NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Terry Lau Shue Chien

MA3233 Algorithmic Graph Theory
AY 2009/2010 Sem 1

Question 1

(a) (i) $d(v_5, v_7) = 4$

(ii) $e(v_3) = \max_{i=1,2,4,5,6,7}\{d(v_i, v_3)\} = 2$

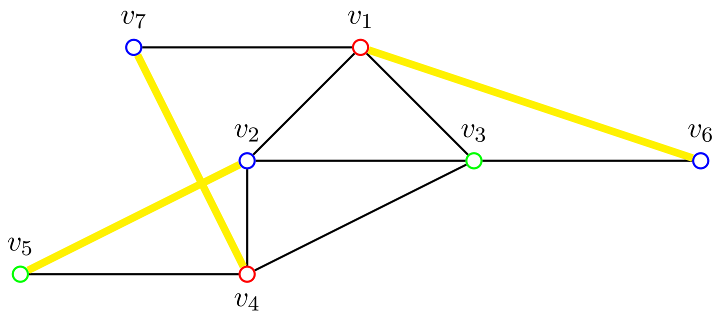
(iii)

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(iv) $diam(H) = \max_{i=1,\ldots,7}\{e(v_i)\} = \{3, 2, 2, 3, 4, 3, 4\} = 4$

(v) $\chi(H) = 3$

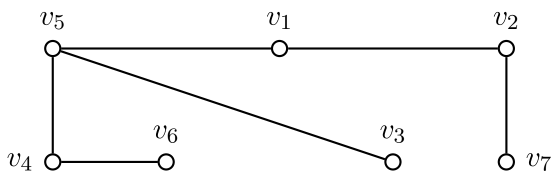
(b) From (3), we have to make sure that G is Eulerian. By observing H, we realize that $v_1, v_2, v_4, v_5, v_6, v_7$ are odd vertices. Therefore by (2), we know that we have to obtain a perfect matching for these 6 vertices. Next, we have to also make sure that $\chi(G) = \chi(H)$. By trial and error, we obtain the following graph, G:



Check that all the vertices now have even degrees, which satisfies (3). Also, $\chi(G) = \chi(H) = 3$ as referring to the above colouring.

Question 2

(i) Using Prim's algorithm, we obtain the following spanning tree, T of G with $w(T) = 12$.



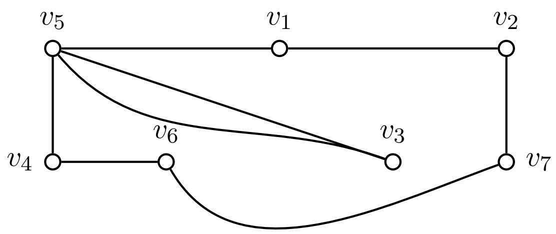
(ii) Applying Christophide's algorithm, we first identify odd vertices in T: v_3, v_5, v_6, v_7. Then, we find a minimum matching from K_4 that formed by these odd vertices. There are C_2^3 matchings in total.

$$M_1 = \{v_3v_5, v_6v_7\} \quad w(M_1) = 7$$

$$M_2 = \{v_3v_7, v_6v_5\} \quad w(M_2) = 7$$

$$M_3 = \{v_3v_6, v_5v_7\} \quad w(M_3) = 8$$

We choose M_1 or M_2 since they have the same lowest weight. Without loss of generality, we choose M_1 and we have H



We now construct an Eulerian path from H,

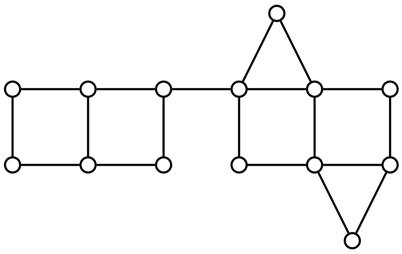
$$v_1v_2v_7v_6v_4v_5v_3v_5$$

Then we eliminate repeated vertices, and obtain a spanning cycle C of G with $w(C) = 23$

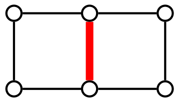
$$v_1v_2v_7v_6v_4v_5v_3v_1$$

Question 3

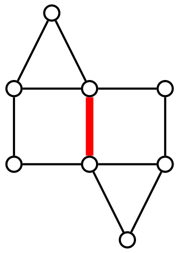
Let G



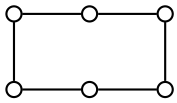
Note that $\tau(G) = \tau(G_1) \times \tau(G_2)$, where G_1 is



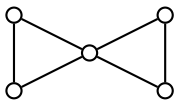
and G_2 is



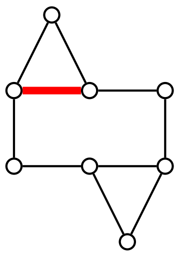
Contracting at the red edge of G_1, $\tau(G_1) = \tau(G_3) + \tau(G_4)$, where G_3 is



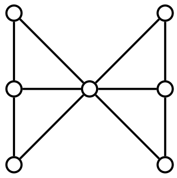
and G_4 is



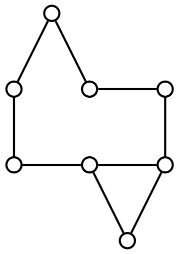
Therefore $\tau(G_1) = 6 + 3 \times 3 = 15$.
Contracting at the red edge of G_2, we have $\tau(G_2) = \tau(G_5) + \tau(G_6)$, where G_5 is



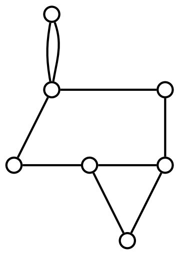
and G_6 is



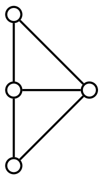
For G_5, contracting the red edge, we have $\tau(G_5) = \tau(G_7) + \tau(G_8)$, where G_7 is



and G_8 is



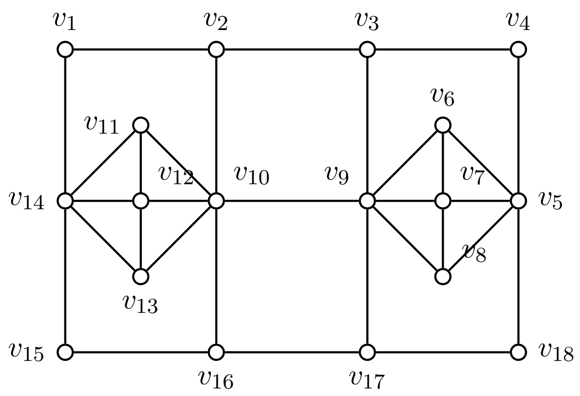
Since $\tau(G_7) = 7 + 3 - 2 + 6 \times 2 = 20$ and $\tau(G_8) = 2 \times (5 + 3 - 2 + 4 \times 2) = 28$, therefore $\tau(G_5) = 20 + 28 = 48$.
For G_6, $\tau(G_6) = (\tau(G_9))^2$, where G_9 is



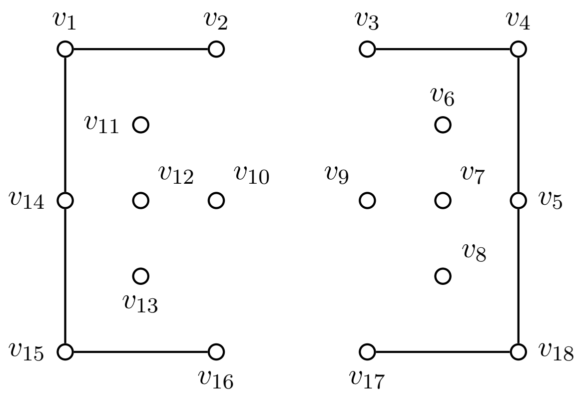
Therefore $\tau(G_6) = (3+3-2+2\times 2)^2 = 64$. Hence we obtain $\tau(G_2) = \tau(G_5)+\tau(G_6) = 48+64 = 112$.
Finally $\tau(G) = \tau(G_1) \times \tau(G_2) = 15 \times 112 = 1680$.

Question 4

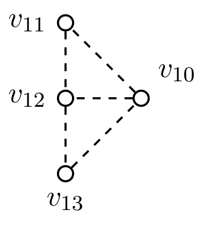
(a) Let G be the graph



We want to show that G is not Hamiltonian. Assume to the contrary that G is Hamiltonian, the all edges of vertices with degree 2 must be included in the Hamiltonian cycle, C. These edges are as labeled in the following graph:



Without loss of generality, let us consider the vertices of $v_{10}, v_{11}, v_{12}, v_{13}$. For C to be complete, v_{11}, v_{12}, v_{13} must be included. Note that v_{10} must be passed through to get to v_{11}, v_{12}, v_{13}. Either v_9v_{10},v_2v_{10} or $v_{16}v_{10}$ must be chosen.



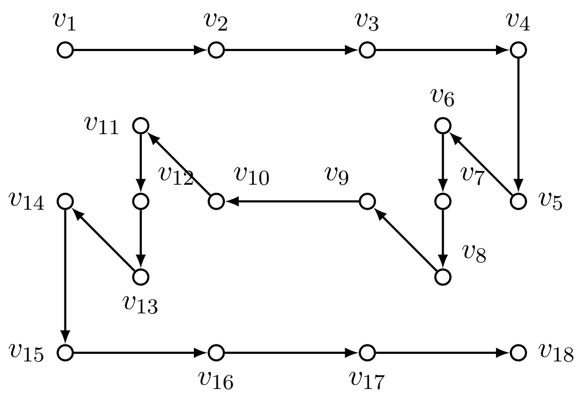
There are 2 cases in total for the edge to be chosen.

(i) When $v_{10}v_{12}$ is chosen, the next edge to be chosen must be either $v_{12}v_{11}$ or $v_{12}v_{13}$, which both cases will leave out another vertex to be excluded in C, therefore gives us a contradiction.

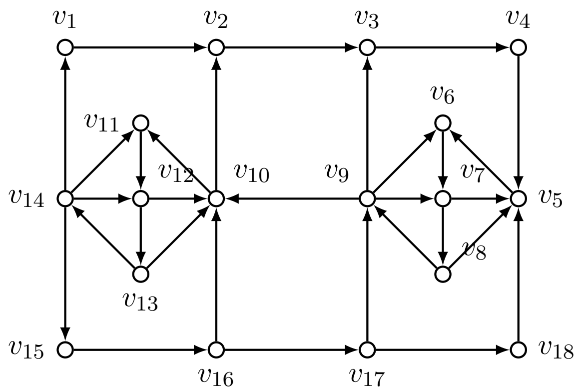
(ii) Without loss of generality, $v_{10}v_{11}$ is chose, therefore $v_{11}v_{12}$ and $v_{12}v_{13}$ must be chosen as well. At this stage, there is no more edge to be chosen as all the vertices that are adjacent to v_{13} has appeared in C already, which gives us a contradiction.

Therefore G is not Hamiltonian.

(b) Using depth first search algorithm on G as labeled, we first obtain a tree and assign a direction from v_i to v_{i+1}.

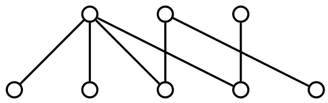


We now add direction for all $e = v_iv_j \in E(G)$ from v_i to v_j if $i > j$, giving us an orientable graph.

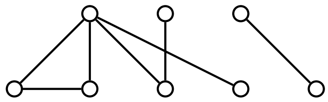


Question 5

(a) To find two non isomorphic graphs, G_1 and G_2, we might want to construct these two graphs in a way that G_1 is a bipartite graph while G_2 is not bipartite, which will solve the problem of isomorphism easily. The following are the construction of G_1

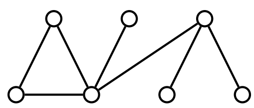


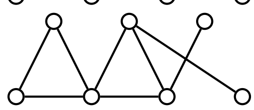
and G_2

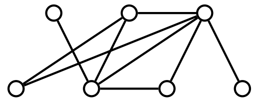


(b) (i) H cannot be a tree. Note that $x \geq 1$ since H is connected. Assume to the contrary that H is a tree, then $e(H) = 7 - 1 = 6$. Therefore by Handshaking Lemma, $\sum_{i=1}^{7} d(v_i) = 13 + x \geq 14 \neq 2 \times 6 = 12$, a contradiction.

(ii) Note that x has an upper bound of 6, i.e v_7 at most is adjacent to 6 other vertices. Calculating the number of odd vertices in H, we realize that there are 3 vertices which are odd. As a result, x must be a odd number to make sure H is a graph.

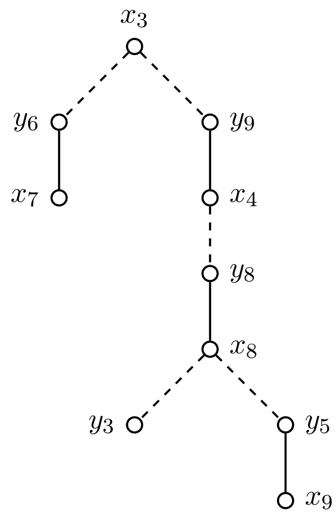
Case 1: $x = 1$ 

Case 1: $x = 3$ 

Case 1: $x = 5$ 

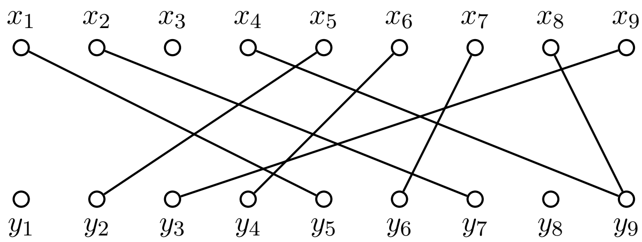
Question 6

(i) Given the matching M, we thus construct an alternating tree rooted at x_3. Note that solid lines represented edges in M while dashed lines represented edges in $E(G)\backslash M$



We augment along the path x_3y_3 obtaining a new matching $M' = \{x_9y_5, x_8y_3, x_7y_6, x_6y_7, x_5y_2, x_4y_8, x_3y_9\}$.

(ii) Since G does not have a perfect matching, therefore a maximum matching, $M_{\max}$ must have $|M_{\max}| \leq 8$. Thus we have to construct a matching with $|M_{\max}| = 8$ if possible. By construction, an example of $M_{\max}$ is



Question 7

Applying Dijkstra's algorithm, Therefore we have

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	update
$(0,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	v_1
$[0,-]$	$(\infty,-)$	$(4,1)$	$(2,1)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	v_4
	$(7,4)$	$(4,1)$	$[2,1]$	$(5,4)$	$(\infty,-)$	$(3,4)$	$(\infty,-)$	v_7
	$(7,4)$	$(4,1)$		$(5,4)$	$(\infty,-)$	$[3,4]$	$(7,7)$	v_3
	$(6,3)$	$[4,1]$		$(5,4)$	$(7,3)$		$(7,7)$	v_5
	$(6,3)$			$[5,4]$	$(7,3)$		$(7,7)$	v_2
	$[6,3]$				$(7,3)$		$(7,7)$	v_6
					$[7,3]$		$(7,7)$	v_8
							$[7,7]$	

$$
\begin{aligned}
&v_1v_2 \text{ path} : v_1v_3v_2 \quad d(v_1,v_2) = 6\\
&v_1v_3 \text{ path} : v_1v_3 \quad d(v_1,v_3) = 4\\
&v_1v_4 \text{ path} : v_1v_4 \quad d(v_1,v_4) = 2\\
&v_1v_5 \text{ path} : v_1v_4v_5 \quad d(v_1,v_5) = 5\\
&v_1v_6 \text{ path} : v_1v_3v_6 \quad d(v_1,v_6) = 7\\
&v_1v_7 \text{ path} : v_1v_4v_7 \quad d(v_1,v_7) = 3\\
&v_1v_8 \text{ path} : v_1v_4v_7v_8 \quad d(v_1,v_8) = 7
\end{aligned}
$$

Question 8

(a) Suppose $\overline{G}$ is not $C_5 - free$, there exists $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5}$ such that subgraph of $\overline{G}$ induced by these five vertices is isomorphic to C_5. Taking complement of $\overline{G}$, subgraph of G induced by these five vertices also isomorphic to C_5, therefore G is not $C_5 - free$.

(b) (i) Assume to the contrary that there exist adjacent vertices u, v in G such that $d(u)+d(v) > n$, i.e $d(u) + d(v) \geq n + 1$. If we are able to prove that $N(u)\backslash\{v\} \cap N(v)\backslash\{u\} \neq \phi$, then we are done. Note that $|N(u)\backslash\{v\} \cup N(v)\backslash\{u\}| \leq n - 2$. Therefore

$$
\begin{aligned}
|N(u)\backslash\{v\} \cap N(v)\backslash\{u\}| &= |N(u)\backslash\{v\}| + |N(u)\backslash\{v\} - |N(u)\backslash\{v\} \cup N(v)\backslash\{u\}|\\
&\geq n + 1 - 2 - (n - 2)\\
&= 1
\end{aligned}
$$

Therefore $N(u)\backslash\{v\}\cap N(v)\backslash\{u\} \neq \phi$, there exists $w \in N(u)\backslash\{v\}\cap N(v)\backslash\{u\}$ such that vw and uw are in $e(G)$. Recall that $uv \in e(G)$, therefore uv, uw, vw forms a C_3, contradicts the fact that G is $C_3 - free$.

(ii) Using (i), we know that there are total m edges in G. For an edge uv, we use (i) and we obtain $d(u) + d(v) \leq n$. This implies that the summing of m times of n is due to every edges in G. For each vertex v, it is adjacent to $d(v)$ number of other vertices. Therefore $d(u)$ will appear exactly $d(u)$ times. Summing them all up will give the inequality as desired.

Question 9

(a) Suppose G is Eulerian, for all $v \in G$, it has even degrees, i.e there exists $p \in \mathbb{Z}^+$ such that $d(v) = 2p$. Note that G is of odd order, which implies that $d(u) = 2k$ where $k \in \mathbb{Z}^+$ in K_{2k+1}. Therefore $d(v) = 2k - 2p$ in $\overline{G}$, therefore every vertices has even degree in $\overline{G}$, giving us $\overline{G}$ is Eulerian. The converse of the statement could be proven similarly by replacing G with $\overline{G}$ and $\overline{G}$ with G.

(b) Let ϕ be a k-colouring of G. For $i = 1, 2, \ldots, k$, let $V_i = \{v \in V(H) | \phi(v) = i\}$. Without loss of generality, assume $|V_1| = \max_{i=1,\ldots,k}\{|V_i|\}$, hence $|V_1| \geq \frac{n}{k}$. Note that V_1 is an independent set in H, therefore

$$\alpha \geq |V_1| \geq \frac{n}{k} \quad \Rightarrow k = \chi(G) \geq \frac{n}{\alpha}$$

(c) Suppose a tree T has two distinct perfect matching. Now consider the symmetric difference of the two perfect matchings. By a theorem in lecture notes, every vertex in the symmetric difference of the two perfect matchings is of degree zero or two. Since every vertex is incident to exactly one edge in each matching, then there will be an alternating cycle, which is a contradiction that T is a tree.