MA2101S - Linear Algebra II(S) Suggested Solutions

(Semester 2 : AY2017/18)

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Question 1

(a) Note that

$$\begin{aligned} X \in C(A) &\to AX = XA \\ &\to (Q^{-1}AQ)(Q^{-1}XQ) = (Q^{-1}XQ)(Q^{-1}AQ) \\ &\to Q^{-1}XQ \in C(Q^{-1}AQ). \end{aligned}$$

Define the map $T : C(A) \to C(Q^{-1}AQ)$ by:

$$T(X) = Q^{-1}XQ.$$

Claim 1 : T is a linear transformation.

Proof : Let $X, Y \in \mathbb{M}_n(\mathbb{C})$ and $a, b \in \mathbb{C}$. Then

$$\begin{aligned} T(aX + bY) &= Q^{-1}(aX + bY)Q \\ &= Q^{-1}(aX)Q + Q^{-1}(bY)Q \\ &= aQ^{-1}XQ + bQ^{-1}XQ \\ &= aT(X) + bT(Y). \end{aligned}$$

Claim 2 : T is injective.

Proof : Let $X \in \ker(T)$. Then $T(X) = 0_V \to Q^{-1}XQ = 0_{n\times n} \to X = 0_{n\times n}$. Thus $\ker(T) = \{0_V\}$ so T is injective.

Claim 3 : T is surjective.

Proof : Let $X \in C(Q^{-1}AQ)$. Then

$$\begin{aligned} Q^{-1}AQX = XQ^{-1}AQ &\to AQXQ^{-1} = QXQ^{-1}A \\ &\to QXQ^{-1} \in C(A). \end{aligned}$$

Choose $QXQ^{-1} \in C(A)$ and $T(QXQ^{-1}) = X$.

Since T is a bijective linear transformation from $C(A)$ to $C(Q^{-1}AQ)$, the two subspaces are isomorphic.

(b) We will prove the statement via mathematical induction.

Base case $(n = 1) : \forall A, X \in \mathbb{M}_1(\mathbb{C}),\ AX = XA$.

Thus $\dim_{\mathbb{C}} C(A) = \dim(\mathbb{M}_1(\mathbb{C})) = 1 \geq n$.

Inductive step: Let J be the jordan canonical form of A. Consider 2 cases:

Case 1: J has a single jordan block.

$$J = \begin{pmatrix} \lambda & 1 & 0 & ... & 0 \\ 0 & \lambda & 1 & ... & 0 \\ 0 & 0 & \lambda & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \lambda \end{pmatrix} \text{ for some } \lambda \in \mathbb{C}.$$

Then $\deg(m_A(x)) = n$.

Claim : $\{I, A, A^2, ..., A^{n-1}\}$ is a linearly independent set.

Proof: Assume that $\{I, A, A^2, ..., A^{n-1}\}$ is not a linearly independent set. Then $\exists a_0, a_1, ..., a_{n-1}$, not all zero, such that:

$$a_0 I + a_1 A + a_2 A^2 ... + a_{n-1} A^{n-1} = 0_{n \times n}.$$

This means that the non-zero polynomial $p(x) = a_0 + a_1 x + ... + a_{n-1} x^{n-1}$ satisfies the property $p(A) = 0_V$ so $\deg(m_A(x)) < n$. This is a contradiction, since $\deg(m_A(x)) = n$.

Since $\text{span}\{I, A, A^2, ..., A^{n-1}\} \subseteq C(A)$, $\dim_{\mathbb{C}} C(A) \geq \dim(\text{span}\{I, A, A^2, ..., A^{n-1}\}) = n$.

Case 2: J has two or more jordan blocks. Write J as:

$$J = \begin{pmatrix} J_k(\mu) & 0_{k \times (n-k)} \\ 0_{(n-k) \times k} & D \end{pmatrix} \text{ for some } \mu \in \mathbb{C},\ D \in \mathbb{M}_{n-k}(\mathbb{C}).$$

By our induction hypothesis, $\exists$ basis $\{B_1, B_2, ..., B_j\}$ for $C(J_k(\mu))$, where $j \geq k$. Similarly, $\exists$ basis $\{C_1, C_2, ..., C_i\}$ for $C(D)$, where $i \geq n-k$.

$\forall 1 \leq x \leq j,\ 1 \leq y \leq i$:

$$\begin{aligned} \begin{pmatrix} J_k(\mu) & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} B_x & 0 \\ 0 & C_y \end{pmatrix} &= \begin{pmatrix} J_k(\mu) B_x & 0 \\ 0 & D C_y \end{pmatrix} \\ &= \begin{pmatrix} B_x J_k(\mu) & 0 \\ 0 & C_y D \end{pmatrix} \\ &= \begin{pmatrix} B_x & 0 \\ 0 & C_y \end{pmatrix} \begin{pmatrix} J_k(\mu) & 0 \\ 0 & D \end{pmatrix}. \end{aligned}$$

Let $D = \{\begin{pmatrix} B_x & 0 \\ 0 & 0 \end{pmatrix} \mid 1 \leq x \leq j\} \cup \{\begin{pmatrix} 0 & 0 \\ 0 & C_y \end{pmatrix} \mid 1 \leq y \leq i\}$.
Then $\text{span}(D) \subseteq C(J)$. Since $\dim(\text{span}(D)) = i + j \geq n$, $\dim_{\mathbb{C}} C(J) \geq n$.
By (a), $\dim_{\mathbb{C}} C(A) = \dim_{\mathbb{C}} C(J) \geq n$.

Question 2

(a) & (b) Let $p \in \mathbb{R}[X]$.

Then $p(x) = b_0 + b_1 x + b_2 x^2 + ... + b_n x^n$ for some $b_0, b_1, ..., b_n \in \mathbb{R}$.

$$\int_0^1 p(t)dt = \big[b_0 x + \frac{b_1 x^2}{2} + \frac{b_2 x^3}{3} + ... + \frac{b_n x^{n+1}}{n+1}\big]_0^1 = b_0 + \frac{b_1}{2} + \frac{b_2}{3} + ... + \frac{b_n}{n+1}.$$

$$
\begin{aligned}
\sum_{j=0}^{n} a_j p(x_j) &= a_0 p(x_0) + a_1 p(x_1) + ... + a_n p(x_n) \\
&= b_0(a_0 + a_1 + ... + a_n) \\
&+ b_1(a_0 x_0 + a_1 x_1 + ... + a_n x_n) \\
&+ b_2(a_0 x_0^2 + a_1 x_1^2 + ... + a_n x_n^2) \\
&+ ... \\
&+ b_n(a_0 x_0^n + a_1 x_1^n + ... + a_n x^n).
\end{aligned}
$$

Since the equality $\int_0^1 p(t)dt = \sum_{j=0}^{n} a_j p(x_j)$ holds $\forall\ b_0, b_1, ...b_n \in \mathbb{R}$, we can compare each term separately and set up the linear system:

$$
\begin{aligned}
a_0 + a_1 + ... + a_n &= 1 \\
a_0 x_0 + a_1 x_1 + ... + a_n x_n &= \frac{1}{2} \\
a_0 x_0^2 + a_1 x_1^2 + ... + a_n x_n^2 &= \frac{1}{3} \\
&\vdots \\
a_0 x_0^n + a_1 x_1^n + ... + a_n x_n^n &= \frac{1}{n+1}.
\end{aligned}
$$

In other words, $a_0, a_1, ..., a_n$ satisfy the linear system:

$$
\begin{pmatrix} 1 & 1 & ... & 1 \\ x_0 & x_1 & ... & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & ... & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \vdots \\ \frac{1}{n+1} \end{pmatrix}
$$

Recall that $\begin{pmatrix} 1 & 1 & ... & 1 \\ x_0 & x_1 & ... & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & ... & x_n^n \end{pmatrix}$ is a Vandermonde matrix.

Since $x_0, x_1, ..., x_n$ are all distinct, it is an invertible Vandermonde matrix. Thus the above equation has a unique solution. In other words, $\exists$ unique $a_0, a_1, .., a_n \in \mathbb{R}$ such that $\int_0^1 p(t)dt = \sum_{j=0}^{n} a_j p(x_j)$.

(c) Define $q \in \mathbb{R}[X]$ by:

$$
q(x) = (x - x_1)(x - x_2)...(x - x_n).
$$

Then $\sum_{j=0}^{n} a_j q(x_j) = a_0 q(x_0) = a_0(x_0 - x_1)(x_0 - x_2)...(x_0 - x_n)$.

Using the equality $\int_0^1 q(t)dt = \sum_{j=0}^{n} a_j p(x_j)$:

$$
a_0 = \frac{\int_0^1 (t - x_1)(t - x_2)...(t - x_n)\ dt}{(x_0 - x_1)(x_0 - x_2)...(x_0 - x_n)}.
$$

Question 3

(a) False. Counterexample:

$$
A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}, A + B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
$$

Then $c_A(x) = x^2 - x - 1 = (x - \frac{1+\sqrt{5}}{2})(x - \frac{1-\sqrt{5}}{2}),\ c_B(x) = x(x-1)$.
Since $c_A(x)$ and $c_B(x)$ have no repeated factors, A and B are diagonalisable. Notice that $(A+B)^T$ is a $J_2(0)$ Jordan block, which is not diagonalisable. Thus $A+B$ is not diagonalisable.

(b) False. Counterexample:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$c_A(x) = x^2 - x - 1 = (x - \frac{1+\sqrt{5}}{2})(x - \frac{1-\sqrt{5}}{2})$. Since $c_A(x)$ have no repeated factors, A is diagonalisable. B is obviously diagonalisable since it is a diagonal matrix. AB is obviously not diagonalisable since it is a $J_2(0)$ Jordan block.

(c) True. Consider $f(x) = x^2 - x = x(x-1)$. Since $f(A) = 0_V,\ m_A(x) \mid f(x)$ by definition of minimal polynomial. But $f(x)$ has no repeated factors so $m_A(x)$ has no repeated factors as well. Thus A is diagonalisable.

(d) False. Counterexample:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

A^2 is obviously diagonalizable but A is not diagonalizable since it is a $J_2(0)$ Jordan block.

Question 4

Assume, for the sake of contradiction, that $\exists$ eigenvalue $\mu \in \mathbb{C} \setminus \{0\}$ such that $\mu^l \neq 1\ \forall l \in \mathbb{N}$.

Let $v \in \mathbb{C}^n$ be an eigenvector of M associated with eigenvalue μ.

$$M(v) = \mu v \to M^k(v) = \mu^k v.$$

Then μ^k is an eigenvalue of M^k.

Claim: μ^k is an eigenvalue of M.

Proof: Since μ^k is an eigenvalue of $M^k, \det(M^k - \mu^k I) = 0$. Then:

$$\begin{aligned} \det(A^{-1}(M^k - \mu^k I)A) = 0 &\to \det(A^{-1}M^k A - \mu^k I) = 0 \\ &\to \det(M - \mu^k I) = 0. \end{aligned}$$

By mathematical induction, we can prove that $\mu, \mu^k, \mu^{k^2}, ..., \mu^{k^n}$ are all eigenvalues of M. Note that if $\mu^{k^i} = \mu^{k^j}$ for some $i, j \in \mathbb{N}$ with $i < j$, then $\mu^{k^j - k^i} = 1$, which contradicts our assumption that $\mu^l \neq 1\ \forall l \in \mathbb{N}$.

Thus $\mu, \mu^k, \mu^{k^2}, ..., \mu^{k^n}$ are all distinct eigenvalues of M. This is again a contradiction, as a $n \times n$ matrix cannot have $n+1$ distinct eigenvalues. Thus the assumption is wrong and for any eigenvalue $\lambda \in \mathbb{C} \setminus \{0\}$ of M, there exists an integer $l > 0$ such that $\lambda^l = 1$.

Question 5

Remark: For a matrix M, we will use $\text{Null}(M)$ to denote the nullspace of M.

A is upper triangular so we can deduce that $C_A(x) = (x-1)^6$ by observing the diagonal entries.

Note that

$$A - I = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ (A-I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(A-I)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that $\text{rank}(A - I) = 2$, $\text{rank}(A - I)^2 = 1$ and $\text{rank}(A - I)^3 = 0$. Thus we have a $J_3(1)$ block and 3 $J_1(1)$ blocks.

Note that $(0, 0, 0, 0, 0, 1)^T \in \text{Null}(A - I)^3 \setminus \text{Null}(A - I)^2$. By further computation:

$$(A-I)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ (A-I)^2\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the 3 vectors $(0, 0, 0, 0, 0, 1)^T, (1, 1, 1, 1, 1, 0)^T$ and $(4, 0, 0, 0, 0, 0)^T$ are part of the $J_3(1)$ block.

To find the other 3 vectors that make up the 3 $J_1(1)$ blocks, we extend the set $\{(4, 0, 0, 0, 0, 0)^T\}$ to a basis for $\text{Null}(A - I)$ as follows:

$$\text{Basis for Null}(A-I) = \{\begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}\}$$

The 3 vectors $(0, 1, -1, 0, 0, 0)^T, (0, 1, 0, -1, 0, 0)^T$ and $(0, 1, 0, 0, -1, 0)^T$ will make up the 3 $J_1(1)$ blocks. To conclude:

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ Q = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$