NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lee Yung Hei

MA4291 Undergraduate Topics in Mathematics
AY 2008/2009 Sem 2

This course was specially structured for the semester and the contents covered may not be used in future courses in the same module. Please refer to attached course summary for more details on the topics covered.

Question 1

(i) Formally: $\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$ (**i.e.** All true (valid) statements, can be proven.)

Gödel's completeness theorem says that a deductive system of first-order predicate calculus is 'complete' in the sense that no additional inference rules are required to prove all the logically valid formulas.

Note: The converse to completeness is *soundness*, the fact that only logically valid formulas are provable in the deductive system. Taken together, these theorems imply that a formula is logically valid if and only if it is the conclusion of a formal deduction.

(ii) **Note:** This proof should be covered in any MA4207 course. Thus, we would only briefly highly key points to note.
To show completeness of propositional logic, for every formula we add φ or $\neg\varphi$ to a consistent set Γ.
The main difference with first order is the inclusion of $=$ and $\forall$.

1. Since $=$ is difficult to handle, we change $=$ to a binary relation E. (ie. $c = d$ becomes cEd)
2. Henkin construction's main tool is the introduction of $\neg\forall x\varphi \rightarrow \neg\varphi^c_x$ which simply means that if not all φ is true, then there must be a witnessing constant c to show that it's false.
3. Finally, we take a quotient of our new structure $\mathfrak{A}$. This new structure $\mathfrak{A}/E$ would show our completeness.

It is simple to verify that the structure constructed above is complete.

Question 2

(i) Complement of a recursively enumerable set.

Fix x to check if it is in the complementary set.
$\forall k \leq i$, run $\varphi_x(k)$ for i steps.
Gradually increase i.
If φ_x terminates, then $W_x \neq \emptyset$ and x is in the complementary set.
If $W_x \neq \emptyset$, the process will eventually terminate.

(ii) Recursively enumerable set.

Similar to (i), but this time keep a record of how many outputs have been generated.
Keep repeating until the process terminates from 7 different starting values.
If at least 7 can be found, x is in the set.

(iii) Recursive set.

It is possible to check if $m|n$ for all $m, n \in \mathbb{N}$.
To check for any x, check if $k^2|x$, $\forall k < x$.

(iv) Complement of a recursively enumerable set.

This is similar to (ii), but you repeat the process and keeping a record of all unique outputs generated thus far.
The program terminates if at least 1000 unique outputs are generated.

(v) Recursively Enumerable set.

If $\varphi_x(x) = 1$, then $\varphi_x(x)$ will terminate on finite steps.
However, it is not recursive as there are $\varphi_x(x) = k \neq 1$ or $\varphi_x(x) = \uparrow$.

Question 3

(i) Russell believes that the problem arises from the 'vicious circle principle'.
Thus, his solution is that all statements are given a hierarchy and a statement can only comment on the truth value of a statement of a lower hierarchy. Thus, a statement such as 'This sentence is false.' refers to a statement of the same hierarchy and cannot be possibly made.
An analogous of of this approach is that of set theory where $\forall x, x \notin x$.

(ii) Kripke's solution is based on 'groundedness'.
In Kripke's solution, truth-predicate of sentences may not be totally defined. Thus, there is a need to explain what the word 'true' means and what the word 'false' means. Therefore a sentence such as 'This sentence is false.' is not paradoxical as 'false' is not clearly defined and ungrounded.

(iii) Tarski's solution involves the 'hierarchy of languages'.
This hierarchy includes *object language*, O, *meta-language*, M, *meta-meta-language*, M', ...
So, when a sentence says 'This sentence is false.', it will have to say 'false-in-O'.
Thus, the sentence 'This sentence is false-in-O.' is a sentence of the metalanguage hierarchy and so is no longer paradoxical.

(iv) Each has its pros and cons, you are only required to write 1 set:

Russell's advantage is mainly its simplicity. With the analogous found in ZF set theory, its foundations are well laid and can easily build upon set theory. However, but restricting what is allowed and what is not, many important proves cannot be made and more axioms are needed.

Kripke's advantage is that NOT EVERY well-formed sentence must be either true or false. this means that the formal theory is built about the solution of this paradox. Unfortunately, it is difficult to find a good grounding for 'true'. Most logicians use 'Snow is white.' as the example to explain 'true'. However, some people have never seen snow.

Tarski's solution explores the different hierarchy levels and approaches the problem from a metaphysical direction. This allows statements to comment other lower levels and provides more flexibility with sentence construction. The weakness is that it is impossible to explicitly assign a hierarchy level to every statement. **e.g:** *(Postcard paradox)* 'The sentence on the other side of the card is true.' and 'The sentence on the other side of the card is false.' on each side of the card. No explicit value can be assigned to either sentence.

Question 4

Consider the *sorites paradox.*
'Sorites' is the greek word for heap and the paradox arises from the fact that 1 grain of sand is not a heap.
Adding 1 grain of sand to the collection of n grains, the new collection of $n+1$ grains is not a heap.
However, if we have 1000000 grains, it is a heap.
Thus, somewhere between 1 grain and 1000000 grains, the addition of 1 grain changed a non-heap into a heap.
However, this i not possible.

The solution is very simple if we use a fuzzy logical approach.
In fuzzy logic, the truth value can take any real value between 0 and 1.
Thus, 0 grains is NOT a heap and has a truth value of 0.
13 grains will take a value of a $(0 < a < 1)$ and is slightly a heap, 17 grains would then have a value b. $(0 < a < b < 1)$
Thus, 17 grains is more of a heap than 13 grains.
By the time we reach a million grains, it would be taking a value of 1, which is definitely a heap.

Question 5

(i) Modal logic introduces the concept of necessity and possibility with 2 unary symbols, $\Box$ and $\Diamond$. Any statement p is considered to be necessarily true in $\Box p$ and possibly true in $\Diamond p$. For example, let φ be the statement 'It is a warm day.' $\Box p$ would be 'It MUST be a warm day.' whereas $\Diamond p$ would be 'It could be a warm day.'
Note: In our study of modal logic, only the systems of K, S4 and S5 are considered.

(ii) Modal logic's strength lies in the fact that it can assign necessity and possibility to statements. Thus, allowing for statements to have difference levels of truth.
Let's consider 3 sentences 'Paul is tall.', 'Paul is a human.' and 'Paul could be wearing a green shirt tomorrow.'.
While all 3 are considered to be true statements, 'Paul could be wearing a green shirt.' shows a possibility and it is also possible that Paul would not wear a green shirt. Thus, the symbol $\Diamond$ could be used in the symbolic representation to show the possibility of the statement.
On the other hand, 'Paul is tall.' and 'Paul is human.' are both true. However, there is a difference in necessity. While Paul's existence necessitates that he's a human, he is not necessarily tall. Paul could be amputated, have a spinal degenerative disease or stopped growing at a young age. Thus, while the statement 'Paul is tall.' is true, it is not necessarily true.
These differences cannot be achieved by propositional logic.

Question 6

(i) In first order logic, functions and relations have to be pre-defined.
e.g: In first order, we can define function f such that $\forall x f(x) = x$.
In second order logic, functions and relations can take on any form and the identity must exist.
i.e: $\exists g \forall x g(x) = x$

Furthermore, many results that hold in first order does not hold in second order.
e.g: *The compactness Theorem, the downward Löwenhein-Skolem Theorem, the upward Löwenhein-Skolem Theorem* and *Gödel's Completeness Theorem* do not hold in second order logic.

(ii) 'There are infinitely many elements.'
In second order, if no functions or relations are pre-defined:

$$\exists z \exists u (\forall x u(x) \neq z \& \forall x \forall y (u(x) = u(y) \leftrightarrow x = y))$$

In first order, if no functions or relations are pre-defined:

$$\exists x_0$$

$$\exists x_0 \exists x_1 (x_0 \neq x_1)$$

$$\exists x_0 \exists x_1 \exists x_2 (x_0 \neq x_1 \ \&\ x_0 \neq x_2 \ \&\ x_1 \neq x_2)$$

$$\vdots$$

(iii) Compactness does not hold in 2nd order logic.

Proof. Let P be a complete model of first order logic.
Then, let $\Gamma^* = \{P, c \neq 0, c \neq 1, c \neq 2, \dots\}$.
Every finite subset of $\Gamma \subseteq \Gamma^*$ is consistent. However, there is no model that satisfies Γ^*. □

Question 7

(i) In standard set theory, an object is either in the set, or outside of a set.
e.g: Let A be the set of even numbers. $2 \in A$, $13 \notin A$
In fuzzy set theory, an object can be partially in the set.
e.g: Let B be the set of small numbers. Then f_B will be the characteristic function of B.
$f_B(97) = 0.1$, $f_B(5) = 0.9$; 5 belongs in the set of small numbers more than 97 since it is smaller than 97.

(ii) $A \subseteq B$ means that $\forall x, f_A(x) \leq f_B(x)$.
$A \subset B$ means that $\forall x, f_A(x) < f_B(x)$.

(iii) $(A \cup B)(x) = \max(f_A(x), f_B(x)) = \max(0.5, 0.7) = 0.7$
$(A \cap C)(x) = \min(f_A(x), f_C(x)) = \min(0.5, 0.2) = 0.2$
$\overline{A}(x) = 1 - f_A(x) = 1 - 0.5 = 0.5$