where $n \in \mathbb{Z} \setminus \{0\}$. Let (y_n) be a sequence of rational numbers converging to c. If f is continuous at c, then $f(c) = \lim f(y_n) = \sin c$. On the other hand, by definition, $f(c) = 0$. So $\sin c = 0$ which gives us $c = n\pi$ where $n \in \mathbb{Z} \setminus \{0\}$, contradicting our assumption. So f is NOT continuous at irrational numbers $c \neq n\pi$ where $n \in \mathbb{Z} \setminus \{0\}$.

Now we show that f is continuous at $c = n\pi$ where $n \in \mathbb{Z}$. Let $\epsilon > 0$. Set $\delta = \inf\{0, \epsilon\}$. Note that $f(n\pi) = 0$. Suppose $|x - n\pi| < \delta$.

If x is irrational, then

$$\begin{aligned}
|f(x) - f(n\pi)| &= |0 - 0| \\
&< \delta \leq \epsilon
\end{aligned}$$

If x is rational, then

$$\begin{aligned}
|f(x) - f(n\pi)| &= |\sin x - \sin n\pi| \\
&= 2 \left| \cos \frac{x + n\pi}{2} \sin \frac{x - n\pi}{2} \right| \\
&\leq 2 \left| \sin \frac{x - n\pi}{2} \right| \\
&\leq 2 \left| \frac{x - n\pi}{2} \right| \\
&< \delta \leq \epsilon
\end{aligned}$$

Therefore, f is continuous at all $n\pi$ where $n \in \mathbb{Z}$.

(b) We define $g(x) = f(x+1) - f(x) - \frac{f(x)}{2} + \frac{f(0)}{2}$. This means that g is continuous on $[0,1]$. It suffices to find $k \in [0,1]$ such that $g(k) = 0$, since this will lead us to $a = k+1, b = k$.

If $g(0) = 0$ or $g(1) = 0$, then we have found the k which leads us to the values of a and b.

If $g(0) \neq 0$ and $g(1) \neq 0$, then we see that

$$\begin{aligned}
g(0) &= f(1) - f(0) - \frac{f(2)}{2} + \frac{f(0)}{2} \\
&= -\frac{f(0)}{2} + f(1) - \frac{f(2)}{2} \\
g(1) &= f(2) - f(1) - \frac{f(2)}{2} + \frac{f(0)}{2} \\
&= \frac{f(0)}{2} - f(1) + \frac{f(2)}{2} \\
&= -g(0)
\end{aligned}$$

By Intermediate Value Theorem, there exists a $k \in (0,1)$ such that $g(k) = 0$. Therefore, there exists $k \in [0,1]$ such that $g(k) = 0$. Taking $a = k+1, b = k$, $a, b \in [0,2]$, and we are done.