MA1100 - Basic Discrete Mathematics Suggested Solutions (Semester 2: AY2021/22)

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1. **Solution:**

   - $\bigcup_{n=1}^{\infty} A_n = \mathbb{Z}^+$ because for every $m \in \mathbb{Z}^+$, $1 \le m \le 5k$ for some $k \in \mathbb{Z}^+$, so $m \in A_k \subseteq \bigcup_{n=1}^{\infty} A_n$ and thus $\mathbb{Z}^+ \subseteq \bigcup_{n=1}^{\infty} A_n$. We also have $A_n \subseteq \mathbb{Z}^+$ so $\bigcup_{n=1}^{\infty} A_n \subseteq \mathbb{Z}^+$.
   - $\bigcap_{n=1}^{\infty} A_n = A_1$, because every $k \in A_1$ satisfies $1 \le k \le 5n$ for all $n \in \mathbb{Z}^+$ (hence is a member of all A_n and thus $\bigcap_{n=1}^{\infty} A_n$). Any $k \notin A_1$ will not be in $\bigcap_{n=1}^{\infty} A_n$ by definition of intersection.

2. **Solution:**

   (i) No; $f(0) = 5 = f(2)$ for instance.

   (ii) For all $x \in \mathbb{Q}$, $(x-1)^2 \ge 0$ which implies $4(x-1)^2 \ge 0$ and $4(x-1)^2 + 1 \ge 1$. Hence $f(x) \ge 1$, thus $\mathcal{R}(f) \subseteq [1, \infty)$.

   (iii) No; for any x in the domain, $x = \frac{a}{b}$ for integers a and b, where $b \ne 0$. Therefore

   $$\begin{aligned} f(x) &= f\left(\frac{a}{b}\right) \\ &= 4\left(\frac{a}{b} - 1\right)^2 + 1 \\ &= 4\left(\frac{a-b}{b}\right)^2 + 1 \\ &= \frac{4(a-b)^2}{b^2} + 1 \\ &= \frac{4(a-b)^2 + b^2}{b^2} \in \mathbb{Q}, \end{aligned}$$

   so $\sqrt{2}$ would be in $[1, \infty)$ but not $\mathcal{R}(f)$, for instance.

3. **Solution:**

   (i) If $y = (f \circ g)(x)$ for any $x \in \mathbb{R}$, then

   $$y = 6x + 5 \iff x = \frac{y-5}{6},$$

   therefore $(f \circ g)^{-1}(x) = \frac{x-5}{6}$.

(ii) Since f, $f \circ g$ and h are bijective, we have

$$\begin{aligned} f(x) &= (f \circ g)^{-1} \circ (f \circ g) \circ f(x) \\ &= (f \circ g)^{-1} \circ f \circ (g \circ f)(x) \\ &= (f \circ g)^{-1} \circ h(x) \\ &= (f \circ g)^{-1}(18x + 17) \\ &= \frac{(18x + 17) - 5}{6} \\ &= \frac{18x + 12}{6} \\ &= 3x + 2. \end{aligned}$$

4. **Solution:**

Proof. ($\subseteq$): Suppose $x \in f^{-1}\left[\bigcap_{i \in I} Z_i\right]$. Then $f(x) \in \bigcap_{i \in I} Z_i$ and is thus in Z_i for all $i \in I$. Therefore $x \in f^{-1}[Z_i]$ for all $i \in I$, so $x \in \bigcap_{i \in I} f^{-1}[Z_i]$.

($\supseteq$): Now suppose $x \in \bigcap_{i \in I} f^{-1}[Z_i]$. Then $x \in f^{-1}[Z_i]$ for all $i \in I$, so $f(x) \in Z_i$ for all $i \in I$. Therefore $f(x) \in \bigcap_{i \in I} Z_i$ and so $x \in f^{-1}\left[\bigcap_{i \in I} Z_i\right]$. □

5. **Solution:**

(i) *Proof.* Noting that

$$\begin{gathered} 10 \equiv -1 \mod 11, \\ 10^2 \equiv 1 \mod 11, \\ 10^{k+2n} \equiv 10^k 10^{2n} \equiv 10^k (10^2)^n \equiv 10^k \mod 11, \end{gathered}$$

we have

$$\begin{gathered} 10^k \equiv -1 \mod 11 \text{ if } k \text{ is odd}, \\ 10^k \equiv 1 \mod 11 \text{ if } k \text{ is even}. \end{gathered}$$

Therefore $10^k \equiv (-1)^k \mod 11$, so $\sum_{k=0}^{n} a_k \cdot 10^k \equiv \sum_{k=0}^{n} a_k \cdot (-1)^k \mod 11$. Since an integer N is divisible by 11 if and only if $N \equiv 0 \mod 11$, by the established congruence we have the desired result. □

(ii) Setting $S = \sum_{k=1}^{9} (10 - k) \cdot 10^{k-1}$ for brevity, we have

$$\begin{aligned} 123456789123456789123456789123456789 &\equiv \sum_{j=0}^{3} 10^{9j} \cdot \left(\sum_{k=1}^{9} (10 - k) \cdot 10^{k-1} \right) && \pmod{11} \\ &\equiv 10^0 \cdot S + 10^9 \cdot S + 10^{18} \cdot S + 10^{27} \cdot S && \pmod{11} \\ &\equiv S - S + S - S && \pmod{11} \\ &\equiv 0 && \pmod{11}, \end{aligned}$$

so it is divisible by 11.

6. **Solution:**

(i) *Proof.* We verify that $\sim$ is reflexive, symmetric and transitive:

- (reflexivity): for all (x, y), $(x, y) \sim (x, y)$ because $y - x = y - x$.

- (symmetry): if $(x, y) \sim (x', y')$, then $y - x = y' - x' \iff y' - x' = y = x$, thus $(x', y') \sim (y, x)$.
- (transitivity): if $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$, then $y_1 - x_1 = y_2 - x_2 = y_3 - x_3$, so $(x_1, y_1) \sim (x_3, y_3)$.

$\square$

(ii) For any $(x, y) \in [(a, b)]$, we have $(x, y) \sim (a, b)$. Therefore $y - x = b - a$, so $y = b - a + x$. Thus the points $(x, y) \in [(a, b)]$ form a straight line in $\mathbb{R}^2$ described by the equation.

(iii) *Proof.* The function $f : \mathbb{R} \to X/\sim$ defined by $f(x) = [(0, x)]$ is a bijection; this can easily be verified:

- (injectivity): If $f(x) = f(x')$, then $[(0, x)] = [(0, x')]$. So $(0, x) \sim (0, x')$ implying $x - 0 = x' - 0$ and thus $x = x'$.
- (surjectivity): Any $[(a, b)] \in X/\sim$ is equal to $[(0, b - a)] = f(b - a)$.

$\square$

7. **Solution:**

Proof. Suppose not; then $A \cup B = B \cup (A - B)$ is the union of countable sets and hence countable. By the inclusion injection $\iota : A \hookrightarrow A \cup B$, $A \preceq A \cup B$ and is hence countable, a contradiction. $\square$

8. **Solution:**

(i) *Proof.* Since $p \mid p!$ and $p! = k!(p-k)!\binom{p}{k}$, by the primality of p at least one of $p \mid k!$, $p \mid (p-k)!$ and $p \mid \binom{p}{k}$ holds. Since $k < p$, $p \nmid n$ for any $n \in \{1, \ldots, k\}$, so $p \nmid k!$ again by primality. Since $0 < k$, $p - k < p$ and a similar argument shows that $p \nmid (p-k)!$. Therefore $p \mid \binom{p}{k}$. $\square$

(ii) *Proof.* Fix a prime number p; we shall perform induction on $n \in \mathbb{Z}^+$.

- Base case: $1^p = 1$, so $1^p \equiv 1 \mod p$.
- Inductive step: Suppose $n^p \equiv n \pmod p$ for some $n \in \mathbb{Z}^+$. Then

$$\begin{aligned}
(n+1)^p &\equiv n^p + \binom{p}{1} n^{p-1} + \cdots + \binom{p}{p-1} n + 1 \pmod p \\
&\equiv n^p + 0 + \cdots + 0 + 1 \pmod p \\
&\equiv n^p + 1 \pmod p \\
&\equiv n + 1 \pmod p,
\end{aligned}$$

where the second equivalence follows by the divisibility of $\binom{p}{k}$ by p and the fourth equivalence follows from the inductive hypothesis. Thus $n^p \sim n \pmod p$ for all $n \in \mathbb{Z}^+$ by induction.

$\square$