NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA1102R Calculus
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Question 1

(i) We take the derivative $f'(x) = 12x^5 - 60x^3 = 12x^3\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)$. This is negative in $\left(-\infty, -\sqrt{5}\right)$, positive in $\left(-\sqrt{5}, 0\right)$, negative in $\left(0, \sqrt{5}\right)$, and positive again in $\left(\sqrt{5}, \infty\right)$. Therefore the function is increasing in $\left(-\sqrt{5}, 0\right)$ and $\left(\sqrt{5}, \infty\right)$, and it is decreasing in $\left(-\infty, -\sqrt{5}\right)$ and $\left(0, \sqrt{5}\right)$.

(ii) By the Increasing-Decreasing Test on the previous result, local minima are at $\left(-\sqrt{5}, f\left(-\sqrt{5}\right)\right) = \left(-\sqrt{5}, -121\right)$ and $\left(\sqrt{5}, f\left(\sqrt{5}\right)\right) = \left(\sqrt{5}, -121\right)$. There is a local maximum at $(0, f(0)) = (0, 4)$.

(iii) We take the second derivative $f''(x) = 60x^4 - 180x^2 = 60x^2\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$. We see that this is positive at $\left(-\infty, -\sqrt{3}\right)$ and $\left(\sqrt{3}, \infty\right)$, and so f is concave up in these intervals. f'' is negative at $\left(-\sqrt{3}, \sqrt{3}\right)$, and so f is concave down in that interval.

(iv) From the previous result, the sign of the second derivative changes at $x = \pm\sqrt{3}$. Therefore, we have inflection points $\left(-\sqrt{3}, -77\right)$ and $\left(\sqrt{3}, -77\right)$.

Question 2

(a)

$$\begin{aligned}
\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{x^2}{x-1}\right) &= \lim_{x\to 1}\frac{(x-1) - x^2\ln x}{(x-1)\ln x}\left[\frac{0}{0}\right] \\
&= \lim_{x\to 1}\frac{1 - 2x\ln x - x}{\ln x + \frac{x-1}{x}}\left[\frac{0}{0}\right] \\
&= \lim_{x\to 1}\frac{-2\ln x - 2 - 1}{\frac{1}{x} + \frac{1}{x^2}} \\
&= -\frac{3}{2}
\end{aligned}$$

(b) We wish to show for all $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x+1| < \delta$ implies

$$\left|\frac{2-x}{1+x^2} - \frac{3}{2}\right| \leq \epsilon$$

We choose $\delta = \min\left\{1, \frac{2}{7}\epsilon\right\}$. Then $0 < |x+1| < \delta$ implies

$$\begin{array}{ccccc} -1 & < & x+1 & < & 1 \\ -2 & < & x & < & 0 \\ 1 & < & 1+x^2 & < & 5 \\ \frac{1}{5} & < & \frac{1}{1+x^2} & < & 1 \\ & & \left|\frac{1}{1+x^2}\right| & < & 1 \end{array}$$

and also

$$\begin{array}{ccccc} -6 & < & 3x & < & 0 \\ 1 & < & -3x+1 & < & 7 \\ & & |-3x+1| & < & 7 \end{array}$$

Now we have

$$\begin{aligned} \left|\frac{2-x}{1+x^2} - \frac{3}{2}\right| &= \left|\frac{1-2x-3x^2}{2\left(1+x^2\right)}\right| \\ &= \frac{1}{2}\left|\frac{1}{1+x^2}\right| |-3x+1| \, |1+x| \\ &< \frac{1}{2}(1)(7)\,\delta \\ &\leq \epsilon \end{aligned}$$

as desired.

Question 3

(a) We decompose into partial fractions:

$$\begin{aligned} \int \frac{x^2}{\left(x^2-3x+2\right)^2}dx &= \int \left[\frac{4}{x-1} + \frac{1}{(x-1)^2} - \frac{4}{x-2} + \frac{4}{(x-2)^2}\right]dx \\ &= 4\ln|x-1| - \frac{1}{(x-1)} - 4\ln(x-2) - \frac{4}{x-2} + C \end{aligned}$$

(b) We substitute $u = \sqrt{1-x}$, so that $du = -\frac{dx}{2\sqrt{1-x}}$:

$$\begin{aligned} \int_0^1 \frac{1}{(2-x)\sqrt{1-x}}dx &= -\int_1^0 \frac{2}{\left(1+u^2\right)}du \\ &= 2\int_0^1 \frac{du}{1+u^2} \\ &= 2\left(\tan^{-1}(1) - \tan^{-1}(0)\right) \\ &= \frac{\pi}{2} \end{aligned}$$

Question 4

(i) We take the first integral and substitute $u = 6-x$ so that $du = -dx$:

$$\begin{aligned} \int_2^4 \frac{f(9-x)}{f(9-x)+f(x+3)}dx &= -\int_4^2 \frac{f(u+3)}{f(u+3)+f(9-u)}du \\ &= \int_2^4 \frac{f(x+3)}{f(9-x)+f(x+3)}dx \end{aligned}$$

as desired.

(ii) Continuing from our previous result,

$$\begin{aligned}
2\int_2^4 \frac{f(9-x)}{f(9-x)+f(x+3)}dx &= \int_2^4 \frac{f(9-x)+f(x+3)}{f(9-x)+f(x+3)}dx \\
\int_2^4 \frac{f(9-x)}{f(9-x)+f(x+3)}dx &= \frac{1}{2}\int_2^4 dx \\
&= 1
\end{aligned}$$

Since $f(x) = \sqrt[5]{x}$ is a positive, continuous function on the interval $[5,7]$, we simply plug in to obtain

$$\int_2^4 \frac{\sqrt[5]{9-x}}{\sqrt[5]{9-x}+\sqrt[5]{x+3}}dx = 1$$

Question 5

Let x be the radius of the semicircular portion of the Norman window of perimeter 9. Then the height of the rectangular portion would be $\frac{9-x\pi-2x}{2}$. The area A of the window is

$$A(x) = \frac{1}{2}\pi x^2 + 2x\left(\frac{9-x\pi-2x}{2}\right) = -\frac{\pi+4}{2}x^2 + 9x$$

The derivative $A'(x)$ is $(-\pi-4)x+9$, which is positive when $x \in \left(-\infty, \frac{9}{\pi+4}\right)$ and negative when $x \in \left(\frac{9}{\pi+4}, \infty\right)$. By the Increasing-Decreasing Test, A is maximum at $x = \frac{9}{\pi+4}$. Therefore the Norman window of perimeter 9m of largest area has width $\frac{9}{\pi+4}$ and length (tip of semicircular part to base) $\frac{18}{4+\pi}$.

Question 6

(a) We use cylindrical shells. The curve $y = x + \frac{4}{x}$ intersects $y = 5$ twice - first at $(1,5)$, and then at $(4,5)$. The region enclosed is below the line $y = 5$ and above $y = x + \frac{4}{x}$ from $x = 1$ to $x = 4$. Hence we have

$$\begin{aligned}
2\pi\int_1^4 (x+1)\left(5-x-\frac{4}{x}\right)dx &= 2\pi\int_1^4 \left(-x^2+4x+1-\frac{4}{x}\right)dx \\
&= 2\pi\left[-\frac{1}{3}x^3+2x^2+x-4\ln x\right]_{x=1}^{x=4} \\
&= 2\pi\left[-21+30+3-8\ln 2\right] \\
&= (24-16\ln 2)\pi
\end{aligned}$$

(b) We take first the derivative $\frac{dy}{dx}$, and then integrate $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$:

$$\begin{aligned}
y &= 8\left(\ln\frac{2+\sqrt{x}}{2-\sqrt{x}}-\sqrt{x}\right)\\
&= 8\left(\ln\left(2+\sqrt{x}\right)-\ln\left(2-\sqrt{x}\right)-\sqrt{x}\right)\\
\frac{dy}{dx} &= 8\left(\frac{1}{2+\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}+\frac{1}{2-\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x}}\right)\\
&= \frac{4}{\sqrt{x}}\left(\frac{4}{4-x}-1\right)\\
\left(\frac{dy}{dx}\right)^2 &= \frac{16}{x}\left[\frac{x^2}{(4-x)^2}\right]\\
\sqrt{1+\left(\frac{dy}{dx}\right)^2} &= \sqrt{1+\frac{16x}{(4-x)^2}}\\
&= \sqrt{\frac{(4+x)^2}{(4-x)^2}}\\
&= \frac{8}{4-x}-1\\
\int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx &= \int_0^1\left(\frac{8}{4-x}-1\right)dx\\
&= \left[-8\ln\left(4-x\right)-x\right]_{x=0}^{x=1}\\
&= 16\ln 2-8\ln 3-1
\end{aligned}$$

Question 7

(a) We have

$$\begin{aligned}
F(x) &= \int_0^x f(t)(x-t)^2\,dt\\
&= \int_0^x \left(x^2 f(t)-2f(t)\,xt+t^2 f(t)\right)dt\\
&= x^2\int_0^x f(t)\,dt-2x\int_0^x tf(t)\,dt+\int_0^x t^2 f(t)\,dt\\
F'(x) &= 2x\int_0^x f(t)\,dt+x^2 f(x)-2\int_0^x tf(t)\,dt-2x^2 f(x)+x^2 f(x)\\
&= 2x\int_0^x f(t)\,dt-2\int_0^x tf(t)\,dt\\
F''(x) &= 2\int_0^x f(t)\,dt+2xf(x)-2xf(x)\\
F'''(x) &= 2f(x)
\end{aligned}$$

(b) We substitute $u = a + (b-a)\,x$, $du = (b-a)\,dx$, and obtain the limit through l'Hospital's Rule:

$$\begin{aligned}
\lim_{t\to 0}\left\{\int_0^1 \left[a\,(1-x)+bx\right]^t dx\right\}^{\frac{1}{t}} &= \lim_{t\to 0}\left\{\frac{\int_a^b u^t du}{b-a}\right\}^{\frac{1}{t}} \\
&= \lim_{t\to 0}\left[\frac{b^{t+1}-a^{t+1}}{(t+1)\,(b-a)}\right]^{\frac{1}{t}} \\
&= \lim_{t\to 0}\exp\left[\frac{\ln\left(b^{t+1}-a^{t+1}\right)-\ln\,(t+1)-\ln\,(b-a)}{t}\right] \\
&= \exp\left[\lim_{t\to 0}\frac{\ln\left(b^{t+1}-a^{t+1}\right)-\ln\,(t+1)-\ln\,(b-a)}{t}\right] \\
&= \exp\left[\lim_{t\to 0}\frac{\frac{(\ln b)b^{t+1}-(\ln a)a^{t+1}}{b^{t+1}-a^{t+1}}-\frac{1}{t+1}}{1}\right] \\
&= \exp\left[\frac{b\ln b-a\ln a}{b-a}-1\right] \\
&= \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{b-a}
\end{aligned}$$

(c) We observe that for $n \neq 0, 1$,

$$\begin{aligned}
f'(a) &= \lim_{h\to 0}\frac{f\,(a+h)-f\,(a)}{h} \\
&= \lim_{h\to 0}\frac{f\,(a+nh)-f\,(a)}{nh} \\
&= \lim_{h\to 0}\frac{f\,(a+(n-1)\,h)-f\,(a)}{(n-1)\,h}
\end{aligned}$$

Then we have

$$\begin{aligned}
f'(a) &= nf'(a)-(n-1)\,f'(a) \\
&= \lim_{h\to 0}\frac{f\,(a+nh)-f\,(a)}{h}-\lim_{h\to 0}\frac{f\,(a+(n-1)\,h)-f\,(a)}{h} \\
&= \lim_{h\to 0}\frac{f\,(a+nh)-f\,(a+(n-1)\,h)}{h}
\end{aligned}$$

as desired.

Question 8

(a)

$$\begin{aligned}
\frac{dy}{dx}-(\tan x)\,y &= \exp(-x)\tan x\\
\cos x\frac{dy}{dx}-(\sin x)\,y &= \exp(-x)\sin x\\
\frac{d}{dx}\left[y\cos x\right] &= \exp(-x)\sin x\\
y\cos x &= \int \exp(-x)\sin x dx\\
&= -\exp(-x)\sin x+\int \exp(-x)\cos x dx\\
&= -\exp(-x)\sin x-\exp(-x)\cos x-\int \exp(-x)\sin x dx\\
&= -\frac{1}{2}\exp(-x)(\sin x+\cos x)+C\\
y &= -\frac{1}{2}\exp(-x)(\tan x+1)+\frac{C}{\cos x}
\end{aligned}$$

When $x=0$, $y=1$, hence

$$\begin{aligned}
1 &= -\frac{1}{2}\exp(-0)(\tan 0+1)+\frac{C}{\cos 0}\\
\Rightarrow C &= \frac{3}{2}
\end{aligned}$$

Therefore

$$y = -\frac{1}{2}\left(e^{-x}(\tan x+1)-\frac{3}{\cos x}\right)$$

(b) We solve the differential equation

$$\begin{aligned}
\frac{dT}{dt} &= -k(T-T_S)\\
\frac{dT}{T-T_S} &= -kdt\\
\ln|T-T_S| &= -kt+C\\
T &= T_S+Ae^{-kt}
\end{aligned}$$

where $A=e^C$

We have $T(0)=20$, $T(5)=25$, and $T(10)=28$, giving us

$$\begin{aligned}
20 &= T_S+A\\
25 &= T_S+Ae^{-5k}\\
28 &= T_S+Ae^{-10k}
\end{aligned}$$

We have

$$e^{5k}=\frac{1-e^{-5k}}{e^{-5k}-e^{-10k}}=\frac{20-25}{25-28}=\frac{5}{3}$$

and therefore $20=T_S+A$, and $25=T_S+\frac{3}{5}A$, concluding that $T_S=\frac{5}{2}\left(25-\frac{3}{5}\cdot 20\right)=32.5$. The outdoor temperature is therefore $32.5°$.

Question 9

If we can show that there exists two real numbers c_1 and c_2, $(c_1 \neq c_2)$ such that $f(c_1) = f(c_2)$, then we are done, because by Rolle's Theorem there must exist some c between c_1 and c_2 such that $f'(c) = 0$.

Since f is differentiable on $\mathbb{R}$, it must be defined on $\mathbb{R}$. Now, for any point $x \in \mathbb{R}$, we have either $f(x) < 0, f(x) = 0$ or $f(x) > 0$. There are obviously more than 3 points on the real line, hence by the Pigeonhole Principle, there must exist two numbers $x_1 < x_2$ for which $f(x_1)$ and $f(x_2)$ have the same sign (positive, negative, or zero).

- **Case 1**: $f(x_1)$ and $f(x_2)$ are both zero. Then immediately, we are done.
- **Case 2**: $f(x_1)$ and $f(x_2)$ are either both positive or both negative. If $|f(x_1)| = |f(x_2)|$ then $f(x_1) = f(x_2)$ and we are done. Now, assume that $|f(x_1)| \neq |f(x_2)|$.
  - Suppose $|f(x_1)| > |f(x_2)|$. Since $\lim_{x\to-\infty} f(x) = 0$, there exists some $N \in \mathbb{R}$ such that if $x < N$, $|f(x)| < |f(x_2)|$; clearly $x_1 > N$. Take any $x_3 < N < x_1$. It follows that $f(x_2)$ is between $f(x_1)$ and $f(x_3)$. (We note that it doesn't matter if the sign of $f(x_3)$ is the same as that of $f(x_1)$ or $f(x_2)$.) Since f is continuous, from the Intermediate Value Theorem there must exist $x_4 \in (x_3, x_1)$ such that $f(x_4) = f(x_2)$, and we are done. (Note that $x_4 < x_1 < x_2$ and hence $x_4 \neq x_2$)
  - Suppose $|f(x_1)| < |f(x_2)|$ Similarly, use $\lim_{x\to\infty} = 0$. There exists some $M \in \mathbb{R}$ so that if $x > M$, $|f(x)| < |f(x_1)|$; clearly $x_2 < M$. Take any $x_3 > M > x_2$. $f(x_1)$ must be between $f(x_2)$ and $f(x_3)$. Since f is continuous, from the Intermediate Value Theorem there must exist $x_4 \in (x_2, x_3)$ such that $f(x_4) = f(x_1)$, and we are done. (Note that $x_4 > x_2 > x_1$ and hence $x_4 \neq x_1$)

END OF SOLUTIONS

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