NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Zhuang Linjie

MA2213 Numerical Analysis 1
AY 2008/2009 Sem 2

Question 1

(a) $\max\{|a_{11}|, |a_{21}|\} = \{|2.000|, |3.000|\} = |3.000| = |a_{21}|$
Exchange row 1 and row 2,

$$\begin{aligned} a_{22} &= 0.6525 + \frac{2.000}{3.000} \times 4.000 \\ &= 0.6525 + 0.6667 \times 4.000 \\ &= 0.6525 + 2.667 \\ &= 3.320 \end{aligned}$$

$$\begin{aligned} b_2 &= 5.200 - \frac{2.000}{3.000} \times 3.000 \\ &= 5.200 - 0.6667 \times 3.000 \\ &= 5.200 - 2.000 \\ &= 3.200 \end{aligned}$$

Hence,

$$\begin{aligned} y &= \frac{3.200}{3.320} \\ &= 0.9639 \end{aligned}$$

$$\begin{aligned} x &= \frac{3.000 + 0.9639 \times 4.000}{3.000} \\ &= \frac{3.000 + 3.856}{3.000} \\ &= \frac{6.856}{3.000} \\ &= 2.285 \end{aligned}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2.285 \\ 0.9639 \end{pmatrix}.$$

(b)

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & 3 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Question 2

(a) define $f(x) = x^3 - 40$. $f'(x) = 3x^2$ and apply Newton's method
$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 3 - \frac{3^3-40}{3\times 3^2} = 3 - \frac{-13}{27} = \frac{94}{27} = 3.481.$

(b) Newton's method has second order of convergence, therefore

$$\frac{|p_2 - p|}{|p_1 - p|^2} \approx \frac{|p_3 - p|}{|p_2 - p|^2}, |p_3 - p| \approx 3.31 \times 10^{-7}$$

r_2 is most likely to be the value $|p_3 - p|$.

Question 3

(a)

$$T_1 = \int_0^2 \frac{1}{x+2}dx = \frac{2}{2}\left[\frac{1}{0+2} + \frac{1}{2+2}\right] = 0.75.$$

$$T_2 = \int_0^2 \frac{1}{x+2}dx = \frac{1}{2}\left[\frac{1}{0+2} + \frac{2}{1+2} + \frac{1}{2+2}\right] = 0.7083.$$

$$T_4 = \int_0^2 \frac{1}{x+2}dx = \frac{1}{4}\left[\frac{1}{0+2} + \frac{2}{\frac{1}{2}+2} + \frac{2}{1+2} + \frac{2}{\frac{3}{2}+2} + \frac{1}{2+2}\right] = 0.6970.$$

(b) $R_{1,1} = 0.75$, $R_{2,1} = 0.7083$, $R_{3,1} = 0.6970$
$R_{2,2} = R_{2,1} + \frac{R_{2,1}-R_{1,1}}{4^1-1} = 0.6944$
$R_{3,2} = R_{3,1} + \frac{R_{3,1}-R_{2,1}}{4^1-1} = 0.6932$
$R_{3,3} = R_{3,2} + \frac{R_{3,2}-R_{2,2}}{4^2-1} = 0.6931.$

Question 4

(a) Let $g(x) = F(x) - f(x) = 0.0023x^2 + xe^{-x}$
$g(x=0) = 0, g(x=1) = 0.3702, g(x=2) = 0.2799$

$$\begin{aligned} P_g(x) &= g(x_0)L_{2,0}(x) + g(x_1)L_{2,1}(x) + g(x_2)L_{2,2}(x) \\ &= 0.3702\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + 0.2799\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\ &= -0.3702x(x-2) + \frac{0.2799}{2}x(x-1) = -0.23025x^2 + 0.60045x \end{aligned}$$

The polynomial that interpolates the function $F(x)$ is

$$0.2426x^2 - 0.8344x + 1.0000 - 0.23025x^2 + 0.60045x = 0.01235x^2 - 0.23395x + 1.0000$$

(b) $y = \frac{1}{\sqrt{ax+b}} \Rightarrow y^2 = \frac{1}{ax+b} \Rightarrow \frac{1}{y^2} = ax + b$
Let $\frac{1}{y^2} = z, z(x=0) = 1, z(x=0.5) = 1.4172, z(x=1.5) = 2.1626, m = 3.$

$$a = \frac{m\sum_{i=1}^m x_i z_i - \sum_{i=1}^m x_i \sum_{i=1}^m z_i}{m\sum_{i=1}^m x_i^2 - (\sum_{i=1}^m x_i)^2}, b = \frac{\sum_{i=1}^m x_i^2 \sum_{i=1}^m z_i - \sum_{i=1}^m x_i z_i \sum_{i=1}^m x_i}{m\sum_{i=1}^m x_i^2 - (\sum_{i=1}^m x_i)^2}.$$

$\sum_{i=1}^m x_i = 2, \sum_{i=1}^m x_i^2 = 2.5, \sum_{i=1}^m x_i z_i = 3.9525, \sum_{i=1}^m z_i = 4.5798.$
$a = 0.7708, b = 1.01271.$

Question 5

(a) Divide $[a, b]$ into 96 subintervals.
$T(24) = \frac{x_{96}-x_0}{48}[f(x_0) + 2f(x_4) + 2f(x_8) + \ldots + 2f(x_{92}) + f(x_{96})] = 0.80326$
$T(48) = \frac{x_{96}-x_0}{96}[f(x_0) + 2f(x_2) + 2f(x_4) + \ldots + 2f(x_{92}) + 2f(x_{94}) + f(x_{96})] = 0.80440$
$T(96) = \frac{x_{96}-x_0}{192}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{94}) + 2f(x_{95}) + f(x_{96})] = 0.80468$
$S(48) = \frac{x_{96}-x_0}{144}[f(x_0)+4f(x_2)+2f(x_4)+\ldots+2f(x_{92})+4f(x_{94})+f(x_{96})] = T(48)\times\frac{4}{3} - T(24)\times\frac{1}{3} = 0.80478$
$S(96) = \frac{x_{96}-x_0}{288}[f(x_0)+4f(x_1)+2f(x_2)+\ldots+2f(x_{94})+4f(x_{95})+f(x_{96})] = T(96)\times\frac{4}{3} - T(48)\times\frac{1}{3} = 0.80477.$

(b) $f(x) = 1$

$$\int_0^1 f(x)x^2dx = \int_0^1 x^2dx = \frac{1}{3} = Q(f) = af(0) + bf(\frac{1}{2}) + cf(1) = a + b + c$$

$f(x) = x$

$$\int_0^1 f(x)x^2dx = \int_0^1 x^3dx = \frac{1}{4} = Q(f) = af(0) + bf(\frac{1}{2}) + cf(1) = \frac{1}{2}b + c$$

$f(x) = x^2$

$$\int_0^1 f(x)x^2dx = \int_0^1 x^4dx = \frac{1}{5} = Q(f) = af(0) + bf(\frac{1}{2}) + cf(1) = \frac{1}{4}b + c$$

$a = \frac{-1}{60}, b = \frac{1}{5}, c = \frac{3}{20}.$

$$\begin{aligned}
\int_{-1}^1 x^2(x+1)^2\sin(x+1)dx &= \int_{-1}^0 x^2(x+1)^2\sin(x+1)dx + \int_0^1 x^2(x+1)^2\sin(x+1)dx \\
&= \int_0^1 (u-1)^2u^2\sin(u)du + \int_0^1 x^2(x+1)^2\sin(x+1)dx \\
&= a(0-1)^2\sin(0) + b(\frac{1}{2}-1)^2\sin\left(\frac{1}{2}\right) + c(1-1)^2\sin(1) \\
&+a(0+1)^2\sin(0+1) + b\left(\frac{1}{2}+1\right)^2\sin\left(\frac{1}{2}+1\right) + c(1+1)^2\sin(1+1) \\
&= \frac{1}{20}\sin\left(\frac{1}{2}\right) - \frac{1}{60}\sin(1) + \frac{9}{20}\sin\left(\frac{3}{2}\right) + \frac{3}{5}\sin(2)
\end{aligned}$$

Question 6

(a) $P_n(x) = f(x_0) + \sum_{k=1}^{n} f[x_0, \dots, x_k](x - x_0)\dots(x - x_{k-1})$
$f(x) - f(x_0) = f[x_0, x](x - x_0)$
(Since$f[x_1, x_0, x] = \frac{f[x_0,x]-f[x_1,x_0]}{x-x_1}$)
$f(x) = (f[x_1, x_0, x](x-x_1)+f[x_1, x_0])(x-x_0)+f(x_0) = f[x_0, x_1, x](x-x_0)(x-x_1)+f[x_0, x_1](x-x_0) + f(x_0)$
(Since$f[x_2, x_0, x_1, x] = \frac{f[x_0,x_1,x]-f[x_2,x_0,x_1]}{x-x_2}$)

$$\begin{aligned} f(x) &= (f[x_2, x_0, x_1, x](x - x_2) + f[x_2, x_0, x_1])(x - x_0)(x - x_1) + f[x_0, x_1](x - x_0) + f(x_0) \\ &= f[x_0, x_1, x_2, x](x - x_0)(x - x_1)(x - x_2) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1](x - x_0) + f(x_0) \\ &= \cdots = f[x_0, x_1, x_2, \cdots x_n, x](x - x_0)(x - x_1)\cdots(x - x_n) \\ &+ f[x_0, x_1, x_2, \cdots x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1}) + \cdots + f[x_0, x_1](x - x_0) + f(x_0) \\ &= P_n(x) + f[x_0, x_1, x_2, \cdots x_n, x](x - x_0)(x - x_1)\cdots(x - x_n). \end{aligned}$$

(b) By Question 6 (i),
$|error| = |f(0) - P_3(0)| = |f[-1, 1, 2, 4, 0](0 + 1)(0 - 1)(0 - 2)(0 - 4)| \leq 0.01 \times 8 = 0.08.$