MA1102R AY1718 Sem 2 Answers

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1. i

$$
\begin{aligned}
& \ln \frac{e^x+1}{e^x-1} \text{ exists} \\
\iff & \frac{e^x+1}{e^x-1} > 0 \\
\iff & e^x - 1 > 0 \\
\iff & x > 0
\end{aligned}
$$

$$
\begin{aligned}
& \sin^{-1} \frac{1}{\ln \frac{e^x+1}{e^x-1}} \text{ exists} \\
\iff & -1 < \frac{1}{\ln \frac{e^x+1}{e^x-1}} < 1 \\
\iff & 0 < \frac{1}{\ln \frac{e^x+1}{e^x-1}} < 1, \text{ since } \ln \frac{e^x+1}{e^x-1} > 0 \\
\iff & \ln \frac{e^x+1}{e^x-1} > 1 \\
\iff & \frac{e^x+1}{e^x-1} > e \\
\iff & e^x < \frac{e+1}{e-1} \\
\iff & x < \ln \frac{e+1}{e-1}
\end{aligned}
$$

$\therefore 0 < x < \ln \frac{e+1}{e-1}$

ii $f(x)$ is a one-to-one function on its maximal domain.

$$f^{-1}(\sin^{-1} \frac{1}{\ln \frac{e^x+1}{e^x-1}}) = x$$

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Substitute $x = \ln(\frac{2}{y-1} + 1)$

$$\begin{aligned}
f^{-1}(\sin^{-1}\frac{1}{\ln\frac{e^x+1}{e^x-1}}) &= f^{-1}(\sin^{-1}\frac{1}{\ln(1+\frac{2}{e^x-1})}) \\
&= f^{-1}(\sin^{-1}\frac{1}{\ln(1+\frac{2}{\frac{2}{y-1}+1-1})}) \\
&= f^{-1}(\sin^{-1}\frac{1}{\ln y}) = \ln(\frac{2}{y-1}+1)
\end{aligned}$$

Substitute $y = e^{1/\sin z}$

$$\begin{aligned}
f^{-1}(\sin^{-1}\frac{1}{\ln y}) &= f^{-1}(z) \\
&= \ln(\frac{2}{y-1}+1) \\
&= \ln(\frac{2}{e^{1/\sin z}-1}+1)
\end{aligned}$$

$$f^{-1}(x) = \ln(\frac{2}{e^{1/\sin x}-1}+1)$$

2.

$$\begin{aligned}
\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} \frac{a}{x-1}[3\sin(x-1) - 2\tan(\ln x)] \\
&= \lim_{x\to 1^-} \frac{a}{1}[3\cos(x-1) - 2\sec^2(\ln x)\frac{1}{x}] \\
&= a \\
\therefore a = b & \\
\lim_{x\to 1^-} f'(x) &= \lim_{x\to 1^-} \left[\frac{d}{dx}\left(\frac{a}{x-1}(3\sin(x-1) - 2\tan(\ln x))\right)\right] \\
&= a\lim_{x\to 1^-}\left[-(x-1)^{-2}(3\sin(x-1) - 2\tan(\ln x)) + (x-1)^{-1}\frac{3\cos(x-1) - 2\sec^2(\ln x)}{x}\right] \\
&= a\left[-3(x-1)^{-1} + 3(x-1)^{-1} + 2\lim_{x\to 1^-}\left(\frac{\tan(\ln x)}{(x-1)^2} - \frac{\sec^2(\ln x)}{x(x-1)}\right)\right] \\
&= 2a\lim_{x\to 1^-}\left(\frac{\tan(\ln x)}{(x-1)^2} - \frac{\sec^2(\ln x)}{x(x-1)}\right) \\
&= a
\end{aligned}$$

The last step requires us to prove the limit equals $\frac{1}{2}$. We prove it in the following way. In general, by L'Hospital's Rule,

$$\lim_{x\to a}\frac{f(x) - f(a) - (x-a)f'(x)}{(x-a)^2} = -\frac{f''(a)}{2}.$$

Now, rewrite the limit (which does not need to be one-sided) as follows:

$$\lim_{x\to 1} \frac{\tan(\ln(x)) - \tan(\ln(1)) - (x-1)\frac{\mathrm{d}}{\mathrm{d}x}(\tan(\ln(x)))}{(x-1)^2}$$

where $f(x) := \tan(\ln x)$. Applying the general result, we can easily see the limit goes to $\frac{1}{2}$. Define

$$d = \int_0^1 e^{[\ln(t+1)]^c}\ dx$$

$$\begin{aligned}
\lim_{x\to 1^-} f'(x) &= \frac{d}{dx}\left(\int_{4(x-1)}^{x^2} e^{x+[\ln(t+1)]^c}\ dx\right)_{x=1} \\
&= \left[e^x \frac{d}{dx}\left(\int_{4(x-1)}^{x^2} e^{[\ln(t+1)]^c}\ dx\right)\right]_{x=1} + \left[e^x \left(\int_{4(x-1)}^{x^2} e^{[\ln(t+1)]^c}\ dx\right)\right]_{x=1} \\
&= e^1(2\cdot e^{[\ln(2)]^c]} - 4e^{[\ln 1]^c}) + e^1\cdot d \\
&= e^1(2\cdot e^{[\ln(2)]^c]} - 4) + e^1\cdot d
\end{aligned}$$

On the other hand,

$$\lim_{x\to 1^+} f(x) = e^1\cdot d$$

For f to be continuous,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) \Rightarrow a = e\cdot d$$

Since f is differentiable,

$$\lim_{x\to 1^-} f'(x) = a = \lim_{x\to 1^+} f'(x) = e\cdot d + e\cdot(2\cdot e^{[\ln(2)]^c]} - 4)$$

Therefore

$$2\cdot e^{[\ln(2)]^c]} - 4 = 0 \Rightarrow c = 1$$

$$\begin{aligned}
d = \int_0^1 e^{[\ln(t+1)]^c}\ dx &= \int_0^1 e^{\ln(t+1)}\ dx \\
&= \int_0^1 t+1\ dx \\
&= \frac{3}{2}
\end{aligned}$$

$$\therefore a = b = \frac{3}{2}e, c = 1$$

3.
$$f(1) = f(0) + f'\left(\frac{1}{2}\right) + Af''(c)$$

By Intermediate value theorem, $\exists a \in (0,1)$ such that $f'(a) = f(1) - f(0)$

$$\therefore f'\left(\frac{1}{2}\right) + Af''(c) = f(1) - f(0) = f(a)$$

If $a = \frac{1}{2}$, then we can choose $A = 0$ and $c = 0.123$, and we are done

Otherwise, $a \neq \frac{1}{2}$

By Intermediate value theorem, $\exists b \in (a, \frac{1}{2})$ or $(\frac{1}{2}, a)$ such that $(\frac{1}{2} - a)f''(b) = f'(\frac{1}{2}) - f'(a)$

Choose $c = b$ and $A = \frac{1}{2} - a$

$$\therefore Af''(c) = \left(\frac{1}{2} - a\right) f''(b) = f(a) - f'\left(\frac{1}{2}\right)$$

$$\frac{1}{2} - 1 \leq A = \frac{1}{2} - a \leq \frac{1}{2} - 0$$

$$\therefore -\frac{1}{2} \leq A \leq \frac{1}{2}$$

4. i

$$\int (Ax^2 + B)^{-3/2} dx$$

Let $x = \sqrt{\frac{B}{A}} \tan\theta$, then $dx = \sqrt{\frac{B}{A}} \sec^2\theta d\theta$

$$\begin{aligned}
\int (Ax^2 + B)^{-3/2} dx &= \int \left(A \left(\sqrt{\frac{B}{A}} \tan\theta \right)^2 + B \right)^{-3/2} \sqrt{\frac{B}{A}} \sec^2\theta d\theta \\
&= \int (B\tan^2\theta + B)^{-3/2} \sqrt{\frac{B}{A}} \sec^2\theta d\theta \\
&= \int (B\sec^2\theta)^{-3/2} \sqrt{\frac{B}{A}} \sec^2\theta d\theta \\
&= \int \frac{1}{B\sqrt{A}\sec\theta} d\theta \\
&= \frac{1}{B\sqrt{A}} (\sin\theta) + C \\
&= \frac{1}{B\sqrt{A}} \left(\frac{\tan\theta}{\sqrt{\tan^2\theta + 1}} \right) + C \\
&= \frac{1}{B} \left(\frac{x}{\sqrt{Ax^2 + B}} \right) + C
\end{aligned}$$

ii

$$y = [(2x^2(2+\sin t)^4 + 2 - \sin t)]^{-3/2}$$

$$[(2x^2(2+\sin t)^4 + 2 - \sin t)]^{-3/2} = y = \frac{1}{8}$$

$$\therefore 2x^2(2+\sin t)^4 + 2 - \sin t = 4$$

$$\therefore x^2 = \frac{2+\sin t}{2(2+\sin t)^4} = \frac{1}{2(2+\sin t)^3}$$

$$\therefore x = \pm\sqrt{\frac{1}{2(2+\sin t)^3}}$$

The intersection of the curves has x values $-\sqrt{\frac{1}{2(2+\sin t)^3}}$ and $\sqrt{\frac{1}{2(2+\sin t)^3}}$. Let $p = \sqrt{\frac{1}{2(2+\sin t)^3}}$

$$\begin{aligned}
\text{Area} &= \int_{-p}^{p} y\,dx - \frac{1}{8}(2p) \\
&= \left[\frac{x}{B\sqrt{Ax^2+B}}\right]_{-p}^{p} - \frac{p}{4} \qquad \text{where } A = 2(2\sin t)^4 \text{ and } B = 2-\sin t \\
&= 2\left[\frac{p}{B\sqrt{Ap^2+B}}\right] - \frac{p}{4} \\
&= 2\left[\frac{1}{B\sqrt{A+B/p^2}}\right] - \frac{p}{4} \\
&= \frac{2}{(2-\sin t)\sqrt{2(2+\sin t)^4 + (2-\sin t)(2)(2+\sin t)^3}} - \frac{1}{4\sqrt{2(2+\sin t)^3}} \\
&= \frac{1}{(2-\sin t)(2+\sin t)\sqrt{2(2+\sin t)}} - \frac{1}{4(2+\sin t)\sqrt{2(2+\sin t)}} \\
&= \frac{1}{(2+\sin t)\sqrt{2(2+\sin t)}}\left(\frac{1}{2-\sin t} - \frac{1}{4}\right) \\
&= \frac{1}{(2+\sin t)\sqrt{2(2+\sin t)}}\left(\frac{2+\sin t}{4(2-\sin t)}\right) \\
&= \frac{1}{4(2-\sin t)\sqrt{2(2+\sin t)}}
\end{aligned}$$

iii Find the absolute min and max of $S(t) = \frac{1}{4(2-\sin t)\sqrt{2(2+\sin t)}}$

This is equivalent to finding the max and min of $\frac{1}{4(2-x)\sqrt{2(2+x)}}$ for $x \in [-1, 1]$

Let $f(x) = \frac{1}{4(2-x)\sqrt{2(2+x)}}$

$$f'(x) = \frac{1}{4\sqrt{2}}\left(\frac{1}{(2-x)^2\sqrt{2+x}} - \frac{1}{2(2-x)(2+x)^{3/2}}\right) = 0$$

$$\therefore (2+x) - \frac{1}{2}(2-x) = 0$$

$$\therefore x = -\frac{2}{3}$$

$$f(-1) = \frac{1}{12\sqrt{2}}, f\left(-\frac{2}{3}\right) = \frac{3\sqrt{6}}{128}, f(1) = \frac{1}{4\sqrt{6}}$$

Absolute minimal is $\frac{3\sqrt{6}}{128}$, maximal is $\frac{1}{4\sqrt{6}}$

5. i

$$\begin{aligned}
I_{n+2} &= \int_0^{\pi/2} \sin^{n+2} x \, dx \\
&= \int_0^{\pi/2} \sin x \sin^{n+1} x \, dx \\
&= \left[(-\cos x)\sin^{n+1} x\right]_0^{\pi/2} + (n+1)\int_0^{\pi/2} \cos^2 x \sin^n x \, dx \\
&= (n+1)\int_0^{\pi/2} (1-\sin^2 x)\sin^n x \, dx \\
&= (n+1)(I_n - I_{n+2})
\end{aligned}$$

$$\therefore (n+2)I_{n+2} = (n+1)I_n$$

$$\therefore I_{n+2} = \frac{n+1}{n+2} I_n$$

ii

$$I_0 = \frac{\pi}{2}$$

$$I_1 = 1$$

$$\begin{aligned}
I_9 &= \frac{8}{9} I_7 \\
&= \frac{8}{9} \times \frac{6}{7} I_5 \\
&= \frac{8 \times 6 \times 4 \times 2}{9 \times 7 \times 5 \times 3} \\
&= \frac{128}{315}
\end{aligned}$$

$$\begin{aligned}
I_{10} &= \frac{9}{10} I_8 \\
&= \frac{9}{10} \times \frac{7}{8} I_6 \\
&= \frac{9 \times 7 \times 5 \times 3 \times 1}{10 \times 8 \times 6 \times 4 \times 2} I_0 \\
&= \frac{63\pi}{512}
\end{aligned}$$

6.

$$
\begin{aligned}
\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt &= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt \\
&= \int_0^{2\pi} \sqrt{2 - 2(1 - 2\sin^2(t/2))}\, dt \\
&= \int_0^{2\pi} 2\sin(t/2)\, dt \\
&= [-4\cos(t/2)]_0^{2\pi} \\
&= 8
\end{aligned}
$$

7. Let $y = 2\tanh x - x$

   We want to find all stationary points on y

$$\frac{dy}{dx} = 2\operatorname{sech}^2 x - 1 = 0$$

$$\frac{4}{e^x - e^{-x}} = 1$$

$$e^x + e^{-x} = 4$$

$$e^x = 2 \pm \sqrt{3}$$

   $\therefore$ y has exactly 2 stationary points at $x = \ln(2+\sqrt{3})$ and $x = \ln(2-\sqrt{3})$

   At $x = 100$, $y \approx -98$
   At $x = \ln(2+\sqrt{3})$, $y \approx 0.451$
   At $x = \ln(2-\sqrt{3})$, $y \approx -0.451$
   At $x = -100$, $y \approx 98$

   And since y is monotonous between those 3 intervals, there is exactly 1 solution for each interval.

   $\therefore$ the equation has exactly 3 solutions.

8.

$$(x^2y - y)\frac{dy}{dx} + (xy^2 + x) = 0$$

$$\frac{dy}{dx} = -\frac{x(y^2+1)}{y(x^2-1)}$$

$$\int \frac{y}{y^2+1}\, dy = -\int \frac{x}{x^2-1}\, dx$$

$$\frac{1}{2}\ln(y^2+1) = -\frac{1}{2}\ln(1-x^2) + \frac{1}{2}\ln C$$

$$\ln(y^2+1) = \ln\left(\frac{C}{1-x^2}\right)$$

$$y^2+1 = \frac{C}{1-x^2}$$

$$y = \sqrt{\frac{C}{1-x^2} - 1} \text{ or } y = -\sqrt{\frac{C}{1-x^2} - 1} \text{ (rejected since } y > 0 \text{ at } 0)$$

$$\therefore y = \sqrt{\frac{C}{1-x^2} - 1}$$

Sub $y = 1$ and $x = 0$

$$1 = \sqrt{C-1}$$

$$\therefore C = 2$$

$$\therefore y = \sqrt{\frac{2}{1-x^2} - 1}$$