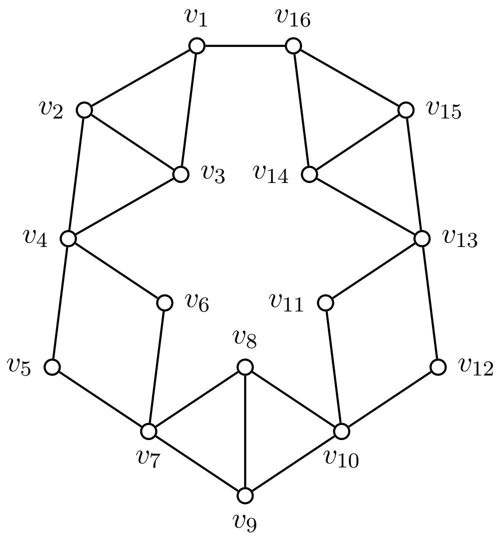
NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Zheng Shaoxuan

MA3233 Algorithmic Graph Theory
AY 2007/2008 Sem 2

Question 1

(i) Observe the following graph G:



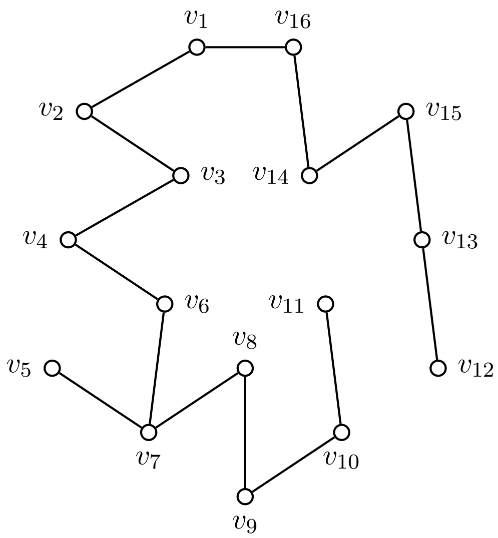
C_3 s and C_4 s can be easily observed from G. Interestingly, it can be observed that there are no other values of n for which C_n is a subgraph of G unless you (in layman's term) 'go around' the whole graph, i.e. find a cycle that passes through the vertices v_1, v_4, v_7, v_{10}, v_{13} and v_{16}, and the edge v_1v_{16}. Such cycles include C_{11}, C_{12}, C_{13} and C_{14}, depending on whether the edges v_2v_3, v_8v_9 and $v_{14}v_{15}$ are traversed by the respective cycles.

Hence the desired values of n are 3, 4, 11, 12, 13, 14.

(ii) A spanning tree of G with the greatest possible number of cut-vertices is equivalent to that with the least possible number of end-vertices.

Ideally a spanning (Hamiltonian) path of G will provide the least possible number of end-vertices. However such a spanning path does not exist on G. To prove this, suppose such a spanning path does exist. Then at least one among v_5 and v_6 has to be an end-vertex of the spanning path, and at least one among v_{11} and v_{12} has to be an end-vertex of the spanning path, because if both among either pair are not end vertices of the spanning path, then the 2 edges they are each incident with have to be included inside the spanning path, but this forms a C_4, a contradiction. Because a spanning path has exactly 2 end-vertices, exactly one among v_5 and v_6, and exactly one among v_{11} and v_{12} have to be an end-vertex of the spanning path. By symmetry, WLOG let v_5 and v_{12} be the stated end-vertices. Then v_4v_6, v_6v_7, $v_{10}v_{11}$ and $v_{11}v_{13}$ are within the spanning path. If v_5v_7 is in the spanning path, then $v_{10}v_8v_9v_{10}$ forms a C_3 within the spanning path, a contradiction. If v_5v_4 is in the spanning path, then $v_1v_2v_3v_1$ forms a C_3 within the spanning path, another contradiction. So v_5 itself is incident with no edges in the spanning path, a contradiction! Hence such a spanning path does not exist on G.

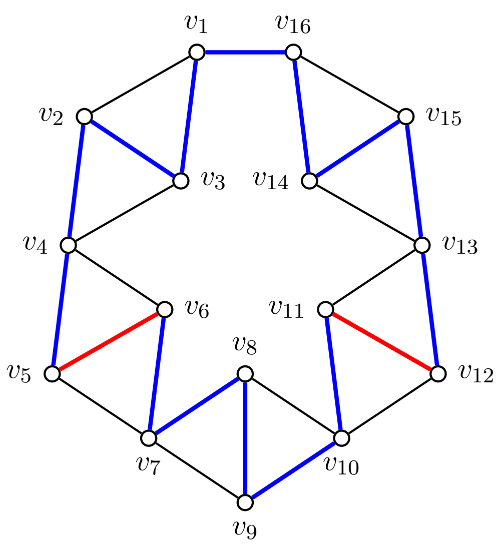
A spanning tree with 3 end-vertices does exist, and hence that is the spanning tree of G with the greatest possible number of cut-vertices. One such graph is shown below (there are other possible solutions):



(iii) Suppose a hamiltonian cycle of G does exist. Since v_5 and v_6 have degree of 2, the edges v_5v_4, v_5v_7, v_6v_4 and v_6v_7 are included within the hamiltonian cycle. However these edges themselves form a C_4! A contradiction. Hence G is not hamiltonian. You may also apply a similar argument starting form the two vertices v_{11} and v_{12}. You may also consider the set of vertices $S = \{v_4, v_7, v_{10}, v_{13}\}$, and since $|S| = 4 < 6 = c(G - S)$, hence G is not hamiltonian.

One additional edge onto G is not sufficient to obtain a resultant hamiltonian graph. To prove this we choose the same set of vertices $S = \{v_4, v_7, v_{10}, v_{13}\}$. Suppose G' is the new graph after adding one more edge to G. Then $c(G' - S)$ is the same, or one less than that of $c(G - S)$, as an additional edge into a disconnected graph can at most merge two components into one component. Hence $|S| = 4 < 5 \leq c(G' - S)$. Hence G' is still not hamiltonian.

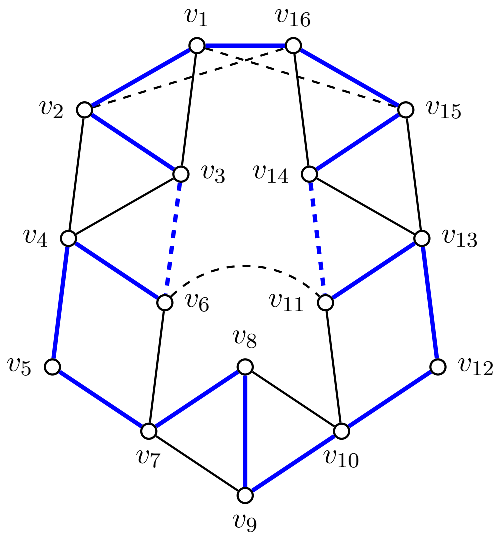
Two new edges onto G is sufficient to obtain a resultant hamiltonian graph and hence 2 is the least number of new edges to be considered. One such graph is shown below, with the hamiltonian cycle bolded blue and red, red being the additional edges onto G.



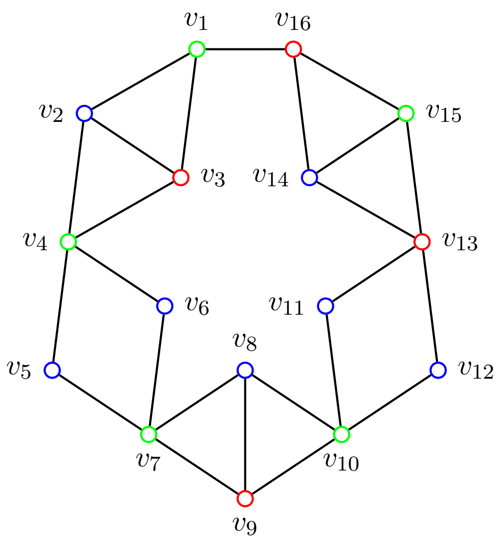
(iv) For the resultant graph to be hamiltonian, at least one of v_5 and v_6 has to be incident to a new edge, otherwise a contradiction occurs as mentioned in (iii). Similarly, at least one of v_{11} and v_{12} has to be incident to a new edge. Without loss of generality we consider adding edges to v_6 and v_{11}. Furthermore G has 8 odd vertices: v_1, v_2, v_3, v_8, v_9, v_{14}, v_{15}, v_{16}. For the resultant graph to be semi-eulerian, the resultant graph must have exactly 2 odd vertices.

Since v_6 and v_{11} are even vertices in G, to become even vertices in the resultant graph, v_6 and v_{11} must each be incident to at least 2 new edges. For the 8 odd vertices, to become even vertices in the resultant graph, they must each be incident to at least 1 new edge. Hence, for the resultant graph to have exactly 2 odd vertices, these vertices must, in total, be incident to at least 10 new edges.

5 new edges is enough, and hence is the minimum number possible to add to G such that the resultant becomes hamiltonian and semi-eulerian. One such graph is shown below, with odd vertices v_8 and v_9, and with the hamiltonian cycle bolded in blue and the additional edges dashed:



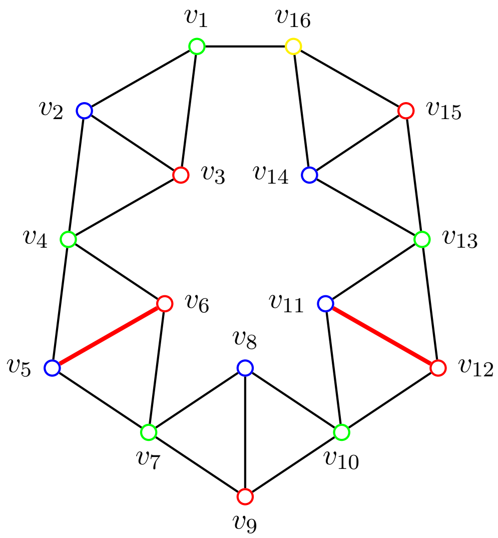
(v) Since an odd cycle exists in G, $\chi(G) \geq 3$. There exists a 3-colouring for G, and one such colouring is shown below:



Hence, $\chi(G) \leq 3$. And therefore, $\chi(G) = 3$.

(vi) We need to add edges into G such that the resultant graph H has $\chi(H) = 4$, without H having a K_4 (otherwise the possible solutions will be fairly obvious).

One such possible H is precisely the graph used in (iii):



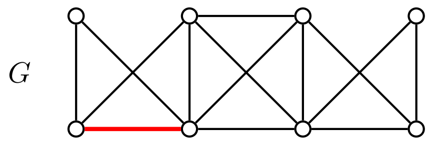
H has a 4-colouring as shown above. Hence $\chi(H) \leq 4$.

To justify that $\chi(H) = 4$, suppose H has a 3-colouring. If v_1 is coloured as '1', then whatever 2 different colours v_2 and v_3 hold, v_4 has to be coloured as '1' too. By a similar argument, v_7, v_{10}, v_{13} and hence v_{16} has to be coloured as '1' as well. But v_1 is adjacent to v_{16}, a contradiction! Hence $\chi(H) \geq 4$.

Therefore $\chi(H) = 4$.

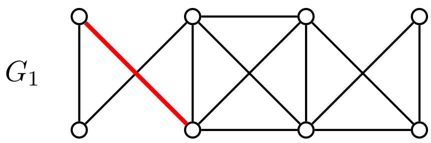
Question 2

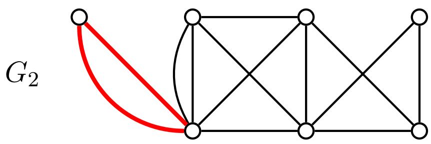
Define the following graph as such:

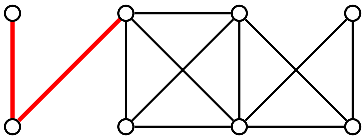


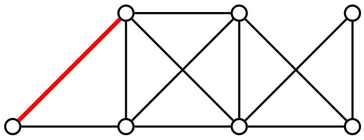
To count the number of spanning trees of G, an intuitive manner to approach the question will be to slowly remove the edges in the center and count the number of spanning trees of all the resultant graph. It does work and will give you the correct answer after a long and tedious process. However, removing the sides first results in a more elegant and less lengthy approach.

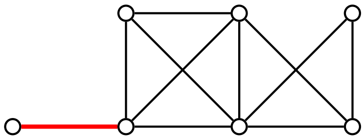
Consider the following graphs:

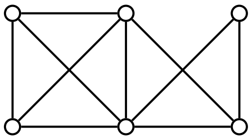


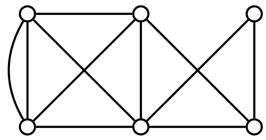


G_3


G_4


G_5


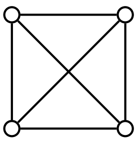
A


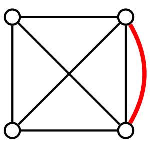
B


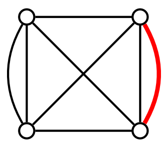
By considering the removal of the bolded edge in G, $\tau(G) = \tau(G_1) + \tau(G_2)$. By considering the removal of the bolded edge in G_1, $\tau(G_1) = \tau(G_3) + \tau(G_4)$. The bolded edges in G_2 is basically a C_2 and meets the rest of the graph at a single vertex, hence $\tau(G_2) = 2 \times \tau(B)$. The bolded edges in G_3 makes no difference to the count of spanning trees in G_3, and hence $\tau(G_3) = \tau(A)$. By considering the removal of the bolded edge in G_4, $\tau(G_4) = \tau(G_5) + \tau(B)$. The bolded edge in G_5 makes no difference to the count opf spanning trees in G_5, and hence $\tau(G_5) = \tau(A)$.

By summing the results up, we find that $\tau(G) = 2 \times \tau(A) + 3 \times \tau(B)$.

Define the following graphs as such:

X


Y


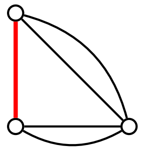
Z


We next perform the identical edge removal process as demonstrated above to the right side of the graphs A and B. We obtain $\tau(A) = 2 \times \tau(X) + 3 \times \tau(Y)$, and $\tau(B) = 2 \times \tau(Y) + 3 \times \tau(Z)$. Hence by summing the results we further simplify the problem to $\tau(G) = 4 \times \tau(X) + 12 \times \tau(Y) + 9 \times \tau(Z)$.

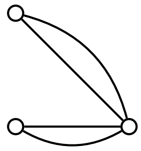
By Cayley's formula, $\tau(X) = \tau(K_4) = 4^{4-2} = 16$.

To find $\tau(Y)$ and $\tau(Z)$, define the following graphs as such:

Y_1

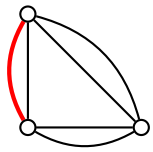


Y_2



Y_3

Z_1

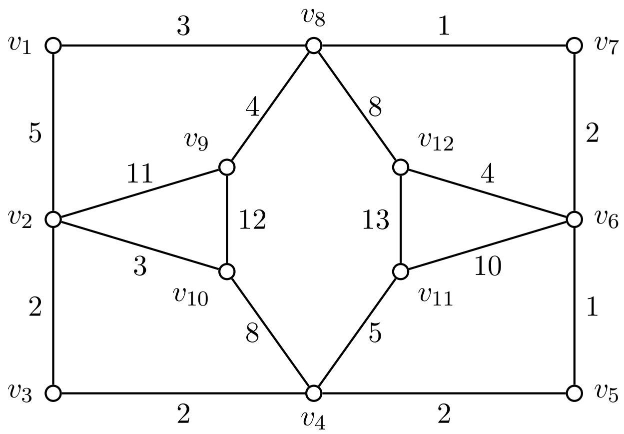


By considering the removal of the bolded edge in Y, $\tau(Y) = \tau(X) + \tau(Y_1)$. By considering the removal of the bolded edge in Y_1, $\tau(Y_1) = \tau(Y_2) + \tau(Y_3)$. Y_2 is basically 2 C_2 s at a common point, hence $\tau(Y_2) = 2 \times 2 = 4$. Also $\tau(Y_3) = \binom{4}{1} = 4$. Hence $\tau(Y_1) = 4 + 4 = 8$, and $\tau(Y) = 16 + 8 = 24$.

By considering the removal of the bolded edge in Z, $\tau(Z) = \tau(Y) + \tau(Z_1)$. By considering the removal of the bolded edge in Z_1, $\tau(Z_1) = \tau(Y_1) + \tau(Y_3) = 8 + 4 = 12$. Hence $\tau(Z) = 24 + 12 = 36$.

Therefore, the number of spanning trees of G, $\tau(G) = 4 \times 16 + 12 \times 24 + 9 \times 36 = 676$.

Question 3



Apply Edmond's algorithm to the above graph:

There are 4 odd vertices in the graph, v_9, v_{10}, v_{11} and v_{12}. The least weight and path of least weight between each pair of these vertices are:

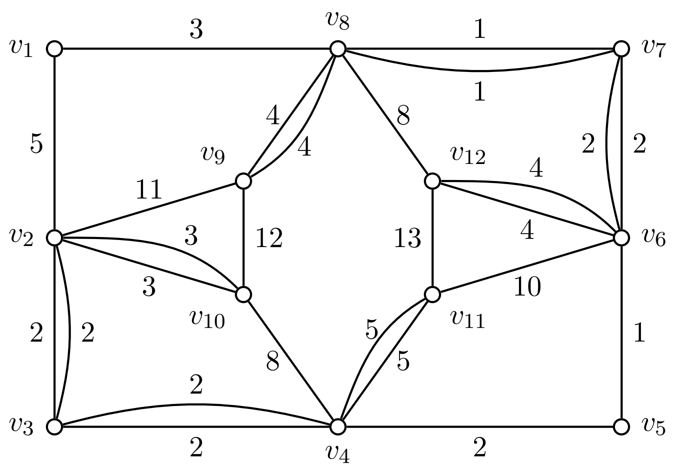
- $v_9 - v_{10}$: 12 (via v_9v_{10}),
- $v_9 - v_{11}$: 15 (via $v_9v_8v_7v_6v_5v_4v_{11}$),
- $v_9 - v_{12}$: 11 (via $v_9v_8v_7v_6v_{12}$),
- $v_{10} - v_{11}$: 12 (via $v_{10}v_2v_3v_4v_{11}$),
- $v_{10} - v_{12}$: 14 (via $v_{10}v_2v_3v_4v_5v_6v_{12}$),
- $v_{11} - v_{12}$: 12 (via $v_{11}v_4v_5v_6v_{12}$).

The weights of the 3 possible pairings between these 4 vertices are:

- $v_9 - v_{10}$ and $v_{11} - v_{12}$: $12 + 12 = 24$,
- $v_9 - v_{11}$ and $v_{10} - v_{12}$: $15 + 14 = 29$,
- $v_9 - v_{12}$ and $v_{10} - v_{11}$: $11 + 12 = 23$.

The minimum weight pairing is $v_9 - v_{12}$ and $v_{10} - v_{11}$.

We append the paths of least weights of the two paths within the minimum weight pairing into the original graph. We obtain:



Using Fluerry's algorithm, we construct a closed trail of this new graph which contains all its edges. One such trail can be $v_1v_2v_{10}v_2v_3v_2v_9v_8v_9v_{10}v_4v_3v_4v_{11}v_4v_5v_6v_{12}v_6v_7v_6v_{11}v_{12}v_8v_7v_8v_1$, and this is also the closed walk with minimum weight which contains all the edges in the original graph. (there are many other possible answers).

The weight of our closed walk is $(23)+(3+1+5+4+8+2+2+8+5+1+2+2+11+3+4+10+12+13) = 119$.

Question 4

Apply Dijkstra's algorithm to the graph as drawn in the question.

We begin with this table listing the shortest distances from u_9 from each of the other vertices, as well as the previous vertex on the path of shortest distance as will be determined when performing the algorithm.

Vertex	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Current shortest distance from u_9	∞	∞	∞	∞	∞	∞	∞	∞	0
Previous vertex on path of shortest distance	-	-	-	-	-	-	-	-	NA
Is it the fixed shortest possible distance?	-	-	-	-	-	-	-	-	Y

Focus on u_9, having shortest distance of 0 from u_9. u_1, u_6, u_7 and u_8 are unfixed vertices adjacent to u_9. Path $u_1 - u_9$ passing through u_1u_9 has distance of $2 + 0 = 2$, which is smaller than its current value of ∞. Path $u_6 - u_9$ passing through u_6u_9 has distance of $3 + 0 = 3$, which is smaller than its current value of ∞. Path $u_7 - u_9$ passing through u_7u_9 has distance of $1 + 0 = 1$, which is smaller than its current value of ∞. Path $u_8 - u_9$ passing through u_8u_9 has distance of $3 + 0 = 3$, which is smaller than its current value of ∞. Hence update table on new shortest distances from u_9, of u_1, u_6, u_7 and u_8:

Vertex	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Current shortest distance from u_9	2	∞	∞	∞	∞	3	1	3	0
Previous vertex on path of shortest distance	u_9	-	-	-	-	u_9	u_9	u_9	NA
Is it the fixed shortest possible distance?	-	-	-	-	-	-	-	-	Y

Among the unfixed shortest distances, u_7 has the shortest distance of 1 from u_9. Hence fix u_7.

Vertex	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Current shortest distance from u_9	2	∞	∞	∞	∞	3	1	3	0
Previous vertex on path of shortest distance	u_9	-	-	-	-	u_9	u_9	u_9	NA
Is it the fixed shortest possible distance?	-	-	-	-	-	-	Y	-	Y

Focus on u_7, having shortest distance of 1 from u_9. u_5, u_6 and u_8 are unfixed vertices adjacent to u_7. Path $u_5 - u_9$ passing through u_5u_7 has distance of $5+1=6$, which is smaller than its current value of ∞. Path $u_6 - u_9$ passing through u_6u_7 has distance of $6+1=7$, which is larger than its current value of 3. Path $u_8 - u_9$ passing through u_8u_7 has distance of $2+1=3$, which is equal to its current value of 3. Hence update table on new shortest distances from u_9, of u_5:

Vertex	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Current shortest distance from u_9	2	∞	∞	∞	6	3	1	3	0
Previous vertex on path of shortest distance	u_9	-	-	-	u_7	u_9	u_9	u_9	NA
Is it the fixed shortest possible distance?	-	-	-	-	-	-	Y	-	Y

Among the unfixed shortest distances, u_1 has the shortest distance of 2 from u_9. Hence fix u_1.

Vertex	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Current shortest distance from u_9	2	∞	∞	∞	6	3	1	3	0
Previous vertex on path of shortest distance	u_9	-	-	-	u_7	u_9	u_9	u_9	NA
Is it the fixed shortest possible distance?	Y	-	-	-	-	-	Y	-	Y

Focus on u_1, having shortest distance of 2 from u_9. u_2, u_3 and u_6 are unfixed vertices adjacent to u_1. Path $u_2 - u_9$ passing through u_2u_1 has distance of $2+2=4$, which is smaller than its current value of ∞. Path $u_3 - u_9$ passing through u_3u_1 has distance of $6+2=8$, which is smaller than its current value of ∞. Path $u_6 - u_9$ passing through u_6u_1 has distance of $8+2=10$, which is larger than its current value of 3. Hence update table on new shortest distances from u_9, of u_2 and u_3:

Vertex	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Current shortest distance from u_9	2	4	8	∞	6	3	1	3	0
Previous vertex on path of shortest distance	u_9	u_1	u_1	-	u_7	u_9	u_9	u_9	NA
Is it the fixed shortest possible distance?	Y	-	-	-	-	-	Y	-	Y

Among the unfixed shortest distances, u_6 has the shortest distance of 3 from u_9 (alternatively you may choose u_8). Hence fix u_6.

Vertex	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Current shortest distance from u_9	2	4	8	∞	6	3	1	3	0
Previous vertex on path of shortest distance	u_9	u_1	u_1	-	u_7	u_9	u_9	u_9	NA
Is it the fixed shortest possible distance?	Y	-	-	-	-	Y	Y	-	Y

Focus on u_6, having shortest distance of 3 from u_9. u_3, u_4 and u_5 are unfixed vertices adjacent to u_6. Path $u_3 - u_9$ passing through u_3u_6 has distance of $5+3=8$, which is equal to its current value of 8. Path $u_4 - u_9$ passing through u_4u_6 has distance of $8+3=11$, which is smaller than its current value of ∞. Path $u_5 - u_9$ passing through u_5u_6 has distance of $3+3=6$, which is equal to its current value of 6. Hence update table on new shortest distances from u_9, of u_4:

Vertex	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Current shortest distance from u_9	2	4	8	11	6	3	1	3	0
Previous vertex on path of shortest distance	u_9	u_1	u_1	u_4	u_7	u_9	u_9	u_9	NA
Is it the fixed shortest possible distance?	Y	-	-	-	-	Y	Y	-	Y

Among the unfixed shortest distances, u_8 has the shortest distance of 3 from u_9. Hence fix u_8.

Vertex	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Current shortest distance from u_9	2	4	8	11	6	3	1	3	0
Previous vertex on path of shortest distance	u_9	u_1	u_1	u_4	u_7	u_9	u_9	u_9	NA
Is it the fixed shortest possible distance?	Y	-	-	-	-	Y	Y	Y	Y

Focus on u_8, having shortest distance of 3 from u_9. u_5 is the unfixed vertex adjacent to u_8. Path $u_5 - u_9$ passing through u_5u_8 has distance of $4+3=7$, which is larger than its current value of 6. Hence there is no update of the table in this step.

Among the unfixed shortest distances, u_2 has the shortest distance of 4 from u_9. Hence fix u_2.

Vertex	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Current shortest distance from u_9	2	4	8	11	6	3	1	3	0
Previous vertex on path of shortest distance	u_9	u_1	u_1	u_4	u_7	u_9	u_9	u_9	NA
Is it the fixed shortest possible distance?	Y	Y	-	-	-	Y	Y	Y	Y

Focus on u_2, having shortest distance of 4 from u_9. u_3 and u_4 are unfixed vertices adjacent to u_2. Path $u_3 - u_9$ passing through u_3u_2 has distance of $3+4=7$, which is smaller than its current value of 8. Path $u_4 - u_9$ passing through u_4u_2 has distance of $2+4=6$, which is smaller than its current value of 11. Hence update table on new shortest distances from u_9, of u_3 and u_4:

Vertex	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Current shortest distance from u_9	2	4	7	6	6	3	1	3	0
Previous vertex on path of shortest distance	u_9	u_1	u_2	u_2	u_7	u_9	u_9	u_9	NA
Is it the fixed shortest possible distance?	Y	Y	-	-	-	Y	Y	Y	Y

Among the unfixed shortest distances, u_4 has the shortest distance of 6 from u_9 (you may also choose u_5). Hence fix u_4.

Vertex	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Current shortest distance from u_9	2	4	7	6	6	3	1	3	0
Previous vertex on path of shortest distance	u_9	u_1	u_2	u_2	u_7	u_9	u_9	u_9	NA
Is it the fixed shortest possible distance?	Y	Y	-	Y	-	Y	Y	Y	Y

Focus on u_4, having shortest distance of 6 from u_9. u_3 and u_5 are unfixed vertices adjacent to u_4. Path $u_3 - u_9$ passing through u_3u_4 has distance of $7+6=13$, which is larger than its current value of 7. Path $u_5 - u_9$ passing through u_5u_4 has distance of $4+6=10$, which is larger than its current value of 6. Hence there is no update of the table in this step.

Among the unfixed shortest distances, u_5 has the shortest distance of 6 from u_9. Hence fix u_5.

Vertex	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Current shortest distance from u_9	2	4	7	6	6	3	1	3	0
Previous vertex on path of shortest distance	u_9	u_1	u_2	u_2	u_7	u_9	u_9	u_9	NA
Is it the fixed shortest possible distance?	Y	Y	-	Y	Y	Y	Y	Y	Y

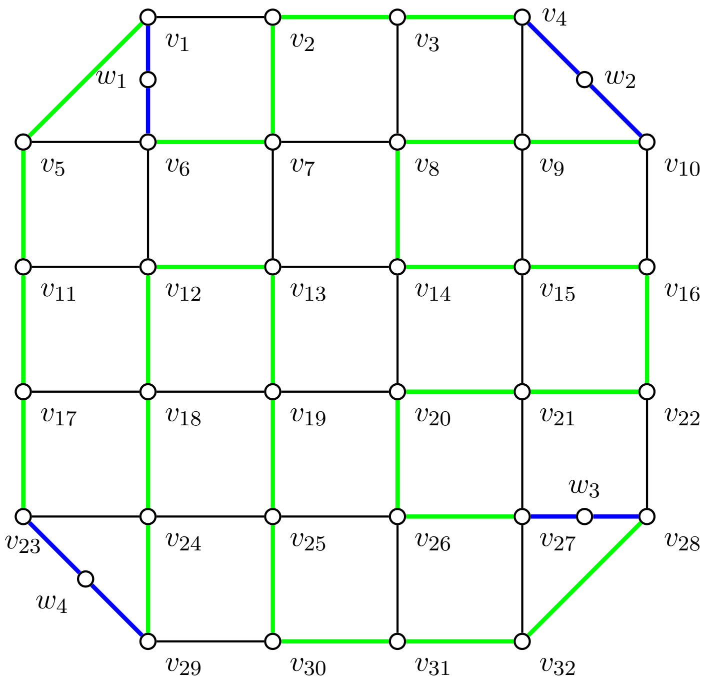
There are no unfixed vertices adjacent to u_5. Hence there is no update of the table in this step.

Only u_3 is unfixed at this point. Hence fix u_3. We now have our final table, which shows the shortest distances from u_9 to all the vertices in G:

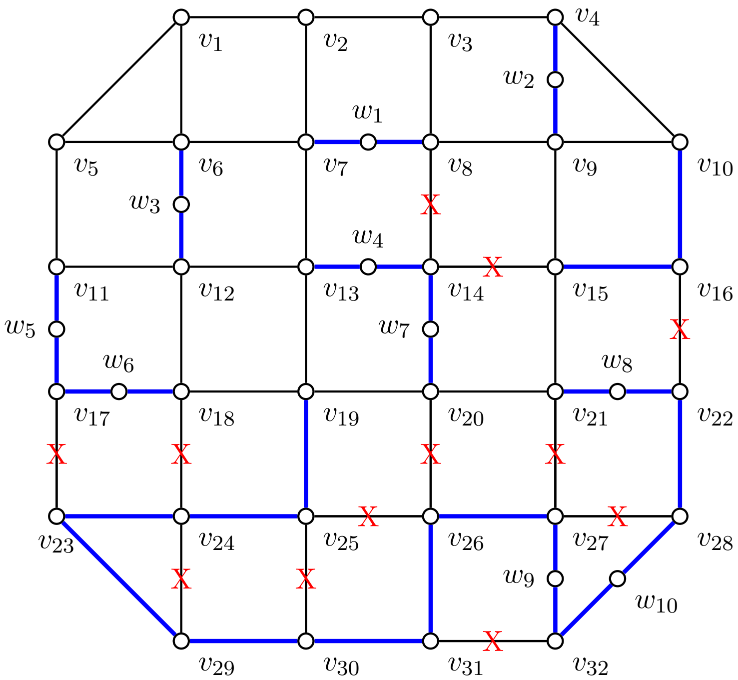
Vertex	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Shortest distance from u_9	2	4	7	6	6	3	1	3	0

Question 5

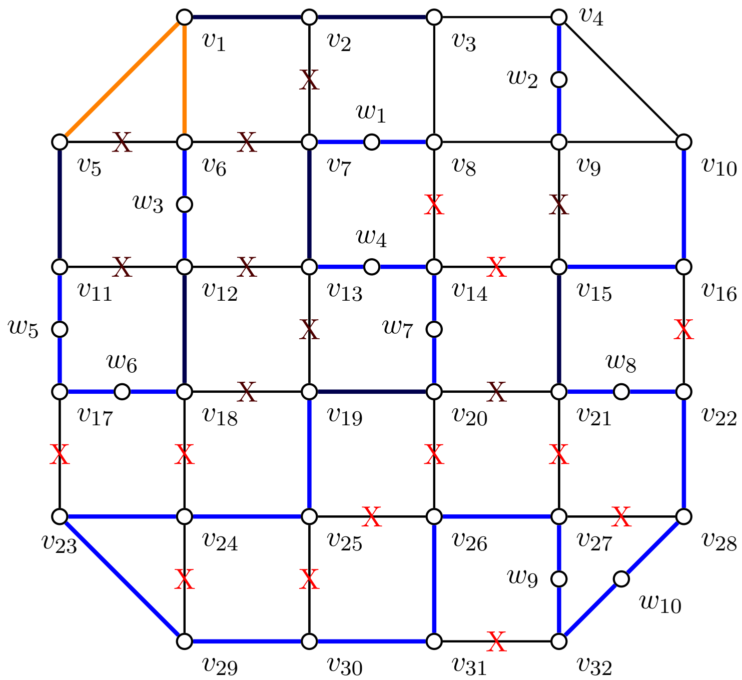
(a) This graph is hamiltonian. The hamiltonian cycle is as shown below, where the blue edges indicate edges that have to be included due to vertices having only two possible adjacent edges left that could be in the hamiltonian cycle, and green edges represents the rest of the edges filled in to complete the hamiltonian cycle (your own hamiltonian cycle can have different such green edges, as long as all vertices are in the cycle).



(b) Suppose a hamiltonian cycle does exist for this graph. Consider the following construction, where the blue edges indicate edges that have to be included due to vertices having only two possible adjacent edges left that could be in the hamiltonian cycle, and red crosses represent edges that are not possible to be in the hamiltonian cycle due to similar logical deductions.



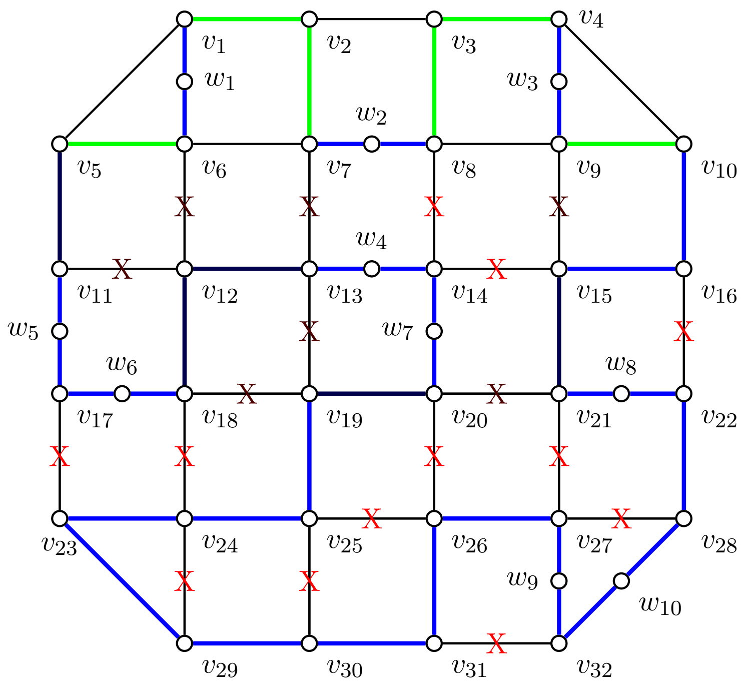
From here, consider the edge v_9v_{15}. This edge cannot be present in the hamiltonian cycle because if it was present, then v_4v_{10} is forced to be present too, forming a C_5 $v_4v_9v_{15}v_{16}v_{10}v_4$, a contradiction. Hence we carry on our plotting of selected and removed edges:



Ultimately you will be forced to plot both orange edges above into the hamiltonian cycle, giving a C_7 $v_1v_5v_{11}v_{17}v_{18}v_{12}v_6v_1$. This is a contradiction!

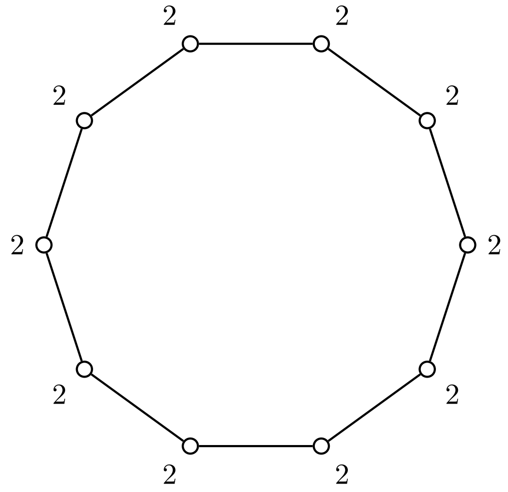
Hence this graph is not hamiltonian.

(c) This graph is hamiltonian. The hamiltonian cycle is as shown below, where the blue edges indicate edges that have to be included due to vertices having only two possible adjacent edges left that could be in the hamiltonian cycle, red crosses represent edges that are not possible to be in the hamiltonian cycle due to similar logical deductions, and green edges represents the rest of the edges filled in to complete the hamiltonian cycle (your own hamiltonian cycle can have different such green edges, as long as all vertices are in the cycle). Take note that v_9v_{15}, and similarly v_6v_{12} have been crossed out with reasons as stated in (ii).



Question 6

(i) Given that G is eulerian, all degrees must be even. Furthermore given that G is hamiltonian, a C_{10} must be a spanning subgraph of G, with 8 more edges to be added to this C_{10} to form G. We first construct the spanning subgraph below, with their degrees as labelled:



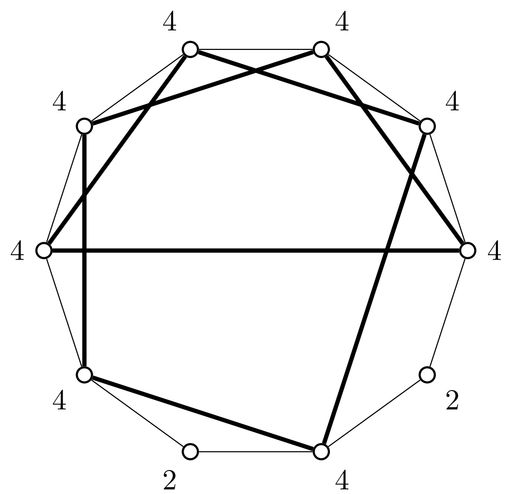
We first determine what the maximum degree of a vertex could be in G. Take note this degree must be even.

A vertex of degree 10 or above is not possible as there exists only 10 vertices in G, and hence the maximum degree cannot exceed 9.

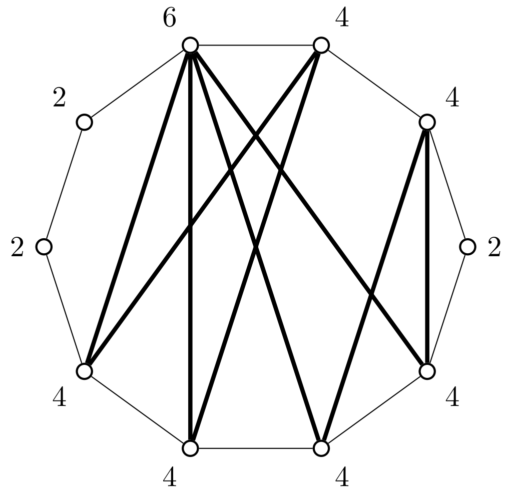
Suppose there exists a vertex of degree 8. This implies that exactly 6 of the 8 additonal edges are incident to this vertex. Since the graph is simple i.e. no pair of edges is incident to the same pair of vertices, therefore by adding these 6 additional edges, it will result in 6 vertices of odd degree 3. The 2 other additional edges will not be able to be incident to all of these 6 odd vertices to cause them to be even, and hence the graph will contain odd vertices, and hence G will not be eulerian, a contradiction. Therefore there does not exist a vertex of degree 8 in G.

It is possible for a vertex to have degree 6, as shown in the subsequent examples. Hence the degrees of the vertices in G are either 2, 4 or 6. We divide the subsequent cases into the number of vertices of degree 6 there are in G:

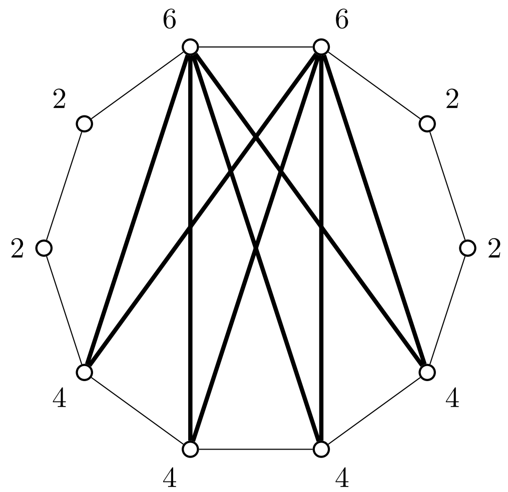
Case 1: No vertices of degree 6. This means that G only contains vertices of degree 2 and 4. For the number of edges to be 18, the sum of degrees of vertices in G must be 36. This means that there are 8 vertices of degree 4 and 2 vertices of degree 2, giving a degree sequence of $(4, 4, 4, 4, 4, 4, 4, 4, 2, 2)$. A eulerian, hamiltonian graph with this degree sequence is shown below:



Case 2: Exactly 1 vertex of degree 6. Similarly this means that there are 6 vertices of degree 4 and 3 vertices of degree 2, giving a degree sequence of $(6, 4, 4, 4, 4, 4, 4, 2, 2, 2)$. A eulerian, hamiltonian graph with this degree sequence is shown below:



Case 3: Exactly 2 vertices of degree 6. Similarly this means that there are 4 vertices of degree 4 and 4 vertices of degree 2, giving a degree sequence of $(6, 6, 4, 4, 4, 4, 2, 2, 2, 2)$. A eulerian, hamiltonian graph with this degree sequence is shown below:

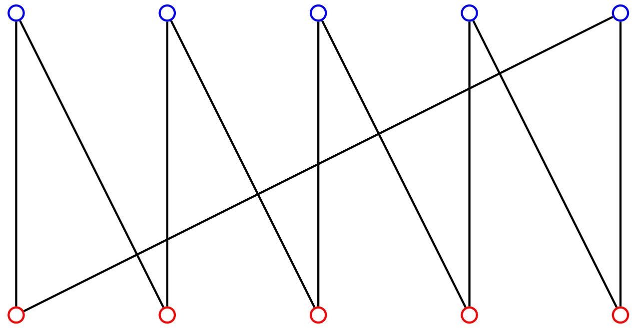


Case 4++: Suppose there are exactly 3 vertices of degree 6. By a similar argument there are only 2 vertices of degree 4 and 5 vertices of degree 2. However, the 3 vertices of degree 6 must each be adjacent to all the other 4 vertices of degree greater than 2 (2 of degree 4, the other 2 of degree 6), a contradiction since the vertices of degree 4 can be adjacent to at most 2 other vertices of degree greater than 2.

And suppose there are exactly 4 vertices of degree 6. The rest of the vertices have to be degree 2. However, the 4 vertices of degree 6 must be adjacent to at least 4 vertices of degree greater than 2, an impossibility.

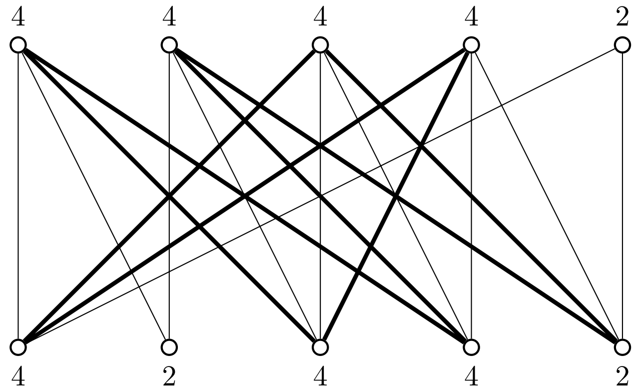
Since there are only 18 edges in G, we do not need to consider more vertices of degree 6 than 4. Hence we have identified all possible degree sequences of G.

(ii) Given also that G is bipartite, the 10 vertices in G have to be coloured so that 5 vertices are each coloured by one colour, and hence each partite set of G has exactly 5 vertices. The spanning subgraph C_{10} is redrawn in the following manner:

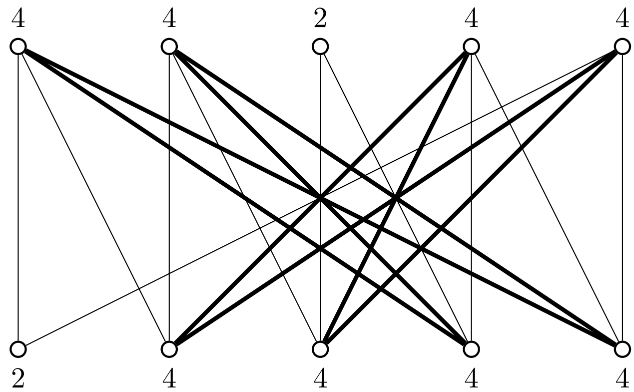


Since each partite set has exactly 5 vertices, the degree of any vertex cannot be greater than 5. Therefore, since all degrees are even, the maximum degree of G is now only 4. Hence, the only degree sequence in (i) that can be considered is Case 1, $(4, 4, 4, 4, 4, 4, 4, 4, 2, 2)$.

To draw two non-isomorphic such G, we consider two different ways the additional 8 edges can be drawn. One way these 8 edges can be drawn is to form a C_8 by themselves, giving rise to the below graph:



Another way the additional 8 edges can be drawn is to form 2 C_4s by themselves, as shown:



Question 7

(a) Suppose that v is a cut-vertex of G. By definition of cut-vertex, $\exists w \in V(G)$ such that all $u - w$ paths contains v.

But since v is the furthest from u, $\forall w \in V(G)$, $d(u, w) \leq d(u, v)$. Hence, $\forall w \in V(G)$, $\exists$ a $u - w$ path that does not contain v (otherwise $d(u, w) > d(u, v)$). This is a direct contradiction to the above definition of v being a cut-vertex.

Hence v is not a cut-vertex of G.

(b) $G - v$ has more than 1 component since v is a cut-vertex of G. It is understood that vertices in different components of $G - v$ are not adjacent, and hence would be adjacent in $\overline{G} - v$.

Consider $\overline{G} - v$. $\forall x, y \in V(\overline{G} - v)$, if x and y are in different components of $G - v$, then $d(x, y) = 1$ by adjacency in $\overline{G} - v$. if x and y are in the same component of $G - v$, then $d(x, y) \leq 2$ since $\exists z$ in another component of $G - v$ where in $\overline{G} - v$, x is adjacent to z and y is adjacent to z, and hence xzy is a $x - y$ path.

Hence $\text{diam}(\overline{G} - v) = 1$ or 2. In particular, $\text{diam}(\overline{G} - v) = 1$ when each component of $G - v$ has only 1 vertex, i.e. G is a star graph.

(c) We claim that G is a bipartite graph.

The direct way to prove it would be to patition $V(G)$ into 2 sets A and B, where $A = \{$all odd vertices$\}$ and $B = \{$all even vertices$\}$. Since no pair of vertices is mutually adjacent in A and no pair of vertices is mutually adjacent in B, we have A, B is a biparitition of G.

Another way to prove it would be to suppose that G is not bipartite i.e. G contains an odd cycle $v_1 v_2 \ldots v_n v_1$, where n is odd. WLOG let v_1 be even. Then vertices of odd subscripts are even and vertices of even subscripts are odd. Hence v_n is even. But v_1 is adjacent to v_n, a contradiction. Hence G is bipartite.