MA3269 - Mathematical Finance I Suggested Solutions

(Semester 1, AY2022/2023)

Written by: Zhang Jingyi
Audited by: Chow Yong Lam

1 (a) (i)

$$\begin{aligned}U(c) = EU(w_0 + X) &= 0.5 \times U(16) + 0.5 \times U(4) \\ 1 - e^{-c} &= 0.5 \times (1 - e^{-16}) + 0.5 \times (1 - e^{-4}) \\ e^{-c} &= 0.5 \times (e^{-4} + e^{-16}) \\ c &= 4.693141036... \\ &= 4.693 (4 \text{ s.f.})\end{aligned}$$

The certainty equivalent of this game is \$4.693.
$RP = w_0 - CE(X; U) = 8 - 4.693141036... = 3.306858964... = 3.307$ (4 s.f.)
The risk premium of this game is \$3.307.
Since $RP > 0$, i.e., $CE \leq w_0$, the investor should not play the game.

(ii) The investor should reject the game when $EU(X + w_0) < U(w_0)$.

$$\begin{aligned}U(w_0) &= U(8) = 1 - e^{-8} \\ EU(X + w_0) &= p(1 - e^{-16}) + (1 - p)(1 - e^{-4}) \\ &< U(w_0) \\ 1 - e^{-8} &> 1 - e^{-4} + (e^{-4} - e^{-16})p \\ e^{-4} - e^{-8} &> p(e^{-4} - e^{-16}) \\ p &< \frac{e^{-4} - e^{-8}}{e^{-4} - e^{-16}} \quad \text{since } (e^{-4} - e^{-16}) > 0 \\ p &< 0.9816903928... = 0.9817 (4 \text{ s.f.}) \\ \therefore 0 &< p < 0.9817\end{aligned}$$

(iii)

$$\begin{aligned}U &= 1 - e^{-w} \\ U' &= e^{-w} \\ U'' &= -e^{-w} \\ R_U &= -\frac{U''}{U'} = -(-1) = 1 \\ 2\frac{U'}{U} &= \frac{2e^{-w}}{1 - e^{-w}} \\ V &= 1 - \frac{1}{1 - e^{-w}} \\ V' &= \frac{e^{-w}}{(1 - e^{-w})^2} \\ V'' &= -\frac{e^{-w}}{(1 - e^{-w})^2} - \frac{2e^{-2w}}{(1 - e^{-w})^3}\end{aligned}$$

$$
\begin{aligned}
R_V = -\frac{V''}{V'} &= \left(\frac{e^{-w}}{(1-e^{-w})^2} + \frac{2e^{-2w}}{(1-e^{-w})^3}\right) \times \frac{(1-e^{-w})^2}{e^{-w}} \\
&= 1 + \frac{2e^{-w}}{1-e^{-w}} \\
&= R_U + 2\frac{U'}{U}
\end{aligned}
$$

Since $w > 0$, $e^{-w} < 1$, $1 - e^{-w} > 0$, and since $2e^{-w} > 0$, $\frac{U'}{U} > 0$,

$$R_V = R_U + 2\frac{U'}{U} > R_U$$

Therefore, we can say that investor B is globally more risk averse than investor A.

(b) (i) Let c be the certainty equivalent of the investment. Since the investment has a convex utility function,

$$
\begin{aligned}
EU(w_0 + X) \geq U(E(w_0 + X)) &= U(w_0 + E(X)) > U(w_0) \\
U(c) &> U(w_0)
\end{aligned}
$$

Since the utility function U is strictly increasing, $c > w_0$.
By the definition of risk premium and certainty equivalent,

$$RP(X; U) = w_0 - CE(X; U) < 0$$

(ii) Let $CE(X; U)$ and $CE(X; V)$ be the certainty equivalents of investors A and B respectively. Since investor B is globally more risk averse than investor A, $V(w) = g(U(w))$ where g is an increasing and strictly concave function. Therefore,

$$
\begin{aligned}
E(g(U(w_0 + X))) &< g(E(U(w_0 + X))) \\
E(V(w_0 + X)) &< g(U(CE(X; U))) \\
V(CE(X; V)) &< V(CE(X; U))
\end{aligned}
$$

Since the utility function V is strictly increasing, $CE(X; V) < CE(X; U)$.
By definitions of certainty equivalent and risk premium,

$$
\begin{aligned}
w_0 - CE(X; U) &< w_0 - CE(X; V) \\
RP(X; U) &< RP(X; V)
\end{aligned}
$$

2 (a) (i) Let weights of stocks A and B be x and $(1-x)$ respectively. The risk of the portfolio can be expressed as

$$\sigma_p^2 = 0.04x^2 + 0.16(1-x)^2 + 2x(1-x)\rho_{AB}(0.2 \times 0.4)$$

To find the weight x when the risk is at its minimum, differentiate σ_p^2 with respect to x

$$
\begin{aligned}
\frac{d\sigma_p^2}{dx} &= 0.08x - 0.32(1-x) + 0.16\rho_{AB}(-2x + 1) \\
&= (0.4 - 0.32\rho_{AB})x + (0.16\rho_{AB} - 0.32)
\end{aligned}
$$

When $\frac{d\sigma_p^2}{dx} = 0$,

$$x = \frac{0.32 - 0.16\rho_{AB}}{0.4 - 0.32\rho_{AB}}$$

Differentiate with respect to x again, $\frac{d^2\sigma_p^2}{dx^2} = 0.4 - 0.32\rho_{AB}$, which is greater than 0 since $-1 \leq \rho_{AB} \leq 1$. At $x = \dfrac{0.32 - 0.16\rho_{AB}}{0.4 - 0.32\rho_{AB}}$, the risk σ_p^2 is at its minimum.

When $x = \dfrac{0.32 - 0.16\rho_{AB}}{0.4 - 0.32\rho_{AB}}$, $(1-x) = \dfrac{0.08 - 0.16\rho_{AB}}{0.4 - 0.32\rho_{AB}}$.

The weight vector of the minimum-risk portfolio is

$$\left(\frac{0.32 - 0.16\rho_{AB}}{0.4 - 0.32\rho_{AB}}, \frac{0.08 - 0.16\rho_{AB}}{0.4 - 0.32\rho_{AB}}\right)^T$$

(ii) Substitute $\rho_{AB} = -0.5$ into the weight vector in 2(a)(i), the weight vector is

$$\left(\frac{0.32+0.08}{0.4+0.16}, \frac{0.08+0.08}{0.4+0.16}\right)^T = \left(\frac{5}{7}, \frac{2}{7}\right)^T$$

The mean of the portfolio is $\frac{5}{7} \times 0.1 + \frac{2}{7} \times 0.15 = \frac{4}{35}$.
The variance of the portfolio is $0.04 \times (\frac{5}{7})^2 + 0.16 \times (\frac{2}{7})^2 + 2 \times \frac{5}{7} \times \frac{2}{7} \times 0.2 \times 0.4 \times (-0.5) = \frac{3}{175}$.

(iii) Given the variance and the mean rate of return of the minimum-variance portfolio, the equation of the minimum variance frontier can be expressed as (for some value of a):

$$\sigma^2 = a\left(\mu - \frac{4}{35}\right)^2 + \frac{3}{175}$$

To find the value of a, substitute the variance and mean corresponding to the weight vector $(0,1)^T$, where $\mu = 0.15$ and $\sigma^2 = 0.16$

$$\frac{1}{784}a + \frac{3}{175} = 0.16$$
$$a = 112$$

The equation of the minimum variance frontier is

$$\sigma^2 = 112\left(\mu - \frac{4}{35}\right)^2 + \frac{3}{175}$$

(iv) The minimum-variance mean is $\frac{4}{35} > 0.1$.
The smallest variance when the portfolio mean is at least 0.1 is the variance of the minimum-risk portfolio, which is $\frac{3}{175}$.
The weight vector is $(\frac{5}{7}, \frac{2}{7})^T$.
The mean is $\frac{4}{35}$, and the variance is $\frac{3}{175}$.

(v) The minimum-variance mean is $\frac{4}{35} < 0.2$.
The smallest variance when the portfolio mean is at least 0.2 is achieved at $\mu = 0.2$.
The variance is $112(0.2 - \frac{4}{35})^2 + \frac{3}{175} = \frac{21}{25}$.
To get the weight of the portfolio, let the weights of stocks A and B be x and $(1-x)$ respectively.

$$0.1x + 0.15(1-x) = 0.2$$

$$x = -1, \quad (1-x) = 2$$

The weight vector is $(-1, 2)^T$.

(b) (i)

$$\begin{aligned}
a &= \mathbf{1}^T\mathbf{C}^{-1}\mathbf{1} = 1+1 = 2 \\
b &= \mathbf{1}^T\mathbf{C}^{-1}\boldsymbol{\mu} = -0.1+0.7-0.3 = \frac{3}{10} \\
c &= \boldsymbol{\mu}^T\mathbf{C}^{-1}\boldsymbol{\mu} = -0.02+0.35-0.03 = \frac{3}{10} \\
\sigma^2 &= \frac{a\mu^2 - 2b\mu + c}{ac - b^2} \\
&= \frac{2\mu^2 - \frac{3}{5}\mu + \frac{3}{10}}{\frac{51}{100}} \\
&= \frac{200}{51}\mu^2 - \frac{20}{17}\mu + \frac{10}{17}
\end{aligned}$$

The equation of the minimum-variance frontier is $\sigma^2 = \dfrac{200}{51}\mu^2 - \dfrac{20}{17}\mu + \dfrac{10}{17}$.

(ii) Completing the squares, $\sigma^2 = \frac{200}{51}(\mu - \frac{3}{20})^2 + \frac{1}{2}$.
The mean of the global minimum-variance portfolio is $\frac{3}{20}$ and the variance is $\frac{1}{2}$.
The weight vector can be found by

$$\mathbf{w}_{GMV} = \frac{C^{-1}1}{1^T C^{-1} 1} = \frac{1}{2}(1,0,1)^T = \left(\frac{1}{2}, 0, \frac{1}{2}\right)^T$$

(iii) Another portfolio that lies on the efficient frontier can be found by

$$\mathbf{w}' = \frac{C^{-1}\mu}{1^T C^{-1}\mu} = \frac{10}{3}(-0.1, 0.7, -0.3)^T = \left(-\frac{1}{3}, \frac{7}{3}, -1\right)^T$$

(iv) To check whether the portfolios are efficient, we check whether the weight vector can be expressed as convex combination $\mathbf{w} = \alpha\mathbf{w}_1 + (1-\alpha)\mathbf{w}_2$ for some $\alpha \in \mathbb{R}$, where $\mathbf{w}_1 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)^T$ and $\mathbf{w}_2 = \left(-\frac{1}{3}, \frac{7}{3}, -1\right)^T$, and the portfolio mean is more than the GMVP mean, i.e., $\mathbf{w}^T\boldsymbol{\mu} > \frac{3}{20}$.

(1)

$$\alpha\left(\frac{1}{2}, 0, \frac{1}{2}\right)^T + (1-\alpha)\left(-\frac{1}{3}, \frac{7}{3}, -1\right)^T = \left(\frac{1}{12}, \frac{7}{6}, -\frac{1}{4}\right)^T$$

There is a solution of $\alpha = \frac{1}{2}$, and $\mathbf{w}^T\boldsymbol{\mu} = \frac{23}{40} > \frac{3}{20}$, the portfolio is efficient.

(2)

$$\alpha\left(\frac{1}{2}, 0, \frac{1}{2}\right)^T + (1-\alpha)\left(-\frac{1}{3}, \frac{7}{3}, -1\right)^T = \left(\frac{11}{12}, -\frac{7}{6}, \frac{5}{4}\right)^T$$

There is a solution of $\alpha = \frac{3}{2}$. However, $\mathbf{w}^T\boldsymbol{\mu} = -\frac{11}{40} < \frac{3}{20}$, the portfolio is not efficient.

(3)

$$\alpha\left(\frac{1}{2}, 0, \frac{1}{2}\right)^T + (1-\alpha)\left(-\frac{1}{3}, \frac{7}{3}, -1\right)^T = \left(-\frac{1}{18}, \frac{5}{3}, -\frac{11}{18}\right)^T$$

There is no solution for α. The portfolio is not efficient.

(v) From the answer obtained in 2(b)(ii), the efficient frontier can be obtained as

$$\sigma^2 = \frac{200}{51}\left(\mu - \frac{3}{20}\right)^2 + \frac{1}{2}$$

$$\mu = \sqrt{\frac{51}{200}\left(\sigma^2 - \frac{1}{2}\right)} + \frac{3}{20}$$

(vi) (1) From the answer obtained in 2(b)(ii),

$$\sigma^2 = \frac{200}{51}\left(\mu - \frac{3}{20}\right)^2 + \frac{1}{2}$$

Substitute the value $\mu = \frac{57}{40}$ into the minimum-variance frontier,

$$\sigma^2 = \frac{200}{51}\left(\frac{57}{40} - \frac{3}{20}\right)^2 + \frac{1}{2} = \frac{55}{8}$$
$$\sigma = \sqrt{\frac{55}{8}}$$

Differentiate both sides of the minimum-variance frontier with respect to σ,

$$2\sigma = \frac{400}{51}\left(\mu - \frac{3}{20}\right)\frac{d\mu}{d\sigma}$$

Substitute the values $\mu = \frac{57}{40}$ and $\sigma = \sqrt{\frac{55}{8}}$,

$$2\sqrt{\frac{55}{8}} = \frac{400}{51}\left(\frac{57}{40} - \frac{3}{20}\right)\frac{d\mu}{d\sigma}$$
$$\frac{d\mu}{d\sigma} = \frac{\sqrt{110}}{20}$$

The equation of the Capital Market Line can be obtained by

$$\mu - \frac{57}{40} = \frac{\sqrt{110}}{20}\left(\sigma - \sqrt{\frac{55}{8}}\right)$$
$$\mu = \frac{\sqrt{110}}{20}\sigma + \frac{1}{20}$$

Substitute $\sigma = 0$, $r_f = \frac{1}{20}$.

(2) As shown in 2(b)(vi)(1), the equation of the Capital Market Line is

$$\mu = \frac{\sqrt{110}}{20}\sigma + \frac{1}{20}$$

(3) As shown in 2(b)(vi)(1), the variance of the market portfolio is $\sigma^2 = \frac{55}{8}$.

(4)

$$\begin{aligned}\mathbf{w} &= \frac{c - b\mu}{ac - b^2}\mathbf{C}^{-1}\mathbf{1} + \frac{a\mu - b}{ac - b^2}\mathbf{C}^{-1}\boldsymbol{\mu} \\ &= -\frac{1}{4}(1, 0, 1)^T + 5(-0.1, 0.7, -0.3)^T \\ &= \left(-\frac{3}{4}, \frac{7}{2}, -\frac{7}{4}\right)^T\end{aligned}$$

(vii)

$$\beta_p = \frac{\sigma_{pm}}{\sigma_m^2}$$
$$\frac{1}{3} = \frac{\sigma_{pm}}{\frac{55}{8}}$$
$$\sigma_{pm} = \frac{55}{24}$$

(viii)

$$\sigma_{pm} = \sigma_p \sigma_m \rho_{pm}$$

Squaring both sides,

$$\begin{aligned} \sigma_{pm}^2 &= \sigma_p^2 \sigma_m^2 \rho_{pm}^2 \\ \frac{55}{8}\sigma_p^2 \rho_{pm}^2 &= \frac{3025}{576} \end{aligned}$$

Since $\rho_{pm}^2 \leq 1$,

$$\begin{aligned} \frac{55}{8}\sigma_p^2 &\geq \frac{3025}{576} \\ \sigma_p^2 &\geq \frac{55}{72} \end{aligned}$$

(ix) $\beta_p = \frac{1}{3}$, beta of the market portfolio m is $\beta_m = 1$, beta of the risk-free asset is $\beta_r = 0$. The beta of the globally minimum-variance portfolio is

$$\beta_{GMV} = \frac{\sigma_{GMV}^2}{\sigma_m^2} = \frac{\frac{1}{2}}{\frac{55}{8}} = \frac{4}{55}$$

For the portfolio equally weighted in the aforementioned four components, the beta can be obtained as

$$\beta = \frac{1}{4}\left(\frac{1}{3} + 1 + 0 + \frac{4}{55}\right) = \frac{58}{165}$$

3 (a) (i)

$$\begin{aligned} \mathbf{w}_1 &= \frac{\mathbf{C}^{-1}\mathbf{1}}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}}, \mathbf{w}_1^T = \frac{\mathbf{1}^T\mathbf{C}^{-1}}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}} \\ \mathbf{w}_1^T\mathbf{C}\mathbf{w}_1 &= \frac{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{C}}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}} \frac{\mathbf{C}^{-1}\mathbf{1}}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}} = \frac{1}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}} = \frac{1}{a} \\ \mathbf{w}_2 &= \frac{\mathbf{C}^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\mathbf{C}^{-1}\boldsymbol{\mu}} \\ \mathbf{w}_1^T\mathbf{C}\mathbf{w}_2 &= \frac{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{C}}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}} \frac{\mathbf{C}^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\mathbf{C}^{-1}\boldsymbol{\mu}} = \frac{1}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}} = \frac{1}{a} \\ \mathbf{w}_2 &= \frac{\mathbf{C}^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\mathbf{C}^{-1}\boldsymbol{\mu}}, \mathbf{w}_2^T = \frac{\boldsymbol{\mu}^T\mathbf{C}^{-1}}{\mathbf{1}^T\mathbf{C}^{-1}\boldsymbol{\mu}} \\ \mathbf{w}_2^T\mathbf{C}\mathbf{w}_2 &= \frac{\boldsymbol{\mu}^T\mathbf{C}^{-1}\mathbf{C}}{\mathbf{1}^T\mathbf{C}^{-1}\boldsymbol{\mu}} \frac{\mathbf{C}^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\mathbf{C}^{-1}\boldsymbol{\mu}} = \frac{\boldsymbol{\mu}^T\mathbf{C}^{-1}\boldsymbol{\mu}}{(\mathbf{1}^T\mathbf{C}^{-1}\boldsymbol{\mu})^2} = \frac{c}{b^2} \end{aligned}$$

(ii)

$$\begin{aligned} Cov(r_1, r_2) &= Cov(\alpha\mathbf{w}_1 + (1-\alpha)\mathbf{w}_2, \beta\mathbf{w}_1 + (1-\beta)\mathbf{w}_2) \\ &= \alpha\beta Cov(\mathbf{w}_1, \mathbf{w}_1) + (1-\alpha)(1-\beta)Cov(\mathbf{w}_2, \mathbf{w}_2) + [\alpha(1-\beta) + \beta(1-\alpha)]Cov(\mathbf{w}_1, \mathbf{w}_2) \\ &= \alpha\beta\sigma_1^2 + (1-\alpha)(1-\beta)\sigma_2^2 + (\alpha + \beta - 2\alpha\beta)Cov(\mathbf{w}_1, \mathbf{w}_2) \\ &= \frac{\alpha\beta}{a} + \frac{(1-\alpha)(1-\beta)c}{b^2} + \frac{\alpha + \beta - 2\alpha\beta}{a} \\ &= \frac{\alpha\beta b^2 + (1-\alpha)(1-\beta)ac + (\alpha + \beta - 2\alpha\beta)b^2}{ab^2} \\ &= \frac{(1-\alpha)(1-\beta)ac - (-\alpha - \beta + \alpha\beta)b^2 - b^2 + b^2}{ab^2} \\ &= \frac{1}{a} + \frac{(1-\alpha)(1-\beta)(ac - b^2)}{ab^2} \end{aligned}$$

(b) (i) Since the global minimum-variance portfolio has variance and mean of σ_g^2 and μ_g respectively, the minimum-variance frontier has the equation of

$$\sigma^2 = \frac{a}{ac-b^2}(\mu-\mu_g)^2 + \sigma_g^2$$

The efficient portfolio q has variance and mean of $\sigma_q^2 = \sigma_p^2$ and μ_q, substituting these values into the equation

$$\sigma_p^2 = \sigma_g^2 + \frac{a}{ac-b^2}(\mu_q-\mu_g)^2$$

(ii) From 3(b)(i),

$$\begin{aligned}\sigma_p^2 &= \sigma_g^2 + \frac{a}{ac-b^2}(\mu_q-\mu_g)^2 \\ (\mu_q-\mu_g)^2 &= \frac{\sigma_p^2-\sigma_g^2}{\frac{a}{ac-b^2}}\end{aligned}$$

The minimum-variance portfolio r has variance and mean of σ_r^2 and $\mu_r = \mu_p$, substituting these values into the equation of the minimum-variance frontier

$$\begin{aligned}\sigma_r^2 &= \sigma_g^2 + \frac{a}{ac-b^2}(\mu_p-\mu_g)^2 \\ (\mu_p-\mu_g)^2 &= \frac{\sigma_r^2-\sigma_g^2}{\frac{a}{ac-b^2}} \\ \Psi_p^2 &= \frac{(\sigma_p^2-\sigma_g^2)^2}{(\mu_q-\mu_g)^2} = \frac{\sigma_r^2-\sigma_g^2}{\frac{a}{ac-b^2}} \times \frac{\frac{a}{ac-b^2}}{\sigma_p^2-\sigma_g^2} \\ &= \frac{\sigma_r^2-\sigma_g^2}{\sigma_p^2-\sigma_g^2}\end{aligned}$$

4 (a) (i) From $K_3 - K_2 = K_2 - K_1$, $2K_2 = K_1 + K_3$.
Suppose for the sake of contradiction that $C_2 > \frac{1}{2}(C_1 + C_3)$, i.e., $2C_2 > C_1 + C_3$.
To construct an arbitrage strategy, we

* Long 1 K_1-call
* Long 1 K_3-call
* Short 2 K_2-call

Let S_T = asset price at time of maturity, T, and r = annual interest rate.
The initial value of the strategy is $C_1 + C_3 - 2C_2 < 0$ since $2C_2 > C_1 + C_3$.
The profit table of the strategy is

	$S_T < K_1$	$K_1 < S_T < K_2$	$K_2 < S_T < K_3$	$S_T > K_3$
long 1 K_1-call	$-C_1e^{rT}$	$S_T - K_1 - C_1e^{rT}$	$S_T - K_1 - C_1e^{rT}$	$S_T - K_1 - C_1e^{rT}$
short 2 K_2-call	$2C_2e^{rT}$	$2C_2e^{rT}$	$2(K_2 - S_T + C_2e^{rT})$	$2(K_2 - S_T + C_2e^{rT})$
long 1 K_3-call	$-C_3e^{rT}$	$-C_3e^{rT}$	$-C_3e^{rT}$	$S_T - K_3 - C_3e^{rT}$

When $S_T < K_1$, total profit is $-C_1e^{rT} + 2C_2e^{rT} - C_3e^{rT} = e^{rT}(2C_2 - C_1 - C_3) > 0$ since $2C_2 > C_1 + C_3$.
When $K_1 < S_T < K_2$, total profit is $S_T - K_1 - C_1e^{rT} + 2C_2e^{rT} - C_3e^{rT} = e^{rT}(2C_2 - C_1 - C_3) + S_T - K_1 > 0$ since $2C_2 > C_1 + C_3$ and $S_T > K_1$.
When $K_2 < S_T < K_3$, total profit is $S_T - K_1 - C_1e^{rT} + 2K_2 - 2S_T + 2C_2e^{rT} - C_3e^{rT} = e^{rT}(2C_2 - C_1 - C_3) + 2K_2 - K_1 - S_T = e^{rT}(2C_2 - C_1 - C_3) + K_1 + K_3 - K_1 - S_T = e^{rT}(2C_2 - C_1 - C_3) + K_3 - S_T > 0$ since $2C_2 > C_1 + C_3$ and $K_3 > S_T$.
When $S_T > K_3$, total profit is $S_T - K_1 - C_1e^{rT} + 2K_2 - 2S_T + 2C_2e^{rT} + S_T - K_3 - C_3e^{rT} = e^{rT}(2C_2 - C_1 - C_3) + 2K_2 - K_1 - K_3 = e^{rT}(2C_2 - C_1 - C_3) > 0$ since $2C_2 > C_1 + C_3$ and $2K_2 - K_1 - K_3 = 0$.
This is an arbitrage opportunity. Therefore, it can be proved that $C_2 \le \frac{1}{2}(C_1 + C_3)$.

(ii) Let S_T = asset price at time of maturity, T, and r = annual interest rate.
By put-call parity,

$$\begin{aligned}
C(K) - P(K) &= S_T - Ke^{-rT} \\
P(K) &= C(K) - S_T + Ke^{-rT} \\
2P_2 &= 2C_2 - 2S_T + 2K_2e^{-rT} \\
&\leq (C_1 + C_3) - 2S_T + 2K_2e^{-rT} \quad \text{since } C_2 \leq \frac{1}{2}(C_1 + C_3) \text{ as proven in 4(a)(i)} \\
&= C_1 + C_3 - 2S_T + e^{-rT}(K_1 + K_3) \quad \text{since } 2K_2 = K_1 + K_3 \\
&= (C_1 - S_T + K_1e^{-rT} + (C_3 - S_T + K_3e^{-rT}) \\
&= P_1 + P_3 \quad \text{by put-call parity} \\
2P_2 &\leq P_1 + P_3 \\
P_2 &\leq \frac{1}{2}(P_1 + P_3)
\end{aligned}$$

(b) (i) From the question, $S_0 = 19$, $K = 20$, $C = 1$, $P = 1.5$, $T = \frac{1}{4}$, $r = 0.04$.

$$\begin{aligned}
C - P &= 1 - 1.5 = -0.5 \\
S_0 - Ke^{-rT} &= 10 - 20e^{-0.04\times\frac{1}{4}} = -0.800996675 \\
C - P &> S_0 - Ke^{-rT} \\
C + Ke^{-rT} &> S_0 + P
\end{aligned}$$

To construct an arbitrage strategy, we
- Long 1 share
- Long 1 K-put
- Short Ke^{-rT} worth of risk-free asset
- Short 1 K-call

Initial value is $S_0 + P - (C + Ke^{-rT}) < 0$.
Profit matrix is

	$S_T < 20$	$S_T > 20$
Long 1 share	$S_T - S_0e^{rT}$	$S_T - S_0e^{rT}$
Long 1 K-put	$20 - S_T - Pe^{rT}$	$-Pe^{rT}$
Short Ke^{-rT} worth of risk-free asset	0	0
Short 1 K-call	Ce^{rT}	$Ce^{rT} - S_T + 20$

When $S_T < 20$, total profit is $S_T - S_0e^{rT} + 20 - S_T - Pe^{rT} + Ce^{rT} = e^{0.04\times\frac{1}{4}}(1-1.5) + 20 - 19e^{0.04\times\frac{1}{4}} = 0.3040217419 > 0$.
When $S_T > 20$, total profit is $S_T - S_0e^{rT} - Pe^{rT} + Ce^{rT} - S_T + 20 = e^{0.04\times\frac{1}{4}}(1-1.5) + 20 - 19e^{0.04\times\frac{1}{4}} = 0.3040217419 > 0$.
This is an arbitrage strategy.

(ii) From the question, $S_0 = 80$, $K = 75$, $r = 0.1$, $T = \frac{1}{2}$, $C_E = 8$.

$$\begin{aligned}
S_0 - Ke^{-rT} &= 80 - 75e^{-0.1\times\frac{1}{2}} = 8.657793162 \\
C_E &= 8 \\
S_0 - Ke^{-rT} &> C_E \\
S_0 &> Ke^{-rT} + C_E
\end{aligned}$$

To construct an arbitrage strategy, we
- Short 1 share
- Long Ke^{-rT} worth of risk-free asset
- Long 1 K-call

Initial value is $Ke^{-rT} + C_E - S_0 < 0$.
Profit matrix is

	$S_T < 75$	$S_T > 75$
Short 1 share	$-S_T + S_0 e^{rT}$	$-S_T + S_0 e^{rT}$
Long Ke^{-rT} worth of risk-free asset	0	0
Long 1 K-call	$-C_E e^{rT}$	$-C_E e^{rT} + S_T - 75$

When $S_T < 75$, total profit is $-S_T + S_0 e^{rT} - C_E e^{rT} = 80e^{0.1\times\frac{1}{2}} - 8e^{0.1\times\frac{1}{2}} - S_T = 75.69151894 - S_T > 0$ since $S_T < 75$.
When $S_T > 75$, total profit is $80e^{0.1\times\frac{1}{2}} - 75 - 8e^{0.1\times\frac{1}{2}} = 0.69151894 > 0$.
This is an arbitrage strategy.

(iii) From the question, $S_0 = 58$, $K = 65$, $r = 0.05$, $T = \frac{1}{6}$, $P_E = 6$.

$$P_E = 6$$

$$Ke^{-rT} - S_0 = 65e^{-0.05\times\frac{1}{6}} - 58 = 6.460584022$$

$$P_E < Ke^{-rT} - S_0$$

$$P_E + S_0 < Ke^{-rT}$$

To construct an arbitrage strategy, we

* Short Ke^{-rT} worth of risk-free asset
* Long 1 K-put
* Long 1 share

Initial value is $P_E + S_0 - Ke^{-rT} < 0$.
Profit matrix is

	$S_T < 65$	$S_T > 65$
Short Ke^{-rT} worth of risk-free asset	0	0
Long 1 K-put	$65 - S_T - P_E e^{rT}$	$-P_E e^{rT}$
Long 1 share	$S_T - S_0 e^{rT}$	$S_T - S_0 e^{rT}$

When $S_T < 65$, total profit is $65 - 6e^{0.05\times\frac{1}{6}} - 58e^{0.05\times\frac{1}{6}} = 0.4644382587$.
When $S_T > 65$, total profit is $S_T - 6e^{0.05\times\frac{1}{6}} - 58e^{0.05\times\frac{1}{6}} = S_T - 64.53556174 > 0$ since $S_T > 65$.
This is an arbitrage strategy.