NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA2101 Linear Algebra II
AY 2013/2014 Sem 1
Version 1: November 16, 2014

Written by	**Audited by**	**Contributors**
Le Hoang Van	Henry Morco	–

Question 1

(a) We begin by showing that W exhibits the following properties.

(1) $\mathbf{0} \in W$, since $\mathbf{0}\boldsymbol{A} = \boldsymbol{B}\mathbf{0} = \mathbf{0}$. ($\mathbf{0}$ is the $n \times n$ zero matrix.)

(2) Take any $\boldsymbol{X}, \boldsymbol{Y} \in W$. We have $\boldsymbol{XA} = \boldsymbol{BX}$ and $\boldsymbol{YA} = \boldsymbol{BY}$. Then

$$(\boldsymbol{X} + \boldsymbol{Y})\,\boldsymbol{A} = \boldsymbol{XA} + \boldsymbol{YA} = \boldsymbol{BX} + \boldsymbol{BY} = \boldsymbol{B}\,(\boldsymbol{X} + \boldsymbol{Y}),$$

meaning $\boldsymbol{X} + \boldsymbol{Y} \in W$.

(3) Take any $\boldsymbol{X} \in W$. Then for any $c \in \mathbb{R}$

$$(c\boldsymbol{X})\,\boldsymbol{A} = c\,(\boldsymbol{XA}) = c\,(\boldsymbol{BX}) = \boldsymbol{B}\,(c\boldsymbol{X}),$$

meaning $c\boldsymbol{X} \in W$.

Since W is a subset of $\mathcal{M}_{n\times n}(\mathbb{R})$ satisfying (1)-(3), it is a subspace of $\mathcal{M}_{n\times n}(\mathbb{R})$.

(b) (i) For any $\boldsymbol{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R})$, $\boldsymbol{X} \in W$ if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a & a \\ c & c \end{pmatrix} = \begin{pmatrix} c & d \\ c & d \end{pmatrix},$$

that is, $a = c = d = s$ and $b = t$ for some $s, t \in \mathbb{R}$. In other words, $\boldsymbol{X} = \begin{pmatrix} s & t \\ s & s \end{pmatrix}$. Thus $W = \operatorname{span}\left\{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ and $\dim(W) = 2$.

(ii) $W' = \operatorname{span}\left\{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$.

Question 2

(a)

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1+i & 0 & 1-i \\ 1+i & 0 & 1-i & 0 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1+i & 0 & 1-i \\ 0 & 0 & 1-i & -1+i \end{pmatrix}$$

Therefore $\text{rank}(T) = 3$ and by the *Dimension Theorem for Linear Transformations*, $\text{nullity}(T) = 4 - 3 = 1$.

(b) T is not injective since $\text{nullity}(T) \neq 0$. T is surjective, however, since $\text{rank}(T) = 3 = \dim(W)$.

(c) Let $\boldsymbol{u_1} = \boldsymbol{v_1} - \boldsymbol{v_2}, \boldsymbol{u_2} = \boldsymbol{v_2} - \boldsymbol{v_3}$ and $\boldsymbol{u_3} = \boldsymbol{v_3}$. Then

$$\begin{aligned} \boldsymbol{v_1} &= \boldsymbol{u_1} + \boldsymbol{u_2} + \boldsymbol{u_3}, \\ \boldsymbol{v_2} &= \boldsymbol{u_2} + \boldsymbol{u_3}, \\ \boldsymbol{v_3} &= \boldsymbol{u_3}. \end{aligned}$$

Thus $\boldsymbol{P} = [I_W]_{D,C} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. ($I_W$ is called the identity operator on V and $\boldsymbol{P}$ the transition matrix from C to D.)

Question 3

(a) For all $a, b \in \mathbb{R}$ and $p(x), q(x) \in \mathcal{P}_n(\mathbb{R})$,

$$\begin{aligned} T\left(ap(x) + bq(x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[(x-1)\left(ap(x) + bq(x)\right)\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[(x-1)ap(x)\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[(x-1)bq(x)\right] \\ &= a\frac{\mathrm{d}}{\mathrm{d}x}\left[(x-1)p(x)\right] + b\frac{\mathrm{d}}{\mathrm{d}x}\left[(x-1)q(x)\right] \\ &= aT\left(p(x)\right) + bT\left(q(x)\right). \end{aligned}$$

This shows that T is a linear operator on $\mathcal{P}_n(\mathbb{R})$.

(b) Take the standard basis $C = \{1, x, \cdots, x^n\}$ for $\mathcal{P}_n(\mathbb{R})$, then

$$\boldsymbol{A} = [T]_C = \begin{pmatrix} 1 & -1 & & & \\ & 2 & -2 & & \\ & & \ddots & \ddots & \\ & & & & -n \\ & & & & n+1 \end{pmatrix}.$$

The characteristic polynomial is

$$c_T(x) = \det(x\boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} x-1 & -1 & & & \\ & x-2 & -2 & & \\ & & \ddots & \ddots & \\ & & & & -n \\ & & & & x-(n+1) \end{vmatrix} = (x-1)(x-2)\cdots[x-(n+1)].$$

(c) Since the characteristic polynomial $c_T(x)$ splits and has distinct roots, so does the minimal polynomial. Hence T is diagonalizable.

Question 4

(a) For any polynomial $p(x) = a + bx + cx^2 \in \mathcal{P}_3(\mathbb{R})$,

$$p(x) \in W \quad \Leftrightarrow \quad p(-1) = p(1) \quad \Leftrightarrow \quad a - b + c = a + b + c \quad \Leftrightarrow \quad \begin{cases} a = s \\ b = 0 \\ c = t \end{cases} \quad \text{for } s, t \in \mathbb{R},$$

i.e. $p(x) \in W$ if and only if $p(x) = s + tx^2$ for some $s, t \in \mathbb{R}$. Thus $W = \operatorname{span}(C)$ where $C = \left\{1, x^2\right\}$ is a basis for W.

(b) We have

$$\begin{aligned} \langle 1, 1 \rangle &= \frac{1}{2}\int_{-1}^{1} \mathrm{d}x = 1, \\ \langle x^2, 1 \rangle &= \frac{1}{2}\int_{-1}^{1} x^2 \,\mathrm{d}x = \frac{1}{3}. \end{aligned}$$

By the Gram-Schmidt Process,

$$\begin{aligned} p_1(x) &= 1, \\ p_2(x) &= x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} = -\frac{1}{3} + x^2 \end{aligned}$$

form an orthogonal basis for W. Hence

$$\left\{ \frac{1}{||p_1(x)||} p_1(x), \frac{1}{||p_2(x)||} p_2(x) \right\} = \left\{ 1, \frac{45}{24}(-1 + 3x^2) \right\}$$

is an orthonormal basis for W.

Question 5

(a) Suppose $\exists c_0, \ldots, c_{n-1} \in \mathbb{F}$ such that

$$c_0 I_V + c_1 T + \ldots + c_{n-1} T^{n-1} = O_V,$$

then

$$\begin{aligned} c_0 I_V(\boldsymbol{v}) + c_1 T(\boldsymbol{v}) + \cdots + c_{n-1} T^{n-1}(\boldsymbol{v}) &= O_V(\boldsymbol{v}) \\ c_0 \boldsymbol{v} + c_1 T(\boldsymbol{v}) + \cdots + c_{n-1} T^{n-1}(\boldsymbol{v}) &= O_V. \end{aligned}$$

Since $\left\{\boldsymbol{v}, T(\boldsymbol{v}), \ldots, T^{n-1}(\boldsymbol{v})\right\}$ is a basis for V, $c_0 = c_1 = \cdots = c_{n-1}$. Thus $I_V, T, \ldots, T^{n-1}$ are linearly independent.

(b) Since $S(\boldsymbol{v}) \in V$ and $\left\{\boldsymbol{v}, T(\boldsymbol{v}), \ldots, T^{n-1}(\boldsymbol{v})\right\}$ is a basis for V, there exist $b_0, b_1, \ldots, b_{n-1} \in \mathbb{F}$ such that

$$\begin{aligned} S(\boldsymbol{v}) &= b_0 \boldsymbol{v} + b_1 T(\boldsymbol{v}) + \cdots + b_{n-1} T^{n-1}(\boldsymbol{v}) \\ &= \left(b_0 I_V + b_1 T + \cdots + b_{n-1} T^{n-1}\right)(\boldsymbol{v}) \end{aligned}$$

For $i = 0, 1, \dots, n-1$,

$$\begin{aligned}
S(T^i(\boldsymbol{v})) &= (S \circ T^i)(\boldsymbol{v}) \\
&= (T^i \circ S)(\boldsymbol{v}) \qquad (\because S \circ T = T \circ S) \\
&= T^i(S(\boldsymbol{v})) \\
&= T^i \left(b_0 I_V + b_1 T + \cdots + b_{n-1} T^{n-1}\right)(\boldsymbol{v}) \\
&= \left(b_0 T^i + b_1 T^{i+1} + \cdots + b_{n-1} T^{i+n-1}\right)(\boldsymbol{v}) \\
&= \left(b_0 I_V + b_1 T + \cdots + b_{n-1} T^{n-1}\right)(T^i(\boldsymbol{v})).
\end{aligned}$$

As $\left\{\boldsymbol{v}, T(\boldsymbol{v}), \dots, T^{n-1}(\boldsymbol{v})\right\}$ is a basis for V, for any $\boldsymbol{u} \in V$, there exist $c_0, c_1, \dots, c_{n-1} \in \mathbb{F}$ such that $\boldsymbol{u} = \sum\limits_{i=0}^{n-1} c_i T^i(\boldsymbol{v})$ and hence

$$\begin{aligned}
S(\boldsymbol{u}) &= \sum_{i=0}^{n-1} c_i S(T^i(\boldsymbol{v})) \\
&= \sum_{i=0}^{n-1} c_i \left(b_0 I_V + b_1 T + \cdots + b_{n-1} T^{n-1}\right)(T^i(\boldsymbol{v})) \\
&= \left(b_0 I_V + b_1 T + \cdots + b_{n-1} T^{n-1}\right)\left(\sum_{i=0}^{n-1} c_i T^i(\boldsymbol{v})\right) \\
&= \left(b_0 I_V + b_1 T + \cdots + b_{n-1} T^{n-1}\right)(\boldsymbol{u}).
\end{aligned}$$

This shows that $S = b_0 I_V + b_1 T + \cdots + b_{n-1} T^{n-1}$.

(c) Take our given basis $\left\{\boldsymbol{v}, T(\boldsymbol{v}), \dots, T^{n-1}(\boldsymbol{v})\right\}$. The representation of T with respect to this basis is

$$C\left(m_T(x)\right) = \begin{pmatrix} & & & & -a_0 \\ 1 & & & & -a_1 \\ & 1 & & & -a_2 \\ & & \ddots & & \vdots \\ & & & 1 & -a_{n-1} \end{pmatrix}$$

where $0 = T^n(\boldsymbol{v}) + a_{n-1} T^{n-1}(\boldsymbol{v}) + \cdots + a_0 \boldsymbol{v}$, so that $m_T(x)$ must be $x^n + a_{n-1} x^{n-1} + \cdots + a_0$. It is apparent that the characteristic polynomial must be

$$\det \begin{pmatrix} x & & & & a_0 \\ -1 & x & & & a_1 \\ & -1 & x & & a_2 \\ & & \ddots & \ddots & \vdots \\ & & & -1 & x + a_{n-1} \end{pmatrix}$$

which can be shown through induction/cofactor expansion along the first row to be $m_T(x)$. Hence the characteristic and minimal polynomials coincide, and so a Jordan canonical form for T is given by

$$\begin{pmatrix} \boldsymbol{J}_{r_1}(\lambda_1) & & & \\ & \boldsymbol{J}_{r_2}(\lambda_2) & & \\ & & \ddots & \\ & & & \boldsymbol{J}_{r_k}(\lambda_k) \end{pmatrix},$$

where $\boldsymbol{J}_r(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$.

Question 6

(a) By the determinant property $\det(\boldsymbol{A})\det(\boldsymbol{B}) = \det(\boldsymbol{AB})$ (for square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of equal size), we have

$$\begin{aligned} c_{\boldsymbol{A}^{-1}}(x) = \det(x\boldsymbol{I} - \boldsymbol{A}^{-1}) &= \frac{1}{\det(\boldsymbol{A})}\det(x\boldsymbol{A} - \boldsymbol{I}) \\ &= \frac{1}{(-1)^n \det(\boldsymbol{A})} x^n \det(\boldsymbol{I}x^{-1} - \boldsymbol{A}) \end{aligned}$$

Rewriting the terms using the facts that $\frac{1}{(-1)^n \det(\boldsymbol{A})} = \frac{1}{0\boldsymbol{I}-\boldsymbol{A}} = [c_{\boldsymbol{A}}(0)]^{-1}$ and $\det(\boldsymbol{I}x^{-1} - \boldsymbol{A}) = c_{\boldsymbol{A}}(x^{-1})$, we get the desired equality.

(b) Let $m_{\boldsymbol{A}}(x) = a_0 + a_1 x + \cdots + a_{k-1}x^{k-1} + x^k$. Suppose $m_{\boldsymbol{A}}(0) = a_0 = 0$, that is

$$m_{\boldsymbol{A}}(x) = (a_1 + \cdots + a_{k-1}x^{k-2} + x^{k-1})x$$

so that

$$\mathbf{0} = m_{\boldsymbol{A}}(\boldsymbol{A}) = (a_1 + \cdots + a_{k-1}\boldsymbol{A}^{k-2} + \boldsymbol{A}^{k-1})\boldsymbol{A}.$$

Now $g(x) = a_1 + \cdots + a_{k-1}\boldsymbol{A}^{k-2} + \boldsymbol{A}^{k-1}$ must be nonzero otherwise this would contradict minimality of $m_{\boldsymbol{A}}(x)$, but then $\boldsymbol{A}$ cannot be invertible, which is not true. The conclusion thus follows.

(c) Let $f(x) = x^k[m_{\boldsymbol{A}}(0)]^{-1}m_{\boldsymbol{A}}(x^{-1})$. Note that $\deg(f(x)) = k$ and $f(\boldsymbol{A}^{-1}) = 0$.
Suppose some polynomial g' exists with degree $j' < k$ such that $g'(\boldsymbol{A}^{-1}) = 0$. Then by the same logic as the previous item, there must exist a polynomial g with degree $j \leq< j' < k$ such that $g(\boldsymbol{A}^{-1}) = 0$ and $g(0) \neq 0$. Then $h(x) = x^j g(x^{-1})$ would be a polynomial of degree j such that $h(\boldsymbol{A}) = 0$, contradicting minimality of $m_{\boldsymbol{A}}(x)$. Hence $m_{\boldsymbol{A}^{-1}}(x) = f(x)$.

(d) Note, using the formulae given in previous questions, that $c_{\boldsymbol{A}^{-1}}(x) = (x+1)^3(x+\frac{1}{2})^2$ and $m_{\boldsymbol{A}^{-1}}(x) = (x+1)(x+\frac{1}{2})^2$. Therefore, a Jordan canonical form of $\boldsymbol{A}^{-1}$ is given by

$$\boldsymbol{J} = \begin{pmatrix} -\frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Question 7

(a) (i) For any $\boldsymbol{u} \in V$,

$$\begin{aligned} (F_{\boldsymbol{n}} \circ F_{\boldsymbol{n}})(\boldsymbol{u}) &= F_{\boldsymbol{n}}(F_{\boldsymbol{n}}(\boldsymbol{u})) \\ &= F_{\boldsymbol{n}}(\boldsymbol{u} - 2\langle \boldsymbol{u}, \boldsymbol{n}\rangle \boldsymbol{n}) \\ &= \boldsymbol{u} - 2\langle \boldsymbol{u}, \boldsymbol{n}\rangle \boldsymbol{n} - 2\langle \boldsymbol{u} - 2\langle \boldsymbol{u}, \boldsymbol{n}\rangle \boldsymbol{n}, \boldsymbol{n}\rangle \boldsymbol{n} \\ &= \boldsymbol{u} - 2\langle \boldsymbol{u}, \boldsymbol{n}\rangle \boldsymbol{n} - 2\langle \boldsymbol{u}, \boldsymbol{n}\rangle \boldsymbol{n} + 4\langle \boldsymbol{u}, \boldsymbol{n}\rangle\langle \boldsymbol{n}, \boldsymbol{n}\rangle \boldsymbol{n} \\ &= \boldsymbol{u} - 4\langle \boldsymbol{u}, \boldsymbol{n}\rangle \boldsymbol{n} + 4\langle \boldsymbol{u}, \boldsymbol{n}\rangle \boldsymbol{n} \qquad (\because \langle \boldsymbol{n}, \boldsymbol{n}\rangle = 1) \\ &= \boldsymbol{u}. \end{aligned}$$

Therefore $F_{\boldsymbol{n}} \circ F_{\boldsymbol{n}} = I_V$.

(ii) For any $\boldsymbol{u} \in V$,

$$\begin{aligned}
\langle F_{\boldsymbol{n}}(\boldsymbol{u}), F_{\boldsymbol{n}}(\boldsymbol{u})\rangle &= \langle \boldsymbol{u} - 2\langle \boldsymbol{u}, \boldsymbol{n}\rangle \boldsymbol{n}, \boldsymbol{u} - 2\langle \boldsymbol{u}, \boldsymbol{n}\rangle \boldsymbol{n}\rangle \\
&= \langle \boldsymbol{u}, \boldsymbol{u}\rangle - 2\langle \boldsymbol{u}, \boldsymbol{n}\rangle\langle \boldsymbol{n}, \boldsymbol{u}\rangle - 2\langle \boldsymbol{u}, \boldsymbol{n}\rangle\langle \boldsymbol{u}, \boldsymbol{n}\rangle + 4\langle \boldsymbol{u}, \boldsymbol{n}\rangle^2\langle \boldsymbol{n}, \boldsymbol{n}\rangle \\
&= \langle \boldsymbol{u}, \boldsymbol{u}\rangle - 4\langle \boldsymbol{u}, \boldsymbol{n}\rangle^2 + 4\langle \boldsymbol{u}, \boldsymbol{n}\rangle^2 \\
&= \langle \boldsymbol{u}, \boldsymbol{u}\rangle.
\end{aligned}$$

It follows that $||F_{\boldsymbol{n}}(\boldsymbol{u})|| = ||\boldsymbol{u}||$ and thus $F_{\boldsymbol{n}}$ is orthogonal.

(b) (i) Since $F_{\boldsymbol{n}} \circ F_{\boldsymbol{n}} = I_V$, $F_{\boldsymbol{n}}(S(\boldsymbol{w})) = \boldsymbol{w}$ implies $S(\boldsymbol{w}) = F_{\boldsymbol{n}}(\boldsymbol{w})$.
Observe that it is possible to recover the direction vector $\boldsymbol{n}$ from $\boldsymbol{w}$ and its reflected image, $F_{\boldsymbol{n}}(\boldsymbol{w})$, by subtracting $F_{\boldsymbol{n}}(\boldsymbol{w})$ from $\boldsymbol{w}$ followed by a normalization. In other words, $\boldsymbol{n}$ is given by

$$\boldsymbol{n} = \frac{\boldsymbol{w} - F_{\boldsymbol{n}}(\boldsymbol{w})}{||\boldsymbol{w} - F_{\boldsymbol{n}}(\boldsymbol{w})||} = \frac{\boldsymbol{w} - S(\boldsymbol{w})}{||\boldsymbol{w} - S(\boldsymbol{w})||}.$$

(ii) Take any $\boldsymbol{v} \in W$. Since $S(\boldsymbol{v}) = \boldsymbol{v}$,

$$\begin{aligned}
(F_{\boldsymbol{n}} \circ S)(\boldsymbol{v}) &= (F_{\boldsymbol{n}}(S(\boldsymbol{v})) \\
&= F_{\boldsymbol{n}}(\boldsymbol{v}) \\
&= \boldsymbol{v} - \langle \boldsymbol{v}, \boldsymbol{n}\rangle \boldsymbol{n} \\
&= \boldsymbol{v} - \frac{1}{||\boldsymbol{w} - S(\boldsymbol{w})||}\langle \boldsymbol{v}, \boldsymbol{w} - S(\boldsymbol{w})\rangle \boldsymbol{n}.
\end{aligned}$$

Now because

$$\langle \boldsymbol{v}, \boldsymbol{w} - S(\boldsymbol{w})\rangle = \langle \boldsymbol{v}, \boldsymbol{w}\rangle - \langle \boldsymbol{v}, S(\boldsymbol{w})\rangle = \langle S(\boldsymbol{v}), S(\boldsymbol{w})\rangle - \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \langle S(\boldsymbol{v}) - \boldsymbol{v}, \boldsymbol{w}\rangle = \langle \boldsymbol{0}, \boldsymbol{w}\rangle = 0,$$

we must have $(F_{\boldsymbol{n}} \circ S)(\boldsymbol{v}) = \boldsymbol{v}$, meaning $\boldsymbol{v} \in E_1(F_{\boldsymbol{n}} \circ S)$. Hence $W \subseteq E_1(F_{\boldsymbol{n}} \circ S)$.
However, we know that $\boldsymbol{w} \notin W$ but $\boldsymbol{w} \in E_1(F_{\boldsymbol{n}} \circ S)$. It can therefore be concluded that $W \subsetneq E_1(F_{\boldsymbol{n}} \circ S)$.

END OF SOLUTIONS

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