NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

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MA3209 Mathematical Analysis III
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SECTION A

Question 1

(a) $\rho(z,w)$ is well-defined, since $\rho(z,w) = \sum_{k=1}^{\infty} |z_k - w_k| \leq \sum_{k=1}^{\infty} |z_k| + \sum_{k=1}^{\infty} |w_k| < \infty$.

Note that $\rho(z,w) = \sum_{k=1}^{\infty} |z_k - w_k| \geq 0$.

Suppose $\rho(z,w) = 0$, then $\sum_{k=1}^{\infty} |z_k - w_k| = 0 \Rightarrow \forall k \in \mathbb{N}, z_k = w_k$. So $z = w$.

Conversely, suppose $z = w$. Then $\forall k \in \mathbb{N}, z_k = w_k$. So $\rho(z,w) = \sum_{k=1}^{\infty} |z_k - w_k| = 0$.

We also have symmetry: $\rho(z,w) = \sum_{k=1}^{\infty} |z_k - w_k| = \sum_{k=1}^{\infty} |w_k - z_k| = \rho(w,z)$.

To show triangle inequality, take $z = (z_k), w = (w_k), v = (v_k) \in \ell^{\infty}$.
Then $\rho(z,w) = \sum_{k=1}^{\infty} |z_k - w_k| \leq \sum_{k=1}^{\infty} |z_k - v_k| + \sum_{k=1}^{\infty} |v_k - w_k| = \rho(z,v) + \rho(v,w)$.

Therefore, ρ is a metric on ℓ^{∞}.

(b) Since $x_1 \neq x_2$, so $d(x_1,x_2) > 0$. Take $\epsilon_1 = \frac{d(x_1,x_2)}{2}, \epsilon_2 = \frac{d(x_1,x_2)}{2} > 0$.

Suppose $D(x_1,\epsilon_1) \cap D(x_2,\epsilon_2) \neq \phi$. Take an element $p \in D(x_1,\epsilon_1) \cap D(x_2,\epsilon_2)$.

Then $d(p,x_1) < \epsilon_1$ and $d(p,x_2) < \epsilon_2$.

So $d(x_1,x_2) \leq d(x_1,p) + d(x_2,p) < \epsilon_1 + \epsilon_2 = \frac{d(x_1,x_2)}{2} + \frac{d(x_1,x_2)}{2} = d(x_1,x_2)$, which is a contradiction.

Question 2

(i) We know that $A = \{y \in \mathbb{R}^n : d(w,y) \leq 1\} = \{y \in \mathbb{R}^n : \|w-y\|_2 \leq 1\}$ is closed and bounded in $\mathbb{R}^n$. By the Heine-Borel Theorem, A is compact.

(ii) It suffices to show that $A = \{y \in \ell^{\infty} : d(w,y) \leq 1\}$ is not sequentially compact.

Given $w = (w_1, w_2, w_3, ...)$ where w is a bounded sequence in $\mathbb{C}$, define the following sequences:

$$\begin{aligned} z^{(1)} &= (w_1+1, w_2, w_3, ...) \\ z^{(2)} &= (w_1, w_2+1, w_3, ...) \\ z^{(3)} &= (w_1, w_2, w_3+1, ...) \\ &\vdots \end{aligned}$$

Since w is bounded, so all the $z^{(k)}$'s are bounded as well, i.e. $\forall k \in \mathbb{N}, z^{(k)} \in \ell^\infty$.

Note that for all k, $d(z^{(k)}, w) = 1$, so $z^{(k)} \in A$. So $\{z^{(1)}, z^{(2)}, z^{(3)}, ...\}$ is a sequence in A.

Furthermore, note that if $m \neq n$, then $d(z^{(m)}, z^{(n)}) = 1$, so any subsequence of $\{z^{(1)}, z^{(2)}, z^{(3)}, ...\}$ cannot be Cauchy, and hence cannot be convergent.

Thus, $\{z^{(1)}, z^{(2)}, z^{(3)}, ...\}$ is a sequence in A that has no convergent subsequence, so A is not sequentially compact, and hence not compact.

Question 3

(a) Write $\lim\limits_{k\to\infty} x_k = x$, $\lim\limits_{k\to\infty} y_k = y$. Note that x and y exist as (M, d) is complete.

($\Rightarrow$) Assume $x = y$. So given any $\varepsilon > 0$, there exists K_1, $K_2 \in \mathbb{N}$ such that $\forall k \geq K_1, d(x_k, x) < \frac{\varepsilon}{2}$, and $\forall k \geq K_2, d(y_k, x) < \frac{\varepsilon}{2}$. Then $\forall k \geq \max\{K_1, K_2\}, d(x_k, y_k) \leq d(x_k, x) + d(y_k, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
$\therefore d(x_k, y_k) \to 0$ as $k \to \infty$.

($\Leftarrow$) Assume that $d(x_k, y_k) \to 0$ as $k \to \infty$.

Then given any $\varepsilon > 0$, there exists $K_3 \in \mathbb{N}$ such that $\forall k \geq K_3, d(x_k, y_k) < \frac{\varepsilon}{3}$.
There exists $K_4 \in \mathbb{N}$ such that $\forall k \geq K_4, d(x_k, x) < \frac{\varepsilon}{3}$.
Similarly, there exists $K_5 \in \mathbb{N}$ such that $\forall k \geq K_5, d(y_k, y) < \frac{\varepsilon}{3}$.
Let $K_0 = \max\{K_3, K_4, K_5\}$. Then $d(x, y) \leq d(x, x_{K_0}) + d(x_{K_0}, y_{K_0}) + d(y_{K_0}, y) < \varepsilon$. Since ε is arbitrary, $d(x, y) = 0$. Hence $x = y$.

(b) Since A is closed in N, so its complement $N \setminus A$ is open in N. Note that:

- $A \cap N = A \neq \phi$ (given)
- $(N \setminus A) \cap N = (N \setminus A) \neq \phi$ (since $A \neq N$)
- $A \cap (N \setminus A) \cap N = \phi$ (since $A \cap (N \setminus A) = \phi$)
- $N = A \cup (N \setminus A)$.

Therefore, N is disconnected.

Question 4

(i) Given any $\varepsilon > 0$, choose $\delta = \varepsilon$. Then $\forall \mathbf{x} = (x_1, x_2, ..., x_n), \mathbf{y} = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, whenever $\|\mathbf{x} - \mathbf{y}\|_2 < \delta$, we have $\sqrt{\sum_{k=1}^{n} |x_k - y_k|^2} < \delta$.

Hence, $\|f(\mathbf{x}) - f(\mathbf{y})\|_2 = \sqrt{\sum_{k=1}^{s} |x_k - y_k|^2} \leq \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2} < \varepsilon$.

So f is uniformly continuous on $\mathbb{R}^n$.

(ii) Since f is continuous from part (i), and A is closed in $\mathbb{R}^s$, so $B = f^{-1}(A)$ is closed in $\mathbb{R}^n$.

Question 5

(i) For all $k \in \mathbb{N}$, define $g_k : \mathbb{R}^2 \to \mathbb{R}, g_k(x, y) = \frac{(-1)^k}{(k!)^2} e^{-k(x^2+y^2)}$. Note that for each $k \in \mathbb{N}$ and for all $(x, y) \in \mathbb{R}^2$, $|g_k(x, y)| = \frac{1}{(k!)^2} e^{-k(x^2+y^2)} \leq \frac{1}{k!}$. Since $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges by ratio test, so $\sum_{k=1}^{\infty} g_k(x, y)$ converges uniformly on $\mathbb{R}^2$ by Weierstrass M-test.

(ii) Let $g : \mathbb{R}^2 \to \mathbb{R},\ g(x, y) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} e^{-k(x^2+y^2)} = \sum_{k=1}^{\infty} g_k(x, y)$. Note that since g_k is continuous for all k, and by (i), $\sum_{k=1}^{\infty} g_k(x, y)$ converges uniformly on $\mathbb{R}^2$, so g is continuous on $\mathbb{R}^2$.

Define $f : [0, 1] \to \mathbb{R}^2,\ f(t) = (t, \cos t)$. Note that f is continuous on [0,1], since it is continuous at each of its components. Hence $h = g \circ f$ is continuous on [0,1].

By the Weierstrass Approximation Theorem, given any $\epsilon > 0$, there exists a polynomial p on [0,1] such that $|h(t) - p(t)| < \epsilon$ for all $t \in [0, 1]$.

SECTION B

Question 6

(a)(i) True.

Take any $p \in \operatorname{int}\left(\bigcap_{k=1}^{\infty} A_k\right)$. Then there exists $\varepsilon > 0$ such that $D(p, \varepsilon) \subseteq \bigcap_{k=1}^{\infty} A_k$. Since $D(p, \varepsilon) \subseteq A_k$ for all $k \in \mathbb{N}$, so $p \in \operatorname{int}(A_k)$ for all $k \in \mathbb{N}$. Thus, $p \in \bigcap_{k=1}^{\infty} \operatorname{int}(A_k)$.

Hence, $\operatorname{int}\left(\bigcap_{k=1}^{\infty} A_k\right) \subseteq \bigcap_{k=1}^{\infty} \operatorname{int}(A_k)$.

(a)(ii) False. Let $M = \{3, 4\}$, $x = 3$, $r = 1$. Consider the discrete metric $d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$.

Then $\operatorname{cl}(\{y \in M : d(3, y) < 1\}) = \operatorname{cl}(\{3\}) = \{3\}$ since any singleton set is closed.
However, $\{y \in M : d(3, y) \leq 1\} = \{3, 4\}$.

(b) Suppose $\mathbb{Q}$ can be expressed as a countable intersection of open subsets of $\mathbb{R}$.
For simplicity of notation, $\forall B \subseteq \mathbb{R}$, denote $\mathbb{R} \setminus B$ as B^c.

Write $\mathbb{Q} = \bigcap\limits_{k=1}^{\infty} A_k$, where each A_k is an open subset of $\mathbb{R}$.

So $\mathbb{R} - \mathbb{Q} = \left(\bigcap\limits_{k=1}^{\infty} A_k\right)^c = \bigcup\limits_{k=1}^{\infty} (A_k)^c$.

Recall the following theorem:
Let (M, d) be a metric space, and $A \subseteq M$. Then A is nowhere dense in M if and only if $M \setminus [\mathrm{cl}(A)]$ is dense in M.

Since each A_k contains $\mathbb{Q}$, and $\mathbb{Q}$ is dense in $\mathbb{R}$, so A_k is dense in $\mathbb{R}$.
Furthermore, each $(A_k)^c$ is closed in $\mathbb{R}$, so $\mathrm{cl}((A_k)^c) = (A_k)^c$.

Using the above theorem with $M = \mathbb{R}, A = (A_k)^c$, and the fact that $[\mathrm{cl}((A_k)^c)]^c = [(A_k)^c]^c = A_k$ is dense in $\mathbb{R}$, we conclude that $(A_k)^c$ is nowhere dense in $\mathbb{R}$.

Let $r_1, r_2, \ldots$ be an enumeration of $\mathbb{Q}$.

Note that $\forall k \in \mathbb{N}$, $(A_k)^c \cup \{r_k\}$ is nowhere dense. Otherwise, if $\exists\ x \in \mathrm{int}[\mathrm{cl}((A_k)^c \cup \{r_k\})] = \mathrm{int}((A_k)^c \cup \{r_k\})$, then $\exists\ \varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq (A_k)^c \cup \{r_k\}$ which is a contradiction as the interval contains more than 2 rational numbers.

Then $\mathbb{R} = (\mathbb{R} - \mathbb{Q}) \cup \mathbb{Q} = \bigcup\limits_{k=1}^{\infty} (A_k)^c \cup \bigcup\limits_{k=1}^{\infty} \{r_k\} = \bigcup\limits_{k=1}^{\infty} [(A_k)^c \cup \{r_k\}]$ is a countable union of nowhere dense subsets of $\mathbb{R}$. This is impossible because of Baire Category Theorem and that $\mathbb{R}$ is complete.

Therefore, $\mathbb{Q}$ cannot be expressed as a countable intersection of open subsets of $\mathbb{R}$.

Question 7

(a) Substitute $y = 0$ into the given inequality. Then for each fixed $x_0 \in [-1, 1]$, we obtain for all $k \in \mathbb{N}$,

$$\begin{aligned} |f_k(x_0) - f_k(0)| &\leq C|x_0|. \\ |f_k(x_0) - 1| &\leq C|x_0|. \\ |f_k(x_0)| &\leq 1 + C|x_0|. \end{aligned}$$

Hence, for each fixed $x_0 \in [-1, 1]$, the sequence $\{f_k(x_0)\}_{k=1}^{\infty}$ is bounded.

$\therefore \{f_k\}$ is pointwise bounded.

Furthermore, given any $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{C}$. Then for all $k \in \mathbb{N}$, and for all $x, y \in [-1, 1]$, whenever $|x - y| < \delta$, we have $|f_k(x) - f_k(y)| \leq C|x - y| \leq C(\frac{\varepsilon}{C}) < \varepsilon$.

$\therefore \{f_k\}$ is equicontinuous on $[-1, 1]$.

Lastly, $[-1, 1]$ is a compact subset of $\mathbb{R}$, so by the Arzelà-Ascoli Theorem, $\{f_k\}$ has a uniformly convergent subsequence.

(b) Note that if $(x, y) \in A$, then $(-x, -y) \in A$. So define $h : A \to \mathbb{R}, h(x, y) = f(x, y) - f(-x, -y)$. Since f is continuous on A, so h is also continuous on A. Suppose there is no $(x_0, y_0) \in A$ such that $f(x_0, y_0) = f(-x_0, -y_0)$. Take $(0, 1), (0, -1) \in A$. Then either $f(0, 1) > f(0, -1)$ or $f(0, 1) < f(0, -1)$. Without loss of generality, assume $f(0, 1) > f(0, -1)$. Then $h(0, 1) =$

$f(0,1) - f(0,-1) > 0$, and $h(0,-1) = f(0,-1) - f(0,1) < 0$. By the intermediate value theorem, there exists a point $(x_1, y_1) \in A$ such that $h(x_1, y_1) = 0$. This implies that $f(x_1, y_1) = f(-x_1, -y_1)$, which is a contradiction. Thus, there exists $(x_0, y_0) \in A$ such that $f(x_0, y_0) = f(-x_0, -y_0)$.

Question 8

(a) Let $\varepsilon > 0$ be given. Since f is uniformly continuous on A, so there exists $\delta > 0$ such that for all $x, y \in A$, if $d(x,y) < \delta$, then $\rho(f(x), f(y)) < \varepsilon$. Since $A \subseteq M$ is totally bounded, so there exists a finite subset $\{x_1, x_2, ..., x_n\} \subseteq M$ such that $A \subseteq \bigcup_{i=1}^{n} D_d(x_i, \delta)$, where $D_d(x_i, \delta) = \{y \in M : d(y, x_i) < \delta\}$.

Claim: $f(A) \subseteq \bigcup_{i=1}^{n} D_\rho(f(x_i), \varepsilon)$, where $D_\rho(f(x_i), \varepsilon) = \{z \in N : \rho(z, f(x_i)) < \varepsilon\}$.

Proof: For any $p \in f(A)$, there exists $q \in A$ such that $f(q) = p$. But $A \subseteq \bigcup_{i=1}^{n} D_d(x_i, \delta)$ implies that there exists $i_0 \in \{1, 2, .., n\}$ such that $q \in D_d(x_{i_0}, \delta) \Rightarrow d(q, x_{i_0}) < \delta$. By the uniform continuity of f, we have $\rho(f(q), f(x_{i_0})) = \rho(p, f(x_{i_0})) < \varepsilon$. So $p \in D_\rho(f(x_{i_0}), \varepsilon)$. Hence, $p \in \bigcup_{i=1}^{n} D_\rho(f(x_i), \varepsilon)$. Thus, $f(A) \subseteq \bigcup_{i=1}^{n} D_\rho(f(x_i), \varepsilon)$.

$\therefore f(A) \subseteq N$ is totally bounded.

(b)(i) Since Φ^r is a contraction, so there exists $\lambda \in (0,1)$ such that for all $x, y \in M$, we have

$$d(\Phi^r(x), \Phi^r(y)) \leq \lambda \ d(x,y).$$

Suppose that for some positive integer m, we have $d(\Phi^{rm}(x), \Phi^{rm}(y)) \leq \lambda^m d(x,y)$ for all $x, y \in M$. So for any $x, y \in M$, since $\Phi^r(x), \Phi^r(y) \in M$, we have:

$$\begin{aligned} d(\Phi^{r(m+1)}(x), \Phi^{r(m+1)}(y)) &= d(\Phi^{rm}(\Phi^r(x)), \Phi^{rm}(\Phi^r(y))) \\ &\leq \lambda^m d(\Phi^r(x), \Phi^r(y)) \\ &\leq \lambda^{m+1} d(x,y). \end{aligned}$$

So by induction, for any $\ell \geq 1$,

$$d(\Phi^{r\ell}(x), \Phi^{r\ell)}(y)) \leq \lambda^\ell d(x,y) \text{ for all } x, y \in M.$$

(b)(ii) <u>*Lemma:* Φ has a unique fixed point.</u>

Proof: Since Φ^r is a contraction mapping on the complete metric space M, so by the contraction mapping principle, Φ^r has a unique fixed point which we shall denote it by x_0.
From the definition of contraction of Φ^r, we have

$$\begin{aligned} d(\Phi^r(\Phi(x_0)), \Phi^r(x_0)) &\leq \lambda \ d(\Phi(x_0), x_0) \\ d(\Phi(\Phi^r(x_0)), \Phi^r(x_0)) &\leq \lambda \ d(\Phi(x_0), x_0) \\ d(\Phi(x_0), x_0) &\leq \lambda \ d(\Phi(x_0), x_0) \end{aligned}$$

Suppose $\Phi(x_0) \neq x_0$, then $d(\Phi(x_0), x_0) > 0$. Dividing both sides by $d(\Phi(x_0), x_0)$, we obtain $1 \leq \lambda$ which is a contradiction. Therefore, $\Phi(x_0) = x_0$, i.e. x_0 is also a fixed point of Φ.

To prove uniqueness, suppose Φ has two fixed points x_0, y_0. From the definition of contraction of Φ^r, we have

$$\begin{aligned} d(\Phi^r(x_0), \Phi^r(y_0)) &\leq \lambda\, d(x_0, y_0) \\ d(x_0, y_0) &\leq \lambda\, d(x_0, y_0) \end{aligned}$$

Suppose $x_0 \neq y_0$, then $d(x_0, y_0)$. Dividing both sides by $d(x_0, y_0) > 0$, we obtain $1 \leq \lambda$ which is a contradiction. Therefore, $x_0 = y_0$, i.e. Φ has a unique fixed point.$\square$

Returning to the main problem, fix $x \in M$. If x is the fixed point x_0, then the sequence $\{\Phi^k(x_0)\}_{k=1}^{\infty}$ is just the constant sequence $\{x_0, x_0, ...\}$, which converges in M. So we shall assume that x is not a fixed point of M. Let $B = \max\{d(x, x_0), d(\Phi(x), x_0), d(\Phi^2(x), x_0), ..., d(\Phi^{r-1}(x), x_0)\}$. Note that B is positive because $x \neq x_0$ implies $d(x, x_0) > 0$.

Now, given any $\varepsilon > 0$, choose L large enough such that $\lambda^{\lfloor \frac{L}{r} \rfloor} < \dfrac{\varepsilon}{B}$, where $\lfloor \frac{L}{r} \rfloor$ is the quotient when L is divided by r. For any k, using the division algorithm, write $k = rm_1 + m_2$, where m_1 is the quotient, and $m_2 \in \{0, 1, 2, .., r-1\}$ is the remainder. Then for all $k \geq L$,

$$\begin{aligned} d(\Phi^k(x), x_0) &= d(\Phi^{rm_1+m_2}(x), x_0) \\ &= d(\Phi^{rm_1}(\Phi^{m_2}(x)), \Phi^{rm_1}(x_0)) \\ &\leq \lambda^{m_1}\, d(\Phi^{m_2}(x), x_0) \\ &\leq \lambda^{\lfloor \frac{L}{r} \rfloor}\, d(\Phi^{m_2}(x), x_0) \qquad (1) \\ &< \left(\frac{\varepsilon}{B}\right)(B) \\ &= \varepsilon \end{aligned}$$

Note that (1) is derived from $m_1 = \lfloor \frac{k}{r} \rfloor \geq \lfloor \frac{L}{r} \rfloor$ and since $\lambda < 1$, $\lambda^{\lfloor \frac{k}{r} \rfloor} \leq \lambda^{\lfloor \frac{L}{r} \rfloor}$.
Therefore, the sequence $\{\Phi^k(x)\}_{k=1}^{\infty}$ converges to x_0 for any fixed $x \in M$.