NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lau Tze Siong

MA2101 Linear Algebra II
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SECTION A

Question 1

(i) $\begin{pmatrix} 1-i & i \\ 1 & 1+i \end{pmatrix} \in W$, but $i\begin{pmatrix} 1-i & i \\ 1 & 1+i \end{pmatrix} = \begin{pmatrix} i+1 & -1 \\ i & i-1 \end{pmatrix} \notin W$.
Hence W is not closed under scalar multiplication in $\mathbb{C}$, i.e. not a complex subspace of $M_{22}(\mathbb{C})$.

(ii) for any $\begin{pmatrix} z+\overline{w} & w \\ z & \overline{z}+w \end{pmatrix}, \begin{pmatrix} z_1+\overline{w_1} & w_1 \\ z_1 & \overline{z_1}+w_1 \end{pmatrix} \in W$, and any $r \in \mathbb{R}$, we have
Closure under vector addition:

$$\begin{aligned}
\begin{pmatrix} z+\overline{w} & w \\ z & \overline{z}+w \end{pmatrix} + \begin{pmatrix} z_1+\overline{w_1} & w_1 \\ z_1 & \overline{z_1}+w_1 \end{pmatrix} &= \begin{pmatrix} z+z_1+\overline{w}+\overline{w_1} & w+w_1 \\ z+z_1 & \overline{z}+\overline{z_1}+w+w_1 \end{pmatrix} \\
&= \begin{pmatrix} z+z_1+\overline{w+w_1} & w+w_1 \\ z+z_1 & \overline{z+z_1}+w+w_1 \end{pmatrix} \in W.
\end{aligned}$$

Closure under scalar multiplication:

$$r\begin{pmatrix} z+\overline{w} & w \\ z & \overline{z}+w \end{pmatrix} = \begin{pmatrix} rz+r\overline{w} & w \\ rz & r\overline{z}+w \end{pmatrix} = \begin{pmatrix} rz+\overline{rw} & rw \\ rz & \overline{rz}+rw \end{pmatrix} \in W.$$

Hence W is a real subspace of $M_{22}(\mathbb{C})$.

Claim: $S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -i & i \\ 0 & i \end{pmatrix}, \begin{pmatrix} i & 0 \\ i & -i \end{pmatrix} \right\}$ is a basis for W.
Spanning:
For any $\begin{pmatrix} z+\overline{w} & w \\ z & \overline{z}+w \end{pmatrix} \in W$. Let $z = a+bi$ and $w = c+di$, $a, b, c, d \in \mathbb{R}$.

$$\begin{aligned}
\begin{pmatrix} a+bi+\overline{c+di} & c+di \\ a+bi & \overline{a+bi}+c+di \end{pmatrix} &= \begin{pmatrix} a+bi+c-di & c+di \\ a+bi & a-bi+c+di \end{pmatrix} \\
&= \begin{pmatrix} a+c+(b-d)i & c+di \\ a+bi & (a+c)+(d-b)i \end{pmatrix} \\
&= c\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + a\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + d\begin{pmatrix} -i & i \\ 0 & i \end{pmatrix} + b\begin{pmatrix} i & 0 \\ i & -i \end{pmatrix}.
\end{aligned}$$

Linear independence:
Suppose $a, b, c, d \in \mathbb{R}$ such that,

$$\begin{aligned}
a\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + c\begin{pmatrix} -i & i \\ 0 & i \end{pmatrix} + d\begin{pmatrix} i & 0 \\ i & -i \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\begin{pmatrix} a+b-ci+di & a+ci \\ b-di & a+b+c-di \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Solving, we have $a, b, c, d = 0$. Hence S is linearly independent.

Question 2

(i) Since

$$\begin{aligned} T(1) &= 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \\ T(x) &= 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \\ T(x^2) &= 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

we have $[T]_{\mathcal{B}_2,\mathcal{B}_1} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix}$.

(ii) $a + bx + cx^2 \in \ker(T)$ iff $\begin{pmatrix} a+2c & a+b \\ a+b & b-2c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
Hence we have the following set of equations:-

$$\begin{aligned} a + 2c &= 0; \\ a + b &= 0; \\ b - 2c &= 0. \end{aligned}$$

Solving, we have $a = a, b = -a, c = -\frac{a}{2}$. Hence $\left\{1 - x - \frac{1}{2}x^2\right\}$ is a basis for $\ker(T)$.

Since $S = \left\{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$ is a spanning set for $\mathcal{R}(T)$ and $S = \left\{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right\}$ is a linearly independent set that spans $\mathcal{R}(T)$. S is a basis for $\mathcal{R}(T)$.

(iii) Let $\mathcal{B}_3 = \left\{1, x, 1 - x - \frac{1}{2}x^2\right\}$, $\mathcal{B}_4 = \left\{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$
The above $\mathcal{B}_3$ and $\mathcal{B}_4$ would give us the required matrix.

Question 3

(i) The characteristic equation of A is $(x-2)(x+1)^2$.
Since $(A - 2I)(A + I) = 0$. The minimal polynomial $m_A(x) = (x-2)(x+1)$.

(ii) Yes. Since the minimal polynomial is a product of distinct linear factors, A is diagonalizable.

(iii) Since $\deg(m_A(x)) = 2$, $\dim(W) = 2$.

Question 4

(i) There are only 2 possible Jordan Canonical Form.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

(ii) For either Jordan Canonical Form for T, there are 2 Jordan blocks of eigenvalue 1. Thus we have $\dim(E_1) = |\text{Jordan blocks of eigenvalue } 1| = 2$.

SECTION B

Question 5

(a) Let $a_1, a_2, \ldots, a_n \in F$ such that $a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0_V$. Since T is a linear transformation, we have

$$\begin{aligned} a_1w_1 + a_2w_2 + ... + a_nw_n &= a_1T(v_1) + a_2T(v_2) + ... + a_nT(v_n) \\ &= T(a_1v_1 + a_2v_2 + ... + a_nv_n) = T(0_V) = 0_W. \end{aligned}$$

Hence we have $a_1 = a_2 = \cdots = a_n = 0_F$, i.e. $\{v_1, v_2, \ldots, v_n\}$ is linearly independent.

(b) Let $\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $\mathcal{B}_2 = \{1, x, x^2\}$. Hence,

$$[T_1]_{\mathfrak{B}_2,\mathfrak{B}_1} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix};$$

$$[T_2]_{\mathfrak{B}_2,\mathfrak{B}_1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & -1 \end{pmatrix};$$

$$[T_3]_{\mathfrak{B}_2,\mathfrak{B}_1} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ -1 & -1 \end{pmatrix};$$

$$[T_4]_{\mathfrak{B}_2,\mathfrak{B}_1} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since $T_1 + T_3 = T_2$ and $2T_1 + T_3 = T_4$, $\text{Span}(\{T_1, T_2, T_3, T_4\}) = \text{Span}(\{T_1, T_3\})$. And $\{T_1, T_3\}$ is linearly independent. Hence $\{T_1, T_3\}$ is a basis for U.
Therefore $\dim(U) = 2$.

Question 6

(i) For any $X, Y \in M_{nn}(\mathbb{R})$ and $a \in \mathbb{R}$. We have

$$\begin{aligned} T(X + aY) &= A(X + aY) \\ &= AX + aAY \\ &= T(X) + aT(Y) \in M_{nn}(\mathbb{R}). \end{aligned}$$

Hence T is a linear operator.

(ii) Let $V = M_{nn}(\mathbb{R})$. Notice that for all polynomial $p(x) \in \mathbb{R}[x]$, we have $p(T) : V \to V$ to be a linear operator such that $p(T)(X) = p(A)X$ for all $X \in V$.
Now for all $X \in V$, $m_A(T)(X) = m_A(A)X = 0_V X = 0_V$, i.e. $m_A(T) = 0_{L(V,V)}$.
Thus $m_T(x) \mid m_A(x)$.

Since A is diagonalisable, $m_A(x)$ consist only of distinct linear factors, and thus so is $m_T(x)$, i.e. T is diagonalisable.

Question 7

(a) (i) Let the the set of 3×3 skew-symmetric matrix be $S = \{A \in M_{33}(\mathbb{R}) | A^T = -A\}$.
Notice that S is a subspace of V.
Claim: $W^\perp = S$.
Proof:
Let $A \in W$ and $X \in S$. From commutativity of inner products, we have $\langle A, X\rangle = \langle X, A\rangle$.
Now since $\mathrm{Tr}(A^T X) = \mathrm{Tr}(AX) = \mathrm{Tr}(XA) = -\mathrm{Tr}(X^T A)$, we have $\langle X, A\rangle = -\langle A, X\rangle$.
Hence $\langle A, X\rangle = -\langle A, X\rangle$, i.e. $\langle A, X\rangle = 0$. Therefore $X \in W^\perp$, i.e. $S \subseteq W^\perp$.
Since $S \oplus W = V$, we have $\dim(S) = \dim(W^\perp)$, and so $S = W^\perp$.

Since $\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$ is a basis for $S = W^\perp$ and

$$\begin{aligned} \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle \\ &= 0, \end{aligned}$$

We only need to normalise each of the elements in the basis. Hence an orthonormal basis for $W^\perp$ is $\left\{ \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \right\}$.

(ii) Let $P = \frac{F+F^T}{2}$ and $Q = \frac{F-F^T}{2}$.
We have $P + Q = \frac{F+F^T+F-F^T}{2} = F$ and $P^T = \frac{F+F^T}{2} = P$ and $Q^T = \frac{F^T-F}{2} = -Q$.
Hence $P = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 5 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$.

(b) Since the range of T is W, for all $\boldsymbol{w} \in W$ there exists $\boldsymbol{v} \in V$ such that $T(\boldsymbol{v}) = \boldsymbol{w}$.
Since $T^2 = T$, we have $T(\boldsymbol{v}) = T(T(\boldsymbol{v}))$, which gives us $T(\boldsymbol{w}) = \boldsymbol{w}$.
Hence, for all $\boldsymbol{w} \in W$, we have $T(\boldsymbol{w}) = \boldsymbol{w}$.

Now suppose for some $\mathbf{z} \in W^\perp$ there exist $\mathbf{w} \in W \backslash \{0_V\}$ such that , $T(\mathbf{z}) = \mathbf{w}$.
Let $k \in \mathbb{R}$ be large enough such that $\|\boldsymbol{v}\| < \sqrt{2k+1}\|\boldsymbol{w}\|$.
Hence we have $T(\boldsymbol{z} + k\boldsymbol{w}) = (k+1)\boldsymbol{w}$. However,

$$\|\boldsymbol{z} + k\boldsymbol{w}\| = \sqrt{\|\boldsymbol{z}\|^2 + k^2\|\boldsymbol{w}\|^2} \quad < \quad \sqrt{(2k+1)\|\boldsymbol{w}\|^2 + k^2\|\boldsymbol{w}\|^2} = (k+1)\|\boldsymbol{w}\| = \|T(\boldsymbol{z} + k\boldsymbol{w})\|,$$

contradicting $\|T(\boldsymbol{v})\| \leq \|\boldsymbol{v}\|$.
Hence, for any $\boldsymbol{z} \in W^\perp$, $T(\boldsymbol{z}) = 0_V$.
Therefore for any $\boldsymbol{v} \in V$, which we can write as $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{z}$ such that $\boldsymbol{w} \in W$ and $\boldsymbol{z} \in W^\perp$, we have $T(\boldsymbol{v}) = T(\boldsymbol{w} + \boldsymbol{z}) = T(\boldsymbol{w}) + T(\boldsymbol{z}) = \boldsymbol{w}$. Hence T is the orthogonal projection on W.