NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Prof Wong Yan Loi

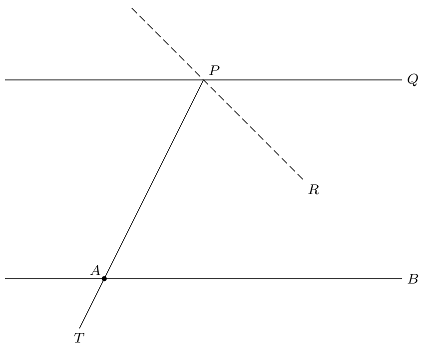
solutions prepared by Tay Jun Jie

MA2219 Introduction to Geometry
AY 2009/2010 Sem 1

Question 1

(a) That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

(b) Let AB be a line and P a point not on AB. Join PA and extend it to a point T. Construct a point Q on the same side of the line AB as P such that $\angle QPA = \angle BAT$. Then by proposition 28, PQ is parallel to AB. If PR is a line with R inside the region bounded by PQ, PA and AB. Then $\angle APR < \angle APQ$. Thus $\angle APR + \angle PAB < 180°$. By Euclid's 5th axiom, the line PR meets the line AB. Thus PQ is the only line through P parallel to AB.



Question 2

By angle bisector theorem,

$$\frac{BX}{XC} = \frac{PB}{PC}, \qquad \frac{CY}{YA} = \frac{PC}{PA}, \qquad \text{and} \qquad \frac{AZ}{ZB} = \frac{PA}{PB}$$
$$\Rightarrow \frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = \frac{PB}{PC}\frac{PC}{PA}\frac{PA}{PB} = 1$$

Therefore, by Ceva's theorem, AX, BY, and CZ are concurrent.

Question 3

(a) Treating $AA_1B_1C_1CB$ as a hexagon, it's 6 vertices lie on a circle and the 3 pairs of opposite sides intersect at M, N, and I. By Pascal's theorem, M, N, and I are collinear.

(b) Let K be the intersection of AA_1 and B_1C_1. Since $\angle BB_1C_1 = \angle BCC_1 = \frac{1}{2}\angle C$ and $\angle ABB_1 = \frac{1}{2}\angle B$,

$$\Rightarrow \angle BMB_1 = 180^\circ - \frac{1}{2}\angle B - \frac{1}{2}\angle C = 90^\circ + \frac{1}{2}\angle A$$
$$\Rightarrow \angle AKM = 90^\circ.$$

Hence AA_1 is perpendicular to B_1C_1. Furthermore,

$$\angle CC_1B_1 = \angle CBB_1 = \angle B_1BA = \angle B_1C_1A.$$

Thus triangle C_1KI is congruent to triangle C_1KA by AAS.

$$\Rightarrow KI = KA$$
$$\Rightarrow \triangle MKI \cong \triangle MKA \qquad \text{by SAS}$$
$$\Rightarrow \angle MIK = \angle MAK = \angle IAC$$

That is, the alternate angles are equal. We conclude that MN is parallel to AC.

Question 4

(a)

$$I = (a : b : c) \qquad \text{and} \qquad N = (s-a : s-b : s-c)$$

(b) Recall that $G = (1 : 1 : 1)$. Consider the following matrix.

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ s-a & s-b & s-c \end{pmatrix}$$

By observation, $sR_1 = R_2 + R_3$ where R_i denotes the ith row. Hence its determinant is 0 and we conclude that I, G, N be the collinear. Since I, G, N are collinear, G divides IN in some ratio $\alpha : \beta$. Now we standardize the coordinates of I and N.

$$I = \frac{(a : b : c)}{a+b+c} = \left(\frac{a}{2s} : \frac{b}{2s} : \frac{c}{2s}\right) \text{ and } N = \frac{(s-a : s-b : s-c)}{(s-a)+(s-b)+(s-c)} = \left(\frac{s-a}{s} : \frac{s-b}{s} : \frac{s-c}{s}\right)$$

Hence $G = \left(\alpha\frac{a}{2s} + \beta\frac{s-a}{s} : \alpha\frac{b}{2s} + \beta\frac{s-b}{s} : \alpha\frac{c}{2s} + \beta\frac{s-c}{s}\right) = (1 : 1 : 1)$. Solving, $\alpha : \beta = 1 : 2$.

Question 5

(a) Let $OP = p$. Hence $OP' = \frac{r^2}{p}$.

$$\Rightarrow \{AB, PP'\} = \frac{AP \times BP'}{AP' \times BP} = \frac{(r+p)\left(\frac{r^2}{p} - r\right)}{\left(\frac{r^2}{p} + r\right)(r-p)} = 1$$

(b) Firstly observe that $\angle BOQ$ is a common angle. Furthermore, since $OP \cdot OP' = r^2$, we have $\frac{OP}{OQ} = \frac{OP}{r} = \frac{r}{OP'} = \frac{OQ}{OP'}$. Therefore the triangles OPQ and OQP' are similar.

(c) Since the triangles OPQ and OQP' are similar, $\frac{QP}{QP'} = \frac{OP}{OQ} = \frac{p}{r}$ a constant.

(d)

$$\frac{PB}{BP'} = \frac{r-p}{\frac{r^2}{p} - r} = \frac{p}{r} = \frac{QP}{QP'}$$

Therefore QB bisects $\angle PQP'$ by the angle bisector theorem.

Question 6

(a) Rewriting the equations for ω, α, and β, we have

$$\omega : x^2 + y^2 = 10^2, \qquad \alpha : x^2 + y^2 - 4x = 0, \qquad \text{and} \qquad \beta : x^2 + y^2 - 8y = 0$$

Using the formula,

$$\alpha' : 0x^2 + 0y^2 - 400x + 0y + 10000 = 0 \Leftrightarrow \alpha' : x = 25$$

Similarly, $\beta' : y = \frac{25}{2}$.

(b) Any circle centred at P will be orthogonal to both α' and β', therefore upon inversion with respect to ω, it is either a circle or a straight line orthogonal to both α and β.

(c) The equation of γ is

$$\gamma : (x-a)^2 + (y-b)^2 = \left(\sqrt{a^2+b^2}\right)^2 \Leftrightarrow \gamma : x^2 + y^2 - 2ax - 2by = 0$$

However, since $\alpha' : x = 25$ and $\beta' : y = \frac{25}{2}$, we have $P = \left(25, \frac{25}{2}\right)$.

$$\Rightarrow \gamma : x^2 + y^2 - 50x - 25y = 0$$

Using the formula,

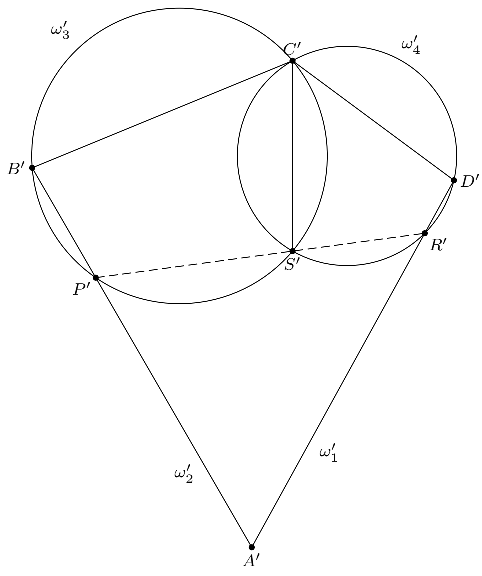
$$\gamma' : 0x^2 + 0y^2 - 5000x - 2500y + 10000 = 0 \Leftrightarrow \gamma' : 2x + y = 4$$

Question 7

Consider inversion with respect to some circle centred at Q. The inverses will be in the configuration below. Now observe that $B'P'S'C'$ and $D'R'S'C'$ are concyclic, hence $\angle C'B'P' + \angle C'S'P' = \angle C'D'R' + \angle C'S'R' = 180°$.

$$\Rightarrow \angle C'B'A' + \angle C'D'A' = 180°$$

Thus $A'B'C'D'$ is a cyclic quadrilateral. Therefore, by inverting backwards, A, B, C, D are concyclic.



Question 8

(a) A Saccheri quadrilateral is a quadrilateral $ABCD$ such that AB forms the base, AD and BC the sides such that $AD = BC$, and the angles at A and B are right angles.

(b) Firstly, observe that triangle PAR and triangle QBS are congruent by AAS.

$$\Rightarrow PR = QS$$

Hence $PRSQ$ is a saccheri quadrilateral. Let C and D be the midpoint of PQ and RS respectively. Then $PCDR$ and $QCDS$ are Lambert quadrilaterals.

$$\Rightarrow PC > RD \qquad \text{and} \qquad CQ > DS$$
$$\Rightarrow PQ = PC + CQ > RD + DS = RS$$

Since triangle PAR and triangle QBS are congruent, we have

$$RS = BS \pm RB = AR \pm RB = AB$$

where the plus is for case of R between A and B and the minus is for the case of R between B and S. Therefore we conclude that $PQ > AB$.