MA2001 - Linear Algebra Suggested Solutions

(Semester 1: AY2022/23)

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Note on Notations:

- 0 refers to the real value zero, $0 \in \mathbb{R}$
- $\tilde{0}$ refers to the zero vector, $\tilde{0} \in \mathbb{R}^n$
- $\mathbf{0}$ refers to the zero matrix, $\mathbf{0} \in \mathbb{R}^m \times \mathbb{R}^n$

Question 1

Let $B = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{pmatrix}$

(a) (6 marks) Use the Gauss-Jordan Elimination to reduce B to the reduced row-echelon form. (Indicate the elementary row operations used in each step.)

Solution:

$$B = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 + R_2} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{pmatrix} \xrightarrow{R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{pmatrix}$$

(b) Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ be two bases for a vector space V where

$$\boldsymbol{v_1} = \boldsymbol{u_1} + \boldsymbol{u_2},\ \boldsymbol{v_2} = -\boldsymbol{u_1} + 2\boldsymbol{u_3},\ \boldsymbol{v_3} = \boldsymbol{u_1} + \boldsymbol{u_2} + \boldsymbol{u_3}$$

(i) (3 marks) Write down the transition matrix from T to S.

Solution: First, we find the coordinate vectors of $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$ relative to the basis S. There ase given by:

$$[\boldsymbol{v_1}]_S = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^\top,\ [\boldsymbol{v_2}]_S = \begin{pmatrix} -1 & 0 & 2 \end{pmatrix}^\top,\ [\boldsymbol{v_3}]_S = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^\top$$

The required transition matrix is given by:

$$P = \begin{pmatrix} [\boldsymbol{v_1}]_S & [\boldsymbol{v_2}]_S & [\boldsymbol{v_3}]_S \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

(ii) (3 marks) Write down the transition matrix from S to T. (Hint: Use the result in (a).)

Solution: The transition matrix from S to T will be the inverse of the transition matrix from T to S. Thus, the required transition matrix is given by:

$$P' = P^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 3 & -1 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$$

Note: The inverse was obtained while performing Gauss-Jordan Elimination in part (i).

Question 2

Let $V = \{(a+b, a+2b+d, b+c+d, a+b+c) | a,b,c,d \in \mathbb{R}\}$ which is a subspace in $\mathbb{R}^4$

(a) (5 marks) Find a basis for V and determine the dimension of V.
Solution: We may express V as follows:

$$V = \left\{ a\begin{pmatrix}1\\1\\0\\1\end{pmatrix} + b\begin{pmatrix}1\\2\\1\\1\end{pmatrix} + c\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + d\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \middle| a,b,c,d \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix} \right\}$$

Next, we check for linear independence of the vectors by setting up the following augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{\text{Gauss Elimination}} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, the matrix system has non-trivial solutions implying linear dependence of the vectors. We pick the vectors corresponding to the pivotal columns of the row echelon form, giving us the required basis:

$$\left\{ \begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\}$$

Since the dimension of a subspace is given by the number of basis vectors, the dimension of V is 3.

(b) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation and define $W = \{T(u^\top) | u \in V\}$
(i) (4 marks) Show that W is a subspace of $\mathbb{R}^3$
Solution: Let A be the standard matrix of T.

- Check for zero vector:
$$\tilde{0} \in V \text{ (} V \text{ is a subspace)} \implies A\tilde{0} \in W \implies \tilde{0} \in W$$
- Closure under linear combination: Let $v_1 \in W$ and $v_2 \in W$
$$\begin{aligned} &\implies v_1 = Au_1,\ v_2 = Au_2 \text{ for some } u_1, u_2 \in V \\ &\implies \alpha u_1 + \beta u_2 \in V,\ \forall \alpha, \beta \in \mathbb{R} \text{ (as } V \text{ is a subspace)} \\ &\implies A(\alpha u_1 + \beta u_2) \in W \\ &\implies \alpha A u_1 + \beta A u_2 \in W \\ &\implies \alpha v_1 + \beta v_2 \in W \end{aligned}$$

Thus, W contains the zero vector and is closed under vector addition and scalar multiplication. It is therefore a subspace.

(ii) (3 marks) Suppose the standard matrix for T is $\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ Determine the dimension of W.

Solution: Any vector $u \in V$ can be written as:

$$v = a\begin{pmatrix}1\\1\\0\\1\end{pmatrix} + b\begin{pmatrix}1\\2\\1\\1\end{pmatrix} + c\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \begin{pmatrix} a+b \\ a+2b \\ b+c \\ a+b+c \end{pmatrix} \text{ for some } a,b,c \in \mathbb{R}$$

Thus, any vector in W is given by:

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a+b \\ a+2b \\ b+c \\ a+b+c \end{pmatrix} = \begin{pmatrix} c \\ 2a+3b+c \\ 2a+3b+2c \end{pmatrix} = a\begin{pmatrix}0\\2\\2\end{pmatrix} + b\begin{pmatrix}0\\3\\3\end{pmatrix} + c\begin{pmatrix}1\\1\\2\end{pmatrix} \text{ for some } a,b,c \in \mathbb{R}$$

Thus,

$$W = \text{span}\left\{\begin{pmatrix}0\\2\\2\end{pmatrix}, \begin{pmatrix}0\\3\\3\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}\right\}$$

Clearly, the above two vectors are independent. Since the dimension of a subspace is given by the number of basis vectors, the dimension of W is 2.

Question 3

Let $A = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 2 & -1\\0 & -1 & 1\end{pmatrix}$ and $b = \begin{pmatrix}9\\0\\0\\0\end{pmatrix}$

(a) (4 marks) Is the linear system $Ax = b$ consistent? Justify your answer.
Solution: On performing Gauss Elimination, we observe that:

$$\left(\begin{array}{ccc|c}1 & 0 & 0 & 9\\0 & 1 & 1 & 0\\1 & 2 & -1 & 0\\0 & -1 & 1 & 0\end{array}\right) \xrightarrow{R_3-R_1} \left(\begin{array}{ccc|c}1 & 0 & 0 & 9\\0 & 1 & 1 & 0\\0 & 2 & -1 & -9\\0 & -1 & 1 & 0\end{array}\right) \xrightarrow{R_3-2R_2, R_4+R_2} \left(\begin{array}{ccc|c}1 & 0 & 0 & 9\\0 & 1 & 1 & 0\\0 & 0 & -3 & -9\\0 & 0 & 2 & 0\end{array}\right) \xrightarrow{R_4+2/3R_3} \left(\begin{array}{ccc|c}1 & 0 & 0 & 9\\0 & 1 & 1 & 0\\0 & 0 & -3 & -9\\0 & 0 & 0 & -6\end{array}\right)$$

Since the last column of the augmented matrix is a pivotal column, the system is inconsistent.

(b) (5 marks) Find a least square solution to $Ax = b$.
Solution: In order to obtain the least square solution, we transform the problem to solving the following linear system:

$$A^\top A\hat{x} = A^\top b \implies \begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 2 & -1\\0 & -1 & 1\end{pmatrix}^\top \begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 2 & -1\\0 & -1 & 1\end{pmatrix} \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 2 & -1\\0 & -1 & 1\end{pmatrix}^\top \begin{pmatrix}9\\0\\0\\0\end{pmatrix} \implies \begin{pmatrix}2 & 2 & -1\\2 & 6 & -2\\-1 & -2 & 3\end{pmatrix} \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}9\\0\\0\end{pmatrix}$$

On solving the above linear system, we obtain the least square solution:

$$\hat{x} = \begin{pmatrix}7\\-2\\1\end{pmatrix}$$

(c) (3 marks) Use the result in (b) to find the projection of b on to the column space of A.
Solution: The projection of b onto the column space of A is given by:

$$\text{proj}_A b = A\hat{x}, \text{ where } \hat{x} \text{ is least square solution to } Ax = b$$

(Intuitively, $\text{proj}_A b$ is the vector on the column space of A such that the perpendicular distance between b and $\text{proj}_A b$ is minimum. The least square solution minimizes the residual error of the linear system and should also, therefore, give the coefficients that determine $\text{proj}_A b$.) Thus,

$$\text{proj}_A b = A\hat{x} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 2 & -1\\0 & -1 & 1\end{pmatrix} \begin{pmatrix}7\\-2\\1\end{pmatrix} = \begin{pmatrix}7\\-1\\2\\3\end{pmatrix}$$

Question 4

Let $C = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$. It is known that C has only two eigenvalues 0 and 3.

(a) (4 marks) Find an orthonormal basis for the eigenspace E_0 of C.
Solution: E_0 is the solution space for the homogeneous system:

$$Cx = \tilde{0} \implies \begin{pmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus, we consider the following augmented matrix:

$$\left(\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right) \xrightarrow[R_2-0.5R_1]{R_3+0.5R_1} \left(\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 0 \\ 0 & 1.5 & 1.5 & 0 & 0 \\ 0 & 1.5 & 1.5 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right) \xrightarrow[R_4-R_2]{R_3\leftrightarrow R_4} \left(\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 0 \\ 0 & 1.5 & 1.5 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, we have the following solution set ($t \in \mathbb{R}$):

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} t \\ -t \\ t \\ 0 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

We normalise the above basis vector in order to obtain the required orthonormal basis for E_0:

$$\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(b) (4 marks) Find an orthonormal basis for the eigenspace E_3 of C.
Solution: E_3 is the solution space for the homogeneous system:

$$(C - 3I)x = \tilde{0} \implies \begin{pmatrix} -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus, we consider the following augmented matrix:

$$\left(\begin{array}{cccc|c} -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_2+R_1]{R_3-R_1} \left(\begin{array}{cccc|c} -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We let $x_2 = r, x_3 = s, x_4 = t,\ r, s, t \in \mathbb{R}$. Thus, we have the following solution set:

$$\begin{pmatrix} r-s \\ r \\ s \\ t \end{pmatrix} = r\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} = \text{span}\{u_1, u_2, u_3\}$$

We use Gram Schmidt to obtain an orthogonal basis:

$$v_1 = u_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$v_2 = u_2 - \frac{v_1 \cdot u_2}{\|v_1\|^2} v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \tilde{0} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_3 = u_3 - \frac{v_1 \cdot u_3}{\|v_1\|^2} v_1 - \frac{v_2 \cdot u_3}{\|v_2\|^2} v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \tilde{0} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0.5 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$$

Thus, the required orthonormal basis is:

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

(c) (2 marks) Based on the results of (a) and (b), write down the characteristic polynomial of C.
Solution: From a and b, it is evident that the dimensions of E_0 and E_3 are 1 and 3 respectively. Since these dimensions determine the multiplicity of the eigenvalue root in the characteristic polynomial, the required polynomial is given by:

$$P(\lambda) = (\lambda - 0)^{\dim E_0}(\lambda - 3)^{\dim E_3} = \lambda(\lambda - 3)^3$$

(d) (4 marks) Write down two 4×4 matrices P and D such that P is an orthogonal matrix, D is a diagonal matrix and $D = P^\top CP$.
Solution: It is evident that C is a symmetric matrix and therefore it must be orthogonally diagonalizable. The orthogonal matrix can be obtained by letting the columns be the eigenvectors in the orthonormal basis of E_0 and E_3. Then, D can be the diagonal matrix whose diagonal entries are the corresponding eigenvalues. Since C is symmetric, it need not be shown that the basis vectors for E_0 and E_3 are orthogonal to one another. Thus, we have:

$$C = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & 0 & \frac{2}{\sqrt{6}} \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & 0 & \frac{2}{\sqrt{6}} \\ 0 & 1 & 0 & 0 \end{pmatrix}^\top$$

Thus, we have:

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & 0 & \frac{2}{\sqrt{6}} \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Question 5

Let V be a subspace of $\mathbb{R}^n$. Define $V^\perp = \{u \in \mathbb{R}^n | u \text{ is orthogonal to } V\}$, i.e. $V^\perp = \{u \in \mathbb{R}^n | v \cdot u = 0 \ \ \forall\, v \in V\}$.

(a) (5 marks) Show that $V^\perp$ is a subspace of $\mathbb{R}^n$.
Solution:

- Check for zero vector: For any $v \in V$,

$$v \cdot \tilde{0} = 0 \implies \tilde{0} \text{ is orthogonal to } V \implies \tilde{0} \in V^\perp$$

- Closure under linear combination: Let $v_1 \in V^\perp$ and $v_2 \in V^\perp$

$$\begin{aligned}
&\implies v_1 \cdot u = 0,\ v_2 \cdot u = 0 \;\forall\, u \in V \\
&\implies \alpha v_1 \cdot u = 0,\ \beta v_2 \cdot u = 0 \;\forall\, u \in V;\ \alpha, \beta \in \mathbb{R} \\
&\implies \alpha v_1 \cdot u + \beta v_2 \cdot u = 0 \;\forall\, u \in V \\
&\implies (\alpha v_1 + \beta v_2) \cdot u = 0 \;\forall\, u \in V \\
&\implies \alpha v_1 + \beta v_2 \in V^\perp
\end{aligned}$$

Thus, $V^\perp$ contains the zero vector and is closed under vector addition and scalar multiplication. It is therefore a subspace.
Note: Another way to prove this statement would be to show that the vectors in $V^\perp$ must constitute the solution set of a homogeneous system given by a matrix whose rowspace is V. An idea of this proof is demonstarted in part c).

(b) (5 marks) Let $\{v_1, v_2, \cdots, v_k\}$ be a basis for V and $\{u_1, u_2, \cdots, u_m\}$ a basis for $V^\perp$. Show that $\{v_1, v_2, \cdots, v_k, u_1, u_2, \cdots, u_m\}$ is a basis for $\mathbb{R}^n$.
Solution: First, we show that $n = k + m$. We construct a $k \times n$ matrix A, whose row space is given by V:

$$A = \begin{pmatrix} \text{—}\ v_1\ \text{—} \\ \text{—}\ v_2\ \text{—} \\ \vdots \\ \text{—}\ v_k\ \text{—} \end{pmatrix}$$

Then, its null space must be $V^\perp$, because the null space is the orthogonal complement of the row space.

Theorem: Every vector in the null space of A is orthogonal to the row space of A i.e., the null space of A is the orthogonal complement of the row space of A i.e.,

$$\text{nullsp}(A) = \text{rowsp}(A)^\perp$$

Proof. We use the element-chasing method to prove our theorem: Let A be an $m \times n$ matrix:

$$A = \begin{pmatrix} a_1^\top \\ a_2^\top \\ \vdots \\ a_m^\top \end{pmatrix}$$

where $a_i \in \mathbb{R}^n \ \forall i \in \{1, 2, \cdots, m\}$. Then,

$$x \in \text{nullsp}(A) \iff Ax = \tilde{0} \iff \begin{pmatrix} a_1^\top \\ a_2^\top \\ \vdots \\ a_m^\top \end{pmatrix} x = \tilde{0} \iff \begin{pmatrix} a_1^\top x \\ a_2^\top x \\ \vdots \\ a_m^\top x \end{pmatrix} = \tilde{0} \iff \begin{pmatrix} a_1 \cdot x \\ a_2 \cdot x \\ \vdots \\ a_m \cdot x \end{pmatrix} = \tilde{0}$$

$$\begin{gathered}
\iff x \cdot a_i = 0 \;\forall i \in \{1, 2, \cdots, m\} \iff x \perp a_i \;\forall i \in \{1, 2, \cdots, m\} \iff x \in \text{rowsp}(A)^\perp \\
\implies \text{nullsp}(A) = \text{rowsp}(A)^\perp
\end{gathered}$$

□

Thus, by the dimension theorem of matrices,

$$\text{rank}(A) + \text{nullity}(A) = \text{ncols}(A) = n \implies \dim \text{rowsp}(A) + \dim \text{nulsp}(A) = n \implies \dim V + \dim V^{\perp} = n \implies k + m = n$$

Next, we show that the vectors $\{v_1, v_2, \cdots, v_k, u_1, u_2, \cdots, u_m\}$ are linearly independent. Consider the following homogenous vector equation:

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k + d_1u_1 + d_2u_2 + \cdots + d_mu_m = \tilde{0} \implies v + u = \tilde{0}$$

$$\text{where } v = c_1v_1 + c_2v_2 + \cdots + c_kv_k \in V \text{ and } u = d_1u_1 + d_2u_2 + \cdots + d_mu_m \in V^{\perp}$$

$$\implies v = -u$$

Since $v \in V, u \in V^{\perp}$,

$$v \cdot u = 0 \implies -u \cdot u = 0 \implies \|u\|^2 = 0 \implies u = \tilde{0} = v$$

Thus,

$$v = c_1v_1 + c_2v_2 + \cdots c_kv_k = \tilde{0},\ u = d_1u_1 + d_2u_2 \cdots d_mu_m = \tilde{0} \implies c_1 = c_2 = \cdots = c_k = d_1 = d_2 = \cdots = d_m = 0$$

The above reasoning comes from the fact that $\{v_1, v_2, \cdots, v_k\}$ and $\{u_1, u_2, \cdots, u_m\}$ are respectively the basis for V and $V^{\perp}$. Thus the homogeneous system has only the trivial solution, implying the linear independence of the n vectors. Consequently $\{v_1, v_2, \cdots, v_k, u_1, u_2, \cdots, u_m\}$ is a set of $k + m = n$ vectors in $\mathbb{R}^n$ that are linearly independent and therefore should constitute a basis for $\mathbb{R}^n$.

(c) (4 marks) Suppose $n = 4$ and $V = \text{span}\{(1, 1, 1, 1), (-1, 1, -3, 1), (1, 3, -1, 3)\}$. Find a basis for $V^{\perp}$.
Solution: We need to find $V^{\perp}$, which is the set of vectors of the form (a, b, c, d) such that $(a, b, c, d) \cdot (1, 1, 1, 1) = 0$, $(a, b, c, d) \cdot (-1, 1, -3, 1) = 0$, $(a, b, c, d) \cdot (1, 3, -1, 3) = 0$ In other words, we find the solution space of the homogeneous linear system represented by the augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ -1 & 1 & -3 & 1 & 0 \\ 1 & 3 & -1 & 3 & 0 \end{array}\right) \xrightarrow[R_2+R_1]{R_3-R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 \\ 0 & 2 & -2 & 2 & 0 \end{array}\right) \xrightarrow{R_3-R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, we have the following general solution ($s, t \in \mathbb{R}$):

$$\begin{pmatrix} -2t \\ s \\ t \\ t-s \end{pmatrix} = s\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Therefore, we have the following solution space:

$$\text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

Since the two vectors are linearly independent, the required basis is:

$$\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

Question 6

(All vectors in this question are written as column vectors.)

(a) (3 marks) Let P and Q be two $m \times n$ matrices. Suppose $Pv_i = Qv_i$, for $i = 1, 2, \cdots, n$, where $\{v_1, v_2, \cdots, v_n\}$ is a basis for $\mathbb{R}^n$. Show that $P = Q$.
(Hint: Let $\{e_1, e_2, \cdots, e_n\}$ be the standard basis for $\mathbb{R}^n$ Then for any $m \times n$ matrix R, Re_j is the jth column of R for $j = 1, 2, \cdots, n$.)
Solution: Let there be some $i \in \{1, 2, \cdots, n\}$. Further, let p_i and q_i denote the ith column of P and Q respectively. Since, $\{v_1, v_2, \cdots, v_n\}$ is a basis for $\mathbb{R}^n$, we must be able to combine them linearly to obtain the standard basis vector:

$$e_i = c_{1i}v_1 + c_{2i}v_2 + \cdots + c_{ni}v_n$$

It is evident that:

$$\begin{aligned} p_i = Pe_i &= P(c_{1i}v_1 + c_{2i}v_2 + \cdots + c_{ni}v_n) = c_{1i}Pv_1 + c_{2i}Pv_2 + \cdots + c_{ni}Pv_n \\ &= c_{1i}Qv_1 + c_{2i}Qv_2 + \cdots + c_{ni}Qv_n = Q(c_{1i}v_1 + c_{2i}v_2 + \cdots + c_{ni}v_n) = Qe_i = q_i \end{aligned}$$

Since the above result holds $\forall\ i \in \{1, 2, \cdots, n\}$, P and Q must share the same columns and are consequently the same. Thus, we have that $P = Q$.

(b) Let A and B be two square matrices of order n.

(i) (3 marks) Suppose $AB + BA = \mathbf{0}$. Show that if u is an eigenvector of A associated with an eigenvalue λ, then either $Bu = \tilde{0}$ or Bu is an eigenvector of A associated with $-\lambda$.
Solution: Since u is an eigenvector of A associated with an eigenvalue λ,

$$Au = \lambda u$$

Using the equation provided and post-multiplying both sides with u:

$$AB + BA = \mathbf{0} \implies ABu + BAu = \tilde{0} \implies ABu + \lambda Bu = \tilde{0} \implies A(Bu) = -\lambda(Bu) \implies Aw = -\lambda w$$

where $w = Bu$. The above equation only implies two possibilities:

- $w = Bu$ is an eigenvector of A with eigenvalue of $-\lambda$
- w is not an eigenvector, which is possible only if $w = \tilde{0} \implies Bu = \tilde{0}$

(ii) (6 marks) Let A be diagonalizable. Suppose B has the property that for any eigenvalue λ of A and eigenvector u of A associated with λ, either $Bu = \tilde{0}$ or Bu is an eigenvector of A associated with $-\lambda$. Prove that $AB + BA = \mathbf{0}$.
Solution: Since A is given to be diagonalizable, there must be a set of eigenvectors $\{v_1, v_2, \cdots, v_n\}$ that form a basis for $\mathbb{R}^n$. For any of these eigenvectors v_i associated with eigenvalue λ_i, this would mean that

$$ABv_i = -\lambda_i Bv_i \implies ABv_i + \lambda_i Bv_i = \tilde{0} \implies ABv_i + B(\lambda_i v_i) = \tilde{0} \implies ABv_i + BAv_i = \tilde{0} \implies (AB + BA)v_i = \tilde{0}$$

Consider some arbitrary vector $w \in \mathbb{R}^n$. Since, the eigenvectors of A form a basis in $\mathbb{R}^n$, we must be able to combine them linearly to obtain w:

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \text{ for some } \alpha_1, \cdots, \alpha_n \in \mathbb{R}^n$$

We may now compute $(AB + BA)w$ as follows:

$$\begin{aligned} (AB + BA)w &= (AB + BA)(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) \\ &= \alpha_1(AB + BA)v_1 + \alpha_2(AB + BA)v_2 + \cdots + \alpha_n(AB + BA)v_n \\ &= \tilde{0} + \tilde{0} + \cdots + \tilde{0} = \tilde{0} \end{aligned}$$

Thus, $\forall\ w \in \mathbb{R}^n$, $(AB+BA)w = \tilde{0}$. In other words, the nullspace of $AB+BA$ is the entire $\mathbb{R}^n$ i.e., nullity$(AB+BA) = n$. Since, this is true only for the zero matrix, we can conclude that $AB + BA = \mathbf{0}$.

(iii) (4 marks) Suppose A, B are symmetric and $AB + BA = \mathbf{0}$. Prove that if μ is an eigenvalue of AB, then $-\mu$ is also an eigenvalue of AB.
Solution: Let v be an eigenvector of AB associated with eigenvalue μ It is given that:

$$AB + BA = \mathbf{0} \implies ABv + BAv = \tilde{0} \text{ (post multiplying by } v) \implies \mu v + BAv = \tilde{0}$$

$$\implies BAv = -\mu v \implies (A^\top B^\top)^\top v = -\mu v \implies (AB)^\top v = -\mu v \text{ (since } A = A^\top, B = B^\top)$$

Thus, $-\mu$ is an eigenvalue of $(AB)^\top$. But, a matrix and its transpose must share the same eigenvalues. In particular,

$$(AB)^\top v = -\mu v \implies \det((AB)^\top + \mu I) = 0 \implies \det((AB + \mu I)^\top) = 0 \implies \det(AB + \mu I) = 0 \implies ABv' = -\mu v'$$

for some $v' \in \mathbb{R}^n$. Thus, $-\mu$ is also an eigenvalue of AB.