NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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**MA2101 Linear Algebra II**
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## SECTION A

### Question 1

(a) A **Jordan Block** is a matrix $J_{r_i}(\lambda_i) \in M_{r_i}(\mathbb{F})$ of the form,

$$J_{r_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{bmatrix},$$

where $\lambda_i \in \mathbb{F}$.

The **Jordan Canonical Form** of a matrix $A \in M_m(\mathbb{F})$ where $\mathbb{F}$ is a algebraically closed field, is a matrix $J \in M_m(\mathbb{F})$ of the form,

$$J = \begin{bmatrix} J_{r_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{r_2}(\lambda_2) & & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_{r_m}(\lambda_m) \end{bmatrix},$$

which is similar to $A$.

(b) The **Smith normal form** $S$ of a matrix $M$(need not be square) with entries in a principal ideal domain (PID) is the diagonal matrix obtained by pre-multiplying and post-multiplying by invertible square matrices. Hence the Smith normal form is obtain by performing a series of elementary row and column operations (equivalent to pre-multiplying and post-multiplying by invertible square matrices).

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{C_4 - C_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_4 - R_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = S.$$

The characteristic polynomial of $M$ is $\chi_M = \det(xI_4 - M) = x(x-1)^2(x+1)$.

Consider $I_4 - A = \begin{pmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$.

Since $\mathrm{rk}(I_4 - A) = 3$, by Dimension Theorem for Matrices, we get

$$\dim(E_1) = \mathrm{null}(I_4 - A) = 4 - \mathrm{rk}(I_4 - A) = 1 \neq 2.$$

Thus the Jordan block of eigenvalue 1 is at least of size 2.
Hence we can conclude that the Jordan Canonical Form of $M$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

## Question 2

(a) A **basis** for vector space $V$ is a set of vectors in $V$ that is linearly independent and spans $V$.
The **dimension** of the zero space is 0 and the **dimension** of any non-zero vector space $V$ is the cardinality of any basis for $V$.

(b) Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for $V$ over $\mathbb{C}$

Claim:
$\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\}$ where $v_{n+j} = i(v_j)$ for $j \in \{1, \ldots, n\}$, is a basis for $V$ over $\mathbb{R}$.

Proof:
**Linear Independence**
Let $c_i \in \mathbb{R}$ such that,

$$\begin{aligned} \sum_{i=1}^{2n} c_i v_i &= 0 \\ \sum_{j=1}^{n} (c_j + i c_{n+j}) v_j &= 0 \end{aligned}$$

Since $\{v_1, \ldots, v_n\}$ is a linearly independent set in $V$ over $\mathbb{C}$, $c_j + i c_{n+j} = 0$ for all $j \in \{1, \ldots, n\}$.
Hence we have $c_i = 0$ for all $i \in \{1, \ldots, 2n\}$.

**Spanning**
Let $v \in V$.
Since $v \in \text{span}(\{v_1, v_2, \ldots, v_n\})$ over $\mathbb{C}$, We have $v = \displaystyle\sum_{j=1}^{n} c_j v_j$ for some $c_j \in \mathbb{C}$ for all $j \in \{1, \ldots, n\}$.

Since $\exists\ a_i \in \mathbb{R}$ for all $i \in \{1, \ldots, 2n\}$ such that $a_j + i(a_{n+j}) = c_j$, we have $v = \displaystyle\sum_{j=1}^{n} c_j v_j = \sum_{j=1}^{2n} a_j v_j$.

Hence $\{v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$ is a basis of $V$ over $\mathbb{R}$.
Therefore $V$ over $\mathbb{R}$ is of dimension $2n$.

The converse is not true.
We have $\mathbb{R}^2$ to be a vector space over $\mathbb{R}$ of dimension 2. We notice that $i \in \mathbb{C}$, $(0,1) \in \mathbb{R}^2$, however $i(0,1) = (0,i) \notin \mathbb{R}^2$. Thus $\mathbb{R}^2$ does not satisfy the Axiom on closure of scalar multiplication, and so is not a vector space over $\mathbb{C}$.

## Question 3

(a) Let $W$ be a vector subspace of $V$(vector space of dimension $n$ over $\mathbb{C}$) and let $S \subseteq M_n(\mathbb{C})$.
Then $W$ is called $S$-invariant iff for all $X \in S$, we have $Xw \in W$ for all $w \in W$.

(b) As $\mathbb{C}$ is algebraically closed, the characteristic polynomial of $A$ has at least 1 root, say $\lambda \in \mathbb{C}$. Thus $\lambda$ is an eigenvalue, and so we have the eigenspace of $\lambda$, $E_\lambda \neq \{0\}$. Now let, $w \in E_\lambda$.
Then for any $X \in S$, we have $AXw = XAw = X(\lambda w) = \lambda(Xw)$, i.e. $Xw \in E_\lambda$.
Hence $E_\lambda$ is $S$-invariant.

This give us $E_\lambda = V$, i.e. for all $v \in V$, $Av = \lambda v$. Hence $A = \lambda I$.

**Question 4**

(a) Let $\mathbb{F}$ be a subfield of $\mathbb{C}$.
A **scalar product** on a vector space $V$ over $\mathbb{F}$, is a function $\langle \cdot \mid \cdot \rangle : V \times V \to \mathbb{F}$ such that:-
1.) For all $u, v \in V$, $\langle u \mid v \rangle = \langle v \mid u \rangle$.
2.) For all $u, v, w \in V$ and $r, s \in \mathbb{F}$ we have $\langle ru + sv \mid w \rangle = r\langle u \mid w \rangle + s\langle v \mid w \rangle$.

Note: The definition above may be different from some other sources, but it is the definition given to the student taking this module that semester. Also (4b.) is solvable only under the above definition.

(b) WLOG, let $\langle e_j \mid e_j \rangle = 0$ iff $j \in \{1, 2, \ldots, r\}$, $r \leq n$.
Since $\{e_1, e_2, \ldots, e_n\}$ is an orthogonal basis, we have $\langle e_i \mid e_j \rangle \neq 0$ iff $i = j$ and $i \in \{r+1, \ldots, n\}$.

Claim:
$B = \{e_1, e_2, \ldots, e_r\}$ is a basis for $V_0$.

Proof:
Since $B$ to be linearly independent, it suffice to prove that $\text{span}(B) = V_0$.

Let $v \in \text{span}(B)$, i.e. $\exists\ \lambda_i \in \mathbb{R}$, $i \in \{1, 2, \ldots, r\}$ such that $v = \sum_{i=1}^{r} \lambda_i e_i$. Then for all $x \in V$,

$$\begin{aligned}
\langle v \mid x \rangle &= \left\langle \sum_{i=1}^{r} \lambda_i e_i \mid x \right\rangle \\
&= \sum_{i=1}^{r} \lambda_i \langle e_i \mid x \rangle \\
&= \sum_{i=1}^{r} \lambda_i \langle x \mid e_i \rangle \\
&= \sum_{i=1}^{r} \lambda_i \left\langle \sum_{j=0}^{n} \mu_j e_j \mid e_i \right\rangle, \qquad \text{for some } \mu_j \in \mathbb{R},\ j \in \{1, 2, \ldots, n\}, \\
&= \sum_{i=1}^{r} \sum_{j=1}^{n} \mu_j \lambda_i \langle e_j \mid e_i \rangle = 0.
\end{aligned}$$

Thus $\text{span}(B) \subseteq V_0$.

Now instead let $v \in V_0$. Since $\{e_1, e_2, \ldots, e_n\}$ spans $V$, there exists $\lambda_i \in \mathbb{R}$, $i \in \{1, 2, \ldots, n\}$ such that $v = \sum_{i=1}^{n} \lambda_i e_i$. Now consider for $k \in \{r+1, r+2, \ldots, n\}$, we have,

$$\begin{aligned}
0 = \langle v \mid e_k \rangle &= \left\langle \sum_{i=1}^{n} \lambda_i e_i \mid e_k \right\rangle \\
&= \sum_{i=1}^{n} \lambda_i \langle e_i \mid e_k \rangle \\
&= \lambda_k \langle e_k \mid e_k \rangle
\end{aligned}$$

Since $\langle e_k \mid e_k \rangle \neq 0$, $\lambda_k = 0$ for all $k \in \{r+1, r+2, \ldots, n\}$, and so $v \in \operatorname{span}(B)$, i.e. $V_0 \subseteq \operatorname{span}(B)$.

Hence $V_0 = \operatorname{span}(B)$, which give us $\dim(V_0) = |B| = r$.

## SECTION B

### Question 5

Let $V$ and $W$ be vector spaces over the same field $\mathbb{F}$.
A function $T : V \to W$ is a **linear transformation** iff

$$T(\mu u + \lambda v) = \mu T(u) + \lambda T(v)$$

for all $\mu, \lambda \in \mathbb{F}$ and $u, v \in V$.

Yes, such a basis $B$ exists.
Since $T$ has domain $\mathbb{R}^2$ of dimension 2 over $\mathbb{R}$, $\chi_T(x) \in \mathbb{R}[x]$ is of degree 2. Also as $T$ fixes the line $4x + 5y = 0$, there exists an eigenvalue $\lambda \in \mathbb{R}$. Thus $\chi_T(x) = (x - \lambda)(x - \mu)$ for some $\lambda, \mu \in \mathbb{R}$.

Hence there exist a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is in the Jordan Canonical Form with diagonal entries $\lambda, \mu$. This give us the possible forms to be $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$, $\lambda \neq \mu$.
The only matrix that is not in the form we want is if $[T]_{\mathcal{B}} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.
Since $\lambda \neq \mu$, we have $\begin{pmatrix} \mu & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda - \mu & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda - \mu & 0 \end{pmatrix}^{-1}$, and thus there exists another basis $\mathcal{B}'$ such that $[T]_{\mathcal{B}'} = \begin{pmatrix} \mu & 1 \\ 0 & \lambda \end{pmatrix}$, and we are done.

### Question 6

(a) The empty set is defined to be linearly independent.
A non-empty set $S$ of vectors in $V$ is **linearly independent** iff for any $v_1, \ldots, v_n \in S$, we have $r_1 v_1 + \cdots + r_n v_n = 0 \Rightarrow r_1 = \cdots = r_n = 0$. If a set of vectors is not linearly independent then it is linearly dependent.

(b) Let $z \in \mathbb{C}$, then $z = x + iy$ for some $x, y \in \mathbb{R}$. This give us $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$.
Hence $e^z = u(x, y) + iv(x, y)$ where $u(x, y) = e^x \cos y$ and $v = e^x \sin y$.
Then we have $\frac{\partial u}{\partial x} = e^x \cos y$, $\frac{\partial v}{\partial x} = e^x \sin y$, $\frac{\partial u}{\partial y} = -e^x \sin y$, $\frac{\partial v}{\partial x} = e^x \cos y$.
Since all the partial derivatives are continuous, with $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, the Cauchy-Riemann Equations are satisfied.
It follows from Complex Analysis that $e^z$ is entire and so $\frac{d(e^{nz})}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = ne^{nz}$ for all $n \in \mathbb{Z}^+$.

Let $P_n$ be the statement "$\{e^x, e^{2x}, \ldots, e^{nx}\}$ is a linearly independent set over $\mathbb{C}$", $n \in \mathbb{Z}^+$.
$\{e^x\}$ is trivially linearly independent, and thus $P_1$ is true.

Assume that $P_k$ is true for some $k \in \mathbb{Z}^+$. Consider $P_{k+1}$.
Let $c_1, c_2, \ldots, c_{k+1} \in \mathbb{C}$, such that $\sum_{m=1}^{k+1} c_m e^{mx} = 0$. By applying the linear transformation $(\frac{d}{dx} - (k+1)\mathrm{id})$ to the equation, we get, $\sum_{m=1}^{k} (mc_m - (k+1)c_m)e^{mx} = 0$. Since $\{e^x, \ldots, e^{kx}\}$ is linearly independent. Hence we have $(m - (k+1))c_m = 0$ for all $m \in \{1, \ldots, k\}$. Since $m - (k+1) \neq 0$, we must have $c_m = 0$ for all $m \in \{1, \ldots, k\}$. Putting these values back into the original equation, we get $c_{k+1}e^{(k+1)x} = 0$. Since $e^{(k+1)x}$ is not a zero-function, we get $c_{k+1} = 0$. Therefore $P_{k+1}$ is true.

Hence by Mathematical Induction, $P_n$ is true for all $n \in \mathbb{Z}^+$.