MA1100T - Basic Discrete Mathematics (T) Suggested Solutions

(Semester 1, AY2021/2022)

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1 True or False

Problem 1. For any uncountable set A, the power set $\mathcal{P}(A)$ is uncountable.

Ans. True. □

Problem 2. For any positive integers a, b, $c \in \mathbb{Z}_{>0}$, if a is relatively prime to b and a is relatively prime to c, ten a is relatively prime to bc.

Ans. True. □

Problem 3. For any infinite set A and any infinite set B, the set $A \times B$ is infinite.

Ans. True. □

Problem 4. Let I be an uncountable indexing set, and suppose for each $i \in I$, the set X_i is an uncountable set. Then $\bigcup_{i\in I} X_i$ is uncountable.

Ans. True. □

Problem 5. For any countable set S and for any map $f : S \to S$ from S to itself, if f is surjective, then f is injective.

Ans. False. Consider $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(n) = \begin{cases} 1 & n = 1 \\ n-1 & n > 1. \end{cases}$$

for $n \in \mathbb{N}$. Then f is surjective since $\forall y \in \mathbb{N}$, $f(y+1) = y$. But f is not injective since $f(1) = f(2)$ and $1 \neq 2$. □

Problem 6. For any finite set A and any infinite set B, the set $A \times B$ is infinite.

Ans. False. Let A be the empty set which is finite. Then $A \times B = \{(a, b) : a \in A \wedge b \in B\}$ is also empty. $\square$

Problem 7. There exists a countable set A and a countable set B such that the set $\mathrm{Maps}(A, B)$ is uncountable.

Ans. True. $\square$

Problem 8. Let I be a countable indexing set, and suppose for each $i \in I$, the set X_i is a countable set. Then $\bigcup_{i \in I} X_i$ is countable.

Ans. True. Axiom of Choice needed. $\square$

Problem 9. There are only finitely many prime numbers p such that for any positive integer $a \in \mathbb{Z}_{>0}$, one has $p \mid a$.

Ans. True. $\square$

Problem 10. There exists a set B such that for any set A, there exists an injective map $f : A \to B$.

Ans. False. $\mathcal{P}(B)$ does not inject into B. $\square$

Problem 11. For any finite set A, the power set $\mathcal{P}(A)$ is finite.

Ans. True. $\square$

Problem 12. There exist integers $x, y \in \mathbb{Z}$ such that $15x^2 - 7y^2 = 9$.

Ans. False. Note that $9 \equiv 15x^2 - 7y^2 \equiv 3y^2 \pmod 5 \Rightarrow y^2 \equiv 3 \pmod 5$. It is easy to see that 3 is not a quadratic residue mod 5. $\square$

Problem 13. For any finite set S and for any map $f : S \to S$ from S to itself, if f is injective, then f is surjective.

Ans. True. $\square$

Problem 14. For any finite set S and for any map $f : S \to S$ from S to itself, if f is surjective, then f is injective.

Ans. True. $\square$

Problem 15. There exists a set B such that for any set A, every map $f : A \to B$ is surjective.

Ans. True. Let $B = \emptyset$. If $f : A \to B$ is a function, then $A = \emptyset$. Hence, f is vacuously surjective. $\square$

Problem 16. Let $a \in \mathbb{Z}_{>0}$ be a positive integer with the following property:
$(\forall d \in \mathbb{Z}_{>0})\,[(d \mid a) \Leftrightarrow ((d = 1) \vee (d = a))]$. Then a is a prime number.

Ans. False. This holds for $a = 1$. $\square$

Problem 17. There exists a set B such that for any set A, every map $f : A \to B$ is injective.

Ans. True. Let $B = \emptyset$. If $f : A \to B$ is a function, then $A = \emptyset$. Therefore, f is vacuously injective. $\square$

Problem 18. There exists an integer $n \in \mathbb{Z}$ such that $17 \mid (n^2 + 1)$.

Ans. True. It holds for $n = 4$. $\square$

Problem 19. Let I be a finite indexing set, and suppose for each $i \in I$, the set X_i is a finite set. Then $\bigcup_{i \in I} X_i$ is finite.

Ans. True. $\square$

Problem 20. Let $a \in \mathbb{Z}_{>0}$ be a prime number. Then a has the following property: $(\forall d \in \mathbb{Z}_{>0})\,[(d \mid a) \Leftrightarrow ((d = 1) \vee (d = a))]$.

Ans. True. $\square$

Problem 21. For any infinite set A, the power set $\mathcal{P}(A)$ is infinite.

Ans. True. $\square$

Problem 22. For any integers a, b, $c \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$, then for any integers m, $n \in \mathbb{Z}$, one has $a \mid (bm + cn)$.

Ans. True. $\square$

Problem 23. For any finite set A and any infinite set B, the set $\mathrm{Maps}(A, B)$ is countable.

Ans. False. Pick B to be uncountable. $\square$

Problem 24. Let $a \in \mathbb{Z}_{>0}$ be a prime number. Then a has the following property: $(\forall b, c \in \mathbb{Z}_{>0})\,[(a \mid bc) \Leftrightarrow ((a \mid b) \vee (a \mid c)]$.

Ans. True. $\square$

Problem 25. There exist integers $x, y \in Z$ such that $15x - 7y = 9$.

Ans. True. Pick $x = 2$, $y = 3$. $\square$

Problem 26. There are only finitely many prime numbers p for which there exists a positive integer $a \in \mathbb{Z}_{>0}$ such that $p \mid a$.

Ans. False. For each prime number p, choose $a = p$. $\square$

Problem 27. For any positive integers $a, b, c \in \mathbb{Z}_{>0}$, if a is relatively prime to bc, then a is relatively prime to b and a is relatively prime to c.

Ans. True. $\square$

Problem 28. For any integers $m, n \in \mathbb{Z}$, if $5 \mid (m^2 + n^2)$, then $5 \mid m$ and $5 \mid n$.

Ans. False. Pick $m = 1$ and $n = 2$. $\square$

Problem 29. For any positive integers $a, b, d \in \mathbb{Z}_{>0}$, if $d \mid \gcd(a, b)$, then for any integers $m, n \in \mathbb{Z}$, one has $d \mid am + bn$.

Ans. True. $\square$

Problem 30. For any countable set A and any countable set B, the set $A \times B$ is countable.

Ans. True. $\square$

Problem 31. For any infinite set A and any finite set B, the set $\mathrm{Maps}(A, B)$ is countable.

Ans. False. Pick A to be uncountable. $\square$

Problem 32. For any $n \in \mathbb{Z}$, there exists $k \in \mathbb{Z}$ such that $n^2 = 4k$ or $n^2 = 4k - 1$.

Ans. False. Pick $n = 1$. Then $4 \nmid n^2 = 1$ and $4 \nmid n^2 + 1 = 2$. $\square$

Problem 33. For any $n \in \mathbb{Z}$, there exists $k \in \mathbb{Z}$ such that $n^2 = 8k$ or $n^2 = 8k + 1$ or $n^2 = 8k + 4$.

Ans. True. $\square$

Problem 34. There are only finitely many positive integers $a \in \mathbb{Z}_{>0}$ for which there exists a prime numbers p such that $p \mid a$.

Ans. False. All even positive integers are divisible by $p = 2$ and there are infinitely many even positive integers. $\square$

Problem 35. For any countable set A, the power set $\mathcal{P}(A)$ is countable.

Ans. False. Cantor's Theorem. $\square$

Problem 36. There exists a countable set A and an uncountable set B such that the set $\text{Maps}(A, B)$ is uncountable.

Ans. True. $\square$

Problem 37. Let I be an uncountable indexing set, and suppose for each $i \in I$, the set X_i is a countable set. Then $\bigcup_{i \in I} X_i$ is uncountable.

Ans. True. $\square$

Problem 38. Let I be an infinite indexing set, and suppose for each $i \in I$, the set X_i is an infinite set. Then $\bigcup_{i \in I} X_i$ is infinite.

Ans. True. $\square$

Problem 39. There exists a set B such that for any set A, there exists a surjective map $f : A \to B$.

Ans. False. If A is empty, then all elements in the codomain (nonempty) will not be reached by f. If the codomain is empty, then for $A \neq \emptyset$ there exist no map $f : A \to B$. $\square$

Problem 40. For any integer $n \in \mathbb{Z}$ with $n > 4$, if n is prime, then n does not divide $(n-1)!$.

Ans. True. $\square$

Problem 41. There exists an uncountable set A and a countable set B such that the set $\text{Maps}(A, B)$ is uncountable.

Ans. True. $\square$

Problem 42. For any integer $n \in \mathbb{Z}$ with $n > 4$, if n is not prime, then n divides $(n-1)!$.

Ans. True. $\square$

Problem 43. For any integers $l, m, n \in \mathbb{Z}$, if $7 \mid (l^2 + m^2 + n^2)$, then $7 \mid l$ or $7 \mid m$ or $7 \mid n$.

Ans. False. Pick $l = 1$, $m = 2$, and $n = 3$. Then $7 \mid 1^2 + 2^2 + 3^2 = 14$ however $7 \nmid 1, 2, 3$. $\square$

Problem 44. For any finite set A and finite set B, the set $A \times B$ is finite.

Ans. True. $\square$

Problem 45. For any integers $m, n \in \mathbb{Z}$, if $7 \mid (m^2 + n^2)$, then $7 \mid m$ and $7 \mid n$.

Ans. True. $\square$

Problem 46. Let $a \in \mathbb{Z}_{>0}$ be a positive integer with the following property: $(\forall b, c \in \mathbb{Z}_{>0})\,[(a \mid bc) \Leftrightarrow (a \mid b) \vee (a \mid c)]$. Then a is a prime number.

Ans. False. This also holds for $a = 1$. $\square$

Problem 47. There exists an uncountable set A and an uncountable set B such that the set $\mathrm{Maps}(A, B)$ is uncountable.

Ans. True. $\square$

Problem 48. There are only finitely many positive integers $a \in \mathbb{Z}_{>0}$ such that for any prime numbers p, one has $p \mid a$.

Ans. True. $\square$

Problem 49. There exists an integer $n \in \mathbb{Z}$ such that $19 \mid (n^2 + 1)$.

Ans. False. Note that 18 is not a quadratic residue modulo 19. $\square$

Problem 50. For any uncountable set A and any uncountable set B, the set $A \times B$ is uncountable.

Ans. True. $\square$

Problem 51. For any countable set A and any uncountable set B, the set $A \times B$ is uncountable.

Ans. True. $\square$

Problem 52. Let I be a finite indexing set, and suppose for each $i \in I$, the set X_i is an infinite set. Then $\bigcup_{i \in I} X_i$ is infinite.

Ans. False. If I is empty, $\bigcup_{i \in I} X_i$ is empty. $\square$

Problem 53. For any finite set A and any finite set B, the set $\mathrm{Maps}(A, B)$ is countable.

Ans. False. Let $A = B = \{1\}$. Then $\mathrm{Maps}(A, B)$ has cardinality 1. $\square$

Problem 54. For any integers $l, m, n \in \mathbb{Z}$, if $5 \mid (l^2 + m^2 + n^2)$,then $5 \mid l$ or $5 \mid m$ or $5 \mid n$.

Ans. True. $\square$

Problem 55. For any infinite set A and any infinite set B, the set $\mathrm{Maps}(A, B)$ is uncountable.

Ans. True. $\square$

Problem 56. Let I be an infinite indexing set, and suppose for each $i \in I$, the set X_i is a finite set. Then $\bigcup_{i\in I} X_i$ is infinite.

Ans. False. $X_i = \emptyset$ for all $i \in I$ means that $\bigcup_{i\in I} X_i =$ is finite. $\square$

Problem 57. For any integers $a, b, c \in \mathbb{Z}$, if for any integers $m, n \in \mathbb{Z}$, one has $a \mid (bm+cn)$, then $a \mid b$ and $a \mid c$.

Ans. True. $\square$

Problem 58. For any countable set S and for any map $f : S \to S$ from S to itself, if f is injective, then f is surjective.

Ans. False. Consider $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$. $\square$

Problem 59. Let I be a countable indexing set, and suppose for each $i \in I$, the set X_i is an uncountable set. Then $\bigcup_{i\in I} X_i$ is uncountable.

Ans. True. $\square$

Problem 60. For any positive integers $a, b, d \in \mathbb{Z}_{>0}$, if for any integers $m, n \in \mathbb{Z}$, one has $d \mid am + bn$, then $d \mid \gcd(a, b)$.

Solution. True. Direct application of Bezout's Identity. $\square$

2 Prove or Disprove/Proving Questions

Problem 1. [10 points] Prove or disprove: For any sets A and B, there exists a unique set X with the following property:

$$\text{For any set } T\text{, one has } T \subseteq X \text{ if and only if } T \cup B \subseteq A$$

Solution. Disprove by counterexample. Take $A = \{1\}$ and $B = \{1, 2\}$ and suppose there exists such a set X. If T is the empty set, then $T \subseteq X$ is true. It follows from the property of X that $T \cup B \subseteq A$. But $T \cup B = \{1, 2\} \not\subseteq A$ which is a contradiction. $\square$

Problem 2. [10 points] Prove or disprove: For any sets A and B, there is a unique set X with the following property:

$$\text{For any set } T\text{, one has } T \supseteq X \text{ if and only if } T \cup B \supseteq A$$

Solution. The statement is true, let $X = A - B$. We will show that X has the desired properties.

($\Rightarrow$) Suppose $T \supseteq A - B$. Let $x \in A$. If $x \in B$ then $x \in T \cup B$. Else, $x \notin B \Rightarrow x \in A - B \subseteq T \subseteq T \cup B$. This shows that $T \cup B \supseteq A$.

($\Leftarrow$) Suppose $T \cup B \supseteq A$. Let $x \in A - B$, then $x \in A$ and $x \notin B$. Since $T \cup B \supseteq A$, $x \in A \Rightarrow x \in T \vee x \in B$. But $x \notin B$ hence $x \in T$. This shows that $T \supseteq A - B$.

To show that X is unique, suppose that another set Y satisfies the given conditions. Then $T \supseteq X$ if and only if $T \cup B \supseteq A$ if and only if $T \supseteq Y$. Then picking $T = X$ and $T = Y$ yields $X \supseteq Y$ and $Y \supseteq X$. Therefore, we must have $X = Y$. $\square$

Problem 3. [10 points] Let X be any set such that $\emptyset \in X$ and such that for any $x \in X$, one has $\{x\} \in X$. The sequence $A_1, A_2, \ldots$ of elements of X is defined recursively as follows:

$$A_1 := \emptyset, \text{ and for each } n \in \mathbb{N}, \text{ we let } A_{n+1} := \{A_n\}.$$

Show that for any $i, j \in \mathbb{N}$ with $i \neq j$ one has $A_i \neq A_j$.

Solution. Let $P(n)$ be the proposition that for any $i, j \in \mathbb{N}$ with $i, j \leq n$ and $i \neq j$, $A_i \neq A_j$. We shall prove $P(n)$ for all n via induction.

(Base case) If $n = 1$, then the proposition is vacuously true as i and j must both be equal to 1. If $n = 2$ then $i = 1$ and $j = 2$ without loss of generality. Then $A_1 = \emptyset$ and $A_2 = \{\emptyset\}$ so $A_1 \neq A_2$. This proves the base case.

(Inductive Step) Suppose that $P(n)$ is true for some positive integer $n \geq 2$. We will prove $P(n+1)$ is true. Let $i, j \leq n+1$. Note if $i, j \leq n$ then by assumption, $A_i \neq A_j$. Hence, i or j must be $n+1$. Without loss of generality, let $i = n+1$. Since $j = n+1 \Rightarrow i = j$, we must have $j \leq n$. If $j = 1$ then $A_{n+1} \neq A_1$ since A_1 is empty and A_{n+1} is not. Otherwise, $j > 1$. Suppose toward a contradiction that $A_{n+1} = A_j$. Since A_n and A_{j-1} are the only elements of the sets A_{n+1} and A_j respectively, $A_n = A_{j-1}$. But $n, j-1 \in \mathbb{N}$ and $n, j-1 \leq n$ with $n \neq j-1$ therefore by assumption $A_n \neq A_{j-1}$. This is a contradiction, hence, $A_{n+1} \neq Aj$. We have shown that $P(n+1)$ is true which completes the inductive step.

Now for any $i, j \in \mathbb{N}$, take $n = \max i, j$. Since $i, j \leq n$ and $i \neq j$, $P(n)$ witnesses $A_i \neq A_j$ as desired. $\square$

Problem 4. [10 points] Let X, Y be sets and let $f : X \to Y$ be a map. Prove or disprove: f is injective if and only if for any set T, the "post-composition with f" map

$$\Phi_T : \mathrm{Maps}(T, X) \longrightarrow \mathrm{Maps}(T, Y), \quad \phi \longmapsto f \circ \phi, \quad \text{is injective.}$$

Solution. True.

($\Rightarrow$) Suppose f is injective. Let $\phi_1, \phi_2 \in \mathrm{Maps}(T, X)$ such that $\Phi_T(\phi_1) = \Phi(\phi_2)$. This implies that $f \circ \phi_1 = f \circ \phi_2$. Hence, for all $t \in T$, $f(\phi_1(t)) = f(\phi_2(t))$. Since f is injective, $\phi_1(t) = \phi_2(t)$ for all $t \in T$. Therefore, ϕ_1 and ϕ_2 are the same function. This proves that Φ_T is injective.

($\Leftarrow$) Suppose Φ_T is injective for any set T. Pick $T = \{0\}$, then Φ_T is injective. For all $x, y \in X$ such that $f(x) = f(y)$, choose functions $\phi_x, \phi_y \in$ Maps(T, X) such that $\phi_x(0) = x$ and $\phi_y(0) = y$. Then $f(x) = f(y) \Rightarrow f(\phi_x(t)) = f(\phi_y(t))$ for all $t \in T = \{0\}$, i.e. $f \circ \phi_x = f \circ \phi_y$. But then $\Phi_T(\phi_x) = f \circ \phi_x = f \circ \phi_y = \Phi_T(\phi_y)$. Since Φ_T is injective, it follows that $\phi_x = \phi_y$. Hence, $x = \phi_x(0) = \phi_y(0) = y$. We conclude that f is injective. $\square$

Problem 5. [10 points] Let X, Y be sets and let $f : X \to Y$ be a map. Prove or disprove: f is surjective if and only if for any set T, the "pre-composition with f" map

$$\Psi_T : \mathrm{Maps}(Y, T) \longrightarrow \mathrm{Maps}(X, T), \quad \psi \longmapsto \psi \circ f, \quad \text{is surjective.}$$

Solution. False. Consider the sets $X = \{1, 2\}$, $Y = \{3\}$. We will prove that for set $T = X$, Ψ_T is not surjective. Define the function $f : X \to Y$ by $f(x) = 3$. Since $f(X) = \{3\} = Y$, so f is surjective. For any $\psi \in \mathrm{Maps}(Y, T)$, note that $\psi(f(1)) = \psi(3) = \psi(f(2))$ but $1 \neq 2$. Hence, $\psi \circ f$ is not injective, and is therefore not the identity function, id_X. This proves that $\mathrm{id}_X \notin \mathrm{Range}(\Psi_T)$. Since, $T = X$, $\mathrm{id}_X \in \mathrm{Maps}(X, T)$ hence, Ψ_T is not surjective. We conclude that the forward direction does not hold, hence the statement is false. $\square$