NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1101R Linear Algebra 1
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Question 1

(a) (i) $(w, x, y, z) = (w, x, w - x, 2w + x)$, $(w, x \in \mathbb{R})$.

(ii) $(w, x, w - x, 2w + x) = x(0, 1, -1, 1) + w(1, 0, 1, 2)$
So $V = \text{span}\{(0, 1, -1, 1), (1, 0, 1, 2)\}$.
So V is a subspace of $\mathbb{R}^4$.

(iii) For any a in V, $a = x(0, 1, -1, 1) + w(1, 0, 1, 2)$.
$(0, 1, -1, 1)$ and $(1, 0, 1, 2)$ are linearly independent.
$\{(0, 1, -1, 1), (1, 0, 1, 2)\}$ is a basis for V.
So dim $V = 2$.

(iv) Because dim $V = 2$, it is not a linearly independent set as v_3 could be expressed as linear combinations of v_1 and v_2 since dim $V = 2$.

(b) (i) Since $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ are linearly independent. Therefore for the equality to hold

$$c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3} = \mathbf{0}$$

the only solution is $c_1 = c_2 = c_3 = 0$.

To prove the linear independency $\mathbf{u_1} + \mathbf{u_2}, \mathbf{u_1} - \mathbf{u_2}, \mathbf{u_3}$,

$$d_1(\mathbf{u_1} + \mathbf{u_2}) + d_2(\mathbf{u_1} - \mathbf{u_2}) + d_3\mathbf{u_3} = \mathbf{0} \tag{1}$$

Rearranging the above term, we have

$$(d_1 + d_2)\mathbf{u_1} + (d_1 - d_2)\mathbf{u_2} + d_3\mathbf{u_3} = \mathbf{0}$$

Comparing the coefficients of $\mathbf{u_i}$ with (1), $d_1 + d_2 = 0$, $d_1 - d_2 = 0$ and $d_3 = 0$. Therefore $d_1 = d_2 = d_3 = 0$ which implies that $\mathbf{u_1} + \mathbf{u_2}, \mathbf{u_1} - \mathbf{u_2}, \mathbf{u_3}$ are linearly independent.

Now let $\mathbf{x} \in \mathbb{R}^3$

$$\begin{aligned}\mathbf{x} &= c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3} \\ &= \frac{c_1 + c_2}{2}(\mathbf{u_1} + \mathbf{u_2}) + \frac{c_1 - c_2}{2}(\mathbf{u_1} - \mathbf{u_2}) + c_3\mathbf{u_3} \\ &= d_1(\mathbf{u_1} + \mathbf{u_2}) + d_2(\mathbf{u_1} - \mathbf{u_2}) + d_3\mathbf{u_3}\end{aligned}$$

Therefore T is a basis for $\mathbb{R}^3$.

(ii) Let $v_1 = u_1 + u_2, v_2 = u_1 - u_2, v_3 = u_3$. So $u_1 = \frac{v_1+v_2}{2}$ and $u_2 = \frac{v_1-v_2}{2}$, $u_3 = v_3$. The transition matrix from S to T is

$$\begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & -0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) Let $a_i = (0, 0 \ldots, 0, 1, 0, \ldots, 0)$, with the ith entry being 1. Let $V_i = \text{span}\{a_1, a_2, \ldots, a_i\}$.
Because $a_1, a_2, \ldots, a_i$ are linearly independent, dim $V_i = i$.
$\{a_1, a_2, \ldots, a_i\}$ is a subset of $\{a_1, a_2, \ldots, a_{i+1}\}$. So it satisfies the condition.

Question 2

(a) (i) The reduced row-echelon form of A is

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

A basis for the row space of A is $\{(1, 0, 1, 0, 1), (0, 1, 0, 1, 0)\}$.

(ii) A basis for the column space of A is $\{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix}\}$

(iii) Let $Ax = 0$. Referring to (2)

We get $x = s \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} + r \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$.
with $r, s, t \in \mathbb{R}$.

A basis for the null space is $\{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}\}$.

(iv) Add two vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ into it.

(b) (i) if rank(C)=1,

$$x - 2 = 0 \quad \& \quad x^2 - x - 2 = 0 \quad \& \quad x + 1 = 0$$

There is no real x that satisfy the simultaneous solution.

(ii) If rank(C) =2, either $x - 2 \neq 0$ or $x^2 - x - 2 \neq 0$ or $x^2 - x - 2 = 0$ and $x + 1 = 0$.
Case(i) $x \neq 2$, then $x^2 - x - 2$ must be zero. Then $x = -1$.
Case(ii) $x = 2$, then $x + 1 = 0$.
Therefore $x = 2$ or -1.

(iii) if $x \neq 2$ and $x \neq -1$, rank(C)=3.

(c) let B be $\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$, where $\beta_1, \beta_2, \beta_3$ are the row vectors.

For any $\alpha = (x, y, z)$ in the null space of B, $B\alpha = 0$.

So $\beta_1\alpha = 0, \beta_2\alpha = 0, \beta_3\alpha = 0$, (a, b, c) belongs to the row space of B.

$(a, b, c) = x_1\beta_1 + x_2\beta_2 + x_3\beta_3$. So $(a, b, c)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$.

So the nullspace of B is a subset of the plane $ax + by + cz = 0$.

Question 3

(a) (i) $r_2 = r_1 + r_3$, ie. r_2 is a redundant vector as a basis.
$r_1 \cdot r_3 = 0$.
So S is an orthogonal basis for the vector space V.

(ii) Let $u = x_1r_1 + x_3r_3$. Solving, $x_1 = 5, x_3 = 4$.
$(u)_s = (5, 4)$.

(iii) The projection of v onto V is

$$\frac{v \cdot r_1}{|r_1|^2}r_1 + \frac{v \cdot r_3}{|r_3|^2}r_3 = r_1 + \frac{1}{3}r_3 = (\frac{4}{3}, \frac{4}{3}, \frac{10}{3})$$

(b) (i) Consider $\left(\begin{array}{ccc|c} 2 & 0 & -2 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 4 & 1 & 1 \end{array}\right)$.

The reduced row-echelon form is $\left(\begin{array}{ccc|c} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & -\frac{3}{8} \end{array}\right)$

$Ax = b$ is inconsistent.

(ii) Consider $A^TAx = A^Tb$.

$$\begin{pmatrix} 14 & 14 & 0 \\ 14 & 20 & 6 \\ 0 & 6 & 6 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 6 \\ 6 \\ 0 \end{pmatrix}$$

The least squares solutions are $\begin{pmatrix} s + \frac{3}{7} \\ -s \\ s \end{pmatrix}$, $(s \in \mathbb{R})$.

(c) Consider $B^TBx = 0$. rank$(B^TB)\leq$rank$(B)\leq m < n$.
So it has infinitely many solutions. So $Bx = b$ has infinitely many least squares solutions.

Question 4

(a) (i) $\lambda I - A = \begin{pmatrix} \lambda - 4 & 0 & 0 \\ -1 & \lambda - 4 & 0 \\ 0 & 0 & \lambda - 5 \end{pmatrix}$

$\det(\lambda I - A) = (\lambda - 4)^2(\lambda - 5)$

The eigenvalues of A are 4 and 5.

(ii) $\lambda_1 = 4$. Consider $(4I - A)x = 0$.

$E_4 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$

Similarly, we get $E_5 = \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

(iii) $\dim E_4 = 1$, $\dim E_5 = 1$. A is not a diagonalizable matrix.

(iv) Let $B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$A + B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$,

$A + B$ has eigenvalues 4 with respect to $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, 3 with respect to $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and 5 with respect to $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ Hence. it's diagonalizable.

(b) Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. $A^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$.

$C = ABA^{-1}$.

The eigenvalues of C are the same as B. The eigenvalues of C^T are the same as C. So the eigenvalues of C^T are 2 and 3.

Consider $(\lambda I - C^T)x = 0$.

The eigenvector corresponding to 2 is $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, and the eigenvector corresponding to 3 is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Alternatively, $C^T = (A^{-1})^T B^T A^T = (A^T)^{-1} B^T A^T$, therefore we have

$$A^T C^T (A^T)^{-1} = B^T$$

The eigenvalues are 2 with respect to eigenvectors $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, and 3 with respect to eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(c) X is diagonalizable matrix with one eigenvalue λ. $\dim E_\lambda = n$.

Consider $(\lambda I - X)x = 0$, nullity$(\lambda I - X) = n$. rank$(\lambda I - X) = 0$. So $\lambda I - X = 0$, $X = \lambda I$.

Question 5

(a) (i) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$.

(ii) Consider $Ax = 0$, ker(T)= $\{(s, -s, -s) | s \in \mathbb{R}\}$.

(iii) rank(T)=rank(A)=2. nullity(T) = nullity(A)=1

(iv) $A^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$.

rank(T)=rank(A^T)=2, nullity(T)=nullity(A^T)=0.

(v) $R(T)$= the column space of A. Both $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$,$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ belong to R(T). So $R(T) = \mathbb{R}^2$.

(vi) Let the standard matrix of T and S be A and B.
The standard matrix of $S \cdot T$ is I. $I = BA$. rank(I)$\leq$rank(A)≤ 2.
It contradicts rank(I)=3. So it is not possible.

Alternatively, one can let $B = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$, and compute BA,

$$BA = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} a & b & a-b \\ c & d & c-d \\ e & f & e-f \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

But if $a = 1$, then $b = 1$ and $b = 0$, which does not make sense. Hence it is impossible to find such a S.

(b) Let the standard matrix of F be C. Let $U = (u_1\ u_2\ \dots\ u_n)$, $V = (F(u_1)\ F(u_2)\ \dots\ F(u_n))$.
U, V are orthogonal matrices. So $UU^T = I_n, VV^T = I_n$, $F(u_i) = Cu_i$ for $1 \leq i \leq n$.
$V = CU$. $C = VU^{-1}$ and therefore $CC^T = I$. So the standard matrix of F is an orthogonal matrix.