NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA2216 Probability
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Question 1

(a) (i) Let $f(x)$ and $F(x)$ be the probability density function and cumulative density function of X respectively.
By symmetry, we have $f(x) = f(-x)$. Therefore, $P(X \leq 0) = \frac{1}{2}$.
Now, for any $x \in \mathbb{R}^+$,

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= P(X \leq 0) + P(0 \leq X \leq x) \\ &= \frac{1}{2} + \frac{1}{2}P(-x \leq X \leq x) \\ &= \frac{1}{2} + \frac{1}{2}P(X^2 \leq x^2) \\ &= \frac{1}{2} + \frac{1}{2}\frac{\frac{1}{2}^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\int_0^{x^2} t^{-\frac{1}{2}}e^{-\frac{1}{2}t}dt. \end{aligned}$$

$\therefore$

$$\begin{aligned} f(x) &= \frac{d}{dx}F(x) \\ &= \frac{d}{dx}(\frac{1}{2} + \frac{1}{2}\frac{\frac{1}{2}^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\int_0^{x^2} t^{-\frac{1}{2}}e^{-\frac{1}{2}t}dt) \\ &= \frac{1}{2}\frac{\frac{1}{2}^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}(2e^{-\frac{1}{2}x^2}) \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}, x \in \mathbb{R}. \end{aligned}$$

(ii) Let $f(x)$ and $F(x)$ be the probability density function and cumulative density function of X respectively.
Note that $X > 0$, $\therefore f(x) = 0$, $\forall x \leq 0$.

Now, for any $x > 0$,

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= P(e^{-\frac{Y}{2}} < x) \\ &= P(Y > -2\ln x) \\ &= e^{\left(-\frac{1}{2}\right)-2\ln x} \\ &= x. \end{aligned}$$

$\therefore$

$$\begin{aligned} f(x) &= \frac{d}{dx}F(x) \\ &= \frac{d}{dx}(x) \\ &= 1, x \in (0,1). \end{aligned}$$

(iii) Let $Y = \frac{1}{2\pi}\tan^{-1}(\frac{V}{U})$, then we have

$$\begin{cases} u = \sqrt{-2\ln x}\cos(2\pi y) \\ v = \sqrt{-2\ln x}\sin(2\pi y) \end{cases}$$

Note that, since $U, V \in \mathbb{R}$, $\therefore X > 0$ and $0 < Y < 1$.

The Jacobian of the transformation is

$$J(u,v) = \begin{vmatrix} -ue^{-\frac{u^2+v^2}{2}} & -ve^{-\frac{u^2+v^2}{2}} \\ \frac{1}{2\pi}\frac{-v}{u^2+v^2} & \frac{1}{2\pi}\frac{u}{u^2+v^2} \end{vmatrix} = -\frac{1}{2\pi}e^{-\frac{u^2+v^2}{2}} = -\frac{x}{2\pi}$$

So,

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{2\pi}{x} f_{U,V}(\sqrt{-2\ln x}\cos(2\pi y), \sqrt{-2\ln x}\sin(2\pi y))) \\ &= \frac{2\pi}{x}\frac{1}{2\pi}e^{-\frac{-2\ln x\cos^2(2\pi y)-2\ln x\sin^2(2\pi y)}{2}} \\ &= 1,\ 0 < x < 1, 0 < y < 1. \end{aligned}$$

It is easy to see that X and Y are independent uniform distribution over $(0,1)$.
i.e $X \sim U(0,1)$.

(b) Note that, $\{X < Y\} = \{(x,y) \mid 0 < x < y < \infty\}$,

$\therefore$

$$\begin{aligned}
P(Y > X) &= \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} p(1-p)^{j-1}(e^{-\lambda}\frac{\lambda^i}{i!}) \\
&= \sum_{i=0}^{\infty} \frac{p(1-p)^i}{p}(e^{-\lambda}\frac{\lambda^i}{i!}) \\
&= \sum_{i=0}^{\infty} e^{-\lambda}\frac{[\lambda(1-p)]^i}{i!} \\
&= e^{-\lambda}e^{\lambda(1-p)} \\
&= e^{-\lambda p}.
\end{aligned}$$

Question 2

(a) (i) The transformation is given by:

$$\begin{cases} w = g_1(x,y) = \frac{x}{y} \\ z = g_2(x,y) = y \end{cases}$$

The inverse transformation can be deduced as:

$$\begin{cases} x = h_1(x,y) = wz \\ y = h_2(x,y) = z \end{cases}$$

Note that, since $W, Z > 0$, $\therefore X, Y > 0$. Hence, we conclude that $f_{X,Y}(x,y) = 0$ for $-\infty < x < 0, -\infty < y < 0$.

The Jacobian of the transformation is

$$J(w,z) = \begin{vmatrix} z & w \\ 0 & 1 \end{vmatrix} = z = y.$$

So,

$$\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{y} f_{W,Z}(\frac{x}{y}, y) \\
&= \frac{1}{y} e^{-\frac{x}{y}} e^{-y} \\
&= \frac{1}{ye^{\frac{x}{y}+y}},\ 0 < x < \infty, 0 < y < \infty.
\end{aligned}$$

(ii) By assumption that $W, Z \sim \text{Exp}(1)$, we have $E(W) = E(Z) = 1$ and $E(W^2) = 2$.
$\therefore E(XW) = E(W^2Z) = E(W^2)E(Z) = 2$.
Note that the second equality follows from the independence of W and Z.

(iii) Observe that $Y, Z \sim \text{Exp}(1)$ and hence $E(Y) = 1$.
Also $E(X) = E(WZ) = E(W)E(Z) = 1$ and $E(Z^2) = 2$.

$$\begin{aligned}\text{Cov}(X,Y) &= E(XY) - E(X)E(Y)\\ &= E(WZ^2) - E(X)E(Y)\\ &= E(W)E(Z^2) - E(X)E(Y)\\ &= 1.\end{aligned}$$

Question 3

(a) Let

$$X_i = \begin{cases} 1, \text{ if collision occurs at } i^{th} \text{ position} \\ 0, \text{ Otherwise} \end{cases}$$

Then $E(X_i) = \sum_{j=1}^{m} P(X_i = 1| \text{ placed in cell } j) \cdot p_j$.
To calculate $P(X_i = 1| \text{ placed in cell } j)$, use the complement of the event. Observe that a collision will NOT occur when i^{th} item is placed in cell j, if none of the previous $i-1$ items were put in cell j.
i.e. $P(X_i = 1| \text{ placed in cell } j) = 1 - (1-p_j)^{i-1}$.
(One can easily deduce this equation using Geometric distribution.)
$\therefore$

$$\begin{aligned}E(X) &= \sum_{i=1}^{m} E(X_i)\\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} p_j(1-(1-p_j)^{i-1})\\ &= \sum_{i=1}^{m}\left[\sum_{j=1}^{m} p_j - \sum_{j=1}^{m} p_j(1-p_j)^{i-1}\right]\\ &= \sum_{i=1}^{m}\left[1 - \sum_{j=1}^{m} p_j(1-p_j)^{i-1}\right]\end{aligned}$$

($\sum_{i=1}^{m} p_j = 1$ as it is a probability)

$$\begin{aligned}&= \sum_{i=1}^{m} 1 - \sum_{i=1}^{m}\sum_{j=1}^{m} p_j(1-p_j)^{i-1}\\ &= m - \sum_{j=1}^{m}\sum_{i=1}^{m} p_j(1-p_j)^{i-1}\end{aligned}$$

(We can switch the order of summation by Tonelli's/Fubini's Theorem.)

$$
\begin{aligned}
&= m - \sum_{j=1}^{m} \frac{p_j[1-(1-p_j)]^m}{p_j} \\
&= m - \sum_{j=1}^{m} 1 + \sum_{j=1}^{m} (1-p_j)^m \\
&= \sum_{j=1}^{m} (1-p_j)^m
\end{aligned}
$$

(b) Let X denote the random variable of the distance measured by the astronomer. By assumption, $E(X) = d$ and $Var(X) = 4$.
Denote $\overline{X_n}$ as the average value of n measurements. Note that, $\overline{X_n}$ is also the estimate of the distance.
By the Central Limit Theorem, $\overline{X_n} \sim N(d, [\frac{2}{\sqrt{n}}]^2)$
To have $P(|X - \overline{X_n}| < .0.5) \geq 0.95$, we need

$$
\begin{aligned}
P(|\frac{X - \overline{X_n}}{2/\sqrt{n}}| < 0.25\sqrt{n}) &\geq 0.95 \\
P(|Z| < 0.25\sqrt{n}) &\geq 0.95 \\
P(Z < 0.25\sqrt{n}) &\geq 0.975 \\
0.25\sqrt{n} &\geq 1.960 \\
n &\geq 61.4656
\end{aligned}
$$

.
$\therefore n_{\min} = 62.$

Question 4

(i)

$$
\begin{aligned}
\int_0^{\infty} \int_0^{x} K \frac{e^{-x^2/2}}{x} dy dx &= 1 \\
K \int_0^{\infty} e^{-x^2/2} dx &= 1 \\
K \sqrt{\frac{\pi}{2}} &= 1 \\
K &= \sqrt{\frac{2}{\pi}}.
\end{aligned}
$$

(ii)

$$\begin{aligned} f_X(x) &= \int_0^x K \frac{e^{-x^2/2}}{x} dy \\ &= \sqrt{\frac{2}{\pi}} e^{-x^2/2}, 0 < x < \infty. \end{aligned}$$

(iii)

$$\begin{aligned} E(X) &= \sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-x^2/2} dx \\ &= \sqrt{\frac{2}{\pi}} \left[-e^{-x^2/2} \right]_0^\infty \\ &= \sqrt{\frac{2}{\pi}}. \end{aligned}$$

$$\begin{aligned} E(X^2) &= \sqrt{\frac{2}{\pi}} \int_0^\infty x^2 e^{-x^2/2} dx \\ &= 1. \end{aligned}$$

$\therefore$

$$Var(X) = E(X^2) - [E(X)]^2 = 1 - \frac{2}{\pi}.$$

(iv) For $0 < y < x$,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X}(x) = \frac{\sqrt{\frac{2}{\pi}} \frac{e^{-x^2/2}}{x}}{\sqrt{\frac{2}{\pi}} e^{-x^2/2}} = \frac{1}{x}.$$

(v)

$$E(Y|X = x) = \int_0^x y f_{Y|X}(y|x) dy = \int_0^x \frac{y}{x} dy = \frac{x}{2}.$$

$$E(Y|X) = \frac{X}{2}$$

$\therefore$

$$E(Y) = E(E(Y|X)) = E\left(\frac{X}{2}\right) = \frac{1}{2}E(X) = \frac{1}{\sqrt{2\pi}}.$$

(vi)

$$E(XY) = E(E(XY|X)) = E(X \cdot (Y|X)) = E\left(\frac{X^2}{2}\right) = \frac{1}{2}.$$

$\therefore$

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{1}{2} - \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{2\pi}} = \frac{1}{2} - \frac{1}{\pi}.$$