NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

Written by: Chua Xin Rong
Audited by: Chua Hongshen

MA3110 Mathematical Analysis II

AY 2011/2012 Sem 2

Question 1

(a) We may assume that f is strictly increasing or strictly decreasing. If f is not monotone, then since f is continuous, f can be broken into subintervals that are strictly increasing or strictly decreasing, each of which must satisfy the conditions given in the question. Without loss of generality, consider the case where f is strictly increasing.

Note that $h'(x) = \frac{(x-a)f'(x)-f(x)+f(a)}{(x-a)^2}$. Therefore, we can see that $h'(c) = 0$ for some $c \in (a, b)$ implies that $\frac{f(c)-f(a)}{c-a} = f'(c)$.

Since f is strictly increasing, $\therefore h(a + \delta a) = \frac{f(a+\delta a)-f(a)}{\delta a} > 0 = f'(a) = h(a)$ for $\delta a \in (0, b - a)$. Hence, h is strictly increasing near a and exists $a' \in (a, b)$ such that $h'(a') > 0$.

Now, $h'(b) = \frac{(b-a)f'(b)-f(b)+f(a)}{(b-a)^2} = \frac{-f(b)+f(a)}{(b-a)^2} < 0$.

Since $h'(x)$ is continuous and experience a change of sign from a to b, by Intermediate Value Theorem, there exist a point $c \in (a, b)$ such that $h'(c) = 0$, which is what we desired.

(b) By Mean Value Theorem, for any $x \in (0, 1)$ there exist $y \in (0, x)$ such that $|\frac{g'(x)-g'(0)}{x-0}| = |g''(y)| < K$.
$\therefore |g'(x)| < K|x| + |g'(0)| < K + |g'(0)|, \forall x \in (0, 1)$, *i.e.* g' is bounded in $(0, 1)$ by some $K_1 = K + |g'(0)| \in \mathbb{R}$.

Given any $a, b \in (0, 1)$, by Mean Value Thoerem again, there exists $c \in (a, b)$ such that $\frac{g(a)-g(b)}{a-b} = |g'(c)| < K_1$.
$\therefore |g(a)-g(b)| \leq K_1|a-b|$. Thus, taking $\delta = \epsilon/K_1$, we have $|a-b| < \delta$ implies that $|g(a)-g(b)| < \epsilon$. Thus, g is uniformly continuous on $(0, 1)$.

Question 2

(a) (i) We know that $\sin x$ is integrable and $\int_0^\pi \sin x = 2$. Since $\sin x$ is integrable, given $\epsilon > 0$, there exists δ such that for any partition $\dot{P}$ such that $||\dot{P}|| < \delta, |S(\sin, \dot{P})| < \epsilon$. Let P_1 be one such partition where all the points in $P_1 \in \mathbb{Q}$, i.e. the points C_n are the points in P_1. Then $S(h, P_1) = S(\sin, P_1)$, which implies that $|S(h, P_1) - 2| < \epsilon$, as desired.

(ii) Similarly, we know that $\int_0^\pi \cos x = 0$. Since $\cos x$ is integrable, given $\epsilon > 0$, there exists δ such that for any partition P such that $||P|| < \delta, |S(\cos, P)| < \epsilon$. Let P_2 be one such partition where all the points in $P_2 \notin \mathbb{Q}$, i.e. the points D_n are the points in P_2. Then $S(h, P_2) = S(\cos, P_2)$, which implies that $|S(h, P_2) - 0| < \epsilon$, as desired.

One explicit selection of points is as follows: Let the kth partition of (Q_n) be $= [\frac{(k-1)\pi}{n}, \frac{k\pi}{n}]$. If the right endpoint of the partition is smaller than $\pi/2$, choose D_k as the left endpoint of the kth partition. If the kth partition contains $\pi/2$ and k is an odd number, let $D_k = \pi/2$. Otherwise, choose D_k to be the right endpoint of the kth partition. Note that all the D_k specified are irrational, thus all the values are cosines. The summation of these values cancel due to the symmetry of the cosine function.

Since two Riemann sums converge to different values, the function is not Riemann integrable.

(b) (i) By Cauchy Criterion, for every $\epsilon > 0$, there exist $M(\epsilon)$ such that if $m > n \geq M(\epsilon)$, then $|S_m - S_n| < \epsilon, \forall x \in I$.
In particluar, take $m = n + 1$, then $|f_{n+1}| = |S_{n+1} - S_n| < \epsilon, \forall x \in I$.
$\therefore$ $(f_n(x))$ converges uniformly on I to the zero constant function.

(ii) For $x \in [-K, K], |\sin(\frac{x}{n^2})| < \frac{x}{n^2} \leq \frac{K}{n^2}$ and $\sum_{n=1}^{\infty} \frac{K}{n^2}$ converges by the Integral Test. Hence, by Weierstrass M-Test, $\sum_{n=1}^{\infty} \sin(\frac{x}{n^2})$ converges uniformly on $[-K, K]$.

For any $x \in \mathbb{R}, \sum_{n=1}^{\infty} |\sin(\frac{x}{n^2})| < \sum_{n=1}^{\infty} |\frac{x}{n^2}|$ and $\sum_{n=1}^{\infty} |\frac{x}{n^2}|$ converges by the Integral Test, hence by Comparison Test, $\sum_{n=1}^{\infty} \sin(\frac{x}{n^2})$ converges pointwise on $\mathbb{R}$ to a function f. Since each function is continuous, f must be continuous.

Taking $x = n^2$, then $\sum_{n=1}^{\infty} \sin(\frac{x}{n^2}) = \sum_{n=1}^{\infty} \sin 1$ diverges. Hence, $\sum_{n=1}^{\infty} \sin(\frac{x}{n^2})$ does not converge uniformly on $\mathbb{R}$.

Question 3

(i) We know that $\frac{d}{dx}(\sin x - x) = \cos x - 1 < 0, \forall x \in \mathbb{R}$. Since $\sin 0 = 0$, $\therefore \sin x < x$ for $x > 0$.

Let $g_1(x) = \cos x - 1 + \frac{1}{2!}x^2, x > 0$. By Mean Value Theorem, $\frac{g_1(x) - g_1(0)}{x - 0} = g_1'(c) = -\sin c + c > 0$ for some $c \in (0, x)$. $\therefore g_1(x) > 0, \forall x > 0$.

Let $f_1(x) = \sin x - x + \frac{1}{3!}x^3, x > 0$. By Mean Value Theorem, $\frac{f_1(x) - f_1(0)}{x - 0} = f_1'(c) = g_1(c) > 0$ for some $c \in (0, x)$. $\therefore \sin x > x - \frac{1}{3!}x^3, \forall x > 0$.

Similarly, let $g_2(x) = \cos x - 1 + \frac{1}{2!}x^2 - \frac{1}{4!}x^4, x > 0$. By Mean Value Theorem, $\frac{g_2(x) - g_2(0)}{x - 0} = g_2'(c) = -\sin c + c - \frac{1}{3!}x^3 < 0$ for some $c \in (0, x)$. $\therefore g_2(x) < 0, \forall x > 0$.

Let $f_2(x) = \sin x - x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5, x > 0$. By Mean Value Theorem, $\frac{f_2(x) - f_2(0)}{x - 0} = f_2'(c) = g_2(c) < 0$ for some $c \in (0, x)$. $\therefore \sin x < x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5, \forall x > 0$

Combining both, we have $x - \frac{1}{3!}x^3 < \sin x < x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5, \forall x > 0$.

(ii) From (i), we know that $\sin(\frac{x}{\sqrt{n}}) - \frac{x}{\sqrt{n}} > -\frac{1}{3!}(\frac{x}{\sqrt{n}})^3$.

The series $\sum_{n=1}^{\infty} |-\frac{1}{3!}(\frac{x}{\sqrt{n}})^3| \leq \frac{a^3}{6}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges on $[0,a]$ by the Integral Test. So, $\sum_{n=1}^{\infty} -\frac{1}{3!}(\frac{x}{\sqrt{n}})^3$ converges uniformly on $[0,a]$ by Weierstrass M-Test. Since each partial sum is continuous, the series converges to a continuous function $f:[0,a]\to\mathbb{R}$.

For any $x\in[0,\infty)$, $\sum_{n=1}^{\infty}|\sin(\frac{x}{\sqrt{n}})-\frac{x}{\sqrt{n}}| \leq \sum_{n=1}^{\infty}|\sin(\frac{x}{\sqrt{n}})|+|\frac{x}{\sqrt{n}}| < 2\sum_{n=1}^{\infty}\frac{x}{\sqrt{n}}$ and $\sum_{n=1}^{\infty}\frac{x}{\sqrt{n}}$ converges by the Integral Test, hence by Comparison Test, $\sum_{n=1}^{\infty}\sin(\frac{x}{\sqrt{n}})-\frac{x}{\sqrt{n}}$ converges pointwise on $[0,\infty)$ to a function f. Since each function is continuous, f must be continuous.

(iii) Note that $|\frac{1}{\sqrt{n}}\cos(\frac{x}{\sqrt{n}})-\frac{1}{\sqrt{n}}| = |\frac{1}{\sqrt{n}}(\cos(\frac{x}{\sqrt{n}})-1)| < \frac{2}{\sqrt{n}}$ and $\sum_{n=1}^{\infty}\frac{2}{\sqrt{n}}$ converges. Hence by Weierstrass M-Test, $\sum_{n=1}^{\infty}\cos(\frac{x}{\sqrt{n}})-\frac{1}{\sqrt{n}}$ converges uniformly to a function g on $[0,a]$ for any $a>0$.

Similar to the argument above, we can conclude that $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\cos(\frac{x}{\sqrt{n}})-\frac{1}{\sqrt{n}}$ converges pointwise to a continuous function g on $[0,\infty)$.

(iv) Since $S_k(x)=\sum_{n=1}^{k}\sin(\frac{x}{\sqrt{n}})-\frac{x}{\sqrt{n}}$ converges for some $x_0\in[a,\infty)$ and $S_k'(x)$ exists on $[a,\infty]$ and converges uniformly on $[a,\infty]$ to a function g, hence $S_k(x)$ converges on $[a,\infty]$ to f, where f has a derivative on $[a,\infty]$ and $f'=g$.

Question 4

(a) (i) $|\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{e^{nx}}| = |\sum_{n=1}^{\infty}(\frac{-1}{e^x})^n| = |\frac{e^{-x}}{1+e^{-x}}| = |\frac{1}{e^x+1}| < 1$ for all $x\in[0,\infty)$.

Hence, $|S_n|<|S_\infty|<1$ is uniformly bounded for all $x\in[0,\infty), n\in[1,\infty)$.

(ii) $a_n=\frac{1}{\sqrt{n^2+x^2}}\to 0$ as $n\to\infty$ and $\{a_n\}$ is decreasing.

Thus, by Dirichlet's Test, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}e^{-nx}}{\sqrt{n^2+x^2}}$ converges uniformly on $[0,\infty)$.

(b) (i) Using the Ratio Test, $\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty}|\frac{1}{(4n+4)(4n+3)(4n+2)(4n+1)}| = 0 < 1$.

Hence, the series converges.

(ii) $f^{(i)}(x)=\sum_{n=0}^{\infty}\frac{x^{4n-i}}{(4n-i)!}$, for $i=0,1,2,3$, where $f^{(0)}=f$.
$\therefore$

$$\begin{aligned} f(x)+f'(x)+f''(x)+f'''(x) &= \sum_{i=1}^{4}(\sum_{n=0}^{\infty}\frac{x^{4n-i}}{(4n-i)!}) \\ &= \sum_{m=0}^{\infty}\frac{x^m}{m!} \\ &= e^x. \end{aligned}$$