NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Associate Professor Victor Tan

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MA1100 Fundamental Concepts of Mathematics
AY 2008/2009 Sem 1

Question 1

(a) Let $P(n)$ be the proposition that $3 + 3^2 + \cdots + 3^n = \frac{3^{n+1}-3}{2}$.
Consider $P(1)$:
LHS $= 3$.
RHS $= \frac{3^{1+1}-3}{2} = 3$.
So, P(1) is true.
Assume $P(k)$ is true, consider $P(k+1)$.
LHS $= 3 + 3^2 + \cdots + 3^n + 3^{n+1} = \frac{3^{k+1}-3}{2} + 3^{k+1} = \frac{3^{k+1}+2\times 3^{k+1}-3}{2} = \frac{3^{k+2}-3}{2}$.
RHS $= \frac{3^{k+2}-3}{2}$.
Therefore, $P(k+1)$ is true whenever $P(k)$ is true.
By Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

(b) Let $P(n)$ be the proposition that $S_1, S_2, \cdots, S_n$ are divisible by 4.
By the conditions given, $P(1)$ and $P(2)$ are true.
Assume $P(k)$ is true, Consider S_{k+1}.
$S_{k+1} = 5S_k + S_{k-1}^2 \equiv 5 \times 0 + 0^2 \equiv 0 (\text{mod } 4)$
Therefore, $P(k+1)$ is true whenever $P(k)$ is true.
By Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Question 2

(a) Since $7 - 2 = 5$ and $3 \nmid 5$, $7 \nsim 2$.
Since $2 - 5 = -3$ and $3 \mid -3$, $2 \sim 5$.
Since $8 - 8 = 0$ and $3 \mid 0$, $8 \sim 8$.

(b) $x \sim y \iff \{(x, y) \in Z \times Z | x \equiv y \text{ mod } 3\}$

(c) Since $x - x = 0$ and $3 \mid 0$, $\sim$ is reflexive.
Since $x \sim y \Leftrightarrow 3 \mid x - y \Leftrightarrow 3 \mid y - x \Leftrightarrow y \sim x$, $\sim$ is symmetric.
Since $x \sim y$ and $y \sim z \Leftrightarrow 3 \mid x - y$ and $3 \mid y - z$
$\Rightarrow 3 \mid (x - y) - (y - z) \Leftrightarrow 3 \mid x - z \Leftrightarrow x \sim z$, $\sim$ is transitive.

(d)

$$[2]_3 = \{2, 5, 8\}$$

$$[3]_3 = \{3, 6\}$$

$$[4]_3 = \{4, 7\}$$

Question 3

(a) f is injective since $f(x) = f(y) \Leftrightarrow 16x - 5 = 16y - 5 \Leftrightarrow x = y$.
f is surjective since for any real number y, $\exists x = \frac{y+5}{16}$ such that $f(x) = y$.

(b) Let $y = f(x) = 16x - 5$. Then $x = f^{-1}(y) = \frac{y+5}{16}$.

(c) g is an injection since $g(x) = g(y) \Leftrightarrow 16x - 5 = 16y - 5 \Leftrightarrow x = y$.

(d) g is not a surjection since 0 is not mapped to by any x in the domain of g.

Question 4

(a) By Euclidean algorithm,

$$\begin{aligned} 284 &= 168 + 116 \\ &= 52 + 2 \times 116 \\ &= 5 \times 52 + 2 \times 12 \\ &= 5 \times 4 + 22 \times 12 \\ &= 71 \times 4 + 0 \end{aligned}$$

$$\begin{aligned} \gcd(284, 168) &= \gcd(116, 168) \\ &= \gcd(116, 52) \\ &= \gcd(12, 52) \\ &= \gcd(12, 4) \\ &= \gcd(0, 4) = 4 \end{aligned}$$

(b) $\{n \mid n \equiv 0 (\bmod 4)\}$

(c) $4 = 52 \times 1 - 12 \times 4 = 52 \times 1 - (116 - 52 \times 2) \times 4$
$= 52 \times 9 - 116 \times 4 = (168 - 116) \times 9 - 116 \times 4 = 168 \times 9 - 116 \times 13$
$= 168 \times 9 - (284 - 168) \times 13 = 168 \times 22 - 284 \times 13$.

Therefore the general solution to $284x + 168y = 4$ is: $x = -13 + 42a, y = 22 - 71a, a \in \mathbb{Z}$.
The smallest integer x is 29.

Question 5

(a) $0^2 \equiv 0 (\bmod 7)$, $1^2 \equiv 1 (\bmod 7)$, $2^2 \equiv 4 (\bmod 7)$, $3^2 \equiv 2 (\bmod 7)$,
$4^2 \equiv 2 (\bmod 7)$, $5^2 \equiv 4 (\bmod 7)$, $6^2 \equiv 1 (\bmod 7)$.
Therefore the congruence classes are $[0]_7, [1]_7, [2]_7, [4]_7$.

(b) From part (a), we know that $\forall k \in \mathbb{Z}$,

$$n^2, m^2 \in [0]_7 \text{ or } [1]_7 \text{ or } [2]_7 \text{ or } [4]_7$$

We list out all the possibilities of $n^2 + m^2$, we have

$$\begin{aligned} 0+0 &\equiv 0 \bmod 7, & 0+1 &\equiv 1 \bmod 7 \\ 0+2 &\equiv 2 \bmod 7, & 0+4 &\equiv 4 \bmod 7 \\ 1+1 &\equiv 2 \bmod 7, & 1+2 &\equiv 3 \bmod 7 \\ 1+4 &\equiv 5 \bmod 7, & 2+2 &\equiv 1 \bmod 7 \\ 2+4 &\equiv 6 \bmod 7, & 4+4 &\equiv 1 \bmod 7 \end{aligned}$$

Since only $n^2 \in [0]_7$ and $m^2 \in [0]_7$ gives a sum of 0, from part (a), $n \equiv 0 \bmod 7$ and $m \equiv 0 \bmod 7$, implies that m and n are both divisible by 7.

(c) No. For $a = 1, b = 2, c = 3$, $a^2 + b^2 + c^2 \equiv 1 + 4 + 2 \equiv 0 (\text{mod } 7)$.

Question 6

(a) Since a function that maps to itself is one to one and onto, $f : A \to A$ is a bijection
$\Rightarrow A \sim A$, $\sim$ is reflexive.
If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.
Therefore, $A \sim B \Leftrightarrow B \sim A$. $\sim$ is symmetric.
If $f : A \to B$ and $g : B \to C$ are both bijections, Let $h = g \circ f$. We want to show that $h : A \to C$ is also a bijection.

(Injectivity) Let $a_1, a_2 \in A$, we want to show that if $h(a_1) = h(a_2)$, then $a_1 = a_2$.

$$\begin{aligned} h(a_1) = h(a_2) \quad &\Rightarrow g \circ f(a_1) = g \circ f(a_2) \\ &\Rightarrow f(a_1) = f(a_2) \quad (\text{since } g \text{ is injective}) \\ &\Rightarrow a_1 = a_2 \quad (\text{since } f \text{ is injective}) \end{aligned}$$

Therefore h is an injective function from A to C.

(Surjectivity) Let $c \in C$, since g is surjective function, there exists $b \in B$ such that $g(b) = c$. Since f is surjective function, there exists $a \in A$ such that $f(a) = b$. Therefore $\forall c \in C$, there exists $a \in A$ such that

$$h(a) = g \circ f(a) = g(b) = c$$

Therefore h is an surjective function from A to C.

Therefore $\sim$ is an equivalence relation.

(b) First of all, we try to list the elements of $S(U)$.

$$S(U) = \{\{[0]_3\}, \{[1]_3\}, \{[2]_3\}, \{[0]_3, [1]_3\}, \{[0]_3, [2]_3\}, \{[1]_3, [2]_3\}, \{[0]_3, [1]_3, [2]_3\}\}$$

If there exists a bijection, $f : A \to B$, then the cardinality of A and B must be the same. Hence there are in total 3 different equivalence class on $S(U)$. They are

$$[\{[0]_3\}] = \{\{[0]_3\}, \{[1]_3\}, \{[2]_3\}\}$$

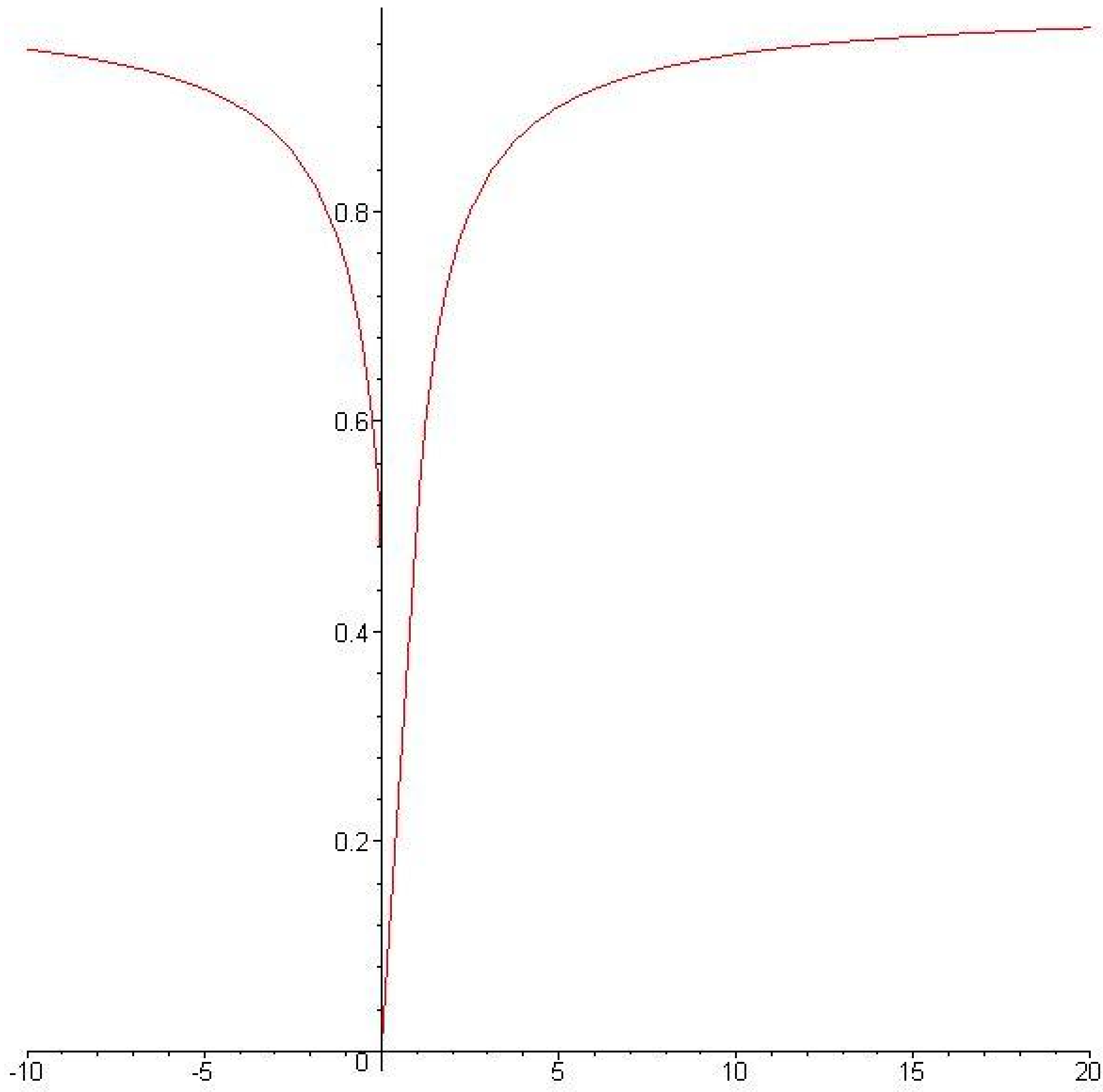
$$[\{[0]_3, [1]_3\}] = \{\{[0]_3, [1]_3\}, \{[0]_3, [2]_3\}, \{[1]_3, [2]_3\}\}$$
$$[\{[0]_3, [1]_3, [2]_3\}] = \{\{[0]_3, [1]_3, [2]_3\}\}$$

Question 7

(a) Yes. For all $x \in h(A \cup B)$, either $x \in h(A)$ or $x \in h(B)$, and in the first case $h(x) = f(x)$ covers the entire range of A, since f is surjective, and in the second case $h(x) = g(x)$ covers the entire range of B, since g is surjective, the range of h covers the union of A and B, h is a surjection.

(b) No. First of all, note that the statement is true when A and B are both finite sets. Thus, we want to find some sets A and B such that they are infinite sets, so that we can arrive to a function h which is not injective.
Consider $A = [0, \infty)$ and $B = (-\infty, 1]$. We define f and g as follows

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in [0,1], \\ 1 - \frac{1}{2x} & \text{if } x \in (1, \infty). \end{cases} \quad \& \quad g(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in [0,1], \\ 1 + \frac{1}{2(x-1)} & \text{if } x \in (-\infty, 0). \end{cases}$$



Note that f and g are both injective functions. However, h is not injective.

$$h(x) = \begin{cases} 1 + \frac{1}{2(x-1)} & \text{if } x \in (-\infty, 0) \\ \frac{1}{2}x & \text{if } x \in [0,1], \\ 1 - \frac{1}{2x} & \text{if } x \in (1, \infty). \end{cases}$$

For instance, $x_1 = 2 \neq -1 = x_2$ but $h(x_1) = \frac{3}{4} = h(x_2)$.

Question 8

(a) Since p is a prime greater than 3, $p \equiv 1 \bmod 3$ or $p \equiv 2 \bmod 3$. When $p \equiv 1 \bmod 3$, there exists $k \in \mathbb{Z}$ such that $p - 1 = 3k$,

$$2p - 2 = 6k \Rightarrow 2p + 1 = 2p - 2 + 3 = 6k + 3 = 3(2k + 1)$$

$2p+1$ is divisible by 3, thus it is not a prime number. When $p \equiv 2 \bmod 3$, there exists $m \in \mathbb{Z}$ such that $p - 2 = 3m$,

$$4p - 8 = 12m \Rightarrow 4p + 1 = 4p - 8 + 9 = 12m + 9 = 3(4m + 1)$$

$4p + 1$ is divisible by 3, thus it is not a prime number.
Therefore they cannot be prime at the same time.

(b) Consider an integer $n \in \mathbb{Z}^+$. We split the cases of n into two.

(Case 1) n is not a perfect square.
Since n is not a perfect square, i.e $\sqrt{n} \notin \mathbb{Z}^+$, by considering the property of divisors, if a is a divisor of n, then there exists $b \in \mathbb{Z}$ such that $a \times b = n$. WLOG, we assume that $a < b$. In other words, b is the other divisor of n that pairs up with a. Now, we might want to study how many pairs of such divisors are there for n. Since we know that there is a b divisors corresponding to a, then we just need to consider those divisors that are less than $\sqrt{n}$. By the following claim:

Claim: There is no 2 paired-distinct divisors of n such that both of them larger than $\sqrt{n}$.
Assume to the contrary that there are 2 divisors, a and b of n such that both of them larger than $\sqrt{n}$, ie.

$$a > \sqrt{n} \quad \& \quad b > \sqrt{n}, \quad \Rightarrow \quad a \times b > \sqrt{n} \times \sqrt{n} = n$$

We obtain a contradiction. Therefore one of the paired divisors must be less than $\sqrt{n}$ and the other divisor must be greater than $\sqrt{n}$.
Now consider the set of divisors that are less than $\sqrt{n}$, there are at most $\sqrt{n}$ of divisors of n. By the 'pairing' that we have discussed earlier, there are in total at most $2\sqrt{n}$ divisors of n.

(Case 2) n is a perfect square.
Since n is a perfect square, there exists $k \in \mathbb{Z}^+$ such that $\sqrt{n} = k$. As similar in argument in Case 1, we now need to count divisors of n that are not equal to k. By similar argument as above, there are at most $k - 1$ divisors of n that are less than k. Therefore, considering the paired up divisors of n, there are in total at most $2(k - 1) + 1$ divisors of n. The counting of the '1' is the counting of k as a divisor of n. Therefore the number of divisors of n is at most

$$2k - 2 + 1 = 2k - 1 \leq 2k = 2\sqrt{n}$$