NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1102R Calculus
AY 2007/2008 Sem 2

Question 1

(a) We have $\lim_{x\to\infty} \frac{\sqrt{x}-\sqrt[3]{x}}{\sqrt{x}+\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1-x^{\left(\frac{1}{3}-\frac{1}{2}\right)}}{1+x^{\left(\frac{1}{3}-\frac{1}{2}\right)}} = \lim_{x\to\infty} \frac{1-x^{\frac{-1}{6}}}{1+x^{\frac{-1}{6}}} = \frac{1-0}{1+0} = 1.$

(b) Since $\lim_{x\to 0} \cos x - 1 = 0$ and $\lim\limits_{x\to 0} e^{x^2} - 1 = 0$, we apply L'Hôpital's rule to get,

$$\lim_{x\to 0} \frac{\cos x - 1}{e^{x^2}-1} = \lim_{x\to 0} \frac{-\sin x}{2xe^{x^2}} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{-1}{2e^{x^2}} = 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}.$$

(c) Applying L'Hôpital's rule, we have,

$$\begin{aligned}
\lim_{x\to 0} \ln(\cos x)^{\frac{1}{\ln(1+x^2)}} = \lim_{x\to 0} \frac{\ln\cos x}{\ln(1+x^2)} &= \lim_{x\to 0} \frac{\frac{-\sin x}{\cos x}}{\frac{2x}{1+x^2}} \\
&= \frac{1}{2} \cdot \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1+x^2}{-\cos x} \\
&= \frac{1}{2} \cdot 1 \cdot \frac{1+0^2}{-\cos 0} = -\frac{1}{2}.
\end{aligned}$$

Since $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = e^x$ is continuous on $\mathbb{R}$, we have,

$$\lim_{x\to 0} (\cos x)^{\frac{1}{\ln(1+x^2)}} = \lim_{x\to 0} f\left(\ln(\cos x)^{\frac{1}{\ln(1+x^2)}}\right) = f\left(\lim_{x\to 0} \ln(\cos x)^{\frac{1}{\ln(1+x^2)}}\right) = f\left(\frac{-1}{2}\right) = e^{\frac{-1}{2}}.$$

Question 2

(a) We have,

$$\begin{aligned}
\int x^2 \ln x \; dx &= \frac{x^3}{3} \ln x - \int \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right) \; dx \\
&= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} \; dx \\
&= \frac{x^3}{3} \ln x - \frac{x^3}{9} + c.
\end{aligned}$$

(b) We have,

$$\begin{aligned}
\int \frac{4x-2}{(x-1)(x^2+1)} \; dx &= \int \frac{1}{x-1} - \frac{1}{2}\left(\frac{2x}{x^2+1}\right) + 3\left(\frac{1}{x^2+1}\right) \; dx \\
&= \ln(x-1) - \frac{1}{2}\ln\left(x^2+1\right) + 3\tan^{-1} x + c.
\end{aligned}$$

Question 3

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = \dfrac{\ln x}{x}$, and $b_n = f(n)$, $n \in \mathbb{Z}^+$. We have $b_n > 0$ for all $n > 1$. Since $\dfrac{d}{dx}f(x) = \dfrac{1-\ln x}{x^2} < 0$ for all $x > e$, we have $b_n > b_{n+1}$ for all $n > 2$. Also, $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0$.
Therefore by Alternating Series test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ is convergent.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = \dfrac{1}{x(\ln x)^2}$.
Since $\dfrac{d}{dx}f(x) = \dfrac{-\ln x - 2}{x^2(\ln x)^3} < 0$ for all $x \geq 2$, we have f to be a continuous, positive decreasing function.
We have $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \int_2^{\infty} \frac{1}{(\ln x)^2}\, d(\ln x) = \left[\frac{-1}{\ln x}\right]_2^{\infty} = \lim_{x\to\infty} \frac{-1}{\ln x} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$.
Therefore, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent by Integral Test.

(c) Let $a_n = \dfrac{1}{n} - \ln\left(1 + \dfrac{1}{n}\right)$ for all $n \in \mathbb{Z}^+$, and $S_N = \displaystyle\sum_{n=1}^{N} a_n$ for all $N \in \mathbb{Z}^+$.
Then $a_n > \displaystyle\int_n^{n+1} \frac{1}{t}\, dt - (\ln(n+1) - \ln n) = (\ln(n+1) - \ln n) - (\ln(n+1) - \ln n) = 0$.
This give us $(S_N)_{N\in\mathbb{Z}^+}$ to be an increasing sequence. Also,

$$\begin{aligned}
S_N = \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{N} \ln\left(1 + \frac{1}{n}\right) &< 1 + \sum_{n=2}^{N}\left(\int_{n-1}^{n} \frac{1}{t}\, dt\right) - \sum_{n=1}^{N} (\ln(n+1) - \ln n) \\
&= 1 + \int_1^N \frac{1}{t}\, dt - (\ln(N+1) - \ln 1) \\
&= 1 + \ln N - \ln(N+1) \\
&< 1 + \ln(N+1) - \ln(N+1) = 1,
\end{aligned}$$

and so $(S_N)_{N\in\mathbb{Z}^+}$ is a bounded sequence.
Therefore by Monotone Convergence Theorem, $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$ converges.

Question 4

WLOG, let the coordinate of C be $(3\sin t, 2\cos t)$, $0 \leq t \leq \pi$. Let the area of ABC be S.
Then we have, $S = (2 - 2\cos t)(3\sin t) = 6\sin t - 6\sin t\cos t = 6\sin t - 3\sin 2t$.
This give us $\frac{dS}{dt} = 6(\cos t - \cos 2t) = -6\left(2\cos^2 t - \cos t - 1\right) = -6(2\cos t + 1)(\cos t - 1)$.
Thus when $\cos t = 1$ or $\cos t = -\dfrac{1}{2}$, we have $\dfrac{dS}{dt} = 0$.
Since $0 \leq t \leq \pi$, $\cos t = 1$ give us $t = 0$, and $\cos t = -\dfrac{1}{2}$ give us $t = \dfrac{2\pi}{3}$.
Therefore $t = 0$, $t = \pi$ and $t = \dfrac{2\pi}{3}$ are the only critical point.
When $t = 0$ and $t = \pi$, $S = 0$. When $t = \dfrac{2\pi}{3}$, $S = 6\left(\dfrac{\sqrt{3}}{2}\right) - 3\left(-\dfrac{\sqrt{3}}{2}\right) = \dfrac{9\sqrt{3}}{2}$.
Therefore, we obtain the largest area of such triangle to be $\dfrac{9\sqrt{3}}{2}$.

Question 5

(a) Suppose the tangent point is $(a, \ln a)$. Then the tangent line is $y - \ln a = \dfrac{1}{a}(x-a)$, which passes through $(0,0)$. Hence $-\ln a = \frac{1}{a}(-a) = -1$, which give us $a = e$.
Substitute back into the equation, we obtain $y - 1 = y - \ln e = \dfrac{1}{e}(x-e) = \dfrac{x}{e} - 1$, i.e. $y = \dfrac{x}{e}$.

(b) Using Cylindrical Shells method, we get,

$$\begin{aligned}
\text{Volume} &= \int_0^1 2\pi x\left(\frac{x}{e}\right)\,dx + \int_1^e 2\pi x\left(\frac{x}{e} - \ln x\right)\,dx \\
&= \frac{2\pi}{e}\int_0^e x^2\,dx - 2\pi\int_1^e x\ln x\,dx \\
&= \frac{2\pi}{e}\left[\frac{x^3}{3}\right]_0^e - 2\pi\left[\frac{x^2\ln x}{2} - \frac{x^2}{4}\right]_1^e \\
&= \frac{2\pi e^2}{3} - 2\pi\left[\left(\frac{e^2}{2} - \frac{e^2}{4}\right) - \left(0 - \frac{1}{4}\right)\right] \\
&= \frac{2\pi e^2}{3} - \frac{\pi e^2}{2} - \frac{\pi}{2} = \frac{\pi e^2}{6} - \frac{\pi}{2}.
\end{aligned}$$

Question 6

(a) We have the Maclaurin series $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$.

This give us $\cos(x^4) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{8n}}{(2n)!}$, and so $f(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{8n+8}}{(2n)!}$.

Hence,

$$\begin{aligned}
f^{2008}(0) &= 2008!\cdot\big(\text{coefficient of } x^{2008} \text{ in Maclaurin series of } f(x)\big) \\
&= 2008!\cdot(\text{coefficient of the } n = 250 \text{ term in Maclaurin series of } f(x)) \\
&= 2008!\cdot\left(\frac{(-1)^{250}}{(2\cdot 250)!}\right) = \frac{2008!}{500!}.
\end{aligned}$$

(b) Since $f(t)$ and $f(t)t$ are continuous function in t over $\mathbb{R}$, by the Fundamental Theorem of Calculus, we have,

$$\begin{aligned}
F'(x) = \frac{d}{dx}\left(\int_a^x f(t)(x-t)\,dt\right) &= \frac{d}{dx}\left(x\int_a^x f(t)\,dt\right) - \frac{d}{dx}\int_a^x f(t)t\,dt \\
&= \frac{dx}{dx}\cdot\int_a^x f(t)\,dt + x\cdot\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) - \frac{d}{dx}\left(\int_a^x f(t)t\,dt\right) \\
&= \int_a^x f(t)\,dt + xf(x) - f(x)x = \int_a^x f(t)\,dt.
\end{aligned}$$

Therefore, also by the Fundamental Theorem of Calculus, $F''(x) = \dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$.

Question 7

(a) We have,

$$\begin{aligned} f(x) = \int_x^{x+1} \frac{1}{2t} \cdot 2t\cos t^2 \; dt &= \left[\frac{1}{2t} \cdot \sin t^2 \; dt\right]_x^{x+1} - \int_x^{x+1} \frac{-1}{2t^2} \cdot \sin t^2 \; dt \\ &= \frac{\sin(x+1)^2}{2(x+1)} - \frac{\sin x^2}{2x} + \int_x^{x+1} \frac{\sin t^2}{2t^2} \; dt. \end{aligned}$$

(b) Since $-1 \le \sin y \le 1$ for all $y \in \mathbb{R}$, we have $\dfrac{-1}{2t^2} \le \dfrac{\sin t^2}{2t^2} \le \dfrac{1}{2t^2}$ for all $t \in [x, x+1]$.
This give us $\left[\dfrac{1}{2t}\right]_x^{x+1} = \displaystyle\int_x^{x+1} \frac{-1}{2t^2} \; dt \le \int_x^{x+1} \frac{\sin t^2}{2t^2} \; dt \le \int_x^{x+1} \frac{1}{2t^2} \; dt = \left[\frac{-1}{2t}\right]_x^{x+1}$.
Also, we have $\dfrac{-1}{2(x+1)} \le \dfrac{\sin(x+1)^2}{2(x+1)} \le \dfrac{1}{2(x+1)}$ and $\dfrac{-1}{2x} \le \dfrac{\sin x^2}{2x} \le \dfrac{1}{2x}$.
Thus we have $\dfrac{-1}{x} = \dfrac{-1}{2(x+1)} - \dfrac{1}{2x} + \left[\dfrac{1}{2t}\right]_x^{x+1} \le f(x) \le \dfrac{1}{2(x+1)} - \dfrac{-1}{2x} + \left[\dfrac{-1}{2t}\right]_x^{x+1} = \dfrac{1}{x}$.
Since $\displaystyle\lim_{x\to\infty} \frac{-1}{x} = 0 = \lim_{x\to\infty} \frac{1}{x}$, by Squeeze Theorem, we have $\displaystyle\lim_{x\to\infty} f(x) = 0$.

Question 8

We have $g'(x) = f'(x) - f'(a) - 2M(x-a)$ and $g''(x) = f''(x) - 2M$.
We observe that $g(a) = 0$ and $g'(a) = 0$.
Since $a < b$, we can let $M = \dfrac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2}$ so that we also have $g(b) = 0$.
Thus by Rolle's Theorem, there exists $c' \in (a, b)$ such that $g'(c') = 0$.
Again, since $g'(a) = 0$, there exists $c \in (a, c')$ such that $g''(c) = 0$, i.e. $f''(c) - 2M = 0$.
Since $c \in (a, c')$, we have $c \in (a, b)$, and so we shown that there exists $c \in (a, b)$ such that

$$\begin{aligned} f''(c) = 2M &= 2\left(\frac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2}\right) \\ \frac{(b-a)^2 f''(c)}{2} &= f(b) - f(a) - (b-a)f'(a) \\ f(b) &= f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2} f''(c). \end{aligned}$$