NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA3110 Mathematical Analysis II
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Question 1

(a) Since g can be rewritten as the $g(x) = \min\{\max\{f(x), -1\}, 1\}$. Since $f(x), 1, -1$ are continous functions on $\mathbb{R}$ and $\max\{f, g\}$ is continuous if f and g are continuous, also $\min\{f, g\}$ is continuous if f and g are continuous, therefore we can conclude that g is a continuous function.

(b) Since $f(x_1 + x_2) \leq f(x_1) + f(x_2)$ we have $-f(x_2) \leq f(x_1) - f(x_1 + x_2)$. Letting $x_2 = h$ and $x_1 = x - h$, we have $-f(h) \leq f(x - h) - f(x)$. Also, $f(x_1 + x_2) - f(x_1) \leq f(x_2)$ now let $x_2 = -h$ and $x_1 = x$, hence we have $f(x - h) - f(x) \leq f(-h)$. Therefore we have $|f(x - h) - f(x)| \leq \max\{|f(-h)|, |f(h)|\}$.
Given $\epsilon > 0$,
since f is continuous, there exist $\delta_1 > 0$ such that $|f(h)| < \epsilon$ for $|h| < \delta_1$.
By leting $h = x - y$ we have for all $x \in \mathbb{R}$, if $|x - y| < \delta_1$ then

$$|f(y) - f(x)| \leq \max\{|f(x - y)|, |f(|y - x|)\} < \epsilon.$$

Hence f is uniformly continuous on $\mathbb{R}$.

Question 2

(a) Since f is twice differentiable on the open interval I then for any $c \in I$ there exists a small enough $\delta > 0$ such that $f(c + h), f(c - h)$ is twice differentiable for all $h \in (c - \delta, c + \delta)$.
Since $\lim_{h\to 0} f(c + h) - 2f(c) + f(c - h) = 0$ and $\lim_{h\to 0} h^2 = 0$, $\lim_{h\to 0} f'(c + h) - f'(c - h) = 0$ and $\lim_{h\to 0} 2h = 0$, by L'Hospital's Rule, we have

$$\begin{aligned}\lim_{h\to 0} \frac{f(c + h) - 2f(c) + f(c - h)}{h^2} &= \lim_{h\to 0} \frac{f'(c + h) - f'(c - h)}{2h} \\ &= \lim_{h\to 0} \frac{f''(c + h) + f''(c - h)}{2} \\ &= \frac{2f''(c)}{2} \\ &= f''(c)\end{aligned}$$

(b) For $x \neq n\pi$ for $n \in \mathbb{Z}$, we have $\sin(x) \neq 0$.

Dividing both sides by $|\sin(x)|$ and apply L'Hospital's Rule we get,

$$\begin{aligned}
\frac{|a_1\sin(x)+...+a_n\sin(nx)|}{|\sin(x)|} &\leq 1\\
\lim_{x\to 0}\frac{|a_1\sin(x)+...+a_n\sin(nx)|}{|\sin(x)|} &\leq 1\\
\lim_{x\to 0}\left|a_1\frac{\sin(x)}{\sin(x)}+...+a_n\frac{\sin(nx)}{\sin(x)}\right| &\leq 1\\
\lim_{x\to 0}\left|a_1\frac{\cos(x)}{\cos(x)}+...+na_n\frac{\cos(nx)}{\cos(x)}\right| &\leq 1\\
|a_1+2a_2+3a_3+...+na_n| &\leq 1
\end{aligned}$$

Question 3

(a) Since the limit

$$\begin{aligned}
\lim_{n\to\infty}\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} &= \lim_{n\to\infty}\frac{x}{n+1}\\
&= \lim_{n\to\infty}\frac{x}{n+1}\\
&= 0
\end{aligned}$$

Therefore radius of convergence is ∞. Hence the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges uniformly hence pointwise on the whole line $\mathbb{R}$.

(b) Since f is differentiable at 2.

$$\begin{aligned}
\lim_{x\to 2}\frac{6x-2f(x)}{x-2} &= \lim_{x\to 2}\frac{6x-12+12-2f(x)}{x-2}\\
&= \lim_{x\to 2}\left(\frac{6x-12}{x-2}-\frac{2f(x)-2(6)}{x-2}\right)\\
&= 6-2f'(2)\\
&= 0
\end{aligned}$$

Question 4

(a) Since f is continuous on $[0,1]$, f^2 is continuous on $[0,1]$. Hence f^2 is Riemann Integrable. Applying the Fundamental Theorem of Calculus, we have

$$F_1(x)=\int_0^x [f(t)]^2\,dt.$$

is differentiable on $[0,1]$.
Since $h(x) = x^3$ is differentiable on $[0,1]$ and $h([0,1]) = [0,1]$, therefore we have $F(x) = F_1(h(x))$ is a composition of differentiable functions. Hence we have $F(x)$ is differentiable on $[0,1]$ and $F'(x) = 3x^2\left[\left(f(x^3)\right)^2 - (f(0))^2\right]$.

(b) Let $M = \sup_{x\in[0,1]}(f_0(x)) < \infty$. Then we have $|f_0(x)| \leq M$ for all $x \in [0,1]$. Therefore we have $-Mx \leq f_1(x) = \int_0^x f_0(x)\ dx \leq Mx$ for all $x \in [0,1]$.
Then for all f_n we have $\frac{-M}{n!} \leq \frac{-M}{n!}x^n \leq f_n(x) = \int_0^x f_{n-1}(x)\ dx \leq \frac{M}{n!}x^n \leq \frac{M}{n!}$ for all $n \in \mathbb{N}$.
Hence we can see that $f_n \to 0$ uniformly.

Question 5

(a) Since for all $0 \leq x \leq \frac{\pi}{4} < 1-\epsilon$ for some $\epsilon > 0$. Hence we have $0 \leq x^n < (1-\epsilon)^n$. Therefore we have $0 \leq x^n \sin(nx) \leq (1-\epsilon)^n \sin(nx) \leq (1-\epsilon)^n$.
Hence we can see that $x^n \sin(nx)$ converges uniformly to 0. Hence we have

$$\begin{aligned}\lim_{n\to\infty}\int_0^{\frac{\pi}{4}} x^n \sin(nx)\ dx &= \int_0^{\frac{\pi}{4}} \lim_{n\to\infty} x^n \sin(nx)\ dx \\ &= \int_0^{\frac{\pi}{4}} 0\ dx \\ &= 0\end{aligned}$$

(b) By Taylor's Theorem, for all $x \in [0,1]$ there exists $\alpha_x \in (0,1)$ such that we have

$$f(x) = f(1) + (x-1)f'(1) + f''(\alpha_x)\frac{(x-1)^2}{2}.$$

Hence we obtain the following equations

$$\begin{aligned}0 = f(0) &= 1 - f'(1) + \frac{1}{2}f''(\alpha_1) \\ 2 = f(2) &= 1 + f'(1) + \frac{1}{2}f''(\alpha_2)\end{aligned}$$

Hence by taking the sum of the two equations , we have

$$\begin{aligned}2 &= 2 + \frac{1}{2}f''(\alpha_1) + \frac{1}{2}f''(\alpha_2) \\ -f''(\alpha_1) &= f''(\alpha_2)\end{aligned}$$

Therefore either $f''(\alpha_1) = 0$ or $f''(\alpha_1)$ and $f''(\alpha_2)$ differs in sign. Hence by Daboux Theorem there exists a $c \in (\alpha_1, \alpha_2)$ such that $f''(c) = 0$.

(c) Claim: $\lim_{x\to\infty} f'(x) = 0$.
Let $M > 0$ be such that $|f''(x)| \leq M$ for all $x \in (0, \infty)$. Suppose we are given $\epsilon > 0$. There exists N such that if $x > N$, then $|f(x)| = |f(x) - 0| < \epsilon^2/8M$.

Fix $x > N$. Let h be $\frac{\epsilon}{2M}$. We perform the Taylor expansion of degree 2 at $x + h$ about x:

$$\begin{aligned} f(x+h) &= f(x) + f'(x)h + \frac{f''(c)}{2}h^2 \\ f'(x) &= \frac{f(x+h) - f(x)}{h} - \frac{f''(c)h}{2} \end{aligned}$$

for some $x < c < x + h$. Thus

$$\begin{aligned} |f'(x)| &< \left|\frac{f(x+h) - f(x)}{h}\right| + \left|\frac{f''(c)h}{2}\right| \\ &< \frac{|f(x+h)| + |f(x)|}{|h|} + \left|\frac{M \cdot \epsilon/2M}{2}\right| \\ &< \frac{\frac{\epsilon^2}{8M} + \frac{\epsilon^2}{8M}}{\frac{\epsilon}{2M}} + \frac{\epsilon}{4} = \frac{\epsilon}{2} + \frac{\epsilon}{4} < \epsilon. \end{aligned}$$

Given $\epsilon > 0$, we have produced N such that if $x > N$, then $|f'(x) - 0| = |f'(x)| < \epsilon$.

Question 6

(a) (i) For any $x \in \mathbb{R}_{\leq 0}$ we have

$$\lim_{n\to\infty} ne^{-nx} = \infty \neq 0$$

therefore the series diverges.
Now for all $x \in \mathbb{R}_{>0}$ we have $e^{-x} < 1$, there exist an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $e^{\frac{nx}{2}} \geq n$. Hence we may rewrite the sum

$$\begin{aligned} \sum_{n=1}^{\infty} ne^{-nx} &= \sum_{n=1}^{N} ne^{-nx} + \sum_{m=N}^{\infty} me^{-mx} \\ &\leq \sum_{n=1}^{N} ne^{-nx} + \sum_{m=N}^{\infty} e^{\frac{mx}{2}} e^{-mx} \\ &= \sum_{n=1}^{N} ne^{-nx} + \sum_{m=N}^{\infty} \left(\frac{1}{e^{\frac{x}{2}}}\right)^m \end{aligned}$$

Since $x > 0$, we have $\frac{1}{e^{\frac{x}{2}}} < 1$, therefore the last sum converges. Therefore for all $x \in \mathbb{R}_{>0}$, the series converges pointwise.

(ii) For all $\epsilon > 0$, the series converges pointwise in the closed interval $[\epsilon, \infty)$. Also we note that for all $x \in [\epsilon, \infty)$, we also note that for all $x \in [\epsilon, \infty)$ we have $ne^{-nx} \leq ne^{-n\epsilon}$. Hence by the Weistrass M-test,the series converges uniformly on the interval $[\epsilon, \infty)$. Since ϵ is arbitrary, we can let ϵ go to zero, therefore the series converges uniformly on the open interval $(0, \infty)$.

(iii) Since f is a uniform limit of continuous functions on $(0, \infty)$, hence f is continuous on $(0, \infty)$.

(iv) Since the series converges uniformly on $[1, 2]$, we have

$$\begin{aligned}\int_1^2 \sum_{n=1}^{\infty} ne^{-nx}\,dx &= \sum_{n=1}^{\infty} \int_1^2 ne^{-nx}\,dx \\ &= \sum_{n=1}^{\infty} [-e^{-2n} + e^{-n}] \\ &= \frac{-e^{-2}}{1-e^{-2}} + \frac{e^{-1}}{1-e^{-1}}\end{aligned}$$

The last equality is true due to the absolute convergence of the series

$$\sum_{n=1}^{\infty} -e^{-2n} + e^{-n}$$

since $\left|-e^{-2n} + e^{-n}\right| \leq e^{-2n} + e^{-n}$.

(b) Since f is twice differentiable on $[0, 1]$, by Taylor's Theorem we get

$$f(x) = f(0) + (x - \frac{1}{3})f'(0) + \frac{(x-\frac{1}{3})^2}{2} f''(\alpha)$$

for some $\alpha \in (0, 1)$.
Since $g(x) = x^2$ maps $[0, 1]$ to $[0, 1]$ we have

$$\begin{aligned}f(x^2) = fg(x) &= f\left(\frac{1}{3}\right) + (x^2 - \frac{1}{3})f'\left(\frac{1}{3}\right) + \frac{(x^2-\frac{1}{3})^2}{2} f''(\alpha) \\ &\leq f\left(\frac{1}{3}\right) + (x^2 - \frac{1}{3})f'\left(\frac{1}{3}\right)\end{aligned}$$

Integrating from 0 to 1, we get

$$\begin{aligned}\int_0^1 f(x^2)\,dx &\leq \int_0^1 f\left(\frac{1}{3}\right) + \left(x^2 - \frac{1}{3}\right) f'\left(\frac{1}{3}\right)\,dx \\ &= f\left(\frac{1}{3}\right) + f'\left(\frac{1}{3}\right)\left[\frac{x^3}{3} - \frac{x}{3}\right]_0^1 \\ &= f\left(\frac{1}{3}\right)\end{aligned}$$

(c) By the definition of derivatives, we may do the following computations,

$$\begin{aligned}&\lim_{x\to 0} \frac{f(x) + f\left(\frac{x}{2}\right) + ... + f\left(\frac{x}{10}\right)}{x} \\ =\ &\lim_{x\to 0} \frac{f(x) - f(0)}{x} + \frac{f\left(\frac{x}{2}\right) - f(0)}{x} + ... + \frac{f\left(\frac{x}{10}\right) - f(0)}{x} + 10\frac{f(0)}{x} \\ =\ &\lim_{x\to 0} \frac{f(x) - f(0)}{x} + \frac{1}{2}\frac{f\left(\frac{x}{2}\right) - f(0)}{\frac{x}{2}} + ... + \frac{1}{10}\frac{f\left(\frac{x}{10}\right) - f(0)}{\frac{x}{10}} + 10\frac{f(0)}{x} \\ =\ &f'(0) + \frac{1}{2}f'(0) + \cdots + \frac{1}{10}f'(0) + 10\lim_{x\to 0}\frac{0}{x} \\ =\ &\sum_{n=1}^{10} \frac{1}{n} =\end{aligned}$$