NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lau Tze Siong

MA2108 Mathematical Analysis I
AY 2007/2008 Sem 1

Question 1

(a) (i)

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{n+3-5n^2}{n^2-3n+6}\right) &= \lim_{n\to\infty}\left(\frac{\frac{1}{n}+\frac{3}{n^2}-5}{1-\frac{3}{n}+\frac{6}{n^2}}\right)\\ &= -5\end{aligned}$$

(ii)

$$\begin{aligned}\text{Since } \lim_{n\to\infty} 1+\frac{1}{n+4} &= 1\\ \lim_{n\to\infty}\left(1+\frac{1}{n+4}\right)^{n-4} &= \lim_{n\to\infty}\left(1+\frac{1}{n+4}\right)^{n+4}\cdot\left(1+\frac{1}{n+4}\right)^{-8}\\ &= \lim_{n\to\infty}\left(1+\frac{1}{n+4}\right)^{n+4}\cdot 1^{-8}\\ &= e\end{aligned}$$

(iii) Since $1 \le (n!) \le n^n$, we have $(1)^{\frac{1}{n^2}} \le (n!)^{\frac{1}{n^2}} \le n^{\frac{1}{n}}$.
By Squeeze Theorem we have $\lim\limits_{n\to\infty}(1)^{\frac{1}{n^2}} \le \lim\limits_{n\to\infty}(n!)^{\frac{1}{n^2}} \le \lim\limits_{n\to\infty} n^{\frac{1}{n}}$.
Therefore, $1 \le \lim\limits_{n\to\infty}(n!)^{\frac{1}{n^2}} \le 1$.
Hence $\lim\limits_{n\to\infty}(n!)^{\frac{1}{n^2}} = 1$

(b) Let $a = \inf(S+T)$, $b = \inf(S)$ and $c = \inf(T)$.
Hence we have $b \le s$ for all $s \in S$ and $c \le t$ for all $t \in T$.
Therefore we have $b + c \le s + t$ for all $s \in S$ and $t \in T$.
Since $a = \inf(S+T)$, we have $a \ge b + c$.
Also since $a \le s + t$ for all $s \in S$ and $t \in T$, we have $a - s \le t$ for all $t \in T$.
Since $c = \inf(T)$, we have $c \ge a - s$ for all $s \in S$, which leads to $a - c \le s$ for all $s \in S$.
Since $b = \inf(S)$, we have $b \ge a - c$. Hence we have $a \le c + b$.
Together we have $a = c + b$

Question 2

(a) (i) By Limit Comparison Test, since

$$\begin{aligned}\lim_{n\to\infty}\frac{\frac{n^2+2n}{n^3+n+1}}{\frac{1}{n}} &= \lim_{n\to\infty}\frac{n^3+2n^2}{n^3+n+1}\\ &= \lim_{n\to\infty}\frac{1+\frac{2}{n}}{1+\frac{1}{n^2}+\frac{1}{n^3}}\\ &= 1\end{aligned}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+2n}{n^3+n+1}$ diverges.

(ii) By Root Test, since

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{2^n\left(\frac{n}{n+1}\right)^n} &= \lim_{n\to\infty}2\left(\frac{n}{n+1}\right)^n\\ &= \lim_{n\to\infty}2\left(1-\frac{1}{n+1}\right)^n\\ &= \lim_{n\to\infty}2\left(1-\frac{1}{n+1}\right)^{n+1}\cdot\left(1-\frac{1}{n+1}\right)^{-1}\\ &= 2e^{-1}\\ &< 1\end{aligned}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty}2^n\left(\frac{n}{n+1}\right)^{n^2}$ converges.

(iii) By Ratio Test, since

$$\begin{aligned}\lim_{n\to\infty}\frac{((n+1)!)^2}{(2n+2)!}\frac{(2n)!}{(n!)^2} &= \lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+1)(2n+2)}\\ &= \frac{1}{4}\\ &< 1\end{aligned}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{2n!}$ converges.

(iv) By AM-GM inequality we have

$$\begin{aligned}
\frac{(\sum_{i=1}^{n-1}\frac{n}{n-1})+1}{n} &\geq \left(\left(\frac{n}{n-1}\right)^{n-1}\cdot 1\right)^{\frac{1}{n}} \\
\frac{n+1}{n} &\geq \left(\frac{n}{n-1}\right)^{\frac{n-1}{n}} \\
\left(\frac{n+1}{n}\right)^{n} &\geq \left(\frac{n}{n-1}\right)^{n-1} \\
\left(\frac{n}{n+1}\right)^{n} &\leq \left(\frac{n-1}{n}\right)^{n-1} \\
\frac{n^n}{(n+1)^{n+1}} \leq \frac{n^n}{n(n+1)^n} &\leq \frac{(n-1)^{n-1}}{n^n}
\end{aligned}$$

Hence $\frac{n^n}{(n+1)^{n+1}}$ is decreasing and $\lim_{n\to\infty}\frac{n^n}{(n+1)^{n+1}}=0$.
Hence by Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^n}{(n+1)^{n+1}}$ converges.

(b) Let S_k denote the kth-partial sum of $1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+...$
Note that

$$\begin{aligned}
S_{3k} &= \sum_{n=1}^{k}\left(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}\right) \\
&= \sum_{n=1}^{k}\frac{1}{4n-2}-\frac{1}{4n} \\
&= \frac{1}{2}\left(\sum_{n=1}^{k}\frac{1}{2n-1}-\frac{1}{2n}\right) \\
&= \frac{1}{2}t_{2k}
\end{aligned}$$

where t_k is the kth-partial sum of $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$.
Since $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is convergent, the sequence (t_k) is Cauchy.
Let $\epsilon>0$. Then there exists $N\in\mathbb{N}$ such that for all $m,n\geq N$,

$$|t_m-t_n|<\epsilon$$

. In particular, for all $m,n\geq N$,

$$\begin{aligned}
|S_{3m}-S_{3n}| &= \left|\frac{1}{2}t_{2m}-\frac{1}{2}t_{2n}\right| \\
&= \frac{1}{2}\left|t_{2m}-t_{2n}\right|<\frac{\epsilon}{2}
\end{aligned}$$

Consider the following sum

$$
\begin{aligned}
S_{3m} - S_{3n+1} &= S_{3m} - \left(S_{3n} + \frac{1}{2n+1}\right) \\
S_{3m} - S_{3n+2} &= S_{3m} - \left(S_{3n} + \frac{1}{2n+1} - \frac{1}{4n+2}\right) \\
&= S_{3m} - \left(S_{3n} + \frac{1}{4n+2}\right) \\
S_{3m+1} - S_{3n} &= \left(S_{3m} + \frac{1}{2m+1}\right) - S_{3n} \\
S_{3m+1} - S_{3n+1} &= \left(S_{3m} + \frac{1}{2m+1}\right) - \left(S_{3n} + \frac{1}{2n+1}\right) \\
S_{3m+1} - S_{3n+2} &= \left(S_{3m} + \frac{1}{2m+1}\right) - \left(S_{3n} + \frac{1}{4n+2}\right) \\
S_{3m+2} - S_{3n} &= \left(S_{3m} + \frac{1}{4m+2}\right) - S_{3n} \\
S_{3m+2} - S_{3n+1} &= \left(S_{3m} + \frac{1}{4m+2}\right) - \left(S_{3n} + \frac{1}{2n+1}\right) \\
S_{3m+2} - S_{3n+2} &= \left(S_{3m} + \frac{1}{4m+2}\right) - \left(S_{3n} + \frac{1}{4n+2}\right)
\end{aligned}
$$

Let $i \in \{3m, 3m+1, 3m+2\}$, $j \in \{3n, 3n+1, 3n+2\}$. It is easy to verify that

$$|S_i - S_j| < |S_{3m} - S_{3n}| + \frac{1}{n}$$

By Archimedean Property of Real Numbers, there exist $N' \in \mathbb{N}$ such that $\frac{1}{N'} < \frac{\epsilon}{2}$.
Set $M = \max\{N', 3N\}$
Then for all $m, n \geq M$, we have

$$
\begin{aligned}
|S_m - S_n| &< |S_m - S_n| + \frac{1}{N'} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}
$$

Therefore the sequence of partial sums of the series is Cauchy, so the series is convergent.

Question 3

(a) (i) Since $-1 \leq \sin\left(\frac{1}{(x-1)^2}\right) \leq 1$ for all $x \in \mathbb{R}$.
Hence we have $-\left|\frac{x-1}{x+1}\right| \leq \frac{x-1}{x+1}\sin\left(\frac{1}{(x-1)^2}\right) \leq \left|\frac{x-1}{x+1}\right|$ for all $\mathbb{R}$.
Therefore by Squeeze Theorem we have $0 \leq \lim\limits_{x\to 1} \dfrac{x-1}{x+1}\sin\left(\dfrac{1}{(x-1)^2}\right) \leq 0$.
So, $\lim\limits_{x\to 1} \dfrac{x-1}{x+1}\sin\left(\dfrac{1}{(x-1)^2}\right) = 0$.

(ii)

$$
\begin{aligned}
\lim_{x\to 8^-} \frac{x-8}{|x-8|} &= \lim_{x\to 8^-} \frac{x-8}{-(x-8)} \\
&= \lim_{x\to 8^-} -1 \\
&= -1
\end{aligned}
$$

(iii) Since

$$\begin{aligned}\lim_{x\to 9^-}\left([5x]-\left[\frac{4x}{9}\right]\right) &= 44-3\\ &= 41\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 9^+}\left([5x]-\left[\frac{4x}{9}\right]\right) &= 45-4\\ &= 41\end{aligned}$$

we have $\lim_{x\to 9}\left([5x]-\left[\frac{4x}{9}\right]\right)=41$

(b) Claim:$g:\mathbb{R}\to\mathbb{R}$ is continuous for all $x\in(n,n+1)$ for all $n\in\mathbb{Z}$.
Proof:
Since $[x]:(n,n+1)\to\mathbb{R}$ is a constant function with value for $[x]=n$ for all $x\in(n,n+1)$, it is continuous in the interval $(n,n+1)$ for all $n\in\mathbb{Z}$.
Similarly, $\left[\frac{x}{2}\right]:\mathbb{R}\to\mathbb{R}$ is continuous on the interval $(n,n+1)$ for all $n\in\mathbb{Z}$. Since $x:\mathbb{R}\to\mathbb{R}$ and $\left[\frac{x}{2}\right]:\mathbb{R}\to\mathbb{R}$ and $[x]:(n,n+1)\to\mathbb{R}$ are continuous functions on the interval $(n,n+1)$ for all $n\in\mathbb{Z}$ and the product and sum of continuous functions is a continuous function. We have $g:\mathbb{R}\to\mathbb{R}$ being continuous for all $x\in(n,n+1)$ for all $n\in\mathbb{Z}$.

Claim: For all $n\in\mathbb{Z}$, g is not continuous at $x=2n+1$.
Proof:
Since

$$\begin{aligned}\lim_{x\to 2n+1^-} g(x) &= \lim_{x\to 2n+1^-} 2[x]-x\left[\frac{x}{2}\right]\\ &= \lim_{x\to 2n+1^-} 2(2n)-(2n+1)(n)\\ &= \lim_{x\to 2n+1^-} 4n-2n^2-n\\ &= 3n-2n^2\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 2n+1^+} g(x) &= \lim_{x\to 2n+1^+} 2[x]-x\left[\frac{x}{2}\right]\\ &= \lim_{x\to 2n+1^+} 2(2n+1)-(2n+1)(n)\\ &= \lim_{x\to 2n+1^+} 4n+2-2n^2-n\\ &= 3n-2n^2+2\end{aligned}$$

Since $3n-2n^2\neq 3n-2n^2+2$ for all $n\in\mathbb{Z}$, $\lim_{x\to 2n+1^-} g(x)\neq\lim_{x\to 2n+1^+} g(x)$ for all $n\in\mathbb{Z}$, $\lim_{x\to 2n+1} g(x)$ does not exist for all $n\in\mathbb{Z}$.
Hence g is not continuous at $x=2n+1$ for all $n\in\mathbb{Z}$.

Claim: For all $n \in \mathbb{Z}\backslash\{1\}$, g is not continuous at $x = 2n$ and g is continuous at $x = 2$.
Proof:

$$\begin{aligned}
\lim_{x\to 2n^-} g(x) &= \lim_{x\to 2n+1^-} 2[x] - x\left[\frac{x}{2}\right] \\
&= \lim_{x\to 2n^-} 2(2n-1) - (2n)(n-1) \\
&= \lim_{x\to 2n^-} 4n - 2 - 2n^2 + 2n \\
&= 6n - 2 - 2n^2
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 2n^+} g(x) &= \lim_{x\to 2n+1^+} 2[x] - x\left[\frac{x}{2}\right] \\
&= \lim_{x\to 2n^+} 2(2n) - (2n)(n) \\
&= \lim_{x\to 2n^+} 4n - 2n^2 \\
&= 4n - 2n^2
\end{aligned}$$

Since $6n - 2 - 2n^2 \neq 4n - 2n^2$ for all $n \in \mathbb{Z}\backslash\{1\}$, $\lim_{x\to 2n^-} g(x) \neq \lim_{x\to 2n^+} g(x)$ for all $n \in \mathbb{Z}\backslash\{1\}$.
Hence g is not continuous at $x = 2n$ for all $n \in \mathbb{Z}\backslash\{1\}$.
Since at $x = 2$, $n = 1$, $\lim_{x\to 2n^-} g(x) = \lim_{x\to 2n^+} g(x) = 2$, we have $\lim_{x\to 2} g(x) = 2$.
Also since $g(2) = 2(2) - 2(1) = 2$. g is continuous at $x = 2$.
Therefore, $g : \mathbb{R} \to \mathbb{R}$ is continuous for all $x \in \left(\bigcup_{n\in\mathbb{Z}} I_n\right) \cup \{2\}$, where $I_n = (n, n+1)$.

Question 4

(a) By AM-GM inequality we have $x_{n+1} \geq y_{n+1}$ for all $n \in \mathbb{N}$.
Hence for $n \in \mathbb{N}$ such that $n \geq 2$, we have $x_{n+1} = \frac{x_n + y_n}{2} \leq x_n$ and $y_{n+1} = \sqrt{x_n y_n} \geq y_n$.
Therefore we have

$$y_2 \leq y_3 \leq y_4 \leq \ldots \leq y_n \leq x_n \leq \ldots \leq x_4 \leq x_3 \leq x_2$$

Hence $\{y_n\}$ is bounded above by $\max(y_1, x_2)$ and $\{x_n\}$ is bounded below by $\min(x_1, y_2)$.
By the Completeness of the $\mathbb{R}$, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ exist.
Since $x_{n+1} = \frac{x_n + y_n}{2}$, we have $x = \frac{x+y}{2}$.
Therefore we have $x = y$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

(b) Since $f(1) = -1 < 0$ and $f(2) = 2\ln(2) + \sqrt{2} - 2 > 0$ and f is continuous on the interval$[1, 2]$.
By Intermediate Value Theorem, there exist $c \in [1, 2]$ such that $f(c) = 0$.

(c) Let $S = \{f(x) | x \in [0, p]\}$ and $M = \sup(S)$ and $m = \inf(S)$.
Then by Extreme Value Theorem there exist $a, b \in [0, p]$ such that $f(a) = M$ and $f(b) = m$.
Hence $m \leq f(x) \leq M$ for all $[0, p]$.
Since for all $x \in \mathbb{R}$, $f(x) = f(x - np)$ such that $x - np \in [0, p]$ for some $n \in \mathbb{Z}$.
Hence for all $x \in \mathbb{R}$, we have $m \leq f(x) \leq M$. Therefore f is bounded.
Since any continuous function on a closed bounded interval $[a, b]$ is uniformly convergent.
Hence we have $f : [0, p] \to \mathbb{R}$ is uniformly continuous.
However since f is periodic, $f : [np, (n+1)p] \to \mathbb{R}$ is uniformly continuous for all $n \in \mathbb{Z}$.
Hence $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

Question 5

(a) Since $\lim_{n\to\infty} = n^2 x_n$ exists, $(n^2 x_n)$ is bounded,say

$$|n^2 x_n| \leq M$$

for some $M \in \mathbb{R}$.
Hence $|x_n| < \frac{M}{n^2}$.
Since $\displaystyle\sum_{n=1}^{\infty} \frac{M}{n^2}$ is convergent, by Comparison Test,$\displaystyle\sum_{n=1}^{\infty} |x_n|$ is convergent.
So $\displaystyle\sum_{n=1}^{\infty} x_n$ is absolutely convergent.

(b) Given any $\epsilon \in \mathbb{R}_{>0}$, there exist a continuous $g_{\frac{\epsilon}{3}}$ such that $|f(x) - g_{\frac{\epsilon}{3}}(x)| < \frac{\epsilon}{3}$ for all $x \in \mathbb{R}$.
For any $x_1 \in \mathbb{R}$,
Since $g_{\frac{\epsilon}{3}}$ is continuous, there exist a $\delta \in \mathbb{R}_{>0}$ such that $|g_{\frac{\epsilon}{3}}(x) - g_{\frac{\epsilon}{3}}(x_1)| < \frac{\epsilon}{3}$ whenever $|x - x_1| < \delta$.
Hence we have

$$\begin{aligned}
|f(x) - f(x_1)| &= |f(x) - g_{\frac{\epsilon}{3}}(x) + g_{\frac{\epsilon}{3}}(x) - f(x_1)| \\
&< |f(x) - g_{\frac{\epsilon}{3}}(x)| + |g_{\frac{\epsilon}{3}}(x) - g_{\frac{\epsilon}{3}}(x_1)| + |g_{\frac{\epsilon}{3}}(x_1) - f(x_1)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

whenever $|x - x_1| < \delta$.
Hence f is continuous on $\mathbb{R}$.

(c) Assume that g is continuous on $[0, 1]$.
Let $c_1 < c_2$ be the two points on $[0, 1]$ where g attains its absolute maximum.

If $0 < c_1$, choose a_1, a_2 such that $0 < a_1 < c_1 < a_2 < c_2$.
Let k satisfy

$$\max\{g(a_1), g(a_2)\} < k < g(c_1)$$

.
Then there exist b_1, b_2, b_3 where
$a_1 < b_1 < c_1 < b_2 < a_2 < b_3 < c_2$ such that $g(b_1) = g(b_2) = g(b_3)$ which is a contradiction. So we have $c_1 = 0$.

By a similar argument, we have $c_2 = 1$.

Now consider the points where g attains its absolute minimum. Using a similar argument we deduce that the absolute minimum points are 0 and 1. This implies that g is a constant function on $[0, 1]$ which is a contradiction.