NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Zheng Shaoxuan

MA2214 Combinatorial Analysis
AY 2003/2004 Sem 2

Question 1

(a) (i) If there are no restrictions, consider the 40 workers as 40 objects to be positioned among 7 barriers, where the positions of the objects before, after and between each barrier represents the queues in front of each of the food stalls. If the objects are identical, these 40 objects can be arranged with the 7 barriers in $\binom{47}{7}$ ways.

Since the objects (workers in this case) are distinct, the number of ways for the workers to queue in front of the food stalls without restrictions is

$$\binom{47}{7} \cdot 40!.$$

(ii) If there are at least 3 workers in each queue, consider 16 free objects to be positioned among 7 barriers, since 24 of the objects have already been positioned. If the objects are identical, these 16 objects can be arranged with the 7 barriers in $\binom{23}{7}$ ways.

Since the objects are distinct, the number of ways this time is

$$\binom{23}{7} \cdot 40!.$$

(b) (i) For each distinct sweet, there are 6 different ways to assign it, since there are 6 distinct children. Hence, for 20 distinct sweets, the number of ways to distribute them among 6 distinct children is

$$6^{20}.$$

(ii) There are $\binom{20}{5}$ ways to choose which 5 of the 20 distinct sweets to give to the particular boy. After giving this boy the 5 sweets, there are a total of 15 sweets left to be distributed among 5 children in which no child is empty handed. This second task can be performed in $F(15, 5)$ ways, which is 5! times the Sterling number of the second kind. Hence, the total number of ways is the product of the number of ways to do each task, which is

$$\binom{20}{5} \sum_{k=0}^{5} (-1)^k \binom{15}{k} (5-k)^{15}.$$

Question 2

(a) (i) By fixing a particular child at a position on the table and considering the linear arrangement of the remaining 24 children from one side of the child to the other, looping around the table, the number of ways is

$$24!.$$

(ii) Within the block, the girls can be arranged in 5! ways. Fix the block at a position on the table, and consider the similar linear arrangement among the remaining 20 boys, which by itself has 20! ways. Hence, the total number of ways is the product of the two which is

$$5! \cdot 20!.$$

(iii) Consider the block formed by G with B_1 and B_2 adjacent to G. There are exactly 2 ways to arrange this block by itself. Fix the block at a position on the table, and consider the similar linear arrangement among the remaining 22 children, which by itself has 22! ways. Hence, the total number of ways is the product of the two which is

$$2 \cdot 22!.$$

(iv) Fix the position of a particular girl at a position on the table. The problem is now identical to arranging 20 boys with the 4 other girls in a row such that there are at least 2 boys between, before and after each girl. If the boys are identical, this is similar to the problem of arranging 10 free objects amongst 4 barriers, which has $\binom{14}{4}$ number of ways. Since the boys are distinct, arrangement amongst themselves contributes a factor of 20! ways. Since the 4 girls can too arrange among themselves, this contributes another 4! ways. Hence, the total number of ways is

$$\binom{14}{4} \cdot 20! \cdot 4!.$$

(b) (i) It suffices to show that the expression given in the question is the number of ways to arrange $(n+1)!$ distinct objects into $n+1$ identical boxes, each containing $n!$ objects, and hence must be an integer for every positive integer n.

Indeed by first arranging the $(n+1)!$ objects in a row (in $((n+1)!)!$ ways), then by discounting all the different arrangements of the $n!$ objects in each of the $n+1$ boxes of objects, where the objects in each box is chosen by taking the first or subsequent $n!$ objects in the row (this contributes a division by a factor of $((n!)!)^{n+1}$ ways), and finally by considering that the boxes are all identical and the objects can be interchangeably positioned in the other boxes while being considered as the same case (this contributes a division by a factor of $(n+1)!$ ways), we obtain the expression as given in the question, which is

$$\frac{((n+1)!)!}{((n!)!)^{n+1}(n+1)!}.$$

(ii) Observe that

$$\begin{aligned} & 3\binom{7n+4}{4n+2} - 4\binom{7n+4}{4n+3} \\ = \; & 3\frac{(7n+4)!}{(4n+2)!(3n+2)!} - 4\frac{(7n+4)!}{(4n+3)!(3n+1)!} \\ = \; & \frac{(3(4n+3)-4(3n+2))(7n+4)!}{(4n+3)!(3n+2)!} \\ = \; & \frac{(7n+4)!}{(4n+3)!(3n+2)!}. \end{aligned}$$

Since each of the binomial coefficients in the first expression represents a way to choose a number of objects from a set of distinct objects, each binomial coefficient is an integer, and hence any linear combination of them is an integer for every positive integer n. Hence the expression as given in the question is an integer for every positive integer n.

Question 3

(a) Let

- P_1, P_2, P_3 and P_4 be the properties that the integer in concern is divisible by 4, 6, 10 and 25 respectively;
- $\omega(P_{i_1}P_{i_2}\dots P_{i_m})$ be the number of elements of A possessing the properties $P_{i_1}, P_{i_2}, \dots, P_{i_m}$, where $1 \leq m \leq 4$;
- $\omega(m) = \sum(\omega(P_{i_1}P_{i_2}\dots P_{i_m}))$, $\omega(0) = |A| = 1000$.

The number of integers between 1 to 1000 that is divisible by a positive integer n is simply $\lfloor \frac{1000}{n} \rfloor$, while the number of integers between 1 to 1000 that is divisible by all of a number of integers with the lowest common multiple (lcm) between these positive integers being q, is $\left\lfloor \frac{1000}{q} \right\rfloor$, since a number is divisible by all of a number of positive integers if and only the number is divisible by the lowest common multiple of all the positive integers. This is because ($\Rightarrow$) if each of these positive integers divides a certain number N, for each prime factor in each of these positive integers, N must contain at least as many of the prime factor, and hence N contains all the prime factors of q in enough quantity by the definition of q, and hence q divides N, and because ($\Leftarrow$) if q divides N, N is divisible by all factors of q, and hence is divisible by each of the stated positive integers whose lcm is q.

Hence, to work out $E(m)$ for m from 0 to 4, it is necessary to find the corresponding $\omega(m)$ values. By performing the below calculations using the reasoning in the above paragraph:

- $\omega(P_1) = \lfloor \frac{1000}{4} \rfloor = 250$;
- $\omega(P_2) = \lfloor \frac{1000}{6} \rfloor = 166$;
- $\omega(P_3) = \lfloor \frac{1000}{10} \rfloor = 100$;
- $\omega(P_4) = \lfloor \frac{1000}{25} \rfloor = 40$;
- $\omega(P_1P_2) = \left\lfloor \frac{1000}{lcm(4,6)} \right\rfloor = \lfloor \frac{1000}{12} \rfloor = 83$;
- $\omega(P_1P_3) = \left\lfloor \frac{1000}{lcm(4,10)} \right\rfloor = \lfloor \frac{1000}{20} \rfloor = 50$;
- $\omega(P_1P_4) = \left\lfloor \frac{1000}{lcm(4,25)} \right\rfloor = \lfloor \frac{1000}{100} \rfloor = 10$;
- $\omega(P_2P_3) = \left\lfloor \frac{1000}{lcm(6,10)} \right\rfloor = \lfloor \frac{1000}{30} \rfloor = 33$;
- $\omega(P_2P_4) = \left\lfloor \frac{1000}{lcm(6,25)} \right\rfloor = \lfloor \frac{1000}{150} \rfloor = 6$;
- $\omega(P_3P_4) = \left\lfloor \frac{1000}{lcm(10,25)} \right\rfloor = \lfloor \frac{1000}{50} \rfloor = 20$;
- $\omega(P_1P_2P_3) = \left\lfloor \frac{1000}{lcm(4,6,10)} \right\rfloor = \lfloor \frac{1000}{60} \rfloor = 16$;
- $\omega(P_1P_2P_4) = \left\lfloor \frac{1000}{lcm(4,6,25)} \right\rfloor = \lfloor \frac{1000}{300} \rfloor = 3$;
- $\omega(P_1P_3P_4) = \left\lfloor \frac{1000}{lcm(4,10,25)} \right\rfloor = \lfloor \frac{1000}{100} \rfloor = 10$;
- $\omega(P_2P_3P_4) = \left\lfloor \frac{1000}{lcm(6,10,25)} \right\rfloor = \lfloor \frac{1000}{150} \rfloor = 6$;
- $\omega(P_1P_2P_3P_4) = \left\lfloor \frac{1000}{lcm(4,6,10,25)} \right\rfloor = \lfloor \frac{1000}{300} \rfloor = 3$.

We obtain:

- $\omega(1) = 250 + 166 + 100 + 44 = 556$;
- $\omega(2) = 83 + 50 + 10 + 33 + 6 + 20 = 202$;

- $\omega(3) = 16 + 3 + 10 + 6 = 35$;
- $\omega(4) = 3$.

And hence, by the Principle of Inclusion and Exclusion:

- $E(0) = 1000 - 556 + 202 - 35 + 3 = 614$;
- $E(1) = 556 - 2 \times 202 + 3 \times 35 - 4 \times 3 = 245$;
- $E(2) = 202 - \binom{3}{2} \times 35 + \binom{4}{2} \times 3 = 115$;
- $E(3) = 35 - \binom{4}{3} \times 3 = 23$;
- $E(4) = 3$.

(b) (i) Without any restrictions, the number of ways needed is simply

$$\frac{9!}{2! \times 2! \times 2! \times 3!}.$$

(ii) Let

- S be the set of all possible integers formed by the 9 letters 3 1s, 2 2s, 2 3s and 2 4s;
- P_i be the property that the (i)-th and $(i+1)$-th number are identical, for $1 \leq i \leq 8$;
- $E(m)$ be the number of elements of S possessing exactly m of the 8 properties for $0 \leq m \leq 8$;
- $\omega(P_{i_1}P_{i_2}\ldots P_{i_m})$ be the number of elements of S possessing the properties P_{i_1}, P_{i_2}, $\ldots$, P_{i_m}, where $1 \leq m \leq 8$;
- $\omega(m) = \sum(\omega(P_{i_1}P_{i_2}\ldots P_{i_m}))$, $\omega(0) = |S|$.

$\omega(0)$ is simply the answer to the above question part, which turns out to be 7560.

$\omega(1)$, by definition, is the sum of $\binom{8}{1}$ cases where for each case, at a different position, the 2 consecutive numbers are identical. If the number 1 is chosen, there are $\frac{7!}{2!2!2!}$ ways to arrange the remaining 7 numbers in the remaining available slots. If number 2, 3 or 4 is chosen, there are $\frac{7!}{3!2!2!}$ ways to arrange the remaining 7 numbers in the remaining available slots. Hence, $\omega(1) = \binom{8}{1} \times \left(1 \times \frac{7!}{2!2!2!} + 3 \times \frac{7!}{3!2!2!}\right) = 10080$.

$\omega(2)$ is the sum of two scenarios: the $\binom{7}{1}$ cases where for each case, at a different position, the 3 consecutive numbers are identical (i.e. they are all 1), and the $\binom{7}{2}$ cases where for each case, at different positions, 2 pairs of consecutive numbers are identical. For the first scenario, there are $\frac{6!}{2!2!2!}$ ways to arrange the remaining 6 numbers in the remaining available slots. For the second scenario, if one of the numbers chosen is 1, there are 3 ways to choose the other number, 2 ways for a pair of numbers to be positioned relative to the other, and $\frac{5!}{2!2!}$ ways to arrange the remaining 5 numbers in the remaining available slots. If none of the numbers chosen is 1, there are $\binom{3}{2}$ ways to choose the two numbers, 2 ways for a pair of numbers to be positioned relative to the other, and $\frac{5!}{3!2!}$ ways to arrange the remaining 5 numbers in the remaining available slots. Hence, $\omega(2) = \binom{7}{1} \times \frac{6!}{2!2!2!} + \binom{7}{2}\left(3 \times 2 \times \frac{5!}{2!2!} + \binom{3}{2} \times 2 \times \frac{5!}{3!2!}\right) = 5670$.

$\omega(3)$ is the sum of two scenarios: the $\binom{6}{2}$ cases where for each case, at different positions, there are 3 consecutive 1s and a different pair of consecutive numbers, and the $\binom{6}{3}$ cases where for each case, at different positions, 3 pairs of consecutive numbers are identical. For the first scenario, there are 3 ways to choose the other number, 2 ways for the relative positioning of the pair and trio of numbers and $\frac{4!}{2!2!}$ ways to arrange the remaining 4 numbers in the remaining available slots. For the second scenario, if one of the numbers chosen is 1, there are $\binom{3}{2}$ ways to choose the other 2 numbers, 3! ways for the relative positioning of the 3 pairs of numbers, and $\frac{3!}{2!}$ ways to arrange the remaining 3 numbers in the remaining available slots. If none of the numbers chosen is 1, there is only 1 way to choose the 3 numbers, 3! ways for the relative positioning of the numbers, and $\frac{3!}{3!}$ ways to arrange the remaining 3 numbers in the remaining available slots. Hence, $\omega(3) = \binom{6}{2} \times 3 \times 2 \times \frac{4!}{2!2!} + \binom{6}{3}\left(\binom{3}{2} \times 3! \times \frac{3!}{2!} + 1 \times 3! \times \frac{3!}{3!}\right) = 1740$.

$\omega(4)$ is the sum of two scenarios: the $\binom{5}{3}$ cases where for each case, at different positions, there are 3 consecutive 1s and two different pair of consecutive numbers, and the $\binom{5}{4}$ cases where for each case, at different positions, 4 pairs of consecutive numbers (2 1s, 2 2s, 2 3s, 2 4s) are identical. For the first scenario, there are $\binom{3}{2}$ ways to choose the other 2 numbers, 3! ways for the relative positioning of the 2 pairs and 1 trio of numbers and $\frac{2!}{2!}$ ways to arrange the remaining 2 numbers in the remaining available slots. For the second scenario, there are 4! ways for the relative positioning of the 4 pairs of numbers, and 1 way to arrange the remaining 1 number in the remaining available slot. Hence, $\omega(4) = \binom{5}{3}\times\binom{3}{2}\times 3!\times\frac{2!}{2!}+\binom{5}{4}\times 4!\times 1 = 300$.

$\omega(5)$ is simply the case where the 3 1s, 2 2s, 2 3s and 2 4s are all consecutive among themselves. There are 4! ways for the relative positions of them and hence $\omega(5) = 4! = 24$. Also $\omega(6) = \omega(7) = \omega(8) = 0$.

$E(0)$ is the desired result, where exactly none of the adjacent digits are identical. Hence, by the Principle of Inclusion and Exclusion,

$$E(0) = 7560 - 10080 + 5670 - 1740 + 300 - 24 = 1686.$$

Question 4

(a) The characteristic equation of the homogenous equation of a_n is

$$\begin{aligned} x^3 - 2x^2 - x + 2 &= 0 \\ (x-1)(x^2 - x - 2) &= 0 \\ (x-1)(x-2)(x+1) &= 0. \end{aligned}$$

Therefore, the homogenous equation of a_n is

$$\begin{aligned} a_n^{(h)} &= A(1)^n + B(2)^n + C(-1)^n \\ &= A + B(2)^n + C(-1)^n. \end{aligned}$$

where A, B, and C are constants to be determined.

Let the particular solution $a_n^{(p)}$ be $Dn^2 + En$ (since the characteristic equation has the constant term A), where D and E are constants to be determined. By substituting the particular solution into the recurrence relation,

$$(Dn^2 + En) - 2(D(n-1)^2 + E(n-1)) - (D(n-2)^2 + E(n-2)) + 2(D(n-3)^2 + E(n-3)) = -4n.$$

By comparing the coefficient of n,

$$\begin{aligned} E + 4D - 2E + 4D - E - 12D + 2E &= -4 \\ -4D &= -4 \\ D &= 1. \end{aligned}$$

By comparing the constant term,

$$\begin{aligned} -2D + 2E - 4D + 2E + 18D - 6E &= 0 \\ 12D - 2E &= 0 \\ E &= 6. \end{aligned}$$

Therefore,

$$a_n^{(p)} = n^2 + 6n.$$

Hence,

$$\begin{aligned} a_n &= a_n^{(h)} + a_n^{(p)} \\ &= A + B(2)^n + C(-1)^n + n^2 + 6n. \end{aligned}$$

By substituting the three given conditions $a_1 = 14$, $a_2 = 29$ and $a_3 = 46$,

$$\begin{cases} A + 2B - C = 7; \\ A + 4B + C = 13; \\ A + 8B - C = 19. \end{cases}$$

By solving the above system of simultaneous equations, $A = 4$, $B = 2$ and $C = 1$. Therefore,

$$a_n = 4 + 2^{n+1} + (-1)^n + n^2 + 6n.$$

(b) (i) First let b_n denote the number of ways of paving a 2 meters by n meters floor, missing a 1 meter by 1 meter corner, using the 3 types of tiles as mentioned in the question. To count a_n, it is necessary to simplify the question by breaking a_n and b_n down into smaller quantities of the same nature (and hence forming a system of recurrence relations) by considering the different cases in which a tile can be placed in the top left or bottom left corner respectively.

The top half of the diagram below shows the 7 different cases in which a_n can be counted in terms of smaller quantities of a_n and b_n. For instance for the first case, if a 1 by 1 tile is placed on the top left corner, the question is simplified into counting b_n. For the subsequent 6 cases of different ways of placing tiles in the top left corner, the problem is respectively broken down into cases of counting b_{n-1}, a_{n-2}, a_{n-1}, b_{n-1}, b_{n-1} and a_{n-2}. Therefore the following recurrence relation can be constructed:

$$a_n = a_{n-1} + 2a_{n-2} + b_n + 3b_{n-1}. \qquad (1)$$

The bottom half of the diagram below shows the 3 different cases in which b_n can be counted in terms of smaller quantities of a_n and b_n. Following a similar manner of placing tiles in the bottom left corner, the problem is respectively broken down into cases of counting a_{n-1}, b_{n-1} and a_{n-2}. Therefore the following recurrence relation can be constructed:

$$b_n = a_{n-1} + a_{n-2} + b_{n-1}.$$

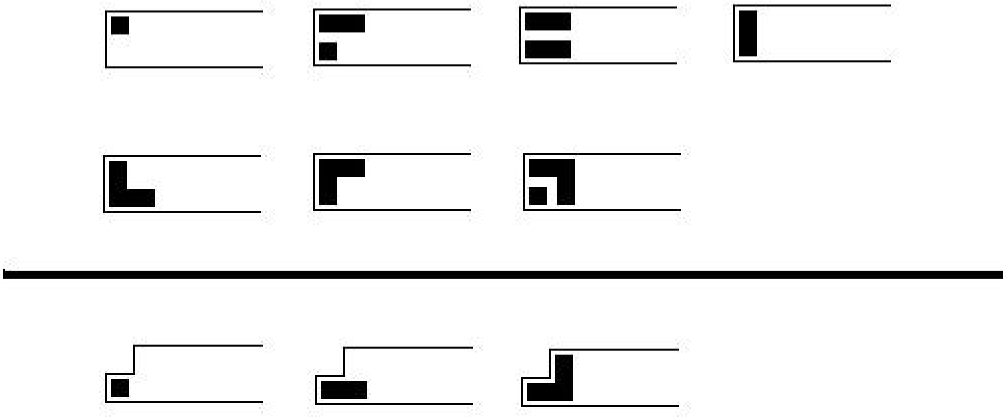
which can be rearranged to form

$$b_{n-1} = b_n - a_{n-1} - a_{n-2}. \qquad (2)$$

Substituting (2) into (1),

$$\begin{aligned} a_n &= a_{n-1} + 2a_{n-2} + b_n + 3b_n - 3a_{n-1} - 3a_{n-2} \\ 4b_n &= a_n + 2a_{n-1} + a_{n-2} \\ b_n &= \frac{1}{4}(a_n + 2a_{n-1} + a_{n-2}). \end{aligned}$$

AY0304 Sem 2 Qn 4b picture.jpg



Substituting b_n into (1),

$$\begin{aligned}
a_n &= a_{n-1} + 2a_{n-2} + \frac{1}{4}(a_n + 2a_{n-1} + a_{n-2}) + \frac{3}{4}(a_{n-1} + 2a_{n-2} + a_{n-3}) \\
4a_n &= 4a_{n-1} + 8a_{n-2} + a_n + 2a_{n-1} + a_{n-2} + 3a_{n-1} + 6a_{n-2} + 3a_{n-3} \\
3a_n &= 9a_{n-1} + 15a_{n-2} + 3a_{n-3} \\
a_n &= 3a_{n-1} + 5a_{n-2} + a_{n-3}.
\end{aligned}$$

By manual counting, $a_1 = 2$, $a_2 = 11$, $b_1 = 1$, $b_2 = 4$. By (2), $b_3 = a_2 + a_1 + b_2 = 17$. By (1), $a_3 = a_2 + 2a_1 + b_3 + 3b_2 = 44$.
Hence a recurrence relationship for a_n is

$$a_n = 3a_{n-1} + 5a_{n-2} + a_{n-3}.$$

with $a_1 = 2$, $a_2 = 11$, $a_3 = 44$.

(ii) The characteristic equation for a_n is

$$\begin{aligned}
x^3 - 3x^2 - 5x - 1 &= 0 \\
(x+1)(x^2 - 4x - 1) &= 0 \\
x &= -1, 2 + \sqrt{5}, 2 - \sqrt{5}.
\end{aligned}$$

Therefore,

$$a_n = A(-1)^n + B(2+\sqrt{5})^n + C(2-\sqrt{5})^n.$$

where A, B and C are constants to be determined.
Be aware that substituting the case where $n = 3$ into the above equation will increase the tedium of solving this question, so it will make my life easier by evaluating a_0 (which is not a quantity that holds meaning in the context of the question, but rather to simplify our calculations), for which a_0, a_1 and a_2 will lead to the expression of a_3 via the recurrence relation. By using the recurrence relation,

$$a_0 = a_3 - 3a_2 - 5a_1 = 1.$$

By substituting the 3 conditions $a_0 = 1$, $a_1 = 2$ and $a_2 = 11$ into the equation of a_n,

$$\begin{cases}
A + B + C = 1; \\
-A + B(2+\sqrt{5}) + C(2-\sqrt{5}) = 2; \\
A + B(9+4\sqrt{5}) - C(9-4\sqrt{5}) = 11.
\end{cases}$$

By solving the above system of simultaneous equations (this is a tedious but mechanical process), $A = \frac{1}{2}$, $B = \frac{5+3\sqrt{5}}{20}$ and $C = \frac{5-3\sqrt{5}}{20}$. Therefore,

$$a_n = \frac{1}{2}(-1)^n + \frac{5+3\sqrt{5}}{20}(2+\sqrt{5})^n + \frac{5-3\sqrt{5}}{20}(2-\sqrt{5})^n.$$

Question 5

(a) (i) A suitable exponential generating function for a_n is

$$\begin{aligned} & \left(\frac{e^{4x}+e^{-4x}}{2}\right)\left(\frac{e^{4x}-e^{-4x}}{2}\right)(e^x - 1 - x - \frac{x^2}{2})(e^{3x}) \\ = \; & \frac{1}{4}e^{3x}(e^{8x}-e^{-8x})(e^x - 1 - x - \frac{x^2}{2}) \\ = \; & \frac{1}{4}\left(e^{12x} - e^{-4x} - e^{11x} + e^{-5x} + x(-e^{11x} + e^{-5x}) + \frac{1}{2}x^2(-e^{11x}+e^{-5x})\right). \end{aligned}$$

(ii) Note that for any positive integer a, the x^n term of the expansion of e^{ax} is $\frac{a^n}{n!}$. Since a^n is $n!$ times the coefficient of x^n in the above exponential generating function,

$$\begin{aligned} a_n \; = \; & \frac{1}{4}\Big((12)^n - (-4)^n - (11)^n + (-5)^n + n(-(11)^{n-1} + (-5)^{n-1}) \\ & + \frac{1}{2}(n)(n-1)(-(11)^{n-2} + (-5)^{n-2})\Big). \end{aligned}$$

(b) (i) A suitable ordinary generating function for a_n is

$$\begin{aligned} & \left(\frac{(1-x)^{-4}+(1+x)^{-4}}{2}\right)\left(\frac{(1-x)^{-4}-(1+x)^{-4}}{2}\right)(x^3)(1-x)^{-1}(1-x)^{-3} \\ = \; & \frac{1}{4}x^3(1-x)^{-4}((1-x)^{-8} - (1+x)^{-8}) \\ = \; & \frac{1}{4}x^3((1-x)^{-12} - (1-x)^{-4}(1+x)^{-8}) \\ = \; & \frac{1}{4}x^3\left(\sum_{i=0}^{\infty}\binom{11+i}{11}x^i - \sum_{i=0}^{\infty}\binom{3+i}{3}x^i \sum_{j=0}^{\infty}(-1)^j\binom{7+j}{7}x^j\right). \end{aligned}$$

(ii) a_n is the coefficient of x_n in the above generating function. Therefore,

$$\begin{aligned} a_n \; &= \; \frac{1}{4}\left(\binom{11+n-3}{11} - \sum_{i=0}^{n-3}(-1)^i\binom{7+i}{7}\binom{3+n-3-i}{3}\right) \\ &= \; \frac{1}{4}\left(\binom{8+n}{11} - \sum_{i=0}^{n-3}(-1)^i\binom{7+i}{7}\binom{n-i}{3}\right). \end{aligned}$$