MA2101S - Linear Algebra II(S) Suggested Solutions

(Semester 2 : AY2018/19)

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Question 1

(a)(i) Write A and B as:

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \ , \ B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}.$$

Where $A_{1,1}, B_{1,1} \in M_{r\times r}(\mathbb{F})$. Then we have:

$$\begin{aligned}
& J(r) \in \ker(\alpha) \\
\iff & \alpha(J(r)) = 0 \\
\iff & AJ(r) - J(r)B = 0 \\
\iff & \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\iff & \begin{pmatrix} A_{1,1} & 0 \\ A_{2,1} & 0 \end{pmatrix} - \begin{pmatrix} B_{1,1} & B_{1,2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\iff & A_{1,1} = B_{1,1} \ \wedge \ B_{1,2} = 0_{r\times(n-r)} \ \wedge \ A_{2,1} = 0_{(m-r)\times r} \\
\iff & A = \begin{pmatrix} C & * \\ 0 & * \end{pmatrix} \ \wedge \ B = \begin{pmatrix} C & 0 \\ * & * \end{pmatrix} \text{ for some } C \in M_{r\times r}(\mathbb{F}).
\end{aligned}$$

(ii) Recall that if A and B are of the form:

$$A = \begin{pmatrix} D & A_{1,2} \\ 0 & A_{2,2} \end{pmatrix} \ , \ B = \begin{pmatrix} D & 0 \\ B_{2,1} & B_{2,2} \end{pmatrix}.$$

Then $c_A(x) = c_D(x)c_{A_{2,2}}(x) \ \wedge \ c_B(x) = c_D(x)c_{B_{2,2}}(x)$. Thus $c_D(x)$ is a common factor of degree r of both $c_A(x)$ and $c_B(x)$. (Since D is a block matrix of size $r \times r$)

(b) Let R be the reduced row echelon form of X. Note that:

$$X \xrightarrow[\text{Operations}]{\text{Elementary Row}} R \xrightarrow[\text{Operations}]{\text{Elementary Column}} J(r).$$

Then $\exists$ invertible matrices $P \in M_{m\times m}(\mathbb{F}),\ Q \in M_{n\times n}(\mathbb{F})$ such that:

$$PXQ = J(r). \ (P \text{ and } Q \text{ are simply the product of elementary matrices})$$

Define $\alpha' : M_{m\times n}(\mathbb{F}) \to M_{m\times n}(\mathbb{F})$ by:

$$\alpha'(X) = PAP^{-1}X - XQ^{-1}BQ.$$

Then

$$\begin{aligned}
\alpha'(PXQ) &= PAP^{-1}(PXQ) - (PXQ)Q^{-1}BQ \\
&= PAXQ - PXBQ \\
&= P(AX - XB)Q \\
&= P(0_{m\times n})Q \\
&= 0_{m\times n}.
\end{aligned}$$

Thus $J(r) \in \ker(\alpha')$.

By (a)(ii), the characteristic polynomials of PAP^{-1} and $Q^{-1}BQ$ have a common factor of degree r. Since $c_A(x) = c_{PAP^{-1}}(x) \ \wedge \ c_B(x) = c_{Q^{-1}BQ}(x)$, it follows that the characteristic polynomials of A and B also have a common factor of degree r.

(c) Assume that α is not injective.

$\exists X \in M_{m\times n}(\mathbb{F})$ such that $X \neq 0_{m\times n} \ \wedge \ X \in \ker(\alpha)$.

Let $\text{rank}(X) = r$. Since $X \neq 0_{m\times n}, \ r \geq 1$ so by (b), the characteristic polynomials of A and B have a common factor of degree r. This is a contradiction as the characteristic polynomials of A and B are coprime. Thus the assumption is false and α is injective.

Question 2

(a) Since $\gcd(f(x), m(x)) = 1, \ \exists s_1(x), s_2(x) \in F[x]$ such that:

$$s_1(x)f(x) + s_2(x)m(x) = 1.$$

Then

$$s_1(\alpha)f(\alpha)(v) + s_2(\alpha)m(\alpha)(v) = v \to s_1(\alpha)(u) = v.$$

Simply choose $g(x) = s_1(x)$ and the proof is complete.

(b)(i) Let $f_{gcd}(x) = \gcd(p(x), m(x))$. To prove that $p(x) \mid m(x)$, it suffice to prove that $f_{\text{gcd}}(x) = p(x)$.

$\exists \ t_1(x), t_2(x) \in F[x]$ such that $t_1(x)p(x) + t_2(x)m(x) = f_{gcd}(x)$. Then:

$$t_1(\alpha)p(\alpha)(w) + t_2(\alpha)m(\alpha)(w) = f_{gcd}(\alpha)(w) \to t_1(\alpha)p(\alpha)(w) = f_{gcd}(\alpha)(w).$$

Observe that

$$\begin{aligned} p(\alpha)(w) \in \langle v\rangle_\alpha &\to t_1(\alpha)p(\alpha)(w) \in \langle v\rangle_\alpha \\ &\to f_{gcd}(\alpha)(w) \in \langle v\rangle_\alpha. \end{aligned}$$

Then $\deg(f_{\text{gcd}}(x)) \geq \deg(p(x))$ so $f_{\text{gcd}}(x) = p(x)$. (Recall that $f_{\text{gcd}}(x) \mid p(x)$)

(ii) Since $p(x) \mid m(x)$, write $m(x) = k(x)p(x)$ for some $k(x) \in F[x]$.

Then $k(\alpha)q(\alpha)(v) = k(\alpha)p(\alpha)(w) = m(\alpha)(w) = 0_V$.

Thus $m(x) \mid k(x)q(x)$ and so $k(x)p(x) \mid k(x)q(x)$. Thus $p(x) \mid q(x)$.

(iii) Claim: $p(x) \mid h(x)$.

Proof: Let $j_{\text{gcd}}(x) = \gcd(p(x), h(x))$. Similar to b(i), we show that $p(x) \mid h(x)$ by showing that $j_{\text{gcd}}(x) = p(x)$. $\exists l_1, l_2 \in F[x]$ such that:

$$l_1(x)p(x) + l_2(x)h(x) = j_{\text{gcd}}(x).$$

Then $l_1(\alpha)p(\alpha)(w) + l_2(\alpha)h(\alpha)(w) = j_{\text{gcd}}(\alpha)(w)$. Since $l_1(\alpha)p(\alpha)(w) + l_2(\alpha)h(\alpha)(w) \in \langle v\rangle_\alpha$, $j_{\text{gcd}}(\alpha)(w) \in \langle v\rangle_\alpha$.
Then $\deg(j_{\text{gcd}}(x)) \geq \deg(p(x))$ so $j_{\text{gcd}}(x) = p(x)$. (Recall that $j_{\text{gcd}}(x) \mid p(x)$)

Write $h(x) = n(x)p(x)$ for some $n(x) \in F[x]$. Then $p(\alpha)(w) = q(\alpha)(v)$.

By (b)(ii), $q(x) = p(x)r(x)$ for some $r(x) \in F[x]$. Thus:

$$\begin{aligned} h(\alpha)(w) &= n(\alpha)p(\alpha)(w) \\ &= n(\alpha)q(\alpha)(v) \\ &= n(\alpha)p(\alpha)r(\alpha)(v) \\ &= h(\alpha)r(\alpha)(v). \end{aligned}$$

Question 3

(a)(i) Let $A, B \in M_{n\times n}(\mathbb{F})$ and let $x, y \in \mathbb{F}$.

Then $\text{tr}(xA + yB) = \text{tr}(xA) + \text{tr}(yB) = x\text{tr}(A) + y\text{tr}(B)$.

Thus tr is a linear functional from $M_{n\times n}(\mathbb{F})$ to $\mathbb{F}$ so $\text{tr} \in (M_{n\times n}(\mathbb{F}))^*$.

(ii) First note that the i, j entry of AB is:

$$(AB)_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$$

Then:

$$\begin{aligned}
\text{tr}(AB) &= \sum_{i=1}^{n} (AB)_{i,i} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{i,k} b_{k,i} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{k,i} a_{i,k} \\
&= \sum_{k=1}^{n} (BA)_{k,k} \\
&= \text{tr}(BA)
\end{aligned}$$

(b)(i) Claim 1 : $\forall\ 1 \le i \le n, 1 \le j \le n,\ f(E_{i,j}) = 0$ if $i \neq j$.

Proof : $f(E_{i,j}E_{j,j}) = f(E_{j,j}E_{i,j}) \to f(E_{i,j}) = f(0_{n\times n}) = 0$.

Remark : $f(0_{n\times n}) = 0$ since f is a linear functional.

Claim 2 : $\forall\ 1 \le i \le n,\ f(E_{i,i}) = f(E_{1,1})$.

Proof : $f(E_{1,i}E_{i,1}) = f(E_{i,1}E_{1,i}) \to f(E_{1,1}) = f(E_{i,i})$.

Write $A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} E_{i,j}$.

$$\begin{aligned}
f(A) &= f(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} E_{i,j}) \\
&= f(\sum_{i=1}^{n} a_{i,i} E_{i,i}) \\
&= f(\sum_{i=1}^{n} a_{i,i} E_{1,1}) \\
&= \sum_{i=1}^{n} a_{i,i} f(E_{1,1}) \quad \text{(Since } f \text{ is a linear functional)} \\
&= f(E_{1,1}) \sum_{i=1}^{n} a_{i,i} \\
&= f(E_{1,1}) \text{tr}(A).
\end{aligned}$$

(ii) Claim 1 : $\text{span}(\{AB - BA \mid A, B \in M_{n\times n}(\mathbb{F})\}) \subseteq \ker(f)$.

Proof : Let $D \in \text{span}(\{AB - BA | A, B \in M_{n\times n}(\mathbb{F})\})$. Then $\exists A, B \in M_{n\times n}(\mathbb{F})$ such that $D = AB - BA$.

$$\begin{aligned}\text{tr}(D) &= \text{tr}(AB - BA)\\ &= \text{tr}(AB) - \text{tr}(BA)\\ &= 0.\end{aligned}$$

By (b)(i), $f(D) = f(E_{1,1})\text{tr}(D) = 0$. Thus $D \in \ker(f)$.

Claim 2 : $\dim(\ker(f)) = n^2 - 1$.

Proof : Since $f \neq 0$, $\ker(f) \neq M_{n\times n}(\mathbb{F}) \to \dim(\ker(f)) < n^2$. $- (*)$

Let $E_i' = E_{i,i} - E_{i+1,i+1}$.
Note that $\text{tr}(E_{i,j}) = 0$ for $i \neq j$ and $\text{tr}(E_i') = 0$ for $1 \leq i < n$.

Then let $B = \{E_{i,j} \in M_{n\times n}(\mathbb{F}) \mid i \neq j\} \cup \{E_i' \in M_{n\times n}(\mathbb{F}) \mid 1 \leq i < n\}$.

It is easy to check that $B \subseteq \ker(f)$ since every matrix in B has trace 0. Thus $\text{span}(B) \subseteq \ker(f)$. Since $\dim(\text{span}(B)) = n^2 - 1$, $\dim(\ker(f)) \geq n^2 - 1$. Together with (*), we conclude that $\dim(\ker(f)) = n^2 - 1$.

Claim 3 : $\dim(\text{span}(\{AB - BA \mid A, B \in M_{n\times n}(\mathbb{F})\})) \geq n^2 - 1$.

Proof : Using the same set B as in claim 2, choose arbitrary $D \in B$ and consider 2 cases:

Case 1: $D \in \{E_{i,j} \in M_{n\times n}(\mathbb{F}) \mid i \neq j\}$.

Then $D = E_{i,j}$ for some $i \neq j$. Write D as:

$$D = E_{i,j} = E_{i,j} - 0_{n\times n} = E_{i,j}E_{j,j} - E_{j,j}E_{i,j} \quad \text{for } i \neq j.$$

Thus $D \in \text{span}(\{AB - BA \mid A, B \in M_{n\times n}(\mathbb{F})\})$.

Case 2: $D \in \{E_i' \in M_{n\times n}(\mathbb{F}) \mid 1 \leq i < n\}$.

Then $D = E_k'$ for some $1 \leq k < n$. Write D as:

$$D = E_k' = E_{k,k} - E_{k+1,k+1} = E_{k,k+1}E_{k+1,k} - E_{k+1,k}E_{k,k+1}.$$

Thus $D \in \text{span}(\{AB - BA \mid A, B \in M_{n\times n}(\mathbb{F})\})$.

We thus conclude that $B \subseteq \text{span}(\{AB - BA \mid A, B \in M_{n\times n}(\mathbb{F})\})$ so $\dim(\text{span}(\{AB - BA \mid A, B \in M_{n\times n}(\mathbb{F})\})) \geq \dim(\text{span}(B)) = n^2 - 1$.

From claim 1, we know that $\text{span}(\{AB - BA \mid A, B \in M_{n\times n}(\mathbb{F})\}) \subseteq \ker(f)$.
From claim 2, we know that $\dim(\ker(f)) = n^2 - 1$.
From claim 3, we know that $\dim(\text{span}(\{AB - BA \mid A, B \in M_{n\times n}(\mathbb{F})\})) \geq n^2 - 1$.
Combining the 3 claims, we have: $\ker(f) = \text{span}(\{AB - BA \mid A, B \in M_{n\times n}(\mathbb{F})\})$.

Question 4

(a) To prove $\alpha - \lambda I_V$ is invertible:

Assume that $\alpha - \lambda I_V$ is not invertible.

Then $\text{nullity}(\alpha - \lambda I_V) > 0$.
$\exists$ non-zero $v \in V$ such that $(\alpha - \lambda I_V)(v) = 0_V$. But then $\alpha(v) - \lambda v = 0_V$.
This implies that $\alpha(v) = \lambda v$. which is a contradiction as λ is not an eigenvalue of α.

To prove that α commutes with $(\alpha - \lambda I_V)^{-1}$:

$$\begin{aligned}
\alpha \circ I_V &= I_V \circ \alpha \\
\alpha \circ (\alpha - \lambda I_V)^{-1} \circ (\alpha - \lambda I_V) &= (\alpha - \lambda I_V)^{-1} \circ (\alpha - \lambda I_V) \circ \alpha \\
\alpha \circ (\alpha - \lambda I_V)^{-1} \circ (\alpha - \lambda I_V) &= (\alpha - \lambda I_V)^{-1} \circ \alpha \circ (\alpha - \lambda I_V) \\
\alpha \circ (\alpha - \lambda I_V)^{-1} \circ (\alpha - \lambda I_V) \circ (\alpha - \lambda I_V)^{-1} &= (\alpha - \lambda I_V)^{-1} \circ \alpha \circ (\alpha - \lambda I_V) \circ (\alpha - \lambda I_V)^{-1} \\
\alpha \circ (\alpha - \lambda I_V)^{-1} &= (\alpha - \lambda I_V)^{-1} \circ \alpha.
\end{aligned}$$

(b)(i)

$$\begin{aligned}
\phi(\beta(v), v) &= \phi(v, \beta^\star(v)) \\
&= \phi(v, -\beta(v)) \\
&= \phi(-\beta(v), v) \text{ - (Since } \mathbb{F} = \mathbb{R}) \\
&= -\phi(\beta(v), v)
\end{aligned}$$

Thus $\phi(\beta(v), v) = 0$

(ii) Assume that $\exists \lambda \in \mathbb{R} \setminus \{0\}$ such that λ is an eigenvalue of β. Note that $\lambda \neq 0$.

$\exists$ nonzero $w \in V$ such that $\beta(w) = \lambda w$.
Then $\phi(\beta(w), w) = \phi(\lambda w, w) = \lambda \phi(w, w)$.
Since ϕ is positive definite, $\phi(w, w) > 0$. $\lambda \neq 0 \ \wedge \ \phi(w, w) \neq 0 \to \phi(\beta(w), w) \neq 0$. This is a contradiction to b(i).

(c) First note that:

$$\begin{aligned}
\gamma^\star &= [(I_V - \beta) \circ (I_V + \beta)^{-1}]^\star \\
&= [(I_V + \beta)^{-1}]^\star \circ (I_V - \beta)^\star \\
&= [(I_V + \beta)^\star]^{-1} \circ (I_V - \beta)^\star \\
&= (I_V^\star + \beta^\star)^{-1} \circ (I_V^\star - \beta^\star) \\
&= (I_V - \beta)^{-1} \circ (I_V + \beta).
\end{aligned}$$

Then:

$$\begin{aligned}
\gamma^\star \circ \gamma &= (I_V - \beta)^{-1} \circ (I_V + \beta) \circ (I_V - \beta) \circ (I_V + \beta)^{-1} \\
&= (I_V - \beta)^{-1} \circ (I_V - \beta) \circ (I_V + \beta) \circ (I_V + \beta)^{-1} \\
&= I_V.
\end{aligned}$$

Thus $\gamma^\star = \gamma^{-1}$.

(d) Claim: $(I_V + \eta)^{-1} \circ (I_V - \eta) = (I_V - \eta) \circ (I_V + \eta)^{-1}$.

Proof:

$$\begin{aligned}
(I_V + \eta)^{-1} \circ (I_V - \eta) &= (I_V + \eta)^{-1} \circ (I_V - \eta) \circ (I_V + \eta) \circ (I_V + \eta)^{-1} \\
&= (I_V + \eta)^{-1} \circ (I_V + \eta) \circ (I_V - \eta) \circ (I_V + \eta)^{-1} \\
&= (I_V - \eta) \circ (I_V + \eta)^{-1}.
\end{aligned}$$

Since -1 is not an eigenvalue of ζ, $I_V + \zeta$ is invertible.

To prove existence: Choose $\eta = (I_V - \zeta) \circ (I_V + \zeta)^{-1}$. We first check that $\eta^\star = -\eta$:

Note that by (a)(i), ζ commutes with $(\zeta + I_V)^{-1}$. (Choose $\lambda = -1$)
By our claim above, $(I_V - \zeta) \circ (I_V + \zeta)^{-1} = \eta = (I_V + \zeta)^{-1}(I_V - \zeta)$.

$$\begin{aligned}
\eta^\star &= [(I_V - \zeta) \circ (I_V + \zeta)^{-1}]^\star \\
&= [(I_V + \zeta)^\star]^{-1} \circ (I_V - \zeta)^\star \\
&= (I_V^\star + \zeta^\star)^{-1} \circ (I_V^\star - \zeta^\star) \\
&= (I_V + \zeta^{-1})^{-1} \circ (I_V - \zeta^{-1}) \\
&= \zeta \circ \zeta^{-1} \circ (I_V + \zeta^{-1})^{-1} \circ (I_V - \zeta^{-1}) \\
&= \zeta \circ (\zeta + I_V)^{-1} \circ (I_V - \zeta^{-1}) \\
&= (\zeta + I_V)^{-1} \circ \zeta \circ (I_V - \zeta^{-1}) \\
&= (\zeta + I_V)^{-1} \circ (\zeta - I_V) \\
&= -(\zeta + I_V)^{-1} \circ (I_V - \zeta) \\
&= -\eta.
\end{aligned}$$

We now check that our choice of η satisfies the inequality:

$$\begin{aligned}
\eta &= (I_V - \zeta) \circ (I_V + \zeta)^{-1} \\
\eta \circ (I_V + \zeta) &= (I_V - \zeta) \\
\eta + \eta \circ \zeta &= I_V - \zeta \\
(I_V + \eta) \circ \zeta &= I_V - \eta \\
\zeta &= (I_V + \eta)^{-1} \circ (I_V - \eta) \\
&= (I_V - \eta) \circ (I_V + \eta)^{-1} \text{ as desired.}
\end{aligned}$$

To prove uniqueness: Let η_1 and η_2 be 2 linear operators satisfying:

$$(I_V - \eta_1) \circ (I_V + \eta_1)^{-1} = \zeta = (I_V - \eta_2) \circ (I_V + \eta_2)^{-1}.$$

By our claim, $(I_V + \eta_2)^{-1} \circ (I_V - \eta_2) = (I_V - \eta_2) \circ (I_V + \eta_2)^{-1}$. Then:

$$\begin{aligned}
(I_V - \eta_1) \circ (I_V + \eta_1)^{-1} &= (I_V + \eta_2)^{-1} \circ (I_V - \eta_2) \\
(I_V + \eta_2) \circ (I_V - \eta_1) &= (I_V - \eta_2) \circ (I_V + \eta_1) \\
I_V - \eta_1 + \eta_2 - \eta_2 \circ \eta_1 &= I_V + \eta_1 - \eta_2 - \eta_2 \circ \eta_1 \\
2\eta_2 &= 2\eta_1 \\
\eta_2 &= \eta_1.
\end{aligned}$$

(ii) Recall that $\forall$ linear operators α on finite dimensional vector spaces,
$\det(\alpha) = \det(\alpha^\star)$.

$$\begin{aligned}
\det(\zeta) &= \det((I_V - \eta) \circ (I_V + \eta)^{-1}) \\
&= \det(I_V - \eta) \det((I_V + \eta)^{-1}) \\
&= \frac{\det(I_V + \eta^\star)}{\det(I_V + \eta)} \\
&= \frac{\det((I_V^\star + \eta)^\star)}{\det(I_V + \eta)} \\
&= \frac{\det(I_V + \eta)}{\det(I_V + \eta)} \\
&= 1.
\end{aligned}$$