NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Teo Wei Hao

MA2101S Linear Algebra II (version S)
AY 2007/2008 Sem 1

Question 1

(a) For $t \in F$, let $f_1(x), f_2(x) \in F[x]$, $\lambda \in F$. We have

$$\begin{aligned}\varepsilon_t(f_1(x) + \lambda f_2(x)) &= \varepsilon_t((f_1 + \lambda f_2)(x)) \\ &= (f_1 + \lambda f_2)(t) \\ &= f_1(t) + \lambda f_2(t) \\ &= \varepsilon_t(f_1(x)) + \lambda \varepsilon_t(f_2(x)).\end{aligned}$$

Therefore ε_t is a linear transformation from $F[x] \to F$, i.e. $\varepsilon_t \in F[x]^*$.

(b) Assume on the contrary that $\{\varepsilon_t \mid t \in F\}$ is linearly dependent subset of $F[x]^*$.
Then there exists $\lambda_i, \mu_i \in F$, $i = 1, 2, \ldots, n$, such that not all λ_i are zero, and

$$\sum_{i=1}^{n} \lambda_i \varepsilon_{\mu_i} = 0_{F[x]^*}.$$

Let $f_{\mu_k} \in F[x]$ such that $f_{\mu_k} = (x - \mu_k)^{-1} \prod_{i=1}^{n}(x - \mu_i)$, $k = 1, 2, \ldots, n$.
This give us $f_{\mu_k}(x) = 0_F$ for $x = \mu_i$, $i \in \{1, 2, \ldots, n\} - \{k\}$, and $f_{\mu_k}(k) \neq 0_F$. Thus we get,

$$\begin{aligned}\left(\sum_{i=1}^{n} \lambda_i \varepsilon_{\mu_i}\right)(f_{\mu_k}(x)) &= 0_{F[x]^*}(f_{\mu_k}(x)) \\ \sum_{i=1}^{n} \lambda_i (f_{\mu_k}(\mu_i)) &= 0_F \\ \lambda_k f_{\mu_k}(\mu_k) &= 0_F.\end{aligned}$$

Since $f_{\mu_k}(\mu_k) \neq 0_F$, we have $\lambda_k \neq 0_F$ for all $k = 1, 2, \ldots, n$, a contradiction.
Therefore $\{\varepsilon_t \mid t \in F\}$ is a linearly independent subset of $F[x]^*$.

(c) (i) Similar argument as 1b.) works, with suitable substitution, and so we can immediately conclude that $\{\phi_t \mid t \in T\}$ is linearly independent in P_n^*. Now since P_n is a finite-dimensional vector subspace of $F[x]$, we have $\dim(P_n^*) = \dim(P_n) = n + 1$.
Since $\{\phi_t \mid t \in T\}$ has $n + 1$ elements, we conclude that $\{\phi_t \mid t \in T\}$ is a basis for P_n^*.

(ii) Notice that $\phi : P_n \to F$ such that $\phi(f(x)) = \int_0^1 f(x)\, dx$ is a linear transformation, i.e. $\phi \in P_n^*$.
Since $\{\phi_t \mid t \in T\}$ is a basis of P_n^*, we have,

$$\begin{aligned}\phi(f(x)) &= \left(\sum_{t \in T} \lambda_t \phi_t\right)(f(x)) \\ \int_0^1 f(x)\, dx &= \sum_{t \in T} \lambda_t f(t),\end{aligned}$$

where $\lambda_t \in F$ is unique for each $t \in T$.

Question 2

(a) (i) For all $v \in \mathrm{Im}(\alpha+\beta)$, we have $v = (\alpha+\beta)(v')$ for some $v' \in V$.
Thus $v = \alpha(v') + \beta(v') \in \mathrm{Im}(\alpha) + \mathrm{Im}(\beta)$. This give us,

$$\begin{aligned} \mathrm{rk}(\alpha+\beta) &\leq \dim(\mathrm{Im}(\alpha)+\mathrm{Im}(\beta)) \\ &= \mathrm{rk}(\alpha)+\mathrm{rk}(\beta) - \dim(\mathrm{Im}(\alpha)\cap\mathrm{Im}(\beta)) \\ &\leq \mathrm{rk}(\alpha)+\mathrm{rk}(\beta). \end{aligned}$$

(ii) Let $v \in \ker(\alpha\beta)$, then $\beta(v) \in \ker(\alpha)$, and so $\mathrm{Im}(\beta|_{\ker(\alpha\beta)}) \subseteq \ker(\alpha)$.
Now by applying Rank-Nullity Theorem on $\beta|_{\ker(\alpha\beta)}$, we have

$$\begin{aligned} \dim(\ker(\alpha\beta)) &= \mathrm{null}(\beta|_{\ker(\alpha\beta)}) + \mathrm{rk}(\beta|_{\ker(\alpha\beta)}) \\ \mathrm{null}(\alpha\beta) &= \mathrm{null}(\beta) + \mathrm{rk}(\beta|_{\ker(\alpha\beta)}) \\ &\leq \mathrm{null}(\beta) + \mathrm{null}(\alpha). \end{aligned}$$

(b) Let us be given that $\alpha+\beta$ is bijective, and $\alpha\beta = 0$.
This implies that $\mathrm{rk}(\alpha+\beta) = \mathrm{null}(\alpha\beta) = \dim(V)$.
By Rank-Nullity Theorem,

$$2\dim(V) = \mathrm{rk}(\alpha+\beta) + \mathrm{null}(\alpha\beta) \quad \leq \quad (\mathrm{rk}(\alpha)+\mathrm{rk}(\beta)) + (\mathrm{null}(\beta)+\mathrm{null}(\alpha)) = 2\dim(V).$$

Thus equality holds for the above equation, which give us equality to holds in both inequalities.

Instead let us be given that $\mathrm{rk}(\alpha+\beta) = \mathrm{rk}(\alpha)+\mathrm{rk}(\beta)$ and $\mathrm{null}(\alpha\beta) = \mathrm{null}(\beta)+\mathrm{null}(\alpha)$.
Then similarly by Rank-Nullity Theorem, $\mathrm{rk}(\alpha+\beta) + \mathrm{null}(\alpha\beta) = 2\dim(V)$.
Since $\mathrm{rk}(\alpha+\beta), \mathrm{null}(\alpha\beta) \leq \dim(V)$, equality holds for both equations.
Thus we have $\alpha+\beta$ to be bijective, and $\alpha\beta = 0$.

Question 3

(a) Let us be given that α is bijective. Then α is injective, and so $E_{0_F} = \ker(\alpha) = \{0_V\}$.
This implies that 0_F is not an eigenvalue, i.e. $x = x - 0_F \nmid m_\alpha(x)$.

Instead let us be given that $x \nmid m_\alpha(x)$. The reverse argument give us that α is injective.
Let $v \in V$. Since $\gcd(m_\alpha(x), x) = 1_F$, there exists $p(x), q(x) \in F[x]$ such that

$$\begin{aligned} 1_F &= m_\alpha(x)p(x) + xq(x) \\ \mathrm{id}_V(v) &= (p(\alpha)m_\alpha(\alpha) + q(\alpha)\alpha)(v) \\ v &= p(\alpha)m_\alpha(\alpha)(v) + \alpha q(\alpha)(v) \\ &= \alpha[q(\alpha)(v)]. \end{aligned}$$

Thus $q(\alpha)(v) \in V$ is a pre-image of v, i.e. α is surjective.
Therefore α is bijective.

(b) Let W be a vector subspace of V such that $V = U \oplus W$.
For all $v \in V$, there exists $u \in U$ and $w \in W$ such that $v = u + w$.
Now, we have $m_\beta(\alpha)(u) = m_\beta(\beta)(u) = 0_V$.
Also, $m_\gamma(\alpha)(w) + U = m_\gamma(\gamma)(w + U) = U$. Thus $m_\gamma(\alpha)(w) \in U$.
We note that $m_\gamma(\alpha)(w) \in W$, and since $U \cap W = \{0_V\}$, we have $m_\gamma(\alpha)(w) = 0_V$. Thus,

$$\begin{aligned} m_\beta(\alpha)m_\gamma(\alpha)(v) &= m_\beta(\alpha)m_\gamma(\alpha)(u+w) \\ &= m_\gamma(\alpha)m_\beta(\alpha)(u) + m_\beta(\alpha)m_\gamma(\alpha)(w) \\ &= 0_V. \end{aligned}$$

Therefore α satisfy $m_\beta(x)m_\gamma(x)$, and so $m_\alpha(x) \mid m_\beta(x)m_\gamma(x)$.

Since U is α-invariant, we have $m_\beta(x) = m_{\alpha|_U}(x) \mid m_\alpha(x)$.
Now for all $v \in V$, we have $m_\alpha(\gamma)(v + U) = m_\alpha(\alpha)(v) + U = U$.
Thus γ satisfy $m_\alpha(x)$, and so $m_\gamma(x) \mid m_\alpha(x)$.
Therefore $\mathrm{lcm}(m_\beta(x), m_\gamma(x)) \mid m_\alpha(x)$.

Question 4

(i) Since $\mathrm{Im}(\alpha)$ is finite-dimensional, $m_{\alpha|_{\mathrm{Im}(\alpha)}}(x)$ exists.
Now for all $v \in V$, we have $\alpha(v) \in \mathrm{Im}(\alpha)$, and so $m_{\alpha|_{\mathrm{Im}(\alpha)}}(\alpha)(\alpha(v)) = 0_V$.
Thus α satisfies $xm_{\alpha|_{\mathrm{Im}(\alpha)}}(x) \in F[x]\backslash\{0_F\}$.

(ii) Since $\mathrm{Im}(\alpha)$ is finite, we have $\mathrm{rk}(\alpha) = \mathrm{rk}(\alpha^2)$. Thus by Rank-Nullity Theorem on $\alpha|_{\mathrm{Im}(\alpha)}$, we get

$$\begin{aligned} \mathrm{rk}(\alpha) &= \mathrm{null}(\alpha|_{\mathrm{Im}(\alpha)}) + \mathrm{rk}(\alpha|_{\mathrm{Im}(\alpha)}) \\ \mathrm{rk}(\alpha) &= \dim(\ker(\alpha) \cap \mathrm{Im}(\alpha)) + \mathrm{rk}(\alpha^2). \end{aligned}$$

Thus $\dim(\ker(\alpha) \cap \mathrm{Im}(\alpha)) = 0$, i.e. $\ker(\alpha) \cap \mathrm{Im}(\alpha) = \{0_V\}$.

Now since $\mathrm{Im}(\alpha) = \mathrm{Im}(\alpha^2)$, for all $v \in V$, there exists $v' \in V$ such that,

$$\begin{aligned} \alpha(v) &= \alpha^2(v') \\ \alpha(v - \alpha(v')) &= 0_V \\ v - \alpha(v') &\in \ker(\alpha) \\ v &\in \ker(\alpha) + \mathrm{Im}(\alpha). \end{aligned}$$

Thus $V = \ker(\alpha) \oplus \mathrm{Im}(\alpha)$.

Question 5

(i) Let X be the largest definite subspace (either positive or negative).
Thus we have $\dim(X) = \frac{1}{2}(\mathrm{rk}(\phi) + |s|)$.

Now, if $u \in U \cap X - \{0_V\}$, then $\phi(u, u) \neq 0_F$. Thus $\phi|_{(U\cap X)\times(U\cap X)}$ is non-degenerate.
Since $U \cap X$ is a subspace of U, which is a subspace of V, we have

$$\dim(U \cap X) = \mathrm{rk}(\phi|_{(U\cap X)\times(U\cap X)}) \leq \mathrm{rk}(\phi|_{U\times U}) \leq \mathrm{rk}(\phi).$$

Together with the fact that $U + X \subseteq V$, we have,

$$\begin{aligned} \dim(X) + \dim(U) &= \dim(U + X) + \dim(U \cap X) \\ \frac{1}{2}(\mathrm{rk}(\phi) + |s|) + \dim(U) &\leq \dim(V) + \mathrm{rk}(\phi|_{U\times U}). \end{aligned}$$

And so we combined results to get $\frac{1}{2}(\mathrm{rk}(\phi) + |s|) + \dim(U) - \dim(V) \leq \mathrm{rk}(\phi|_{U\times U}) \leq \mathrm{rk}(\phi)$.

(ii) Let $P, Q, X, Y \subseteq V$ be the set of symmetric matrices, skew-symmetric matrices, upper triangular matrices with diagonal entries 0_F, and diagonal matrices respectively.

Let $p = (p_{ij}) \in P$. Then we have $\phi(p, p) = \sum_{i=1}^n \sum_{j=1}^n p_{ij}p_{ji} = \sum_{i=1}^n \sum_{j=1}^n (p_{ij})^2 \geq 0_F$.
Let $q = (q_{ij}) \in Q$. Then we have $\phi(q, q) = \sum_{i=1}^n \sum_{j=1}^n q_{ij}q_{ji} = \sum_{i=1}^n \sum_{j=1}^n -(q_{ij})^2 \leq 0_F$.
Thus P and Q are positive definite and negative definite respectively.

Next, we notice that $X + Y$ is direct, with $\dim(X) = \frac{1}{2}(n^2 - n)$ and $\dim(Y) = n$.
We observe that $p \in P$ iff there exists $x \in X$ and $y \in Y$ such that $p = x + x^T + y$.
Also, $q \in Q$ iff there exists $x \in X$ such that $q = x - x^T$.
This implies that $\dim(P) = \dim(X) + \dim(Y) = \frac{1}{2}(n^2 + n)$ and $\dim(Q) = \dim(X) = \frac{1}{2}(n^2 - n)$.
Thus ϕ is non-degenerate, with $\dim(P) + \dim(Q) = \dim(V)$.

Now since $\phi(u, u) = 0_F$ for all $u \in U$, we have $U \cap P = U \cap Q = \{0_V\}$.
Thus $U + P$ and $U + Q$ are direct.
So by Rank-Nullity Theorem, we have $\dim(U) \leq \dim(V) - \dim(P) = \dim(Q)$.
Similarly $\dim(U) \leq \dim(P)$. Thus $\dim(U) \leq \min(\dim(P), \dim(Q)) = \frac{1}{2}(n^2 - n)$.

Now let $x = (x_{ij}) \in X$. Then we have $\phi(x, x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}x_{ji} = 0_F$.
Thus X satisfy the condition of being U and since $\dim(X) = \frac{1}{2}(n^2 - n)$, we have the largest possible dimension of U to be $\frac{1}{2}(n^2 - n)$.