NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Stefanus Lie

MA1102R Calculus
AY 2012/2013 Sem 1

Question 1

(a)

$$\begin{aligned}
\lim_{x\to 1}\left(\frac{5}{1-x^5}-\frac{4}{1-x^4}\right) &= \lim_{x\to 1}\frac{5(1-x^4)-4(1-x^5)}{(1-x^5(1-x^4))} \\
&= \lim_{x\to 1}\frac{1-5x^4-4x^5}{(1-x^5)(1-x^4)} \\
&= \lim_{x\to 1}\frac{-20x^3-20x^4}{-5x^4(1-x^4)-4x^3(1-x^5)} \\
&= \lim_{x\to 1}\frac{-20-20x}{-4-5x+9x^5} \\
&= \lim_{x\to 1}\frac{-20}{-5+45x^4} \\
&= -\frac{1}{2}
\end{aligned}$$

(b) Given $\epsilon > 0$, we want to prove that there exists $\delta > 0$ such that $0 < |x+1| < \delta$ implies $|x^3+x^2+x+1| < \epsilon$. Choose $\delta = \min(1, \epsilon/5)$. Since $\delta < 1$, then $-2 < x < -1$ and hence $|x^2+1| < 5$. Then, $|x^3+x^2+x+1| = |x^2+1|\cdot|x+1| < 5|x+1| < 5\cdot\frac{\epsilon}{5} = \epsilon$. Q.E.D.

Question 2

Note that $f'(x) = e^x(x-1)^2$. Then, f is increasing everywhere and no local maximum or minimum exists.

Note that $f''(x) = e^x(x-1)^2 + 2(x-1)e^x = e^x(x-1)(x+1)$. Hence, for both open intervals $(-\infty,-1)$ and $(1,\infty)$, f is concave up, while on the open interval $(-1,1)$, f is concave down. The inflection points occur when $x = -1, 1$, which is exactly the points $(-1, 10e^x)$ and $(1, 2e^x)$.

Question 3

(a) Note that $p(1) = p(3) = 0$, $p(4) < 0$, $p(6) > 0$, $q(1) < 0$, $q(3) > 0$, and $q(4) = q(6) = 0$. Hence, $p(1)+q(1) < 0$, $p(3)+q(3) > 0$, $p(4)+q(4) < 0$, and $p(6)+q(6) > 0$. Using IVT on three intervals $(1,3)$, $(3,4)$, and $(4,6)$, we could easily get that $p(x)+q(x)$ has three real roots.

(b) Note that $\ln y = x\cdot\ln|4-x^2| - (4-x^2)\ln|x|$. Hence, $\frac{y'}{y} = \ln|4-x^2| + x\cdot\frac{-2x}{4-x^2} - \frac{4-x^2}{x} - (-2x)\ln|x|$. So, $y'(1) = 3(\ln 3 - \frac{2}{3} - 3) = 3\ln 3 - 11.$. The equation of its tangent line is therefore $y = (3\ln 3 - 11)x - (3\ln 3 - 14)$

Question 4

Using the Sine Rule, the base of that triangle is $2\sin 2\theta$ and the length of foot is $2\sin(\pi/2-\theta) = 2\cos\theta$. Now, using usual trigonometry, the height is $2\cos\theta\cos\theta$. Hence, the area of that triangle is $1/2{\cdot}base{\cdot}height = 2\sin 2\theta\cos^2\theta = 4\sin\theta\cos^3\theta$. This proves part (i). Now, let $f(\theta) = 4\sin\theta\cos^3\theta$, with $0<\theta<$
$pi/2$. Then, $f'(\theta) = 4\cos^4\theta + 4\sin\theta\cdot 3\cos^2\theta\cdot -\sin\theta = 4\cos^2\theta(\cos^2\theta - 3\sin^2\theta)$. Since for $0<\theta<\pi/6$, $f'(\theta)>0$ and for $\pi/6<\theta<\pi/2$, $f'(\theta)<0$, then the maximum area is attained when $\theta=\pi/6$, which is exactly when the triangle is equilateral.

Question 5

(a) Let $u=\tan^{-1}\sqrt{x}$. Then, $\frac{du}{dx} = \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}$. Then,

$$\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}(1+x)}dx = \int 2u\ du = u^2 + c = (\tan^{-1}\sqrt{x})^2 + C$$

(b) Let $x=\sin t$. Then, $\frac{dx}{dt} = \cos t$. Hence,

$$\int \frac{x^2}{\sqrt{1-x^2}}dx = \int \sin^2 t\ dt = \int \frac{1-\cos 2t}{2}dt = \frac{t}{2} - \frac{\sin 2t}{4} + C = \frac{\sin^{-1}x}{2} - \frac{x\sqrt{1-x^2}}{2} + C$$

Question 6

(a) Note that

$$\begin{aligned} LHS &= exp\left(\lim_{x\to 0}\frac{2a}{x^2}\cdot\ln|\cos x|\right) \\ &= exp\left(\lim_{x\to 0} 2a\cdot\frac{\ln|\cos x|}{x^2}\right) \\ &= exp\left(\lim_{x\to 0} 2a\cdot\frac{-\tan x}{2x}\right) \\ &= exp(-a) \\ &= e^{-a} \end{aligned}$$

Note also that using tabular integration,

$$\begin{aligned} RHS &= 2\left[-e^{-x}(x+1)\right]_a^\infty \\ &= 2\left(\lim_{x\to\infty} -\frac{x+1}{e^x}\right) + 2\cdot\frac{a+1}{e^a} \\ &= 2\left(\lim_{x\to\infty} -\frac{1}{e^x}\right) + 2\cdot\frac{a+1}{e^a} \\ &= 2\cdot\frac{a+1}{e^a} \end{aligned}$$

Since $LHS=RHS$, then $2(a+1)=1$ which implies that $a=-\frac{1}{2}$.

(b) Note that

$$\begin{aligned}\int_0^x tf(x^2-t^2)dt &= -\frac{1}{2}\cdot\int_0^x f(x^2-t^2)d(x^2-t^2)\\ &= -\frac{1}{2}\cdot\int_{x^2}^0 f(u)d(u)\\ &= \frac{1}{2}\cdot\int_0^{x^2} f(u)du\end{aligned}$$

Hence, the derivative of that expression is $\frac{1}{2}\cdot 2x\cdot f(x) = xf(x)$.

Question 7

(a) Note that the graph $y=\sqrt{x}$ and $y=ax$ cuts at $x=0$ and $x=\frac{1}{a^2}$. The volume is therefore

$$\begin{aligned}\pi\int_0^{1/a^2}(x-a^2x^2)dx &= \pi\left[\frac{x^2}{2}-\frac{a^2x^3}{3}\right]_0^{1/a^2}\\ &= \frac{\pi}{6a^4}\end{aligned}$$

The volume is 6π when $a=1/\sqrt{6}$.

(b) Note that $\frac{dx}{dy}=\frac{3}{4}(\frac{2y^{1/3}}{3}-\frac{2y^{-1/3}}{3})=\frac{1}{2}(y^{1/3}-y^{-1/3})$. So, $1+(\frac{dx}{dy})^2=1+\frac{(y^{1/3}-y^{-1/3})}{4}=\frac{y^{1/3}+y^{-1/3}}{4}$. The length of the curve is therefore

$$\int_1^{27}\sqrt{1+\frac{dx}{dy}}dy=\int_1^{27}\frac{y^{1/3}-y^{-1/3}}{2}=\left[\frac{3y^{4/3}}{8}-\frac{3y^{2/3}}{4}\right]_1^{27}=24$$

Question 8

Note that

$$(1+e^x)\frac{dz}{dx}+e^x\cdot z=\sin x$$

Hence,

$$\frac{d(1+e^x)\cdot z}{dx}=\sin x$$

So,

$$z=\frac{C-\cos x}{1+e^x}$$

This solves part (i). For part (ii), let $z=y^{-3}$, then $\frac{dz}{dy}=-3y^{-4}\cdot\frac{dy}{dx}$. Hence,

$$\frac{dz}{dx}+(\frac{e^x}{1+e^x})z=-3y^{-4}\cdot\frac{dy}{dx}+(\frac{e^x}{1+e^x})y^{-3}=\frac{\sin x}{1+e^x}$$

This is exactly the part(i), and hence

$$y=\sqrt[3]{\frac{1}{z}}=\sqrt[3]{\frac{1+e^x}{C-\cos x}}$$

Now, using the fact that $y = 1$ is $x = 0$, then $C = 3$. So,

$$y = \sqrt[3]{\frac{1}{z}} = \sqrt[3]{\frac{1 + e^x}{3 - \cos x}}$$

Question 9

From the known equation,

$$\int \frac{dv}{(v - 70)(v + 70)} = -0.002\ dt$$

So,

$$\frac{1}{140}\left(\int \frac{dv}{v - 70} - \int \frac{dv}{v + 70}\right) = -0.002t + C$$

Hence,

$$\ln\left|\frac{v - 70}{v + 70}\right| = 140C - 0.28t$$

Note that $\frac{v-70}{v+70}$ is always negative for $-70 < v < 70$, and when $t = 0$, $v = 0$. From this, we get $C = 0$, so

$$\frac{70 - v}{70 + v} = e^{-0.28t}$$

Now, by using algebraic manipulation, we get

$$v = 70\left(\frac{1 - e^{-0.28t}}{1 + e^{-0.28t}}\right)$$

When t is going to infinity, v is going to 70, which is its terminal velocity.

Question 10

Define well-defined function h, with $h(x) = \frac{g(x)}{f(x)}$ for all $x \in (a, b)$.

If g has 2 real roots on (a, b), say $g(p) = g(q) = 0$, then $h(p) = h(q) = 0$. Since h is continuous, by Rolle's, there exists $c \in (p, q)$ such that $h'(c) = 0$, which implies that $c \in (a, b)$ and $f'(c)g(c) = f(c)g'(c)$. A contradiction. Until here, we can conclude that g has at most one real root in (a, b).

Now, assume the contrary that g has no root on (a, b). By Intermediate Value Theorem on (a, b), then either g is definite positive in (a, b) OR g is negative definite in (a, b). Using the same argument, either f is definite positive in (a, b) OR f is negative definite in (a, b).

We left with four possible combination of cases, but we only consider the case when f and g are both positive definite in (a, b), since the other cases can be done in the same way by considering functions $f_1 = -f$ or $g_1 = -g$.

Note that $\lim\limits_{x\to a^+} h(x) = \lim\limits_{x\to a^+} \dfrac{g(x)}{f(x)} = +\infty$ and in the same way $\lim\limits_{x\to b^-} h(x) = -\infty$. Now, let $h(\frac{a+b}{2}) = s$. Since $\lim\limits_{x\to b^-} h(x) = -\infty$ and h is continuous, then there exists $r_1 \in (\frac{a+b}{2}, b)$ such that $h(r_1) = s + 1$. In the same way, there exists $r_2 \in (a, \frac{a+b}{2})$ such that $h(r_2) = s + 1$. By Rolle's, there exists $r \in (r_1, r_2)$ such that $h'(r) = 0$, which means, $f'(r)g(r) = f(r)g'(r)$ for $r \in (a, b)$. A contradiction.

Hence, g has exactly one real root in (a, b). Q.E.D.