NATIONAL UNIVERSITY OF SINGAPORE
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Question 1

For each $n \in \mathbb{N}$, we define $x_n = \frac{n+1}{n} \sin \frac{n\pi}{2}$. Firstly, we assert that $C(x_n) = \{-1, 0, 1\}$. To this end, let (x_{n_k}) be a convergent subsequence of (x_n), and let $c := \lim\limits_{k\to\infty} x_{n_k} = \lim\limits_{k\to\infty} \frac{n_k+1}{n_k} \sin \frac{n_k\pi}{2}$. Since $\lim\limits_{n\to\infty} \frac{n+1}{n} = 1$, it follows that $\lim\limits_{k\to\infty} \frac{n_k+1}{n_k} = 1$, and hence $\lim\limits_{k\to\infty} \frac{n_k}{n_k+1} = 1$. This implies that $\lim\limits_{k\to\infty} \sin \frac{n_k\pi}{2}$ exists. As $\sin \frac{n_k\pi}{2} = -1, 0$, or 1, this implies that $c = \lim\limits_{k\to\infty} \frac{n_k+1}{n_k} \sin \frac{n_k\pi}{2} = -1, 0$, or 1, and this implies that $C(x_n) \subseteq \{-1, 0, 1\}$.

Conversely, for all $n \in \mathbb{N}$, we have $x_{2n} = \frac{2n+1}{2n} \sin n\pi = 0$, $x_{4n+1} = \frac{4n+2}{4n+1} \sin \frac{(4n+1)\pi}{2} = \frac{4n+2}{4n+1}$, and $x_{4n+3} = \frac{4n+4}{4n+3} \sin \frac{(4n+3)\pi}{2} = -\frac{4n+4}{4n+3}$. This implies that $\lim\limits_{n\to\infty} x_{2n} = 0$, $\lim\limits_{n\to\infty} x_{4n+1} = 1$ and $\lim\limits_{n\to\infty} x_{4n+3} = -1$. Hence, we have $\{-1, 0, 1\} \subseteq C(x_n)$, and this completes the claim.

Next, we assert that $V = (1, \infty)$. To this end, let us take any $v > 1$. Since $\lim\limits_{n\to\infty} \frac{n+1}{n} = 1$, there exists some $M \in \mathbb{N}$, such that for all $n \geq M$, we have $\left|\frac{n+1}{n} - 1\right| < v - 1$. This implies that $\frac{n+1}{n} < v$ for all $n \geq M$, and hence we have $x_n = \frac{n+1}{n} \sin \frac{n\pi}{2} \leq \frac{n+1}{n} < v$ for all $n \geq M$. Consequently, this shows that $v \in V$. Conversely, let us take any $v' \leq 1$. Since $\frac{n+1}{n} > 1$ for all $n \in \mathbb{N}$, we have $v' \leq 1 < \frac{4n+2}{4n+1} = x_{4n+1}$ for all $n \in \mathbb{N}$. So by definition of V, we must have $v \notin V$, and this completes the claim. By a similar argument as above, we have $W = (-\infty, -1)$.

Finally, we shall show for each $m \in \mathbb{N}$ that $u_{4m-2} = u_{4m-1} = u_{4m} = u_{4m+1} = x_{4m+1}$, and $u_1 = x_1$. To see that this is indeed the case, we first note that $x_n > 0$ if and only if $n = 4k - 3$ for some $k \in \mathbb{N}$. Hence, we observe that $u_m = \sup\{x_n : n \geq m\} = \sup\{x_{4k-3} : 4k - 3 \geq m\}$. As the sequence $\left(\frac{n+1}{n}\right)$ is a decreasing sequence, this show that $u_{4m-2} = u_{4m-1} = u_{4m} = u_{4m+1} = x_{4m+1}$ for all $m \in \mathbb{N}$, and $u_1 = x_1$ as desired. By symmetry, and by a similar argument as above, we see that $v_{4m} = v_{4m+1} = v_{4m+2} = v_{4m+3} = x_{4m+3}$ for all $m \in \mathbb{N}$, and $v_1 = v_2 = v_3 = x_3$.

Now, by definition, we have $\limsup x_n = \sup C(x_n) = \inf V = \inf\{u_m : m \in \mathbb{N}\}$, and similarly, we have $\liminf x_n = \inf C(x_n) = \sup V = \sup\{v_m : m \in \mathbb{N}\}$. It is easy to see that $\sup C(x_n) = \inf V = 1$. Next, we shall show that $\inf\{u_m : m \in \mathbb{N}\} = 1$. Note that $\{u_m : m \in \mathbb{N}\} = \{x_{4m+1} : m \geq 0\}$. Since $x_{4m+1} = \frac{4m+2}{4m+1} > 1$, this shows that 1 is a lower bound for $\{u_m : m \in \mathbb{N}\}$. Conversely, if u is a lower bound for $\{u_m : m \in \mathbb{N}\} = \{x_{4m+1} : m \geq 0\}$, then we must have $u \leq x_{4m+1}$ for all $m \in \mathbb{N}$. This implies that $u \leq \lim\limits_{m\to\infty} x_{4m+1} = \lim\limits_{m\to\infty} \frac{4m+2}{4m+1} = 1$, so this implies that 1 is the greatest lower bound for $\{u_m : m \in \mathbb{N}\}$ as desired. Hence, the three definitions of limit superior match for (x_n). By a similar argument as above, we see that the three definitions of limit inferior match for (x_n) as well.

Question 2

For each $n \in \mathbb{N}$, we clearly have $x_n > 0$ (since $\lambda > 0$), and

$$|x_{n+2} - x_{n+1}| = \left|\frac{1}{3 + x_{n+1}} - \frac{1}{3 + x_n}\right|$$

$$\begin{aligned}
&= \left|\frac{(3+x_n)-(3+x_{n+1})}{(3+x_{n+1})(3+x_n)}\right| \\
&= \left|\frac{x_n - x_{n+1}}{(3+x_{n+1})(3+x_n)}\right| \\
&< \left|\frac{x_n - x_{n+1}}{3\cdot 3}\right| \\
&= \frac{1}{9}|x_{n+1}-x_n|.
\end{aligned}$$

This implies that (x_n) is a contractive sequence. Hence (x_n) is a Cauchy sequence, so it is convergent. Let $x := \lim\limits_{n\to\infty} x_n$. Then we have $x = \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} \frac{1}{3+x_n} = \frac{1}{3+x}$. Equivalently, this implies that $x^2+3x-1=0$. Solving for the roots this equation yields $x = \frac{-3+\sqrt{13}}{2}$ or $x = \frac{-3-\sqrt{13}}{2}$. Furthermore, since $x_n > 0$ for all $n \in \mathbb{N}$, we have $x = \lim\limits_{n\to\infty} x_n \geq 0$. So $x = \frac{-3+\sqrt{13}}{2}$.

Question 3

(i) For all $n \geq 100$, we have $0 \leq \frac{\sqrt{2n+1}}{n^2-n+100} \leq \frac{\sqrt{2n+1}}{n^2} \leq \frac{\sqrt{4n}}{n^2} = \frac{2}{n\sqrt{n}}$. As the series $\sum\limits_{n=1}^{\infty} \frac{2}{n\sqrt{n}}$ is convergent, we have the series $\sum\limits_{n=1}^{\infty} \frac{\sqrt{2n+1}}{n^2-n+100}$ to be convergent by the Comparison Test.

(ii) Clearly, the sequence $\left(\frac{1}{\sqrt[3]{n}}\right)$ is a decreasing sequence, $\frac{1}{\sqrt[3]{n}} > 0$ for all $n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$. Thus, we have the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$ to be convergent by the Alternating Series Test.

Question 4

(i) Let $\varepsilon > 0$ be given, and set $\delta = \min\{3, 5\varepsilon\}$. Then for all $x \in \mathbb{R}$ such that $0 < |x-3| < \delta$, we must have $x - 3 > -\delta \geq -3$, which implies that $|x+2| \geq x+2 = (x-3)+5 > 2$. Hence, for all $x \in \mathbb{R}$ such that $0 < |x-3| < \delta$, we have

$$\left|\frac{x}{x+2} - \frac{3}{5}\right| = \left|\frac{5x-3(x+2)}{5(x+2)}\right| = \left|\frac{2(x-3)}{5(x+2)}\right| = \frac{2|x-3|}{5|x+2|} < \frac{2\cdot(5\varepsilon)}{5\cdot 2} = \varepsilon.$$

By the $\varepsilon - \delta$ definition of limit, we must have $\lim\limits_{x\to 3} \frac{x}{x+2} = \frac{3}{5}$ as desired.

(ii) Note that $\frac{x^2+1}{x+1} = \frac{x^2+x-x-1+2}{x+1} = x - 1 + \frac{2}{x+1}$ for all $x > 0$. This implies that

$$\frac{x^2+1}{x+1} - \alpha x - \beta = (1-\alpha)x - (1+\beta) + \frac{2}{x+1}$$

for all $x > 0$. Note that

$$\lim_{x\to+\infty}(1-\alpha)x = \begin{cases} 0 & \text{if } \alpha = 1, \\ -\infty & \text{if } \alpha > 1, \\ \infty & \text{if } \alpha < 1. \end{cases}$$

As $\lim\limits_{x\to+\infty} -(1+\beta) = -(1+\beta)$ and $\lim\limits_{x\to+\infty} \frac{2}{x+1} = 0$, this implies that

$$\lim_{x\to+\infty}\left(\frac{x^2+1}{x+1} - \alpha x - \beta\right) = \begin{cases} -(1+\beta) & \text{if } \alpha = 1, \\ -\infty & \text{if } \alpha > 1, \\ \infty & \text{if } \alpha < 1. \end{cases}$$

By assumption, we have $\lim\limits_{x\to+\infty}\left(\frac{x^2+1}{x+1} - \alpha x - \beta\right) = 0$, so we must have $\alpha = 1$ and $\beta = -1$.

Question 5

Let $\lim_{x\to+\infty} f(x) = c$, where c is a finite (real) number. By definition, it follows that there exists some $N > 0$, such that $N > a$, and for all $x > N$, we have $|f(x) - c| < 1$. This implies that for all $x > N$, we have $|f(x)| \leq |f(x) - c| + |c| < 1 + |c|$. Furthermore, since f is continuous on $[a, +\infty)$ (hence continuos on $[a, N]$), it is bounded on $[a, N]$, so there exists some $K > 0$, such that $|f(x)| \leq K$ for all $x \in [a, N]$. By setting $M = \max\{1 + |c|, K\}$, it is easy to see that $|f(x)| \leq M$ for all $x \in [a, +\infty)$, so this shows that f is bounded on $[a, +\infty)$ as desired.

Question 6

Without loss of generality, let us assume that $f(x_1) = \min\{f(x_1), f(x_2), \cdots, f(x_n)\}$, $f(x_n) = \max\{f(x_1), f(x_2), \cdots, f(x_n)\}$, and $x_1 < x_n$. As λ_i is positive for all $i = 1, 2, \cdots, n$, it follows that $\lambda_i f(x_1) \leq \lambda_i f(x_i) \leq \lambda_i f(x_n)$ for all $i = 1, 2, \cdots, n$. Furthermore, since $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, we must have

$$f(x_1) = (\lambda_1 + \cdots + \lambda_n)f(x_1) \leq \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n) \leq (\lambda_1 + \cdots + \lambda_n)f(x_n) = f(x_n).$$

As f is continuous on $[x_1, x_n]$, it follows from the Intermediate Value Theorem that there exists some $\xi \in [x_1, x_n] \subseteq (a, b)$, such that $f(\xi) = \lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n)$, and we are done.

Question 7

(i) Let $\varepsilon > 0$ be given. For each $x \in \mathbb{R}$, let us write $f(x) = f_1(x)f_2(x)\cdots f_n(x)$, and let us define $g_i(x) = f_1(x)f_2(x)\cdots f_i(x)$ and $h_i(x) = f_i(x)f_{i+1}(x)\cdots f_n(x)$ for all $x \in \mathbb{R}$ and $i = 1, 2, \cdots, n$. Then it is easy to check that

$$\begin{aligned} & f(x) - f(y) \\ = \; & (f_1(x) - f_1(y))h_2(x) + \left(\sum_{i=2}^{n-1} g_{i-1}(y)(f_i(x) - f_i(y))h_{i+1}(x)\right) + g_{n-1}(y)(f_n(x) - f_n(y)) \end{aligned}$$

for all $x, y \in \mathbb{R}$. As $f_i(x)$ is a bounded function on $\mathbb{R}$ for all $i = 1, 2, \cdots, n$, it follows that there exist $M_1, M_2, \cdots, M_n > 0$, such that $|f_i(x)| \leq M_i$ for all $x \in \mathbb{R}$ and $i = 1, 2, \cdots, n$. Let us define $M := \max\{M_1, M_2, \cdots M_n\}$. Then it is easy to check that $|h_2(x)| \leq M^{n-1}$, $|g_{n-1}(y)| \leq M^{n-1}$, and $|g_{i-1}(y)||h_{i+1}(x)| \leq M^{n-1}$ for all $x, y \in \mathbb{R}$ and $i = 2, 3, \cdots, n-1$.

Next, let us fix a $j \in \{1, 2, \cdots, n\}$. As $f_j(x)$ is uniformly continuous on $\mathbb{R}$, it follows that there exists some $\delta_j > 0$, such that for all $x, y \in \mathbb{R}$ that satisfies $|x - y| < \delta_j$, we have $|f_j(x) - f_j(y)| < \frac{\varepsilon}{nM^{n-1}}$. Let $\delta = \min\{\delta_1, \delta_2, \cdots, \delta_n\}$. Then for all $x, y \in \mathbb{R}$ that satisfies $|x - y| < \delta$, we have

$$\begin{aligned} & |f(x) - f(y)| \\ \leq \; & |(f_1(x) - f_1(y))h_2(x)| + |g_{n-1}(y)(f_n(x) - f_n(y))| + \sum_{i=2}^{n-1} |g_{i-1}(y)(f_i(x) - f_i(y))h_{i+1}(x)| \\ \leq \; & M^{n-1}|f_1(x) - f_1(y)| + M^{n-1}|f_n(x) - f_n(y)| + \left(\sum_{i=2}^{n-1} M^{n-1}|f_i(x) - f_i(y)|\right) \\ < \; & M^{n-1} \cdot \frac{\varepsilon}{nM^{n-1}} + M^{n-1} \cdot \frac{\varepsilon}{nM^{n-1}} + \left(\sum_{i=2}^{n-1} M^{n-1} \cdot \frac{\varepsilon}{nM^{n-1}}\right) \\ = \; & \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, this shows that the function $f(x) = f_1(x)f_2(x)\cdots f_n(x)$ is a uniformly continuous function on $\mathbb{R}$ as desired.

(ii) Let $\varepsilon > 0$ be given. By assumption, we have $f(x) \neq 0$, and $\frac{1}{|f(x)|} < \frac{1}{M}$ for all $x \in \mathbb{R}$. As $f(x)$ is a uniformly continuous function on $\mathbb{R}$, it follows that there exists some $\delta > 0$, such that for all $x, y \in \mathbb{R}$ that satisfies $|x - y| < \delta$, we have $|f(x) - f(y)| < M^2\varepsilon$. Then for all $x, y \in \mathbb{R}$ that satisfies $|x - y| < \delta$, we have

$$\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| = \left|\frac{f(y) - f(x)}{f(x)f(y)}\right| = \frac{|f(x) - f(y)|}{|f(x)||f(y)|} < M^2\varepsilon \cdot \frac{1}{M^2} = \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, this shows that the function $\frac{1}{f(x)}$ is a uniformly continuous function on $\mathbb{R}$ as desired.

Question 8

Let $\varepsilon > 0$ be given. As f is uniformly continuous on $[0, +\infty)$, it follows that there exists some $\delta > 0$, such that for all $x, y \in [0, +\infty)$ that satisfies $|x - y| < \delta$, we have $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Let us choose some $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$. Next, let us fix an $i \in \{0, 1, \cdots, N-1\}$. By assumption, we have $\lim_{n\to\infty} f\left(\frac{i}{N} + n\right) = 0$, so there exists $M_i \in \mathbb{N}$, such that $\left|f\left(\frac{i}{N} + n\right)\right| < \frac{\varepsilon}{2}$ for all $n \geq M_i$.

Let $M = \max\{M_0, M_1, \cdots, M_{N-1}\}$, and let us show that $|f(x)| < \varepsilon$ for all $x \in [M, \infty)$. To this end, let us take any $y \in [M, +\infty)$. Then there exists some $K \in \mathbb{N}$ and $j \in \{0, 1, \cdots, N-1\}$, such that $K \geq M$, and $y \in \left[K + \frac{j}{N}, K + \frac{j+1}{N}\right)$. This implies that $\left|y - \left(K + \frac{j}{N}\right)\right| < \frac{1}{N} < \delta$, so we have $\left|f(y) - f\left(K + \frac{j}{N}\right)\right| < \frac{\varepsilon}{2}$. As $K \geq M \geq M_j$, we must have

$$|f(y)| \leq \left|f(y) - f\left(K + \frac{j}{N}\right)\right| + \left|f\left(K + \frac{j}{N}\right)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

as desired. By the $\varepsilon - \delta$ definition of limits, we must have $\lim_{x\to+\infty} f(x) = 0$ as desired.