NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lau Tze Siong

MA2108 Mathematical Analysis I
AY 2004/2005 Sem 2

Question 1

(a) (i)

$$\begin{aligned}\lim_{n\to\infty}\frac{(n+1)!+n^2+\ln n}{2n(3^n+n!)} &= \lim_{n\to\infty}\frac{1+\frac{n^2}{(n+1)!}+\frac{\ln n}{(n+1)!}}{\frac{2n3^n}{(n+1)!}+\frac{2n(n!)}{(n+1)!}}\\ &= \frac{1+\lim\limits_{n\to\infty}\frac{n^2}{(n+1)!}+\lim\limits_{n\to\infty}\frac{\ln n}{(n+1)!}}{\lim\limits_{n\to\infty}\frac{2n3^n}{(n+1)!}+\lim\limits_{n\to\infty}\frac{2n}{n+1}}\\ &= \frac{1+0+0}{0+2}=\frac{1}{2}\end{aligned}$$

(ii)

$$\lim_{n\to\infty}\left(\frac{3n^3}{3n^3-2}\right)^{2n^3} = \lim_{n\to\infty}\left(1+\frac{2}{3n^3-2}\right)^{2n^3}$$

Let $\frac{1}{m}=\frac{2}{3n^3-2}$

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{2}{3n^3-2}\right)^{2n^3} &= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{\frac{4m+4}{3}}\\ &= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^{\frac{4}{3}}\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{\frac{4}{3}}\\ &= e^{\frac{4}{3}}\end{aligned}$$

(iii) Since

$$\sqrt{n}\le\sqrt{n}+1\le\sqrt{2n}$$

for all $n\in\mathbb{N}$. We have

$$\left(\sqrt{n}\right)^{\frac{1}{1+3\ln n}}\le\left(\sqrt{n}+1\right)^{\frac{1}{1+3\ln n}}\le\left(\sqrt{2n}\right)^{\frac{1}{1+3\ln n}}$$

. Since

$$\begin{aligned}\lim_{n\to\infty}\ln\sqrt{n}^{\frac{1}{1+3\ln n}} &= \lim_{n\to\infty}\frac{1}{1+3\ln n}\ln\sqrt{n}\\ &= \lim_{n\to\infty}\frac{\ln n}{2(1+3\ln n)}\\ &= \lim_{n\to\infty}\frac{1}{\frac{2}{\ln n}+6}\\ &= \frac{1}{6}.\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{n\to\infty} \ln \sqrt{2n}^{\frac{1}{1+3\ln n}} &= \lim_{n\to\infty} \frac{1}{1+3\ln n} \ln \sqrt{2n} \\
&= \lim_{n\to\infty} \frac{\ln 2 + \ln n}{2(1+3\ln n)} \\
&= \lim_{n\to\infty} \frac{\frac{\ln 2}{\ln n} + 1}{\frac{2}{\ln n} + 6} \\
&= \frac{1}{6}
\end{aligned}$$

By Squeeze Theorem, we have

$$\lim_{n\to\infty} \ln(\sqrt{n}+1)^{\frac{1}{1+3\ln n}} = \frac{1}{6}$$

Since $\ln : \mathbb{R}_{>0} \to \mathbb{R}$ is a continuous function, we have

$$\lim_{n\to\infty} (\sqrt{n}+1)^{\frac{1}{1+3\ln n}} = e^{\frac{1}{6}}.$$

(iv)

$$\begin{aligned}
\lim_{n\to\infty} (4^n \ln n + 3^n \sin n)^{\frac{1}{n}} &= \lim_{n\to\infty} 4\,(\ln n)^{\frac{1}{n}} \left(1 + \left(\frac{3}{4}\right)^n \frac{\sin n}{\ln n}\right)^{\frac{1}{n}} \\
&= 4(1)(1) = 4
\end{aligned}$$

(b) If $n = 4m$ for some $m \in \mathbb{N}$, we have $a_n = (-1)^{4m}2 + \cos(2m\pi) = 3$.
If $n = 4m+1$ for some $m \in \mathbb{N}$, we have $a_n = (-1)^{4m+1}2 + \cos\left(2m\pi + \frac{\pi}{2}\right) = -2$.
If $n = 4m+2$ for some $m \in \mathbb{N}$, we have $a_n = (-1)^{4m+2}2 + \cos(2m\pi + \pi) = 1$.
If $n = 4m+3$ for some $m \in \mathbb{N}$, we have $a_n = (-1)^{4m+3}2 + \cos\left(2m\pi + \frac{3\pi}{2}\right) = -2$.

Hence $\lim_{n\to\infty} \inf a_n = -2$

Question 2

(a) (i) By Limit Comparison Test, since,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{\sqrt{n^2+n+1}-\sqrt{n^2-n-1}}{n}}{\frac{1}{n}} &= \lim_{n\to\infty} \sqrt{n^2+n+1} - \sqrt{n^2-n-1} \\
&= \lim_{n\to\infty} \frac{2n+2}{\sqrt{n^2+n+1} + \sqrt{n^2-n-1}} \\
&= \lim_{n\to\infty} \frac{2 + \frac{2}{n}}{\sqrt{1 + \frac{1}{n} + \frac{1}{n^2}} + \sqrt{1 - \frac{1}{n} - \frac{1}{n^2}}} \\
&= 1
\end{aligned}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+n+1} - \sqrt{n^2-n-1}}{n}$ diverges.

(ii) By Ratio Test, since,

$$\begin{aligned}
\lim_{n\to\infty} \frac{((2n+2)!)^2(2^{n+1})}{(3n+3)!(n+1)!} \frac{(3n)!(n!)}{((2n)!)^2(2^n)} &= \lim_{n\to\infty} 2\frac{2n+1}{3n+1}\frac{2n+1}{3n+2}\frac{2n+2}{3n+3}\frac{2n+2}{n+1} \\
&= 2\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)(2) \\
&= \frac{32}{27} > 1
\end{aligned}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{((2n)!)^2(2^n)}{(3n)!(n)!}$ diverges.

(iii) Since for $n > e^{(e^4)}$, we have $\ln(\ln n) > 4$. Hence we have $\ln n \ln(\ln n) > 4 \ln n$ and $(\ln n)^{\ln n} > n^4$. Therefore $0 \le \frac{n^2}{(\ln n)^{\ln n}} < \frac{n^2}{n^4} = \frac{1}{n^2}$ for $n > e^{(e^4)}$. Hence $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(\ln n)^{\ln n}}$ is convergent.

(b) Claim: $0 \le a_n \le \frac{1}{2}$ for all $n \in \mathbb{N}$.
Proof:
We have $0 \le a_1 \le \frac{1}{2}$.
Suppose for some $k \in \mathbb{N}$ such that $0 \le a_k \le \frac{1}{2}$. Since $a_k \le \frac{1}{2}$, $(1 - a_k) \ge 0$. Also since $a_k \ge 0$, $a_{k+1} = a_k(1 - a_k) \ge 0$.
Since $a_k \ge 0$, we have $1 - a_k \le 1$. Also since $a_k \le \frac{1}{2}$, $a_{k+1} = a_k(1 - a_k) \le \frac{1}{2}$.

By induction, we have $0 \le a_n \le \frac{1}{2}$ for all $n \in \mathbb{N}$.

Claim: (a_n) is decreasing.
Proof:
We have $a_2 = (0.5)(1 - 0.5) = 0.25 < 0.5 = a_1$.
Suppose for some $k \in \mathbb{N}$ that $a_k > a_{k+1}$. Hence we have $a_k - a_{k+1} > 0$. Since $a_n \le \frac{1}{2}$ for all $n \in \mathbb{N}$, we have $a_k + a_{k+1} < 1$. Hence we have,

$$\begin{aligned}
(a_k + a_{k+1})(a_k - a_{k+1}) &< 1(a_k - a_{k+1}) \\
a_k^2 - a_{k+1}^2 &< a_k - a_{k+1} \\
a_k - a_k^2 &> a_{k+1} - a_{k+1}^2 \\
a_{k+1} &> a_{k+2}
\end{aligned}$$

Hence by induction, (a_n) is decreasing and $\lim\limits_{n\to\infty} a_n = a$ satisfies the equation $a = a(1 - a)$.
Hence $\lim\limits_{n\to\infty} a_n = 0$

Question 3

(a) (i) Since for $x \in (1, \infty)$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{x^n + n\ln x}{x^n + \ln x} &= \lim_{n\to\infty} \frac{1 + n\frac{\ln x}{x^n}}{1 + \frac{\ln x}{x^n}} \\
&= 1
\end{aligned}$$

$f_n(x) \to 1$ for all $x \in (1, \infty)$.
Since,

$$\begin{aligned}
\lim_{n\to\infty} \sup_{x\in(1,\infty)} \left| \frac{x^n + n\ln x}{x^n + \ln x} - 1 \right| &= \lim_{n\to\infty} \sup_{x\in(1,\infty)} \left| \frac{(n-1)\ln x}{x^n + \ln x} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n-1)}{e^n + 1} \right| \\
&= 0
\end{aligned}$$

Hence $\{f_n\}$ converges uniformly on the interval $(1, \infty)$.

(ii) Since $\{f_n\}$ converges uniformly on the interval $(1, \infty)$ and

$$0 \le \lim_{n\to\infty} \sup_{x\in[3,5]} \left| \frac{x^n + n\ln x}{x^n + \ln x} - 1 \right| \le \lim_{n\to\infty} \sup_{x\in(1,\infty)} \left| \frac{x^n + n\ln x}{x^n + \ln x} - 1 \right| \le 0$$

$\{f_n\}$ converges uniformly on the interval $[3, 5]$.

(iii) Since $\{f_n\}$ is converges uniformly to 1 on the interval $[3, 5]$,

$$\begin{aligned}
\lim_{n\to\infty} \int_3^5 \frac{x^n + n\ln x}{x^n + \ln x}\, dx &= \int_3^5 \lim_{n\to\infty} \frac{x^n + n\ln x}{x^n + \ln x}\, dx \\
&= \int_3^5 1\, dx = 2
\end{aligned}$$

(b) True.
Since $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{a_n^2}$ converges, we have $\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{a_n^2} = 0$. Letting $\epsilon = 1$, there exist a $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\ge N}$, we have $\frac{\sqrt{n}}{a_n^2} < 1$. Hence we have $a_n^2 > \sqrt{n}$. Therefore we have $a_n^5 > n^{\frac{5}{4}} > n$. Hence we have $a_n^5 - n > 0$ for all $n \in \mathbb{N}_{\ge N}$. Therefore $\frac{a_n^5}{n^2} > \frac{1}{n}$ for all $n \in \mathbb{N}_{\ge N}$. Hence $\displaystyle\sum_{n=1}^{\infty} \frac{a_n^5}{n^2}$ diverges.

Question 4

(a) Let $y = x + 4$, then $x = y - 4$.

$$f(y) = \frac{1}{(y-4+2)(y-4+3)} = \frac{1}{(y-2)(y-1)} = \frac{1}{y-2} - \frac{1}{y-1}$$

Let $g(y) = \frac{1}{y-2}$ and $h(y) = \frac{1}{y-1}$.

$$
\begin{aligned}
f(y)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)(y)^n}{n!} &= \sum_{n=0}^{\infty}\frac{g^{(n)}(0)(y)^n}{n!}-\sum_{n=0}^{\infty}\frac{h^{(n)}(0)(y)^n}{n!} \\
&= -\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-n!)(0-2)^{-n-1}(y)^n}{n!}+1-\sum_{n=1}^{\infty}\frac{(-n!)(0-1)^{-n-1}(y)^n}{n!} \\
&= \frac{1}{2}+\sum_{n=1}^{\infty}-\frac{y^n}{2^{n+1}}-\sum_{n=1}^{\infty}-y^n \\
&= \frac{1}{2}-\sum_{n=1}^{\infty}\frac{y^n}{2^{n+1}}+\sum_{n=1}^{\infty}y^n \\
&= \frac{1}{2}+\sum_{n=1}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)y^n
\end{aligned}
$$

(b) Since

$$
\begin{aligned}
n &> \left(1+\frac{1}{n}\right)^n \\
n &> \frac{(n+1)^n}{n^n} \\
n^{n+1} &> (n+1)^n \\
(n+1)\ln n &> n\ln(n+1) \\
(n+1)\ln n+\ln n\ln(n+1) &> n\ln(n+1)+\ln n\ln(n+1) \\
(\ln n)\left((n+1)+\ln(n+1)\right) &> \ln(n+1)\,(n+\ln n) \\
\frac{\ln n}{n+\ln n} &> \frac{\ln(n+1)}{(n+1)+\ln(n+1)}
\end{aligned}
$$

for all $n \geq 2$.

Hence $\frac{\ln n}{n+\ln n}$ is eventually decreasing. Therefore $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n+\ln n}$ converges.

Since,

$$
\begin{aligned}
\lim_{n\to\infty}\frac{\frac{\ln n}{n+\ln n}}{\frac{\ln n}{n}} &= \lim_{n\to\infty}\frac{n}{n+\ln n} \\
&= \lim_{n\to\infty}\frac{1}{1+\frac{\ln n}{n}} \\
&= 1
\end{aligned}
$$

by Limit Comparison Test, $\sum_{n=1}^{\infty}\frac{\ln n}{n+\ln n}$ diverges.

Hence $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n+\ln n}$ is conditionally convergent.

(c)

Question 5

(a) The sum $\sum_{n=1}^{\infty} \frac{(2x-9)^n}{3^n n}$ converges on the interval

$$\begin{aligned}
\lim_{n\to\infty} \sup \sqrt[n]{\frac{|2x-9|^n}{|3^n n|}} &< 1 \\
\frac{|2x-9|}{3} &< 1 \\
|2x-9| &< 3 \\
3 < \quad x &< 6
\end{aligned}$$

Since $\sum^{\infty} \frac{(2x-9)^n}{3^n n}$ converges due to Alternating Series Test at $x = 3$ and $\sum^{\infty} \frac{(2x-9)^n}{3^n n}$ diverges at $x = 6$ due to Comparison Test.
Hence the interval of convergence is $3 \le x < 6$.

(b) For $x \in [3, 4]$,

$$\begin{aligned}
\frac{x + e^n}{x^n + \ln x} &= \frac{\frac{1}{x^{n-1}} + \left(\frac{e}{x}\right)^n}{1 + \frac{\ln x}{x^n}} \\
&\le \frac{1}{3^{n+1}} + \left(\frac{e}{3}\right)^n
\end{aligned}$$

Since $\frac{1}{3}, \frac{e}{3} < 1$, Hence $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \left(\frac{e}{3}\right)^n$ converges. Therefore by Weierstrass M-Test, $\sum_{n=1}^{\infty} \frac{x+e^n}{x^n + \ln x}$ converges uniformly on the interval $[3, 4]$. Since for all $m \in \mathbb{N}$, $\sum_{n=1}^{m} \frac{x+e^n}{x^n + \ln x}$ a finite sum of continuous functions is continuous on $[3, 4]$. $F(x)$ is continuous on $[3, 4]$.

(c) True.
Since $\lim_{n\to\infty} (a_n - a_{n-1}) = 2$. For any given $\epsilon \in \mathbb{R}_{>0}$, there exist a $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}_{\ge N}$,

$$\begin{array}{rcl}
2 - \epsilon < & a_m - a_{m-1} & < 2 + \epsilon \\
\sum_{m=1}^{n} (2-\epsilon) < & \sum_{m=1}^{n} (a_m - a_{m-1}) & < \sum_{m=1}^{n} (2+\epsilon) \\
\frac{\sum_{m=1}^{n} (2-\epsilon)}{n} < & \frac{\sum_{m=1}^{n} (a_m - a_{m-1})}{n} & < \frac{\sum_{m=1}^{n} (2+\epsilon)}{n} \\
2 - \epsilon < & \frac{a_n}{n} & < 2 + \epsilon \\
\left| \frac{a_n}{n} - 2 \right| & < & \epsilon
\end{array}$$

Hence $\lim_{n\to\infty} \frac{a_n}{n} = 2$.