NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Zheng Shaoxuan

MA2214 Combinatorial Analysis
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Question 1

(a) Fix the person A at a position and consider the arrangements of the remaining 15 people around the circular table relative to person A. The total number of arrangements under the given condition is now equivalent to the total number of ways to arrange the remaining 15 people in a straight line, such that there is at least 3 people between B and C, at least 1 person on the other side of B and at least 2 people on the other side of C.

In this new stated problem, B can be either to the left or to the right of C. By symmetry the number of ways in either cases is the same and hence only the case where B is to the left of C shall be considered. With 6 other people assigned to their roles in their respective positions, there remains 7 free people, to be arranged with B and C. The number of ways to do this latter arrangement is $\binom{9}{2}$, if the people in concern were identical.

Since the 13 other people in concern are distinct, their positions may be permuted among themselves. Hence, by considering both cases of B to the left and right of C, the total number of ways to perform the arrangement is

$$2 \times \binom{9}{2} \times 13!.$$

(b) (i) Divide the 30 people into 3 distinct groups of 10 people each (for the front, middle and back row). The number of ways to do that, by considering the 30 people standing in a row and allocating the first 10 to group 1 (of which the order of the 10 in group 1 does not matter), the second 10 to group 2 and the last 10 to group 2, is $\frac{30!}{(10!)^3}$.

Note that for each chosen row of 10 people, there is exactly 1 way to arrange the people in each row in ascending order of height from left to right. Hence, the total number of ways to perform the mentioned arrangement is simply

$$\frac{30!}{(10!)^3}.$$

(ii) Divide the 30 people into 10 distinct groups of 3 people each (for each of the 10 columns of the block of people). The number of ways to do that, by considering the 30 people standing in a row, allocating the first 3 to group 1 (of which their ordering in the group does not matter), the second 3 to group 2 and so on, is $\frac{30!}{(3!)^{10}}$.

Note that for each chosen column of 3 people, there is exactly 1 way to arrange the people in each column such that the person in the middle is taller than the one in front and shorter than the one behind. Hence the total number of ways to perform the mentioned arrangement is simply

$$\frac{30!}{(3!)^{10}}.$$

Question 2

(a) (i) Each of the 15 distinct sweets has 5 ways to be allocated. Hence the total number of ways is

$$5^{15}.$$

(ii) The total number of ways is $F(15,5)$, which is the 5! times the Stirling number of the second kind. This is given by

$$\sum_{k=0}^{5}(-1)^k\binom{5}{k}(5-k)^{15}.$$

(iii) Divide the enumeration into 9 disjoint cases, where case k represents the condition that the first child is given exactly $k+2$ sweets, where $1 \leq k \leq 9$. For each case k, the number of ways to choose the $k+2$ sweets that go to the first child is $\binom{15}{k+2}$ and the number of ways to assign the remaining $13-k$ sweets to the 4 children is $F(13-k,4)$. Hence, by adding up the number of ways in each of the 9 cases, the total number of ways is

$$\begin{aligned}
&\sum_{k=1}^{9}\binom{15}{k+2}F(13-k,4)\\
= &\sum_{k=1}^{9}\left(\binom{15}{k+2}\sum_{j=0}^{4}(-1)^j\binom{4}{j}(4-j)^{13-k}\right).
\end{aligned}$$

(b) (i) This is akin to arranging 20 identical objects with the 4 barriers between the boxes, and the total number of ways is given by

$$\binom{24}{4}.$$

(ii) Given that each box must contain at least one object, there are 15 free objects left to be arranged with the 4 barriers between the boxes. The total number of ways to do this is given by

$$\binom{19}{4}.$$

(iii) Given the new condition, there are 13 free objects left to be arranged with the 4 barriers between the boxes. The total number of ways to do this is

$$\binom{17}{4}.$$

Question 3

(a) The characteristic equation for the homogenous equation of a_n is

$$\begin{aligned}
x^3-3x^2+4 &= 0\\
(x+1)(x^2-4x+4) &= 0\\
(x+1)(x-2)^2 &= 0.
\end{aligned}$$

Therefore, the homogenous equation of a_n is

$$a_n^{(h)} = A(-1)^n + (Bn+C)2^n.$$

where A, B and C are constants to be determined.

Let the particular solution $a_n^{(p)}$ be $Dn+E$, where D and E are constants to be determined. By substituting the particular solution into the recurrence relation,

$$\begin{aligned}(Dn+E)-3(D(n-1)+E)+4(D(n-3)+E) &= 2n-7\\ n(D-3D+4D)+(E+3D-3E-12D+4E) &= 2n-7\\ n(2D)+(-9D+2E) &= 2n-7.\end{aligned}$$

By comparing coefficients of n,

$$D=1.$$

By comparing the constant term,

$$E=1.$$

Therefore,

$$a_n^{(p)}=n+1.$$

Hence,

$$\begin{aligned}a_n &= a_n^{(h)}+a_n^{(p)}\\ &= A(-1)^n+(Bn+C)2^n+n+1.\end{aligned}$$

By substituting the three given conditions $a_1=0$, $a_2=7$ and $a_3=18$,

$$\left\{\begin{array}{rcrcrcr} -A & + & 2B & + & 2C & = & -2;\\ A & + & 8B & + & 4C & = & 4;\\ -A & + & 24B & + & 8C & = & 14.\end{array}\right.$$

By solving the above system of equations, we obtain $A=\frac{10}{9}$, $B=\frac{7}{6}$, $C=-\frac{29}{18}$. Therefore,

$$a_n=\frac{10}{9}(-1)^n+\left(\frac{7}{6}n-\frac{29}{18}\right)2^n+n+1.$$

(b) (i) Of the numbers counted in a_n, let b_n be the number of those that ends with 1. By symmetry of the given conditions, b_n also represents the number of those that ends with 2, and also represents the number of those that ends with 3.

Constructing 2 recurrence relations involving a_n and b_n,

$$\begin{aligned}a_n &= a_{n-1}+3b_n; &(1)\\ b_n &= a_{n-2}+2b_{n-1}. &(2)\end{aligned}$$

The RHS of (1) represents the quantity of such numbers counted in a_n which ends with 4 (a_{n-1}) and which ends with 1, 2 or 3 ($3b_n$). The RHS of (2) represents the quantity of such numbers counted in a_n which ends with 41 (a_{n-2}) and which ends with 21 or 31 ($2b_{n-1}$).

From (1),

$$b_n=\frac{1}{3}a_n-\frac{1}{3}a_{n-1}. \qquad (3)$$

Substituting (3) into (2),

$$\begin{aligned}\frac{1}{3}a_n-\frac{1}{3}a_{n-1} &= a_{n-2}+2\left(\frac{1}{3}a_{n-1}-\frac{1}{3}a_{n-2}\right)\\ \frac{1}{3}a_n &= a_{n-1}+\frac{1}{3}a_{n-2}\\ a_n-3a_{n-1}-a_{n-2} &= 0.\end{aligned}$$

$a_1=4$ since a single digit can be either 1, 2, 3, 4. For a_2, we count all permutations of 1, 2, 3 and 4 apart from 11, 22 and 33, hence $a_2=4\times 4-3=13$.

(ii) The characteristic equation for a_n is

$$\begin{aligned} x^2 - 3x - 1 &= 0 \\ x &= \frac{3 \pm \sqrt{13}}{2}. \end{aligned}$$

Hence,

$$a_n = A\left(\frac{3+\sqrt{13}}{2}\right)^n + B\left(\frac{3-\sqrt{13}}{2}\right)^n.$$

where A and B are constants to be determined.
By substituting the two conditions $a_1 = 4$ and $a_2 = 13$ into the above equation,

$$\begin{cases} 8 = A(3+\sqrt{13}) + B(3-\sqrt{13}); \\ 52 = A(3+\sqrt{13})^2 + B(3-\sqrt{13})^2. \end{cases}$$

By solving the tedious simultaneous equations, $A = \frac{1}{2} + \frac{5}{26}\sqrt{13}$ and $B = \frac{1}{2} - \frac{5}{26}\sqrt{13}$. Therefore,

$$a_n = \left(\frac{1}{2} + \frac{5}{26}\sqrt{13}\right)\left(\frac{3+\sqrt{13}}{2}\right)^n + \left(\frac{1}{2} - \frac{5}{26}\sqrt{13}\right)\left(\frac{3-\sqrt{13}}{2}\right)^n.$$

Question 4

(a) This question shall be solved using the Principle of Inclusion and Exclusion.

Let

- S be the set of all possible 12-letter words formed by the 12 letters 4 a's, 4 b's and 4 c's;
- P_i be the property that the (i)-th, $(i+1)$-th and $(i+2)$-th letters are identical, for $1 \le i \le 10$;
- $E(m)$ be the number of elements of S possessing exactly m of the 10 properties for $0 \le m \le 10$;
- $\omega(P_{i_1}P_{i_2}\ldots P_{i_m})$ be the number of elements of S possessing the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_m}$, where $1 \le m \le 10$;
- $\omega(m) = \sum(\omega(P_{i_1}P_{i_2}\ldots P_{i_m}))$, $\omega(0) = |S|$.

$\omega(0)$ is simply the number of ways to arrange the 12 above mentioned letters in a row. Hence $\omega(0) = \frac{12!}{4!4!4!} = 34650$.

$\omega(1)$, by definition, is the sum of $\binom{10}{1}$ cases where for each case, at a different position, the 3 consecutive letters are identical. There are 3 ways to choose which letter is being referred to. Among the 9 other letters, there are $\frac{9!}{4!4!}$ ways to arrange them as there are 4 of one letter, 4 of another letter, and 1 of the third letter remaining. Hence, $\omega(1) = \binom{10}{1} \times 3 \times \frac{9!}{4!4!} = 18900$.

$\omega(2)$ is the sum of two different scenarios: the $\binom{9}{1}$ cases where for each case at a different position, the 4 consecutive letters are identical, and the $\binom{8}{2}$ cases where for each case at different positions, 2 groups of 3 consecutive letters are identical within their groups. For the former scenario, there are 3 ways to choose the consecutive letter, and $\frac{8!}{4!4!}$ ways to arrange the remaining 8 letters. For the latter scenario, there are 3×2 ways to choose the 2 groups of consecutive letters, and $\frac{6!}{4!}$ ways to arrange the remaining 6 letters. Hence, $\omega(2) = \binom{9}{1} \times 3 \times \frac{8!}{4!4!} + \binom{8}{2} \times 3 \times 2 \times \frac{6!}{4!} = 6930$.

$\omega(3)$ is the sum of two different scenarios: the $\binom{7}{1} \times \binom{6}{1}$ cases where for each case at different positions, a group of 4 consecutive letters are identical and a group of 3 consecutive letters are identical, and the $\binom{6}{3}$ cases where for each case at different positions, 3 groups of 3 consecutive letters are identical within their groups. For the former scenario, there are 3×2 ways to choose the 2 groups of consecutive letters, and $\frac{5!}{4!}$ ways to arrange the remaining 5 letters. For the latter scenario, there are 3×2 ways to choose the 3 groups of consecutive letters, and 3! ways to arrange the remaining 3 letters. Hence, $\omega(3) = \binom{7}{1} \times \binom{6}{1} \times 3 \times 2 \times \frac{5!}{4!} + \binom{6}{3} \times 3 \times 2 \times 3! = 1980$.

$\omega(4)$ is the sum of two different scenarios: the $\binom{6}{2}$ cases where for each case at different positions, 2 group of 4 consecutive letters are identical within their groups, and the $\binom{5}{1} \times \binom{4}{2}$ cases where for each case at different positions, a group of 4 consecutive letters are identical and 2 groups of 3 consecutive letters are identical within their groups. For the former scenario, there are 3×2 ways to choose the 2 groups of consecutive letter, and $\frac{4!}{4!}$ ways to arrange the remaining 4 letters. For the latter scenario, there are 3×2 ways to choose the 3 groups of consecutive letters, and 2! ways to arrange the remaining 2 letters. Hence, $\omega(4) = \binom{6}{2} \times 3 \times 2 \times \frac{4!}{4!} + \binom{5}{1} \times \binom{4}{2} \times 3 \times 2 \times 2! = 450$.

$\omega(5)$ is the sum of $\binom{4}{2} \times \binom{2}{1}$ cases where for each case at different positions, 2 groups of 4 consecutive letters are identical within their groups and a group of 3 consecutive letters are identical. There are 3×2 ways to choose the 3 groups of consecutive letters, and 1 way to arrange the remaining letter. Hence, $\omega(5) = \binom{4}{2} \times \binom{2}{1} \times 3 \times 2 = 72$.

$\omega(6)$ is the sum of the cases where the 3 types of letters are consecutive within themselves. Hence, $\omega(6) = 3 \times 2 = 6$.

$\omega(7) = \omega(8) = \omega(9) = \omega(10) = 0$.

$E(0)$ is the desired solution where no three consecutive letters are identical. Therefore, by the Principle of Inclusion and Exclusion,

$$\begin{aligned} E(0) &= \omega(0) - \omega(1) + \omega(2) - \omega(3) + \omega(4) - \omega(5) + \omega(6) \\ &= 21084. \end{aligned}$$

(b) The total number of ways is $S(12, 4)$, which is the Stirling number of the second kind. This is given by

$$\frac{1}{4!} \sum_{k=0}^{4} (-1)^k \binom{4}{k} (4-k)^{12}.$$

Question 5

(a) (i) A suitable exponential generating function for a_n is

$$\begin{aligned} & \left(\frac{e^{6x} + e^{-6x}}{2}\right)\left(e^x - 1 - x - \frac{x^2}{2}\right) \\ = \; & \frac{1}{2}(e^{7x} + e^{-5x} - e^{6x} - e^{-6x}) - \frac{1}{2}x(e^{6x} + e^{-6x}) - \frac{1}{4}x^2(e^{6x} + e^{-6x}). \end{aligned}$$

(ii) Note that for any positive integer a, the x^n term of the expansion of e^{ax} is $\frac{a^n}{n!}$. Since a^n is $n!$ times the coefficient of x^n in the above exponential generating function,

$$a_n = \frac{1}{2}(7^n + (-5)^n - 6^n - (-6)^n) - \frac{1}{2}n(6^{n-1} + (-6)^{n-1}) - \frac{1}{4}n(n-1)(6^{n-2} + (-6)^{n-2}).$$

(b) (i) A suitable ordinary generating function for b_n is

$$\begin{aligned} & \left(\frac{(1-x)^{-6}+(1+x)^{-6}}{2}\right)\left(\frac{x^3}{1-x}\right) \\ = \; & \frac{1}{2}x^3((1-x)^{-7}+(1+x)^{-6}(1-x)^{-1}) \\ = \; & \frac{1}{2}x^3\left(\sum_{i=0}^{\infty}\binom{6+i}{6}x^i+\sum_{i=0}^{\infty}\binom{5+i}{5}(-1)^i x^i\sum_{j=0}^{\infty}x^j\right). \end{aligned}$$

(ii) b_n is the coefficient of x_n in the above generating function. Therefore,

$$\begin{aligned} a_n &= \frac{1}{2}\left(\binom{6+n-3}{6}+\sum_{i=0}^{n-3}\binom{5+i}{5}(-1)^i\right) \\ &= \frac{1}{2}\left(\binom{n+3}{6}+\sum_{i=0}^{n-3}\binom{5+i}{5}(-1)^i\right). \end{aligned}$$