NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA2108S Mathematical Analysis I (Special Version)
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Question 1

(1) For any positive real number ϵ, we choose $\delta = \min\{\frac{1}{2}, \frac{\epsilon^2}{256}\}$. Then for all $x \in (1-\delta, 1)$ we have,

$$\frac{\sqrt{1-x^2}}{x^2} < 4\sqrt{1-x^2} < 4\sqrt{2\delta - \delta^2} < 16\sqrt{\delta} = \epsilon.$$

Hence by definition of limit, $\lim_{x \to 1^-} \frac{\sqrt{1-x^2}}{x^2} = 0$.

(2) Let $f(x) = \frac{x^2 - 10x}{2x^2 - 17}$. Now,

$$f(x) - \frac{1}{2} = \frac{17 - 20x}{2(2x^2 - 17)}.$$

For any positive real number ϵ, we choose $\delta = \frac{10}{\sqrt{\epsilon}} + 3$. Then for any $x > \delta$, we have

$$\left| f(x) - \frac{1}{2} \right| = \left| \frac{17 - 20x}{2(2x^2 - 17)} \right| < \frac{17}{2(2x^2 - 17)} < \frac{100}{x^2 - 9} < \frac{100}{(x-3)^2} = \epsilon.$$

Hence by definition, the limit is established.

Question 2

Let $A = \left\{ \sum_{k \in I} a_k : I \text{ is a finite subset of } \mathbb{N} \right\}$. Define $x^+ = \max\{x, 0\}$ and $x^- = \min\{x, 0\}$. Since $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, both $\sum_{k=1}^{\infty} a_k^+$ and $\sum_{k=1}^{\infty} a_k^-$ are convergent. Therefore,

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k^+ + \sum_{k=1}^{\infty} a_k^-.$$

Let $U = \left\{ \sum_{k=1}^{n} a_k^+ : n \in \mathbb{N} \right\}$ and $L = \left\{ \sum_{k=1}^{n} a_k^- : n \in \mathbb{N} \right\}$. By monotone convergence theorem, $\sum_{k=1}^{\infty} a_k^+ = \sup U$ and $\sum_{k=1}^{\infty} a_k^- = \inf L$. Since $L, U \subseteq A$,

$$\sum_{k=1}^{\infty} a_k^+ \leq \sup A \text{ and } \sum_{k=1}^{\infty} a_k^- \geq \inf A. \tag{1}$$

Now for any finite subset I of $\mathbb{N}$, we have $\inf L \le \sum_{k=1}^{\sup I} a_k^- \le \sum_{k\in I} a_k \le \sum_{k=1}^{\sup I} a_k^+ \le \sup U$. Hence $\sup U$ and $\inf L$ is an upper bound and a lower bound of A respectively. Therefore,

$$\sup A \le \sum_{k=1}^{\infty} a_k^+ \text{ and } \inf A \ge \sum_{k=1}^{\infty} a_k^-. \tag{2}$$

Combining (1) and (2) we get,

$$\sum_{k=1}^{\infty} a_k^+ = \sup A \text{ and } \sum_{k=1}^{\infty} a_k^- = \inf A.$$

Therefore we have,

$$\sum_{k=1}^{\infty} a_k = \sup \left\{ \sum_{k\in I} a_k : I \text{ is a finite subset of } \mathbb{N} \right\} + \inf \left\{ \sum_{k\in I} a_k : I \text{ is a finite subset of } \mathbb{N} \right\}.$$

Question 3

(1) $\displaystyle\lim_{x\to 0^+} \frac{2\sin x^2 \tan\sqrt{x}}{\sqrt{x}\sin^2 x} = 2\lim_{x\to 0^+}\frac{\sin x^2}{x^2}\lim_{x\to 0^+}\frac{x^2}{\sin^2 x}\lim_{x\to 0^+}\frac{\sin\sqrt{x}}{\sqrt{x}}\lim_{x\to 0^+}\frac{1}{\cos\sqrt{x}} = 2.$

(2) Observe that, $1 \le (1+2^2+\cdots+n^n)^{\frac{1}{n^2}} \le (n\cdot n^n)^{1/n^2}$. Since $\lim_{n\to\infty} 1 = \lim_{n\to\infty}(n\cdot n^n)^{1/n^2} = 1$, by squeeze theorem, we conclude that $\lim_{n\to\infty}(1+2^2+\cdots+n^n)^{1/n^2} = 1$.

(3) First, we will show that $\displaystyle\lim_{n\to\infty}\left(\frac{\frac{1}{n}}{\sin(\frac{1}{n})}\right)^{\frac{1}{n^2}} = 1$. We know that $\sin x \le x$ for all $0 \le x \le \frac{\pi}{2}$. So it follows that,

$$1 \le \left(\frac{\frac{1}{n}}{\sin(\frac{1}{n})}\right)^{\frac{1}{n^2}}.$$

Now it is also known that $\displaystyle\lim_{x\to 0}\frac{x}{\sin x} = 1$. Hence we have, $\displaystyle\lim_{n\to\infty}\frac{\frac{1}{n}}{\sin(\frac{1}{n})} = 1$. Therefore the sequence $\displaystyle\left(\frac{\frac{1}{n}}{\sin(\frac{1}{n})} : n\in\mathbb{N}\right)$ is bounded. Let $u \ge 1$ be an upper bound. Then we have,

$$1 \le \left(\frac{\frac{1}{n}}{\sin(\frac{1}{n})}\right)^{\frac{1}{n^2}} \le u^{\frac{1}{n^2}}.$$

Since $\lim_{n\to\infty} u^{\frac{1}{n^2}} = \lim_{n\to\infty} 1 = 1$, by squeeze theorem we get $\displaystyle\lim_{n\to\infty}\left(\frac{\frac{1}{n}}{\sin(\frac{1}{n})}\right)^{\frac{1}{n^2}} = 1$.

Now $\displaystyle\lim_{n\to\infty}\sin\left(\frac{1}{n}\right)^{\frac{1}{n^2}} = \lim_{n\to\infty}\left(\frac{\sin(\frac{1}{n})}{\frac{1}{n}}\right)^{\frac{1}{n^2}}\cdot\lim_{n\to\infty}\left(\frac{1}{n}\right)^{\frac{1}{n^2}} = 1\cdot 1 = 1.$

(4) Observe that $\forall x \in (0,1)$,

$$\left(\frac{1}{x}-1\right)\sin x \le \left\lfloor \frac{1}{x} \right\rfloor \sin x \le \left(\frac{1}{x}\right)\sin x$$

and $\forall x \in (-1,0)$,

$$\left(\frac{1}{x}\right)\sin x \le \left\lfloor \frac{1}{x} \right\rfloor \sin x \le \left(\frac{1}{x}-1\right)\sin x.$$

Now define $f(x) = \dfrac{\sin x}{x}$ for all $x > 0$ and $f(x) = \dfrac{\sin x}{x} - \sin x$ for all $x < 0$. And let $g(x) = \dfrac{\sin x}{x} - \sin x$ for all $x > 0$ and $g(x) = \dfrac{\sin x}{x}$ for all $x < 0$. Clearly $\forall x \in (-1,1) - \{0\}$,

$$g(x) \le \left\lfloor \frac{1}{x} \right\rfloor \sin x \le f(x).$$

Since $\lim_{x\to 0} f(x) = \lim_{x \to 0} g(x) = 1$, by squeeze theorem we have $\lim_{x\to 0} \left\lfloor \frac{1}{x} \right\rfloor \sin x = 1$.

(5) First note that,

$$0 \le \frac{2^{\lfloor x \rfloor} x^4}{\left(1+\frac{1}{x}\right)^{(\lfloor x \rfloor)^2}} \le \frac{2^x x^4}{\left(1+\frac{1}{\lceil x \rceil}\right)^{(\lfloor x \rfloor)^2}} \le \frac{2^x x^4 e^{\lceil x \rceil}}{e^{\lceil x \rceil}\left(1+\frac{1}{\lceil x \rceil}\right)^{(\lfloor x \rfloor)^2}}.$$

Let $x_n = \dfrac{e^n}{\left(1+\frac{1}{n}\right)^{(n-1)^2}}$. Observe that for each x there exists $n \in \mathbb{N}$ such that,

$$\frac{e^{\lceil x \rceil}}{\left(1+\frac{1}{\lceil x \rceil}\right)^{(\lfloor x \rfloor)^2}} \le \frac{e^n}{\left(1+\frac{1}{n}\right)^{(n-1)^2}}.$$

Claim: There exists $N \in \mathbb{N}$ such that, $\forall n \ge N$ we have, $e < (5/4)\left(1+\dfrac{1}{n}\right)^n$.

Proof: If not then there exists a subsequence (n_k) such that, $(5/4)\left(1+\dfrac{1}{n_k}\right)^{n_k} \le e$ for all $k \in \mathbb{N}$. Then we have the following contradiction,

$$(5/4)\lim_{k\to\infty}\left(1+\frac{1}{n_k}\right)^{n_k} = (5/4)\, e \le e.$$

The above claim gives us (we will assume n and x is large enough from now on),

$$x_n \le \frac{(5/4)^n \left(1+\frac{1}{n}\right)^{n^2}}{\left(1+\frac{1}{n}\right)^{(n-1)^2}} = (5/4)^n \left(1+\frac{1}{n}\right)^{2n-1}.$$

Since $\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{2n-1} = e^2$, we can get $x_n \le 10\,(5/4)^n$. Hence our first inequality becomes,

$$0 \le \frac{2^{\lfloor x \rfloor} x^4}{\left(1+\frac{1}{x}\right)^{(\lfloor x \rfloor)^2}} \le \frac{10\,(5/4)^{\lceil x \rceil} 2^x x^4}{e^{\lceil x \rceil}} \le \frac{10\,(5/4)^{\lceil x \rceil} 2^{\lceil x \rceil} \lceil x \rceil^4}{e^{\lceil x \rceil}} = \frac{10\,(5/2)^{\lceil x \rceil} \lceil x \rceil^4}{e^{\lceil x \rceil}}.$$

Now let $y_n = \dfrac{(5/2)^n n^4}{e^n}$. Since $\lim\limits_{n\to\infty} \dfrac{y_{n+1}}{y_n} = \lim\limits_{n\to\infty} \dfrac{5}{2e} \times \dfrac{(n+1)^4}{n^4} = \dfrac{5}{2e} < 1$, we have $\lim\limits_{n\to\infty} y_n = 0$. Therefore $\lim\limits_{x\to\infty} \dfrac{10\,(5/2)^{\lceil x\rceil} \lceil x\rceil^4}{e^{\lceil x\rceil}} = 0$. Hence by squeeze theorem, we conclude

$$\lim_{x\to\infty} \frac{2^{\lfloor x\rfloor} x^4}{\left(1+\frac{1}{x}\right)^{(\lfloor x\rfloor)^2}} = 0.$$

Question 4

By definition of limit, for any positive real number $\epsilon > 0$ we can find a $K > 0$ such that,

$$y > K \Rightarrow |g(y) - l| < \epsilon.$$

Again by definition of limit, given any K we can find a $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow f(x) > K.$$

Hence, we find that for any ϵ we can find a δ so that

$$|x - a| < \delta \Rightarrow |(g \circ f)(x) - l| < \epsilon.$$

This completes the proof.

Question 5

(1) Since $\lim\limits_{k\to\infty} \sin\left(k + \dfrac{1}{k}\right) \neq 0$, the series does not converge.

(2) By addition formula,

$$\frac{\sin\left(k+\frac{1}{k}\right)}{\sqrt{k}} = \frac{\sin(1/k)\cos k}{\sqrt{k}} + \frac{\sin(k)\cos(1/k)}{\sqrt{k}}.$$

Since $\left|\dfrac{\sin(1/k)\cos(k)}{\sqrt{k}}\right| \le \dfrac{1}{k\sqrt{k}}$, by comparison test $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(1/k)\cos(k)}{\sqrt{k}}$ converges absolutely.

Now $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k)\cos(1/k)}{\sqrt{k}}$ converges by Abel's Test since $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k)}{\sqrt{k}}$ is convergent and $\cos(1/k)$ monotonically converges to 1.

However, $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k)\cos(1/k)}{\sqrt{k}}$ does not converge absolutely since for large enough k,

$$\left|\frac{\sin(k)\cos(1/k)}{\sqrt{k}}\right| \ge \frac{1}{2} \cdot \frac{\sin^2 k}{k} = \frac{1-\cos 2k}{4k}.$$

Here $\displaystyle\sum_{k=1}^{\infty} \frac{-\cos 2k}{4k}$ converges wheareas $\displaystyle\sum_{k=1}^{\infty} \frac{1}{4k}$ diverges so the entire series diverges to ∞. Therefore, $\displaystyle\sum_{k=1}^{\infty} \frac{\sin\left(k+\frac{1}{k}\right)}{\sqrt{k}}$ converges conditionally.

(3) Let $a_k = \left(\frac{1}{k+1}\right)^{1/k} \sin(k/(k+1))\sin(1/k)$. Now we know that $\forall x \geq 0$,

$$x - \frac{1}{6}x^3 \leq \sin x \leq x.$$

So we get,

$$k\left(\frac{1}{k+1}\right)^{1/k} \sin(k/(k+1))\sin(1/k) \geq \frac{\sin(1/k)}{1/k}\left(\frac{k}{k+1} - \frac{1}{6}\left(\frac{k}{k+1}\right)^3\right).$$

Since $\lim_{k\to\infty} \frac{\sin(1/k)}{1/k}\left(\frac{k}{k+1} - \frac{1}{6}\left(\frac{k}{k+1}\right)^3\right) = \frac{5}{6}$, we can get $a_k \geq \frac{1}{2k}$ for large enough k. Therefore by comparison test, $\sum_{k=1}^{\infty} a_k$ diverges.

(4) Let $a_k = (-1)^k \sin(10/k)\log(10/k)$. Note that for large enough k we have that, $\sin(10/k)\log(10/k)$ is monotone and,

$$-\frac{\sqrt{k}}{10}\sin(10/k) \leq \sin(10/k)\log(10/k) \leq 0.$$

Since $\lim_{k\to\infty} 0 = \lim_{k\to\infty} -\frac{1}{\sqrt{k}}\frac{\sin(10/k)}{10/k} = 0$, by squeeze theorem we have

$$\lim_{k\to\infty} \sin(10/k)\log(10/k) = 0.$$

Since $\sum_{k=1}^{n}(-1)^k \leq 3$ for all $n \in \mathbb{N}$, by Dirichlet's Test we conclude $\sum_{k=1}^{\infty} a_k$ is convergent.

Since we know that $\lim_{k\to\infty} \frac{\sin(10/k)}{10/k} = 1$ and $|\log(10/k)|$ is unbounded, we can get for large enough k,

$$|(-1)^k \sin(10/k)\log(10/k)| = \left|\frac{10}{k}\frac{\sin(10/k)}{10/k}\log(10/k)\right| \geq \frac{10}{k}.$$

Therefore by comparison test, $\sum_{k=1}^{\infty} a_k$ is not absolutely convergent.

Question 6

Since $a_k \to a$, for all $\epsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that,

$$|a_k - a| < \frac{\epsilon}{2} \quad \forall k \geq N_1.$$

Therefore $\forall n \geq N_1 + 1$ we have,

$$\begin{aligned}
\left|\sum_{k=1}^{n} a_k b_{n+1-k} - a\right| &= \left|\sum_{k=1}^{n}(a_k - a)b_{n+1-k} - \left(1 - \sum_{k=1}^{n} b_k\right) a\right| \\
&\leq \sum_{k=1}^{n} |a_k - a| b_{n+1-k} + \left(1 - \sum_{k=1}^{n} b_k\right)|a| \\
&\leq \sum_{k=N_1+1}^{n} |a_k - a| b_{n+1-k} + \sum_{k=1}^{N_1} |a_k - a| b_{n+1-k} + \left(1 - \sum_{k=1}^{n} b_k\right)|a| \\
&< \frac{\epsilon}{2}\left(\sum_{k=N_1+1}^{n} b_{n+1-k}\right) + \sum_{k=1}^{N_1} |a_k - a| b_{n+1-k} + \left(1 - \sum_{k=1}^{n} b_k\right)|a| \\
&< \frac{\epsilon}{2} + \sum_{k=1}^{N_1} |a_k - a| b_{n+1-k} + \left(1 - \sum_{k=1}^{n} b_k\right)|a|.
\end{aligned}$$

Note that if N_1 is fixed then $\displaystyle\sum_{k=1}^{N_1} |a_k - a|$ is also fixed. Since $b_k \to 0$, there exists $N_2 \in \mathbb{N}$ such that,

$$b_{m+1-N_1} < \frac{\epsilon}{4\sum_{k=1}^{N_1} |a_k - a|} \quad \forall n \geq \max\{N_1 + 1, N_2\}.$$

Therefore $\forall n \geq \max\{N_1 + 1, N_2\}$ we get,

$$\left|\sum_{k=1}^{n} a_k b_{n+1-k} - a\right| < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \left(1 - \sum_{k=1}^{n} b_k\right)|a|.$$

Since $\displaystyle\lim_{n\to\infty} \left(1 - \sum_{k=1}^{n} b_k\right)|a| = 0$, we can find $N_3 \in \mathbb{N}$ such that,

$$\left(1 - \sum_{k=1}^{n} b_k\right)|a| < \frac{\epsilon}{4}.$$

Therefore $\forall n \geq \max\{N_1 + 1, N_2, N_3\}$ we get,

$$\left|\sum_{k=1}^{n} a_k b_{n+1-k} - a\right| < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.$$

This completes the proof.

Question 7

(i) False, a counterexample would be $x_n = a_n = (-1)^n$. Then $\displaystyle\lim_{n\to\infty} \sup x_n = 1$ and $\displaystyle\lim_{n\to\infty} \inf a_n = -1$.

(ii) False. Define the sequence (a_n) as $a_1 = 0, a_2 = \dfrac{1}{2}, a_3 = 1$ and for $n > 2$:

$$a_{n+1} = \begin{cases} a_n + (a_n - a_{n-1}) & \text{if } a_n \neq 0, 1 \\ \dfrac{a_{n-1}}{2} & \text{if } a_n = 0 \\ a_{n-1} & \text{if } a_n = 1 \end{cases}$$

Here (a_n) is bounded and $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=0$ but (a_n) does not converge since it has the subsequences $(1,1,1,\cdots)$ and $(0,0,0,\cdots)$ converging to 1 and 0 respectively.

(iii) True. Define $g(x)=f(x)-x$. Note that $f(2K)\leq 2K$ and $f(-2K)\geq -2K$. That is, $g(2K)\leq 0$ and $g(-2K)\geq 0$. Since g is continuous, by Intermediate Value Theorem $\exists c\in[-2K,2K]$ such that $g(c)=0$. That is, $f(c)=c$.

(iv) Note that $\lim\limits_{k\to\infty}\cos\left(\frac{1}{\sqrt{k}}\right)=1$ and $\cos\left(\frac{1}{\sqrt{k}}\right)$ is monotone. Hence by Abel's Test, the statement is true.

(v) False. Consider the function $f(x)=\sqrt{x}$, which is uniformly continuous on [0,1]. Then for any $L>0$ we need:

$$|\sqrt{x}-\sqrt{y}|\leq L\,|x-y|\qquad\forall x,y\in[0,1].$$

That is,

$$1\leq L\,(\sqrt{x}+\sqrt{y})\qquad\forall x,y\in[0,1].$$

Consider (x,y) such that $\sqrt{y}=2\sqrt{x}$. Then we have the following contradiction,

$$1\leq L\lim_{x\to 0^+}3\sqrt{x}=0.$$

(vi) True, since $S(a_n)$ is a closed set.

(vii) True. Let $h(x)=f(x)-g(x)$. Then $\lim\limits_{x\to c}h(x)=\lim\limits_{x\to c}f(x)-\lim\limits_{x\to c}f(x)$. Define (x_n) to be the sequence containing all (countably many) points where $h(x)>0$.

Case 1: If (x_n) has a subsequence which converges to c then from the fact that $\lim\limits_{x\to c}h(x)$ exists, we conclude $\lim\limits_{x\to c}h(x)\geq 0$. Since (a,b) is uncountable we can find another sequence (y_n) with $h(y_n)\leq 0\ \ \forall n\in\mathbb{N}$ and make it converge to c. Then we will have $\lim\limits_{x\to c}h(x)\leq 0$. In order for the limit to exist, we must have $\lim\limits_{x\to c}h(x)=0$.

Case 2: If (x_n) has no subsequence which converges to c then we can find an open set A containing c so that $\forall x\in A-\{c\}$ we have $h(x)\leq 0$. Therefore, $\lim\limits_{x\to c}h(x)\leq 0$.

Combining both cases, we conclude that $\lim\limits_{x\to c}f(x)\leq\lim\limits_{x\to c}g(x)$.

(viii) Let the set be A. Since A is infinite it has a denumerable subset. Let $f:\mathbb{N}\to A$ be an enumeration of such a denumerable subset. Define the sequence (x_n) by $x_n=f(n)$. Since (x_n) is bounded (an upper bound would be 1 and a lower bound is 0), by Bolzano-Weirstrass theorem, it has a convergent subsequence. Since $[0,1]$ is closed, the limit of this subsequence must be in $[0,1]$. Hence the given statement is true.