NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1101R Linear Algebra I
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Question 1

(a) (i) Note that the reduced row echelon form of $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ is the identity matrix. Thus, S is linearly independent. Moreover, S has 3 elements and $\mathbb{R}^3$ has dimension 3. Hence S is a basis for $\mathbb{R}^3$.

(ii) We need to find $x, y, z \in \mathbb{R}$ satisfying

$$x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}.$$

Since the reduced row echelon form of $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 6 \end{array}\right)$ is $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$, we see that $[\boldsymbol{u_4}]_S = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

(iii) For all $k \in \mathbb{R}$, we have

$$k\boldsymbol{u_4} = 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + k \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + 2k \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

Thus,

$$[k\boldsymbol{u_4}]_S = \begin{pmatrix} 0 \\ k \\ 2k \end{pmatrix} = k \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = k[\boldsymbol{u_4}]_S.$$

(iv) Note that $\boldsymbol{u_4} = \boldsymbol{u_2} + 2\boldsymbol{u_3}$. Thus, $\text{span}\{\boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} = \text{span}\{\boldsymbol{u_2}, \boldsymbol{u_3}\}$. Since there is no $k \in \mathbb{R}$ satisfying $\boldsymbol{u_2} = k\boldsymbol{u_3}$, it follows that $\{\boldsymbol{u_2}, \boldsymbol{u_3}\}$ is linearly independent, and hence is a basis for $\text{span}\{\boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$. Since the basis contains two vectors, it has dimension 2.

(b) Since $\boldsymbol{u_3}$ is a solution to the linear system, we have

$$1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, $k_1 = k_2 = -\frac{1}{3}$ and $k_3 = 0$.

(c) Note that $\boldsymbol{v_1} = -2\boldsymbol{u_1} + (-1)\boldsymbol{u_2} + 2\boldsymbol{u_3} = \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix}$.

Since $\boldsymbol{u_4} = \boldsymbol{v_1} + 2\boldsymbol{v_3}$, it follows that $\boldsymbol{v_3} = \frac{1}{2}(\boldsymbol{u_4} - \boldsymbol{v_1}) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

Question 2

(a) (i) Note that the reduced row echelon form of $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Thus, $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for the row space of $\boldsymbol{A}$ and the rank is 3.

(ii) We need to check whether there exist $x, y, z \in \mathbb{R}$ such that $\boldsymbol{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$. As the reduced row echelon form of $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & -2 & 1 & 1 \end{array}\right)$ is $\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$, we see that $x = -1, y = 0, z = 2$ satisfies the equation. So, $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$ is in the range of T_1.

(iii) Since the reduced row echelon form of $\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right)$ is $\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$, we conclude that $\ker T_2 = \{(s, s, -s, 0)^T : s \in \mathbb{R}\}$. Hence $\{(1, 1, -1, 0)^T\}$ is a basis for the kernel of T_2, which has dimension 1.

(iv) We claim that it is impossible to find such $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.
Suppose such vectors exist. Since $\boldsymbol{v_1}$ lies in the column space of $\boldsymbol{A}$, we may write $\boldsymbol{v_1} = \boldsymbol{A}\boldsymbol{x_1}$ for some $\boldsymbol{x_1} \in \mathbb{R}^3$. Similarly, $\boldsymbol{v_2} = \boldsymbol{A}\boldsymbol{x_2}$ for some $\boldsymbol{x_2} \in \mathbb{R}^3$. Now,

$$\begin{aligned} T_2(\boldsymbol{v_1}) = T_2(\boldsymbol{v_2}) &\Rightarrow \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x_1} = \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x_2} \\ &\Rightarrow \boldsymbol{A}^T\boldsymbol{A}(\boldsymbol{x_1} - \boldsymbol{x_2}) = \boldsymbol{0}. \end{aligned}$$

Observe that $\det(\boldsymbol{A}^T\boldsymbol{A}) \neq 0$ so that the nullity of $\boldsymbol{A}^T\boldsymbol{A}$ is 0. Hence $\boldsymbol{x_1} - \boldsymbol{x_2} = \boldsymbol{0}$, which implies that $\boldsymbol{v_1} = \boldsymbol{v_2}$, contradiction.

(b) Since $\boldsymbol{C}$ is of full rank, the row space of $\boldsymbol{C}$ has dimension 4. Since the row space of $\boldsymbol{C}$ is a subset of $\mathbb{R}^4$, it follows that the row space of $\boldsymbol{C}$ (and so is the row space of $\boldsymbol{B}$) is $\mathbb{R}^4$. Note that $\boldsymbol{X} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & x & -1 \\ 0 & 0 & 0 & x \end{pmatrix}$ is row equivalent to $\boldsymbol{B}$.

Since,

$$\text{row space of } \boldsymbol{B} = \mathbb{R}^4 \Leftrightarrow \text{rank } \boldsymbol{B} = 4 \Leftrightarrow \text{rank } \boldsymbol{X} = 4 \Leftrightarrow x \in \mathbb{R}\backslash\{0\},$$

we conclude that for all $x \in \mathbb{R}\backslash\{0\}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ have the same row space.

(c) For $x \in \mathbb{R}\backslash\{0\}$, the row space of $\boldsymbol{B}$ is $\mathbb{R}^4$. So, clearly column space of $\boldsymbol{A}$ is subset of row space of $\boldsymbol{B}$.

For $x = 0$, it suffices to show that $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ are contained in the row space of $\boldsymbol{B} =$

$\text{span}\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0,\end{pmatrix}, \begin{pmatrix}1\\0\\1\\-1\end{pmatrix}\right\}$. Since the reduced row echelon form of $\left(\begin{array}{ccc|c|c|c}1 & 0 & 1 & 1 & 0 & 1\\0 & 1 & 0 & -1 & 1 & 0\\1 & 1 & 1 & 0 & 1 & 1\\0 & 0 & -1 & 1 & -2 & 1\end{array}\right)$ is $\left(\begin{array}{ccc|c|c|c}1 & 0 & 0 & 2 & -2 & 2\\0 & 1 & 0 & -1 & 1 & 0\\0 & 0 & 1 & -1 & 2 & -1\\0 & 0 & 0 & 0 & 0 & 0\end{array}\right)$, we see that $\left\{\begin{pmatrix}1\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\-2\end{pmatrix}, \begin{pmatrix}1\\0\\1\\1\end{pmatrix}\right\} \subseteq$ row space of $\boldsymbol{B}$.

Question 3

(a) (i) The characteristic polynomial of $\boldsymbol{A}$ is

$$\begin{aligned}\det\,(x\boldsymbol{I} - A) &= \det\begin{pmatrix}x-2 & 0 & 0\\0 & x-3 & 1\\0 & 1 & x-3\end{pmatrix}\\&= (x-2)(x-3)^2 - (x-2)\\&= (x-2)^2(x-4).\end{aligned}$$

Thus, the eigenvalues of $\boldsymbol{A}$ are 2 and 4.

(ii) For $x = 2$,

$$\begin{aligned}(x\boldsymbol{I} - \boldsymbol{A})\begin{pmatrix}a\\b\\c\end{pmatrix} = \boldsymbol{0} &\Leftrightarrow \begin{pmatrix}2-2 & 0 & 0\\0 & 2-3 & 1\\0 & 1 & 2-3\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}\\&\Leftrightarrow \begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}s\\t\\t\end{pmatrix} = s\begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\1\end{pmatrix}, \quad s, t \in \mathbb{R}.\end{aligned}$$

Hence a basis for the eigenspace of $\boldsymbol{A}$ associated with the eigenvalue 2 is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$. We can further get an orthonormal basis $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}\right\}$.

For $x = 4$,

$$\begin{aligned}(x\boldsymbol{I} - \boldsymbol{A})\begin{pmatrix}a\\b\\c\end{pmatrix} = \boldsymbol{0} &\Leftrightarrow \begin{pmatrix}4-2 & 0 & 0\\0 & 4-3 & 1\\0 & 1 & 4-3\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}\\&\Leftrightarrow \begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\s\\-s\end{pmatrix} = s\begin{pmatrix}0\\1\\-1\end{pmatrix}, \quad s \in \mathbb{R}.\end{aligned}$$

Hence a basis for the eigenspace of $\boldsymbol{A}$ associated with the eigenvalue 4 is $\left\{\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$. We can further get an orthonormal basis $\left\{\begin{pmatrix}0\\\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}\right\}$.

(iii) Following the working from (ii), let $\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

Then $\boldsymbol{P}$ is orthogonal and $\boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$.

(iv) Let $\boldsymbol{C} = \boldsymbol{P} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{pmatrix} \boldsymbol{P}^T$. Then it is easy to check that $\boldsymbol{C}$ is symmetric and $\boldsymbol{C}^2 = \boldsymbol{A}$.

(b) First, we claim that $\{\boldsymbol{u_1}+\boldsymbol{u_2}, \boldsymbol{u_3}+\boldsymbol{u_4}\}$ and $\{\boldsymbol{u_1}-\boldsymbol{u_2}, \boldsymbol{u_3}-\boldsymbol{u_4}\}$ are linearly independent. We will just prove the linear independence of $\{\boldsymbol{u_1}+\boldsymbol{u_2}, \boldsymbol{u_3}+\boldsymbol{u_4}\}$ as the linear independence of the other set can be deduced similarly. Suppose $a(\boldsymbol{u_1}+\boldsymbol{u_2})+b(\boldsymbol{u_3}+\boldsymbol{u_4}) = \boldsymbol{0}$. Then $a\boldsymbol{u_1}+a\boldsymbol{u_2}+b\boldsymbol{u_3}+b\boldsymbol{u_4} = \boldsymbol{0}$ which implies that $a = b = 0$ since $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ is linearly independent. Thus, $\{\boldsymbol{u_1} + \boldsymbol{u_2}, \boldsymbol{u_3} + \boldsymbol{u_4}\}$ is linearly independent.
Now observe that $\boldsymbol{B}(\boldsymbol{u_1}+\boldsymbol{u_2}) = 1(\boldsymbol{u_1}+\boldsymbol{u_2})$ and $\boldsymbol{B}(\boldsymbol{u_3}+\boldsymbol{u_4}) = 1(\boldsymbol{u_3}+\boldsymbol{u_4})$. Thus, $\{\boldsymbol{u_1}+\boldsymbol{u_2}, \boldsymbol{u_3}+\boldsymbol{u_4}\}$ is linearly independent subset of the eigenspace of $\boldsymbol{B}$ associated with the eigenvalue 1. Similarly, $\{\boldsymbol{u_1} - \boldsymbol{u_2}, \boldsymbol{u_3} - \boldsymbol{u_4}\}$ is linearly independent subset of the eigenspace of $\boldsymbol{B}$ associated with the eigenvalue -1. Each of those eigenspaces has dimension at least 2 since it contains a linearly independent subset consisting of two vectors. Since the sum of dimension of all eigenspaces of $\boldsymbol{B}$ is at most 4, we conclude that 1 and -1 are all the eigenvalues of $\boldsymbol{B}$. The dimension of each eigenspace is exactly two and (since the dimension add up to 4) $\boldsymbol{B}$ is diagonalizable.

(c) Let A_1 and A_{-1} be the eigenspaces of $\boldsymbol{A}$ associated with eigenvalue 1 and -1 respectively. Let B_1 and B_{-1} be the eigenspaces of $\boldsymbol{B}$ associated with eigenvalue 1 and -1 respectively. Note that $A_1 \cap B_1 = \{\boldsymbol{0}\}$ (otherwise, let $\boldsymbol{v} \in A_1 \cap B_1, \boldsymbol{v} \neq \boldsymbol{0}$. Then $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{v} = \boldsymbol{A}\boldsymbol{v} + \boldsymbol{B}\boldsymbol{v} = \boldsymbol{v} + \boldsymbol{v} = 2\boldsymbol{v}$ so that 2 is an eigenvalue of $\boldsymbol{A}+\boldsymbol{B}$, contradiction). Similarly, $A_{-1} \cap B_{-1} = \{\boldsymbol{0}\}$.
Note that one of A_1 and A_{-1} has dimension 2, and one of B_1 and B_{-1} has dimension 2 as well. Their intersection has dimension of at least 1. Since $A_1 \cap B_1 = A_{-1} \cap B_{-1} = \{\boldsymbol{0}\}$, then either $A_1 \cap B_{-1}$ or $A_{-1} \cap B_1$ has dimension at least 1. If $A_1 \cap B_{-1}$ has dimension at least 1, take $\boldsymbol{v} \neq \boldsymbol{0}, \boldsymbol{v} \in A_1 \cap B_{-1}$. Then $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{v} = \boldsymbol{A}\boldsymbol{v} + \boldsymbol{B}\boldsymbol{v} = \boldsymbol{v} - \boldsymbol{v} = \boldsymbol{0}$. Thus 0 is an eigenvalue of $\boldsymbol{A}+\boldsymbol{B}$, and hence $\boldsymbol{A}+\boldsymbol{B}$ is singular. The case $A_{-1} \cap B_1$ has dimension at least 1 is done in a similar manner.

Question 4

(a) (i) Note that $\boldsymbol{u_1} \cdot \boldsymbol{u_3} = (2)(2)+(0)(-1)+(1)(-4) = 0$ and $\boldsymbol{u_2} \cdot \boldsymbol{u_3} = (1)(2)+(2)(-1)+(0)(-4) = 0$. Any vector in V is of the form $a\boldsymbol{u_1} + b\boldsymbol{u_2}$ for some $a, b \in \mathbb{R}$. Then $(a\boldsymbol{u_1} + b\boldsymbol{u_2}) \cdot \boldsymbol{u_3} = a(\boldsymbol{u_1} \cdot \boldsymbol{u_3}) + b(\boldsymbol{u_2} \cdot \boldsymbol{u_3}) = 0$. So, V is orthogonal to $\boldsymbol{u_3}$.

(ii) Using Gram-Schmidt process, we obtain

$$\boldsymbol{v_1} = \frac{\boldsymbol{u_1}}{||\boldsymbol{u_1}||} = \left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)^T$$

and

$$\boldsymbol{v_2} = \frac{\boldsymbol{u_2} - (\boldsymbol{u_2} \cdot \boldsymbol{v_1})\boldsymbol{v_1}}{||\boldsymbol{u_2} - (\boldsymbol{u_2} \cdot \boldsymbol{v_1})\boldsymbol{v_1}||} = \frac{(\frac{1}{5}, 2, -\frac{2}{5})^T}{\sqrt{105}/5} = \left(\frac{1}{\sqrt{105}}, \frac{10}{\sqrt{105}}, -\frac{2}{\sqrt{105}}\right)^T.$$

(iii) The projection is

$$\boldsymbol{p} = (\boldsymbol{w} \cdot \boldsymbol{v_1})\boldsymbol{v_1} + (\boldsymbol{w} \cdot \boldsymbol{v_2})\boldsymbol{v_2} = \left(\frac{16}{21}, -\frac{8}{21}, \frac{10}{21}\right)^T.$$

(iv) Note that $\boldsymbol{p}$ and $\boldsymbol{w}$ are on the plane. Let vector $\boldsymbol{n} = (a, b, c)^T$ be perpendicular to the plane. Thus $\boldsymbol{n} \perp \boldsymbol{p}$ and $\boldsymbol{n} \perp \boldsymbol{w}$. Mathematically,

$$\boldsymbol{n} \cdot \boldsymbol{p} = 0$$
$$\boldsymbol{n} \cdot \boldsymbol{w} = 0$$

After substitution,

$$a\frac{16}{21} + b(-\frac{8}{21}) + c\frac{10}{21} = 0$$
$$2c = 0$$

Therefore, $\boldsymbol{n} = (1, 2, 0)$ is a solution to satisfy above equations and the equation of the plane is

$$x + 2y = 0.$$

(v) Let $\boldsymbol{v_3} = \frac{\boldsymbol{u_3}}{||\boldsymbol{u_3}||}$. Since $\boldsymbol{u_3}$ is orthogonal to V, then so are $\boldsymbol{v_3}$ and $-\boldsymbol{v_3}$. Thus, $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ and $\{\boldsymbol{v_1}, \boldsymbol{v_2}, -\boldsymbol{v_3}\}$ are both orthonormal. So, $\begin{pmatrix}\boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3}\end{pmatrix}$ and $\begin{pmatrix}\boldsymbol{v_1} & \boldsymbol{v_2} & -\boldsymbol{v_3}\end{pmatrix}$ are orthogonal matrices.

(b) The least square solutions can be obtained by solving the equation $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$, i.e.

$$\begin{pmatrix} 3 & -1 & 2 \\ -1 & 3 & 2 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

whose solution is given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s + \frac{1}{4} \\ s - \frac{1}{4} \\ -s \end{pmatrix} \quad , s \in \mathbb{R}.$

Thus, the least square solutions are $\begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad , s \in \mathbb{R}.$

(c) Let $\boldsymbol{A} = \begin{pmatrix}\boldsymbol{a_1} & \boldsymbol{a_2} & \cdots & \boldsymbol{a_n}\end{pmatrix}$. Then we use Gram-Schmidt process on $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}\}$ as follows:

$$\boldsymbol{u_1} = \boldsymbol{a_1}, \qquad \boldsymbol{v_1} = \frac{\boldsymbol{u_1}}{||\boldsymbol{u_1}||},$$
$$\boldsymbol{u_2} = \boldsymbol{a_2} - (\boldsymbol{a_2} \cdot \boldsymbol{v_1})\boldsymbol{v_1}, \qquad \boldsymbol{v_2} = \frac{\boldsymbol{u_2}}{||\boldsymbol{u_2}||},$$
$$\boldsymbol{u_{k+1}} = \boldsymbol{a_{k+1}} - (\boldsymbol{a_{k+1}} \cdot \boldsymbol{v_1})\boldsymbol{v_1} - \cdots - (\boldsymbol{a_{k+1}} \cdot \boldsymbol{v_k})\boldsymbol{v_k}, \quad \boldsymbol{v_{k+1}} = \frac{\boldsymbol{u_{k+1}}}{||\boldsymbol{u_{k+1}}||}.$$

Let $\boldsymbol{B} = \begin{pmatrix}\boldsymbol{v_1} & \boldsymbol{v_2} & \cdots & \boldsymbol{v_n}\end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} \boldsymbol{a_1} \cdot \boldsymbol{v_1} & \boldsymbol{a_2} \cdot \boldsymbol{v_1} & \cdots & \boldsymbol{a_n} \cdot \boldsymbol{v_1} \\ 0 & \boldsymbol{a_2} \cdot \boldsymbol{v_2} & \cdots & \boldsymbol{a_n} \cdot \boldsymbol{v_2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{a_n} \cdot \boldsymbol{v_n} \end{pmatrix}$.

Note that

$$\begin{aligned}
(\boldsymbol{a_i} \cdot \boldsymbol{v_i})\boldsymbol{v_i} &= \frac{1}{||\boldsymbol{u_i}||^2}(\boldsymbol{a_i} \cdot \boldsymbol{u_i})\boldsymbol{u_i} \\
&= \frac{1}{||\boldsymbol{u_i}||^2}\left[(\boldsymbol{u_i} + (\boldsymbol{a_i} \cdot \boldsymbol{v_1})\boldsymbol{v_1} + (\boldsymbol{a_i} \cdot \boldsymbol{v_2})\boldsymbol{v_2} + \cdots + (\boldsymbol{a_i} \cdot \boldsymbol{v_{i-1}})\boldsymbol{v_{i-1}}) \cdot \boldsymbol{u_i}\right]\boldsymbol{u_i} \\
&= \frac{1}{||\boldsymbol{u_i}||^2}(\boldsymbol{u_i} \cdot \boldsymbol{u_i} + 0 + 0 + \cdots + 0)\boldsymbol{u_i} \qquad (\text{Since } \boldsymbol{u_i} \cdot \boldsymbol{v_j} = 0 \text{ if } i \neq j) \\
&= \frac{1}{||\boldsymbol{u_i}||^2}||\boldsymbol{u_i}||^2\boldsymbol{u_i} \\
&= \boldsymbol{u_i}
\end{aligned}$$

Then by expanding $\boldsymbol{BC}$, it is obvious to see that $\boldsymbol{A} = \boldsymbol{BC}$.

Since $\boldsymbol{B}$ is orthogonal (because $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ is orthonormal) and $\boldsymbol{C}$ is upper triangular, we have found an expression for $\boldsymbol{B}$ and $\boldsymbol{C}$ that satisfy $\boldsymbol{A} = \boldsymbol{BC}$ for any invertible matrix $\boldsymbol{A}$.