MA1101R - Linear Algebra I Suggested Solutions

(Semester 1 : AY2020/21)

Written by : Matthew Fan Xin Yu
Audited by : Pan Jing Bin

1. i) Using Matlab, $\det(\boldsymbol{A}) = b - a - c + d$. For $\boldsymbol{Ax} = \boldsymbol{0}$ to have non-trivial solutions, $\boldsymbol{A}$ must be non-invertible. So $b - a - c + d = 0$.

   ii) If $\boldsymbol{Ax} = \boldsymbol{0}$ has only trivial solution, $\det(\boldsymbol{A}) \neq 0$, which gives $b - a - c + d \neq 0$. Hence $(0, 0, 0, 0)$ does not satisfy the conditions and is not in S. So S is not a subspace of $\mathbb{R}^4$.

   iii) $\text{rank}(\boldsymbol{A}) = 3$. Since the first three rows of $\boldsymbol{A}$ are independent, last row can be written as linear combination of first three rows. Hence a REF of $\boldsymbol{A}$ is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

   Nullspace of $\boldsymbol{A}$ is $\left\{ s \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

   iv) Looking at first row, $\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 2$, is twice of 1.

   Hence the eigenvalue of this eigenvector is 2. $\implies \begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 2 \implies a+b+c+d = 2$

   v) If a, b, c, d are all equal, $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ a & a & a & a \end{pmatrix}$ which has $\text{rref}(\boldsymbol{A}) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Hence first 3 columns form a basis. A basis would be $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ a \end{pmatrix} \right\}$.

2. ai) Writing the vectors as columns vectors of $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 1 \\ 0 & 0 & 3 & 3 & -3 \end{pmatrix}$, the rref is $\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Hence the first and the third columns of $\boldsymbol{A}$ form basis for V.
$S' = (1,1,2,0), (0,0,1,3), \dim V = 2$.

aii) Yes. We consider $\left(\begin{array}{cc|ccc} 1 & 0 & 2 & 2 & 1 \\ 1 & 0 & 2 & 2 & 1 \\ 2 & 1 & 5 & 3 & 0 \\ 0 & 3 & 3 & -3 & -6 \end{array}\right)$. The rref of this is $\left(\begin{array}{cc|ccc} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$.
Hence all the vectors $(2,2,5,3), (2,2,3,-3), (1,1,0,-6)$ belong to S.
Furthermore, within these three vectors, the first two are linearly independent.
Hence span $\{(2,2,5,3), (2,2,3,-3), (1,1,0,-6)\}$ is a subspace of V with dimension 2 so it must be equal to V itself.

aiii) The vectors of V are of the form $a(1,1,2,0) + b(0,0,1,3) = (a, a, 2a+b, 3b)$.
For them to be in W, they have to satisfy $a - a + (2a+b) - 3b = 0 \implies a = b$.
Hence vectors of $V \cap W$ are of the form $a(1,1,2,0) + a(0,0,1,3) = a(1,1,3,3)$.
$W \cap V = \text{span}\{(1,1,3,3)\}$.

bi) Let $\boldsymbol{u}$ be any vector in R^n. Then $\boldsymbol{u} \in W \cap W^{\perp} \implies \boldsymbol{u} \cdot \boldsymbol{u} = \boldsymbol{0} \implies ||\boldsymbol{u}||^2 = \boldsymbol{0} \implies \boldsymbol{u} = \boldsymbol{0}$.

bii) Let W be the row space and $W^{\perp}$ be the nullspace of M for some M. Hence $\dim W + \dim W^{\perp} = n$. Now

$$\begin{aligned} \dim(W + W^{\perp}) &= \dim W + \dim W^{\perp} - \dim(W \cap W^{\perp}) \\ &= n - 0 \\ &= n. \end{aligned}$$

Hence every vector $\boldsymbol{v}$ is in $W + W^{\perp}$ and hence can be written as $\boldsymbol{v_1} + \boldsymbol{v_2}$, where $\boldsymbol{v_1} \in W$, $\boldsymbol{v_2} \in W^{\perp}$.

Suppose that there is another way of writing it as $\boldsymbol{a} + \boldsymbol{b}$ where $\boldsymbol{a} \in W, \boldsymbol{b} \in W^{\perp}$. Then

$$\boldsymbol{v} = \boldsymbol{v_1} + \boldsymbol{v_2} = \boldsymbol{a} + \boldsymbol{b} \implies (\boldsymbol{v_1} - \boldsymbol{a}) + (\boldsymbol{v_2} - \boldsymbol{b}) = \boldsymbol{0},$$

where $\boldsymbol{v_1} - \boldsymbol{a} \in W$ and $\boldsymbol{v_2} - \boldsymbol{b} \in W^{\perp}$. Hence $\boldsymbol{v_1} - \boldsymbol{a} = -(\boldsymbol{v_2} - \boldsymbol{b})$ is a scalar multiple of a vector in $W^{\perp}$ and hence is in $W^{\perp}$. So from $\boldsymbol{v_1} - \boldsymbol{a} \in W \cap W^{\perp}$ we get $\boldsymbol{v_1} - \boldsymbol{a} = \boldsymbol{0}$ and therefore $\boldsymbol{v_2} - \boldsymbol{b} = \boldsymbol{0}$. Thus $v_1 = a$ and $v_2 = b$ so there is no second solution.

3. ai) rref of $(\boldsymbol{A}|\boldsymbol{u_1}|\boldsymbol{u_2}|\boldsymbol{u_3})$ is $\left(\begin{array}{cccc|c|c|c} 1 & 0 & 0 & 1 & 1 & -3 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & 2 & -1 \\ 0 & 0 & 1 & \frac{4}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$. Hence $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ form basis for column space of $\boldsymbol{A}$.

$$\boldsymbol{u_1} \cdot \boldsymbol{u_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0, \qquad \boldsymbol{u_2} \cdot \boldsymbol{u_3} = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ -1 \\ -1 \end{pmatrix} = 0, \qquad \boldsymbol{u_3} \cdot \boldsymbol{u_1} = \begin{pmatrix} 0 \\ 2 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0.$$

Hence S is an orthogonal basis for V.

ii) $\boldsymbol{v_1} = \dfrac{\boldsymbol{u_1}}{||\boldsymbol{u_1}||} = \dfrac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\boldsymbol{v_2} = \dfrac{\boldsymbol{u_2}}{||\boldsymbol{u_2}||} = \dfrac{1}{2\sqrt{3}}\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\boldsymbol{v_3} = \dfrac{\boldsymbol{u_2}}{||\boldsymbol{u_3}||} = \dfrac{1}{\sqrt{6}}\begin{pmatrix} 0 \\ 2 \\ -1 \\ -1 \end{pmatrix}$.

$$T = \left\{ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2\sqrt{3}}\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 0 \\ 2 \\ -1 \\ -1 \end{pmatrix} \right\}$$

iii) $\boldsymbol{A^T A x} = \boldsymbol{A^T b}$.

$\begin{pmatrix} 4 & 6 & 6 & 9 \\ 6 & 12 & 12 & 16 \\ 6 & 12 & 18 & 24 \\ 9 & 16 & 24 & 33 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$. The rref of that is $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{4}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$.

Let 4th component of $\boldsymbol{x}$ be 0 to get a specific solution $\begin{pmatrix} 1 \\ -1 \\ \frac{1}{3} \\ 0 \end{pmatrix}$.

The nullspace of $\boldsymbol{A^T A}$ is $s\begin{pmatrix} -1 \\ \frac{1}{2} \\ -\frac{4}{3} \\ 1 \end{pmatrix}$. Hence the least square solutions are $s\begin{pmatrix} -1 \\ \frac{1}{2} \\ -\frac{4}{3} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ \frac{1}{3} \\ 0 \end{pmatrix}$.

iv) Let $\boldsymbol{u} = \begin{pmatrix} 1 \\ -1 \\ \frac{1}{3} \\ 0 \end{pmatrix}$ be the one of the least square solutions found in (iii). We consider $\boldsymbol{Au} - \boldsymbol{b} = \boldsymbol{A}\begin{pmatrix} 1 \\ -1 \\ \frac{1}{3} \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$. This vector is orthogonal to V and hence orthogonal to $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$. Let $\boldsymbol{v_4} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$. Then $T' = \left\{ \boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \dfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$.

bi) False. Since V is a proper subspace of $\mathbb{R}^n$, $m < n$. Thus $\mathbf{C}$ and $\mathbf{D}$ are not square matrices so they cannot be orthogonal.

bii) False. Transition matrices must be square matrices but $\mathbf{P}$ is not a square matrix.

biii) True. Choose arbitrary $\mathbf{v_i} \in T$. Then

$$\begin{aligned}
\mathbf{C^T D}[\mathbf{v_i}]_T &= \mathbf{C^T D}\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \text{(1 at the i-th entry and 0 elsewhere)} \\
&= \mathbf{C^T v_i} \\
&= \begin{pmatrix} \mathbf{u_1} \cdot \mathbf{v_i} \\ \mathbf{u_2} \cdot \mathbf{v_i} \\ \vdots \\ \mathbf{u_m} \cdot \mathbf{v_i} \end{pmatrix} \\
&= [\mathbf{v_i}]_S
\end{aligned}$$

where the last equality follows from the given condition that S is orthonormal.

4. ai)

$$\begin{aligned}
\det(\lambda \boldsymbol{I} - \boldsymbol{C}) &= \det \begin{pmatrix} \lambda-1 & 0 & 0 & -1 \\ 0 & \lambda-2 & -2 & 0 \\ 0 & -2 & \lambda-2 & 0 \\ -1 & 0 & 0 & \lambda-1 \end{pmatrix} \\
&= (\lambda-1) \begin{vmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-2 & -2 \\ 0 & -2 & \lambda-2 \end{vmatrix} - (-1) \begin{vmatrix} 0 & \lambda-2 & -2 \\ 0 & -2 & \lambda-2 \\ -1 & 0 & 0 \end{vmatrix} \\
&= (\lambda-1)^2 \begin{vmatrix} \lambda-2 & -2 \\ -2 & \lambda-2 \end{vmatrix} - (-1)^2 \begin{vmatrix} \lambda-2 & -2 \\ -2 & \lambda-2 \end{vmatrix} \\
&= ((\lambda-1)^2 - 1)((\lambda-2)^2 - 4) \\
&= \lambda^2(\lambda-2)(\lambda-4)
\end{aligned}$$

Eigenvalues are 0, 2, 4.

ii) For 0, $0\boldsymbol{I} - \boldsymbol{C} = -\boldsymbol{C} = \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}$ has nullspace span $\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and hence a basis would be $\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

For 2, $2\boldsymbol{I} - \boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ has nullspace span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and hence a basis would be $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

For 4, $4\boldsymbol{I} - \boldsymbol{C} = \begin{pmatrix} 3 & 0 & 0 & -1 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix}$ has nullspace span $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ and hence a basis would be $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

iii) Firstly, note that the 2 basis vectors for the eigenspace associated with eigenvalue 0, $\left\{\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}\right\}$, are already orthogonal to each other. After normalisation, the 4 orthogonal eigenvectors are $\left\{\begin{pmatrix}0\\-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}, \begin{pmatrix}-\frac{1}{\sqrt{2}}\\0\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}, \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}, \begin{pmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}\right\}$.

We set eigenvalues as the values of the diagonal matrix $\boldsymbol{D}$ and their corresponding eigenvectors as the columns of $\boldsymbol{P}$ to get $\boldsymbol{D} = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&2&0\\0&0&0&4\end{pmatrix}$ and $\boldsymbol{P} = \begin{pmatrix}0&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\-\frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\end{pmatrix}$.

Then $\boldsymbol{C} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{\boldsymbol{T}}$.

bi) Let the column space of $\boldsymbol{M}$ be C. For any vector $\boldsymbol{v} \in C$, there exists $\boldsymbol{u} \in \mathbb{R}^n$ such that $\boldsymbol{v} = \boldsymbol{M}\boldsymbol{u}$. Then $(\boldsymbol{M} - \boldsymbol{I})\boldsymbol{v} = (\boldsymbol{M}^2 - \boldsymbol{M})\boldsymbol{u} = \boldsymbol{0}$ implies that $\boldsymbol{M}\boldsymbol{v} = \boldsymbol{v}$ and $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{M}$ with eigenvalue 1. Thus $\boldsymbol{v} \in E_1$.

At the same time, if $\boldsymbol{v} \in E_1$, then $\boldsymbol{M}\boldsymbol{v} = \boldsymbol{v}$ so $\boldsymbol{v} \in C$. Thus we have $C = E_1$.

bii) For any vector $\boldsymbol{w}$ in the nullspace, $\boldsymbol{A}\boldsymbol{w} = \boldsymbol{0}$ implies $\boldsymbol{w}$ is an eigenvector associated with eigenvalue 0.

Hence we can find $\boldsymbol{v_1}, \ldots, \boldsymbol{v_r} \in C$ and $\boldsymbol{w_1}, \ldots, \boldsymbol{w_{n-r}}$ in the nullspace of $\boldsymbol{M}$ such that

$$\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_r}, \boldsymbol{w_1}, \ldots, \boldsymbol{w_{n-r}}\}$$

is linearly independent. We have found a set of n linearly independent eigenvectors of $\boldsymbol{M}$ so $\boldsymbol{M}$ is diagonalizable.

5. ai) Let $\boldsymbol{A}$ be the standard matrix.

$$\boldsymbol{A}\begin{pmatrix}0&1&1\\1&0&2\\2&2&0\end{pmatrix} = \begin{pmatrix}0&2&0\\1&0&2\\2&4&4\end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix}0&2&0\\1&0&2\\2&4&4\end{pmatrix}\begin{pmatrix}0&1&1\\1&0&2\\2&2&0\end{pmatrix}^{-1} = \begin{pmatrix}\frac{4}{3}&-\frac{2}{3}&\frac{1}{3}\\0&1&0\\\frac{8}{3}&\frac{2}{3}&\frac{2}{3}\end{pmatrix}$$

ii) $\text{rref}(\boldsymbol{A}) = \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So solutions are $s\begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$ and kernel $= \text{span}\left\{\begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}\right\}$

iii) If vector maps to itself, then $\boldsymbol{Av} = \boldsymbol{v}$. Hence we are looking for eigenspace of $\boldsymbol{A}$ associated with eigenvalue 1. $\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 \\ -\frac{8}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$ with rref being $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$. Hence eigenspace is $\text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right\}$, is the largest possible.

iv) $\text{rref}\left(\begin{array}{c|c} \boldsymbol{A} & \begin{matrix} 1 \\ 2 \\ 0 \end{matrix} \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$. Hence $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ is not in the col space of $\boldsymbol{A}$ and hence no such vector $\boldsymbol{v} \in \mathbb{R}^3$ such that $T(\boldsymbol{v}) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

bi) Let the standard matrix of T be $\boldsymbol{A}$. $R(T^{k+1}) =$ col space of $\boldsymbol{A}^{k+1}$.

If $\boldsymbol{v} \in$ col space of $\boldsymbol{A}^{k+1}$, then $\boldsymbol{v} = \boldsymbol{A}^{k+1}\boldsymbol{u} = \boldsymbol{A}^k(\boldsymbol{Au}) \in$ col space of $\boldsymbol{A}^k$. Hence $R(T^{k+1}) \subseteq R(T^k)$.

bii) We will show that if $R(T^k) = R(T^{k+1})$ for some $k \geq 0$, then $R(T^k) = R(T^h)$ for all $h \geq k$ using induction. (Here we treat $R(T^0)$ as $\mathbb{R}^n$ and $\boldsymbol{A}^0$ as $\boldsymbol{I}$.)

Proof. If $R(T^k) = R(T^{k+1})$, then for all $\boldsymbol{u}$, there exists a $\boldsymbol{v}$ such that $\boldsymbol{A}^k\boldsymbol{u} = \boldsymbol{A}^{k+1}\boldsymbol{v}$. Now there exists a $\boldsymbol{w}$ such that $\boldsymbol{A}^k\boldsymbol{v} = \boldsymbol{A}^{k+1}\boldsymbol{w}$.

Hence for all $\boldsymbol{u}$ there exists $\boldsymbol{v}$ and $\boldsymbol{w}$ such that $\boldsymbol{A}^k\boldsymbol{u} = \boldsymbol{A}^{k+1}\boldsymbol{v} = \boldsymbol{A}^{k+2}\boldsymbol{w}$. Hence $R(T^k)$ is a subspace of $R(T^{k+2})$. Together with part (i), this means $R(T^k) = R(T^{k+2})$.

Since we have shown that $R(T^k) = R(T^{k+1}) \implies R(T^{k+1}) = R(T^{k+2})$, by induction $R(T^{k+i}) = R(T^{k+i+1})$ for all $i \geq 0$ and therefore $R(T^k) = R(T^h)$ for all $h \geq k$ $\square$

Now consider the sequence $(\dim(R(T^0)), \dim(R(T^1)), \ldots, \dim(R(T^n)))$. By part (i), this sequence is non-increasing. Suppose otherwise that $\dim(R(T^n)) \geq 1$, then there are $n+1$ terms and only n possible values of the dimension, from 1 to n. Hence by pigeonhole principle, there are two terms having the same dimension and hence all terms in between them have the same dimension.

This means that we can always find two consecutive terms with the same dimension and together with part (i), that implies $R(T^k) = R(T^{k+1})$ for some $k \leq n$ and hence $R(T^k) = R(T^h)$ for all $h \geq k$. In particular, $R(T^m) = R(T^k)$, a contradiction since $R(T^m)$ is the zero-space and $R(T^k)$ has dimension at least 1.

Hence $\dim(R(T^n)) = 0$ and T^n is the zero transformation.