NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Alan Chee

MA2216 Probability
AY 2011/2012 Sem 1

Question 1

(i) If $u = xy$, $v = x/y$, then $J(u, v) = \begin{vmatrix} y & x \\ \frac{1}{y} & \frac{-x}{y^2} \end{vmatrix} = -2\dfrac{x}{y} = -2v$

In addition, $y = \sqrt{u/v}, x = \sqrt{uv}$. Hence we have

$$f_{U,V}(u, v) = \frac{1}{2v} f_{X,Y}(\sqrt{uv}, \sqrt{u/v}) = \frac{1}{2vu^2}, \qquad u \geq 1, \frac{1}{u} < v < u$$

(ii) Integrating, we get

$$f_U(u) = \int_{1/u}^{u} \frac{1}{2vu^2} \mathrm{d}v = \frac{1}{u^2} \log u, \qquad u \geq 1$$

$$f_V(v) = \begin{cases} \displaystyle\int_v^{\infty} \frac{1}{2vu^2} \mathrm{d}u = \frac{1}{2v^2} & , v > 1 \\ \displaystyle\int_{\frac{1}{2}}^{\infty} \frac{1}{2vu^2} \mathrm{d}u = \frac{1}{2} & , 0 < v \leq 1 \end{cases}$$

(iii) Observe that $UV = X^2$. Hence $E[\frac{1}{UV}] = E[\frac{1}{X^2}]$. Now

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) \mathrm{d}y \\ &= \int_1^{\infty} \frac{1}{x^2 y^2} \mathrm{d}y \\ &= \frac{1}{x^2} \end{aligned}$$

Hence

$$\begin{aligned} E[\frac{1}{X^2}] &= \int_{-\infty}^{\infty} \frac{1}{x^2} f_X(x) \mathrm{d}x \\ &= \int_1^{\infty} \frac{1}{x^4} \mathrm{d}x \\ &= \frac{1}{3} \end{aligned}$$

Question 2

(i) The domain can be rewritten as $-y < x < y$, $0 < y < \infty$. Integrating over the domain, we get,

$$\begin{aligned}
\int_0^\infty \int_{-y}^{y} \frac{e^{-y/2}}{y^{3/2}} \mathrm{d}x\mathrm{d}y &= \int_0^\infty \frac{2e^{-y/2}}{\sqrt{y}} \mathrm{d}y \\
&= \int_0^\infty \frac{2e^{-x^2/2}}{x} 2x\mathrm{d}x \qquad \text{(By using the substituition } y = x^2\text{)} \\
&= \int_0^\infty 4e^{-x^2/2}\mathrm{d}x \\
&= \int_{-\infty}^\infty 2e^{-x^2/2}\mathrm{d}x \\
&= 2\sqrt{2\pi}
\end{aligned}$$

Hence $K = \dfrac{1}{2\sqrt{2\pi}}$.

(ii) Integrating with respect to x, we get

$$\begin{aligned}
f_Y(y) &= \int_{-y}^{y} K \frac{e^{-y/2}}{y^{3/2}} \mathrm{d}x \\
&= K \frac{2e^{-y/2}}{\sqrt{y}} \\
&= \frac{e^{-y/2}}{\sqrt{2\pi y}}
\end{aligned}$$

(iii)

$$\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \frac{\left(\frac{1}{2\sqrt{2\pi}} \frac{e^{-y/2}}{y^{3/2}}\right)}{\left(\frac{e^{-y/2}}{\sqrt{2\pi y}}\right)} \\
&= \frac{1}{2y}
\end{aligned}$$

$$\begin{aligned}
E[X|Y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\mathrm{d}x \\
&= \int_{-y}^{y} x \cdot \frac{1}{2y}\mathrm{d}x \\
&= 0
\end{aligned}$$

(iv)

$$\begin{aligned}
E[X] &= E[E[X|Y]] \\
&= \int_{-\infty}^{\infty} E[X|Y=y] f_Y(y)\mathrm{d}y \\
&= \int_0^\infty 0 \cdot \frac{e^{-y/2}}{\sqrt{2\pi y}}\mathrm{d}y \\
&= 0
\end{aligned}$$

(iv)

$$\begin{aligned} E[XY|Y] &= YE[X|Y] \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} E[XY] &= E[XY|Y] \\ &= E[0] \\ &= 0 \end{aligned}$$

Hence $Cov(X, Y) = E[XY] - E[X]E[Y] = 0$.

Question 3

(a)(i) The kth urn remains empty as the first $n - (k+1) + 1 = n - k$ balls are deposited into the other urns. On the $n - k + 1$th drop, urn k must remain empty. Since a ball can land in any of the $n - k + 1$ urns with equal probability, the probability that the $n - k + 1$th ball will not land in urn k is $1 - \frac{1}{n-k+1}$. This will be similar for the subsequent drops.

Therefore,

$$\begin{aligned} P\{I_k = 0\} &= \left(1 - \frac{1}{n-k+1}\right)\left(1 - \frac{1}{n-k+2}\right) \ldots\ldots \left(1 - \frac{1}{n}\right) \\ &= \left(\frac{n-k}{n-k+1}\right)\left(\frac{n-k+1}{n-k+2}\right) \ldots\ldots \left(\frac{n-1}{n}\right) \\ &= \left(\frac{n-k}{n}\right) \end{aligned}$$

Hence,

$$\begin{aligned} P\{I_k = 1\} &= 1 - P\{I_k = 0\} \\ &= 1 - \left(\frac{n-k}{n}\right) \\ &= \frac{k}{n} \end{aligned}$$

(ii) Let I be the number of non-empty urns. Notice that $E(I_k) = P(I_k = 1)$. Therefore,

$$\begin{aligned} E(I) &= \sum_{k=1}^{n} E(I_k) \\ &= \sum_{k=1}^{n} \frac{k}{n} \\ &= \frac{1}{n} \cdot \frac{n(n+1)}{2} \\ &= \frac{n+1}{2} \end{aligned}$$

Hence, the expected number of non-empty urns is $\dfrac{n+1}{2}$.

(b) We let X denote the number of rolls required until the number 6 appears 100 times.

Hence $X \sim NB(100, p)$, where $p = \dfrac{\sqrt{5}-1}{2}$.

Now $E[X] = \dfrac{100}{p}$, and $Var[X] = \dfrac{100(1-p)}{p^2}$.
Therefore X is approximately $N\left(\dfrac{100}{p}, \dfrac{100(1-p)}{p^2}\right)$ Hence

$$\begin{aligned} P(X > 183) &= P\left(Z > \frac{183 - \frac{100}{p}}{\sqrt{\frac{100(1-p)}{p^2}}}\right) \\ &= P\left(Z > \frac{183 - \frac{100}{p}}{\sqrt{\frac{100(1-p)}{p^2}}}\right) \\ &\approx P(Z > 2.12) \\ &\approx 0.017 \end{aligned}$$

Hence the probability that at least 184 rolls will be necessary is approximately 0.017.

Question 4

(i) Yes. Let $U = Z + W$ and $V = Z - W$.because U and V are linear functions of the same two independent normal random variables Z and W, their joint p.d.f is a bivariate normal distribution.

(ii) We see that since Z and W are identically distributed. We have $E[(Z+W)(Z-W)] = E[Z^2 - W^2] = 0$. Hence $Z + W$ and $Z - W$ are uncorrelated. In addition, since $Z + W$ and $Z - W$ are jointly normal, $Z + W$ and $Z - W$ will also be independent.

(iii) The two roots of the quadratic are real iff $4Z^2 - 4W^2 \geq 0$ or $Z^2 - W^2 \geq 0$. Since Z and W are identically distributed, we have $P(Z^2 \geq W^2) = P(W^2 \geq Z^2)$ by symmetry. Furthermore $P(Z^2 \geq W^2) + P(W^2 \geq Z^2) = 1$. Hence $P(Z^2 \geq W^2) = \frac{1}{2}$, and consequently the probability that all of the roots of the given quadratic equation being real is $\frac{1}{2}$.