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SECTION A

Question 1

(a) We shall label this row echelon form $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ of $\boldsymbol{A}$ as $\boldsymbol{R}$.

(i) Since $\boldsymbol{A}$ and $\boldsymbol{R}$ are row equivalent matrices, row space of $\boldsymbol{A}$ = row space of $\boldsymbol{R}$.
So, the row space of $\boldsymbol{A}$ can be given by span$\{(1,0,1,0),(0,0,1,1)\}$. Since the two vectors are linearly independent, $\{(1,0,1,0),(0,0,1,1)\}$ forms a basis for the row space of $\boldsymbol{A}$.

The dimension of the row space of $\boldsymbol{A}$ is the number of vectors in a basis for the row space of $\boldsymbol{A}$. Thus, the dimension of the row space of $\boldsymbol{A}$ is 2.

(ii) The nullspace of $\boldsymbol{A}$ is the solution space of the homogeneous system of equations $\boldsymbol{Ax} = \boldsymbol{0}$.
A row echelon form of the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{0})$ of the homogeneous system of equations $\boldsymbol{Ax} = \boldsymbol{0}$ is given by $(\boldsymbol{R} \mid \boldsymbol{0})$.
We let

$$\boldsymbol{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

From $(\boldsymbol{R} \mid \boldsymbol{0})$, we have

$$\begin{cases} a + c = 0 \\ c + d = 0 \end{cases} \longrightarrow \begin{cases} a = d \\ c = -d. \end{cases}$$

Therefore, the general solution of the homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ is

$$\begin{aligned} \boldsymbol{x} &= \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} d \\ b \\ -d \\ d \end{pmatrix} = \begin{pmatrix} t \\ s \\ -t \\ t \end{pmatrix} \qquad (\text{let } s = b,\ t = d) \\ &= s\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \quad s, t \in \mathbb{R}. \end{aligned}$$

Hence, $\{(0,1,0,0),(1,0,-1,1)\}$ forms a basis for the nullspace of $\boldsymbol{A}$.
Thus, the dimension of the nullspace of $\boldsymbol{A}$ is 2.

(iii) Let $\boldsymbol{u}$ be a vector belonging to the intersection of the row space and nullspace of $\boldsymbol{A}$. Then

$$\boldsymbol{u} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \text{ for some } \alpha_1, \alpha_2 \in \mathbb{R}$$

and

$$\boldsymbol{u} = \beta_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \text{ for some } \beta_1, \beta_2 \in \mathbb{R}.$$

Equating the two expressions, we have

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \beta_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \beta_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \beta_2 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{array}{ccccccc} \alpha_1 & & & -\beta_2 & = & 0 \\ & & -\beta_1 & & = & 0 \\ \alpha_1 & +\alpha_2 & & +\beta_2 & = & 0 \\ & \alpha_2 & & -\beta_2 & = & 0 \end{array} \right.$$

Solving the system of homogenous equation, we have

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.$$

Therefore, $\boldsymbol{u} = \boldsymbol{0}$. Hence the intersection of the row space and nullspace of $\boldsymbol{A}$ is $\{\boldsymbol{0}\}$.

(iv) By result of (1iii.), we have $\{(1,0,1,0),(0,0,1,1),(0,1,0,0),(1,0,-1,1)\}$ to be a linearly independent set in $\mathbb{R}^4$. Thus it is a basis of $\mathbb{R}^4$ extended from a basis of the row space of $\boldsymbol{A}$.

(b) (i) According to Dimension Theorem for Matrices, we have

$$\begin{aligned} 5 + \text{nullity}(\boldsymbol{A}) &= 9 \\ \text{nullity}(\boldsymbol{A}) &= 4. \end{aligned}$$

(ii) Similarly, we have,

$$\begin{aligned} 5 + \text{nullity}(\boldsymbol{A}^T) &= 6 \qquad (\because \text{rank}(\boldsymbol{A}^T) = \text{rank}(\boldsymbol{A})) \\ \text{nullity}(\boldsymbol{A}^T) &= 1. \end{aligned}$$

(iii) No.

$$6 = \text{rank}(\boldsymbol{A} \mid \boldsymbol{b}) \neq \text{rank}(\boldsymbol{A}) = 5.$$

The 6^{th} row of the reduced row echelon form of $(\boldsymbol{A} \mid \boldsymbol{b})$ will be of the form

$$(\underbrace{0 \quad 0 \quad \cdots \quad 0}_{\text{9 zeros}} \mid 1).$$

Thus, the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent.

Question 2

(i) We see that S is a set of vectors in $\mathbb{R}^3$ such that $|S| = 3 = \dim(\mathbb{R}^3)$. It follows that to show S is an orthonormal basis for $\mathbb{R}^3$, we just need to check S is orthogonal (i.e. $\boldsymbol{u}_i \cdot \boldsymbol{u}_j = 0$ for each $i \neq j$) and each vector in S is a unit vector. And so,

$$\begin{aligned}
\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 &= \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(-\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = 0, \\
\boldsymbol{u}_2 \cdot \boldsymbol{u}_3 &= \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(-\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right) = 0, \\
\boldsymbol{u}_3 \cdot \boldsymbol{u}_1 &= \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(-\frac{2}{3}\right)\left(\frac{2}{3}\right) = 0, \\
||\boldsymbol{u}_1|| &= \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = 1, \\
||\boldsymbol{u}_2|| &= \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = 1, \\
||\boldsymbol{u}_3|| &= \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = 1.
\end{aligned}$$

$\therefore S$ is an orthonormal basis for $\mathbb{R}^3$.

(ii) Since S is an orthonormal basis for $\mathbb{R}^3$, we have

$$\begin{aligned}
\boldsymbol{w} &= (\boldsymbol{w} \cdot \boldsymbol{u}_1)\boldsymbol{u}_1 + (\boldsymbol{w} \cdot \boldsymbol{u}_2)\boldsymbol{u}_2 + (\boldsymbol{w} \cdot \boldsymbol{u}_3)\boldsymbol{u}_3 \\
&= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \boldsymbol{u}_1 + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \boldsymbol{u}_2 + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} \boldsymbol{u}_3 \\
&= \left(\frac{1}{3} + \frac{2}{3} + \frac{2}{3}\right)\boldsymbol{u}_1 + \left(\frac{2}{3} - \frac{2}{3} + \frac{1}{3}\right)\boldsymbol{u}_2 + \left(\frac{2}{3} + \frac{1}{3} - \frac{2}{3}\right)\boldsymbol{u}_3 \\
&= \frac{5}{3}\boldsymbol{u}_1 + \frac{1}{3}\boldsymbol{u}_2 + \frac{1}{3}\boldsymbol{u}_3.
\end{aligned}$$

(iii) Since $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is an orthonormal basis for V, the orthogonal projection of $\boldsymbol{v}$ onto V is given by

$$\boldsymbol{p} = (\boldsymbol{v} \cdot \boldsymbol{u}_1)\boldsymbol{u}_1 + (\boldsymbol{v} \cdot \boldsymbol{u}_2)\boldsymbol{u}_2.$$

So

$$\begin{aligned}
\boldsymbol{p} &= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \boldsymbol{u}_1 + \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \boldsymbol{u}_2 \\
&= \left(\frac{1}{3} + \frac{4}{3} - \frac{2}{3} \right) \boldsymbol{u_1} + \left(\frac{2}{3} - \frac{4}{3} - \frac{1}{3} \right) \boldsymbol{u_2} \\
&= \boldsymbol{u}_1 - \boldsymbol{u}_2 \\
&= \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}.
\end{aligned}$$

Then, the shortest distance from $\boldsymbol{v}$ to V is given by

$$\begin{aligned}
& ||\boldsymbol{v} - \boldsymbol{p}|| \\
=& \left\| \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix} \right\| \\
=& \left\| \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ -\frac{4}{3} \end{pmatrix} \right\| \\
=& \sqrt{\left(\frac{4}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{4}{3}\right)^2} = 2.
\end{aligned}$$

(iv) Recall that the least squares solution to $\boldsymbol{Ax} = \boldsymbol{v}$ is when $\boldsymbol{Ax}$ equals to the projection of $\boldsymbol{v}$ onto the column space of $\boldsymbol{A}$.
Therefore we have the least squares solution to be

$$\begin{aligned}
\boldsymbol{Ax} &= \boldsymbol{p} \\
&= \boldsymbol{u}_1 - \boldsymbol{u}_2 \\
&= \boldsymbol{A} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\end{aligned}$$

(v) It is clear that the desired plane is span$\{\boldsymbol{p}, \boldsymbol{n}\}$ for some $\boldsymbol{n}$ perpendicular to V. Since the set $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is orthonormal, $\boldsymbol{u}_3$ is perpendicular to span$\{\boldsymbol{u}_1, \boldsymbol{u}_2\} = V$. Hence, the desired plane is span$\{\boldsymbol{p}, \boldsymbol{u}_3\}$.

Question 3

(a) (i)

$$\begin{aligned}
T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) &= T\left(\frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \\
&= \frac{1}{2} T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) + \frac{1}{2} T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \\
&= \frac{1}{2} \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) &= T\left(\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \\
&= \frac{1}{2}T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) - \frac{1}{2}T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \\
&= \frac{1}{2}\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.
\end{aligned}$$

Thus the standard matrix of T is

$$[T] = (T(\boldsymbol{e_1})\ T(\boldsymbol{e_2})) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}.$$

(ii) Since $R(T)$ is the column space of $[T]$, we have $R(T) = \text{span}\{(1,2,3),(2,4,6)\}$. We observe that $(2,4,6)$ is a scalar multiple of $(1,2,3)$, and thus is a redundant vector.
Therefore, we have $\{(1,2,3)\}$ to be a basis for $R(T)$.

(iii) Let $\boldsymbol{u} = \begin{pmatrix} x \\ y \end{pmatrix} \in \ker(T)$.
Then

$$\begin{aligned}
T(\boldsymbol{u}) &= \boldsymbol{0} \\
\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

Since $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{array}\right) \underset{R_3-3R_1}{\overset{R_2-2R_1}{\longrightarrow}} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$, we get $\boldsymbol{u} = s\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $s \in \mathbb{R}$.
Therefore $\{(-2,1)\}$ forms a basis for $\text{Ker}(T)$.

(iv) No.
Assume on the contrary that such a linear transformation S exists, with standard matrix $[S]$. Then since $[S \circ T]$, the standard matrix of $S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$, is invertible, by Dimension Theorem for Matrices, we have $\text{nullity}([S \circ T]) = 0$, and so $\ker([S \circ T]) = \{\boldsymbol{0}\}$.

Now let $\boldsymbol{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Since $\boldsymbol{u} \in \text{Ker}(T)$, we have

$$\begin{aligned}
[S \circ T]\boldsymbol{u} &= [S][T]\boldsymbol{u} \\
&= [S]\boldsymbol{0} = \boldsymbol{0},
\end{aligned}$$

a contradiction. Hence, such a linear transformation S does not exist.

(b) Let $\boldsymbol{A}$ be the standard matrix that represents the linear transformation T.
Since $R(T)$ is given by the line through the origin and $(1,1,1)$, the column space of $\boldsymbol{A}$ is $\text{span}\{(1,1,1)\}$. Let $\boldsymbol{u}$ be in the row space of $\boldsymbol{A}$. Then $\forall \boldsymbol{v} \in \text{Ker}(\boldsymbol{A})$, $\boldsymbol{u}^T\boldsymbol{v} = 0$. Since $\text{Ker}(\boldsymbol{A})$ is represented by the plane perpendicular to $(1,-1,1)$, $\boldsymbol{u}$ is parallel to $(1,-1,1)$. Therefore, the row space of $\boldsymbol{A}$ is $\text{span}\{(1,-1,1)\}$.
This led us to consider the matrix $\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ as a possible candidate for $\boldsymbol{A}$. It can be easily checked that a linear transformation represented by such $\boldsymbol{A}$ satisfies the required condition.

Question 4

(a) (i) The characteristic polynomial of $\boldsymbol{A}$ is

$$\begin{aligned}\det(\lambda \boldsymbol{I} - \boldsymbol{A}) &= \begin{vmatrix} \lambda-2 & 0 & 2 \\ 0 & \lambda-3 & 0 \\ 0 & 0 & \lambda-3 \end{vmatrix} \\ &= (\lambda-2)(\lambda-3)^2.\end{aligned}$$

Hence, $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$ if and only if $\lambda = 2$ or $\lambda = 3$. Thus the eigenvalues of $\boldsymbol{A}$ are 2 and 3.

Note: Observe that $\boldsymbol{A}$ is a triangular matrix. In this case, the eigenvalues of $\boldsymbol{A}$ are just the diagonal entries of $\boldsymbol{A}$.

(ii) Let $\boldsymbol{x} \in E_2$. Then,

$$\begin{aligned}(2\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} &= 0 \\ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{x} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},\end{aligned}$$

solving which, we get $\boldsymbol{x} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $t \in \mathbb{R}$. So $\{(1,0,0)\}$ forms a basis for E_2.

Let $\boldsymbol{y} \in E_3$. Then,

$$\begin{aligned}(3\boldsymbol{I} - \boldsymbol{A})\boldsymbol{y} &= 0 \\ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{y} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},\end{aligned}$$

solving which, we get $\boldsymbol{y} = s \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $s, t \in \mathbb{R}$. So $\{(2,0,-1),(0,1,0)\}$ forms a basis for E_3.

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.

Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is diagonal.

(b) (i) From the given information, it is clear that 2 and 0 are eigenvalues of $\boldsymbol{B}$.
Next, we observe that,

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{u}_3 + \boldsymbol{u}_4) &= B(\boldsymbol{u}_3) + B(\boldsymbol{u}_4) \\ &= \boldsymbol{u}_4 + \boldsymbol{u}_3 \\ &= 1(\boldsymbol{u}_3 + \boldsymbol{u}_4).\end{aligned}$$

Therefore, 1 is an eigenvalue of $\boldsymbol{B}$.

Also we observe that,

$$\begin{aligned}B(\boldsymbol{u}_3 - \boldsymbol{u}_4) &= B(\boldsymbol{u}_3) - B(\boldsymbol{u}_4) \\ &= \boldsymbol{u}_4 - \boldsymbol{u}_3 \\ &= (-1)(\boldsymbol{u}_3 - \boldsymbol{u}_4).\end{aligned}$$

Therefore, -1 is an eigenvalue of $\boldsymbol{B}$.

Since $\boldsymbol{B}$ is a 4×4 matrix, it has at most 4 eigenvalues.
Therefore the eigenvalues of $\boldsymbol{B}$ are $2, 0, 1$ and -1.

(ii) From (4bi.), we can see that $\boldsymbol{u}_1, \boldsymbol{u}_2, (\boldsymbol{u}_3 + \boldsymbol{u}_4), (\boldsymbol{u}_3 - \boldsymbol{u}_4)$ are eigenvectors corresponding to eigenvalue $2, 0, 1, -1$ respectively.

(iii) Yes.
$\boldsymbol{B}$ is a 4×4 matrix and and has 4 distinct eigenvalues. Hence, $\boldsymbol{B}$ is diagonalisable.

SECTION B

Question 5

(a) Suppose $\det(\boldsymbol{A}) = 1$.
Then,

$$\begin{aligned}
\begin{vmatrix} 1 & 1 & 1 \\ a & 1 & 0 \\ b & b & 1 \end{vmatrix} &= 1 \\
1(1-0) - 1(a-0) + 1(ab-b) &= 1 \\
1 - a + ab - b &= 1 \\
ab - a - b &= 0 \\
b &= \frac{a}{a-1}, a \neq 1.
\end{aligned}$$

Hence, $\{(a,b)\} = \left\{ \left(t, \frac{t}{t-1}\right) \mid t \in \mathbb{R}\backslash\{1\} \right\}$.

(b) Observe that (1,1,1) is in the row space of $\boldsymbol{A}$ but not in the row space of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, $\boldsymbol{A}$ and $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ are not row equivalent. Hence, $\{(a,b)\} = \emptyset$.

(c) The matrix $\boldsymbol{A}$ can be row reduced to $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1-a & -a \\ 0 & 0 & 1-b \end{pmatrix}$. Let this matrix be $\boldsymbol{R}$.

Since $\operatorname{rank}(\boldsymbol{A}) \neq 3$, we have

$$\begin{aligned}
\det(\boldsymbol{A}) &= 0 \\
1 - a + ab - b &= 0 \\
(1-a)(1-b) &= 0.
\end{aligned}$$

Therefore, $a = 1$ or $b = 1$.
If $a = 1$, then,

$$\boldsymbol{R} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1-b \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, $\operatorname{rank}(\boldsymbol{A}) = 2$.

If $b = 1$, then,

$$\boldsymbol{R} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1-a & -a \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $(0, 1-a, -a) \neq \mathbf{0}$ for any $a \in \mathbb{R}$, we have $\text{rank}(\boldsymbol{A}) = 2$.

Hence $\{(a, b)\} = \{(x, y) \in \mathbb{R}^2 \mid x = 1 \text{ or } y = 1\}$.

(d) If $\det(\boldsymbol{A}) \neq 0$, then the column space of $\boldsymbol{A}$ spans $\mathbb{R}^3$, and thus will contains $(1, 2, 3)$. This correspond to the case $a, b \in \mathbb{R}$ such that $a \neq 1$, $b \neq 1$.

Consider the following augmented matrix,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ a & 1 & 0 & 2 \\ b & b & 1 & 3 \end{array}\right).$$

For $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ to belong to the column space of A, the system of equations represented by the above augmented matrix must be consistent.

Consider the case when $b = 1$. Then the augment matrix can be row reduced to

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1-a & -a & 2-a \\ 0 & 0 & 0 & 2 \end{array}\right).$$

The system is not consistent.
Therefore, $(1, 2, 3)$ does not belong to the column space of $\boldsymbol{A}$ when $b = 1$.

Consider the case when $a = 1$ and $b \neq 1$. Then the augment matrix becomes

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1-b & 3-b \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 4-2b \end{array}\right).$$

The system is consistent if $4 - 2b = 0$ i.e. $b = 2$.
Therefore, $(1, 2, 3)$ belongs to the column space of $\boldsymbol{A}$ when $a = 1$ and $b = 2$.

Hence, $\{(a, b)\} = \{(x, y) \in \mathbb{R}^2 \mid x \neq 1,\ y \neq 1\} \cup \{(1, 2)\}$.

(e) For the nullspace of $\boldsymbol{A}$ to be orthogonal to $(0, 0, 1)$, we must have $(0, 0, 1)$ in the row space of $\boldsymbol{A}$. Now let us consider situations where $(0, 0, 1)$ is not in the row space of $\boldsymbol{A}$.

Since row space of $\boldsymbol{A}$ does not span $\mathbb{R}^3$, we must have $\det(\boldsymbol{A}) = 0$, i.e. $a = 1$ or $b = 1$.
Since we have,

$$\begin{pmatrix} 1 & 1 & 1 \\ a & 1 & 0 \\ b & b & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1-a & -a \\ 0 & 0 & 1-b \end{pmatrix},$$

we must have $a \neq 1$ to prevent the second row from giving a scalar multiple of $(0, 0, 1)$. This leaves us with $b = 1$.

Finally, we check that when $a \neq 1$ and $b = 1$, we get $\{(1, 1, 1), (0, 1-a, -a), (0, 0, 1)\}$ to be a linearly independent set, and thus the row space of $\boldsymbol{A}$ does not contain $(0,0,1)$.

Therefore $(0,0,1)$ is not in row space of $\boldsymbol{A}$ iff $a \neq 1$ and $b = 1$.
Hence, for nullspace of $\boldsymbol{A}$ to be orthogonal to (0,0,1), we have $\{(a,b) \in \mathbb{R}^2 \mid a = 1 \text{ or } b \neq 1\}$.

(f) Suppose 1 is an eigenvalue of $\boldsymbol{A}$. Then,

$$\begin{aligned} \det(\boldsymbol{I} - \boldsymbol{A}) &= 0 \\ \begin{vmatrix} 0 & -1 & -1 \\ -a & 0 & 0 \\ -b & -b & 0 \end{vmatrix} &= 0 \\ ab &= 0. \end{aligned}$$

Hence, for 1 to be an eigenvalue of $\boldsymbol{A}$, $a = 0$ or $b = 0$.

Question 6

(a) Observe that each $\boldsymbol{A}\boldsymbol{u}_i$ belongs to $R(\boldsymbol{A})$ for $1 \leq i \leq n$. Since $\{\boldsymbol{A}\boldsymbol{u}_1, \boldsymbol{A}\boldsymbol{u}_2, \ldots, \boldsymbol{A}\boldsymbol{u}_n\}$ is linearly independent, we have $\text{rank}(\boldsymbol{A}) \geq n$.
Since $\boldsymbol{A}$ is a an $n \times n$ matrix, we also have $\text{rank}(\boldsymbol{A}) \leq n$.
Therefore, $\text{rank}(\boldsymbol{A}) = n$ and hence, $\boldsymbol{A}$ is invertible.

(b) Since $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the projections of $\boldsymbol{u}$ and $\boldsymbol{v}$ onto the vector space V, there exists $\boldsymbol{n}_1, \boldsymbol{n}_2$ with $\boldsymbol{n}_i \cdot \boldsymbol{w} = 0$ for all $\boldsymbol{w} \in V$, $i = 1, 2$, such that $\boldsymbol{u} = \boldsymbol{p}_1 + \boldsymbol{n}_1$ and $\boldsymbol{v} = \boldsymbol{p}_2 + \boldsymbol{n}_2$.
This give us

$$\begin{aligned} \boldsymbol{u} + \boldsymbol{v} &= \boldsymbol{p}_1 + \boldsymbol{n}_1 + \boldsymbol{p}_2 + \boldsymbol{n}_2 \\ &= (\boldsymbol{p}_1 + \boldsymbol{p}_2) + (\boldsymbol{n}_1 + \boldsymbol{n}_2). \end{aligned}$$

It is clear that $\boldsymbol{p}_1 + \boldsymbol{p}_2$ lies in V.
Also, since $(\boldsymbol{n}_1 + \boldsymbol{n}_2) \cdot \boldsymbol{w} = \boldsymbol{n}_1 \cdot \boldsymbol{w} + \boldsymbol{n}_2 \cdot \boldsymbol{w} = 0$, therefore $\boldsymbol{n}_1 + \boldsymbol{n}_2$ is orthogonal to V.
Hence, $\boldsymbol{p}_1 + \boldsymbol{p}_2$ is the projections of $\boldsymbol{u} + \boldsymbol{v}$ onto the vector space V.

(c) Suppose λ is an eigenvalue of $\boldsymbol{A}$. Then

$$\begin{aligned} \det(\lambda\boldsymbol{I} - \boldsymbol{A}) &= 0 \\ \det((\lambda\boldsymbol{I} - \boldsymbol{A})^T) &= 0 \\ \det((\lambda\boldsymbol{I})^T - \boldsymbol{A}^T) &= 0 \\ \det(\lambda\boldsymbol{I} - \boldsymbol{A}^T) &= 0. \end{aligned}$$

Therefore, λ is also an eigenvalue of $\boldsymbol{A}^T$.
Applying dimension theorem on $(\lambda\boldsymbol{I} - \boldsymbol{A})$, we have

$$\text{rank}(\lambda\boldsymbol{I} - \boldsymbol{A}) + \text{nullity}(\lambda\boldsymbol{I} - \boldsymbol{A}) = n \text{ , where } n \text{ is the order of matrix } \boldsymbol{A}.$$

Since the dimension of $E_\lambda(\boldsymbol{A})$ and $E_\lambda(\boldsymbol{A}^T)$ are the nullity of $(\lambda\boldsymbol{I} - \boldsymbol{A})$ and $(\lambda\boldsymbol{I} - \boldsymbol{A}^T)$ respectively, we have

$$\begin{aligned} \text{rank}(\lambda\boldsymbol{I} - \boldsymbol{A}) + \dim(E_\lambda(\boldsymbol{A})) &= n \\ \text{rank}(\lambda\boldsymbol{I} - \boldsymbol{A}^T) + \dim(E_\lambda(\boldsymbol{A}^T)) &= n. \end{aligned}$$

Observe that

$$\begin{aligned}\text{rank}(\lambda \boldsymbol{I} - \boldsymbol{A}) &= \text{rank}((\lambda \boldsymbol{I} - \boldsymbol{A})^T) \\ &= \text{rank}((\lambda \boldsymbol{I})^T - \boldsymbol{A}^T) \\ &= \text{rank}(\lambda \boldsymbol{I} - \boldsymbol{A}^T).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\dim(E_\lambda(\boldsymbol{A})) &= n - \text{rank}(\lambda \boldsymbol{I} - \boldsymbol{A}) \\ &= n - \text{rank}(\lambda \boldsymbol{I} - \boldsymbol{A}^T) \\ &= \dim(E_\lambda(\boldsymbol{A}^T)).\end{aligned}$$

(d) We are given that $||T(\boldsymbol{u})|| = 1$ for any unit vector $\boldsymbol{u}$.
This implies that for any $\boldsymbol{x} \in \mathbb{R}^n$ with $||\boldsymbol{x}|| = x$, we have

$$\begin{aligned}||T(\boldsymbol{x})|| &= ||T(x\frac{\boldsymbol{x}}{x})|| \\ &= ||xT(x\frac{\boldsymbol{x}}{x})|| \\ &= x||T(\frac{\boldsymbol{x}}{x})|| \qquad (\text{since } x \geq 0) \\ &= x = ||\boldsymbol{x}||.\end{aligned}$$

We also have,

$$||\boldsymbol{v} + \boldsymbol{w}||^2 = ||\boldsymbol{v}||^2 + 2\boldsymbol{v} \cdot \boldsymbol{w} + ||\boldsymbol{w}||^2.$$

Next,

$$\begin{aligned}||T(\boldsymbol{v} + \boldsymbol{w})||^2 &= ||T(\boldsymbol{v}) + T(\boldsymbol{w})||^2 \\ &= ||T(\boldsymbol{v})||^2 + 2T(\boldsymbol{v}) \cdot T(\boldsymbol{w}) + ||T(\boldsymbol{w})||^2.\end{aligned}$$

Since $||T(\boldsymbol{v})|| = ||\boldsymbol{v}||$, $||T(\boldsymbol{w})|| = ||\boldsymbol{w}||$ and $||T(\boldsymbol{v} + \boldsymbol{w})|| = ||\boldsymbol{v} + \boldsymbol{w}||$, we conclude that

$$\begin{aligned}\boldsymbol{v} \cdot \boldsymbol{w} &= \frac{1}{2}(||\boldsymbol{v} + \boldsymbol{w}||^2 - ||\boldsymbol{v}||^2 - ||\boldsymbol{w}||^2) \\ &= \frac{1}{2}(||T(\boldsymbol{v} + \boldsymbol{w})||^2 - ||T(\boldsymbol{v})||^2 - ||T(\boldsymbol{w})||^2) \\ &= T(\boldsymbol{v}) \cdot T(\boldsymbol{w}).\end{aligned}$$