NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Kenny Sng, Poh Wei Shan Charlotte

MA3110 Mathematical Analysis II
AY 2008/2009 Semester 1

Question 1

(a) Since f is continuous at c, we have that:
$\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall x \in I$, $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$ — $(*)$.
We are also given that g is differentiable at c, thus we also have that:
$\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall x \in I$, $\left|\dfrac{g(x) - g(c)}{x - c} - g'(c)\right| < \epsilon$ whenever $|x - c| < \delta$ — $(**)$.

Suppose that $f(c) > 0$. Letting $\epsilon = \frac{1}{2}f(c) > 0$, we have from $(*)$:
$\exists \delta_1 > 0$ such that $\forall x \in I \cap (c - \delta_1, c + \delta_1)$, $|f(x) - f(c)| < \frac{1}{2}f(c)$ whenever $|x - c| < \delta_1$.
Thus, $\forall x \in I \cap (c - \delta_1, c + \delta_1)$, $f(x) > \frac{1}{2}f(c) > 0$.
Hence, $\forall x \in I \cap (c - \delta_1, c + \delta_1)$, $f(x) = |f(x)| = g(x)$, and thus f is differentiable at c since g is given to be differentiable at c.

Suppose that $f(c) < 0$. Letting $\epsilon = -\frac{1}{2}f(c) > 0$, we have from $(*)$:
$\exists \delta_2 > 0$ such that $\forall x \in I \cap (c - \delta_2, c + \delta_2)$, $|f(x) - f(c)| < -\frac{1}{2}f(c)$ whenever $|x - c| < \delta_2$.
Thus, $\forall x \in I \cap (c - \delta_2, c + \delta_2)$, $f(x) < \frac{1}{2}f(c) < 0$.
Hence, $\forall x \in I \cap (c - \delta_2, c + \delta_2)$, $f(x) = -|f(x)| = -g(x)$, and since g is given to be differentiable at c, $f = -g$ is also differentiable at c.

Suppose that $f(c) = 0$. Then, $g(c) = |f(c)| = 0$.

We suppose for a contradiction that $g'(c) > 0$.
Then, from $(**)$, letting $\epsilon = \frac{1}{2}g'(c) > 0$, we have:
$\exists \delta_3 > 0$ such that $\forall x \in I \cap (c - \delta_3, c + \delta_3)$, $\left|\dfrac{g(x) - g(c)}{x - c} - g'(c)\right| < \dfrac{1}{2}g'(c)$.
Thus, $\forall x \in I \cap (c - \delta_3, c + \delta_3)$ such that $x < c$, we have $\dfrac{g(x) - g(c)}{x - c} > \dfrac{1}{2}g'(c)$, which implies that $g(x) < \frac{1}{2}(x - c)g'(c) < 0$, which is a contradiction as $g(x) = |f(x)| \geq 0$ $\forall x \in I$.

Similarly, if $g'(c) < 0$, letting $\epsilon = -\frac{1}{2}g'(c) > 0$, we have:
$\exists \delta_4 > 0$ such that $\forall x \in I \cap (c - \delta_4, c + \delta_4)$, $\left|\dfrac{g(x) - g(c)}{x - c} - g'(c)\right| < -\dfrac{1}{2}g'(c)$.
Thus, $\forall x \in I \cap (c - \delta_4, c + \delta_4)$ such that $x > c$, we have $\dfrac{g(x) - g(c)}{x - c} < \dfrac{1}{2}g'(c)$, which implies that $g(x) < \frac{1}{2}(x - c)g'(c) < 0$, which is a contradiction as $g(x) = |f(x)| \geq 0$ $\forall x \in I$.

Thus we can conclude that if $f(c) = g(c) = 0$, we must have $g'(c) = 0$ as well.

Hence, from $(**)$, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall x \in I$, if $|x - c| < \delta$,

$$\begin{aligned} \left| \frac{f(x) - f(c)}{x - c} - g'(c) \right| &= \left| \frac{f(x)}{x - c} \right| \\ &= \left| \frac{|f(x)|}{x - c} \right| \\ &= \left| \frac{g(x)}{x - c} \right| \\ &= \left| \frac{g(x) - g(c)}{x - c} - g'(c) \right| \\ &< \epsilon. \end{aligned}$$

Thus, f is differentiable at c if $f(c) = 0$, in fact, $f'(c) = g'(c) = 0$.

(b) We shall assume that such a function exists.
Since we are given that $f'(0) = 1$, by definition, we have:
$\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall x \in (-1, 1)$, $\left| \dfrac{f(x) - f(0)}{x - 0} - 1 \right| < \epsilon$ whenever $|x - 0| < \delta$.
Choosing $\epsilon = \frac{1}{2}$, then, $\exists \delta_1 > 0$, $0 < \delta_1 < 1$, such that $\forall x \in (-1, 1)$, $\left| \dfrac{f(x) - f(0)}{x - 0} - 1 \right| < \dfrac{1}{2}$ whenever $|x - 0| < \delta_1$.
Thus, choosing $c \in (-\delta_1, \delta_1) \cap (0, 1)$, then $c > 0$, and from above, $\dfrac{f(c) - f(0)}{c - 0} < \dfrac{3}{2}$.
Now, by the Mean Value Theorem, $\exists d \in (0, c)$ such that

$$f'(d) = \frac{f(c) - f(0)}{c - 0} < \frac{3}{2}$$

which is a contradiction to property (iii). Hence, such a function cannot exist.

Question 2

(a) Let $P = \{0 = x_0 < x_1 < ... < x_n = 1\}$ be a partition for $[0, 1]$.
For each $k = 0, 1, ..., n$, we have:

$$\begin{aligned} M_k &= \sup\{h(x) : x \in [x_{k-1}, x_k]\} \\ &= 2x_k \\ m_k &= \inf\{h(x) : x \in [x_{k-1}, x_k]\} \\ &= -1. \end{aligned}$$

If $M_k \neq 2x_k$, then let $M_k = a < 2x_k$. By the density of irrational numbers, $\exists 2b \in \mathbb{R} \backslash \mathbb{Q}$ such that $a < 2b < 2x_k$. Then, $b \in \mathbb{R} \backslash \mathbb{Q}$, and thus $2b = h(b)$, contradicting the fact that a is the supremum. A similar reasoning will yield $m_k = -1$. Hence,

$$\begin{aligned} U(h, P) &= \sum_{k=1}^{n} M_k (x_k - x_{k-1}) \\ &= 2 \sum_{k=1}^{n} x_k (x_k - x_{k-1}) \\ &\geq 2 \sum_{k=1}^{n} \frac{x_k + x_{k-1}}{2} (x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} x_k^2 - x_{k-1}^2 \\ &= 1. \end{aligned}$$

$$\begin{aligned}
L(h,P) &= \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} (-1)(x_k - x_{k-1}) \\
&= -\sum_{k=1}^{n} x_k - x_{k-1} \\
&= -1.
\end{aligned}$$

Thus,

$$U(h,P) - L(h,P) \geq 2.$$

Choosing $\epsilon = 2$ in Riemann Integrability Criterion, there does not exist a partition P for $[0,1]$ for which $U(h,P) - L(h,P) < 2$, and thus h is not Riemann integrable on $[0,1]$.

(b) (i) Since f is continuous on $[a,b]$, a closed and bounded interval, by the Extreme Value Theorem, $\exists \alpha \in [a,b]$ such that $f(\alpha) = M$.
Since f is continuous at α, we have that for any $\epsilon > 0$, $\epsilon \leq M$, and for all $x \in [a,b]$, $|f(x) - M| < \epsilon$ whenever $|x - \alpha| < \delta$.
Thus, by letting $[c,d] \subseteq [a,b] \cap (\alpha - \delta, \alpha + \delta) \subseteq [a,b]$, for all x in $[c,d]$, $|f(x) - M| < \epsilon$, that is, $f(x) > M - \epsilon$.

(ii) Hence,

$$\begin{aligned}
f(x) &> M - \epsilon \\
\Rightarrow (f(x))^n &> (M - \epsilon)^n \qquad \text{since } M - \epsilon \geq 0 \\
\Rightarrow \int_a^b (f(x))^n dx &> \int_a^b (M - \epsilon)^n dx \\
\Rightarrow \int_a^b (f(x))^n dx &> (b - a)(M - \epsilon)^n.
\end{aligned}$$

Letting $K = (b - a) > 0$, we are done.

(iii) Also, we have that $f(x) \leq M$. Thus,

$$\begin{aligned}
(f(x))^n &\leq M^n \\
\Rightarrow \int_a^b (f(x))^n dx &\leq \int_a^b M^n dx \\
\Rightarrow \int_a^b (f(x))^n dx &\leq (b - a)M^n.
\end{aligned}$$

Combining the result in (ii), we have

$$(b-a)^{\frac{1}{n}}(M - \epsilon) \leq \left(\int_a^b (f(x))^n dx \right)^{\frac{1}{n}} \leq M(b-a)^{\frac{1}{n}}.$$

Hence,

$$\lim_{n\to\infty} (b-a)^{\frac{1}{n}}(M - \epsilon) \leq \lim_{n\to\infty} \left(\int_a^b (f(x))^n dx \right)^{\frac{1}{n}} \leq \lim_{n\to\infty} M(b-a)^{\frac{1}{n}}$$

$$\Rightarrow M - \epsilon \leq \lim_{n\to\infty} \left(\int_a^b (f(x))^n dx \right)^{\frac{1}{n}} \leq M.$$

Thus, we have

$$-\epsilon \leq \lim_{n\to\infty}\left(\int_a^b (f(x))^n dx\right)^{\frac{1}{n}} - M \leq 0.$$

Since ϵ is arbitrary, we have

$$\begin{aligned} \lim_{n\to\infty}\left(\int_a^b (f(x))^n dx\right)^{\frac{1}{n}} - M = 0 \\ \Rightarrow \lim_{n\to\infty}\left(\int_a^b (f(x))^n dx\right)^{\frac{1}{n}} = M. \end{aligned}$$

Question 3

(a) Given that the sequence of functions converges uniformly on $[a, b]$, we have that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|f_n(t) - f(t)| < \frac{\epsilon}{b-a}$$

for all $t \in [a, b]$.

Then, $\forall n \geq N$,

$$\begin{aligned} \left|\int_a^b f_n(t)dt - \int_a^b f(t)dt\right| &= \left|\int_a^b f_n(t) - f(t)dt\right| \\ &\leq \int_a^b |f_n(t) - f(t)|dt \\ &\leq \int_a^b \frac{\epsilon}{b-a}dt \\ &= \epsilon. \end{aligned}$$

Hence, $\int_a^b f_n(t)dt \to \int_a^b f(t)dt$.

(b) Given that h is a bounded function on A, we have that $|h| \leq M$ for some $M \geq 0$.
Given also that $(g_n : A \to \mathbb{R})$ is a uniformly convergent sequence, say, it converges to $g : A \to \mathbb{R}$, we have that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|g_n(x) - g(x)| < \frac{\epsilon}{M}$$

for all $x \in A$.

Hence, for all $x \in A$ and for all $n \geq N$, we have

$$\begin{aligned} |h(x)g_n(x) - h(x)g(x)| &= |h(x)||g_n(x) - g(x)| \\ &\leq M\frac{\epsilon}{M} \\ &= \epsilon. \end{aligned}$$

Thus, the sequence $(hg_n : A \to \mathbb{R})$ converges uniformly on A.

(c) Now, for all $x \in [\delta, \infty)$, $\delta > 1$, and for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\frac{1}{1+x^n} &\leq \frac{1}{x^n} \\ &\leq \frac{1}{\delta^n}.\end{aligned}$$

Since $\delta > 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\delta^n}$ converges.

Hence, by the Weierstrass M-Test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+x^n}$ converges uniformly on $[\delta, \infty)$.

Question 4

(a)

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}} \right| &= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)} \\ &= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} \\ &= \frac{1}{4}\end{aligned}$$

Hence, the power series is convergent on $(-4, 4)$.

Now, when $x = 4$, the power series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2 4^n}{(2n)!}$, and when $x = -4$, power series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2(-4)^n}{(2n)!}$.

We have that

$$\begin{aligned}(2n)! &= 1.2...2n \\ &= (1.2)(3.4)(5.6)...(2n-1.2n) \\ &\leq (2.2)(4.4)(6.6)...(2n.2n) \\ &= (2^2.1^2)(2^2.2^2)(2^2.3^2)...(2^2.n^2) \\ &= 4^n(n!)^2.\end{aligned}$$

Hence,

$$\frac{4^n(n!)^2}{(2n)!} \geq 1.$$

Thus, $\displaystyle\lim_{n\to\infty} \frac{(n!)^2 4^n}{(2n)!} \neq 0$ and $\displaystyle\lim_{n\to\infty} \frac{(n!)^2(-4)^n}{(2n)!} \neq 0$.

Hence, both $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2 4^n}{(2n)!}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2(-4)^n}{(2n)!}$ diverges, and the region of convergence is $(-4, 4)$.

(b) (i) We are given that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, and thus it converges uniformly on the interval $[0, 1]$ since it is independent of x.

Now, for $x \in (0,1)$, (x^n) is a decreasing sequence of function. For $x = 0$ and $x = 1$, (x^n) is a constant sequence which is still monotone. Hence, for $x \in [0,1]$, (x^n) is a monotone sequence. It is also uniformly bounded above by 1.

Hence, by Abel's Test, $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[0,1]$.

(ii) From (b)(i), $f(x)$ converges uniformly on $[0,1]$. Thus f is continuous on $[0,1]$ since it is a series of continuous functions.
In particular, $f(x)$ is continuous at $x = 1$. Hence,

$$\begin{aligned}\lim_{x\to 1^-} f(x) &= f(1) \\ &= \sum_{n=1}^{\infty} a_n\end{aligned}$$

(c) Since

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

for $|t| < 1$, we have that for $|x| < 1$,

$$\begin{aligned}\int_0^x \frac{1}{1-t}dt &= \ln\left(\frac{1}{1-x}\right) \\ &= \int_0^x \sum_{n=0}^{\infty} t^n dt \\ &= \sum_{n=0}^{\infty} \int_0^x t^n dt \\ &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n}\end{aligned}$$

as we can integrate a power series term by term within its radius of convergence.
Thus,

$$\ln\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Hence, for $0 < a < 1$,

$$\int_0^a \ln\left(\frac{1}{1-x}\right) dx = \int_0^a \sum_{n=1}^{\infty} \frac{x^n}{n} dx.$$

Now, since $\sum_{n=0}^{\infty} x^n$ has radius of convergence 1, the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ will also have radius of convergence 1. Thus, $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges uniformly on $[-c, c]$ for all $c < 1$.

Hence, for all $0 < a < 1$,

$$\begin{aligned}\int_0^a \sum_{n=1}^{\infty} \frac{x^n}{n} dx &= \sum_{n=1}^{\infty} \int_0^a \frac{x^n}{n} dx \\ &= \sum_{n=1}^{\infty} \frac{a^{n+1}}{n(n+1)}.\end{aligned}$$

Now, let $f(a) = \sum_{n=1}^{\infty} \frac{a^{n+1}}{n(n+1)}$. Since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is a convergent series, by (b), we deduce that $\lim_{a \to 1^-} f(a) = f(1) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Hence, combining all the above results,

$$\begin{aligned}\lim_{a \to 1^-} \int_0^a \ln\left(\frac{1}{1-x}\right) dx &= \lim_{a \to 1^-} \int_0^a \sum_{n=1}^{\infty} \frac{x^n}{n} dx \\ &= \lim_{a \to 1^-} \sum_{n=1}^{\infty} \frac{a^{n+1}}{n(n+1)} \\ &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \\ &= \lim_{k \to \infty} \sum_{n=1}^{k} \frac{1}{n(n+1)} \\ &= \lim_{k \to \infty} \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \lim_{k \to \infty} \left(1 - \frac{1}{k+1}\right) \\ &= 1.\end{aligned}$$