MA1101R - Linear Algebra I Suggested Solutions

(Semester 2 : AY2020/21)

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1. Using MATLAB, we have

$$\mathbf{A} = \begin{pmatrix} -5 & -4 & 3 & -25 & 27 \\ 4 & 14 & 12 & -7 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 21 & 9 & -6 & 0 \\ 2 & -20 & -3 & 10 & 9 \end{pmatrix} \xrightarrow{RREF} \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Using the column method, we have the pivot columns of $\mathbf{R}$ are the first four columns. Therefore, the basis of V is obtained from the first four columns of $\mathbf{A}$, namely

$$S = \left\{ \begin{pmatrix} -5 \\ 4 \\ 0 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 14 \\ 0 \\ 21 \\ -20 \end{pmatrix}, \begin{pmatrix} 3 \\ 12 \\ 0 \\ 9 \\ -3 \end{pmatrix}, \begin{pmatrix} -25 \\ -7 \\ 0 \\ -6 \\ 10 \end{pmatrix} \right\}$$

(b) Let a_i be the i-th vector in S. Using the Gram-Schmidt process, we have an orthogonal basis consisting of four vectors u_1, u_2, u_3, u_4 as shown.

$$\begin{aligned}
u_1 &= a_1 \\
&= \begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top
\end{aligned}$$

$$\begin{aligned}
u_2 &= a_2 - \frac{u_1 \cdot a_2}{u_1 \cdot u_1} u_1 \\
&= \begin{pmatrix}-4 & 14 & 0 & 21 & 20\end{pmatrix}^\top - \frac{\begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top \cdot \begin{pmatrix}-4 & 14 & 0 & 21 & 20\end{pmatrix}^\top}{\begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top \cdot \begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top} \begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top \\
&= \begin{pmatrix}-4 & 14 & 0 & 21 & 20\end{pmatrix}^\top - \frac{162}{81} \begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top \\
&= \begin{pmatrix}-4 & 14 & 0 & 21 & 20\end{pmatrix}^\top - 2 \begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top \\
&= \begin{pmatrix}6 & 6 & 0 & 9 & -24\end{pmatrix}^\top
\end{aligned}$$

$$\begin{aligned}
u_3 &= a_3 - \frac{u_1 \cdot a_3}{u_1 \cdot u_1} u_1 - \frac{u_2 \cdot a_3}{u_2 \cdot u_2} u_2 \\
&= \begin{pmatrix}3 & 12 & 0 & 9 & -3\end{pmatrix}^\top - \frac{\begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top \cdot \begin{pmatrix}3 & 12 & 0 & 9 & -3\end{pmatrix}^\top}{\begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top \cdot \begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top} \begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top \\
&\quad - \frac{\begin{pmatrix}6 & 6 & 0 & 9 & -24\end{pmatrix}^\top \cdot \begin{pmatrix}3 & 12 & 0 & 9 & -3\end{pmatrix}^\top}{\begin{pmatrix}6 & 6 & 0 & 9 & -24\end{pmatrix}^\top \cdot \begin{pmatrix}6 & 6 & 0 & 9 & -24\end{pmatrix}^\top} \begin{pmatrix}6 & 6 & 0 & 9 & -24\end{pmatrix}^\top \\
&= \begin{pmatrix}3 & 12 & 0 & 9 & -3\end{pmatrix}^\top - \frac{81}{81} \begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top - \frac{243}{729} \begin{pmatrix}6 & 6 & 0 & 9 & -24\end{pmatrix}^\top \\
&= \begin{pmatrix}3 & 12 & 0 & 9 & -3\end{pmatrix}^\top - \begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top - \frac{1}{3} \begin{pmatrix}6 & 6 & 0 & 9 & -24\end{pmatrix}^\top \\
&= \begin{pmatrix}6 & 6 & 0 & 0 & 3\end{pmatrix}^\top
\end{aligned}$$

$$\begin{aligned}
u_4 &= a_4 - \frac{u_1 \cdot a_4}{u_1 \cdot u_1} u_1 - \frac{u_2 \cdot a_4}{u_2 \cdot u_2} u_2 - \frac{u_3 \cdot a_4}{u_3 \cdot u_3} u_3 \\
&= \begin{pmatrix}-25 & -7 & 0 & -6 & 10\end{pmatrix}^\top - \frac{\begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top \cdot \begin{pmatrix}-25 & -7 & 0 & -6 & 10\end{pmatrix}^\top}{\begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top \cdot \begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top} \begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top \\
&\quad - \frac{\begin{pmatrix}6 & 6 & 0 & 9 & -24\end{pmatrix}^\top \cdot \begin{pmatrix}-25 & -7 & 0 & -6 & 10\end{pmatrix}^\top}{\begin{pmatrix}6 & 6 & 0 & 9 & -24\end{pmatrix}^\top \cdot \begin{pmatrix}6 & 6 & 0 & 9 & -24\end{pmatrix}^\top} \begin{pmatrix}6 & 6 & 0 & 9 & -24\end{pmatrix}^\top \\
&\quad - \frac{\begin{pmatrix}6 & 6 & 0 & 0 & 3\end{pmatrix}^\top \cdot \begin{pmatrix}-25 & -7 & 0 & -6 & 10\end{pmatrix}^\top}{\begin{pmatrix}6 & 6 & 0 & 0 & 3\end{pmatrix}^\top \cdot \begin{pmatrix}6 & 6 & 0 & 0 & 3\end{pmatrix}^\top} \begin{pmatrix}6 & 6 & 0 & 0 & 3\end{pmatrix}^\top \\
&= \begin{pmatrix}-25 & -7 & 0 & -6 & 10\end{pmatrix}^\top - \frac{81}{81} \begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top - \frac{(-486)}{729} \begin{pmatrix}6 & 6 & 0 & 9 & -24\end{pmatrix}^\top \\
&\quad - \frac{(-162)}{81} \begin{pmatrix}6 & 6 & 0 & 0 & 3\end{pmatrix}^\top \\
&= \begin{pmatrix}3 & 12 & 0 & 9 & -3\end{pmatrix}^\top - \begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top + \frac{2}{3} \begin{pmatrix}6 & 6 & 0 & 9 & -24\end{pmatrix}^\top + 2 \begin{pmatrix}6 & 6 & 0 & 0 & 3\end{pmatrix}^\top \\
&= \begin{pmatrix}-4 & 5 & 0 & -6 & -2\end{pmatrix}^\top
\end{aligned}$$

Therefore,

$$
\begin{aligned}
T &= \left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}, \frac{u_4}{\|u_4\|} \right\} \\
&= \left\{ \frac{\begin{pmatrix}-5 & 4 & 0 & 6 & 2\end{pmatrix}^\top}{9}, \frac{\begin{pmatrix}6 & 6 & 0 & 9 & -24\end{pmatrix}^\top}{27}, \frac{\begin{pmatrix}6 & 6 & 0 & 0 & 3\end{pmatrix}^\top}{9}, \frac{\begin{pmatrix}-4 & 5 & 0 & -6 & -2\end{pmatrix}^\top}{9} \right\} \\
&= \left\{ \begin{pmatrix} -5/9 \\ 4/9 \\ 0 \\ 2/3 \\ 2/9 \end{pmatrix}, \begin{pmatrix} 2/9 \\ 2/9 \\ 0 \\ 1/3 \\ -8/9 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 2/3 \\ 0 \\ 0 \\ 1/3 \end{pmatrix}, \begin{pmatrix} -4/9 \\ 5/9 \\ 0 \\ -2/3 \\ -2/9 \end{pmatrix} \right\}
\end{aligned}
$$

(c) Let t_1, t_2, t_3, t_4 be the vectors in T. The projection $\mathbf{p}$ of $\mathbf{q}$ onto V is

$$
\begin{aligned}
\mathbf{p} &= (\mathbf{q} \cdot t_1)t_1 + (\mathbf{q} \cdot t_2)t_2 + (\mathbf{q} \cdot t_3)t_3 + (\mathbf{q} \cdot t_4)t_4 \\
&= (-15+4+0+0+2)t_1 + (6+2+0+0-8)t_2 + (18+6+0+0+3)t_3 + (-12+5+0+0-2)t_4 \\
&= -9t_1 + 27t_3 - 9t_4 \\
&= \begin{pmatrix}5 & -4 & 0 & -6 & -2\end{pmatrix}^\top + \begin{pmatrix}18 & 18 & 0 & 0 & 9\end{pmatrix}^\top + \begin{pmatrix}4 & -5 & 0 & 6 & 2\end{pmatrix}^\top \\
&= \begin{pmatrix}27 & 9 & 0 & 0 & 9\end{pmatrix}^\top
\end{aligned}
$$

(d) The least square solutions to $\mathbf{Ax} = \mathbf{q}$ are the solutions to

$$
\mathbf{A}^\top \mathbf{A} \mathbf{x} = \mathbf{A}^\top \mathbf{q}
$$

Taking the RREF of the augmented matrix $\left(\mathbf{A}^\top \mathbf{A} \;\middle|\; \mathbf{A}^\top \mathbf{q}\right)$ yields

$$
\left(\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & -1 & -1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

Therefore, we have exactly one parameter for the solution $\mathbf{x}$. Suppose $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ and $x_5 = t$ for some real number t. Then, $x_1 = 1 - t, x_2 = -1 + t, x_3 = 1 - t, x_4 = -1 + t$. Finally, we conclude that

$$
\mathbf{x} = \begin{pmatrix} 1-t \\ -1+t \\ 1-t \\ -1+t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \; \forall\, t \in \mathbb{R}
$$

2. (a) Let $\mathbf{M}$ be the standard matrix of T. Then, $T(x) = \mathbf{M}x$. By combining the given information into a form

of a matrix we have

$$\mathbf{M}\begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & 2 \\ -5 & 3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & 2 \\ -5 & 3 & -3 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & -1 \\ 4 & 1 & 2 \\ 3 & 2 & 2 \\ 3 & 0 & 1 \end{pmatrix}$$

(b) Note that $Ker(T) = Null(\mathbf{M})$. Typing null(M,'r') into MATLAB gives you a 3 x 0 empty double matrix, meaning that the basis is the **empty set** and thus nullity$(T) = 0$.

(c) Note that $R(T) = Col(\mathbf{M})$. Therefore, the basis for $R(T)$ is the basis for $Col(\mathbf{M})$. With the column method that we used on Question 1, we obtained

$$\text{rref}(\mathbf{M}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which means the basis for $Col(\mathbf{M})$ is taken from all three columns of $\mathbf{M}$, which is

$$\left\{ \begin{pmatrix} -2 & 4 & 3 & 3 \end{pmatrix}^\top, \begin{pmatrix} 0 & 1 & 2 & 0 \end{pmatrix}^\top, \begin{pmatrix} -1 & 2 & 2 & 1 \end{pmatrix}^\top \right\}$$

Hence, rank$(T) = \dim(S) = 3$.

(d) Let $\mathbf{N}$ be the standard matrix of S. Since $Ker(S) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, the RREF of $\mathbf{N}$ must be

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Next, from the fact that $(S \circ T)(\mathbf{w}) = 2\mathbf{w}$, we have $\mathbf{NM} = 2\mathbf{I}_3$.

Since $\mathbf{N}$ has a RREF, there must be an invertible matrix $\mathbf{E}$ such that $\mathbf{ER} = \mathbf{N} \Rightarrow \mathbf{ERM} = 2\mathbf{I}_3$. Thus, $\mathbf{RM}$ is invertible and we can therefore find $\mathbf{E}$ and finally $\mathbf{N}$.

$$\begin{aligned} \mathbf{ERM} &= 2\mathbf{I}_3 \\ \mathbf{E} &= 2(\mathbf{RM})^{-1} \\ \mathbf{ER} &= 2(\mathbf{RM})^{-1}\mathbf{R} \\ \mathbf{N} &= 2\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 0 & -1 \\ 4 & 1 & 2 \\ 3 & 2 & 2 \\ 3 & 0 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{(MATLAB)} \\ &= \begin{pmatrix} 4 & 4 & -2 & 0 \\ 4 & 2 & 0 & 0 \\ -10 & -8 & 4 & 0 \end{pmatrix} \end{aligned}$$

To reverify if this is indeed the standard matrix, we can check using MATLAB that null(N,'r') indeed gives you $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ and $\mathbf{NM}$ is indeed $2\mathbf{I}_3$.

3. Note that $\lambda\mathbf{I}-\mathbf{A} = \begin{pmatrix}\lambda-b & -a & -a\\ -a & \lambda-b & -a\\ -a & -a & \lambda-b\end{pmatrix}$

(a)

$$\begin{aligned}
\det(\lambda\mathbf{I}-\mathbf{A}) &= (\lambda-b)\begin{vmatrix}\lambda-b & -a\\ -a & \lambda-b\end{vmatrix} + a\begin{vmatrix}-a & -a\\ -a & \lambda-b\end{vmatrix} - a\begin{vmatrix}-a & -a\\ \lambda-b & -a\end{vmatrix}\\
&= (\lambda-b)((\lambda-b)^2-a^2) + a(-a)(\lambda-b+a) - a(-a)(-a-\lambda+b)\\
&= (\lambda-b)(\lambda-b-a)(\lambda-b+a) - a^2(\lambda-b+a) - a^2(\lambda-b+a)\\
&= (\lambda-b)(\lambda-b-a)(\lambda-(b-a)) - 2a^2(\lambda-(b-a))\\
&= (\lambda-(b-a))((\lambda-b)(\lambda-b-a)-2a^2)\\
&= (\lambda-(b-a))(\lambda^2-(2b+a)\lambda+(b^2+ab-2a^2))\\
&= (\lambda-(b-a))(\lambda-(2a+b))(\lambda-(b-a))\\
&= (\lambda-(b-a))^2(\lambda-(2a+b)) = 0
\end{aligned}$$

Which means the eigenvalues are $b-a$ and $2a+b$.

(b) When $\lambda = b-a$, $\lambda\mathbf{I}-\mathbf{A} = \begin{pmatrix}-a & -a & -a\\ -a & -a & -a\\ -a & -a & -a\end{pmatrix}$.

Since $a \neq 0$, we have $\text{rref}(\lambda\mathbf{I}-\mathbf{A}) = \begin{pmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ which has two non-pivot columns.

To find the eigenspace, we shall solve the augmented matrix $\left(\text{rref}(\lambda\mathbf{I}-\mathbf{A}) \mid \mathbf{0}\right)$ where the solution is $\begin{pmatrix}x\\y\\z\end{pmatrix}$.

Suppose $y = p$ and $z = q$, then $x = -p-q$, meaning

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = p\begin{pmatrix}-1\\1\\0\end{pmatrix} + q\begin{pmatrix}-1\\0\\1\end{pmatrix}\ \forall\, p,q \in \mathbb{R}$$

Therefore, the basis to the eigenspace E_{b-a} is $\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$.

(c) When $\lambda = 2a+b$, $\lambda\mathbf{I}-\mathbf{A} = \begin{pmatrix}2a & -a & -a\\ -a & 2a & -a\\ -a & -a & 2a\end{pmatrix}$.

Since $a \neq 0$, we have $\text{rref}(\lambda\mathbf{I}-\mathbf{A}) = \begin{pmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}$ which has one non-pivot column.

To find the eigenspace, we shall solve the augmented matrix $\left(\text{rref}(\lambda\mathbf{I}-\mathbf{A}) \mid \mathbf{0}\right)$ where the solution is $\begin{pmatrix}x\\y\\z\end{pmatrix}$.

Suppose $z = t$, then $x = y = t$, meaning

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = t\begin{pmatrix}1\\1\\1\end{pmatrix}\ \forall\, t \in \mathbb{R}$$

Therefore, the basis to the eigenspace E_{2a+b} is $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$.

(d) Combining the bases obtained from the two previous parts we have

$$\mathbf{P} = \begin{pmatrix}-1 & -1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix}b-a & 0 & 0\\ 0 & b-a & 0\\ 0 & 0 & 2a+b\end{pmatrix}$$

(e) From the previous part, if $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{M}$ for some diagonal matrix $\mathbf{M}$, then $\mathbf{P}^{-1}\mathbf{A}^3\mathbf{P} = \mathbf{M}^3$ and vice versa.
Substitute $a = 3$ and $b = 2$, we have

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^{-1}\begin{pmatrix}2 & 3 & 3\\ 3 & 2 & 3\\ 3 & 3 & 2\end{pmatrix}\mathbf{P} = \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 8\end{pmatrix}$$

$$\mathbf{P}^{-1}\mathbf{C}^3\mathbf{P} = \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 8\end{pmatrix} = \mathbf{M}^3$$

$$\mathbf{M} = \begin{pmatrix}\sqrt[3]{-1} & 0 & 0\\ 0 & \sqrt[3]{-1} & 0\\ 0 & 0 & \sqrt[3]{8}\end{pmatrix} = \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{pmatrix}$$

$$\begin{aligned}\mathbf{C} &= \mathbf{P}\mathbf{M}\mathbf{P}^{-1}\\ &= \begin{pmatrix}-1 & -1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}-1 & -1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}^{-1}\\ &= \begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}\end{aligned}$$

4. (a) Let $\mathbf{A} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$. Thus,

$$\begin{aligned}\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix}\lambda - a & -b\\ -c & \lambda - d\end{vmatrix} &= (\lambda - a)(\lambda - d) - bc\\ &= \lambda^2 - (a+d)\lambda + (ad - bc)\\ &= \lambda^2 - t(\mathbf{A})\lambda + \det(\mathbf{A})\end{aligned}$$

(b) Let $\mathbf{A} = \begin{pmatrix}a_{11} & \cdots & a_{1n}\\ & \vdots & \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}b_{11} & \cdots & b_{1n}\\ & \vdots & \\ b_{n1} & \cdots & b_{nn}\end{pmatrix}$.

Then,

$$\mathbf{AB} = \begin{pmatrix}a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} & \cdots & \cdots\\ & \ddots & \\ \cdots & \cdots & a_{n1}b_{1n} + a_{n2}b_{2n} + \cdots + a_{nn}b_{nn}\end{pmatrix}$$

and

$$\mathbf{BA} = \begin{pmatrix}b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1n}a_{n1} & \cdots & \cdots\\ & \ddots & \\ \cdots & \cdots & b_{n1}a_{1n} + b_{n2}a_{2n} + \cdots + b_{nn}a_{nn}\end{pmatrix}$$

Note that

$$t(\mathbf{AB}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji}$$

(inner sigma iterates the terms in the same diagonal entry, outer sigma iterates the entries along the diagonal)

However,

$$\begin{aligned}
t(\mathbf{BA}) &= \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}a_{ij} \qquad \text{(swap the sigmas)} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} \\
&= t(\mathbf{AB}).
\end{aligned}$$

(c) From the previous part we have $t(\mathbf{AB}) = t(\mathbf{BA})$. Therefore,

$$t(\mathbf{P}^{-1}\mathbf{AP}) = t(\mathbf{APP}^{-1}) = t(\mathbf{A})$$

(d) No, take $\mathbf{A} = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $\mathbf{B}$ is the RREF of $\mathbf{A}$ hence row equivalent, but they have different values of t.

5. (a) Since $\mathbf{w} \in V$, we can write $\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$ for some real constants $c_1, c_2, \cdots, c_k$. Thus,

$$\begin{aligned}
\|\mathbf{w}\|^2 = \mathbf{w}\cdot\mathbf{w} &= (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k)\cdot(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) \\
&= c_1^2(\mathbf{u}_1\cdot\mathbf{u}_1) + c_2^2(\mathbf{u}_2\cdot\mathbf{u}_2) + \cdots + c_k^2(\mathbf{u}_k\cdot\mathbf{u}_k) && (\mathbf{u}_i\cdot\mathbf{u}_j = 0 \text{ if } i \neq j) \\
&= c_1^2 + c_2^2 + \cdots + c_k^2 && (\mathbf{u}_i\cdot\mathbf{u}_i = 1 \text{ for } i = 1, 2, \cdots, k)
\end{aligned}$$

However, note that for $i = 1, 2, \cdots, k$, we have

$$\begin{aligned}
\mathbf{w}\cdot\mathbf{u}_i &= (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k)\cdot\mathbf{u}_i \\
&= c_i\mathbf{u}_i\cdot\mathbf{u}_i \\
&= c_i
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathbf{w}\|^2 &= c_1^2 + c_2^2 + \cdots + c_k^2 \\
&= |\mathbf{w}\cdot\mathbf{u}_1|^2 + |\mathbf{w}\cdot\mathbf{u}_2|^2 + \cdots + |\mathbf{w}\cdot\mathbf{u}_k|^2
\end{aligned}$$

(b) Let $\mathbf{p}$ be the projection of $\mathbf{v}$ onto V. Then $\|\mathbf{v}\| \geq \|\mathbf{p}\|$ with equality if $\mathbf{v} \in V$. Thus,

$$\begin{aligned}
\|\mathbf{v}\|^2 &\geq \|\mathbf{p}\|^2 \\
&= \|(\mathbf{v}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{v}\cdot\mathbf{u}_k)\mathbf{u}_k\|^2
\end{aligned}$$

Let $d_i = \mathbf{v}\cdot\mathbf{u}_i$ for $i = 1, 2, \cdots, k$. Therefore, according to part (a),

$$\begin{aligned}
\|\mathbf{v}\|^2 &\geq \|d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \cdots + d_k\mathbf{u}_k\|^2 \\
&= d_1^2 + d_2^2 + \cdots + d_k^2 \\
&= |\mathbf{v}\cdot\mathbf{u}_1|^2 + |\mathbf{v}\cdot\mathbf{u}_2|^2 + \cdots + |\mathbf{v}\cdot\mathbf{u}_k|^2
\end{aligned}$$

6. (a) We are going to use induction on this problem. For $n = 1$,

$$\mathbf{A}_1^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \times \mathbf{I}_{2^1}$$

Assume for $n = k$, $\mathbf{A}_k^2 = k\mathbf{I}_{2^k}$. Then, for $n = k+1$, we can multiply the submatrices as follows.

$$\begin{aligned}
\mathbf{A}_{k+1}^2 &= \begin{pmatrix} \mathbf{A}_k & \mathbf{I}_{2^k} \\ \mathbf{I}_{2^k} & -\mathbf{A}_k \end{pmatrix}^2 \\
&= \begin{pmatrix} \mathbf{A}_k^2 + \mathbf{I}_{2^k}^2 & \mathbf{A}_k\mathbf{I}_{2^k} - \mathbf{I}_{2^k}\mathbf{A}_k \\ \mathbf{I}_{2^k}\mathbf{A}_k - \mathbf{A}_k\mathbf{I}_{2^k} & \mathbf{I}_{2^k}^2 + (-\mathbf{A}_k)^2 \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{A}_k^2 + \mathbf{I}_{2^k} & 0 \\ 0 & \mathbf{A}_k^2 + \mathbf{I}_{2^k} \end{pmatrix} \\
&= \begin{pmatrix} k\mathbf{I}_{2^k} + \mathbf{I}_{2^k} & 0 \\ 0 & k\mathbf{I}_{2^k} + \mathbf{I}_{2^k} \end{pmatrix} \\
&= \begin{pmatrix} (k+1)\mathbf{I}_{2^k} & 0 \\ 0 & (k+1)\mathbf{I}_{2^k} \end{pmatrix} \\
&= (k+1)\mathbf{I}_{2^{k+1}} && \left(\text{Note that } \begin{pmatrix} \mathbf{I}_{2^k} & 0 \\ 0 & \mathbf{I}_{2^k} \end{pmatrix} = \mathbf{I}_{2^{k+1}}\right)
\end{aligned}$$

We have completed the induction and thus the statement is proven.

(b) If λ is an eigenvalue of $\mathbf{A}_n$, then there exists a nonzero vector $\mathbf{u} \in \mathbb{R}^{2^n}$ such that $\mathbf{A}_n\mathbf{u} = \lambda\mathbf{u}$. Therefore,

$$\begin{aligned}
\mathbf{A}_n^2\mathbf{u} &= \mathbf{A}_n \cdot (\mathbf{A}_n\mathbf{u}) \\
&= \mathbf{A}_n \cdot (\lambda\mathbf{u}) \\
&= \lambda \cdot (\mathbf{A}_n\mathbf{u}) \\
&= \lambda \cdot (\lambda\mathbf{u}) \\
&= \lambda^2\mathbf{u}
\end{aligned}$$

From part (a), we have $\mathbf{A}_n^2 = n\mathbf{I}_{2^n}$. Thus,

$$\begin{aligned}
\mathbf{A}_n^2\mathbf{u} &= n\mathbf{I}_{2^n}\mathbf{u} \\
\lambda^2\mathbf{u} &= n\mathbf{u} \\
&\Rightarrow \lambda^2 = n \qquad (\text{since } \mathbf{u} \neq \mathbf{0}) \\
&\Rightarrow \lambda = \pm\sqrt{n}
\end{aligned}$$

which proves the statement.