NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Poh Wei Shan Charlotte

MA3110 Mathematical Analysis II
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Question 1

(a) $\because g^{(2007)}(x_0) = 10 > 0$,
By the continuity of $g^{(2007)}$ on $(x_0 - 1, x_0 + 1)$, $\exists\ 0 < \delta_0 < 1$ such that $g^{(2007)}(u) > 0$, $\forall u \in (x_0 - \delta_0, x_0 + \delta_0) \subset (x_0 - 1, x_0 + 1)$.

By Taylor's Theorem, for $x \in (x_0 - \delta_0, x_0 + \delta_0)$,
$\exists\ c_x$ between x_0 & x (hence $c_x \in (x_0 - \delta_0, x_0 + \delta_0)$) such that

$$\begin{aligned} g(x) &= \sum_{i=0}^{2006} \frac{g^{(i)}(x_0)}{i!}(x - x_0)^i + \frac{g^{(2007)}(c_x)}{2007!}(x - x_0)^{2007} \\ &= g(x_0) + \frac{g^{(2007)}(c_x)}{2007!}(x - x_0)^{2007} \end{aligned}$$

since $g^{(1)}(x_0) = \cdots = g^{(2006)}(x_0) = 0$.

$\therefore$ We have

$$g(x) - g(x_0) = \frac{g^{(2007)}(c_x)}{2007!}(x - x_0)^{2007} \begin{cases} < 0 & \text{if } x \in (x_0 - \delta_0, x_0) \\ > 0 & \text{if } x \in (x_0, x_0 + \delta_0) \end{cases}$$

$\therefore g$ has neither a relative maximum nor a relative minimum at x_0.

(b) $f : I \to \mathbb{R}$ is differentiable at c $\in I$, $f'(c) = L$:
$\forall \varepsilon > 0$, $\exists\ \delta > 0$ such that $\forall x \in I$, $|\frac{f(x)-f(c)}{x-c} - L| < \frac{\varepsilon}{2}$ whenever $0 < |x - c| < \delta$ — $(*)$
$a_n \to c$: $\forall \delta > 0$, $\exists N_a \in \mathbb{N}$ such that $|a_n - c| < \delta$, $\forall n \geq N_a$
$b_n \to c$: $\forall \delta > 0$, $\exists N_b \in \mathbb{N}$ such that $|b_n - c| < \delta$, $\forall n \geq N_b$

Let $N = max(N_a, N_b)$, $\therefore \forall n \geq N$, $|a_n - c| < \delta$ and $|b_n - c| < \delta$, and a_n, $b_n \in I$ which fulfil the sufficient condition of $(*)$. Therefore,

$$|\frac{f(a_n) - f(c)}{a_n - c} - L| < \frac{\varepsilon}{2} \text{ and } |\frac{f(b_n) - f(c)}{b_n - c} - L| < \frac{\varepsilon}{2}$$

At the same time, $a_n < c < b_n \to c - a_n > 0$ and $b_n - c > 0$ $\forall n \in \mathbb{N}$.

$\therefore \forall n \geq N$,

$$\begin{aligned}
|\frac{f(b_n)-f(a_n)}{b_n-a_n}-L| &= |\frac{f(b_n)-f(a_n)-L(b_n-a_n)}{b_n-a_n}| \\
&= |\frac{f(b_n)-f(c)+f(c)-f(a_n)-Lb_n+Lc-Lc+La_n)}{b_n-a_n}| \\
&\leq |\frac{f(b_n)-f(c)-L(b_n-c)}{b_n-a_n}|+|\frac{f(c)-f(a_n)+L(a_n-c)}{b_n-a_n}| \\
&= |\frac{f(b_n)-f(c)-L(b_n-c)}{(b_n-c)+(c-a_n)}|+|\frac{f(a_n)-f(c)-L(a_n-c)}{(b_n-c)+(c-a_n)}| \\
&\leq |\frac{f(b_n)-f(c)-L(b_n-c)}{b_n-c}|+|\frac{f(a_n)-f(c)-L(a_n-c)}{c-a_n}| \\
&= |\frac{f(b_n)-f(c)}{b_n-c}-L|+|\frac{f(a_n)-f(c)}{a_n-c}-L| \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

$\because \varepsilon$ is arbitrary, we have $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|\frac{f(b_n)-f(a_n)}{b_n-a_n} - L| < \varepsilon \ \forall n \geq N$. Therefore,

$$\lim_{n\to\infty} \frac{f(b_n)-f(a_n)}{b_n-a_n} = f'(c)$$

Question 2

(a) (i) When $g(x) \leq f(x) \leq h(x) \ \forall \ x \in [a,b]$, then $\forall$ partitions P of $[a,b]$,

$$U(g,P) \leq U(f,P) \leq U(h,P) \tag{1}$$
$$L(g,P) \leq L(f,P) \leq L(h,P) \tag{2}$$

$\forall \ \varepsilon > 0$, $\exists$ integrable functions g, $h : [a,b] \to \mathbb{R}$ (which may depend on ε) such that $g(x) \leq f(x) \leq h(x) \ \forall \ x \in [a,b]$ and that $\int_a^b h$ - $\int_a^b g < \frac{\varepsilon}{3}$.

By the definition of $U(h)$ and $L(g)$, $\exists$ partitions Q and R of $[a,b]$ such that

$$U(h,Q) < U(h) + \frac{\varepsilon}{3} \text{ and } L(g,R) > L(g) - \frac{\varepsilon}{3}$$

Letting $P = Q \cup R$, we have

$$U(h,P) \leq U(h,Q) < U(h) + \frac{\varepsilon}{3} \text{ and } L(g) \geq L(g,R) > L(g) - \frac{\varepsilon}{3}$$

$\therefore$ By (1) and (2),

$$\begin{aligned}
U(f,P)-L(f,P) &\leq U(h,P)-L(g,P) \\
&< (U(h)+\frac{\varepsilon}{3})-(L(g)-\frac{\varepsilon}{3}) \\
&= \int_a^b h - \int_a^b g + \frac{2\varepsilon}{3} \\
&< \varepsilon
\end{aligned}$$

$\forall \ \varepsilon > 0$, we can find a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.
$\therefore$ By the Riemann Integrability Criterion, f is integrable on $[a,b]$.

(ii) Since ϕ is bounded, let $M > 0$ such that $|\phi(x)| < M$ for all $x \in [a, b]$.
Given any $\varepsilon > 0$, choose $c \in (a, b)$ such that $2(c - a)M < \varepsilon$.
Now, define $h_\varepsilon, g_\varepsilon : [a, b] \to \mathbb{R}$ such that

$$h_\varepsilon(x) = \begin{cases} M & \text{for } x \in [a, c) \\ \phi(x) & \text{for } x \in [c, b] \end{cases}$$

$$g_\varepsilon(x) = \begin{cases} -M & \text{for } x \in [a, c) \\ \phi(x) & \text{for } x \in [c, b] \end{cases}$$

Hence we have $g_\varepsilon(x) \le \phi(x) \le h_\varepsilon(x)$ for all $x \in [a, b]$. Since a piecewise integrable function is integrable, it follows that $h_\varepsilon, g_\varepsilon$ are integrable. Also, since

$$\begin{aligned} \int_a^b h_\varepsilon - \int_a^b g_\varepsilon &= \int_a^c 2M \\ &= 2(c - a)M < \varepsilon \end{aligned}$$

By (2ai), we have $\phi : [a, b] \to \mathbb{R}$ is integrable.

(b) $h(x) \ge 0\ \forall x \in [a, b] \to \forall c \in [a, b]$, $\int_a^b = \int_a^c h + \int_c^b h$ and $\int_a^b h$, $\int_a^c h$, $\int_c^b h \ge 0$
$\therefore \int_a^c h = \int_c^b h = 0\ \forall c \in [a, b]$.
Define $H : [a, b] \to \mathbb{R}$, $H(x) = \int_a^x h$ where $\int_a^a h$ is defined to be 0. $\because h$ is continuous on $[a, b]$, by the Fundamental Theorem of Calculus, H is differentiable at every $c \in [a, b]$ and $H'(c) = h(c)$.
$\because H(c) = \int_a^c h = 0$, $H'(c) = 0 \therefore h(c) = 0$ where $c \in [a, b]$ is arbitrary.
$\therefore h(x) = 0\ \forall x \in [a, b]$.

Question 3

(a) (i) For each $x \in \mathbb{R}$, define $x_n = x + \frac{1}{n}$ and $x_n \to x$.
By the Sequential Criterion on continuity, since g is continuous on $x \in \mathbb{R}$, $g(x_n) \to g(x)$.
$\therefore \{g_n\}$ converges pointwise to g on $\mathbb{R}$.

(ii) $\because g$ is uniformly continuous on $\mathbb{R}$, $\forall \varepsilon > 0$, $\exists\ \delta(\varepsilon) > 0$ such that

$$|g(x) - g(y)| < \varepsilon \text{ whenever } x, y \in \mathbb{R} \text{ and } |x - y| < \delta(\varepsilon) \text{ — } (\triangle)$$

Choose $N(\varepsilon) \in \mathbb{N}$ such that $N > \frac{1}{\delta(\varepsilon)}$.
Then $\forall n \ge N > \frac{1}{\delta(\varepsilon)} \Rightarrow n > \frac{1}{\delta(\varepsilon)}$ and $\forall x \in \mathbb{R}, x + \frac{1}{n} \in \mathbb{R}$ and $|(x + \frac{1}{n}) - x| = |\frac{1}{n}| < \delta(\varepsilon)$.

$$\therefore |g(x + \frac{1}{n}) - g(x)| < \varepsilon \text{ by } (\triangle)$$

Note that $N(\varepsilon)$ depends on $\delta(\varepsilon)$ which depends only on ε.
$\therefore \forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}$ such that $\forall x \in \mathbb{R}$, $|g_n(x) - g(x)| < \varepsilon\ \forall n \ge N(\varepsilon)$.
$\therefore$ By the Cauchy's criterion on sequences of functions, $\{g_n\}$ converges uniformly on $\mathbb{R}$.

(b) (i) For a given $x > 0$, $\frac{1}{1+n^2x} \le \frac{1}{n^2x}\ \forall n \in \mathbb{N}$.
$\because \sum \frac{1}{n^2}$ converges, by the Comparison test, $\sum_{n=1}^{\infty} f_n$ converges for every $x > 0$.

(ii) $\sup_{x \in (0,\infty)} |f_n(x)| \ge |f_n(\frac{1}{n^2})| = \frac{1}{2}$
$\therefore \{f_n\}$ does not converge uniformly to 0 on $(0, \infty)$.
$\therefore \sum_{n=1}^{\infty} f_n$ does not converge uniformly on $(0, \infty)$.

(iii) Fix $r > 0$. Then $|f_n(x)| = \frac{1}{1+n^2x} \leq \frac{1}{n^2r}$
$\because \sum \frac{1}{n^2}$ converges, by the Weierstrass M-test, $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[r, \infty)$.

(iv) For every $n \in \mathbb{N}$, $f_n'(x) = -\frac{n^2}{(1+n^2x)^2}$

Fix $r > 0$. Then $|f_n'(x)| = \frac{n^2}{(1+n^2x)^2} \leq \frac{n^2}{(1+n^2r)^2} \leq \frac{n^2}{n^4r^2} = \frac{1}{n^2r^2}$
$\because \sum \frac{1}{n^2}$ converges, by the Weierstrass M-test, $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[r, \infty)$ — ($\blacktriangle$)

By the theorem on differentiation of series of functions, we have
$\{f_n\}$ is a sequence of differentiable functions on $[r, \infty)$ such that $\sum_{n=1}^{\infty} f_n$ and $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[r, \infty)$ by (iii) and ($\blacktriangle$).
$\therefore$ $f(x)$ is differentiable on $[r, \infty)$ and $\forall x \in [r, \infty)$,

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} f_n'(x) \\ &= -\sum_{n-1}^{\infty} \frac{n^2}{(1+n^2x)^2} \end{aligned} \tag{3}$$

$\because$ (3) is valid for every $r > 0$, it is valid for every $x \in (0, \infty)$.

Question 4

(a) Let $a_n = 3^{-n}(5 + (-1)^n)^n$.

$$|a_n|^{\frac{1}{n}} = \frac{5 + (-1)^n}{3} = \begin{cases} 2 & \text{if } n \text{ is even} \\ \frac{4}{3} & \text{if } n \text{ is odd} \end{cases}$$

Therefore, the radius of convergence of the power series is

$$\begin{aligned} R &= \frac{1}{\overline{\lim}_{n\to\infty} |a_n|^{\frac{1}{n}}} \\ &= \frac{1}{2} \end{aligned}$$

$\therefore$ The series converges for x such that $|x + 2| < \frac{1}{2}$.
For $x + 2 = -\frac{1}{2}$, the series become $\sum_{n=1}^{\infty} (-1)^n \left(\frac{5+(-1)^n}{6}\right)^n$.
The even terms of the series is 1, hence the series do not converge by the nth-term divergence test.
Similarly, the series do not converge for $x + 2 = \frac{1}{2}$.
$\therefore$ The series converges on $(-\frac{5}{2}, -\frac{3}{2})$.

(b) Let $a_n = \frac{1}{n+1} \neq 0 \; \forall n \in \mathbb{N}$.

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{n+1}{n+2}\right| \\ &= 1 \\ \therefore \text{Radius of convergence} &= \frac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|} \\ &= 1 \end{aligned}$$

$\therefore$ The series converges on (-1,1).
Then,

$$\begin{aligned} S(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n+1} \\ xS(x) &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \\ \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \qquad (4) \end{aligned}$$

Integrating both sides of (4) from 0 to x, we have

$$\begin{aligned} \int_0^x \frac{1}{1-t} dt &= \sum_{n=0}^{\infty} \int_0^x t^n dt \\ -\ln(1-x) &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \\ &= xS(x) \\ S(x) &= -\frac{\ln(1-x)}{x} \end{aligned}$$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges by the alternating series test and $\lim_{x\to -1^+} S(x) = S(-1) = \ln 2 \because S$ is continuous at -1.
$\therefore$ By the Abel's Theorem,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} &= \lim_{x\to -1^+} S(x) \\ &= \ln 2 \end{aligned}$$

(c) The power series representation of $\sin x$ about 0 is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$\therefore$ With the uniqueness of power series representation,

$$\begin{aligned} h(x) &= \sin\,(x^3) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!} \end{aligned}$$

This is the Maclaurin series representation of h. Therefore,

$$\begin{aligned} \frac{h^{(2007)}(0)}{2007!} &= \text{coefficient of } x^{2007} \\ &= \frac{(-1)^{334}}{669!} \ (\because 2007 = 6(334)+3) \\ \frac{h^{(2008)}(0)}{2008!} &= \text{coefficient of } x^{2008} \\ &= 0 \end{aligned}$$

$\therefore h^{(2007)}(0) = \frac{2007!}{669!}$ and $h^{(2008)}(0) = 0$.