MA1101R/Linear Algebra I/Semester 1, AY 2015-2016

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1 Question 1 [12 marks]

Let $A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 5 & 0 \end{pmatrix}$

i) Use the Gauss-Jordan Elimination to reduce A to the reduced row-echelon form. (Indicate the elementary row operations used in each step)

ii) Let $T : R^6 \to R^4$ be a linear transformation such that A is the standard matrix for T. Write down a basis for the kernel of T and a basis for the range of T.

Solution:

i)

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 5 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 5 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 4 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 4 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

ii) A basis for ker(T) is equivalent to a basis for null(A). To find the nullspace, we set the 2nd, 5th and 6th columns of the RREF in part i to free parameters r, s and t.

$$x_2 = r \Rightarrow x_1 - r = 0 \Rightarrow x_1 = r$$
$$x_5 = s \Rightarrow x_3 - s = 0 \Rightarrow x_3 = s$$
$$x_5 = s \Rightarrow x_4 + 2s = 0 \Rightarrow x_4 = -2s$$

and so we have $x = \begin{pmatrix} r \\ r \\ s \\ -2s \\ s \\ t \end{pmatrix} = r \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Thus we have $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ as a basis for ker(T).

We know that a basis for R(T) is equivalent to a basis for col(A). To find a basis for col(A), we pick out the pivot columns of RREF(A), which are the 1st, 3rd and 4th columns. Since linear independence of columns is invariant under row operations, it follows that the corresponding columns of A will form a basis for col(A), that is: $\left\{ \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix} \right\}$ is a basis for R(T).

2 Question 2 [12 marks]

Let $S = \{u_1, u_2, u_3\}$ and $T = \{v_1, v_2, v_3\}$ be two bases for R^3.

Suppose $P = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ is the transition matrix from S to T.

i) Find the transition matrix from T to S.

ii) Suppose $u_1 = (1, 1, 1)$, $u_2 = (0, 1, 1)$, $u_3 = (0, 0, 1)$. Find v_1, v_2 and v_3.

Solution:

i) The transition matrix from T to S is simply P^{-1}. Compute P^{-1} by Gauss-Jordan elimination.

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -3 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 3 & -2 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 & -3 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 3 & -2 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 & 3 & -1 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} -1 & 3 & -2 \\ 1 & -2 & 1 \\ -1 & 3 & -1 \end{pmatrix}$$

ii)

$$v_1 = \begin{pmatrix} -1 & 3 & -2 \\ 1 & -2 & 1 \\ -1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_T = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}_S = -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} -1 & 3 & -2 \\ 1 & -2 & 1 \\ -1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_T = \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}_S = 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} -1 & 3 & -2 \\ 1 & -2 & 1 \\ -1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_T = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}_S = -2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}$$

3 Question 3 [21 marks]

Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$, where a is a constant.

i) Find all the eigenvalues of A.
ii) For each of the eigenvalues λ of A, find a basis for the eigenspace associated with λ.
iii) Determine the value of a so that A is diagonalizable.
iv) When A is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Solution:

i) $det(\lambda I - A) = det\begin{pmatrix} \lambda-2 & 0 & -1 \\ -1 & \lambda-1 & -a \\ 0 & 0 & \lambda-1 \end{pmatrix} = (\lambda-1)^2(\lambda-2)$, by expanding along the third row
The eigenvalues are 1 and 2.

ii) $\lambda = 1$:

$$\left(\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ -1 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & a-1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

if $a \neq 1$, we have

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & a-1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Setting $x_2 = t$, and with $x_1 = 0$ and $x_3 = 0$, we have $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t$, and so $E_1 = span\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$
If $a = 1$, we have

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & a-1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Setting $x_2 = s$ and $x_3 = t \Rightarrow x_1 = -t$, and we have $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t$, and so $E_1 = span\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.
$\lambda = 2$:

$$\left(\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ -1 & 1 & -a & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Setting $x_2 = t$, we have $x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} t$, and so $E_2 = span\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.
iii) For A to be diagonalizable, we need 3 eigenvectors. From ii) we see that for this to happen $a = 1$.

iv) $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, by arranging the respective eigenvectors in columns, $P = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

4 Question 4 [21 marks]

a) Let $V = span\{\{u_1, u_2, u_3\}$ where

$$u_1 = (1, 1, 0, 0) \quad u_2 = (0, 2, 1, 1) \quad u_3 = (1, 1, 3, 1)$$

i) Use the Gram-Schmidt Process to transform $\{u_1, u_2, u_3\}$ to an orthogonal basis for V.

ii)Find the projection of $w = (1,0,0,1)$ onto V.

b) Let W be a subspace of R^n and $W^{\perp} = \{w \in R^n | \text{w is orthogonal to W}\}$. Prove that $dim(W)+dim(W^{\perp}) = n$.

Solution:

ai) $v_1 = u_1 = (1,1,0,0)$
$v_2 = u_2 - \frac{u_2 \cdot v_1}{||v_1|| \cdot ||v_1||} v_1 = (0,2,1,1) - \frac{(0,2,1,1)\cdot(1,1,0,0)}{(1,1,0,0)\cdot(1,1,0,0)}(1,1,0,0) = (0,2,1,1) - (1,1,0,0) = (-1,1,1,1)$

$v_3 = u_3 - \frac{u_3 \cdot v_2}{||v_2|| \cdot ||v_2||} v_2 - \frac{u_3 \cdot v_1}{||v_1|| \cdot ||v_1||} v_1 = (1,1,3,1) - \frac{(1,1,3,1)\cdot(-1,1,1,1)}{||(-1,1,1,1)|| \cdot ||(-1,1,1,1)||}(-1,1,1,1) - \frac{(1,1,3,1)\cdot(1,1,0,0)}{||(1,1,0,0)|| \cdot ||(1,1,0,0)||}(1,1,0,0) = (1,1,3,1) - (-1,1,1,1) - (1,1,0,0) = (1,-1,2,0)$

$\{(1,1,0,0), (-1,1,1,1), (1,-1,2,0)\}$ is an orthogonal basis.

aii)

$$\begin{aligned}
proj(w) &= \frac{w \cdot v_1}{||v_1|| \cdot ||v_1||} v_1 + \frac{w \cdot v_2}{||v_2|| \cdot ||v_2||} v_2 + \frac{w \cdot v_3}{||v_3|| \cdot ||v_3||} v_3 \\
&= \frac{(1,0,0,1)\cdot(1,1,0,0)}{||(1,1,0,0)|| \cdot ||(1,1,0,0)||}(1,1,0,0) + \frac{(1,0,0,1)\cdot(-1,1,1,1)}{||(-1,1,1,1)|| \cdot ||(-1,1,1,1)||}(-1,1,1,1) \\
&\quad + \frac{(1,0,0,1)\cdot(1,-1,2,0)}{||(1,-1,2,0)|| \cdot ||(1,-1,2,0)||}(1,-1,2,0) \\
&= (\frac{1}{2}, \frac{1}{2}, 0, 0) + (\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}, 0) \\
&= (\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0)
\end{aligned}$$

b) The "obvious" thing to do is to decompose every vector in R^n into its projection onto W and a respective complement. To be precise, every $v \in R^n$ can be written as $v = (v - proj_w(v)) + proj_w(v)$. One can easily verify that the first term is in $W^{\perp}$ and the second term is in W. Thus we have $W + W^{\perp} = R^n$.

Let $\mathcal{A}$ be a basis for W and $\mathcal{B}$ be a basis for $W^{\perp}$. We know that orthogonal vectors are linearly independent, thus $\mathcal{A} \cup \mathcal{B}$ is a linearly independent set of vectors. Together with the fact that $W + W^{\perp} = R^n$, $\mathcal{A} \cup \mathcal{B}$ is a basis for R^n. Counting the number of vectors in the respective bases we conclude that $dim(W) + dim(W^{\perp}) = dim(R^n) = n$.

Alternatively, choose a basis for W and arrange them in rows to form a matrix X. A basis for $W\perp$ is exactly a basis for null(X). The result follows from the rank-nullity theorem applied to X.

5 Question 5 [17 marks]

(All vectors in this question are written as column vectors.)

Let A be an $n \times n$ matrix such that $A^n = 0$. Suppose there exists a nonzero vector $v \in R^n$ such that $A^{n-1}v \neq 0$.

a) Give an example of a 2×2 matrix A such that $A \neq 0$ but $A^2 = 0$.

b) Prove that $\{v, Av, \ldots, A^{n-1}v\}$ is a basis for R^n.

c) Let $P = \begin{pmatrix} A^{n-1}v & \ldots & Av & v \end{pmatrix}$ which is an invertible matrix of order n.

Show that

$$P^{-1}AP = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

Solution:

a) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
This is also known as a nilpotent matrix.

b) Since the set $\{v, Av, \dots, A^{n-1}v\}$ consists of $n = dim(R^n)$ vectors, for it to form a basis, it suffices to show that they are linearly independent. Consider the equation $\mu_0 v + \mu_1 Av + \dots \mu_{n-1}A^{n-1}v = 0$ (*). Multplying both sides by A^{n-1}, we get

$$\mu_0 A^{n-1}v + \mu_1 A^n v + \dots \mu_{n-1}A^{2n-2}v = A^{n-1}0 \Rightarrow \mu_0 A^{n-1}v = 0$$

Since $A^{n-1} \neq 0$, we have $\mu_0 = 0$, and we have reduced (*) to $\mu_1 Av + \dots \mu_{n-1}A^{n-1}v = 0$.

In general, for the equation $\sum_{i=k}^{n-1} \mu_i A^i v = 0$, multiplying both sides by A^{n-k-1} will kill all terms larger than k, leaving us with $\mu_k A^k v = 0$. Since $A^k v \neq 0$, $\mu_k = 0$.

By induction, we can show that $\mu_i = 0$, where $0 \leq i \leq n-1$, demonstrating linear indepedence of $\{v, Av, \dots, A^{n-1}v\}$.

c) Let $N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & e_1 & e_2 & \dots & e_{n-1} \end{pmatrix}$

Then letting P_i denote the ith column of P, we have

$$\begin{aligned} PN &= \begin{pmatrix} 0 & P_1 & P_2 & \dots & P_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} A^n v & A^{n-1}v & A^{n-2}v & \dots & Av \end{pmatrix} \\ &= A\begin{pmatrix} A^{n-1}v & A^{n-1}v & \dots & v \end{pmatrix} \\ &= AP \end{aligned}$$

Pre-multiplying by P^{-1} on both sides we obtain $N = P^{-1}AP$.

6 Question 6 [17 marks]

(All vectors in this question are written as column vectors.)
Let A be an invertible matrix of order n such that for any nonzero vectors $u, v \in R^n$, the angle between u and v is always equal to the angle between Au and Av.

a) Let $A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$ where a_i is the ith column of A. Show that $a_1, a_2, \dots, a_n$ is an orthogonal basis for R^n.
(Hint: Use the standard basis $E = e_1, e_2, \dots, e_n$ and consider vectors Ae_i for $i = 1, 2, \dots, n$.)

b) Prove that $A = cP$ for some scalar c and orthogonal matrix P.

Solution:

a) Since A is an angle-preserving transformation, we have

$$\frac{u \cdot v}{||u|| \cdot ||v||} = \frac{Au \cdot Av}{||Au|| \cdot ||Av||} \tag{1}$$

and so

$$\begin{aligned} a_i \cdot a_j &= Ae_i \cdot Ae_j \\ &= e_i \cdot e_j \frac{||Ae_i|| \cdot ||Ae_j||}{||e_i|| \cdot ||e_j||} \\ &= 0 \quad \text{for all } 1 \leq i, j \leq n \end{aligned}$$

which shows that $A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$ is an orthogonal set of vectors. Since A is invertible, linear independence of standard basis vectors are preserved, and we can conclude that it the Ae_i's indeed form a basis.

b) From (1), we set $u = e_i$ and $v = e_i + e_j$, and we obtain

$$\begin{aligned} \frac{e_i \cdot (e_i + e_j)}{||e_i|| \cdot ||e_i + e_j||} = \frac{Ae_i \cdot (Ae_i + Ae_j)}{||Ae_i|| \cdot ||Ae_i + Ae_j||} \quad &\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{||a_i||}{||a_i + a_j||} \\ &\Rightarrow \quad \frac{a_i \cdot a_i}{(a_i + a_j) \cdot (a_i + a_j)} = \frac{1}{2} \\ &\Rightarrow \quad 2a_i \cdot a_i = a_i \cdot a_i + a_j \cdot a_j \\ &\Rightarrow \quad a_i \cdot a_i = a_j \cdot a_j \end{aligned}$$

and so all column vectors of A has the are equal in norm. This means we can set $c = \sqrt{a_i \cdot a_i}$. Multiply all entries of A by $\frac{1}{c}$ and we normalize all a_i's, obtaining the orthogonal matrix P.