MA2001 Linear Algebra I AY1819 Sem 1 Final (Solutions)

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December 20, 2022

Question 1

(a) Consider the following matrix

$$\mathbf{C} = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 5 & -3 & 2 \\ 2 & 4 & -2 & 0 \\ 3 & 8 & -5 & 4 \end{pmatrix}.$$

(i) Find a basis for the row space of $\mathbf{C}$.

Solution:

$$\text{rref}(\mathbf{C}) = \begin{pmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, a basis for the row space of C is $\{(1,0,1,-4),(0,1,-1,2)\}$. An alternative solution would be using the original rows, which are $\{(1,2,-1,0),(2,5,-3,2)\}$. $\square$

(ii) Find a basis for the nullspace of $\mathbf{C}$. What is the rank and nullity of $\mathbf{C}$?

Solution: Consider

$$\begin{pmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is equivalent to solving

$$x_1 + x_3 - 4x_4 = 0$$
$$x_2 - x_3 + 2x_4 = 0$$

The solution to this system of equations is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \text{ where } x_3, x_4 \in \mathbb{R}.$$

Hence, a basis for the nullspace of $\mathbf{C}$ is

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

So nullity$(\mathbf{C}) = 2$ and thus, by the Rank-Nullity Theorem, rank$(\mathbf{C}) = 2$. $\square$

(iii) Is the last row of $\mathbf{C}$ a linear combination of the other rows of $\mathbf{C}$? If it is, find such a linear combination. If it is not, explain why.

Solution: Yes, the last row of $\mathbf{C}$ is a linear combination of the other rows since $\operatorname{rank}(\mathbf{C}) = 2 < 4$. Consider

$$\begin{pmatrix} 3 \\ 8 \\ -5 \\ 4 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 5 \\ -3 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ 4 \\ -2 \\ 0 \end{pmatrix}.$$

Solving yields $\alpha = -1 - 2\gamma$, $\beta = 2$ for $\gamma \in \mathbb{R}$. Thus, we can set $\gamma = 0$, resulting in $\alpha = -1$. Therefore,

$$\begin{pmatrix} 3 \\ 8 \\ -5 \\ 4 \end{pmatrix} = - \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 5 \\ -3 \\ 2 \end{pmatrix}.$$

□

(b) Suppose $\mathbf{D}$ is a matrix with k columns such that the linear system $\mathbf{Dx} = \mathbf{r}$ is consistent for all vectors $\mathbf{r} \in \mathbb{R}^n$. For each of the statements below, determine if the statement is true. Justify your answer.

(i) $\mathbf{D}$ has n rows.
Solution: True since $\mathbf{r}$ is of size $n \times 1$. □

(ii) k is at least n
Solution: True as this can be regarded as a system of n equations with k unknowns. □

(iii) $\mathbf{D}$ is of full rank.
Solution: Note that $\operatorname{rank}(\mathbf{D}) \leq \min(n, k)$ because $\mathbf{D}$ cannot have more pivots than rows or columns. From **(ii)**, as $k \geq n$, then $\min(n, k) = n$ so $\operatorname{rank}(\mathbf{D}) \leq n$. Since the linear system $\mathbf{Dx} = \mathbf{r}$ is consistent, then in the augmented matrix $(\mathbf{D}|\mathbf{r})$, there does not exist a pivot in the rightmost column, and so, the row-echelon form of $\mathbf{D}$ has no zero rows. Hence, $\operatorname{rank}(\mathbf{D}) = n$. □

Question 2

(a) $\mathbf{A}$ is a square matrix of order 10 with entries a_{ij} such that $\det(\mathbf{A}) = 2$. Let $\mathbf{B}$ be another square matrix of order 10 such that

$$b_{ij} = \begin{cases} -\frac{1}{2} a_{ij} & \text{if } i \text{ is odd;} \\ 2a_{ij} & \text{if } i \text{ is even.} \end{cases}$$

Find $\det(\mathbf{B})$.

Solution:

$$\det(\mathbf{B}) = \left(-\frac{1}{2} \cdot 2\right)^5 \det(\mathbf{A}) = -\det(\mathbf{A})$$

□

(b) Let

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 10 & 5 \\ 1 & -2 & 3 \end{pmatrix}.$$

Perform three elementary row operations to reduce $\mathbf{B}$ to a row-echelon form. Hence find three elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ such that $\mathbf{B}^{\mathrm{T}}\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3$ is a lower triangular matrix. Write down the elementary row operations that $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ represent respectively.

Solution:

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 10 & 5 \\ 1 & -2 & 3 \end{pmatrix} \xrightarrow{R_1+R_2 \to R_2} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 12 & 9 \\ 1 & -2 & 3 \end{pmatrix} \xrightarrow{-R_1+R_3 \to R_3} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 12 & 9 \\ 0 & -4 & -1 \end{pmatrix} \xrightarrow{\frac{1}{3}R_2+R_3 \to R_3} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 12 & 9 \\ 0 & 0 & 2 \end{pmatrix}$$

Thus,

$$\mathbf{F}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{F}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \text{ and } \mathbf{F}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix}.$$

Since $\mathbf{F}_3\mathbf{F}_2\mathbf{F}_1\mathbf{B}$ is upper triangular, taking transpose, $\mathbf{B}^{\mathrm{T}}\mathbf{F}_1^{\mathrm{T}}\mathbf{F}_2^{\mathrm{T}}\mathbf{F}_3^{\mathrm{T}}$ must be lower triangular. Thus, $\mathbf{E}_i = \mathbf{F}_i^{\mathrm{T}}$ for $i = 1, 2, 3$. That is,

$$\mathbf{E}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{E}_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}.$$

As for what each of the $\mathbf{E}_i$'s represents,

- $\mathbf{E}_1$ denotes adding row 2 to row 1
- $\mathbf{E}_2$ denotes adding negative of row 3 to row 1
- $\mathbf{E}_3$ denotes adding 1/3 of row 3 to row 2

□

Remark: This is similar to the notion of *LU* decomposition, which says that a matrix $\mathbf{A}$ can be *decomposed* into a lower triangular matrix $\mathbf{L}$ and an upper triangular matrix $\mathbf{U}$. The study of this technique is essential in the field of Numerical Analysis.

(c) Let $\mathbf{X}$ and $\mathbf{Y}$ be square matrices of the same order. Prove the following statements.

(i) $\mathbf{X}^{\mathrm{T}}\mathbf{X} = \mathbf{0}$ if and only if $\mathbf{X} = \mathbf{0}$. (Hint: Consider the diagonal entries of $\mathbf{X}^{\mathrm{T}}\mathbf{X}$)

Solution: If $\mathbf{X} = \mathbf{0}$, then $\mathbf{X}^{\mathrm{T}} = \mathbf{0}$, so $\mathbf{X}^{\mathrm{T}}\mathbf{X} = \mathbf{0}$.

Now, we prove that if $\mathbf{X}^{\mathrm{T}}\mathbf{X} = \mathbf{0}$, then $\mathbf{X} = \mathbf{0}$. Suppose $\mathbf{X}$ is of order n. Note that the (i, j)-entry of $\mathbf{X}$ is denoted by x_{ij}, where $1 \le i, j \le n$. Also, the (i, j)-entry of $\mathbf{X}^{\mathrm{T}}$ is denoted by x_{ji}. By matrix multiplication, the (j, j)-entry of $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ is

$$\sum_{i=1}^{n} x_{ij}^2 = 0.$$

That is, each diagonal entry of $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ is represented by the above sum. We have $x_{ij}^2 = 0$ for all $1 \le i, j \le n$, and therefore, $x_{ij} = 0$. The result follows. □

(ii) $\mathbf{XY} = \mathbf{0}$ if and only if $\mathbf{X}^{\mathrm{T}}\mathbf{XY} = \mathbf{0}$. (Hint: Use **(i)**)

Solution: $\mathbf{XY} = \mathbf{0}$ implies $\mathbf{X}^{\mathrm{T}}\mathbf{XY} = \mathbf{0}$ is trivial.

Now, suppose $\mathbf{X}^{\mathrm{T}}\mathbf{XY} = \mathbf{0}$. Then,

$$\begin{aligned} \mathbf{Y}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\mathbf{XY} &= \mathbf{Y}^{\mathrm{T}}\mathbf{0} \\ (\mathbf{XY})^{\mathrm{T}}(\mathbf{XY}) &= \mathbf{0} \end{aligned}$$

The result follows. □

Question 3

(a) Let $V = \{(a-b, a-2b, a+b, a+3b) | a, b \in \mathbb{R}\}$

(i) Show that V is a subspace of $\mathbb{R}^4$.

Solution: Setting $a = b = 0$, the zero vector is contained in V, so V is non-empty.

Let $\mathbf{v}_1 = (a_1 - b_1, a_1 - 2b_1, a_1 + b_1, a_1 + 3b_1)$ and $\mathbf{v}_2 = (a_2 - b_2, a_2 - 2b_2, a_2 + b_2, a_2 + 3b_2)$ be in V. For $\alpha \in \mathbb{R}$,

$$\begin{aligned}
&\alpha\mathbf{v}_1 + \mathbf{v}_2 \\
&= \alpha\left(a_1 - b_1, a_1 - 2b_1, a_1 + b_1, a_1 + 3b_1\right) + \left(a_2 - b_2, a_2 - 2b_2, a_2 + b_2, a_2 + 3b_2\right) \\
&= \left(\alpha\left(a_1 - b_1\right), \alpha\left(a_1 - 2b_1\right), \alpha\left(a_1 + b_1\right), \alpha\left(a_1 + 3b_1\right)\right) + \left(a_2 - b_2, a_2 - 2b_2, a_2 + b_2, a_2 + 3b_2\right) \\
&= \left(\alpha\left(a_1 - b_1\right) + a_2 - b_2, \alpha\left(a_1 - 2b_1\right) + a_2 - 2b_2, \alpha\left(a_1 + b_1\right) + a_2 + b_2, \alpha\left(a_1 + 3b_1\right) + a_2 + 3b_2\right) \\
&= \left(\alpha a_1 + a_2 - \alpha b_1 - b_2, \alpha a_1 + a_2 - 2\alpha b_1 - 2b_2, \alpha a_1 + a_2 + \alpha b_1 + b_2, \alpha a_1 + a_2 + 3\alpha b_1 + 3b_2\right)
\end{aligned}$$

which shows that V is closed under addition and scalar multiplication. □

(ii) Find a basis for V. What is the dimension of V?

Solution: Note that each vector in V can be written as $a(1,1,1,1) + b(-1,-2,1,3)$ so a basis for V is $\{(1,1,1,1), (-1,-2,1,3)\}$. Also, $\dim(V) = 2$. □

(iii) Let W be the solution space of the following homogeneous linear system:

$$\begin{aligned}
x_1 - x_2 + x_3 + x_4 &= 0 \\
x_2 - x_3 + 6x_4 &= 0 \\
x_3 + 3x_4 &= 0
\end{aligned}$$

Find a basis for W and hence show that $W \subseteq V$.

Solution: By backward substitution, $x_1 = -7x_4$, $x_2 = -9x_4$ and $x_3 = -3x_4$. Hence, a basis for W is $\{(-7,-9,-3,1)\}$. As $(-7,-9,-3,1) = -5(1,1,1,1) + 2(-1,-2,1,3)$, then $W \subseteq V$. □

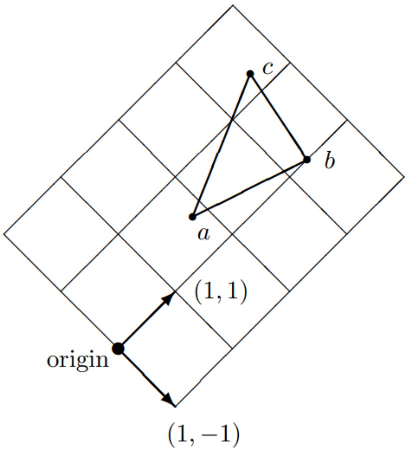
(b) Let $E = \{(1,0), (0,1)\}$ and $S = \{(1,1), (1,-1)\}$.
(i) E is the standard basis for $\mathbb{R}^2$. Is S also a basis for $\mathbb{R}^2$? Justify your answer.

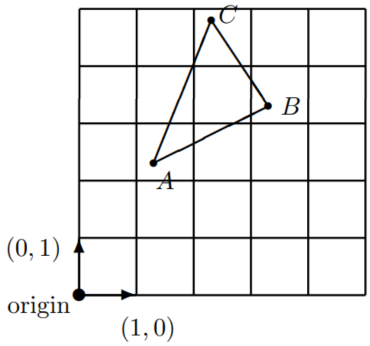
Solution: S spans $\mathbb{R}^2$. This is evident as

$$\left(\begin{array}{cc|c} 1 & 1 & x \\ 1 & -1 & y \end{array}\right) \xrightarrow{-R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & x \\ 0 & -2 & y - x \end{array}\right)$$

which shows that the above system is consistent for any $x, y \in \mathbb{R}$. Consider $\alpha(1,1) + \beta(1,-1) = (0,0)$. Then, $\alpha + \beta = 0$ and $\alpha - \beta = 0$. Thus, $\alpha = \beta = 0$, implying that these vectors are linearly independent. Hence, S is also a basis for $\mathbb{R}^2$. □

(ii) The triangle in the right figure is re-drawn exactly on the left figure as shown. Find the coordinates of the 3 points a, b and c, with respect to the coordinates used in the left figure. The coordinates of A, B and C are given by $(1.2, 2.2), (3.2, 3.2)$ and $(2.2, 4.8)$ respectively.





Solution: Bases for S_1 and S_2 are

$$S_1 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \text{ and } S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

We see that

$$\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]_{S_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] = \left[\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right]_{S_1}$$

and

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]_{S_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] = \left[\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right]_{S_1}$$

so the transition matrix is

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

The position vector representing the point a can be found by

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1.2 \\ 2.2 \end{pmatrix}.$$

The same can be said for b and c.

Hence, $a = (-0.5, 1.7)$, $b = (0, 3.2)$ and $c = (-1.3, 3.5)$. □

Question 4

Let $V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{u}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}.$$

(i) Find a vector $\mathbf{u}$ such that $\|\mathbf{u}\| = 3\sqrt{10}$ and $\mathbf{u}$ is orthogonal to $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3$.

Solution: Let

$$\mathbf{u} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

By considering orthogonality, we can form three equations:

$$\begin{aligned} a+b+c+d&=0\\ b+c&=0\\ b+2d&=0 \end{aligned}$$

Using the norm of $\mathbf{u}$, we have

$$a^2+b^2+c^2+d^2=90.$$

Solving the first three equations by backward substitution yields $a=-d$, $b=-2d$ and $c=2d$. Substituting these into the equation representing the norm of $\mathbf{u}$,

$$d^2+4d^2+4d^2+d^2=90.$$

Thus, $d=3$ ($d=-3$ also works as there is no restriction). So, $a=-3$, $b=-6$ and $d=6$. A vector $\mathbf{u}$ is

$$\begin{pmatrix}-3\\-6\\6\\3\end{pmatrix}.$$

□

(ii) Use the Gram-Schmidt Process to transform $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ into an orthonormal basis for V.

Solution: Note that $\|\mathbf{u}_1\|=4$ so

$$\mathbf{v}_1=\begin{pmatrix}\frac12\\\frac12\\\frac12\\\frac12\end{pmatrix}.$$

Using the Gram-Schmidt Process,

$$\mathbf{v}_2=\begin{pmatrix}0\\1\\1\\0\end{pmatrix}-\left(\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}\frac12\\\frac12\\\frac12\\\frac12\end{pmatrix}\right)\begin{pmatrix}\frac12\\\frac12\\\frac12\\\frac12\end{pmatrix}=\begin{pmatrix}-\frac12\\\frac12\\\frac12\\-\frac12\end{pmatrix}.$$

Using the Gram-Schmidt Process again,

$$\mathbf{v}_3=\begin{pmatrix}0\\1\\0\\2\end{pmatrix}-\left(\begin{pmatrix}0\\1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}\frac12\\\frac12\\\frac12\\\frac12\end{pmatrix}\right)\begin{pmatrix}\frac12\\\frac12\\\frac12\\\frac12\end{pmatrix}-\left(\begin{pmatrix}0\\1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}-\frac12\\\frac12\\\frac12\\-\frac12\end{pmatrix}\right)\begin{pmatrix}-\frac12\\\frac12\\\frac12\\-\frac12\end{pmatrix}=\begin{pmatrix}-\sqrt{\frac25}\\\frac{1}{\sqrt{10}}\\-\frac{1}{\sqrt{10}}\\\sqrt{\frac25}\end{pmatrix}$$

so we conclude that

$$\mathbf{v}_1=\begin{pmatrix}\frac12\\\frac12\\\frac12\\\frac12\end{pmatrix},\mathbf{v}_2=\begin{pmatrix}-\frac12\\\frac12\\\frac12\\-\frac12\end{pmatrix}\text{ and }\mathbf{v}_3=\begin{pmatrix}-\sqrt{\frac25}\\\frac{1}{\sqrt{10}}\\-\frac{1}{\sqrt{10}}\\\sqrt{\frac25}\end{pmatrix},$$

where $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is an orthonormal basis for V.

□

(iii) Find the projection of

$$\mathbf{w}=\begin{pmatrix}-1\\1\\-1\\13\end{pmatrix}$$

onto V.

Solution: The projection is

$$\left(\begin{pmatrix}-1\\1\\-1\\13\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\right)\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}+\left(\begin{pmatrix}-1\\1\\-1\\13\end{pmatrix}\cdot\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}\right)\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}+\left(\begin{pmatrix}-1\\1\\-1\\13\end{pmatrix}\cdot\begin{pmatrix}-\sqrt{\frac{2}{5}}\\\frac{1}{\sqrt{10}}\\-\frac{1}{\sqrt{10}}\\\sqrt{\frac{2}{5}}\end{pmatrix}\right)\begin{pmatrix}-\sqrt{\frac{2}{5}}\\\frac{1}{\sqrt{10}}\\-\frac{1}{\sqrt{10}}\\\sqrt{\frac{2}{5}}\end{pmatrix}$$

$$=6\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}-6\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}+\sqrt{90}\begin{pmatrix}-\sqrt{\frac{2}{5}}\\\frac{1}{\sqrt{10}}\\-\frac{1}{\sqrt{10}}\\\sqrt{\frac{2}{5}}\end{pmatrix}$$

$$=\begin{pmatrix}0\\3\\-3\\12\end{pmatrix}$$

□

(iv) Let $A=\begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3\end{pmatrix}$, where $\mathbf{u}_1,\mathbf{u}_2$ and $\mathbf{u}_3$ are the columns of $\mathbf{A}$. Find a least squares solution to the linear system $\mathbf{A}\mathbf{x}=\mathbf{w}$.

Solution: We have

$$\mathbf{A}=\begin{pmatrix}1&0&0\\1&1&1\\1&1&0\\1&0&2\end{pmatrix}.$$

Consider $\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x}=\mathbf{A}^{\mathrm{T}}\mathbf{w}$, so

$$\begin{pmatrix}4&2&3\\2&2&1\\3&1&5\end{pmatrix}\mathbf{x}=\begin{pmatrix}12\\0\\27\end{pmatrix},$$

hence, a least squares solution is

$$\mathbf{x}=\begin{pmatrix}0\\-3\\6\end{pmatrix}.$$

□

Question 5

(a) Let

$$\mathbf{A}=\begin{pmatrix}2&0&0\\-1&1&3\\1&1&-1\end{pmatrix}.$$

(i) Find all the eigenvalues of $\mathbf{A}$.

Solution:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\det\left(\begin{pmatrix}2-\lambda&0&0\\-1&1-\lambda&3\\1&1&-1-\lambda\end{pmatrix}\right)$$

Setting the determinant to be zero,

$$\begin{aligned}(2-\lambda)((1-\lambda)(-1-\lambda)-3)&=0\\ (2-\lambda)(\lambda+2)(\lambda-2)&=0\end{aligned}$$

Hence, the eigenvalues are -2 and 2. $\square$

(ii) Find a basis for the eigenspace associated with each eigenvalue of $\mathbf{A}$.

For E_{-2}, consider $(\mathbf{A}+2\mathbf{I})\mathbf{x}=\mathbf{0}$, so we have

$$\begin{pmatrix}4&0&0\\-1&3&3\\1&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},$$

whose solution is $x=0$, $y=-z$. Thus, an eigenvector corresponding to $\lambda=-2$ is $(0,1,-1)$, and so a basis for E_{-2} is

$$\left\{\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}.$$

For E_2, consider $(\mathbf{A}-2\mathbf{I})\mathbf{x}=\mathbf{0}$, so we have

$$\begin{pmatrix}0&0&0\\-1&-1&3\\1&1&-3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

This is equivalently $x+y+3z=0$. We see that

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-y-3z\\y\\3z\end{pmatrix}=y\begin{pmatrix}-1\\1\\0\end{pmatrix}+3z\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

so the two eigenvectors corresponding to $\lambda=2$ are $(-1,1,0)$ and $(-1,0,1)$. Therefore, a basis for E_2 is

$$\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}.$$

$\square$

(iii) Find a matrix $\mathbf{P}$ that diagonalises $\mathbf{A}$ and determine $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

$$\mathbf{P}=\begin{pmatrix}0&-1&-1\\1&1&0\\-1&0&1\end{pmatrix}\text{ and }\mathbf{P}^{-1}\mathbf{A}\mathbf{P}=\begin{pmatrix}-2&0&0\\0&2&0\\0&0&2\end{pmatrix}.$$

$\square$

(b) Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of order n. Suppose $\mathbf{AB}=\mathbf{BA}$ and $\mathbf{A}$ has n distinct eigenvalues.

(i) Show that each eigenspace of $\mathbf{A}$ has dimension 1.
Solution: We prove by contradiction. Let the eigenvalues be λ_i for $1\le i\le n$. Suppose on the contrary that for some $1\le i\le n$, $\dim(E_{\lambda_i})>1$. Since the algebraic multiplicity of an eigenvalue is greater than or equal to its geometric multiplicity, claiming that $\dim(E_{\lambda_i})>1$ would imply that the sum of the algebraic multiplicities would be at least $n+1$, which is greater than n. This is a contradiction since $\mathbf{A}$ is of order n. $\square$

Remark: Let $\mathbf{A}$ be a square matrix and λ be an eigenvalue. The **algebraic multiplicity** of λ is the number of times λ appears as a root in the characteristic polynomial of $\mathbf{A}$. The **geometric multiplicity** of λ is the dimension of the eigenspace of λ. That is, $\dim(E_\lambda)$.

(ii) Show that if $\mathbf{u}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{u}$ is also an eigenvector of $\mathbf{B}$.

Solution: Since $\mathbf{Au} = \lambda\mathbf{u}$ with $\mathbf{u} \in E_\lambda$ for some $\lambda \in \mathbb{R}$, then $\mathbf{BAu} = \lambda(\mathbf{Bu})$, so $\mathbf{ABu} = \lambda(\mathbf{Bu})$. This implies that $B_u \in E_\lambda$, so there exists some $\mu \in \mathbb{R}$ such that $\mathbf{Bu} = \mu\mathbf{u}$. $\square$

(iii) Show that $\mathbf{A}$ and $\mathbf{B}$ are simultaneously diagonalisable, i.e. there exists an invertible matrix $\mathbf{P}$ such that $\mathbf{PAP}^{-1}$ and $\mathbf{PBP}^{-1}$ are diagonal.

Solution: Since $\mathbf{A}$ is diagonalisable, then $\mathbf{A} = \mathbf{QD}_1\mathbf{Q}^{-1}$, where $\mathbf{D}_1$ is a diagonal matrix containing the eigenvalues of $\mathbf{A}$ and $\mathbf{Q}$ is a matrix comprising the corresponding eigenvectors. That is, $\mathbf{Q} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{pmatrix}$. So, $\mathbf{Q}^{-1}\mathbf{AQ} = \mathbf{D}_1$. Since $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are the eigenvectors of $\mathbf{B}$ and these n vectors are linearly independent, then $\mathbf{Q}^{-1}\mathbf{BQ} = \mathbf{D}_2$, where $\mathbf{D}_2$ is a diagonal matrix whose diagonal entries are the corresponding eigenvalues of $\mathbf{B}$. Lastly, taking $\mathbf{P} = \mathbf{Q}^{-1}$, the result follows. $\square$

Question 6

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x - y + z \\ 2x - y - z \end{pmatrix} \text{ for all } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

Find 2 different linear transformations S_1 and S_2 such that $(T \circ S_1)$ and $(T \circ S_2)$ are both the identity linear operator on $\mathbb{R}^2$, showing clearly how S_1 and S_2 are derived. Give your answers by providing the formulae for S_1 and S_2.

Solution: The matrix representation of T is

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & -1 \end{pmatrix}.$$

Since this is a 2×3 matrix, the matrices representing S_1 and S_2 must have size 3×2. Say S_1 has a matrix representation of the form

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}.$$

Then,

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Working with the left side of the equation,

$$\begin{pmatrix} a - c + e & b - d + f \\ 2a - c - e & 2b - d - f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence,

$$\begin{aligned} a - c + e &= 1 \\ b - d + f &= 0 \\ 2a - c - e &= 0 \\ 2b - d - f &= 1 \end{aligned}$$

Solving, $a = -1 + 2e$, $b = 1 + 2f$, $c = -2 + 3e$ and $d = 1 + 3f$, where $e, f \in \mathbb{R}$. Without a loss of generality, for the matrix representing S_1, we can set $e = f = 0$, whereas for the matrix representing S_2, we can set $e = f = 1$. Thus, the matrices representing S_1 and S_2 are

$$\begin{pmatrix} -1 & 1 \\ -2 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 3 \\ 1 & 4 \\ 1 & 1 \end{pmatrix}.$$

$\square$