NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Ho Chin Fung

MA4235 Graph Theory
AY 2008/2009 Sem 2

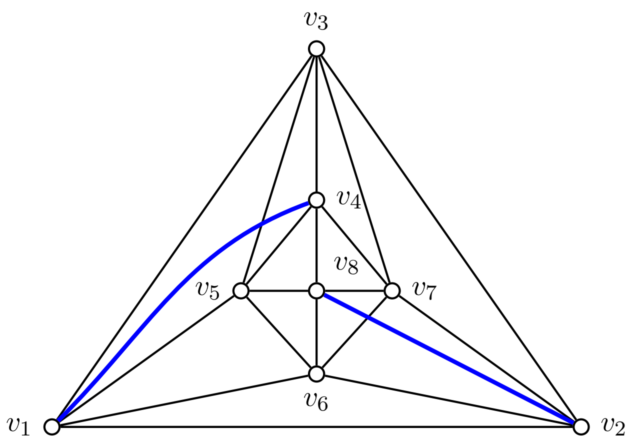
Question 1

(i) Observe that $S = \{v_1, v_3, v_6, v_7\}$, $S = \{v_2, v_3, v_5, v_6\}$, $S = \{v_3, v_5, v_6, v_7\}$, $S = \{v_3, v_5, v_7, v_8\}$ and $S = \{v_4, v_5, v_6, v_7\}$ are examples of cuts of G such that $|S| = 4$. Thus, we have $\kappa(G) \le 4$.
By exhausting all possibilities, we find that there does not exist a cut S of G such that $|S| = 3$. Thus, we have $\kappa(G) > 3$.
Hence, we have $\kappa(G) = 4$.
Now, any of the above S is a cut of G such that $|S| = \kappa(G)$.

(ii) Observe that $F = \{v_1v_2, v_1v_3, v_1v_5, v_1v_6\}$, $F = \{v_1v_2, v_2v_3, v_2v_6, v_2v_7\}$ and $F = \{v_4v_8, v_5v_8, v_6v_8, v_7v_8\}$ are examples of edge-cuts of G such that $|F| = 4$. Thus, we have $\kappa'(G) \le 4$.
By exhausting all possibilities, we find that there does not exist a edge-cut F of G such that $|F| = 3$. Thus, we have $\kappa'(G) > 3$.
Hence, we have $\kappa'(G) = 4$.
Now, any of the above F is a edge-cut of G such that $|F| = \kappa'(G)$.

(iii) We know that $\kappa(G) = 4$. Recall that for any graph G, $\kappa(G) \le \delta(G)$.
Note that removal of any three edges from G will result in a G' with $e(G') = 15$. Thus, G' will be a graph with 8 vertices and a total degree of 30. By pigeonhole principle, we have $3 \ge \delta(G') \ge \kappa(G')$. G' will no longer be 4-connected. Therefore the maximum number of edges that can be removed is at most 2.
Observe that the removal of $\{v_3v_5, v_6v_7\}$ or $\{v_3v_7, v_5v_6\}$ results in a G' that is still 4-connected. Therefore, the maximum number of edges that can be removed is 2.

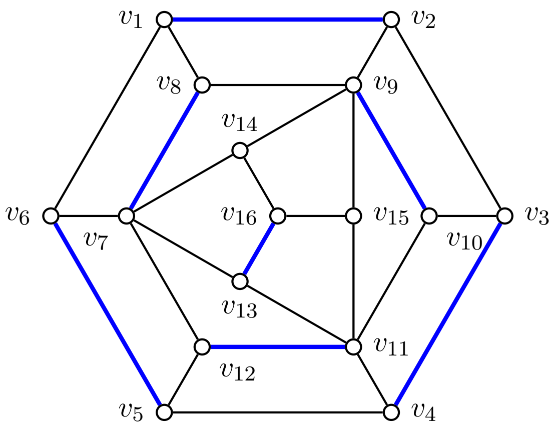
(iv) Note that $d(v_1) = d(v_2) = d(v_4) = d(v_8) = 4 = \delta(G)$. By adding only one edge to G, we can increase the degree of at most two of these four vertices. We still have $4 = \delta(G^*) \ge \kappa(G^*)$. G^* cannot be 5-connected. Therefore the number of new edges to be added is at least 2.
Observe that the addition of $\{v_1v_4, v_2v_8\}$ or $\{v_1v_8, v_2v_4\}$ results in a 5-connected G^*. Therefore, the least number of new edges to be added is 2.
One such G^* is shown below with new edges bolded in blue:



Question 2

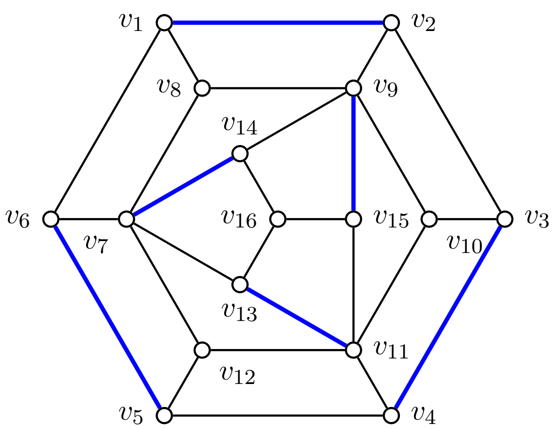
(i) Let $S = \{v_1, v_3, v_5, v_7, v_9, v_{11}, v_{16}\}$.
Now, S is a cut of G with $|S| = 7 < 9 = o(G - S)$. Therefore, G does not contain a perfect matching.
Alternatively, also let $Y = \{v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{13}, v_{14}, v_{15}\}$. Then G is a bipartite graph with bipartite set S and Y, where $|S| = 7 \neq 9 = |Y|$. Therefore G does not contain a perfect matching.

(ii) Let $M = \{v_1v_2, v_3v_4, v_5v_6, v_7v_8, v_9v_{10}, v_{11}v_{12}, v_{13}v_{16}\}$, shown below as blue bold edges:



Now, M is a complete matching of S into Y. Therefore, M is a maximum matching in G.
[There can be other possible answers to this question.]

(iii) Let $M' = \{v_1v_2, v_3v_4, v_5v_6, v_7v_{14}, v_9v_{15}, v_{11}v_{13}\}$, shown below as blue bold edges:

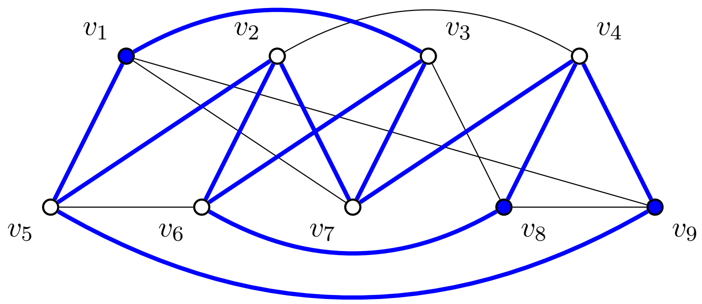


Now, M' is a maximal matching of G. Also, $|M'| = 6 < 7 = |M|$.
Therefore, M' is a maximal matching of G which is not maximum.
[There can be other possible answers to this question.]

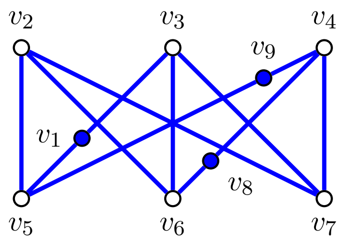
(iv) From results of 2(ii), we have $\alpha'(G) = |M| = 7$.
Using Gallai identities, we have $\beta'(G) = v(G) - \alpha'(G) = 16 - 7 = 9$.
Since G is bipartite, we have $\alpha(G) = \beta'(G) = 9$.
Using Gallai identities again, we have $\beta(G) = v(G) - \alpha(G) = 16 - 9 = 7$.

Question 3

Observe that G contains a subgraph isomorphic to a subdivision of $K_{3,3}$, as shown below:

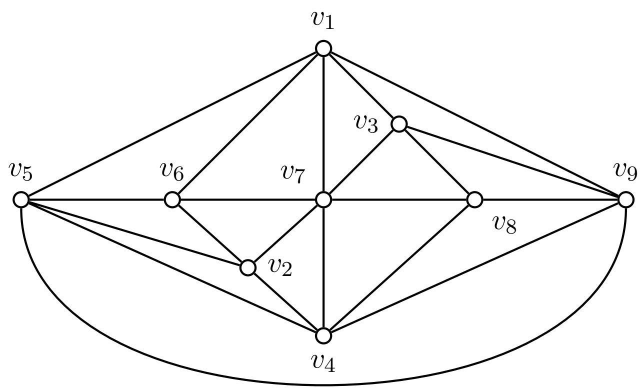


$$\downarrow$$



Therefore, G is not planar.

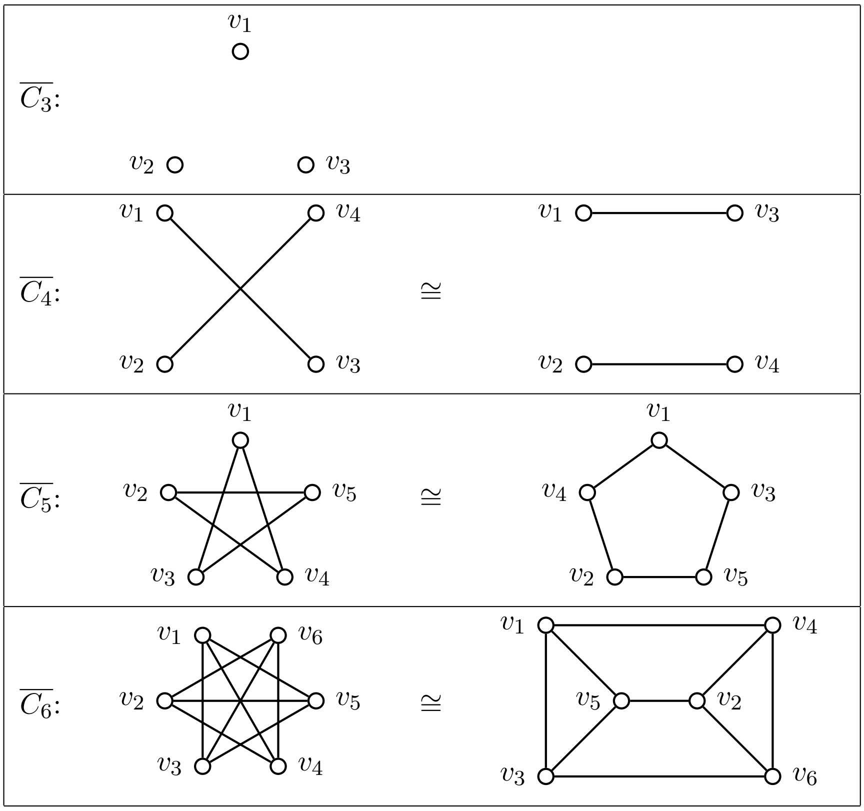
There exists a planar representation of graph H, as shown below:



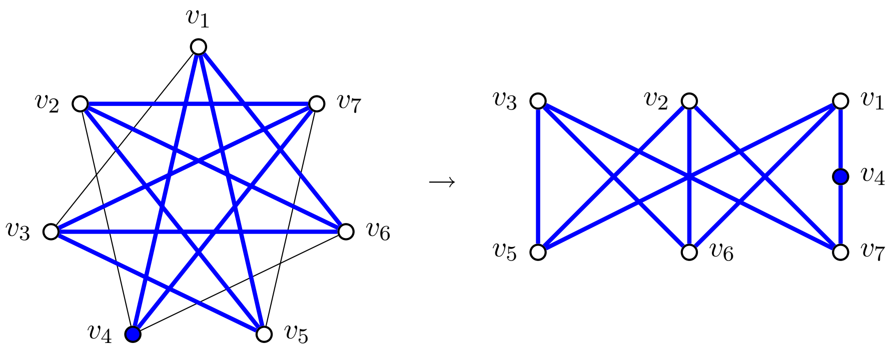
Therefore, H is planar.

Question 4

There exist planar representations of $\overline{C_n}$ for $n = 3, 4, 5$ and 6, as shown below:



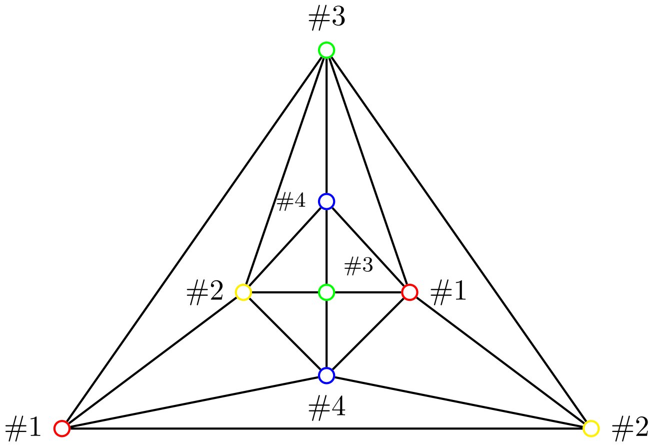
Hence, $\overline{C_3}$, $\overline{C_4}$, $\overline{C_5}$ and $\overline{C_6}$ are planar.
Observe that $\overline{C_7}$ contains a subgraph isomorphic to a subdivision of $K_{3,3}$, as shown below:



Hence, $\overline{C_7}$ is not planar.
Observe that for $n \geq 8$, the subgraph of $\overline{C_n}$ induced by $\{v_1, v_2, v_3, v_5, v_6, v_7\}$ is isomorphic to $K_{3,3}$.
Hence, $\overline{C_n}$ is not planar for $n \geq 8$.
Therefore, 3, 4, 5 and 6 are the only values of n such that $\overline{C_n}$ is planar.

Question 5

(a) (i) Observe that $\langle\{v_1, v_3, v_4, v_5, v_6, v_8\}\rangle$ is isomorphic to W_6. Thus, $\chi(G) \geq 4$.
A 4-colouring of G exists, as shown below:



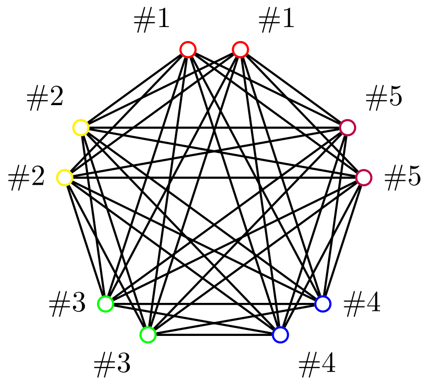
Therefore, we have $\chi(G) = 4$.

(ii) Let $H = G - v_1v_2$ be a subgraph of G. Then, we have $\chi(H) \leq \chi(G) = 4$.
Observe that H also contains a subgraph isomorphic to W_6. Thus, $\chi(H) \geq 4$.
Hence, $\chi(H) = 4$.
Then, there exists a subgraph H of G such that $\chi(H) \nleq \chi(G)$. Therefore, G is not critical.

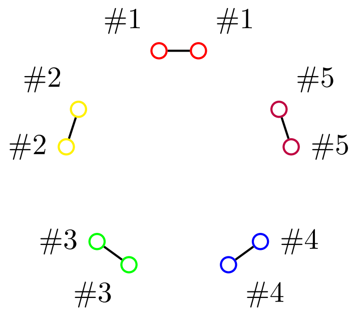
(b) (i) Observe that $e(G) = 40 < 45 = e(K_{10})$. Thus, G is a proper subgraph of K_{10}. Hence, $\max(\chi(G)) < 10$.
Also observe that $e(G) = 40 \geq 36 = e(K_9)$. Thus, it is possible for G to contain a clique of order 9. Hence, $\max(\chi(G)) \geq 9$. Therefore, we have $\max(\chi(G)) = 9$.

(ii) We claim that there does not exist a 4-colourable G.
Suppose not. Let V_1, V_2, V_3, V_4 be the four colour classes of G.
Next, suppose one of the colour classes has 4 or more vertices. WLOG, say $v(V_1) \geq 4$. Then, $e(\overline{V_1}) \geq e(K_4) = 6$. This contradicts to $e(\overline{V_1}) \leq e(\overline{G}) = 45 - 40 = 5$, since $\overline{V_1}$ is a subgraph of $\overline{G}$ and $\overline{G}$ has only 5 edges. Thus, we deduce that if there exists a 4-colourable G, then each colour class can have at most 3 vertices.
Since $v(G) = 10$ and each colour class can have at most 3 vertices, by pigeonhole principle, there are at least two colour classes with 3 vertices. WLOG, say $v(V_1) = v(V_2) = 3$. Then $e(\overline{V_1}) + e(\overline{V_2}) = e(K_3) + e(K_3) = 3 + 3 = 6$. This is also a contradiction since $\overline{G}$ has only 5 edges. Thus, we can conclude that a 4-colouring of G does not exist. Hence, $\chi(G) \geq 5$.
A 5-colourable G exists, as shown below: ($\overline{G}$ is also shown below for easier visualisation)

G:



$\overline{G}$:

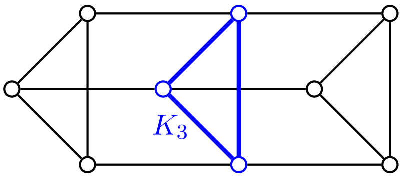


Therefore, we have $\min(\chi(G)) = 5$.

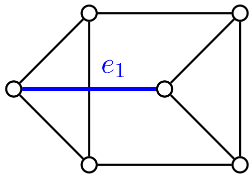
Question 6

(a) Define the following graphs as such:

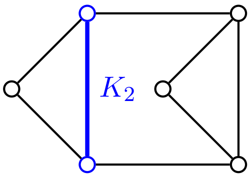
G:



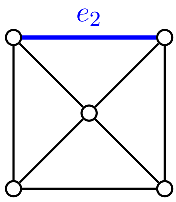
G_1:



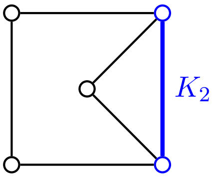
$(G_1 - e_1) \cong G_2$:



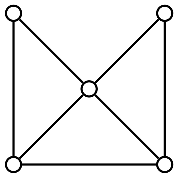
$(G_1 \bullet e_1) \cong G_3$:



G_4:



$(G_3 - e_2) \cong G_5$:



Observe that G is a K_3-gluing of two G_1's. So we have

$$\begin{aligned} P(G,\lambda) &= P(G_1\langle 3\rangle G_1,\lambda) \\ &= \frac{P(G_1,\lambda)P(G_1,\lambda)}{P(K_3,\lambda)} \\ &= \frac{P(G_1,\lambda)^2}{(\lambda)(\lambda-1)(\lambda-2)}. \end{aligned}$$

Consider the edge e_1 of G_1. We have

$$\begin{aligned} P(G_1,\lambda) &= P(G_1 - e_1,\lambda) - P(G_1 \bullet e_1,\lambda) \\ &= P(G_2,\lambda) - P(G_3,\lambda). \end{aligned}$$

Observe that G_2 is an edge-gluing of G_4 and a C_3. So we have

$$\begin{aligned} P(G_2,\lambda) &= P(G_4\langle 2\rangle C_3,\lambda) \\ &= \frac{P(G_4,\lambda)P(C_3,\lambda)}{P(K_2,\lambda)} \\ &= \frac{P(G_4,\lambda)(\lambda)(\lambda-1)(\lambda-2)}{(\lambda)(\lambda-1)} \\ &= (\lambda-2)P(G_4,\lambda). \end{aligned}$$

Observe that G_4 is an edge-gluing of a C_3 and a C_4. So we have

$$\begin{aligned} P(G_4,\lambda) &= P(C_3\langle 2\rangle C_4,\lambda) \\ &= \frac{P(C_3,\lambda)P(C_4,\lambda)}{P(K_2,\lambda)} \\ &= \frac{\big((\lambda)(\lambda-1)(\lambda-2)\big)\big((\lambda)(\lambda-1)(\lambda^2-3\lambda+3)\big)}{(\lambda)(\lambda-1)} \\ &= (\lambda)(\lambda-1)(\lambda-2)(\lambda^2-3\lambda+3). \end{aligned}$$

Consider the edge e_2 of G_3. We have

$$\begin{aligned} P(G_3,\lambda) &= P(G_3 - e_2,\lambda) - P(G_3 \bullet e_2,\lambda) \\ &= P(G_5,\lambda) - P(K_4,\lambda) \\ &= (\lambda)(\lambda-1)(\lambda-2)^3 - (\lambda)(\lambda-1)(\lambda-2)(\lambda-3) \\ &= (\lambda)(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7). \end{aligned}$$

Therefore, we have

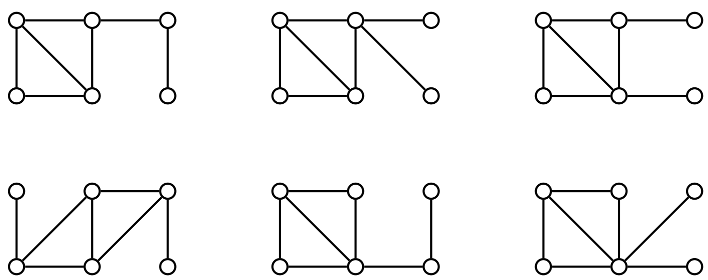
$$\begin{aligned} P(G,\lambda) &= \frac{P(G_1,\lambda)^2}{(\lambda)(\lambda-1)(\lambda-2)} = \frac{(P(G_2,\lambda)-P(G_3,\lambda))^2}{(\lambda)(\lambda-1)(\lambda-2)} \\ &= \frac{\big((\lambda-2)P(G_4,\lambda) - (\lambda)(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)\big)^2}{(\lambda)(\lambda-1)(\lambda-2)} \\ &= \frac{\big((\lambda-2)(\lambda)(\lambda-1)(\lambda-2)(\lambda^2-3\lambda+3) - (\lambda)(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)\big)^2}{(\lambda)(\lambda-1)(\lambda-2)} \\ &= \big((\lambda-2)(\lambda^2-3\lambda+3) - (\lambda^2-5\lambda+7)\big)^2(\lambda)(\lambda-1)(\lambda-2) \\ &= (\lambda^3-6\lambda^2+14\lambda-13)^2(\lambda)(\lambda-1)(\lambda-2). \end{aligned}$$

The equation $P(G,\lambda)=0$ has roots 0, 1, and 2. (Since λ is a positive integer by the definition of chromatic polynomial, we shall omit the non-integer roots.)

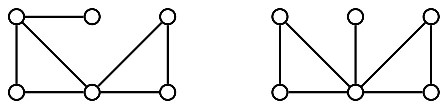
(b) Let H^* be a graph which is χ-equivalent to H. Then we have

$$\begin{aligned} v(H^*) &= v(H) = 6. \\ e(H^*) &= e(H) = 7. \\ \#_{H^*}(C_3) &= \#_H(C_3) = 2. \end{aligned}$$

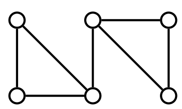
Firstly, H is a possible H^* since H is clearly χ-equivalent to itself.
Next, to avoid omitting or overcounting, we shall list all other possible H^* by cases.
Case 1: The two C_3's of H^* share a common edge:



Case 2: The two C_3's of H^* share a common vertex:



Case 3: The two C_3's of H^* neither share common edges nor vertices:



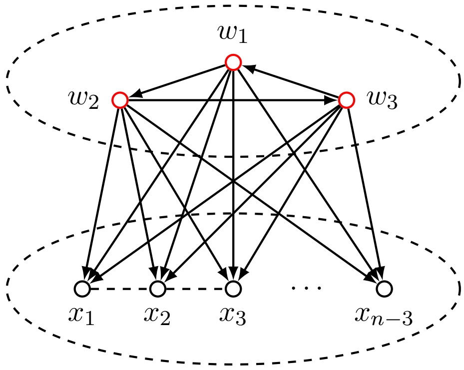
The above 3 cases cover all possible arrangements of the two C_3 of H^*. It can be checked that each of the above listed graphs has the same chromatic polynomial as H, that is $P(H,\lambda) = \lambda(\lambda-1)^3(\lambda-2)^2$. Thus, we have listed all graphs which are χ-equivalent to H.

Question 7

(i) Suppose that there exists a vertex $w \in V(T)$ such that w is an exact-1-king. Observe that w is a source and hence the only source. Thus, $k_1^*(T) \leq 1$. Therefore, it is impossible to construct a T such that $k_1^*(T) \geq 2$.

(ii) Since $k_1^*(T) = 1$, then there exists a vertex $w \in V(T)$ such that w is an exact-1-king. Observe that w is a source and thus not reachable from any other vertices. Hence, all other vertices are not kings. Therefore, $k_r^*(T) = 0$ for each $r \geq 2$.

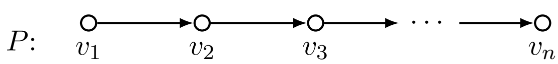
(iii) Since $k_1^*(T) = 0$, T does not contain a source.
Let w_1 be a vertex of maximum score in T. Then w_1 is an exact-2-king of T.
Next, consider the inset and outset of w_1. Let w_2 be a vertex of maximum score in $[I(w_1)]$. Then w_2 is a 2-king of $[I(w_1)]$. Observe that w_2 can reach any vertex in $O(w_1)$ via $w_2 \to w_1 \to O(w_1)$. Hence, w_2 is a exact-2-king of T.
Using similar argument, there exists another exact-2-king $w_3 \in I(w_2)$. Hence, $k_2^*(T) \geq 3$.
For each $n \geq 3$, there exists a tournament T_n with three exact-2-kings, as illustrated below:



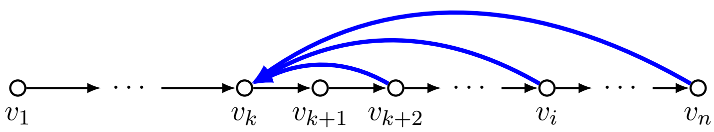
Here, w_1, w_2 and w_3 form a directed C_3. Each x_i is dominated by all three w_i's. The subgraph induced by $\{x_1, x_2, \ldots, x_{n-3}\}$ can be any tournament. It can be checked that each w_i is an exact-2-king. Also, observe that the w_i's are not reachable from any of the x_i's. Thus, the x_i's cannot be kings. Hence, w_1, w_2 and w_3 are the only exact-2-kings.
Since such a construction exists, we can conclude that the minimum value of $k_2^*(T)$ is 3.

(iv) Let T be a tournament such that $k_1^*(T) = 0, k_2^*(T) = 3$ and $k_3^*(T) = 0$. We shall show that $k_4^*(T) = 0$ and hence $k_4^*(T) \ngeq 1$.
Since $k_1^*(T) = 0$, T does not contain a source. Next, since the inset of each exact-2-king contains an exact-2-king and since $k_2^*(T) = 3$, then $\langle K_2(T) \rangle$ forms a directed C_3.
Suppose there exists an arc xw such that $x \in V(T) \backslash K_2(T)$ and $w \in K_2(T)$. Then x will become an exact-3-king, which leads to $k_3^*(T) \geq 1$, a contradiction to $k_3^*(T) = 0$. Hence, no such arc xw exists. Therefore, vertices in $K_2(T)$ is not reachable from any vertices in $V(T) \backslash K_2(T)$. Thus, $V(T) \backslash K_2(T)$ does not contain any kings, in particular, exact-4-kings. Hence, $k_4^*(T) = 0$ and $k_4^*(T) \ngeq 1$.
Therefore, it is not possible to construct T.

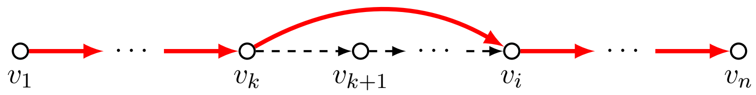
(v) Since $k_{n-1}^*(T) = 1$, there exists vertices w and x such that $d(w, x) = (n-1)$. We shall label the shortest $w - x$ path as P. Note that P is of length $(n-1)$ and it contains all the n vertices of T. Next, we shall re-label the vertices of T such that the path P can be represented as:



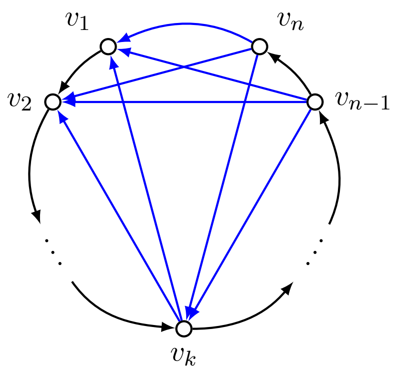
We claim that for each k, v_k is dominated by all v_i's for $(k+2) \leq i \leq n$.



Otherwise, there will exist a v_i such that $v_1 \to \cdots \to v_k \to v_i \to \cdots \to v_n$ is a $v_1 - v_n$ path of length less than $(n-1)$, contrary to that P is the shortest $v_1 - v_n$ path.



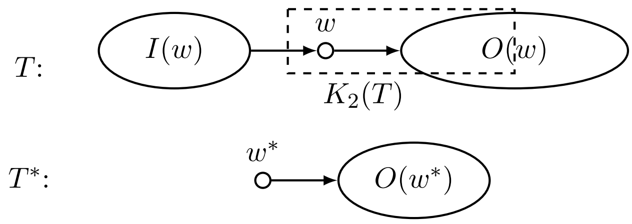
With this claim, we can now deduce the general structure of T as follow:



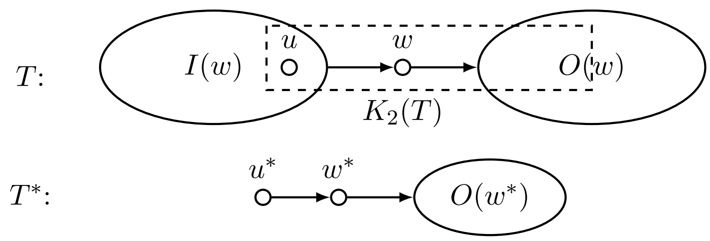
For each (i,j)-pair s.t. $i<j$,
$$\begin{cases} v_i \to v_j & \text{, if } j = i+1, \\ v_j \leftarrow v_i & \text{, otherwise.} \end{cases}$$

Observe that T does not contain a source. Hence, $k_1^*(T) = 0$.
Now consider the vertex v_n. We have $d(v_n, v_1) = d(v_n, v_2) = \cdots = d(v_n, v_{n-2}) = 1$. The shortest $v_n - v_{n-1}$ path is $v_n \to v_{n-2} \to v_{n-1}$, so $d(v_n, v_{n-1}) = 2$. Thus, v_n is an exact-2-king.
Similar argument shows that v_{n-1} and v_{n-2} are also exact-2-kings.
Now consider the vertex v_{n-3}. To reach v_n, the shortest path is $v_{n-3} \to v_{n-2} \to v_{n-1} \to v_n$, so $d(v_{n-3}, v_n) = 3$. By inspection, any other vertex is reachable from v_{n-3} on a path of length 3 or less. Thus, v_{n-3} is an exact-3-king.
Similar argument shows that each of the remaining v_{n-r} is an exact-r-kings.
Therefore, we have $k_1^*(T) = 0$, $k_2^*(T) = 3$, and $k_3^*(T) = k_4^*(T) = \cdots = k_{n-2}^*(T) = 1$.

(vi) Suppose such a T exists. Let w^* be the transmitter(source) in T^* and let w be the corresponding vertex in $T[K_2(T)]$. Since $k_2(T) = v(T[K_2(T)]) = v(T^*) = m \geq 2$, by result of 7(ii), T does not contain a transmitter. Hence, $I(w)$ is not empty.



Since $w \in K_2(T)$, w is a 2-king of T. Using arguments similar to that in 7(iii), there exists a 2-king in $I(w)$. Let this 2-king be u. Then $u \in K_2(T)$ and hence, there exists a corresponding vertex $u^* \in T^*$ such that u^* dominates w^* in T^*.



This contradicts to that w^* is a transmitter.
Therefore, it is not possible to construct T.