MA1301 – Introductory Mathematics
AY2019/20 SEM 1 Solutions

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Question 1

a.

$$\begin{aligned}
\frac{d}{dx}\sin(\ln(x)+x^3+e^{x^2})^{10} &= 10\sin(\ln(x)+x^3+e^{x^2})^9\frac{d}{dx}\sin(\ln(x)+x^3+e^{x^2}) \\
&= 10\sin(\ln(x)+x^3+e^{x^2})^9\cos(\ln(x)+x^3+e^{x^2})\frac{d}{dx}\ln(x)+x^3+e^{x^2} \\
&= 10(\sin(\ln(x)+x^3+e^{x^2}))^9\cos(\ln(x)+x^3+e^{x^2})(\frac{1}{x}+3x^2+2xe^{x^2}).
\end{aligned}$$

b. To find the slope, we need to evaluate $\frac{dy}{dx}$ at $(1,1)$. By implicit differentiation, we get that

$$3x^2+y^2+2xy\frac{dy}{dx}+4y^3\frac{dy}{dx}=0.$$

Plugging $x=1$ and $y=1$ gives us $\frac{dy}{dx}=-\frac{2}{3}$. Since the line passes $(1,1)$, we easily infer that $m=-\frac{2}{3}$ and $c=\frac{5}{3}$.

Question 2

a. First, we note that $\frac{dx}{dt}=te^{t^2}$ and $\frac{dy}{dt}=t$. Hence, $\frac{dy}{dx}=e^{-t^2}$. Now,

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}}=\frac{-2te^{-t^2}}{te^{t^2}}=-2e^{-2t^2}.$$

b. For the base case $n=1$, it is easy to see that the statement is true. Suppose that for $n=k$,

$\sum_{r=1}^{k} 2+3(r-1) = \frac{k}{2}(4+3(k-1))$. Then,

$$\begin{aligned}
\sum_{r=1}^{k+1} 2+3(r-1) &= (\sum_{r=1}^{k} 2+3(r-1)) + 2 + 3(k+1-1) \\
&= \frac{k}{2}(4+3(k-1)) + 2 + 3k \\
&= 2k + \frac{3}{2}k^2 - \frac{3}{2}k + 2 + 3k \\
&= \frac{3}{2}k^2 + \frac{7}{2}k + 2 \\
&= \frac{(k+1)}{2}(4+3k) = \frac{(k+1)}{2}(4+3((k+1)-1))
\end{aligned}$$

Thus, we have verified the induction step. We are done.

Question 3

Our goal is to find $\frac{dx}{dt}$ when $x = 1$. We have $\tan\theta = \frac{x}{3}$. Differentiating w.r.t t, we have that

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{3}\frac{dx}{dt}.$$

Hence,

$$\frac{dx}{dt} = 3\sec^2(\theta)\frac{d\theta}{dt}.$$

When the light is 1 km away, the length is $\sqrt{10}$ km by Pythagorean theorem. Hence, $\sec\theta = \frac{\sqrt{10}}{3}$. Subsstituting, we get our answer is $3 \times \left(\frac{\sqrt{10}}{3}\right)^2 \times 8\pi = \frac{80\pi}{3}$.

Question 4

a. Note that $f'(x) = \frac{5}{2}x^{\frac{3}{2}} - 3x^{\frac{1}{2}}$. Use the linear approximation formula, $f(4.05) \approx f(4) + (4.05 - 4)f'(4) = 17 + (0.05) \times 14 = f(4) + 0.7$. Hence $f(4.05) - f(4.00) \approx 0.7$.

b. First, find we note that the zeroes of f are $\frac{1}{2}, 2$ and 3. we will calculate the second derivative of f. First we prove a lemma.

   Lemma 1 *Let fg denote $f(x)g(x)$. Then, for all differentiable functions a, b, c, d,*

   $$(abcd)' = (a')bcd + a(b')cd + ab(c')d + abc(d').$$

   Proof 1 *Use product rule,*

   $$(abcd)' = (ab)'cd + ab(cd)' = (a'b + ab')cd + ab(c'd + cd') = (a')bcd + a(b')cd + ab(c')d + abc(d').$$

   Now, using lemma 1, we get that $f''(x) = 2(x-2)^2(2x-6)(e^x+1)^{-1} + 2(x-2)(2x-1)(2x-6)(e^x+1)^{-1} + 2(2x-1)(x-2)^2(e^x+1)^{-1} - e^x(e^x+1)^{-2}(2x-1)(x-2)^2(2x-6)$. Note that $f''(2) = 0$, $f''(3) > 0$ and $f''(\frac{1}{2}) < 0$. Hence, we note that a local minima occurs when $x = 3$, a local maxima occurs when $x = \frac{1}{2}$ and a saddle point occurs when $x = 2$.

Question 5

Let $O = (0,0), A = (x,0)$ and $B = (0,y)$. Since A, B and $(2, \sqrt{32})$ are collinear, then

$$\frac{\sqrt{32} - y}{2 - 0} = \frac{0 - y}{x - 0},$$

which rearranges to $y = \frac{\sqrt{32}x}{x-2}$. We aim to minimize $x^2 + y^2$. Substituting y, we find that we want to minimize $x^2 + \frac{32x^2}{(x-2)^2}$, with $x > 2$. Let $f(x) = x^2 + \frac{32x^2}{(x-2)^2}$. Then,

$$f'(x) = \frac{2x(x^3 - 6x^2 + 12x - 72)}{(x-2)^3} = \frac{2x(x-6)(x^2+12)}{(x-2)^3}.$$

From here, it is easy to check that $x = 6$ minimizes $f(x)$, which is equal to 108 when $x = 6$. Hence, the minimum length of the ladder is the minimal value of $\sqrt{x^2 + y^2}$, which is $\sqrt{108}$.

Question 6

a. Let $u = \sqrt{x} + 1$. Then, $du = \frac{1}{2\sqrt{x}}dx$. Hence,

$$\int \frac{1}{x + \sqrt{x}} dx = \int \frac{2}{u} du = 2 \ln u + C = 2 \ln(\sqrt{x} + 1) + C.$$

b. Let the direction of the vector of the line be (i, j, k). To obtain the direction vector, it suffices to solve the system

$$i + j - 2k = 0 \tag{1}$$
$$i + 2j - k = 0. \tag{2}$$

It is easy to solve that $i = 3k$ and $j = -k$, hence the direction vector is $(3, -1, 1)$. Hence, the equation is $v = c(3, -1, 1) + (0, 2, 4)$ where $c \in \mathbb{R}$.

Question 7

a. Firstly, note that when $-1 \leq x \leq 2$, $2 - x^2 > -x$. Hence, $f(x) = 2 - x^2 - (-x) = 2 - x^2 + x$ by the definition of integral. For $g(x)$ and $h(x)$, we consider the area w.r.t the y-axis. In the range $1 \leq y \leq 2$, it only has the portion of the curve $y = 2 - x^2$. Hence, $x = \sqrt{2 - y}$. Since there are two halves of the graph that we want to count, $g(y) = 2\sqrt{2 - y}$. For the range $-2 \leq y \leq 1$, we find that the curve $y = 2 - x^2$ is on the right of the line $y = -x$. Hence, $h(y) = \sqrt{2 - y} - (-y) = y + \sqrt{2 - y}$.

b. We calculate equation of the top part and the bottom part of the ellipse. Note that the top part equation is $y = 1 + \sqrt{\frac{1-x^2}{4}}$ for $-1 \leq x \leq 1$ and the bottom part of the ellipse is $y = 1 - \sqrt{\frac{1-x^2}{4}}$ for $-1 \leq x \leq 1$. Hence, required the volume is

$$\pi \int_{-1}^{1} \left(1 + \sqrt{\frac{1-x^2}{4}} - \frac{1}{4}\right)^2 - \left(1 - \sqrt{\frac{1-x^2}{4}} - \frac{1}{4}\right)^2 dx.$$

Hence, $f(x) = \left(1 + \sqrt{\frac{1-x^2}{4}} - \frac{1}{4}\right)^2 - \left(1 - \sqrt{\frac{1-x^2}{4}} - \frac{1}{4}\right)^2$.