NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to CHEN YUXING, CHEN YUMING

MA4266 Topology
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Question 1

(a) The family of all subsets of the form $O_1 \times O_2$, where O_1, O_2 are open sets in X.

(b) Suppose that D is closed. Let x and y be two distinct points of X. Then (x, y) is not on D, so there is a basis neighborhood $U \times V$ of (x, y) that misses D entirely. So x is in U, y is in V and we claim that U is disjoint from V. If not, say z is in $U \cap V$, then (z, z) would be in $U \times V$, and also in D, which contradicts we way we chose $U \times V$ in the first place. So U and V are the required disjoint neighborhoods of x and y.

Reversely, if (x, y) is any point outside D, we know then that x is unequal to y. So there are disjoint open neighborhoods U of x and V of y that are disjoint. Then we claim that $U \times V$ misses D entirely: if (z, z) from D would also be in $U \times V$ then z would be in U and in V, contradicting the disjointness of U and V. So for any point (x, y) in $X \times X - D$ we have an open neighborhood $U \times V$ (open in $X \times X$) such that $U \times V \subset X \times X - D$, so D is closed.

(c) Let $a \in X$. Then $\{a\} \times X$, as a subspace of Y, is homeomorphic to X under $(a, t) \leftrightarrow t$, $t \in X$.

Then $\{a\} \times X$ is connected and similarly $X \times \{t\}$ is connected. Note that $(a, t) \in (\{a\} \times X) \cap (X \times \{t\})$. By a corollary, $\{a\} \times X \cup \bigcup_{t \in X}(X \times \{t\})$, which is Y, is connected.

(d) Consider a open neighborhood of $(x, -x)$ in $X \times X$ which is $[x, a) \times [-x, b)$, $a > x$, $b > -x$. $(x, -x) = [x, a) \times [-x, b) \cap E$ is open in E. The closure of a subset of X is the subset itself and the only dense set of X is X which is uncountable so X is not separable.

Question 2

(a) Each component of X consists of a single point.

(b) Suppose that there is one component of X which consists of more than one points. Let x, y be two points in this component and p, q be their images under f. Let C be the image of this component. Since f is injective, $p \neq q$. C is connected since f is continuous. Let r be an irrational between p, q. $(-\infty, r) \cap C$ and $(r, \infty) \cap C$ are separated sets of C, which contradicts that C is connected.

(c) The set of all irrationals I with usual topology. Since I is uncountable, there is no injection from I to Q. Similarly, by proof of contradiction, if C is a component of I and C has more that one points, we can find a rational among C and from the rational divide C into separated sets.

(d) The elements in this set has form

$$f = \begin{cases} 0, & x < i; \\ 1, & x \geq i. \end{cases}$$

or

$$f = \begin{cases} 0, & x \leq i; \\ 1, & x > i. \end{cases}$$

where i is an irrational. It is easy to verify that f is continuous. If i is rational, then one of preimages of $\{0\}$, $\{1\}$ is not open, thus f is not continuous.

Question 3

(a) Let $x = -1 + \cos\theta$, $y = \sin\theta$. Then $(x+1)^2 + y^2 = 1$. Because $\theta \epsilon (\pi, 3\pi)$, the graph of the first part of union is a circle except $(-2, 0)$. Similarly, the second part of the union is a circle $(x-1)^2 + y^2 = 1$ except $(2, 0)$. The graph of B is the circle $x^2 + y^2 = 1$ except $(1, 0)$ and $(-1, 0)$. C consists of 4 segments whose one endpoints are all $(0, 0)$, the other endpoints are $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$ and these segments do not contain $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$. We can deform continuously form the graph of A to the graph of C by gluing $(1, 0)$ and $(0, 1)$, $(-1, 0)$ and $(0, -1)$. Notice these endpoints are ont in C so the deformation is surjective and continuous. A and C are homeomorphic.

(b) The closures of A, B, C are the same graph by adding those missing points on circles and endpoints of segments. There are no homeomorphic closures. $\bar{A}$ has one point which is incident to 4 branches, so does $\bar{C}$. This means that only $\bar{A}$ and $\bar{C}$ are possibly homeomorphic to each other. But the original deformation is not injective any more since 4 endpoints of segments belong to $\bar{C}$. $\bar{A}$, $\bar{B}$, $\bar{C}$ are not homeomorphic to each other.

(c) $\tilde{A}$ is formed by gluing $(2, 0)$ and $(-2, 0)$, then filling the gluing point. $\tilde{B}$ is formed by gluing $(1, 0)$ and $(-1, 0)$, then filling the gluing point. $\tilde{C}$ is formed by gluing all 4 endpoints altogether and filling the gluing point. $\tilde{A}$ are homeomorphic to $\tilde{C}$.

(d) $\bar{A}$ is homeomorphic to $\tilde{B}$. Say f is the mapping from $\bar{A}$ to $\tilde{B}$. $f(0, 0) =$ the adding infinity point. The left circle except $(0, 0)$ maps to the upper half of B. The right circle except $(0, 0)$ maps to the lower half of B.

Question 4

(a) Every open set is a union of elements from basis but this is not true for the case of subbasis.

(b) $\phi(s)|s\epsilon S$ is a subbasis of (Y, T').

We can show that T' consists of all $O \subset Y$ s.t. $\phi^{-1}(O)\epsilon T$. Suppose that Q is open in Y, then $P = \phi^{-1}(Q)$ is open in X. Thus $P = \cup_i B_i$ where $B_i = s_1 \cap s_2 \ldots s_n$ which is a finite union of subbasis elements. $Q = \phi(P) = \phi(\cup_i B_i) = \phi(\cup_i(s_1 \cap s_2 \ldots s_n)) = \cup_i(\phi(s_1) \cap \phi(s_2) \ldots phi(s_n))$. This proves that $phi(s)|s\epsilon S$ is a subbasis of (Y, T').

(c) Let a, b belong to X. Define equivalence relation $a \sim b$ if $\phi(a) = \phi(b)$. (Y, T') satisfies T_1 condition iff for every distinctive equivalent class E_1, E_2, there are saturated open sets O_1, O_2 in X s.t. $E_1 \subset O_1$, $E_2 \subset O_2$ and $E_1 \cap O_2 = \emptyset$, $E_2 \cap O_1 = \emptyset$.

Reason: Let y_1, y_2 be two distinct points in Y. Then let $E_1 = \phi(y_1)$, $E_2 = \phi(y_2)$. By condition given above, there exist O_1, O_2 s.t. $E_1 \subset O_1$, $E_2 \subset O_2$ and $E_1 \cap O_2 = \emptyset$, $E_2 \cap O_1 = \emptyset$. $\phi(O_1)$, $\phi(O_2)$ are required open neighborhoods.

(d) Suppose that F, F' are two distinct continuous functions from X to Y whose restrictions on A are f. There exists a point $x \epsilon X$ s.t. $F(x) \neq F'(x)$. Since Y is Hausdorff, there are two disjoint open neighborhoods of $F(x)$, $F'(x)$, denoted as O, P. $F^{-1}(O)$, $F'^{-1}(P)$ are open in X since F, F' are continuous. $F^{-1}(O) \cap F'^{-1}(P)$ is open and nonempty because x is in this intersection, thus this intersection is a neighborhood of x and it should contain some $a \epsilon A$. $F(a) \epsilon O$ and $F'(a) \epsilon P$, but $F(a) = f(a)$, $F'(a) = f(a)$ in which contradicts O and P are disjoint. In conclusion, there can exist at most one extension of f to X.

Question 5

(a) Given $\epsilon > 0$. For each n, let $g_n = f - f_n$ and E_n be those $x \epsilon X$ s.t. $g_n(x) < \epsilon$. Each g_n is continuous and each E_n is open. Since f_n is monotonically increasing, g_n is monotonically decreasing, it follows that E_n is ascending. Since f_n converge pointwisely to f, E_n is an open covering of X. By compactness, there is some positive integer N such that $E_N = X$ i.e. if $n > N$ and then for all x in X we have $|f(x) - f_n(x)| < \epsilon$ as desired.

(b) For convenience, we use a capital letter to represent an element in l^2, a lower class letter to represent a digit of an element in l^2 and letters such as $\mathcal{X}$ to represent a subset in the Hilbert space. Thus X is an element in $\mathcal{X}$.
Let T_C map a bounded set to a set whose closure is compact. We will use proof by contradiction. Suppose c_n does not converge to 0, then there is $\epsilon > 0$ and a subsequence c_{n_i} with $|c_{n_i}| > \epsilon$. Let $Y_i = (0, 0, \ldots 1, 0 \ldots)$, where i occur in the i-th position. All these Y_is form a bounded set $\mathcal{B}$ since $d(Y_i, Y_j) = \sqrt{2}$ when $i \neq j$. If $T_c\bar{(}\mathcal{B})$ is compact, then there is a convergent subsequence of $T_C(Y_i)$. This subsequence is Cauchy. However, two terms in $T_C(Y_i)$ have distance larger that $\sqrt{2}\epsilon$, in contradicts with the existence of a Cauchy subsequence.

Suppose that $c_n \to 0$. Define T_{C_i} as $T_{C_i}(x) = (c_1 x_1, c_2 x_2 \ldots c_i x_i, 0, \ldots)$.
Claim: if T_{C_i} are compact operators for all i,then T_C is a compact operator. Proof of the claim: Suppose that $\mathcal{D}$ is a bounded subset of $\mathcal{X}$. It can be proved that if every sequence in $T_C(\mathcal{D})$ has a convergent subsequence then every sequence (X_i) in $\bar{T}_C(\mathcal{D})$ has a convergent subsequence. We can do this by forming a new sequence (X'_i) in $T_C(\mathcal{D})$ s.t. $||X_i - X'_i|| < 1/i$; this sequence has a convergent subsequence, which provides a convergent subsequence of (X_i).
Suppose that $(T_C(D_i))$ is a sequence in $T_C(\mathcal{D})$. $(T_{C_1}(D_i))$ has a convergent subsequence, denote the index set as $\mathbf{I_1}$. From the sequence $\{T_{C_2}(D_i)|i \in \mathbf{I_1}\}$ we can form a convergent subsequence whose index set is $\mathbf{I_2}$ and $\mathbf{I_2} \subset \mathbf{I_1}$. In particular, the first digit of the limit of $\mathbf{I_2}$ indexed sequence is the same to the first digit of the limit of $\mathbf{I_1}$. Similarly, from subsequence $\{T_{C_2}(D_i)|i \in \mathbf{I_n}\}$ we can form a convergent subsequence whose index set is $\mathbf{I_{n+1}}$ and $\mathbf{I_{n+1}} \subset \mathbf{I_n} \subset \ldots \subset \mathbf{I_1}$ and the first n digits of its limit agreed with the first n digits of the limit of sequence indexed by $\mathbf{I_n}$. Define X whose first n digits are digits we just mentioned. $\lim_{n\to\infty} T_C(D_{i_j}) = X$, where i_1 is the first element in $\mathbf{I_1}$, i_2 is the first element in $\mathbf{I_2}$ or the first element in $\mathbf{I_2}$ -1, etc. The claim is proved.

Similarly, by digonalized method and the convergence of C, we can show that the closure of $T_{C_i}(\mathcal{D})$ is sequentially compact, thus T_C is a compact operator.