NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA3110 Mathematical Analysis II
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Question 1

(a) $f : [a, b] \to \mathbb{R}$, f' exists on $[a, b]$ and f' is continuous on $[a, b]$.
Since f' is continuous on a closed interval, it is uniformly continuous on the interval as well.
$\therefore \forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that

$$|f'(x) - f'(c)| < \varepsilon \text{ whenever } x, c \in [a, b] \text{ and } 0 < |x - c| < \delta(\varepsilon) \text{ — } (*)$$

. Also, f' exists on $[a, b]$ and hence exists on $[x, c]$ or $[c, x]$ $\forall c, x \in [a, b]$.
By the Mean Value Theorem, $\exists u$ between x and c such that

$$\begin{aligned} f(x) - f(c) &= f'(u)(x - c) \\ \therefore f'(u) &= \frac{f(x) - f(c)}{x - c} \end{aligned}$$

. Since u is between x and c, $|u - c| \le |x - c| < \delta(\varepsilon)$.
Therefore, by $(*)$, $|f'(u) - f'(c)| < \varepsilon$ and $\left| \frac{f(x)-f(c)}{x-c} - f'(c) \right| < \varepsilon$.

(b) (i) Since $\lim_{n\to\infty} \frac{1}{n} = 0$ and g is continuous at 0, by the Sequential Criterion on continuity, $g(0) = \lim_{n\to\infty} g(\frac{1}{n}) = 0$

$\forall n \in \mathbb{N}$, by (2), $\because g(\frac{1}{n+1}) = g(\frac{1}{n}) = 0$, by Rolle's Theorem, $\exists x_n^{(1)} \in (\frac{1}{n+1}, \frac{1}{n})$ such that $g'(x_n^{(1)}) = 0$.
$x_n^{(1)} \to 0$ by the Squeeze Theorem. By the continuity of g' at 0 (due to the infinite differentiability of g) and the Sequential Criterion on continuity, $g'(0) = \lim_{n\to\infty} g'(x_n^{(1)}) = 0$.

By a similar argument, there exists a strictly decreasing sequence $\{x_n^{(2)}\}$ such that $x_n^{(2)} \to 0$ and $g''(x_n^{(2)}) = 0$ $\forall n \in \mathbb{N}$. By the continuity of g'' at 0, $g''(0) = \lim_{n\to\infty} g''(x_n^{(2)}) = 0$.

Repeating the above argument, $g^{(k)}(0) = 0$ $\forall k \ge 0$

(ii) Fix $x \in \mathbb{R}$. By the Taylor's Theorem, $\forall n \in \mathbb{N}$ $\exists$ c_n (depending on x and n) between 0 and x such that

$$\begin{aligned} g(x) &= \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} x^k + \frac{g^{(n+1)}(c_n)}{(n+1)!} x^{n+1} \\ &= \frac{g^{(n+1)}(c_n)}{(n+1)!} x^{n+1} \\ \therefore |g(x)| &\le M \frac{x^{n+1}}{(n+1)!} \qquad \text{(since } |g^{(n+1)}(c_n)| \le M\text{)} \end{aligned}$$

$\because \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, $\frac{x^n}{n!} \to 0$ as $n \to \infty$. Hence $g(x) = 0$.

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Question 2

(a) Let $P = \{0 = x_0 < x_1 < ... < x_n = 1\}$ be a partition of $[0, 1]$.
By the density of rational numbers (and also irrational numbers) in $[0, 1]$, for each $k = 1, ..., n$,

$$\begin{aligned} M_k &= \sup\{f(x) : x \in [x_{k-1}, x_k]\} \qquad (1) \\ &= x_k \\ m_k &= \inf\{f(x) : x \in [x_{k-1}, x_k]\} \qquad (2) \\ &= -x_k \end{aligned}$$

. If (1) does not hold, then $a = \sup\{f(x) : x \in [x_{k-1}, x_k]\} < x_k$, then $\exists\ b \in \mathbb{Q}$ such that $x_{k-1} \leq a < b < x_k$, contradicting a being the supremum. The similar reasoning holds for (2).

Hence,

$$\begin{aligned} U(f, P) &= \sum_{k=1}^{n} M_k(f, P)(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} x_k(x_k - x_{k-1}) \\ &\geq \sum_{k=1}^{n} \frac{x_{k-1} + x_k}{2}(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} \frac{x_k^2 - x_{k-1}^2}{2} \\ &= \frac{1}{2} \\ L(f, P) &= \sum_{k=1}^{n} m_k(f, P)(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} -x_k(x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{n} -\frac{x_{k-1} + x_k}{2}(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} -\frac{x_k^2 - x_{k-1}^2}{2} \\ &= -\frac{1}{2} \end{aligned}$$

. By the definitions of upper integral and lower integral,

$$\begin{aligned} U(f) &= \inf\{U(f, P) : \text{P is a partition of } [0, 1]\} \\ &\geq \frac{1}{2} \\ L(f) &= \sup\{L(f, P) : \text{P is a partition of } [0, 1]\} \\ &\leq -\frac{1}{2} \end{aligned}$$

$\therefore L(f) \neq U(f)$ and thus f is not integrable on $[0, 1]$.

(b) Since the logarithmic function $g(x) = \log(x)$ is differentiable on $(0, \infty)$ and $g'(x) = \frac{1}{x}$, by the Mean Value Theorem, $\forall s, t \in \mathbb{R}$, $\exists$ c between s and t such that $\log(s) - \log(t) = \frac{1}{c}(s - t)$.
Hence $\forall x, y \in [a, b]$, $\exists$ d between $h(x)$ and $h(y)$ such that

$$\begin{aligned} |\log(h(x)) - \log(h(y))| &= \left|\frac{1}{d}(h(x) - h(y))\right| \\ &\leq \left|\frac{1}{m}(h(x) - h(y))\right| \qquad \text{(Since } d \geq h(x) \geq m\text{)} \end{aligned}$$

. Next, let $P = \{0 = x_0 < x_1 < ... < x_n = 1\}$ be a partition of $[a, b]$ and

$$\begin{aligned} \triangle x_k &= x_k - x_{k-1} \\ I_k &= [x_{k-1}, x_k] \\ M_k(h, P) &= \sup\{h(x) : x \in I_k\} \\ m_k(h, P) &= \inf\{h(x) : x \in I_k\} \\ M_k(\ln(h), P) &= \sup\{\ln(h(x)) : x \in I_k\} \\ m_k(\ln(h), P) &= \inf\{\ln(h(x)) : x \in I_k\} \end{aligned}$$

. Then,

$$\begin{aligned} M_k(\ln(h), P) - m_k(\ln(h), P) &= \sup\{\ln(h(x)) - \ln(h(y)) : x, y \in I_k\} \\ &= \sup\{|\ln(h(x)) - \ln(h(y))| : x, y \in I_k\} \\ &\leq \frac{1}{m}\sup\{|h(x) - h(y)| : x, y \in I_k\} \\ &= \frac{1}{m}(M_k(h, P) - m_k(h, P)) \end{aligned}$$

.
Then $\forall \varepsilon$ since h is integrable on $[a, b]$, by the Riemann Integrability Criterion, $\exists$ a partition, P, of $[a, b]$ such that

$$\begin{aligned} U(h, P) - L(h, P) &< m\varepsilon \\ \therefore U(\ln(h), P) - L(\ln(h), P) &= \sum_{k=1}^{n}(M_k(\ln(h), P) - m_k(\ln(h), P))\triangle x_k \\ &\leq \frac{1}{m}\sum_{k=1}^{n}(M_k(h, P) - m_k(h, P))\triangle x_k \\ &= \frac{1}{m}(U(h, P) - L(h, P)) \\ &< \varepsilon \end{aligned}$$

. Therefore by the Riemann Integrability Criterion, $\ln(h)$ is integrable on $[a, b]$.

Question 3

(a) Let $g : [0, 1] \to \mathbb{R}$ be continuous on $[0, 1]$ and $g(1) = 0$. For every $n \in \mathbb{N}$, define

$$g_n(x) = g(x)x^n \ , \ x \in [0, 1]$$

.

(i) For $0 \leq x < 1$, $x^n \to 0$ as $n \to \infty$, hence

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} g(x)x^n = g(x) \lim_{n\to\infty} x^n = 0$$

. Next, we also have $g(1)1^n = g_n(1) = 0$, therefore

$$\lim_{n\to\infty} g_n(0) = \lim_{n\to\infty} g_n(1) = 0$$

. Hence, g_n converges pointwise to 0 on $[0, 1]$.

(ii) Let $M = \sup_{x\in[0,1]} g(x)$.
If $M = 0$, then $g(x)$ is the constant zero function and $g_n(x)$ is the zero function sequence and thus the uniform convergence is obvious.

If $M > 0$, since $\lim_{x\to 1^-} g(x) = g(1) = 0$ (due to the continuity of g on $[0, 1]$),$\forall \varepsilon > 0$, $\exists$ $\delta \in (0, 1)$ such that

$$|g(x)| < \varepsilon \ \forall x \in (\delta, 1)$$

. Since $\delta < 1$, $\lim_{n\to\infty} \delta^n = 0$. Hence $\exists$ $N \in \mathbb{N}$ such that

$$\delta^n < \frac{\varepsilon}{M} \ \forall n \geq N$$

. Then $\forall n \in \mathbb{N}$ and $\forall x \in [0, 1]$,
if $x \in (\delta, 1]$, then

$$|g_n(x)| = |g(x)x^n| \leq |g(x)| < \varepsilon$$

. If $x \in [0, \delta]$, then

$$|g_n(x)| = |g(x)x^n| \leq |g(x)\delta^n| < M \cdot \frac{\varepsilon}{M} = \varepsilon$$

. Hence $\{g_n\}$ converges uniformly to 0 on $[0, 1]$.

(b) (i) For a given $x \in \mathbb{R}$, $|f_n(x)| = \frac{x^2}{(x^2+n^2)^2} \leq \frac{x^2}{n^2}$ $\forall n \in \mathbb{N}$
$\because \sum \frac{1}{n^2}$ converges, by the Comparison test, $\sum_{n=1}^{\infty} f_n$ converges for every $x \in \mathbb{R}$.

(ii) $\sup_{x\in\mathbb{R}} |f_n(x)| \geq |f_n(\frac{1}{n})| = \frac{n^2}{n^2+n^2} = \frac{1}{2}$
$\therefore$ $\{f_n\}$ does not converge uniformly to 0 on $\mathbb{R}$.
$\therefore \sum_{n=1}^{\infty} f_n$ does not converge uniformly on $\mathbb{R}$.

(iii) Fix $r > 0$. Then $\forall x \in [-r, r]$ $\forall n \in \mathbb{N}$ $|f_n(x)| = \frac{x^2}{x^2+n^2} \leq \frac{r^2}{n^2}$
$\because \sum \frac{1}{n^2}$ converges, by the Weierstrass M-test, $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[-r, r]$.

(iv) For every $n \in \mathbb{N}$, $f_n'(x) = (-1)^n \frac{2xn^2}{(x^2+n^2)^2}$, $x \in \mathbb{R}$.

Fix $r > 0$. Then $\forall x \in [-r, r]$, $|f_n'(x)| = \frac{2|x|n^2}{(x^2+n^2)^2} \leq \frac{2rn^2}{(n^2)^2} = \frac{2rn^2}{n^4} = \frac{2r}{n^2}$
$\because \sum \frac{1}{n^2}$ converges, by the Weierstrass M-test, $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[-r, r]$ — ($\blacktriangle$).

By the theorem on differentiation of series of functions, we have
$\{f_n\}$ is a sequence of differentiable functions on $[-r, r]$ such that $\sum_{n=1}^{\infty} f_n$ and $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[-r, r]$) by (iii) and ($\blacktriangle$)
$\therefore$ $f(x)$ is differentiable on $[-r, r]$ and $\forall x \in [-r, r]$.

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} f_n'(x) \\ &= \sum_{n-1}^{\infty} \frac{(-1)^n 2xn^2}{(x^2+n^2)^2} \end{aligned} \tag{3}$$

$\because$ (3) is valid for every $r > 0$, it is valid for every $x \in (-\infty, \infty)$.

Question 4

(a) (i) Let $a_n = (-1)^n \frac{n+1}{2^{3n}n^2}$.
Note that $\sum_{n=1}^{\infty} a_n x^{3n+1} = x\sum_{n=1}^{\infty} a_n x^{3n}$, hence $\sum_{n=1}^{\infty} a_n x^{3n+1}$ and $\sum_{n=1}^{\infty} a_n x^{3n}$ have the same radii of convergence.
Let $\sum_{n=1}^{\infty} a_n x^{3n} = \sum_{n=1}^{\infty} a_n y^n$ where $y = x^3$.
Then we have

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{n+1+1}{2^{3n+3}(n+1)^2} \cdot \frac{2^{3n}n^2}{n+1} \\ &= \lim_{n\to\infty} \frac{1}{2^3} \frac{(n+2)n^2}{(n+1)^3} \\ &= \frac{1}{8} \end{aligned}$$

. The radius of convergence of $\sum a_n y^n$ is 8.
$\therefore$ The radius of convergence of $\sum a_n x^{3n+1}$ is 2. $\therefore$ The series converges for x such that $|x| < 2$.
For $x = -2$, the series become $\sum_{n=1}^{\infty} \frac{-2(n+1)}{n^2}$.
Since $\frac{n+1}{n^2} > \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges, the series diverges by the Comparison test.
For $x = 2$, the series become $\sum_{n=1}^{\infty}(-1)^n \frac{2(n+1)}{n^2}$.
Since $\lim_{n\to\infty} \frac{2(n+1)}{n^2} = 0$ and $b_n = \frac{2(n+1)}{n^2}$ is a decreasing sequence, the series converges by the alternating series test. $\therefore$ The series converges on $(-2, 2]$.

(ii) Let $a_n = \frac{1}{(3+(-1)^n)^n}$.

$$|a_n|^{\frac{1}{n}} = \frac{1}{(3+(-1)^n)} = \begin{cases} \frac{1}{2} & \text{if } n \text{ is odd} \\ \frac{1}{4} & \text{if } n \text{ is even} \end{cases}$$

. Therefore, the radius of convergence of the power series is

$$\begin{aligned} R &= \frac{1}{\overline{\lim}_{n\to\infty}|a_n|^{\frac{1}{n}}} \\ &= 2 \end{aligned}$$

. $\therefore$ The series converges for x such that $|x-2| < 2$.
For $x - 2 = -2$, the series become $\sum_{n=1}^{\infty} \frac{1}{(3+(-1)^n)^n}(-2)^n$.
The odd terms of the series is -1, hence the series do not converge by the nth-term divergence test.
Similarly, the series do not converge for $x - 2 = 2$.
$\therefore$ The series converges on $(0, 4)$.

(b) Let $a_n = \frac{(-1)^n}{n(n+1)} \neq 0 \; \forall n \in \mathbb{N}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)(n+2)}{n(n+1)}\right| \\
&= 1 \\
\therefore \text{Radius of convergence} &= \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} \\
&= 1
\end{aligned}$$

. $\therefore$ The series converges on (-1,1).
Then,

$$S'(x) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

. Since $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$, $x \in (-1,1)$.
Integrating both sides from 0 to x, we have

$$\begin{aligned}
\int_0^x \frac{1}{1+t}dt &= \sum_{n=0}^{\infty}\int_0^x (-1)^n x^n dt \\
\log(1+x) &= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{n+1}}{n+1} \\
&= S'(x) \\
S(x) &= \int_0^x \ln(1+t)dt \\
&= (1+x)\ln(1+x) - x
\end{aligned}$$

$\sum_{n=0}^{\infty}(-1)^n\frac{1}{n(n+1)}$ converges by the alternating series test and $\lim_{x\to 1^-} S(x) = S(1) = 2\ln 2 - 1 \because$ S is continuous at 1.
$\therefore$ By the Abel's Theorem,

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n(n+1)} &= \lim_{x\to 1^-} S(x) \\
&= 2\ln 2 - 1
\end{aligned}$$

.

(c) The power series representation of e^x about 0 is

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

. $\therefore$ With the uniqueness of power series representation,

$$\begin{aligned}
f(x) &= e^{x^5} \\
&= \sum_{n=0}^{\infty}\frac{(x^5)^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{x^{5n}}{n!}
\end{aligned}$$

. This is the Maclaurin series representation of f. Therefore,

$$\begin{aligned}
\frac{f^{(2005)}(0)}{2005!} &= \text{coefficient of } x^{2005} \\
&= \frac{1}{401!} \ (\because 2005 = 5(401)) \\
\frac{f^{(2006)}(0)}{2006!} &= \text{coefficient of } x^{2006} \\
&= 0
\end{aligned}$$

. $\therefore f^{(2005)}(0) = \frac{2005!}{401!}$ and $f^{(2006)}(0) = 0$.