NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Professor Lee Soo Teck

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MA2108 Mathematical Analysis I
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Question 1

Firstly, note that $a_n \geq 0 \ \forall n \in \mathbb{N}$. Consider $|a_{n+2} - a_{n+1}|$.

$$\begin{aligned}
|a_{n+2} - a_{n+1}| &= \left| \left(2 + \frac{7}{a_n + 4}\right) - \left(2 + \frac{7}{a_n + 4}\right) \right| \\
&= 7 \left| \frac{a_n - a_{n+1}}{(a_{n+1} + 4)(a_n + 4)} \right| \\
&\leq \frac{7}{16} |a_{n+1} - a_n|
\end{aligned}$$

Therefore (a_n) is contractive and converges. Let $\lim_{n\to\infty} a_n = a$.

$$\begin{aligned}
\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \left(2 + \frac{7}{a_n + 4}\right) \\
\Rightarrow a &= 2 + \frac{7}{a+4} \\
\Rightarrow a^2 + 2a - 15 &= 0 \\
\Rightarrow (a+5)(a-3) &= 0 \\
\Rightarrow a = -5 \quad &\text{or} \quad a = 3
\end{aligned}$$

Since $a_n \geq 0$, we have $a \geq 0$. Therefore, we conclude with $a = 3$.

Question 2

(a) (i)

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{3n+5}{1-n^2+2n^3}}{\frac{1}{n^2}} &= \lim_{n\to\infty} \frac{3n^3 + 5n^2}{1 - n^2 + 2n^3} \\
&= \lim_{n\to\infty} \frac{3 + 5/n}{1/n^3 - 1/n + 2} \\
&= \frac{3}{2} > 0
\end{aligned}$$

By the limit comparision test, we conclude that $\sum_{n=1}^{\infty} \frac{3n+5}{1-n^2+2n^3}$ converges since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(ii)

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{3^n}{\left(1 + \frac{1}{2n}\right)^{4n^2}} \right|^{1/n} &= \lim_{n\to\infty} \frac{3}{\left(1 + \frac{1/2}{n}\right)^{4n}} \\
&= \frac{3}{(e^{1/2})^4} \\
&= \frac{3}{e^2} < 1
\end{aligned}$$

By the root test, we conclude that $\sum_{n=1}^{\infty} \frac{3^n}{\left(1+\frac{1}{2n}\right)^{4n^2}}$ converges.

(b) (i) By hypothesis, $a_n \leq a_{n+1}$ $\forall n \geq K_1$.

$$\Rightarrow a_{K_1} \leq a_{K_1+1} \leq \cdots \leq a_n \qquad \forall n \geq K_1$$

$\Rightarrow \lim_{n\to\infty} a_n \neq 0$. Therefore, we conclude that $\sum_{n=1}^{\infty} a_n$ diverges.

(ii) By hypothesis, $a_{n+1} < ra_n$ $\forall n \geq K_2$.

$$\Rightarrow a_n < ra_{n-1} < \cdots < r^{n-K_2} a_{K_2} = Cr^n \qquad \text{where } C = \frac{a_{K_2}}{r^{K_2}}$$

$\sum_{n=1}^{\infty} Cr^n$ is a geometric series with $0 < r < 1$, thus it converges. Therefore, we conclude that $\sum_{n=1}^{\infty} a_n$ converges by the comparision test.

Question 3

(a) Let $M = \limsup a_n$. Hence $\forall \varepsilon > 0$, there are infinitely many n's such that $M - \varepsilon < a_n \leq M + \varepsilon$. Consider $M - \frac{1}{k} < a_n \leq M + \frac{1}{k}$ where $k \in \mathbb{N}$. For $k = 1$, pick a_n such that $M - 1 < a_n \leq M + 1$ and define $a_{n_1} := a_n$. For $k = 2$, pick a_m such that $M - \frac{1}{2} < a_m \leq M + \frac{1}{2}$ and $m > n_1$. Note that such an m always exists due to existence of infinitely many i's such that $M - \frac{1}{2} < a_i \leq M + \frac{1}{2}$. Define $a_{n_2} := a_m$ and continue to define a_{n_k} inductively. Then $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ with the following property.

$$M - \frac{1}{k} < a_{n_k} \leq M + \frac{1}{k} \qquad \forall k \in \mathbb{N}$$

Since $\lim_{k\to\infty} M - \frac{1}{k} = M = \lim_{k\to\infty} M + \frac{1}{k}$,

$$\lim_{k\to\infty} a_{n_k} = M$$

by squeeze theorem.

(b) (i) For each $n \in \mathbb{N}$, $-1 \leq \sin(n\pi/4) \leq 1$.

$$\Rightarrow -\frac{2n^2+3}{\sqrt{4n^4+5n^3-1}} \leq x_n \leq \frac{2n^2+3}{\sqrt{4n^4+5n^3-1}}$$

Now, let (x_{n_k}) be a convergent subsequence tending to x.

$$\Rightarrow -1 = \lim_{k\to\infty} -\frac{2n_k^2+3}{\sqrt{4n_k^4+5n_k^3-1}} \leq \lim_{k\to\infty} x_{n_k} = x \leq \lim_{k\to\infty} \frac{2n_k^2+3}{\sqrt{4n_k^4+5n_k^3-1}} = 1$$

Hence 1 and -1 are an upper bound and lower bound for $C(x_n)$ respectively. Futhermore,

$$x_n = \begin{cases} \frac{2n^2+3}{\sqrt{4n^4+5n^3-1}} & \text{if } n = 8k+2 \\ -\frac{2n^2+3}{\sqrt{4n^4+5n^3-1}} & \text{if } n = 8k-2 \end{cases}$$

So $x_{8k+2} \to 1$ and $x_{8k-2} \to -1$. Therefore, we conclude that $\limsup x_n = 1$ and $\liminf x_n = -1$.

(ii) Since $\limsup x_n \neq \liminf x_n$, we conclude that (x_n) diverges.

Question 4

(a) Consider $\left|\frac{x^2-3x+1}{2x-1}+1\right|$.

$$\begin{aligned}\left|\frac{x^2-3x+1}{2x-1}+1\right| &= \left|\frac{x^2-x}{2x-1}\right| \\ &= |x-1|\left|\frac{x}{2x-1}\right| \\ &= |x-1|\left|\frac{1}{2}+\frac{1}{2(2x-1)}\right|\end{aligned}$$

If $|x-1|<\frac{1}{4}$, then

$$\begin{aligned} -\frac{1}{4} &< x-1 < \frac{1}{4} \\ \Rightarrow -\frac{1}{2} &< 2x-2 < \frac{1}{2} \\ \Rightarrow \frac{1}{2} &< 2x-1 < \frac{3}{2} \\ \Rightarrow \frac{2}{3} &< \frac{1}{2x-1} < 2 \\ \Rightarrow \frac{4}{3} &< \frac{1}{2}+\frac{1}{2(2x-1)} < \frac{3}{2}\end{aligned}$$

Now, let $\varepsilon>0$ be given and let $\delta=\min\left(\frac{2\varepsilon}{3},\frac{1}{4}\right)$. If $0<|x-1|<\delta$,

$$\Rightarrow \left|\frac{x^2-3x+1}{2x-1}+1\right| < \frac{3}{2}|x-1| < \frac{3}{2}\frac{2\varepsilon}{3}=\varepsilon$$

Therefore, we conclude that $\lim_{x\to1}\frac{x^2-3x+1}{2x-1}=-1$.

(b) (i) Let $f(x)=x\cos\left(\frac{x}{x-1}\right)$. Consider $x_n=1+\frac{1}{2n\pi-1}$.

$$\begin{aligned}&\Rightarrow x_n\to 1 \quad\text{and}\quad \frac{x_n}{x_n-1}=2n\pi \\ &\Rightarrow \cos\left(\frac{x_n}{x_n-1}\right)=1 \\ &\Rightarrow \lim_{n\to\infty} f(x_n)=\lim_{n\to\infty}x_n=1\end{aligned}$$

Now, suppose $y_n=1+\frac{1}{(2n-1)\frac{\pi}{2}-1}$.

$$\begin{aligned}&\Rightarrow y_n\to 1 \quad\text{and}\quad \frac{y_n}{y_n-1}=(2n-1)\frac{\pi}{2} \\ &\Rightarrow \cos\left(\frac{y_n}{y_n-1}\right)=0 \\ &\Rightarrow \lim_{n\to\infty} f(y_n)=\lim_{n\to\infty}0=0\end{aligned}$$

Therefore, by the divergent criterion, we conclude that $\lim_{x\to1}x\cos\left(\frac{x}{x-1}\right)$ does not exist.

(ii)

$$-(x-1)^2 \le (x-1)^2\cos\left(\frac{x}{x-1}\right) \le (x-1)^2$$

Since $\lim_{x\to1}-(x-1)^2=0=\lim_{x\to1}(x-1)^2$, by the squeeze theorem, we conclude that

$$\lim_{x\to1}(x-1)^2\cos\left(\frac{x}{x-1}\right)=0$$

Question 5

(a) Let $c \in \mathbb{R}$, (x_n) be a rational sequence such that $x_n \to c$, (y_n) be an irrational sequence such that $y_n \to c$.

$$\Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n - 1 = c - 1 \quad \text{and}$$
$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 2y_n - 3 = 2c - 3$$

If $c \neq 2$, then $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$. Hence $\lim_{x\to c} f(x)$ does not exist by the divergent criterion and is not continuous on $\mathbb{R}-\{2\}$. Now, let $\varepsilon > 0$ be given and let $\delta = \frac{\varepsilon}{2}$. If $0 < |x-2| < \delta$,

$$\begin{aligned}
\Rightarrow |f(x) - f(2)| &= \begin{cases} |(x-1)-1| & \text{if } x \text{ is rational} \\ |(2x-3)-1| & \text{if } x \text{ is irrational} \end{cases} \\
&= \begin{cases} |x-2| & \text{if } x \text{ is rational} \\ 2\,|x-2| & \text{if } x \text{ is irrational} \end{cases} \\
&< \varepsilon
\end{aligned}$$

Hence, $\lim_{x\to 2} f(x) = f(2)$ and we conclude that f is continuous only at 2.

(b) Since $M - g(a) > 0$, $\exists \delta > 0$ such that

$$|g(x) - g(a)| < M - g(a) \qquad \forall x \in (a-\delta, a+\delta)$$
$$\Rightarrow g(x) < (M - g(a)) + g(a) = M \qquad \forall x \in (a-\delta, a+\delta)$$

Question 6

(a) Let $x_n = n^{1/n} - 1$.

$$\begin{aligned}
\Rightarrow n = (1+x_n)^n &\geq 1 + \frac{n(n-1)}{2}x_n^2 \quad \text{by binomial theorem} \\
\Rightarrow x_n^2 &\leq \frac{2}{n} \\
\Rightarrow |x_n| &\leq \sqrt{\frac{2}{n}}
\end{aligned}$$

Since $\lim_{n\to\infty} \sqrt{\frac{2}{n}} = 0$,

$$\Rightarrow \lim_{n\to\infty} |x_n| = 0$$

by squeeze theorem. Therefore, we conclude that $\lim_{n\to\infty} x_n = 0$, that is

$$\lim_{n\to\infty} n^{1/n} = 1$$

(b) Firstly, observe that $\{x_n\}$ is increasing. Now, suppose that $\{x_n\}$ is bounded above.

$$\Rightarrow \exists M \in \mathbb{R} \quad \text{s.t.} \quad x_n \leq M \ \forall n \in \mathbb{N}$$
$$\Rightarrow \frac{1}{x_n} \geq \frac{1}{M} \ \forall n \in \mathbb{N}$$

Furthermore, by Monotone Convergence Theorem, $\{x_n\}$ converges. For each $k \in \mathbb{N}$, we have:

$$\begin{aligned}
x_{k+1} - x_k &= \frac{1}{kx_k} \\
x_k - x_{k-1} &= \frac{1}{(k-1)x_{k-1}} \\
&\vdots \\
x_2 - x_1 = x_2 - 1 &= \frac{1}{x_1}
\end{aligned}$$

Summing the equations, we get:

$$\begin{aligned}
x_{k+1} - 1 &= \sum_{i=1}^{k} \frac{1}{ix_i} \\
\Rightarrow x_{k+1} &= 1 + \sum_{i=1}^{k} \frac{1}{ix_i} \geq \frac{1}{M} \sum_{i=1}^{k} \frac{1}{i}
\end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i}$ diverges, we deduce that $\{x_{k+1}\}$ is unbounded above, contradicting our assumption. We conclude that $\{x_k\}$ diverges.

Question 7

(a) Observe that h is increasing. Let $x \in (0,1)$ and let $\varepsilon > 0$ be given. Since g is continuous on $[0,1]$, it is continuous on $(0,1)$ too. That is,

$$\exists \delta > 0 \text{ s.t. } \forall y \in (x-\delta, x+\delta),\ |g(y) - g(x)| < \frac{\varepsilon}{2}$$

Consider the following cases on $y \in (x-\delta, x+\delta)$.

Case 1 $y \in [x, x+\delta) \cap (0,1)$

Consequently,

$$h(x) \leq h(y) < h(y) + \varepsilon$$

Now, we are required to prove $h(y) - \varepsilon < h(x)$. Let $t \in [0,y]$. Then

$$\begin{aligned}
\forall t \in [0,x], &\quad g(t) - \varepsilon < g(t) \leq h(x) \qquad \text{and} \\
\forall t \in [x,y], &\quad g(t) - \varepsilon < g(x) - \frac{\varepsilon}{2} < g(x) \leq h(x)
\end{aligned}$$

$\Rightarrow \forall t \in [0,y],\ g(t) - \varepsilon < h(x)$

$$\Rightarrow h(y) - \varepsilon = \sup\{g(t) - \varepsilon \,|\, 0 \leq t \leq y\} \leq h(x)$$

$\therefore |h(x) - h(y)| \leq \varepsilon$

Case 2 $y \in (x-\delta, x]$

Consequently,

$$h(y) \leq h(x) < h(x) + \varepsilon$$

Now, we are required to prove $h(x) - \varepsilon < h(y)$. Let $s \in [0,x]$. Then

$$\begin{aligned}
\forall s \in [0,y], &\quad g(s) - \varepsilon < g(s) \leq h(y) \qquad \text{and} \\
\forall s \in [y,x], &\quad g(s) - \varepsilon < g(x) - \frac{\varepsilon}{2} < g(y) \leq h(y)
\end{aligned}$$

$\Rightarrow \forall s \in [0, x]$, $g(s) - \varepsilon < h(y)$

$$\Rightarrow h(x) - \varepsilon = \sup\{g(s) - \varepsilon \,|\, 0 \le s \le x\} \le h(y)$$

$\therefore |h(x) - h(y)| \le \varepsilon$

$\Rightarrow \forall y \in (x - \delta, x + \delta)$, $|h(x) - h(y)| \le \varepsilon$. Therefore, we conclude that h is continuous on $(0, 1)$.

(b) Let $c_0 \in \mathbb{R}$. Let $P(n) : c_n = 1 + \frac{c_0 - 1}{2^n}$ and $f(c_n) = f(c_{n-1})$ where $n \in \mathbb{N}$. Consider $P(1)$. By definition, $\exists c_1 \in \mathbb{R}$ such that $c_1 = \frac{1 + c_0}{2} = 1 + \frac{c_0 - 1}{2}$ and $f(c_1) = f(c_0)$. Thus $P(1)$ is true. Suppose that $P(k)$ is true for some $k \in \mathbb{N}$. Consider $P(k+1)$. By definition, $\exists c_{k+1} \in \mathbb{R}$ such that $c_{k+1} = \frac{1 + c_k}{2}$ and $f(c_{k+1}) = f(c_k)$.

$$\begin{aligned} \Rightarrow c_{k+1} &= \frac{1 + c_k}{2} \\ &= \frac{1}{2} + \frac{1}{2}\left(1 + \frac{c_0 - 1}{2^k}\right) \quad \text{by induction hypothesis} \\ &= 1 + \frac{c_0 - 1}{2^{k+1}} \end{aligned}$$

$\therefore P(k)$ is true $\Rightarrow P(k+1)$ is true. By induction, $P(n)$ is true $\forall n \in \mathbb{R}$.

$$\Rightarrow c_n \to 1 \quad \text{and} \quad f(c_0) = f(c_1) = f(c_2) = \cdots = f(c_n)$$
$$\Rightarrow f(c_0) = \lim_{n\to\infty} f(c_0) = \lim_{n\to\infty} f(c_n) = f\left(\lim_{n\to\infty} c_n\right) = f(1)$$

Therefore, we conclude that f is a constant function on $\mathbb{R}$.

Question 8

(a) Let $n \in \mathbb{N}$. Define $g : \left[0, 1 - \frac{1}{n}\right] \to \mathbb{R}$ by $g(x) = f(x) - f\left(x + \frac{1}{n}\right)$. Since f is continuous on $[0, 1]$,

$$\Rightarrow g \text{ is continuous on } \left[0, 1 - \frac{1}{n}\right]$$

Suppose $\forall t \in \left[0, 1 - \frac{1}{n}\right]$, $g(t) \ne 0$. If $\exists t_1, t_2 \in \left[0, 1 - \frac{1}{n}\right]$ such that $g(t_1) > 0, g(t_2) < 0$. Then $\exists t_0 \in \left[0, 1 - \frac{1}{n}\right]$ such that $g(t_0) = 0$ by Intermediate Value Theorem. This will contradict $g(t) \ne 0 \forall t \in \left[0, 1 - \frac{1}{n}\right]$, thus

$$\begin{aligned} &\text{either} \qquad g(t) > 0 \qquad \forall t \in \left[0, 1 - \frac{1}{n}\right] \\ &\text{or} \qquad g(t) < 0 \qquad \forall t \in \left[0, 1 - \frac{1}{n}\right] \end{aligned}$$

WLOG, suppose that $g(t) > 0 \; \forall t \in \left[0, 1 - \frac{1}{n}\right]$.

$$\Rightarrow g(0), g\left(\frac{1}{n}\right), \ldots, g\left(1 - \frac{1}{n}\right) > 0$$
$$\Rightarrow f(0) > f\left(\frac{1}{n}\right) > f\left(\frac{2}{n}\right) > \cdots > f(1)$$

Contradicting the definition of f. Thus $\exists x_n \in \left[0, 1 - \frac{1}{n}\right]$ such that $g(x_n) = 0$. That is

$$f(x_n) = f\left(x_n + \frac{1}{n}\right)$$

(b) Let $\varepsilon > 0$ be given. Since $\lim_{x\to\infty} g(x) = 1$,

$$\Rightarrow \exists M > 0 \text{ s.t. } x > M \Rightarrow |g(x) - 1| < \frac{\varepsilon}{2}$$

Since g is continuous at $x = M$,

$$\Rightarrow \exists \delta_1 > 0 \text{ s.t. } |x - M| < \delta_1 \Rightarrow |g(x) - g(M)| < \frac{\varepsilon}{2}$$

Now, f is continuous on $[0, M]$ implies f is uniformly continuous on $[0, M]$.

$$\Rightarrow \exists \delta_2 > 0 \text{ s.t. } \forall y_1, y_2 \in [0, M],\ |y_1 - y_2| < \delta_2 \Rightarrow |g(y_1) - g(y_2)| < \varepsilon$$

Let $\delta = \min(\delta_1, \delta_2)$ and let $u, v \in [0, \infty)$ with $|u - v| < \delta$. Consider the following cases on u and v:

Case 1 $u, v \in [M, \infty)$

$$\Rightarrow |g(u) - g(v)| \leq |g(u) - 1| + |g(v) - 1| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Case 2 $u, v \in [0, M]$

$$\Rightarrow |g(u) - g(v)| < \varepsilon$$

Case 3 $u \in [0, M]$, $v \in (M, \infty)$

$$\Rightarrow M \in [u, v]$$

Since $|u - v| < \delta$,

$$\Rightarrow |u - M| < \delta \leq \delta_2 \quad \text{and} \quad |v - M| < \delta \leq \delta_2$$
$$\Rightarrow |g(u) - g(v)| \leq |g(u) - g(M)| + |g(v) - g(M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Case 4 $v \in [0, M]$, $u \in (M, \infty)$
Similar to **Case 3**.

$\Rightarrow \forall u, v \in [0, \infty)$, $|g(u) - g(v)| < \varepsilon$. That is, g is uniformly continuous on $[0, \infty)$.