MA2108 - Mathematical Analysis I Suggested Solutions

(Semester 2 : AY2020/21)

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Question 1

(a) Determine the convergence or divergence of each of the following series. Justify your answers.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{8^n}\left(1+\frac{2}{n}\right)^{n^2}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n\cos n}{\sqrt{n^6-n^3+3}}$.

Solution

(i)

Let $a_n = \frac{(-1)^{n+1}}{8^n}(1+\frac{2}{n})^{n^2}$.

By taking the root test, $|a_n|^{\frac{1}{n}} = |\frac{1}{8^n}(1+\frac{2}{n})^{n^2}|^{\frac{1}{n}} = \frac{(1+\frac{2}{n})^n}{8}$.

Note that $\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{(1+\frac{2}{n})^n}{8} = \frac{e^2}{8} < 1$

Therefore, $\limsup_{n\to\infty}|a_n|^{\frac{1}{n}} < 1$ and the series converges.

(ii)

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n^2+n\cos n}{\sqrt{n^6-n^3+3}} &\geq \sum_{n=1}^{\infty} \frac{n^2-n}{\sqrt{n^6-n^3+3}} \\
&\geq \sum_{n=1}^{\infty} \frac{n^2-n}{\sqrt{2n^6}} \\
&= \frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{n^2-n}{n^3} \\
&= \frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n^2})
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n^2})$ diverges by p-series test, by Comparison test, the initial series diverges too.

(b) Let $f:(-3,\infty)\to\mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \dfrac{x+11}{x+3} & \text{if } x\in(-3,\infty)\cap\mathbb{Q}, \\ x+6, & \text{if } x\in(-3,\infty)\cap(\mathbb{R}\backslash\mathbb{Q}). \end{cases}$$

Determine the points, if any, at which f is continuous. Justify your answer.

Solution

$$\begin{aligned}
\frac{x+11}{x+3} &= x+6 \\
x^2+9x+18 &= x+11 \\
x^2+8x+7 &= 0 \\
x &= -1 \text{ or } x=-7.
\end{aligned}$$

Let $f_1(x) = \frac{x+11}{x+3}$ and $f_2(x) = x+6$. At $x=-1$, $f_1(x) = f_2(x) = 5$
Since these two functions are both continuous on $(-3,\infty)$, $\forall \epsilon > 0,\ \exists \delta_1, \delta_2$ such that

$$\begin{aligned}
|x+1| < \delta_1 &\implies |f_1(x) - 5| < \epsilon \\
|x+1| < \delta_2 &\implies |f_2(x) - 5| < \epsilon.
\end{aligned}$$

Choose $\delta = \min(\delta_1, \delta_2)$, then

$$|x+1| < \delta \implies |f(x) - 5| < \epsilon.$$

Now, let $g_1(x) = f_1(x) - f(x)$. Then,

$$g_1(x) = \begin{cases} 0, & \text{if } x \in (-3,\infty) \cap \mathbb{Q} \\ f_1(x) - f_2(x), & \text{if } x \in (-3,\infty) \cap (\mathbb{R}\backslash\mathbb{Q}). \end{cases}$$

Observe that $\forall x \in (-3,\infty) \cap (\mathbb{R}\backslash\mathbb{Q})\backslash\{-1\}$, we have $g_1(x) \neq 0$.
Let $y \in ((-3,\infty) \cap (\mathbb{R}\backslash\mathbb{Q}))\backslash\{-1\}$ and choose $\epsilon = \frac{|g_1(y)|}{2}$.
Then, $\forall \delta > 0, \exists p \in \mathbb{Q}$ such that $p \in (y-\delta, y+\delta) \implies |g_1(y) - g_1(p)| = |g_1(y)| > \epsilon$
Therefore, $g_1(x) = f_1(x) - f(x)$ is not continuous for all $x \in ((-3,\infty) \cap (\mathbb{R}\backslash\mathbb{Q}))\backslash\{-1\}$.
This implies that $f(x)$ is not continuous for all $x \in ((-3,\infty) \cap (\mathbb{R}\backslash\mathbb{Q}))\backslash\{-1\}$.
(Note that the conclusion is obtained from the contrapositive of the following statement: If $f(x), g(x)$ are continuous functions, so is $f(x) + g(x)$.)

Therefore, $f(x)$ is only continuous at $x = -1$.

Question 2

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that f is uniformly continuous on $\mathbb{R}$ and

$$3 < f(x) < 5 \text{ for all } x \in \mathbb{R}.$$

Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \frac{f(x)}{f(2x)}, \ x \in \mathbb{R}$$

Is it true that g is uniformly continuous on $\mathbb{R}$? Justify your answer.

Solution
Recall that:

1. f being uniformly continuous in $\mathbb{R}$ means that for all $x, y \in \mathbb{R}$,

$$\forall \epsilon > 0, \ \exists \delta > 0 \text{ s.t. } |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

2. $3 < f(x) < 5$ for all $x \in \mathbb{R}$ implies that $\frac{1}{5} < \frac{1}{f(x)} < \frac{1}{3}$ for all $x \in \mathbb{R}$.

Let $\epsilon > 0$ be arbitrary, from (1), there exists $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Then,

$$\begin{aligned}
|x - y| < \frac{\delta}{2} \implies g(x) - g(y) &= \left| \frac{f(x)}{f(2x)} - \frac{f(y)}{f(2y)} \right| \\
&= \left| \frac{f(x)f(2y) - f(y)f(2x)}{f(2x)f(2y)} \right| \\
&< \frac{1}{9} \Big| f(x)f(2y) - f(y)f(2y) + f(y)f(2y) - f(y)f(2x) \Big| \\
&< \frac{1}{9} \Big| \big[f(2y)f(x) - f(2y)f(y)\big] + \big[f(y)f(2y) - f(y)f(2x)\big] \Big| \\
&< \frac{1}{9} \Big[|f(2y)||f(x) - f(y)| + |f(y)||f(2y) - f(2x)| \Big] \\
&< \frac{1}{9}[5\epsilon + 5\epsilon] \\
&< \frac{10\epsilon}{9}.
\end{aligned}$$

Since ϵ is chosen arbitrarily, we conclude that $g(x)$ is uniformly continuous on $\mathbb{R}$.

(b) Let (S, d) be a metric space, where d is a metric on a non-empty set S. Consider the function $\rho : S \times S \to \mathbb{R}$ given by:

$$\rho(x, y) = d(x, y) + \sqrt{d(x, y)}, x, y \in S$$

Is it true that ρ is a metric on S? Justify your answer.

Solution
Since d is a metric on S, it fulfills commutativity, positive-definiteness and triangle inequality.

1. Commutativity: For all $x, y \in S$, by the commutativity of d,

$$\begin{aligned}\rho(x,y) &= d(x,y) + \sqrt{d(x,y)} \\ &= d(y,x) + \sqrt{d(y,x)} \\ &= \rho(y,x)\end{aligned}$$

   Thus it is a commutative operator.

2. Positive-definiteness:
   If $x = y$, $\rho(x,x) = d(x,x) + \sqrt{d(x,x)} = 0$.
   If $x \neq y$, $\rho(x,y) = d(x,y) + \sqrt{d(x,y)} > 0 + 0 = 0$ by positive-definiteness of d as a metric.
   Therefore, ρ is positive-definite.

3. Triangle Inequality:
   First, note that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for all non-negative real numbers a, b. Squaring terms on both sides can verify this statement. Next, by the triangle inequality for metric d, we have that $d(x,z) \leq d(x,y) + d(y,z)$. Therefore, for any $x, y, z \in S$

$$\begin{aligned}\rho(x,z) &= d(x,z) + \sqrt{d(x,z)} \\ &\leq d(x,y) + d(y,z) + \sqrt{d(x,y) + d(y,z)} \\ &\leq d(x,y) + \sqrt{d(x,y)} + d(y,z) + \sqrt{d(y,z)} \\ &\leq \rho(x,y) + \rho(y,z)\end{aligned}$$

   Therefore, ρ satisfies the triangle inequality.

We conclude that ρ is a metric on S.

Question 3

(a) Consider the sequence (x_n) given by

$$x_1 = 4, \quad \text{and} \quad x_{n+1} = \frac{x_n^2 - 5x_n + 15}{3} \text{ for all } n \in \mathbb{N}.$$

Is it true that (x_n) converges? Find also its limit if it converges. Justify your answers.

Remark: The approximate values of the first few terms of (x_n) are as follows:

$$x_1 = 4,\ x_2 = 3.67,\ x_3 = 3.37,\ x_4 = 3.17,\ x_5 = 3.07,\ x_6 = 3.02,\ x_7 = 3.01, \cdots .$$

Solution
Since $3 < x_1 < 5$,

$$\begin{aligned} x_2 - x_1 &= \frac{11}{3} - 4 < 0 \\ x_2 - 3 &= \frac{11}{3} - 3 > 0 \end{aligned}$$

Therefore, $3 < x_2 < x_1 < 5$. Now, assume that $3 < x_n < 5$ for an arbitrary $n \in \mathbb{N}$, then

$$\begin{aligned} x_{n+1} - x_n &= \frac{x_n^2 - 5x_n + 15 - 3x_n}{3} = \frac{x_n^2 - 8x_n + 15}{3} = \frac{(x_n - 5)(x_n - 3)}{3} < 0 \\ x_{n+1} - 3 &= \frac{x_n^2 - 5x_n + 15 - 9}{3} = \frac{x_n^2 - 5x_n + 6}{3} = \frac{(x_n - 2)(x_n - 3)}{3} > 0 \end{aligned}$$

Therefore, $3 < x_{n+1} < x_n < 5$. By Mathematical Induction, $3 < ... < x_{n+1} < x_n < x_{n-1} < ... < x_1 = 4$. The sequence is bounded and decreasing, so it is convergent. Let $x = \lim n \to \infty(x_n)$. Then,

$$\begin{aligned} x &= \frac{x^2 - 5x + 15}{3} \\ x^2 - 5x + 15 &= 3x \\ x^2 - 8x + 15 &= 0 \\ (x - 5)(x - 3) &= 0 \end{aligned}$$

Since the sequence is decreasing, the limit of the sequence (x_n) is 3.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that f is continuous on R, $f(x) > 0\ \forall x \in \mathbb{R}$ and $f(1) = f(5)$. Is it true that there exists $c \in [1, 2]$ such that

$$\frac{1}{f(c)} + \frac{1}{f(c+2)} = \frac{1}{f(c+1)} + \frac{1}{f(c+3)}?$$

Justify your answer.

Solution
First observe that

$$\begin{aligned} & \frac{1}{f(c)} + \frac{1}{f(c+2)} = \frac{1}{f(c+1)} + \frac{1}{f(c+3)} \\ \iff & \frac{f(c+2) + f(c)}{f(c+2)f(c)} = \frac{f(c+1) + f(c+3)}{f(c+1)f(c+3)} \\ \iff & f(c)f(c+1)f(c+3) + f(c+2)f(c+1)f(c+3) = f(c)f(c+2)f(c+3) + f(c)f(c+2)f(c+1). \end{aligned}$$

Let $g(x) = f(x+1)f(x+3)[f(x)+f(x+2)] - f(x+2)f(x)[f(x+3)+f(x+1)]$. By the continuity of $f(x)$, $g(x)$ is continuous. Then,

$$g(1) = f(2)f(4)[f(1)+f(3)] - f(3)f(1)[f(2)+f(4)]$$
$$g(2) = f(3)f(5)[f(2)+f(4)] - f(4)f(2)[f(5)+f(3)] = f(3)f(1)[f(2)+f(4)] - f(4)f(2)[f(1)+f(3)] = -g(1)$$

If $g(1) = 0$, take $c = 1$, then

$$f(c)f(c+1)f(c+3) + f(c+2)f(c+1)f(c+3) = f(c)f(c+2)f(c+3) + f(c)f(c+2)f(c+1)$$

and the desired equality will be fulfilled.

If $g(1) \neq 0$, WLOG assume $g(1) < 0$, then $g(2) > 0$. By the continuity of $g(x)$ and the Intermediate Value Theorem, there exists $c \in [1, 2]$ such that the desired equality is fulfilled. Therefore, the statement is true.

Question 4

(a) Let (S, d) be a metric space, where d denotes a metric on a non-empty set S. Let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be two functions, both continuous on S. Consider the subset of S given by:

$$G = \big\{x \in S : \max(f(x), g(x)) > 4 \text{ and } \min(f(x), g(x)) > 2\big\}.$$

Is it true that G is open in S? Justify your answer. Here for $a, b \in \mathbb{R}$, $\max(a, b)$ and $\min(a, b)$ denote the maximum and minimum of a and b respectively.

Solution
We define four sets

$$A_1 = \{x \in S \mid f(x) > 4\};\ A_2 = \{x \in S \mid g(x) > 4\};\ B_1 = \{x \in S \mid f(x) > 2\};\ B_2 = \{x \in S \mid g(x) > 2\}.$$

Then, $G = B_1 \cap B_2 \cap (A_1 \cup A_2)$. But observe that $A_1 = f^{-1}((4, \infty))$ is the preimage of an open set under a continuous function. Thus A_1 is open. Similarly, A_2, B_1 and B_2 are also open. It then follows that G is open as well.

(b) Let $(\mathbb{R}^2, d_2)$ be the Euclidean plane, where d_2 denotes the Euclidean metric on $\mathbb{R}^2$. For each $n \in \mathbb{N}$, let A_n be the subset of $\mathbb{R}^2$ given by:

$$A_n = \{(x_1, x_2) \in \mathbb{R}^2 : \frac{1}{2n+1} \leq \sqrt{x_1^2 + x_2^2} \leq \frac{1}{2n}\}.$$

Let $A = \{(0, 0)\} \cup \bigcup_{n \in \mathbb{N}} A_n$. Is it true that A is a compact subset of $\mathbb{R}^2$? Justify your answer.

Solution
We define a sequence of sets as follows:

$$\begin{aligned}
B_1 &= \Big\{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} > \frac{1}{2}\Big\} \\
B_2 &= \Big\{(x_1, x_2) \in \mathbb{R}^2 : \frac{1}{4} < \sqrt{x_1^2 + x_2^2} < \frac{1}{3}\Big\} \\
B_n &= \Big\{(x_1, x_2) \in \mathbb{R}^2 : \frac{1}{2n} < \sqrt{x_1^2 + x_2^2} < \frac{1}{2n-1}\Big\} \text{ for all } n > 2.
\end{aligned}$$

Since B_1 is the complement to $\{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} \leq \frac{1}{2}\}$ which is closed, thus B_1 is an open set.

We will now prove that for all $n \geq 2$, B_n is an open set. Fix arbitrary $n \geq 2$ and $(x, y) \in B_n$. Further denote $d = \sqrt{x^2 + y^2}$. We have $\frac{1}{2n} < d < \frac{1}{2n-1}$. Let $\delta = \min(\frac{1}{2n-1} - d, d - \frac{1}{2n})$.

Then, for all $(p, q) \in N_{\delta/2}((x, y))$, we have

$$\begin{aligned}
\sqrt{p^2 + q^2} &< d + \frac{\delta}{2} \\
&< d + \Big(\frac{1}{2n-1} - d\Big) \\
&= \frac{1}{2n-1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}\sqrt{p^2+q^2} &> d - \frac{\delta}{2} \\ &> d - \left(d - \frac{1}{2n}\right) \\ &\geq \frac{1}{2n}.\end{aligned}$$

Therefore, $N_{\delta/2}((x,y)) \subseteq B_n$.
Thus, B_n is open for all $n \in \mathbb{N}$, which implies that $\bigcup_{n\in\mathbb{N}} B_n$ is open.
Since A is the complement of the union above, A is closed.
Furthermore, A is bounded because $A \subseteq \{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} \leq \frac{1}{2}\}$. Since A is closed and bounded in $\mathbb{R}^2$, A is compact. The statement is true.

Question 5

(a) For each of the following limits, either find the limit or show that the limit does not exist. Justify your answers. Here for $a \in \mathbb{R}$, the floor $\lfloor a \rfloor$ of a denotes the greatest integer less than or equal to a.

(i) $\lim\limits_{x\to 9} \left(2\lfloor 2x \rfloor + \lfloor \frac{12}{\sqrt{x}} \rfloor\right) \sin x.$

(ii) $\lim\limits_{x\to 0} \left(\frac{1}{2x^2+5} - \frac{1}{6x^2+5}\right) \frac{\sin \frac{1}{x}}{x}.$

Solution

(i)

$$\lim_{x\to 9^+} (2\lfloor 2x \rfloor + \lfloor \frac{12}{\sqrt{x}} \rfloor)\text{sin}x = \sin(9)(2\times 18 + 3) = 39\ \sin(9)$$
$$\lim_{x\to 9^-} (2\lfloor 2x \rfloor + \lfloor \frac{12}{\sqrt{x}} \rfloor)\text{sin}x = \sin(9)(2\times 17 + 4) = 38\ \sin(9)$$

The limit does not exist because both one-sided limits are not equal, as shown above.

(ii)

The limit exists, and the value is zero. This is because:

$$\begin{aligned}
\lim_{x\to 0} \left(\frac{1}{2x^2+5} - \frac{1}{6x^2+5}\right) \frac{\text{sin}\frac{1}{x}}{x} &= \lim_{x\to 0} \frac{4x^2}{(2x^2+5)(6x^2+5)} \cdot \frac{x\text{sin}\frac{1}{x}}{x^2} \\
&= \lim_{x\to 0} \left(x \sin \frac{1}{x}\right) \cdot \lim_{x\to 0} \frac{4}{(2x^2+5)(6x^2+5)} \\
&= 0 \cdot \frac{4}{25} \\
&= 0
\end{aligned}$$

since $\lim\limits_{x\to 0} x \sin(\frac{1}{x}) = 0$ by squeeze theorem.

(b) Let (a_n) and (ϵ_n) be two sequences of real numbers such that $a_n \geq 0$ and $\epsilon_n \geq 0$ for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \epsilon_n = 0 \text{ and } a_{n+1} \leq \frac{n^2 a_n}{(n+1)^2} + \frac{\epsilon_n}{n+1}$$

for all $n \in \mathbb{N}$. Is it true that (a_n) converges? Find also its limit if (a_n) converges. Justify your answer.

Solution

Expressing a_{n+1} in terms of a_1 and terms in ϵ_n, we have

$$\begin{aligned}
a_{n+1} &\leq \frac{n^2 a_n}{(n+1)^2} + \frac{\epsilon_n}{n+1} \\
&\leq \frac{n^2}{(n+1)^2} \left[\frac{(n-1)^2 a_{n-1}}{n^2} + \frac{\epsilon_{n-1}}{n}\right] + \frac{\epsilon_n}{n+1} \\
&= \frac{(n-1)^2}{(n+1)^2} a_{n-1} + \frac{n\epsilon_{n-1}}{(n+1)^2} + \frac{(n+1)\epsilon_n}{(n+1)^2}.
\end{aligned}$$

Repeating this process, we obtain

$$a_{n+1} \leq \left(\frac{1}{n+1}\right)^2 a_1 + \left(\frac{1}{n+1}\right)^2 \sum_{i=1}^{n} (i+1)\epsilon_i.$$

Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} \epsilon_n = 0$, $\exists k \in \mathbb{N}$ such that $\forall k' > k$, we have $\epsilon_{k'} < \epsilon$. Then, $\forall n \in \mathbb{Z}_{\geq 1}$,

$$\begin{aligned}
\frac{1}{(k+n+1)^2} \sum_{i=1}^{n} (k+i+1)\epsilon_{k+i} &< \frac{\epsilon}{(k+n+1)^2} \sum_{i=1}^{n} (k+i+1) \\
&= \frac{\epsilon}{(k+n+1)^2} \cdot \frac{(n)(2k+n+3)}{2} \\
&\leq \frac{\epsilon}{(k+n+1)^2} \cdot \frac{(k+n+1)(2k+n+3)}{2} \\
&\leq \epsilon \cdot \frac{2k+n+3}{2k+2n+2} \\
&\leq \epsilon.
\end{aligned}$$

Now, let $M = a_1 + \sum_{i=1}^{k} (i+1)\epsilon_i$. There exists $N \in \mathbb{Z}_{\geq 1}$ such that $\frac{M}{(N+k)^2} < \epsilon$. Then $\forall n \geq (N+k)$,

$$\begin{aligned}
a_{n+1} &\leq \left(\frac{1}{n+1}\right)^2 a_1 + \left(\frac{1}{n+1}\right)^2 \sum_{i=1}^{n} (i+1)\epsilon_i \\
&= \left(\frac{1}{n+1}\right)^2 a_1 + \left(\frac{1}{n+1}\right)^2 \sum_{i=1}^{k} (i+1)\epsilon_i + \left(\frac{1}{n+1}\right)^2 \sum_{i=k+1}^{n} (i+1)\epsilon_i \\
&= \frac{M}{(n+1)^2} + \frac{1}{(n+1)^2} \sum_{i=k+1}^{n} (i+1)\epsilon_i \\
&< \epsilon + \frac{1}{(n+1)^2} \sum_{i=1}^{n-k} (k+i+1)\epsilon_{k+i} \\
&< \epsilon + \epsilon \\
&= 2\epsilon.
\end{aligned}$$

Since the choice of $\epsilon > 0$ is arbitrary, we conclude that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n > N$, $a_{n+1} < 2\epsilon$. Therefore, the sequence $\{a_n\}_{n=1}^{\infty}$ converges, and the limit is zero.

Question 6

(a) Use the $\epsilon - \delta$ definition of limit to show that

$$\lim_{x \to 3} \frac{x^2 - 5}{3x - 7} = 2.$$

Solution
Let $\epsilon > 0$ be arbitrary. By picking $\delta = \min\left\{\frac{1}{3}, 3\epsilon\right\}$, we have:

$$\begin{aligned}
0 < |x - 3| < \delta \implies \left|\frac{x^2 - 5}{3x - 7} - 2\right| &= \left|\frac{x^2 - 6x + 9}{3x - 7}\right| \\
&= |x - 3|\left|\frac{x - 3}{3x - 7}\right| \\
&< |x - 3|\left|\frac{x - 3}{3x - 9}\right| \\
&= \frac{1}{3}|x - 3| \\
&< \epsilon
\end{aligned}$$

And thus the statement is proven. Note: the number $\frac{1}{3}$ is deduced from solving an inequality $\left|\frac{x-3}{3x-7}\right| < \frac{1}{3}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that f is continuous on $\mathbb{R}$ and for all $x \in \mathbb{R}\backslash\{2\}$,

$$|f(x) - 2| < |x - 2|.$$

Consider the sequence (x_n) given by

$$x_1 = 10, \text{ and } x_{n+1} = f(x_n) \text{ for all } n \in \mathbb{N}.$$

Is it true that (x_n) converges? Justify your answer.

Solution
Notice that the sequence $\left\{|x_n - 2|\right\}_{n=1}^{\infty}$ is decreasing. We claim that $\left\{|x_n - 2|\right\}_{n=1}^{\infty}$ converges to zero as $n \to \infty$.

Assume that $\left\{|x_n - 2|\right\}_{n=1}^{\infty}$ converge to some $\epsilon > 0$. Observe that the set $U = [-8, 2 - \epsilon] \cup [2 + \epsilon, 10]$ is compact. Since the function $\left|\frac{f(x)-2}{x-2}\right|$ is continuous in U, by the Extreme Value Theorem, there exists $M \in [0, 1)$ such that:

$$\sup_{x \in U} \left|\frac{f(x) - 2}{x - 2}\right| = M.$$

For all $n \in \mathbb{N}$,

$$\begin{aligned}
|x_{n+1} - 2| &= |f(x_n) - 2| \\
&\leq M|x_n - 2| \\
&= M|f(x_{n-1}) - 2| \\
&\leq \cdots \\
&\leq M^n|x_1 - 2| = 8M^n.
\end{aligned}$$

Since $M \in [0, 1)$, there exists $n \in \mathbb{N}$ such that $M^n < \frac{\epsilon}{8}$. Then $|x_{n+1} - 2| \leq 8M^n < \epsilon$. Therefore, $|x_n - 2|$ converges to zero as $n \to \infty$ (contradiction).
As $n \to \infty$, since $|x_n - 2|$ converges to zero, the sequence $x_n - 2$ converges to zero. Thus, (x_n) converges to 2.