NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Zhuang Linjie

MA3111 Complex Analysis I
AY 2008/2009 Sem 2

Question 1

(a) Let $z = x + iy$.

$$\begin{aligned} f(z) &= (Rez)^3 + \overline{z}^2 - 8i\overline{z} \\ &= x^3 + (x-iy)^2 - 8i(x-iy) \\ &= x^3 + x^2 - 2ixy - y^2 - 8ix - 8y \\ &= x^3 + x^2 - y^2 - 8y + i(-2xy - 8x). \end{aligned}$$

$u(x,y) = x^3 + x^2 - y^2 - 8y, v(x,y) = -2xy - 8x$.
u_x, u_y, v_x, v_y are continuous on $\mathbb{C}$.Solve the CR equations to find points where f is differentiable: $u_x = v_y, u_y = -v_x \Leftrightarrow 3x^2 + 2x = -2x, -2y - 8 = -(-2y-8) \Leftrightarrow x = 0$ or $\frac{-4}{3}, y = -4 \Leftrightarrow z = -4i$ or $\frac{-4}{3} - 4i$.
At $z = -4i$,

$$\begin{aligned} f'(-4i) &= u_x + iv_x \\ &= 3x^2 + 2x + i(-2y-8)|_{x=0,y=-4} \\ &= 0. \end{aligned}$$

At $z = \frac{-4}{3} - 4i$,

$$\begin{aligned} f'(\frac{-4}{3} - 4i) &= u_x + iv_x \\ &= 3x^2 + 2x + i(-2y-8)|_{x=\frac{-4}{3},y=-4} \\ &= \frac{8}{3}. \end{aligned}$$

(b)

$$\begin{aligned} (e^{iz} - e^{3iz})\cos z &= 4e^{2iz} - 2 \\ \Leftrightarrow (e^{iz} - e^{3iz})\frac{(e^{iz} + e^{-iz})}{2} &= 4e^{2iz} - 2 \\ \Leftrightarrow e^{2iz} + 1 - e^{4iz} - e^{2iz} &= 8e^{2iz} - 4 \\ \Leftrightarrow e^{4iz} + 8e^{2iz} - 5 &= 0 \\ \Leftrightarrow e^{2iz} &= \frac{-8 \pm \sqrt{8^2 - 4 \cdot 1 \cdot (-5)}}{2} = -4 \pm \sqrt{21} \end{aligned}$$

Case 1,

$$\begin{aligned} z &= \frac{1}{2i}\log(-4+\sqrt{21}) \\ &= \frac{1}{2i}(\ln|-4+\sqrt{21}| + i\arg(-4+\sqrt{21})) \\ &= \frac{1}{2i}(\ln(-4+\sqrt{21}) + i(0 + 2n\pi)) \\ &= n\pi - \frac{i}{2}\ln(-4+\sqrt{21}), n \in \mathbb{Z}. \end{aligned}$$

Case 2,

$$\begin{aligned} z &= \frac{1}{2i}\log(-4-\sqrt{21}) \\ &= \frac{1}{2i}(\ln|-4-\sqrt{21}|+i\arg(-4-\sqrt{21})) \\ &= \frac{1}{2i}(\ln(4+\sqrt{21})+i(\pi+2n\pi)) \\ &= \frac{\pi}{2}+n\pi-\frac{i}{2}\ln(4+\sqrt{21}), n\in\mathbb{Z}. \end{aligned}$$

The solutions are $z = n\pi - \frac{i}{2}\ln(-4+\sqrt{21})$, or $z = \frac{\pi}{2}+n\pi-\frac{i}{2}\ln(4+\sqrt{21}), n\in\mathbb{Z}$.

Question 2

(a) $f(x,y) = u(x,y)+iv(x,y)$ is an entire function. u is harmonic in $\mathbb{C}$. v is a harmonic conjugate of u.

$$u_x = v_y \Rightarrow v_y = 12x^2y-4y^3+e^y\cos x \Rightarrow v = 6x^2y^2-y^4+e^y\cos x+g_1(x).$$

$$u_y = -v_x \Rightarrow v_x = -4x^3+12xy^2-e^y\sin x \Rightarrow v = -x^4+6x^2y^2+e^y\cos x+g_2(y)$$

Therefore, $v = -x^4+6x^2y^2-y^4+e^y\cos x+c$, for $c\in\mathbb{C}$.
$f(x,y) = 4x^3y-4xy^3+e^y\sin x+i(-x^4+6x^2y^2-y^4+e^y\cos x), x,y\in\mathbb{R}$.

(b) Since $|f(z)+e^z| > |e^z f(z)| \geq 0$, $f(z)+e^z \neq 0, \forall z\in\mathbb{C}$. Consider $g(z) = \frac{e^z f(z)}{f(z)+e^z}$. e^z and $f(z)$ are both entire functions and $f(z)+e^z\neq 0, \forall z\in\mathbb{C}$,so $g(z)$ is also a entire function.

$$|g(z)| = |\frac{e^z f(z)}{f(z)+e^z}| < 1.$$

$g(z)$ is bounded. $g(z)$ is a constant function by Liouville's Theorem. $\exists\alpha\in\mathbb{C}, s.t \frac{e^z f(z)}{f(z)+e^z} \equiv \alpha, \forall z\in\mathbb{C}$.
$f(z) = \frac{e^z\alpha}{e^z-\alpha}.e^z \neq \alpha, \forall z\in\mathbb{C} \Rightarrow \alpha = 0 \Rightarrow f(z) = 0$. f is a constant function.

Question 3

(a) For $-\frac{\pi}{2}\leqslant t\leqslant\frac{\pi}{2}, \gamma(t) = i+e^{it} = i+\cos t+i\sin t = \cos t+i(1+\sin t)$.
$\gamma'(t) = -\sin t+i\cos t$.

$$\int_\gamma[\frac{1}{\overline{z}+i}+\pi\sinh(\frac{\pi z}{4})]dz = \int_\gamma\frac{1}{\overline{z}+i}dz+\int_\gamma\pi\sinh(\frac{\pi z}{4})dz$$

$$\begin{aligned} \int_\gamma\frac{1}{\overline{z}+i}dz &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\cos t-i(1+\sin t)+i}(-\sin t+i\cos t)dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{-\sin t+i\cos t}{\cos t-i\sin t}dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{ie^{it}}{e^{-it}}dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}ie^{2it}dt \\ &= \frac{1}{2}e^{2it}|_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} = \frac{1}{2}e^{i\pi}-\frac{1}{2}e^{-i\pi} = 0. \end{aligned}$$

$$\begin{aligned}\int_\gamma \pi\sinh(\frac{\pi z}{4})dz &= \int_\gamma \pi\frac{e^{\frac{\pi z}{4}}-e^{-\frac{\pi z}{4}}}{2}dz\\ &= \frac{\pi}{2}\int_\gamma e^{\frac{\pi z}{4}}dz+\frac{\pi}{2}\int_\gamma -e^{\frac{-\pi z}{4}}dz\\ &= 2e^{\frac{\pi z}{4}}|_{z=0}^{z=2i}+2e^{\frac{-\pi z}{4}}|_{z=0}^{z=2i}=-4.\end{aligned}$$

$$\int_\gamma[\frac{1}{\overline{z}+i}+\pi\sinh(\frac{\pi z}{4})]dz=-4.$$

(b)

$$P.V.\int_{-\infty}^{\infty}\frac{\cos(6x+5)}{4x^2-4x+17}dx = \lim_{R\to+\infty}\int_{-R}^{R}\frac{\cos(6x+5)}{4x^2-4x+17}dx$$

Let $f(z)=\frac{e^{i(6z+5)}}{4z^2-4z+17}$. The function $f(z)=\frac{e^{i(6z+5)}}{4z^2-4z+17}$ has singular points at $4z^2-4z+17=0\Leftrightarrow z=\frac{1}{2}+2i,\frac{1}{2}-2i$.
For $R>|\frac{1}{2}+2i|$, consider the semi-circular C_R,where $C_R(t)=Re^{it},0\leqslant t\leqslant\pi$.
By Cauchy's Residue Theorem,

$$\int_{[-R,R]}f(x)dx+\int_{C_R}f(z)dz = 2\pi i Res_{z=\frac{1}{2}+2i}f(z).$$

$f(z)=\frac{e^{i(6z+5)}}{4z^2-4z+17}=\frac{p(z)}{q(z)}$, where $p(z)=e^{i(6z+5)},q(z)=4z^2-4z+17$ are both analytic at $z=\frac{1}{2}+2i.q'(z)=8z-4$.
$q(\frac{1}{2}+2i)=0,q'(\frac{1}{2}+2i)=16i\neq0$
$Res_{z=\frac{1}{2}+2i}f(z)=\frac{p(\frac{1}{2}+2i)}{q'(\frac{1}{2}+2i)}=\frac{e^{-12+8i}}{16i}$

$$\int_{[-R,R]}f(x)dx+\int_{C_R}f(z)dz = 2\pi i\frac{e^{-12+8i}}{16i}=\frac{\pi e^{-12+8i}}{8}.$$

Apply ML-inequality to $\int_{C_R}f(z)dz$, $L=\frac{1}{2}2\pi R=\pi R$.
For $z=x+iy\in C_R$

$$|f(z)|=|\frac{e^{i(6z+5)}}{4z^2-4z+17}|=|\frac{e^{6y}}{4z^2-4z+17}|\leqslant\frac{e^{6y}}{|4z^2|-|4z|-|17|}\leqslant\frac{1}{4R^2-4R-17}=M$$
$$0\leqslant|\int_{C_R}f(z)dz|\leqslant ML=\frac{\pi R}{4R^2-4R-17}\to0 \text{ as } R\to\infty.$$

$\lim_{R\to+\infty}|\int_{C_R}f(z)dz|=0\Rightarrow\lim_{R\to+\infty}\int_{C_R}f(z)dz=0.$

$$\lim_{R\to+\infty}\int_{-R}^{R}f(x)dx+\lim_{R\to+\infty}\int_{C_R}f(z)dz = \frac{\pi e^{-12+8i}}{8}$$
$$\lim_{R\to+\infty}\int_{-R}^{R}\frac{\cos(6x+5)+i\sin(6x+5)}{4x^2-4x+17}dx=\frac{\pi}{8}e^{-12}(\cos8+i\sin8)$$

$$P.V.\int_{-\infty}^{\infty}\frac{\cos(6x+5)}{4x^2-4x+17}dx = \lim_{R\to+\infty}\int_{-R}^{R}\frac{\cos(6x+5)}{4x^2-4x+17}dx=\frac{\pi}{8}e^{-12}\cos8.$$

Question 4

(a)

$$f(z) = \frac{13z}{(4z+1)(z-3)} = \frac{1}{4z+1} + \frac{3}{z-3}.$$

$$\frac{1}{4z+1} = \frac{1}{4z} \cdot \frac{1}{1+\frac{1}{4z}} = \frac{1}{4z}\sum_{n=0}^{\infty}(-1)^n(\frac{1}{4z})^n, (|\frac{1}{4z}| < 1).$$

$$\frac{3}{z-3} = (-1)\frac{1}{1-\frac{z}{3}} = (-1)\sum_{n=0}^{\infty}(\frac{z}{3})^n, (|\frac{z}{3}| < 1).$$

The Laurent series of $f(z)$ for the annular domain $\frac{1}{4} < |z| < 3$ is $f(z) = \sum_{n=0}^{\infty}(-1)^n(\frac{1}{4z})^{n+1} - \sum_{n=0}^{\infty}(\frac{z}{3})^n$.

(b)

$$\int_\gamma \frac{1}{z^3(4z+1)(z-3)}dz = \int_\gamma \frac{\frac{13z}{(4z+1)(z-3)}}{13z^4}dz = \frac{1}{13}\cdot 2\pi i \cdot a_3 = \frac{1}{13}\cdot 2\pi i \cdot \frac{-1}{3^3} = \frac{-2\pi i}{351}.$$

(c) Let $u = (z-1)^2$, $\frac{1}{2} < |z-1| < \sqrt{3} \Rightarrow \frac{1}{4} < |u| < 3$

$$\begin{aligned}\frac{z-1}{(4z^2-8z+5)(z^2-2z-2)} &= \frac{z-1}{[4(z-1)^2+1][(z-1)^2-3]} = \frac{z-1}{(4u+1)(u-3)} \\ &= \frac{1}{13(z-1)}\cdot\frac{13u}{(4u+1)(u-3)}\end{aligned}$$

By part(i)

$$\begin{aligned}\frac{z-1}{(4z^2-8z+5)(z^2-2z-2)} &= \frac{1}{13(z-1)}\{\sum_{n=0}^{\infty}(-1)^n(\frac{1}{4u})^{n+1} - \sum_{n=0}^{\infty}(\frac{u}{3})^n\} \\ &= \frac{1}{13(z-1)}\{\sum_{n=0}^{\infty}(-1)^n(\frac{1}{4(z-1)^2})^{n+1} - \sum_{n=0}^{\infty}(\frac{(z-1)^2}{3})^n\} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{13\cdot 4^{n+1}(z-1)^{2n+3}} - \sum_{n=0}^{\infty}\frac{(z-1)^{2n-1}}{3^n\cdot 13}.\end{aligned}$$

Question 5

(a) For any z inside and on γ, consider

$$\begin{aligned}\mathrm{Re}(e^z - \mathrm{Log}z) &= e^x\cos y - \ln|x^2+y^2|, (x\in[2,3], y\in[0,1]) \\ &\geqslant e^2\cos 1 - \ln|3^2+1| \\ &> 0\end{aligned}$$

$e^z - \mathrm{Log}(z)$ is analytic on $\mathbb{C}\backslash(-\infty, 0]$. $\Rightarrow \frac{1}{e^z - \mathrm{Log}z}$ is analytic inside and on γ.
By Cauchy-Goursat Theorem, $\int_\gamma \frac{1}{e^z-\mathrm{Log}z}dz = 0$.

(b) f is analytic in $\mathbb{C}\backslash\{1\}$, therefore f is analytic in $B(0,1)$.
By Taylor's Theorem, f can be written as

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\times z^n\forall z\in B(0,1).$$

f has a simple pole at z = 1, so there exists an entire function g(z) such that

$$\begin{aligned} g(z) &= (z-1)\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\times z^n \\ &= -f(0)+\sum_{n=1}^{\infty}(\frac{f^{(n-1)}(0)}{(n-1)!}-\frac{f^{(n)}(0)}{n!})\times z^n. \end{aligned}$$

When $z=1$,

$$\begin{aligned} g(1) &= -f(0)+\sum_{n=1}^{\infty}(\frac{f^{(n-1)}(0)}{(n-1)!}-\frac{f^{(n)}(0)}{n!})\times 1^n \\ &= \lim_{n\to\infty}[-f(0)+(f(0)-\frac{f^{(1)}(0)}{1!})+\cdots(\frac{f^{(n-1)}(0)}{(n-1)!}-\frac{f^{(n)}(0)}{n!})] \\ &= -\lim_{n\to\infty}\frac{f^{(n)}(0)}{n!} \end{aligned}$$

The limit exists. $\lim_{n\to\infty}\frac{f^{(n)}(0)}{n!}=-g(1)$.

Question 6

(a) $f(z)=(z-1)\exp(\frac{z}{z-2})$ has a singular point at $z=2$ which lies inside the circle $|z|=3$.
By Cauchy's Residue Theorem, $\int_\gamma f(z)dz=2\pi iRes_{z=2}f(z)$.

$$\begin{aligned} f(z) &= (z-1)\exp(\frac{z}{z-2})=[(z-2)+1]\exp(\frac{z-2+2}{z-2})=[(z-2)+1]\exp(1+\frac{2}{z-2}) \\ &= e[(z-2)+1]\exp(\frac{2}{z-2})=e[(z-2)+1]\sum_{n=0}^{\infty}\frac{1}{n!}(\frac{2}{z-2})^n \\ &= e[(z-2)+1](1+\frac{2}{z-2}+\frac{1}{2}(\frac{2}{z-2})^2+\cdots) \\ &= \cdots(2e+2e)\frac{2}{z-2}\cdots \end{aligned}$$

$\int_\gamma f(z)dz=2\pi iRes_{z=2}f(z)=8e\pi i$.

(b) f is analytic in D, then $f'(z)$ is also analytic in D.
$|f'(z)|=|(f'(z)-f'(0))+f'(0)|\leqslant|f'(z)-f'(0)|+|f'(0)|<2|f'(0)|$.
$f'(z)$ is analytic and bounded in $D\Rightarrow f'(z)$ is constant in D.
$|f'(0)|>|f'(z)-f'(0)|\geqslant 0\Rightarrow f'(z)=f'(0)\neq 0$
$f(z)=cz+c'$ for some $c\neq 0, c'\in\mathbb{C}$.
Suppose $z_1,z_2\in D, z_1\neq z_2, f(z_1)-f(z_2)=c(z_1-z_2)\neq 0\Rightarrow f(z_1)\neq f(z_2)$.

Question 7

(a) Let $p(z) = z^2(e^z - 1)$ and $q(z) = \sin^3 z$. Both p and q are analytic on D.$\sin^3 z = 0$ iff $z = n\pi, n \in \mathbb{Z}$. $z \in D, q(z) = 0 \Rightarrow z = 0.\frac{p(z)}{q(z)}$ is analytic on $D\backslash\{0\}$. By Laurent's Theorem,

$$\frac{p(z)}{q(z)} = \sum_{n=1}^{\infty} \frac{b_n}{z^n} + \sum_{n=0}^{\infty} a_n z^n, z \in D\backslash\{0\}.$$

p(z) has a zero of order 3 at $z = 0$. q(z) has a zero of order 3 at $z = 0$. Therefore, $\frac{p(z)}{q(z)}$ has a removable singular point at $z = 0$. Thus,

$$\frac{p(z)}{q(z)} = \sum_{n=0}^{\infty} a_n z^n = f(z), z \in D\backslash\{0\}$$

$$z = 0, \frac{p(z)}{q(z)} = \sum_{n=0}^{\infty} a_n z^n = a_0.$$

$$f(z) = \frac{z^2(e^z - 1)}{\sin^3 z}, \forall z \in D\backslash\{0\}$$

and f is analytic on D.

(b) Let $g(z) = e^{F(z)}$. F is entire, therefore g is also entire. For $R > 2$, let C_R be the circle $|z| = R$. If z is inside the circle,

$$|g(z)| = |e^{F(z)}| = e^{Re[F(z)]} \leqslant e^{4|\text{Log}z|} \leqslant e^{4|\ln|z| + i\text{Arg}z|} \leqslant e^{4|\ln R| + 4\pi} = e^{4\ln R + 4\pi} = e^{4\pi} R^4.$$

To show $|g(z)|$ is bounded, (for each $n \geqslant 5$,) by Cauchy's inequality,

$$|g^{(n)}(0)| \leqslant \frac{n! M_R}{R^n} \leqslant \frac{n! e^{4\pi} R^4}{R^n} = \frac{n! e^{4\pi}}{R^{n-4}} \to 0 \text{ as } R \to \infty.$$

As $R \to \infty, g^{(n)}(0)$ for all $n \geqslant 5$. By Taylor's Theorem,

$$g(z) = \sum_{n=0}^{4} \frac{g^{(n)}(0)}{n!} z^n, \forall z \in \mathbb{C}.$$

g is a polynomial of z. If $\deg(g) \geqslant 1$, $g(z)$ has a solution in $\mathbb{C}$.
However,$g(z) = e^{F(z)} \neq 0 \forall z \in \mathbb{C}$. Therefore, $\deg(g) = 0$, $g(z) = e^{F(z)}$ is a constant function $\Rightarrow$ F is a constant function.