SUGGESTED SOLUTION FOR MA2001 LINEAR ALGEBRA I FINAL (AY21/22 SEM 2)

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Question 1.

Consider the augmented matrix of the linear system in the question. We have

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 4 & 5 & 6 & 2a \\ 0 & -a & 2 & b \end{array}\right) \xrightarrow{R_2-4R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & -3 & 2 & -2a \\ 0 & -a & 2 & b \end{array}\right)$$
$$\xrightarrow{3R_3-aR_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & -3 & 2 & -2a \\ 0 & 0 & 6-2a & 3b+2a^2 \end{array}\right).$$

When $6-2a=0$, i.e. $a=3$, then if $3b+2a^2=0$, i.e. $b=-6$, the system will have infinitely many solutions since there are only two pivotal columns in the REF.

When $a=3$ and $b\neq -6$, then the system will have no solution since the last column of the augmented matrix is a pivotal column.

When $a\neq 3$ and $b\in\mathbb{R}$, the system will have a unique solution since the first three columns are all pivotal.

Question 2.

(i) We have

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & -2 & -3 & 0 \\ 2 & -1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 3 & 1 \end{vmatrix} = (-1)^{4+4} \begin{vmatrix} 1 & -2 & -3 \\ 2 & -1 & 2 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= (-1)^{2+3} \begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix} + 3(-1)^{3+3} \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}$$

$$= 1$$

(ii) Consider $(\mathbf{A}|\mathbf{I})$. We have

$$(\mathbf{A}|\mathbf{I}) = \left(\begin{array}{cccc|cccc} 1 & -2 & -3 & 0 & 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\frac{1}{3}R_2-\frac{2}{3}R_1]{3R_3-R_2+2R_1} \left(\begin{array}{cccc|cccc} 1 & -2 & -3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{8}{3} & 0 & -\frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2-\frac{8}{3}R_3, R_1+2R_2-\frac{7}{3}R_3]{R_4-3R_3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -5 & 3 & -7 & 0 \\ 0 & 1 & 0 & 0 & -6 & 3 & -8 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 & -6 & 3 & -9 & 1 \end{array}\right)$$

$$= (\mathbf{I}|\mathbf{A}^{-1}).$$

(iii) Since we have $\mathbf{B}^T\mathbf{A}\mathbf{B} = \mathbf{A}$ and $\det(\mathbf{A}) \neq 0$ (as $\mathbf{A}$ is invertible), we have

$$\det(\mathbf{B}^T)\det(\mathbf{A})\det(\mathbf{B}) = \det(\mathbf{A}) \implies [\det(\mathbf{B})]^2 = 1 \implies \det(\mathbf{B}) \neq 0,$$

which implies that $\mathbf{B}$ is invertible.

Question 3.

(i) Consider $\det(\mathbf{A}-\lambda\mathbf{I})$. We have

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 0 & -1 & 0 \\ 0 & -1-\lambda & 0 & 0 \\ -1 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} &= (1-\lambda)\begin{vmatrix} -\lambda & 0 & -1 \\ 0 & -1-\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} \\ &= (1-\lambda)(-1-\lambda)\begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} \\ &= (1-\lambda)(-1-\lambda)(\lambda^2-1) \\ &= (\lambda-1)^2(\lambda+1)^2.\end{aligned}$$

Setting $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$, we have $\lambda = \pm 1$.

(ii) Consider the linear system $(\mathbf{A}-\mathbf{I})\mathbf{x} = \mathbf{0}$, i.e., $\lambda = 1$, where $\mathbf{x} = (a\ b\ c\ d)^T$. By Gaussian elimination, we have

$$\begin{aligned}(\mathbf{A}-\mathbf{I})\mathbf{x} = \mathbf{0} \implies \begin{cases} -a-c = 0 \\ -2b = 0 \end{cases} &\implies \begin{cases} a = -c \\ b = 0 \end{cases} \\ &\implies \mathbf{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\end{aligned}$$

so a basis for E_1 is $\{(1\ 0\ -1\ 0)^T,\ (0\ 0\ 0\ 1)^T\}$.

Similarly, consider the linear system $(\mathbf{A}+\mathbf{I})\mathbf{x}=\mathbf{0}$, i.e., $\lambda=-1$. By Gaussian elimination, we have

$$(\mathbf{A}+\mathbf{I})\mathbf{x}=\mathbf{0} \implies \begin{cases} a-c=0 \\ d=0 \end{cases} \implies \begin{cases} a=c \\ d=0 \end{cases}$$

$$\implies \mathbf{x}=\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}=a\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}+b\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

so a basis for E_{-1} is $\{(1\ 0\ 1\ 0)^T,\ (0\ 1\ 0\ 0)^T\}$.

(iii) Since $\mathbf{P}$ is orthogonal, we have $\mathbf{P}^T=\mathbf{P}^{-1}$. From (ii), we know that $\mathbf{A}$ is diagonalisable. Hence, we choose

$$\mathbf{P}=\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which is an orthogonal matrix whose columns form an orthogonal basis of $\mathbb{R}^4$. We now verify that $\mathbf{P}^T\mathbf{A}\mathbf{P}$ is indeed a diagonal matrix:

$$\mathbf{P}^T\mathbf{A}\mathbf{P}=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Question 4.

Substitute $(1,2,0)$, $(0,1,1)$, $(-1,0,1)$, $(1,1,1)$ into the function. We have

$$\begin{cases} a+2b+c=0 \\ b+c=1 \\ a+c=1 \\ a+b+c=1 \end{cases} \implies \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right) \implies \mathbf{A}\mathbf{x}=\mathbf{b},$$

where

$$\mathbf{A}=\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \mathbf{b}=\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}=\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Consider $\mathbf{A}^T\mathbf{A}\mathbf{x}=\mathbf{A}^T\mathbf{b}$. We have

$$\begin{pmatrix} 3 & 3 & 3 \\ 3 & 6 & 4 \\ 3 & 4 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \implies \left(\begin{array}{ccc|c} 3 & 3 & 3 & 2 \\ 3 & 6 & 4 & 2 \\ 3 & 4 & 4 & 3 \end{array}\right)$$

$$\xrightarrow[R_2-R_1]{3R_3-R_2-2R_1} \left(\begin{array}{ccc|c} 3 & 3 & 3 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 3 \end{array}\right)$$

$$\implies \begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{2} \\ \frac{3}{2} \end{pmatrix}.$$

Hence, the best approximation of the function is $z=-\frac{1}{3}x^2-\frac{1}{2}y+\frac{3}{2}$.

Question 5.

(i) To prove $\text{span}(S) = \text{span}(T)$, we just need to prove that

$$\text{span}(S) \subseteq \text{span}(T)$$
$$\text{span}(T) \subseteq \text{span}(S).$$

Since $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ can all be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, we have $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3 \in S$ and thus $\text{span}(T) \subseteq \text{span}(S)$.

We now prove that $\text{span}(S) \subseteq \text{span}(T)$. Since we have

$$\begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{pmatrix},$$

whose coefficient matrix is denoted by $\mathbf{A}$, we have

$$\begin{aligned} \det(\mathbf{A}) = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{vmatrix} &= \begin{vmatrix} 0 & 1 & 1 \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ 1 & 0 & -1 \end{vmatrix} \qquad (R_1 + R_2, R_3 - R_2) \\ &= \begin{vmatrix} 1 & 1 \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -\frac{2}{3} & \frac{2}{3} \end{vmatrix} \\ &= -1 \neq 0, \end{aligned}$$

which implies that $\mathbf{A}$ is invertible. Therefore, we have

$$\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{pmatrix},$$

which implies that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3 \in T$ and thus $\text{span}(S) \subseteq \text{span}(T)$.

We then conclude that $\text{span}(S) = \text{span}(T)$.

(ii) Suppose S is an orthonormal set. We have

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 0 & \text{if } i \neq j; \\ 1 & \text{if } i = j, \end{cases}$$

for $i, j = 1, 2, 3$.

Thus, we have

$$\|\mathbf{u}_1\| = \left(\frac{2}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3\right) \cdot \left(\frac{2}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3\right) = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1;$$
$$\|\mathbf{u}_2\| = \left(-\frac{2}{3}\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3\right) \cdot \left(-\frac{2}{3}\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3\right) = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1;$$
$$\|\mathbf{u}_3\| = \left(\frac{1}{3}\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2 - \frac{2}{3}\mathbf{v}_3\right) \cdot \left(\frac{1}{3}\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2 - \frac{2}{3}\mathbf{v}_3\right) = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1.$$

We also have

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \left(\frac{2}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3\right) \cdot \left(-\frac{2}{3}\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3\right) = -\frac{4}{9} + \frac{2}{9} + \frac{2}{9} = 0;$$
$$\mathbf{u}_2 \cdot \mathbf{u}_3 = \left(-\frac{2}{3}\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3\right) \cdot \left(\frac{1}{3}\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2 - \frac{2}{3}\mathbf{v}_3\right) = -\frac{2}{9} + \frac{4}{9} - \frac{2}{9} = 0;$$
$$\mathbf{u}_1 \cdot \mathbf{u}_3 = \left(\frac{2}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3\right) \cdot \left(\frac{1}{3}\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2 - \frac{2}{3}\mathbf{v}_3\right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0.$$

We therefore conclude that T is also an orthonormal set.

Question 6.

If $\mathbf{A}$ is full rank, then $\text{rank}(\mathbf{A}) = n \implies \text{rank}(\mathbf{A} - \mathbf{I}) = 0 \implies \mathbf{A} - \mathbf{I} = \mathbf{0} \implies \mathbf{A} = \mathbf{I}$, which implies $\mathbf{A}^2 = \mathbf{A}$.

We now suppose $\mathbf{A}$ is not full rank. By the rank-nullity theorem, we have

$$\text{nullity}(\mathbf{A}) = \text{rank}(\mathbf{A} - \mathbf{I});$$
$$\text{rank}(\mathbf{A}) = \text{nullity}(\mathbf{A} - \mathbf{I}).$$

Since we have $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0} \implies \mathbf{A}\mathbf{x} = \mathbf{I}\mathbf{x} = \mathbf{x}$, we conclude that every vector in the null space of $(\mathbf{A} - \mathbf{I})$ is in the column space of $\mathbf{A}$, so the null space of $(\mathbf{A} - \mathbf{I})$ is a subset of the column space of $\mathbf{A}$. This, along with the fact that $\text{rank}(\mathbf{A}) = \text{nullity}(\mathbf{A} - \mathbf{I})$, suggests that

$$\text{the null space of } \mathbf{A} - \mathbf{I} = \text{the column space of } \mathbf{A}.$$

Similarly, with the fact that $\mathbf{A}\mathbf{x} = \mathbf{0} \implies (\mathbf{A} - \mathbf{I})\mathbf{x} = -\mathbf{x}$ and $\text{nullity}(\mathbf{A}) = \text{rank}(\mathbf{A} - \mathbf{I})$, we conclude that

$$\text{the null space of } \mathbf{A} = \text{the column space of } \mathbf{A} - \mathbf{I}.$$

We now prove $\mathbf{A}^2 = \mathbf{A}$. Let $\mathbf{x} \in \mathbb{R}^n$ be arbitrarily chosen. It must satisfy exactly one of the following cases:

(1) $\mathbf{x}$ is in the null space of $\mathbf{A} - \mathbf{I}$:

$$\mathbf{A}(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{A}[(\mathbf{A} - \mathbf{I})\mathbf{x}] = \mathbf{A}\mathbf{0} = \mathbf{0}.$$

(2) $\mathbf{x}$ is not in the null space of $\mathbf{A} - \mathbf{I}$:

$$\begin{aligned}(\mathbf{A} - \mathbf{I})\mathbf{x} \text{ is in the column space of } (\mathbf{A} - \mathbf{I}) &\implies (\mathbf{A} - \mathbf{I})\mathbf{x} \text{ is in the null space of } \mathbf{A} \\ &\implies \mathbf{A}(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0}.\end{aligned}$$

Therefore, for all $\mathbf{x} \in \mathbb{R}^n$, we have $\mathbf{A}(\mathbf{A} - \mathbf{I})\mathbf{x} = 0$, which implies that the null space of $\mathbf{A}(\mathbf{A} - \mathbf{I})$ is $\mathbb{R}^n$. Consequently, we must have

$$\mathbf{A}(\mathbf{A} - \mathbf{I}) = \mathbf{0} \implies \mathbf{A}^2 - \mathbf{A} = 0 \implies \mathbf{A}^2 = \mathbf{A}.$$

We therefore conclude our proof.

Question 7.

We first observe that

$$\begin{aligned}\mathbf{AB} = \mathbf{BA}^{-1} &\implies (\mathbf{AB})^2 = (\mathbf{BA}^{-1})(\mathbf{AB}) = (\mathbf{AB})(\mathbf{BA}^{-1}) \\ &\implies \mathbf{B}^2 = \mathbf{AB}^2\mathbf{A}^{-1} \\ &\implies \mathbf{B}^2\mathbf{A} = \mathbf{AB}^2.\end{aligned}$$

Choose $V := \{\mathbf{u}_i\}$ to be a set of linearly independent eigenvectors of $\mathbf{A}$, where $\mathbf{u}_i \in E_{\lambda_i}, 1 \leq i \leq n$. It is to be noted that V is a basis for $\mathbb{R}^n$.

If $\mathbf{u}_i$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ_i, we will have

$$\mathbf{AB}^2(\mathbf{u}_i) = \mathbf{B}^2(\mathbf{A}\mathbf{u}_i) = \lambda_i \mathbf{B}^2\mathbf{u}_i,$$

which implies that $\mathbf{B}^2\mathbf{u}_i \in E_{\lambda_i}$.

Since $\mathbf{A}$ is diagonalisable and has n distinct eigenvalues, any basis of any eigenspace can only consist of one vector, which suggests that $\{\mathbf{u}_i\}$ is a basis of E_{λ_i}. Therefore, since $\mathbf{B}^2\mathbf{u}_i \in E_{\lambda_i}$, we must have $\mathbf{B}^2\mathbf{u}_i = m\mathbf{u}_i$ for some $m \in \mathbb{R}$, which suggests that $\mathbf{u}_i$ is an eigenvector of $\mathbf{B}^2$. Since this is true for all $\mathbf{u}_i \in V$, we conclude that $\mathbf{B}^2$ has n linearly independent eigenvectors, so it is diagonalisable.

Question 8.

(a) The statement is true.

Since $\mathbf{A}$ is orthogonal, its column space is just $\mathbb{R}^n$. Writing

$$\mathbf{A} = (\mathbf{v}_1\ \mathbf{v}_2\ \dots\ \mathbf{v}_n),$$

we have $V := \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthonormal basis of $\mathbb{R}^n$.

Let E be the standard basis (which is orthonormal). Noticing that

$$(\mathbf{v}_i)_E = \mathbf{v}_i$$

for $1 \le i \le n$, we conclude that

$$\mathbf{A} = ((\mathbf{v}_1)_E\ (\mathbf{v}_2)_E\ \dots\ (\mathbf{v}_n)_E).$$

This implies that $\mathbf{A}$ is the transition matrix from V to E.

(b) The statement is false.

Suppose T is a linear transformation. We have

$$T\left(\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}\right) = T\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) + T\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) \implies 8 = 2 \implies \text{a contradiction.}$$

(c) The statement is true.

By the rank-nullity theorem, we have

$$\operatorname{rank}(\mathbf{A}) + \operatorname{nullity}(\mathbf{A}) = n;$$
$$\operatorname{rank}(\mathbf{BA}) + \operatorname{nullity}(\mathbf{BA}) = n.$$

Since $\mathbf{Ax} = \mathbf{0} \implies \mathbf{BAx} = \mathbf{B}(\mathbf{Ax}) = \mathbf{B0} = \mathbf{0}$, we conclude that the null space of $\mathbf{A}$ is a subset of the null space of $\mathbf{BA}$, so $\operatorname{nullity}(\mathbf{A}) \le \operatorname{nullity}(\mathbf{BA})$. Consequently, we must have

$$\operatorname{rank}(\mathbf{A}) \ge \operatorname{rank}(\mathbf{BA}).$$

Since $\mathbf{BA}$ is full rank, i.e., $\text{rank}(\mathbf{BA}) = \min(m, n)$, we have

$$\min(m, n) = \text{rank}(\mathbf{BA}) \leq \text{rank}(\mathbf{A}) \leq \min(m, n),$$

which implies that $\text{rank}(\mathbf{A}) = \min(m, n)$. We thus conclude that $\mathbf{A}$ is full rank.

(d) The statement is true.

Since $\mathbf{A}$ is diagonalisable, we can write $\mathbf{A} = \mathbf{PDP}^{-1}$ for some invertible matrix $\mathbf{P}$ and diagonal matrix $\mathbf{D}$.

Since $\mathbf{A}$ is nilpotent, there exists $m \in \mathbb{Z}^+$ such that $\mathbf{A}^m = \mathbf{0}$, which implies

$$\mathbf{A}^m = (\mathbf{PDP}^{-1})^m = \mathbf{PD}^m\mathbf{P}^{-1} = \mathbf{0} \implies \mathbf{D}^m = \mathbf{0} \implies \mathbf{D} = 0 \implies \mathbf{A} = 0.$$

We therefore conclude our proof.

(e) The statement is false.

We define $\mathbf{A}$ as

$$\mathbf{A} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Consider $\det(\mathbf{A} - \lambda\mathbf{I})$. We have

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2,$$

which suggests that $\lambda = 0$ is the only eigenvalue of $\mathbf{A}$, which has an algebraic multiplicity of 2. Meanwhile, solving $(\mathbf{A} - 0\mathbf{I})\mathbf{x} = \mathbf{0}$ gives us

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which suggests that $E_0 = \text{span}\{(1\ 0)^T\}$, i.e., geometric multiplicity is 1.

Since $\dim(E_0) = 1 <$ the algebraic multiplicity of $\lambda = 0$, we conclude that $\mathbf{A}$ is not diagonalisable. Yet, clearly, both $\binom{1}{0}$ and $\binom{2}{0}$ are the eigenvectors of $\mathbf{A}$. We therefore conclude that the question statement is false.

Remark: The condition for a square matrix to be diagonalisable is for all eigenvalues, their algebraic multiplicities equal geometric multiplicities.