NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Theo Fanuela Prabowo

MA1102R Calculus
AY 2010/2011 Sem 1

Question 1

(a) Given $\varepsilon > 0$. Take $\delta = \min\{1, \frac{\varepsilon}{19}\}$. Then if $0 < |x+2| < \delta$, we have

$$\begin{aligned} |x^3+8| &= |x+2|\left|x^2-2x+4\right| \\ &\leq |x+2|\left(|x|^2+2|x|+4\right) && \text{(by triangle inequality)} \\ &< 19|x+2| && (\because |x+2| < \delta \leq 1) \\ &< \varepsilon. && (\because |x+2| < \delta \leq \frac{\varepsilon}{19}) \end{aligned}$$

Hence, by definition, $\lim_{x\to -2} x^3 = -8$.

(b) Since f is continuous at $x = 0$, we have $\lim_{x\to 0} f(x) = f(0) \Leftrightarrow \lim_{x\to 0} \dfrac{(\sin x - a)(\cos x - b)}{e^x - 1} = 5$.
Note that

$$\lim_{x\to 0}(\sin x - a)(\cos x - b) = \lim_{x\to 0}\frac{(\sin x - a)(\cos x - b)}{e^x-1} \times \lim_{x\to 0}(e^x - 1) = 5 \times 0 = 0.$$

Since the function $(\sin x - a)(\cos x - b)$ is continuous on $\mathbb{R}$, we have $(\sin 0 - a)(\cos 0 - b) = a(1-b) = 0$.
Since $\lim_{x\to 0}(\sin x - a)(\cos x - b) = 0 = \lim_{x\to 0}(e^x - 1)$, we may apply L'Hopital's Rule to obtain

$$\lim_{x\to 0}\frac{(\sin x - a)(\cos x - b)}{e^x - 1} = \lim_{x\to 0}\frac{\cos x(\cos x - b) - \sin x(\sin x - a)}{e^x} = 1 - b = 5.$$

Thus, $b = -4$.
Recall that $a(1-b) = 0$. Hence, $a = 0$.

Question 2

(a)

$$\begin{aligned} \lim_{x\to\infty} x^{\frac{1}{\ln(x^3+1)}} &= \lim_{x\to\infty} \exp\left(\frac{\ln(x)}{\ln(x^3+1)}\right) \\ &= \exp\left(\lim_{x\to\infty}\frac{\ln(x)}{\ln(x^3+1)}\right) \\ &= \exp\left(\lim_{x\to\infty}\frac{1/x}{3x^2/(x^3+1)}\right) \quad \text{(by L'Hopital's Rule)} \\ &= \exp\left(\lim_{x\to\infty}\frac{x^3+1}{3x^3}\right) \\ &= \exp\left(\lim_{x\to\infty}\frac{1+\frac{1}{x^3}}{3}\right) = \exp\left(\frac{1}{3}\right). \end{aligned}$$

(b) Since

$$\lim_{x\to0^+}\left(\frac{2+e^{1/x}}{1+e^{4/x}}+\frac{\sin x}{|x|}\right)=\lim_{x\to0^+}\left(\frac{2/e^{4/x}+1/e^{3/x}}{1/e^{4/x}+1}\right)+\lim_{x\to0^+}\frac{\sin x}{x}=0+1=1$$

and

$$\lim_{x\to0^-}\left(\frac{2+e^{1/x}}{1+e^{4/x}}+\frac{\sin x}{|x|}\right)=\lim_{x\to0^-}\left(\frac{2+e^{1/x}}{1+e^{4/x}}\right)+\lim_{x\to0^-}\left(-\frac{\sin x}{x}\right)=\frac{2+0}{1+0}-1=1,$$

it follows that $\lim\limits_{x\to0}\left(\dfrac{2+e^{1/x}}{1+e^{4/x}}+\dfrac{\sin x}{|x|}\right)=1.$

Question 3

(a) First, we note that f is not defined when $x<0$.

$$f'(x)=\frac{88}{3}x^{5/3}-\frac{11}{3}x^{8/3}.$$

$$f''(x)=\frac{440}{9}x^{2/3}-\frac{88}{9}x^{5/3}=\frac{88}{9}x^{2/3}(5-x).$$

A necessary condition for inflection point is $f''(x)=0$, so that $x=0$ (omitted since f is not defined when $x<0$) or $x=5$. Observe that $f''(x)>0$ when $x\in(0,5)$ and $f''(x)<0$ when $x\in(5,\infty)$. Thus, the point $(5,f(5))=\left(5,150\cdot5^{2/3}\right)$ is an inflection point.

Remark: According to a more general definition of exponentiation, i.e. $z^a:=\exp\left(a\ln|z|+ia\arg(z)\right)$, $f(x)$ is actually defined to be a non-real number for negative value of x. Please refer to the definition used by your lecturer.

(b) It suffices to show that $g'(x)=0$ for all $x\in\mathbb{R}$.
Given any $x\in\mathbb{R}$. Note that

$$\left|\frac{g(y)-g(x)}{y-x}\right|\le|y-x|\Rightarrow-|y-x|\le\frac{g(y)-g(x)}{y-x}\le|y-x|.$$

Since $\lim\limits_{y\to x}-|y-x|=0=\lim\limits_{y\to x}|y-x|$, by the Squeeze Theorem, $\lim\limits_{y\to x}\dfrac{g(y)-g(x)}{y-x}=g'(x)=0$. Hence, g is a constant function.

(c) Let $u=x-t$. Then

$$\begin{aligned}\frac{d}{dx}\int_0^{3x}\sin\left((x-t)^2\right)dt&=\frac{d}{dx}\int_x^{-2x}-\sin(u^2)du\\&=\frac{d}{dx}\int_{-2x}^{x}\sin(u^2)du\\&=\frac{d}{dx}\int_0^{x}\sin(u^2)du-\frac{d}{dx}\int_0^{-2x}\sin(u^2)du\\&=\sin(x^2)-\frac{d(-2x)}{dx}\cdot\frac{d}{d(-2x)}\int_0^{-2x}\sin(u^2)du\\&=\sin(x^2)+2\sin\left((-2x)^2\right)\\&=\sin(x^2)+2\sin(4x^2).\end{aligned}$$

Question 4

Since the volume of the outside cylinder is 16π ft^3, we have $16\pi = \pi\left(r+\frac{1}{2}\right)^2(h+1)$, so that $h = \frac{16}{\left(r+\frac{1}{2}\right)^2} - 1$.

$$V(r) = \pi r^2\left(\frac{16}{\left(r+\frac{1}{2}\right)^2} - 1\right) = \pi\left(\frac{16r^2}{\left(r+\frac{1}{2}\right)^2} - r^2\right) \qquad , r > 0.$$

Thus,

$$\begin{aligned} V'(r) &= \pi\left(\frac{16\cdot 2r\left(r+\frac{1}{2}\right)^2 - 16r^2\cdot 2\left(r+\frac{1}{2}\right)}{\left(r+\frac{1}{2}\right)^4} - 2r\right) \\ &= \frac{2\pi r\left(r+\frac{1}{2}\right)\left(8-\left(r+\frac{1}{2}\right)^3\right)}{\left(r+\frac{1}{2}\right)^4}. \end{aligned}$$

Critical points are attained when $V'(r) = 0$ or $V'(r)$ is undefined, i.e. when $r = -\frac{1}{2}$ or $r = 0$, or $r = \frac{3}{2}$. Since $r > 0$, the only critical point is $r = \frac{3}{2}$. Note that $V'(r) > 0$ when $r \in (0, \frac{3}{2})$ and $V'(r) < 0$ when $r \in (\frac{3}{2}, \infty)$. Thus, $V(r)$ is maximized when $r = \frac{3}{2}$. Recall that $h = \frac{16}{\left(r+\frac{1}{2}\right)^2} - 1$. Thus, $h = 3$. Hence, the radius and the height of the inside cylinder are 1.5 ft and 3 ft respectively.

Question 5

(a) Let $u = \tan(\ln x) \Rightarrow du = \frac{\sec^2(\ln x)}{x}dx$. Thus,

$$\begin{aligned} \int \frac{\sqrt{\tan(\ln x)}\sec^4(\ln x)}{x}dx &= \int \sqrt{\tan(\ln x)}\left(1+\tan^2(\ln x)\right)\cdot\frac{\sec^2(\ln x)}{x}dx \\ &= \int \sqrt{u}(1+u^2)du \\ &= \int \left(u^{1/2}+u^{5/2}\right)du \\ &= \frac{2}{3}u^{3/2} + \frac{2}{7}u^{7/2} + C \\ &= \frac{2}{3}\left(\tan(\ln x)\right)^{3/2} + \frac{2}{7}\left(\tan(\ln x)\right)^{7/2} + C. \end{aligned}$$

(b) First, we will solve the indefinite integral $\displaystyle\int \frac{3^x}{1+3^{2x}}dx$.
Let $u = 3^x \Rightarrow du = \ln 3\cdot 3^x dx$. Thus,

$$\int \frac{3^x}{1+3^{2x}}dx = \frac{1}{\ln 3}\int\frac{du}{1+u^2} = \frac{1}{\ln 3}\tan^{-1}(u) + C = \frac{1}{\ln 3}\tan^{-1}(3^x) + C.$$

Therefore

$$\begin{aligned} \int_0^\infty \frac{3^x}{1+3^{2x}}dx &= \lim_{t\to\infty}\int_0^t \frac{3^x}{1+3^{2x}}dx \\ &= \lim_{t\to\infty}\frac{1}{\ln 3}\left(\tan^{-1}(3^t) - \tan^{-1}(1)\right) \\ &= \frac{1}{\ln 3}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) \\ &= \frac{\pi}{4\ln 3}. \end{aligned}$$

Question 6

(a)

$$\begin{aligned}
\text{Area} = \int_0^{\ln 2} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx &= 2\pi\int_0^{\ln 2}\cosh x\sqrt{1+\sinh^2 x}dx \\
&= 2\pi\int_0^{\ln 2}\cosh^2 x dx \\
&= 2\pi\int_0^{\ln 2}\left(\frac{e^x+e^{-x}}{2}\right)^2 dx \\
&= 2\pi\int_0^{\ln 2}\left(\frac{1}{4}e^{2x}+\frac{1}{2}+\frac{1}{4}e^{-2x}\right)dx \\
&= 2\pi\left[\frac{1}{8}e^{2x}+\frac{1}{2}x-\frac{1}{8}e^{-2x}\right]_0^{\ln 2} \\
&= \left(\frac{15}{16}+\ln 2\right)\pi.
\end{aligned}$$

(b) Note that

$$\begin{aligned}
V_1 = \pi\int_0^1\left((k(4x-3x^2))^2-(kx^2)^2\right)dx &= \pi k^2\int_0^1\left(16x^2-24x^3+8x^4\right)dx \\
&= \pi k^2\left[\frac{16}{3}x^3-6x^4+\frac{8}{5}x^5\right]_0^1 \\
&= \frac{14}{15}\pi k^2
\end{aligned}$$

and

$$V_2 = 2\pi\int_0^1 x\left(k(4x-3x^2)-kx^2\right)dx = 2\pi k\int_0^1 4(x^2-x^3)dx = 8\pi k\left[\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1 = \frac{2}{3}\pi k.$$

Since V_1 is half of V_2, then $\frac{14}{15}\pi k^2 = \frac{1}{3}\pi k$. Since $k>0$, it follows that $k=\frac{5}{14}$.

Question 7

(a) Integrating factor $= e^{\int \cos x dx} = e^{\sin x}$.

$$\begin{aligned}
&\frac{dy}{dx}\cdot e^{\sin x}+y\cos x\cdot e^{\sin x} = 2x \\
\Rightarrow &\frac{d}{dx}\left(e^{\sin x}\cdot y\right) = 2x \\
\Rightarrow &\int d\left(e^{\sin x}\cdot y\right) = \int 2x dx \\
\Rightarrow &e^{\sin x}\cdot y = x^2+C.
\end{aligned}$$

Substituting $x=\pi$ and $y=0$ to the last equation, we get $C=-\pi^2$. Hence, $e^{\sin x}\cdot y = x^2-\pi^2$, or equivalently

$$y = \frac{x^2-\pi^2}{e^{\sin x}}.$$

(b) (i)

$$
\begin{aligned}
&\frac{dQ}{dt} = k(50-Q)(100-Q) \\
\Rightarrow &\int \frac{dQ}{(Q-50)(Q-100)} = k\int dt \\
\Rightarrow &\frac{1}{50}\left(\int \frac{dQ}{Q-100} - \int \frac{dQ}{Q-50}\right) = kt + C \\
\Rightarrow &\frac{1}{50}\left(\ln|100-Q| - \ln|50-Q|\right) = kt + C \\
\Rightarrow &\ln\left|\frac{100-Q}{50-Q}\right| = 50kt + 50C \\
\Rightarrow &\frac{100-Q}{50-Q} = \pm e^{50C}\cdot e^{50kt} \\
\Rightarrow &\frac{100-Q}{50-Q} = A\cdot e^{50kt} \qquad , A = \pm e^{50C}.
\end{aligned}
$$

Plugging in $Q = t = 0$ to the last equation, we get $A = 2$. Hence,

$$\frac{100-Q}{50-Q} = 2e^{50kt} \Rightarrow Q = \frac{100\left(e^{50kt}-1\right)}{2e^{50kt}-1}.$$

Remark: Here we omit the trivial solutions $Q(t) = 50$ and $Q(t) = 100$ since they do not satisfy the initial condition.

(ii) Let $f(Q) = k(50-Q)(100-Q) = kQ^2 - 150kQ + 5000k$. We want to find the value of Q that minimizes $f(Q)$. Note that $f'(Q) = 2kQ - 150k$. Critical point: $f'(Q) = 0 \Rightarrow Q = 75$. Observe that $f'(Q) < 0$ when $Q < 75$ and $f'(Q) > 0$ when $Q > 75$. Hence, $f(Q)$ is minimized when $Q = 75$.

Question 8

Lemma: For the function f defined in the question, we have

$$\int_0^1 f(x)dx = \int_0^1 \left(\frac{x^2-x}{2}\right) f''(x)dx.$$

Proof of Lemma:

$$
\begin{aligned}
\int_0^1 f(x)dx &= [x\cdot f(x)]_0^1 - \int_0^1 x\cdot f'(x)dx \\
&= -\int_0^1 x\cdot f'(x)dx \\
&= -\int_0^1 x\cdot f'(x)dx + \frac{1}{2}\int_0^1 f'(x)dx \\
&= -\int_0^1 \left(x-\frac{1}{2}\right) f'(x)dx \\
&= -\left[\left(\frac{x^2-x}{2}\right) f'(x)\right]_0^1 + \int_0^1 \left(\frac{x^2-x}{2}\right) f''(x)dt \\
&= \int_0^1 \left(\frac{x^2-x}{2}\right) f''(x)dx.
\end{aligned}
$$

Proof of Question 8:
Since f'' is continuous on $[0,1]$, by the Extreme Value Theorem, it attains minimum and maximum values. Let $m = f''(a)$ and $M = f''(b)$ where $a, b \in [0,1]$ be the minimum and maximum values respectively. Then,

$$
\begin{aligned}
& m \le f''(x) \le M && \forall x \in [0,1] \\
\Rightarrow & M\left(\frac{x^2-x}{2}\right) \le \left(\frac{x^2-x}{2}\right) f''(x) \le m\left(\frac{x^2-x}{2}\right) && \forall x \in [0,1] \\
\Rightarrow & M\int_0^1 \frac{x^2-x}{2}dx \le \int_0^1 \left(\frac{x^2-x}{2}\right) f''(x)dx \le m\int_0^1 \frac{x^2-x}{2}dx \\
\Rightarrow & -\frac{M}{12} \le \int_0^1 \left(\frac{x^2-x}{2}\right) f''(x)dx \le -\frac{m}{12} \\
\Rightarrow & m \le -12\int_0^1 \left(\frac{x^2-x}{2}\right) f''(x)dx \le M \\
\Rightarrow & m \le -12\int_0^1 f(x)dx \le M \\
\Rightarrow & f''(a) \le -12\int_0^1 f(x)dx \le f''(b).
\end{aligned}
$$

Since f'' is continuous on $[0,1]$ and $a, b \in [0,1]$, by the Intermediate Value Theorem, there exists $c \in [0,1]$ such that $f''(c) = -12\int_0^1 f(x)dx$, or equivalently $\int_0^1 f(x)dx = -\frac{1}{12}f''(c)$.