Suggested Solution for MA2001 Linear Algebra Final Exam

Semester 1, AY21/22

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Question 1

(A) False. Take $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ as counterexample.

(B) True. Multiplying $\mathbf{P}$ to the right of $\mathbf{A}$ can be thought of as obtaining linear combinations of columns of $\mathbf{A}$.

(C) False. Taking $\mathbf{A} = \mathbf{I}_n$ and $\mathbf{P} = \mathbf{0}$, we have $\text{null}(\mathbf{B}) = \text{null}(\mathbf{0}) = \mathbb{R}^m \not\subset \mathbf{0} = \text{null}(\mathbf{A})$.

(D) False. By Theorem 4.2.8, $\text{rank}(\mathbf{B}) \leq \text{rank}(\mathbf{A})$. Since $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = \text{rank}(\mathbf{B}) + \text{nullity}(\mathbf{B}) = n$, $\text{nullity}(\mathbf{A}) \leq \text{nullity}(\mathbf{B})$.

(E) True.

Question 2

Recall that for any matrix $\boldsymbol{A}$, the dimension of the row space and column space are the same, which we denote as $\text{rank}(\boldsymbol{A})$.

(A) True. The number of columns of the matrix is equal to the dimension of its column space, which means that all columns are linearly independent.

(B) False. It spans $\mathbb{R}^n$.

(C) False. Take $\mathbf{I}_n$ as counterexample.

(D) False. Take $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ as counterexample.

(E) True. Since the linear system is consistent if and only if the augmented matrix $(\mathbf{A} \mid \mathbf{b})$ is of the same rank as $\mathbf{A}$, we know that $\mathbf{b}$ is a linear combination of columns of $\mathbf{A}$ and that the columns of $\mathbf{A}$ is a basis for $\mathbb{R}^n$. There is thus a unique $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$.

Question 3

Note that eigenvectors are vectors in $\mathbb{R}^4$ in this case. Without loss of generality, either

1. $\dim(E_1) = 1$, $\dim(E_2) = 3$, or
2. $\dim(E_1) = 2$, $\dim(E_2) = 2$.

In any case, we can find orthonormal bases S_1 and S_2 for E_1 and E_2 such that $S_1 \cup S_2$ is again orthonormal and spans $\mathbb{R}^4$. Then, we use such orthonormal basis to form the columns of an orthogonal $\mathbf{P}$ and let $\mathbf{D}$ be the corresponding diagonal matrix with eigenvalues, which gives us

$$\mathbf{A} = \mathbf{PDP}^{-1} = \mathbf{PD}^T\mathbf{P}^T = \mathbf{A}^T.$$

So $\mathbf{A}$ is symmetric.

(A) True.

(B) False. Suppose there is another eigenvalue with an eigenvector $\mathbf{v}$. Then either $\mathbf{v} \in E_1$ or $\mathbf{v} \in E_2$, which implies that the new eigenvalue is one of the two given eigenvalues. Contradiction.

(C) True.

(D) False. Take diag(1,1,1,2) as counterexample. [1]

(E) False. Take diag(1,1,1,2) as counterexample.

Question 4

(A) True.

(B) False.

(C) True. $\mathbf{A}$ and $\mathbf{B}$ being row equivalent indicates that their RREF are the same; therefore, the nullspaces obtained are equivalent.

[1] Note that diag that takes in n arguments is the $n \times n$ diagonal matrix with the n arguments as its diagonal.

(D) False.

(E) False.

Question 5

We have

$$\begin{aligned}\mathbf{A}\mathbf{v} = \lambda\mathbf{v} &\iff \mathbf{A}^T\mathbf{v} = \lambda^{-1}\mathbf{v}, \text{ and}\\ \mathbf{A}^T\mathbf{u} = \mu\mathbf{u} &\iff \mathbf{A}\mathbf{u} = \mu^{-1}\mathbf{u}.\end{aligned}$$

Note also that the characteristic polynomial of any matrix and its transpose are the same. We then have $\lambda, \mu, \lambda^{-1}$ and μ^{-1} as the eigenvalues of both $\mathbf{A}$ and $\mathbf{A}^T$.

(A) False.

(B) True.

(C) True.

(D) False.

(E) False. $\mathbf{u}$ and $\mathbf{v}$ may be the same vector.

Question 6

(A) False. By linearity and linear independence of $\mathbf{u_1}, \cdots, \mathbf{u_n}$,

$$\begin{aligned}T \circ T(\mathbf{v}) &= T(\alpha_1 c_1 \mathbf{u}_1 + \cdots + \alpha_n c_c \mathbf{u}_n)\\ &= T(\alpha_1 c_1 \mathbf{u}_1) + \cdots + T(\alpha_n c_n \mathbf{u}_n)\\ &= \alpha_1^2 c_1 \mathbf{u}_1 + \cdots + \alpha_n^2 c_n \mathbf{u}_n.\end{aligned}$$

(B) True.

(C) False. Take $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ as counterexample, which has $(1,0)^T$ and $(1,1)^T$ as eigenvectors and is not diagonal itself.

(D) True.

(E) False. If $\alpha_0 = \cdots = \alpha_n = 0$, $\mathrm{Ker}(T) = \mathbb{R}^n$.

Question 7

(A) False.

$$\begin{aligned}
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\
&= a(ei - hf) - b(di - gf) + c(dh - ge), \\
-\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} &= -d\begin{vmatrix} h & i \\ b & c \end{vmatrix} + e\begin{vmatrix} g & i \\ a & c \end{vmatrix} - f\begin{vmatrix} g & h \\ a & b \end{vmatrix} \\
&= -d(hc - bi) + e(gc - ai) - f(gb - ah) \\
&= a(-ei + hf) - b(-di + gf) + c(-dh + ge).
\end{aligned}$$

(B) True.

$$\begin{aligned}
\begin{vmatrix} a & b & c \\ a & 2b & 3c \\ a & 4b & 9c \end{vmatrix} &= a\begin{vmatrix} 2b & 3c \\ 4b & 9c \end{vmatrix} - b\begin{vmatrix} a & 3c \\ a & 9c \end{vmatrix} + c\begin{vmatrix} a & 2b \\ a & 4b \end{vmatrix} \\
&= a(18bc - 12bc) - b(9ac - 3ac) + c(4ab - 2ab) \\
&= 2abc, \\
abc\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} &= abc[\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}] \\
&= abc[(18 - 12) - (9 - 3) + (4 - 2)] \\
&= 2abc.
\end{aligned}$$

(C) True.

$$\begin{aligned}
\begin{vmatrix} a+b & c+d & e+f \\ a & b & c \\ d & e & f \end{vmatrix} &= (a+b)(bf - ec) - (c+d)(af - cd) + (e+f)(ae - bd), \\
\begin{vmatrix} a & c & e \\ a & b & c \\ d & e & f \end{vmatrix} + \begin{vmatrix} b & d & f \\ a & b & c \\ d & e & f \end{vmatrix} &= a(bf - ec) - c(af - dc) + e(ae - bd) \\
&\quad + b(bf - ec) - d(af - cd) + f(ae - db) \\
&= (a+b)(bf - ec) - (c+d)(af - cd) + (e+f)(ae - bd).
\end{aligned}$$

(D) False.

$$
\begin{aligned}
\begin{vmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{vmatrix} &= acf, \\
\begin{vmatrix} 0 & 0 & a \\ 0 & c & b \\ f & e & d \end{vmatrix} &= a(0 - cf) \\
&= -acf.
\end{aligned}
$$

(E) False.

$$
\begin{aligned}
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= aei + bfg + cdh - ceg - afh - dbi, \\
\begin{vmatrix} c & f & i \\ b & e & h \\ a & d & g \end{vmatrix} &= ceg + afh + dbi - aei - bfg - cdh.
\end{aligned}
$$

Question 8

(A) True.

(B) False.

(C) True. $\mathbf{A}$ has the second, third and fourth columns as pivot columns.

(D) False. The general solution is given by

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} s \\ -t \\ 0 \\ 1-t \\ t \end{pmatrix} \in \mathbb{R}^5.
$$

(E) False. $\mathbf{0}$ is not in the solution set.

Question 9

(A) Yes.

$$
\frac{1}{3}\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}
$$

(B) Yes.

$$\frac{2}{3}\begin{pmatrix}3\\6\\9\end{pmatrix} = \begin{pmatrix}2\\4\\6\end{pmatrix}$$

(C) Yes.

$$\begin{pmatrix}2\\0\\0\end{pmatrix} + \begin{pmatrix}0\\4\\6\end{pmatrix} = \begin{pmatrix}2\\4\\6\end{pmatrix}$$

(D) No. One can check $\begin{pmatrix}4 & 6 & 2\\6 & 2 & 4\\2 & 4 & 6\end{pmatrix}$ is of full rank.

(E) No. One can check $\begin{pmatrix}2 & 2 & 0\\0 & 6 & 6\\2 & 4 & 6\end{pmatrix}$ is of full rank.

Question 10

(A) Linearly independent.

(B) Linearly dependent. The RREF form is $\begin{pmatrix}1 & 0 & -1 & -2\\0 & 1 & 2 & 3\\0 & 0 & 0 & 0\end{pmatrix}$.

(C) Linearly independent. The RREF form is $\mathbf{I}_4$.

(D) Linearly independent. The RREF form is $\mathbf{I}_4$.

(E) Linearly dependent. The RREF form is $\begin{pmatrix}1 & 0 & \frac{12}{13} & 0\\0 & 1 & \frac{2}{13} & 2\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}$.

Vectors in the following questions are column vectors.

Long Question 1

i.

The RREF of the matrix with S as its rows is

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which has full rank, so S is indeed a basis and $\dim(V) = 3$.

ii.

Equivalently, we can solve for $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$. More explicitly,

$$\mathbf{A}^T\mathbf{A}\mathbf{x} = \begin{pmatrix} 3 & 2 & 2 & 7 \\ 2 & 2 & 1 & 5 \\ 2 & 1 & 3 & 6 \\ 7 & 5 & 6 & 18 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 9 \\ 6 \\ 9 \\ 24 \end{pmatrix}.$$

Using augmented matrix and row operations, we have that $\mathbf{x} \in \operatorname{span}\{(0,0,1,1),(1,1,2,0)\}$.

iii.

Note that the column space of $\mathbf{A}$ is essentially V. We can then substitute any least squares solution into $\mathbf{A}\mathbf{x}$, which gives

$$\mathbf{p} = \mathbf{A} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 2 \end{pmatrix}.$$

iv.

$$(1,1,2).$$

Long Question 2

i.

It is easy to check that

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and}$$
$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

satisfy $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

ii.

No. 0 is an eigenvalue.

iii.

No. With the eigenvalue and eigenvector conditions (nine equations in $\mathbb{R}^9$), one can solve

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \mathbf{v} = \lambda \mathbf{v}.$$

If the eigenspaces and associated eigenvalues were to be the same as those of $\mathbf{A}$'s, it is essentially performing row operations to the augmented matrix formed with the nine equations, leaving us with the same final matrix.

iv.

We know that

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$
$$\implies \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{P}^{-1}.$$

Since the column space of $\mathbf{A}$ is contained in $\operatorname{span}\{(1,1,0),(0,1,1)\}$, $\mathbf{v} \in E_1$, where E_1 is the eigenspace of $\mathbf{A}$ with eigenvalue 1. But then

$$\mathbf{A}^n\mathbf{v} = (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})^n\mathbf{v} = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}\mathbf{v} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{v} = \mathbf{A}\mathbf{v}.$$

Hence, $\mathbf{A}^n\mathbf{v} = \mathbf{v}$.

Long Question 3

i.

The standard matrix of T is given by

$$\mathbf{T} = \begin{pmatrix} 0 & 2 & -2 \\ 2 & 1 & -3 \\ 2 & 3 & -5 \end{pmatrix}.$$

ii.

$$\mathrm{Ker}(T) = \mathrm{span}\{(1,1,1)\}.$$

iii.

Recall that the range of T is the column space, which is spanned by $(2,1,3)$ and $(2,3,5)$. Taking the cross product of the two vectors, we have that vectors in the range must satisfy $x + y - z = d$. Substituting in $(0,1,1) \in R(T)$ gives us $d = 0$. Finally, the equation is $x + y - z = 0$.

iv.

Noting that $(0,1,1)$ is an eigenvector of $\mathbf{T}$, we may take $U = \mathrm{span}\{(0,1,1)\}$ as a proper subspace of $R(T)$ invariant under T. (It is proper since $(1,0,1) \in R(T)$.)

Long Question 4

(a)

i.

Regardless of the value of y, the row and column space of $\mathbf{B}$ are two-dimensional. ($(2,8)^T$ and $(9,2)^T$ are linearly independent, for example.) Since the ranks of row space and column space of a matrix are the same, rank($\mathbf{B}$) = rank($\mathbf{A}$)= 2.

ii.

By i., we know that the column space of $\mathbf{A}$ is spanned by two vectors. We have in particular

$$2 * (2,1,1,-2)^T - (4,0,4,-3)^T = (0,2,-2,-1)^T$$

and so $x = -2$. Similarly,

$$9 * (2,1,1,-2) - 4 * (4,0,4,-3) = (2,9,-7,-6)$$

and so $y = -6$.

(b)

Consider the column space of $\mathbf{C}$. Since the first and last columns are the same, the maximal possible rank is 3. If x neither -1 nor 4, then

$$\mathbf{C} = \begin{pmatrix} ab & a & b & ab \\ 0 & b & a & 0 \\ 0 & a & b & 0 \\ 0 & ab & 0 & 0 \end{pmatrix}$$

for some nonzero a and b, and $\operatorname{rank}(\mathbf{C}) = 3$ in this case. (Note that the first three columns are linearly independent, and the fourth is identical to the first.)

If x is either -1 or 4, then $\mathbf{C}$ has the second and third columns as linearly independent columns with the first and last columns being zero vectors, so $\operatorname{rank}(\mathbf{C}) = 2$.

Long Question 5

(a)

i.

Let $\mathbf{S} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix}$. Since $\mathbf{A}$ is invertible, we have that

$$\mathbf{AS} = \mathbf{E}_n \cdots \mathbf{E}_1 \mathbf{S},$$

where $\mathbf{E}_k$ are elementary matrices for all $1 \leq k \leq n$. We then see that the row space of $\mathbf{AS}$ is exactly the same as that of $\mathbf{S}$, which implies that $\operatorname{rank}(\mathbf{AS}) = \operatorname{rank}(\mathbf{S}) = 3$, and so T is also a basis for $\mathbb{R}^3$.

ii.

Clearly, if $\mathbf{A} = \mathbf{I}_3$, then T is also an orthonormal basis for $\mathbb{R}^3$.

Suppose now that T is an orthonormal basis. Taking the inner product of the vectors in T, we have for example

$$\begin{aligned} 1 = \langle a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \ , \ a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \rangle &= a^2 \langle \mathbf{u} \ , \ \mathbf{u} \rangle + b^2 \langle \mathbf{v} \ , \ \mathbf{v} \rangle + c^2 \langle \mathbf{w} \ , \ \mathbf{w} \rangle \\ &= a^2 + b^2 + c^2 \\ 0 = \langle a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \ , \ d\mathbf{u} + e\mathbf{v} + i\mathbf{w} \rangle &= ad \langle \mathbf{u} \ , \ \mathbf{u} \rangle + be \langle \mathbf{v} \ , \ \mathbf{v} \rangle + ci \langle \mathbf{w} \ , \ \mathbf{w} \rangle \\ &= ad + be + ci, \end{aligned}$$

since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is orthonormal. The same holds for other combinations of vectors in T. Therefore, one can give $\mathbf{AA}^T = \mathbf{I}_3$, i.e. $\mathbf{A}$ is orthogonal, as a necessary condition.

(b)

i.

Let $\mathbf{P} = \mathbf{E}_1 \cdots \mathbf{E}_n$, where $\mathbf{E}_k$ are "Type II" elementary matrices for all $1 \leq k \leq n$. Since such matrices are symmetric and orthogonal, we have $\mathbf{P}^T = (\mathbf{E}_1 \cdots \mathbf{E}_n)^T = \mathbf{E}_n \cdots \mathbf{E}_1$ and that $\mathbf{P}\mathbf{P}^T = (\mathbf{E}_1 \cdots \mathbf{E}_n)(\mathbf{E}_n \cdots \mathbf{E}_1) = \mathbf{I}_n$.

ii.

Suppose that $\mathbf{I}_n - \mathbf{P}$ is not singular, i.e. it is invertible and has nullity 0. By definition, the nullspace of $\mathbf{I}_n - \mathbf{P}$ being trivial (i.e. having dimension 0) is equivalent to the eigenspace of $\mathbf{P}$ associated with the eigenvalue 1 being trivial. On the other hand, we know that $\mathbf{P}$ always admits $(1, \cdots, 1) \in \mathbb{R}^n$ as an eigenvector with eigenvalue 1, which gives us a contradiction and shows that $\mathbf{I}_n - \mathbf{P}$ is singular.