NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA1101R Linear Algebra I
AY 2012/2013 Sem 2
Version 2: August 30, 2014

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Question 1

(i) Consider the matrix $\boldsymbol{K}$ consisting of the elements of S as the row vectors. We have

$$\boldsymbol{K} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Since the reduced row-echelon form has three pivot columns, we can conclude that the rows are linearly independent.

(ii) Since the three vectors of S are linearly independent, V must have dimension 3.

(iii) We solve the system $\boldsymbol{K}^T\boldsymbol{x} = \begin{pmatrix} 7 \\ -1 \\ 3 \\ -5 \end{pmatrix}$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 7 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

It follows that $\boldsymbol{v} = 3\,(1,0,1,0) - 5\,(0,1,0,1) + 4\,(1,1,0,0)$ and so $(\boldsymbol{v})_S = (3,-5,4)$.

(iv) $\boldsymbol{w} = 2\,(1,0,1,0) + 3\,(0,1,0,1) - 6\,(1,1,0,0) = (-4,-3,2,3)$

(v) By the definition of the transition matrix, we have $[\boldsymbol{w}]_T = \boldsymbol{P}\,[\boldsymbol{w}]_S = (-4,-3,2)^T$

(vi) It is known that the matrix $\boldsymbol{L}$ whose columns are the vectors of T must follow

$$\begin{aligned} \boldsymbol{LP} &= \boldsymbol{K}^T \\ \boldsymbol{L} &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \end{aligned}$$

The vectors of T are thus $(1,0,0,-1)$, $(0,1,0,1)$, and $(0,0,1,1)$ in that order.

(vii) No, it is not possible. Since $U \neq \mathbb{R}^4$, U must have dimension less than 4. Since $V \subset U$, U must have dimension greater than 3 (It is absurd for U to have dimension less than that of its subset. Moreover, if $\dim U = \dim V$ we would have $U = V$, which is not true[1]). This is a contradiction because there is no integer greater than 3 and yet less than 4.

Question 2

(i) Elementary row operations do not change rowspace or nullspace, hence the following conclusions:
A basis for the row space is $\{(1,0,0,1,0), (0,1,0,0,1), (0,0,1,1,1)\}$.
The nullspace is a solution space of $(\boldsymbol{R}|\boldsymbol{0})$, a basis for which is $\{(-1,0,-1,1,0), (0,-1,-1,0,1)\}$.
The relative linear independence of the columns is preserved, so a basis for the column space is $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$.

(ii) $(0,0,0,1,0)$ and $(0,0,0,0,1)$ are sufficient.

(iii) Take

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \end{pmatrix}$$

(The last two rows are just linear combinations of the first three rows.)

(iv) The corresponding vectors in $\boldsymbol{R}$ (namely $(0,0,1,0)$,$(1,0,1,0)$, and $(0,1,1,0)$) are linearly independent. Thus $\boldsymbol{a}_3$, $\boldsymbol{a}_4$, and $\boldsymbol{a}_5$ are linearly independent. Since the column space has dimension 3, the three vectors form a basis.

(v) Yes, it is necessarily true. Premultiplication with an invertible matrix is equivalent to performing a series of elementary row operations. Elementary row operations do not change the row space of a matrix.

(vi) We use the linearity of the transformation:

$$\begin{aligned} T\left(\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{pmatrix}\right) &= T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) + 2T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) + 3T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} 5 \\ 14 \\ 8 \\ 5 \end{pmatrix} \end{aligned}$$

[1] Suppose otherwise, that is, some element $\boldsymbol{u}$ of U is not in V. Then $\boldsymbol{u}$ cannot be expressed as a linear combination of vectors in T, and so T and $\boldsymbol{u}$ taken together should be a basis for a subspace of U, but with dimension 4. This cannot be since we took $\dim U = 3$.

(vii) From $\boldsymbol{R}$ we can deduce that $\boldsymbol{a}_4 = \boldsymbol{a}_1 + \boldsymbol{a}_3$ and $\boldsymbol{a}_5 = \boldsymbol{a}_2 + \boldsymbol{a}_3$. From the given transformations (and the fact that $\boldsymbol{A}$ is the standard matrix for T) we have $\boldsymbol{a}_1 = (2,1,3,2)^T$, $\boldsymbol{a}_2 = (0,5,1,0)^T$,$\boldsymbol{a}_3 = (1,1,1,1)^T$. Thus

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 1 & 3 & 1 \\ 1 & 5 & 1 & 2 & 6 \\ 3 & 1 & 1 & 4 & 2 \\ 2 & 0 & 1 & 3 & 1 \end{pmatrix}$$

and $T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}\right) = \begin{pmatrix} 2x_1 + x_3 + 3x_4 + x_5 \\ x_1 + 5x_2 + x_3 + 2x_4 + 6x_5 \\ 3x_1 + x_2 + x_3 + 4x_4 + 2x_5 \\ 2x_1 + x_3 + 3x_4 + x_5 \end{pmatrix}$

Question 3

(a) (i) We have

$$\begin{aligned} \det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} \\ &= -\left(\lambda^3 - 12\lambda - 16\right) \end{aligned}$$

Factoring, we have $-\left(\lambda^3 - 12\lambda - 16\right) = (4-\lambda)(2+\lambda)^2$, and so $\boldsymbol{A}$ has eigenvalues 4 and -2.

(ii) E_4 is the nullspace of $\boldsymbol{A} - 4\boldsymbol{I} = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}$. We have

$$\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

hence a basis for E_4 is $\{(1,1,2)\}$.

E_{-2} is the nullspace of $\boldsymbol{A} + 2\boldsymbol{I} = \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}$. We have

$$\begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

hence a basis for E_{-2} is $\{(1,1,0),(1,0,-1)\}$.

(iii) Yes, $\boldsymbol{A}$ is diagonalizable because sum of number of basis in E_{-2} and number of basis in E_4 is 3. Indeed,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}^{-1}$$

(iv) Take

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -\sqrt[3]{2} & 0 & 0 \\ 0 & -\sqrt[3]{2} & 0 \\ 0 & 0 & \sqrt[3]{4} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}^{-1}$$

(b) Writing the system as

$$(\boldsymbol{A}|\boldsymbol{b}) = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right)$$

The least square solutions to the system is

$$\begin{aligned}
\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} &= \boldsymbol{A}^T\boldsymbol{b} \\
\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 2 & 3 & 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 1 & 1 \\ 0 & -1 & -3 \end{pmatrix}\boldsymbol{x} &= \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 2 & 3 & 1 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 2 & -1 & 1 \\ -1 & 3 & 7 \\ 1 & 7 & 23 \end{pmatrix}\boldsymbol{x} &= \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \\
\boldsymbol{x} &= s\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2/5 \\ -1/5 \\ 0 \end{pmatrix}, \ s \in \mathbb{R}
\end{aligned}$$

(c) (i) $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{AB}$ with eigenvalue λ, hence $\boldsymbol{ABx} = \lambda\boldsymbol{x}$. It follows that $(\boldsymbol{BA})\,\boldsymbol{Bx} = \boldsymbol{B}(\boldsymbol{ABx}) = \lambda\boldsymbol{Bx} \neq \boldsymbol{0}$, and so $\boldsymbol{Bx}$ is an eigenvector of $\boldsymbol{BA}$ associated with eigenvalue λ.

(ii) *$\boldsymbol{Bx}$ is not necessarily an eigenvector of $\boldsymbol{BA}$.* In particular, $\boldsymbol{Bx}$ is an eigenvector if and only if it is not the zero vector. (If it were the zero vector then by definition it cannot be an eigenvector; on the other hand if it were *not* the zero vector, we just proceed as in the previous question).

Question 4

(a) (i) Since we must have $w = x - y + z$, (w, x, y, z) can be rewritten as $(s_1 - s_2 + s_3, s_1, s_2, s_3) = s_1(1,1,0,0) + s_2(-1,0,1,0) + s_3(1,0,0,1)$. It is easy to see that the vectors are linearly independent. We then have $\{(1,1,0,0), (-1,0,1,0), (1,0,0,1)\}$ as a basis for V.

(ii) Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be the basis above. Take

$\boldsymbol{v}_1 = \boldsymbol{u}_1$
$\quad = (1,1,0,0)$.
$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \boldsymbol{v}_1\frac{\boldsymbol{v}_1\cdot\boldsymbol{u}_2}{\boldsymbol{v}_1\cdot\boldsymbol{v}_1}$
$\quad = \left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right)$.
$\boldsymbol{v}_3 = \boldsymbol{u}_3 - \boldsymbol{v}_1\frac{\boldsymbol{v}_1\cdot\boldsymbol{u}_3}{\boldsymbol{v}_1\cdot\boldsymbol{v}_1} - \boldsymbol{v}_2\frac{\boldsymbol{v}_2\cdot\boldsymbol{u}_3}{\boldsymbol{v}_2\cdot\boldsymbol{v}_2}$
$\quad = \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, 1\right)$

(iii) Write $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ as the rows of a matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Now it is clear that with $\boldsymbol{u}_4 = (0,0,0,1)$, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{u}_4\}$ forms a basis for $\mathbb{R}^4$. To get an orthogonal basis for $\mathbb{R}^4$, we take

$$\begin{aligned}
\boldsymbol{v}_4 &= \boldsymbol{u}_3 - \boldsymbol{v}_1\frac{\boldsymbol{v}_1\cdot\boldsymbol{u}_4}{\boldsymbol{v}_1\cdot\boldsymbol{v}_1} - \boldsymbol{v}_2\frac{\boldsymbol{v}_2\cdot\boldsymbol{u}_4}{\boldsymbol{v}_2\cdot\boldsymbol{v}_2} - \boldsymbol{v}_3\frac{\boldsymbol{v}_3\cdot\boldsymbol{u}_4}{\boldsymbol{v}_3\cdot\boldsymbol{v}_3} \\
&= \left(-\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right)
\end{aligned}$$

so that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ forms an orthogonal basis for $\mathbb{R}^4$.

(iv) Letting $\boldsymbol{w} = (2,-2,2,-2)$, the projection is $\boldsymbol{v}_1 \frac{\boldsymbol{v}_1 \cdot \boldsymbol{w}}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} + \boldsymbol{v}_2 \frac{\boldsymbol{v}_2 \cdot \boldsymbol{w}}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2} + \boldsymbol{v}_3 \frac{\boldsymbol{v}_3 \cdot \boldsymbol{w}}{\boldsymbol{v}_3 \cdot \boldsymbol{v}_3} = \boldsymbol{0}$. This result is somewhat expected because $\boldsymbol{w}$ is parallel to $\boldsymbol{v}_4$, which in turn is orthogonal to the three vectors in our orthogonal basis.

(b) (i) We use the identity

$$\begin{aligned} \boldsymbol{x} \cdot \boldsymbol{y} &= \frac{1}{4}\left[\left(\|\boldsymbol{x}\|^2 + 2\boldsymbol{x} \cdot \boldsymbol{y} + \|\boldsymbol{y}\|^2\right) - \left(\|\boldsymbol{x}\|^2 - 2\boldsymbol{x} \cdot \boldsymbol{y} + \|\boldsymbol{y}\|^2\right)\right] \\ &= \frac{1}{4}\left(\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}-\boldsymbol{y}\|^2\right) \end{aligned}$$

We have

$$\begin{aligned} \boldsymbol{Au} \cdot \boldsymbol{Av} &= \frac{1}{4}\left(\|\boldsymbol{Au}+\boldsymbol{Av}\|^2 - \|\boldsymbol{Au}-\boldsymbol{Av}\|^2\right) \\ &= \frac{1}{4}\left(\|\boldsymbol{A}(\boldsymbol{u}+\boldsymbol{v})\|^2 - \|\boldsymbol{A}(\boldsymbol{u}-\boldsymbol{v})\|^2\right) \\ &= \frac{1}{4}\left(\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2\right) \\ &= \boldsymbol{u} \cdot \boldsymbol{v} \end{aligned}$$

as desired.

(ii) Let $\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \ldots, \hat{\boldsymbol{e}}_n$ be the standard basis vectors for $\mathbb{R}^n$ written as column vectors. Then $\boldsymbol{A}\hat{\boldsymbol{e}}_j$ is the j^{th} column of $\boldsymbol{A}$, and $\hat{\boldsymbol{e}}_i^T \boldsymbol{A}^T$ is the i^{th} row of $\boldsymbol{A}^T$. Now consider of the $n \times n$ matrix $\boldsymbol{M} = \boldsymbol{A}^T\boldsymbol{A}$. The i,j entries m_{ij} can be written as

$$\begin{aligned} m_{ij} &= \left(\hat{\boldsymbol{e}}_i^T \boldsymbol{A}^T\right)(\boldsymbol{A}\hat{\boldsymbol{e}}_j) \\ &= (\boldsymbol{A}\hat{\boldsymbol{e}}_i)^T (\boldsymbol{A}\hat{\boldsymbol{e}}_j) \\ &= (\boldsymbol{A}\hat{\boldsymbol{e}}_i) \cdot (\boldsymbol{A}\hat{\boldsymbol{e}}_j) \\ &= \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j \\ &= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \end{aligned}$$

We conclude that $\boldsymbol{M}$ is the $n \times n$ identity matrix and that $\boldsymbol{A}$ is orthogonal.

(c) (i) Suppose λ is an eigenvalue of $\boldsymbol{A}$. Then $\lambda\boldsymbol{u} = \boldsymbol{Au} = \boldsymbol{A}^2\boldsymbol{u} = \boldsymbol{A}(\boldsymbol{Au}) = \lambda\boldsymbol{Au} = \lambda^2\boldsymbol{u}$, and so λ can only have values 0 or 1. Now we show that n linearly independent eigenvectors can be found.

First consider the case where $\operatorname{rank}(\boldsymbol{A}) = 0$, then $\boldsymbol{A}$ is a zero matrix which is already diagonal, and hence trivially diagonalizable.

Now take the case where $\operatorname{rank}(\boldsymbol{A}) = r$ where $0 < r \leq n$. Consider the column space V of $\boldsymbol{A}$. For any vector $\boldsymbol{v} \in V$, there exists $\boldsymbol{u} \in \mathbb{R}^n$ such that $\boldsymbol{v} = \boldsymbol{Au} \implies (\boldsymbol{A} - \boldsymbol{I})\boldsymbol{v} = \left(\boldsymbol{A}^2 - \boldsymbol{A}\right)\boldsymbol{u} = \boldsymbol{0}$, hence $\boldsymbol{v}$ must be an eigenvector of $\boldsymbol{A}$ associated with eigenvalue 1. Next, consider the nullspace N of $\boldsymbol{A}$. For any vector $\boldsymbol{w} \in N$, it is clear that $\boldsymbol{Aw} = \boldsymbol{0}$ and so $\boldsymbol{w}$ is an eigenvector of $\boldsymbol{A}$ associated with eigenvalue 0. Note that V and N have dimensions r and $n - r$ respectively.

Since the vector spaces V and N are disjoint, we can find $\boldsymbol{v}_1, \boldsymbol{v}_2 \ldots, \boldsymbol{v}_r \in V$ and $\boldsymbol{w}_1, \boldsymbol{w}_2 \ldots, \boldsymbol{w}_r \in N$ such that the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n-r}\}$ is linearly independent. We have hence found a set of n linearly independent eigenvectors of $\boldsymbol{A}$, and can conclude that $\boldsymbol{A}$ is diagonalizable. In particular, $\boldsymbol{A} = \boldsymbol{PDP}^{-1}$ where $\boldsymbol{D}$ has entries

$$d_{ij} = \begin{cases} 1 & 1 \leq i = j \leq r \\ 0 & \text{otherwise} \end{cases}$$

(ii) We write $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ as in the previous question. From the form of $\boldsymbol{D}$, it is clear that $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{D}) = \operatorname{tr}(\boldsymbol{D})$. Since $\boldsymbol{A}$ and $\boldsymbol{D}$ are similar, $\operatorname{tr}(\boldsymbol{D}) = \operatorname{tr}(\boldsymbol{A})$, and so we are done.

END OF SOLUTIONS

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