NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1102R Calculus
AY 2008/2009 Sem 2

Question 1

(a) Check that

$$\lim_{x\to\infty}\left(\frac{x^2+2x+3}{4x+5}\sin\left(\frac{6}{7x}\right)\right)=\lim_{y\to 0}\left(\frac{1+2y+3y^2}{4y+5y^2}\sin\left(\frac{6y}{7}\right)\right)$$

Then by L'Hopital's rule, we have

$$\lim_{y\to 0}\frac{(2+6y)\sin\left(\frac{6y}{7}\right)+\frac{6}{7}\cos\left(\frac{6y}{7}\right)(1+2y+3y^2)}{4+10y}=\frac{6/7}{4}=\frac{3}{14}$$

(b) Applying L'Hopital's rule twice, we have

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)&=\lim_{x\to 0}\frac{e^x-x-1}{x(e^x-1)}\\&=\lim_{x\to 0}\frac{e^x-1}{xe^x+e^x-1}\\&=\lim_{x\to 0}\frac{e^x}{xe^x+2e^x}\\&=\lim_{x\to 0}\frac{1}{x+2}=\frac{1}{2}\end{aligned}$$

(c) Let $f(x):=\left(\frac{2^{\frac{1}{x}}+3^{\frac{1}{x}}+4^{\frac{1}{x}}}{3}\right)^x$ and consider

$$\begin{aligned}\lim_{x\to\infty}\ln f(x)&=\lim_{x\to\infty}x\ln\left(\frac{2^{\frac{1}{x}}+3^{\frac{1}{x}}+4^{\frac{1}{x}}}{3}\right)\\&=\lim_{x\to\infty}\frac{\ln\left(\frac{2^{\frac{1}{x}}+3^{\frac{1}{x}}+4^{\frac{1}{x}}}{3}\right)}{\frac{1}{x}}\end{aligned}$$

Then, by L'Hopital's rule we have

$$\begin{aligned}\lim_{x\to\infty}\frac{\frac{3}{2^{\frac{1}{x}}+3^{\frac{1}{x}}+4^{\frac{1}{x}}}\cdot\frac{1}{3}\left(2^{\frac{1}{x}}\ln 2\cdot-\frac{1}{x^2}+3^{\frac{1}{x}}\ln 3\cdot-\frac{1}{x^2}+4^{\frac{1}{x}}\ln 4\cdot-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}&=\lim_{x\to\infty}\frac{2^{\frac{1}{x}}\ln 2+3^{\frac{1}{x}}\ln 3+4^{\frac{1}{x}}\ln 4}{2^{\frac{1}{x}}+3^{\frac{1}{x}}+4^{\frac{1}{x}}}\\&=\frac{\ln 2+\ln 3+\ln 4}{3}=\frac{\ln 24}{3}\end{aligned}$$

Then,

$$\lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x) = \exp\left(\frac{\ln 24}{3}\right) = 24^{1/3} = 2(3)^{1/3}$$

Question 2

(a) Let $u = \ln x$. Then $du = \frac{dx}{x} \Rightarrow e^u du = dx$. Then we have

$$\int_0^1 x^3(u)(e^u\ du) = \int_0^1 ue^{4u}\ du$$

Using Integration by Parts gives us

$$\begin{aligned}\frac{1}{4}\int_0^1 u\ d(e^{4u}) &= \frac{1}{4}\left((ue^{4u})|_0^1 - \int_0^1 e^{4u}\ du\right)\\ &= \frac{1}{4}\left(e^4 - \frac{1}{4}(e^{4u})|_0^1\right)\\ &= \frac{3e^4+1}{16}\end{aligned}$$

(b) Let $u = \sqrt{x}$. Then $du = \frac{dx}{2\sqrt{x}} \Rightarrow 2udu = dx$. Then we have

$$\int_0^1 \tan^{-1}(\sqrt{x})\ dx = \int_0^1 2u\tan^{-1} u\ du$$

Using Integration by Parts gives us

$$\begin{aligned}\int_0^1 2u\tan^{-1} u\ du &= \int_0^1 \tan^{-1} u\ d(u^2)\\ &= (u^2\tan^{-1} u)|_0^1 - \int_0^1 \frac{u^2}{1+u^2}\ du\\ &= \frac{\pi}{4} - \int_0^1 \frac{u^2}{1+u^2}\ du\end{aligned}$$

Here, let $u = \tan v$. Then $du = \sec^2 v dv$. Hence we have

$$\begin{aligned}\frac{\pi}{4} - \int_0^1 \frac{u^2}{1+u^2}\ du &= \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \frac{\tan^2 v}{\sec^2 v}(\sec^2 v dv)\\ &= \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \tan^2 v\ dv\\ &= \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} (\sec^2 v - 1)\ dv\\ &= \frac{\pi}{4} - (\tan v - v)|_0^{\frac{\pi}{4}}\\ &= \frac{\pi}{4} - \tan(\frac{\pi}{4}) + \frac{\pi}{4}\\ &= \frac{\pi}{2} - 1\end{aligned}$$

(c) We have

$$\begin{aligned}\int_1^4 \frac{x^2+4x+4}{x^2(x^2+4)}\,dx &= \int_1^4 \frac{(x+2)^2}{x^2(x^2+4)}\,dx \\ &= \int_1^4 \left(1+\frac{2}{x}\right)^2 \frac{1}{x^2+4}\,dx\end{aligned}$$

Now let $x = 2\tan u$. Then $dx = 2\sec^2 u du$. This gives us

$$\begin{aligned}\int_1^4 \left(1+\frac{2}{x}\right)^2 \frac{1}{x^2+4}\,dx &= \int_{\tan^{-1}\frac{1}{2}}^{\tan^{-1}2} \left(1+\frac{1}{\tan u}\right)^2 \frac{1}{4\sec^2 udu}(2\sec^2 udu) \\ &= \frac{1}{2}\int_{\tan^{-1}\frac{1}{2}}^{\tan^{-1}2} (1+\cot u)^2\,du \\ &= \frac{1}{2}\int_{\tan^{-1}\frac{1}{2}}^{\tan^{-1}2} (1+2\cot u+\cot^2 u)\,du \\ &= \frac{1}{2}\int_{\tan^{-1}\frac{1}{2}}^{\tan^{-1}2} (\csc^2 u+2\cot u)\,du \\ &= \frac{1}{2}(-\cot u + 2\ln|\sin u|)|_{\tan^{-1}\frac{1}{2}}^{\tan^{-1}2} \\ &= \frac{1}{2}(-\frac{1}{2}+2+2(\ln|\sin(\tan^{-1}2)| - \ln|\sin(\tan^{-1}\frac{1}{2})|)) \\ &= \frac{3}{4} + \ln|\sin(\tan^{-1}2)| - \ln|\sin(\tan^{-1}\frac{1}{2})| \\ &= \frac{3}{4} + \ln\left(\frac{2}{\sqrt{5}}\right) - \ln\left(\frac{1}{\sqrt{5}}\right) \\ &= \frac{3}{4} + \ln 2\end{aligned}$$

Question 3

(a) Consider $f(x) := \frac{1}{(x^2+1)^{1/x}} = \left(\frac{1}{x^2+1}\right)^{\frac{1}{x}}$. Then

$$\begin{aligned}\ln \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{\ln\left(\frac{1}{x^2+1}\right)}{x} \\ &= \lim_{x\to\infty} \frac{(x^2+1)\cdot -\frac{2x}{(x^2+1)^2}}{1} \\ &= \lim_{x\to\infty} -\frac{2x}{x^2+1} = 0\end{aligned}$$

by L'Hopital's Rule. Then we have

$$\lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x) = e^0 = 1$$

Hence, we have

$$\lim_{n\to\infty} (-1)^n f(n) \neq 0$$

which shows that the series diverges by the Divergence Test.

(b) Using the Limit Comparison Test and comparing it with $\frac{1}{n^{3/2}}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{\sqrt{n+1}-\sqrt{n-1}}{n}}{\frac{1}{n^{3/2}}} &= \lim_{n\to\infty} \sqrt{n}(\sqrt{n+1}-\sqrt{n-1}) \\
&= \lim_{n\to\infty} \sqrt{n}\left((\sqrt{n+1}-\sqrt{n-1})\cdot\frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}}\right) \\
&= \lim_{n\to\infty} \sqrt{n}\frac{n+1-(n-1)}{\sqrt{n+1}+\sqrt{n-1}} \\
&= \lim_{n\to\infty} \frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}} \\
&= \lim_{n\to\infty} \frac{2}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}} = \frac{2}{1+1} = 1
\end{aligned}$$

Then we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \quad \text{converges if and only if} \quad \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n} \quad \text{converges}$$

(c) Note that

$$\sqrt{(2k-1)(2k+1)} = \sqrt{4k^2-1} < \sqrt{4k^2} = 2k$$

Observe that

$$\begin{aligned}
\frac{1\cdot 3\cdot 5\cdots(2n-1)}{4\cdot 6\cdot 8\cdots(2n+2)} &= \frac{\sqrt{3}\cdot(\sqrt{3\cdot 5})\cdots(\sqrt{(2n-3)(2n-1)})\cdot\sqrt{2n-1}}{4\cdot 6\cdot 8\cdots(2n+2)} \\
&< \frac{\sqrt{3}\cdot 4\cdot 6\cdot\cdots\cdot(2n-2)\cdot\sqrt{2n-1}}{4\cdot 6\cdot 8\cdots(2n+2)} \\
&< \frac{\sqrt{3}\cdot 4\cdot 6\cdot\cdots\cdot(2n-2)\cdot\sqrt{2n}}{4\cdot 6\cdot 8\cdots(2n+2)} \\
&= \frac{\sqrt{3}}{\sqrt{2n}(2n+2)} \\
&< \frac{\sqrt{3}}{\sqrt{2n}2n} \\
&= \sqrt{\frac{3}{8}}\,\frac{1}{n^{\frac{3}{2}}}
\end{aligned}$$

This is dominated by the p-series where $p = \frac{3}{2}$, which converges. Therefore

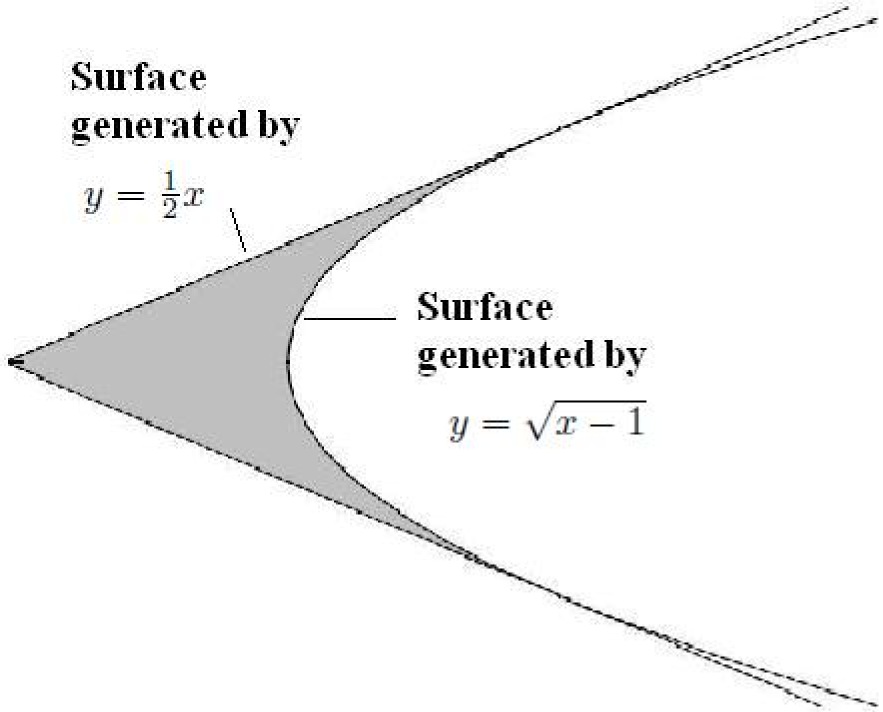
$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{4 \cdot 6 \cdot 8 \cdots (2n+2)}$$

converges.

Question 4

First, we want to find the point of intersection of the two curves.

$$\frac{1}{2}x = \sqrt{x-1} \Rightarrow \frac{1}{4}x^2 - x + 1 = 0 \Rightarrow x = 2$$



So the region of integration will be [0,2].
By observations and visualization of the rotation of the shaded area through the given diagram, the total surface area of solid generated is in fact the surface of the cone,S_1 generated by $y = \frac{1}{2}x$ and surface of the parabola,S_2 generated by $y = \sqrt{x-1}$.
Using the formula

$$A(S) = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

we evaluate $A(S_1)$ and $A(S_2)$.

$$\begin{aligned} A(S_1) &= 2\pi \int_0^2 \frac{1}{2}x\sqrt{1+\frac{1}{4}}\, dx \\ &= \frac{\sqrt{5}}{2}\pi \int_0^2 x\, dx \\ &= \frac{\sqrt{5}}{2}\pi \left[\frac{x^2}{2}\right]_0^2 \\ &= \sqrt{5}\pi \end{aligned}$$

$$
\begin{aligned}
A(S_2) &= 2\pi \int_1^2 \sqrt{x-1}\sqrt{1+\frac{1}{4(x-1)}}\,dx \\
&= 2\pi \int_1^2 \sqrt{x-1+\frac{1}{4}}\,dx \\
&= 2\pi \int_1^2 \sqrt{x-\frac{3}{4}}\,dx \\
&= 2\pi \left[\frac{2}{3}\left(x-\frac{3}{4}\right)^{\frac{3}{2}}\right]_1^2 \\
&= \frac{4}{3}\pi\left(\frac{5\sqrt{5}-1}{8}\right) \\
&= \frac{5\sqrt{5}-1}{6}\pi
\end{aligned}
$$

Therefore

$$A(S) = A(S_1) + A(S_2) = \frac{6\sqrt{5}+3\sqrt{3}+1}{6}\pi$$

Question 5

(a) We have

$$x + y = 1 \Rightarrow y = 1 - x$$

Hence, substituting this into the equation of the curve gives us

$$
\begin{aligned}
1 - x &= px^2 + qx \\
px^2 + (q+1)x - 1 &= 0
\end{aligned}
$$

Since the line is tangent to the curve, the discriminant is 0, i.e.

$$
\begin{aligned}
(q+1)^2 - 4(p)(-1) &= 0 \\
4p + (q+1)^2 &= 0
\end{aligned}
$$

(b) The curve cuts the x-axis at 0 and $-\frac{q}{p}$, and so the area, $A(p,q)$ is given by

$$
\begin{aligned}
A(p,q) &= \int_0^{-\frac{q}{p}} (px^2 + qx)\,dx \\
&= \left(\frac{px^3}{3} + \frac{qx^2}{2}\right)\Big|_0^{-\frac{q}{p}} \\
&= \frac{q^3}{6p^2}
\end{aligned}
$$

Now $p = -\frac{(q+1)^2}{4}$, and so

$$A(q) = \frac{q^3}{6\left(-\frac{(q+1)^2}{4}\right)^2} = \frac{8q^3}{3(q+1)^4}$$

Then we have

$$\begin{aligned} A'(q) &= \frac{3(q+1)^4(24q^2) - 8q^3(12(q+1)^3))}{9(q+1)^8} \\ &= \frac{24q^2(3(q+1) - 4q)}{9(q+1)^5} \\ &= \frac{8q^2(3(q+1) - 4q)}{3(q+1)^5} \\ &= \frac{8q^2(3-q)}{3(q+1)^5} \end{aligned}$$

Now if $A'(q) = 0$, we have $q = 0$ or $q = 3$, but $q > 0$ is given. Then, check that $A'(q) > 0$ if $q < 3$ and $A'(q) < 0$ if $q > 3$. So $q = 3$ is a maximum point. Then, we have $p = -4$.

Question 6

(a) First, note that we can f is one to one is important for us so that $g = f^{-1}$ is in fact well-defined.

We might want to simplify the second part of the integration. Let $x = f(y)$, and thus we have $\frac{dx}{dy} = f'(y)$. Note that also $x = d \Rightarrow y = b$ and $x = c \Rightarrow y = a$. Therefore we have

$$\begin{aligned} \int_c^d g(x)\ dx &= \int_c^d g(g^{-1}(y))\ dx \\ &= \int_a^b y\frac{dx}{dy}\ dy \\ &= \int_a^b yf'(y)\ dy \end{aligned}$$

We perform by parts on the first part, then we can evaluate the given integral

$$\begin{aligned} &\int_a^b f(x)\ dx + \int_c^d g(x)\ dx \\ &= [xf(x)]_a^b - \int_a^b xf'(x)\ dx + \int_a^b yf'(y)\ dy \\ &= bf(b) - af(a) - \int_a^b xf'(x)\ dx + \int_a^b xf'(x)\ dx \\ &= bd - ac \end{aligned}$$

(b) Observe that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $0 < x < 1$. Differentiating both sides gives us

$$\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2} \tag{1}$$

Differentiating it again and rearranging the terms (which is possible because the series is convergent) yields

$$\begin{aligned}\sum_{n=0}^{\infty} n(n-1)x^{n-2} &= \frac{2}{(1-x)^3} \\ \sum_{n=0}^{\infty}(n^2x^{n-2} - nx^{n-2}) &= \frac{2}{(1-x)^3}\end{aligned}$$

By letting $x = 1/3$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} 9\left(\frac{n^2}{3^n} - \frac{n}{3^n}\right) &= \frac{27}{4} \\ \sum_{n=0}^{\infty}\left(\frac{n^2}{3^n} - \frac{n}{3^n}\right) &= \frac{3}{4}\end{aligned}$$

By (1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} n\left(\frac{1}{3}\right)^{n-1} &= \frac{1}{(1-1/3)^2} \\ \sum_{n=0}^{\infty}\frac{n}{3^n} &= \frac{3}{4}\end{aligned}$$

So we have

$$\sum_{n=0}^{\infty}\frac{n^2}{3^n} = \frac{3}{4} + \frac{3}{4} = \frac{3}{2}$$

Question 7

(a) By Mean Value Theorem for $[a, a+\delta]$ for a $\delta > 0$, there exists $c \in (a, a+\delta)$ such that

$$f'(c) = \frac{f(a+\delta) - f(a)}{\delta}$$

Taking limits on both sides gives us

$$\lim_{\delta \to 0} f'(c) = \lim_{\delta \to 0} \frac{f(a+\delta) - f(a)}{\delta}$$

Now $\delta \to 0$ implies $c \to a^+$, and so we have

$$\lim_{x \to a^+} f'(x) = f'(a)$$

Similarly, by Mean Value Theorem for $[a-\delta, a]$ for a $\delta > 0$, there exists $c \in (a-\delta, a)$ such that

$$f'(c) = \frac{f(a) - f(a-\delta)}{\delta}$$

Taking limits on both sides gives us

$$\lim_{\delta \to 0} f'(c) = \lim_{\delta \to 0} \frac{f(a) - f(a-\delta)}{\delta}$$

Now $\delta \to 0$ implies $c \to a^-$, and so we have

$$\lim_{x \to a^-} f'(x) = f'(a)$$

Then note that

$$\lim_{x \to a^-} f'(x) = \lim_{x \to a^+} f'(x) = L = f'(a)$$

and this completes the proof.

(b) By the Extreme Value Theorem, there exists $c_1, c_2 \in [a, b]$ such that c_1 is the global min and c_2 is the global max, i.e.

$$f(c_1) \leq f(x) \leq f(c_2)$$

for all $x \in [a, b]$. Since f is non-constant, we have $f(c_1) \neq f(c_2)$. Now let $c = f(c_1)$ and $d = f(c_2)$. Then the range of f is the closed interval $[c, d]$, as desired.