NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lin Mingyan Simon, Chang Hai Bin

MA1104 Multivariable Calculus
AY 2010/2011 Sem 1

Question 1

(i) Since the plane contain the points $(3, -1, 2)$, $(8, 2, 4)$ and $(-1, -2, -3)$, it follows that the plane must contain the vectors $\langle 3-(-1), -1-(-2), 2-(-3)\rangle = \langle 4, 1, 5\rangle$ and $\langle 8-3, 2-(-1), 4-2\rangle = \langle 5, 3, 2\rangle$. So a vector perpendicular to the plane is $\langle 4, 1, 5\rangle \times \langle 5, 3, 2\rangle = \langle -13, 17, 7\rangle$.
Hence, the equation of the plane W must satisfy the following equation:

$$\langle x - 8, y - 2, z - 4\rangle \cdot \langle -13, 17, 7\rangle = 0.$$

This gives us the equation of the plane to be $-13x + 17y + 7z = -42$.

(ii) Note that U is parallel to W if and only if the vector $\langle -13, 17, 7\rangle$ is perpendicular to U.
So the equation of the plane U must be of the form $-13x + 17y + 7z = d$, $d \in \mathbb{R}$. Now, the distance between U and W is equal to $\left|\frac{d-(-42)}{\sqrt{(-13)^2+17^2+7^2}}\right| = \sqrt{3}$, which after simplification, is equivalent to $|d + 42| = 39$. This gives us $d + 42 = \pm 39$, or equivalently, $d = -3$ or $d = -81$.

So the equations of the planes U which are parallel to the plane W in part (i), and whose distance from U to W is $\sqrt{3}$, are $-13x + 17y + 7z = -3$ and $-13x + 17y + 7z = -81$.

Question 2

(i) Note that parametric equations for the cylinder $x^2+y^2 = 1$ are $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.
So the equation of C is $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t\rangle$, $0 \le t \le 2\pi$.

(ii) Note that $\mathbf{r}'(t) = \langle -\sin t, \cos t, \cos t\rangle$. At $(1, 0, 0)$, we have $\cos t = 1$, so one has $t = 0$.
Hence, the tangent line of C at $(1, 0, 0)$ must be parallel to $\mathbf{r}'(0) = \langle -\sin 0, \cos 0, \cos 0\rangle = \langle 0, 1, 1\rangle$. Therefore, the parametric equations of the tangent line of C at $(1, 0, 0)$ is $x = 1$, $y = k$, $z = k$, $k \in \mathbb{R}$.

(iii)

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) = \langle -\sin t, \cos t, \cos t\rangle \\
|\mathbf{v}(t)| &= \sqrt{(-\sin t)^2 + (\cos t)^2 + (\cos t)^2} = \sqrt{1 + \cos^2 t} \\
|\mathbf{v}(0)| &= \sqrt{1 + \cos^2 0} = \sqrt{2} \\
\mathbf{T}(t) &= \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{1}{\sqrt{1 + \cos^2 t}}\langle -\sin t, \cos t, \cos t\rangle \\
\mathbf{T}'(t) &= \frac{1}{(1 + \cos^2 t)^{3/2}}\langle -2\cos^3 t, -\sin t, -\sin t\rangle \\
\mathbf{T}'(0) &= \langle -\frac{1}{\sqrt{2}}, 0, 0\rangle \\
|\mathbf{T}'(0)| &= \frac{1}{\sqrt{2}} \\
\kappa(0) &= \frac{|\mathbf{T}'(0)|}{|\mathbf{v}(0)|} = \frac{1}{2}.
\end{aligned}$$

So the curvature of C at $(1, 0, 0)$ is $\frac{1}{2}$.

Question 3

(i) Let $f(x,y) = \frac{xy^2}{x^2+4y^4}$ for all $(x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}$. We shall prove by the two-path test that the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Along the path $x = 0$, we see that $f(x,y) = f(0,y) = 0$ for all $y \neq 0$. So as (x,y) approaches $(0,0)$ along the path $x = 0$, we have $f(x,y) \to 0$.
Along the path $x = y^2$, we see that $f(x,y) = f(y^2,y) = \frac{(y^2)y^2}{(y^2)^2+4y^4} = \frac{1}{5}$ for all $y \neq 0$. So as (x,y) approaches $(0,0)$ along the path $x = y^2$, we have $f(x,y) \to \frac{1}{5}$.

Thus, by the two-path test, we see that the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(ii) Let $u = x$, $v = 2y^2$ and $w = 3z^3$. Then one has

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy^2z}{x^2+4y^4+9y^6} = \frac{\sqrt[3]{9}}{6} \lim_{(u,v,w)\to(0,0,0)} \frac{uv\sqrt[3]{w}}{u^2+v^2+w^2}. \tag{1}$$

Lemma: if $a,b,c,d,e,f,g,h \geq 0$ are non-negative real numbers,
then $a+b+c+d+e+f+g+h \geq 8(abcdefgh)^{1/8}$
Proof:

$$\begin{aligned} a+b+c+d+e+f+g+h &\geq 2\left[a^{1/2}b^{1/2} + c^{1/2}d^{1/2} + e^{1/2}f^{1/2} + g^{1/2}h^{1/2}\right] \\ &\geq 4\left[a^{1/4}b^{1/4}c^{1/4}d^{1/4} + e^{1/4}f^{1/4}g^{1/4}h^{1/4}\right] \\ &\geq 8\left[abcdefgh\right]^{1/8} \end{aligned}$$

So,

$$\begin{aligned} \left|\frac{uvw^{1/3}}{u^2+v^2+w^2}\right| &= \frac{|uvw^{1/3}|}{\frac{1}{3}u^2+\frac{1}{3}u^2+\frac{1}{3}u^2+\frac{1}{4}v^2+\frac{1}{4}v^2+\frac{1}{4}v^2+\frac{1}{4}v^2+w^2} \\ &\leq \frac{|uvw^{1/3}|}{8\left[\left(\frac{1}{3}u^2\right)^{1/8}\right]^3\left[\left(\frac{1}{4}v^2\right)^{1/8}\right]^4(w^2)^{1/8}} \\ &\leq \frac{|u|\cdot|v|\cdot|w^{1/3}|}{8\left(\frac{1}{3}\right)^{3/8}\left(\frac{1}{4}\right)^{1/2}|u|^{3/4}|v|^1|w|^{1/4}} \\ &= \frac{1}{8\left(\frac{1}{3}\right)^{3/8}\left(\frac{1}{4}\right)^{1/2}}|u|^{1/4}\cdot|w|^{1/12} \end{aligned}$$

So, by squeeze theorem, $\lim_{(u,v,w)\to(0,0,0)} \frac{uv\sqrt[3]{w}}{u^2+v^2+w^2} = 0$. Hence,

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy^2z}{x^2+4y^4+9y^6} = 0.$$

Question 4

(i) Let $f(x,y,z) = x^2 - 2y^2 + z^2 + yz$. Then one has $\nabla f(x,y,z) = \langle 2x, -4y+z, 2z+y\rangle$, so $\nabla f(2,1,-1) = \langle 2(2), -4(1)+(-1), 2(-1)+1\rangle = \langle 4,-5,-1\rangle$.
Hence, the equation of the tangent plane at $P(1,1,1)$ must satisfy the following equation:

$$\langle x-2, y-1, z-(-1)\rangle \cdot \langle 4,-5,-1\rangle = 0.$$

This gives us the equation of the tangent plane at $P(2,1,-1)$ to be $4x - 5y - z = 4$.

(ii) From part (i), we see that a normal to the tangent plane is the vector $\langle 4, -5, -1\rangle$. As the line is perpendicular to the tangent plane in part (i), it follows that the line must be parallel to the vector $\langle 4, -5, -1\rangle$.
So the equation of the line that is perpendicular to the tangent plane in part (i) and passes through $P(2, 1, -1)$ is $x = 2 + 4t$, $y = 1 - 5t$, $z = -1 - t$, $t \in \mathbb{R}$.

Question 5

Let $V(x, y, z) = xyz$. In order to find the maximum volume of the box, we need to find the maximum value of $V(x, y, z)$, subject to the constraint $xy + 2xz + 2yz = 300$.
Let $R(x, y, z) = xy + 2xz + 2yz - 300$. Then it follows that $\nabla V(x, y, z) = \langle yz, xz, xy\rangle$ and $\nabla R(x, y, z) = \langle y + 2z, x + 2z, 2(x + y)\rangle$. By the Method of Lagrange Multipliers, we have

$$\begin{aligned}
& \nabla V(x,y,z) = \lambda \nabla R(x,y,z) \\
\Rightarrow\ & \langle yz, xz, xy\rangle = \lambda\langle y+2z, x+2z, 2(x+y)\rangle \\
\Rightarrow\ & yz = \lambda(y+2z),\ xz = \lambda(x+2z),\ xy = 2\lambda(x+y) \\
\Rightarrow\ & xyz = \lambda xy + 2\lambda xz = \lambda xy + 2\lambda xy = 2\lambda xz + 2\lambda yz \\
\Rightarrow\ & 2\lambda x(y-z) = 0, \lambda y(x-2z) = 0.
\end{aligned}$$

If $\lambda = 0$, then this would imply that $xy = xz = yz = 0$, so $xy + 2xz + 2yz = 0 \neq 300$, which is a contradiction. Hence, one has $\lambda \neq 0$. Also, if $x = 0$, then we have $\lambda y(x - 2z) = -2\lambda yz = 0$, so we must have $yz = 0$. Thus we also get $xy + 2xz + 2yz = 0 \neq 300$, which is a contradiction. Hence $x \neq 0$, and so one has $y = z$. In particular, $y = z \neq 0$, so we must have $x = 2z$.

Hence, one has $xy + 2xz + 2yz = (2z)z + 2(2z)z + 2(z)z = 8z^2 = 300$, so one has $z = \frac{5\sqrt{6}}{2}$. Thus, $y = z = \frac{5\sqrt{6}}{2}$, $x = 2z = 5\sqrt{6}$, and the maximum volume of the box is equal to $V(x, y, z) = xyz = 5\sqrt{6}\left(\frac{5\sqrt{6}}{2}\right)\left(\frac{5\sqrt{6}}{2}\right) = \frac{375\sqrt{6}}{2}$.

Question 6

Since $f(x, y)$ is differentiable at (a, b), it follows that $f_x(a, b)$ and $f_y(a, b)$ both exist, and one has $\triangle f(x, y) = f_x(a, b)\triangle x + f_y(a, b)\triangle y + \epsilon_1\triangle x + \epsilon_2\triangle y$, in which each of ϵ_1, ϵ_2 tends to 0 as both $\triangle x, \triangle y$ tend to 0. Now, we have $\triangle f(x, y) = f(x, y) - f(a, b)$, $\triangle x = x - a$, $\triangle y = y - b$, so for all $(x, y) \neq (a, b)$, one has

$$\begin{aligned}
\left|\frac{f(x,y) - f(a,b) - \nabla f(a,b)\cdot\langle x-a, y-b\rangle}{\sqrt{(x-a)^2+(y-b)^2}}\right| &= \frac{|\triangle f(x,y) - (f_x(a,b)\triangle x + f_y(a,b)\triangle y)|}{\sqrt{(\triangle x)^2+(\triangle y)^2}} \\
&= \frac{|\epsilon_1\triangle x + \epsilon_2\triangle y|}{\sqrt{(\triangle x)^2+(\triangle y)^2}} \\
&\leq \frac{|\epsilon_1||\triangle x| + |\epsilon_2||\triangle y|}{\sqrt{(\triangle x)^2+(\triangle y)^2}} \\
&\leq \frac{\sqrt{|\epsilon_1|^2+|\epsilon_2|^2}\sqrt{|\triangle x|^2+|\triangle y|^2}}{\sqrt{(\triangle x)^2+(\triangle y)^2}} \\
&\leq \frac{(|\epsilon_1|+|\epsilon_2|)\sqrt{(\triangle x)^2+(\triangle y)^2}}{\sqrt{(\triangle x)^2+(\triangle y)^2}} = |\epsilon_1| + |\epsilon_2|.
\end{aligned}$$

As $\lim\limits_{(x,y)\to(a,b)} \epsilon_1 = \lim\limits_{(x,y)\to(a,b)} \epsilon_2 = 0$, it follows that $\lim\limits_{(x,y)\to(a,b)} (|\epsilon_1| + |\epsilon_2|) = 0$. Hence, it follows from Squeeze Theorem that the limit $\lim\limits_{(x,y)\to(a,b)} \frac{f(x,y)-f(a,b)-\nabla f(a,b)\cdot\langle x-a,y-b\rangle}{\sqrt{(x-a)^2+(y-b)^2}}$ exists, and is equal to 0.

Question 7

(i) By differentating both sides of the equation with respect to t, one has

$$\begin{aligned} \frac{d}{dt}(f(tx,ty)) &= \frac{d}{dt}(t^7 f(x,y)) \\ \Rightarrow \frac{\partial f}{\partial(tx)}\frac{d(tx)}{dt} + \frac{\partial f}{\partial(ty)}\frac{d(ty)}{dt} &= 7t^6 f(x,y) \\ \Rightarrow x\frac{\partial f}{\partial(tx)} + y\frac{\partial f}{\partial(ty)} &= 7t^6 f(x,y). \end{aligned}$$

By setting $t=1$ in the last equation, one has $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 7f(x,y)$.
So $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = Cf(x,y)$ for some constant C as desired, and $C=7$.

(ii) By differentating both sides of the equation with respect to x, one has

$$\frac{\partial f(tx,ty)}{\partial x} = \frac{\partial f(tx,ty)}{\partial(tx)}\frac{\partial(tx)}{\partial x} = tf_x(tx,ty) = t^7 f_x(x,y)$$

So $f_x(tx,ty) = t^n f_x(x,y)$ for some integer n as desired, and $n=6$.

Question 8

(i) It is easy to see that $0 < 1-y^2 < \frac{1}{x^2} - y^2$ and $0 < 1-x^2 < \frac{1}{y^2} - x^2$. Hence, the values $\sin^{-1}\sqrt{\frac{1-y^2}{x^{-2}-y^2}}$ and $\sin^{-1}\sqrt{\frac{1-x^2}{y^{-2}-x^2}}$ are well defined, and hence we may assume from the hint that $0 < u < \frac{\pi}{2}$ and $0 < v < \frac{\pi}{2}$.

Next, from the given condition that $0 < x < 1$, we have $0 < \frac{\sin u}{\cos v} < 1$, or equivalently, $\sin u < \cos v = \sin\left(\frac{\pi}{2} - v\right)$. Since the sine curve is strictly increasing on $\left(0, \frac{\pi}{2}\right)$, one has $u < \frac{\pi}{2} - v$, or equivalently $u + v < \frac{\pi}{2}$.

Now, let G be the region $\{(u,v) \in \mathbb{R}^2 | u > 0, v > 0, u + v < \frac{\pi}{2}\}$. It remains to check that $T : G \to R$ is surjective and hence bijective, leaving the proof to show that T is an injection in part (ii).

Note that from the hint given, one has

$$\sin^2 u + \sin^2 v = \frac{x^2(1-y^2)}{1-x^2y^2} + \frac{y^2(1-x^2)}{1-x^2y^2} = \frac{[1-(1-x^2)(1-y^2)] - x^2y^2}{1-x^2y^2}.$$

Since $0 < x < 1$ and $0 < y < 1$, one has $0 < (1-x^2)(1-y^2) < 1$. Thus one has

$$\sin^2 u + \sin^2 v = \frac{[1-(1-x^2)(1-y^2)] - x^2y^2}{1-x^2y^2} < \frac{1-x^2y^2}{1-x^2y^2} < 1,$$

so this implies that $\sin^2 u < 1 - \sin^2 v = \cos^2 v = \sin^2\left(\frac{\pi}{2} - v\right)$. Hence, one must have $\sin u < \sin\left(\frac{\pi}{2} - v\right)$, so by the earlier argument one has $u + v < \frac{\pi}{2}$. This shows that for a given $(x,y) \in R$, we have found an $(u,v) \in G$, such that $T(u,v) = (x,y)$.

Hence, we conclude that the region G must be $\{(u,v) \in \mathbb{R}^2 | u > 0, v > 0, u + v < \frac{\pi}{2}\}$.

(ii) Suppose there exist (u_1, v_1) and (u_2, v_2) in G such that $T(u_1, v_1) = T(u_2, v_2)$. We would like to show that $u_1 = v_1$ and $u_2 = v_2$, thereby showing that T is injective.

Using the hints, if $x = \dfrac{\sin u}{\cos v}, y = \dfrac{\sin v}{\cos u}$, then

$$\begin{aligned}\frac{1-y^2}{x^{-2}-y^2} = \frac{1-\dfrac{\sin^2 v}{\cos^2 u}}{\dfrac{\cos^2 v}{\sin^2 u}-\dfrac{\sin^2 v}{\cos^2 u}} &= \frac{(\cos^2 u - \sin^2 v)\sin^2 u}{\cos^2 v\cos^2 u - \sin^2 v \sin^2 u} \\ &= \frac{(\cos^2 u - \sin^2 v)\sin^2 u}{(1-\sin^2 v)\cos^2 u - \sin^2 v\sin^2 u} \\ &= \frac{(\cos^2 u - \sin^2 v)\sin^2 u}{\cos^2 u - \sin^2 v} = \sin^2 u\end{aligned}$$

Similarly,

$$\frac{1-x^2}{y^{-2}-x^2} = \frac{1-\dfrac{\sin^2 u}{\cos^2 v}}{\dfrac{\cos^2 u}{\sin^2 v}-\dfrac{\sin^2 u}{\cos^2 v}} = \sin^2 v$$

$$\begin{aligned}& T(u_1,v_1) = T(u_2,v_2) \\ \Rightarrow\ & \frac{\sin u_1}{\cos v_1} = x_1 = x_2 = \frac{\sin u_2}{\cos v_2} \quad \text{and} \quad \frac{\sin v_1}{\cos u_1} = y_1 = y_2 = \frac{\sin v_2}{\cos u_2} \\ \Rightarrow\ & \sin^2 u_1 = \frac{1-(y_1)^2}{(x_1)^{-2}-(y_1)^2} = \frac{1-(y_2)^2}{(x_2)^{-2}-(y_2)^2} = \sin^2 u_2 \quad \text{and} \\ & \sin^2 v_1 = \frac{1-(x_1)^2}{(y_1)^{-2}-(x_1)^2} = \frac{1-(x_2)^2}{(y_2)^{-2}-(x_2)^2} = \sin^2 v_2\end{aligned}$$

Since $u_1, u_2, v_1, v_2 \in (0, \frac{\pi}{2})$ and $\sin^2 u_1 = \sin^2 u_2, \sin^2 v_1 = \sin^2 v_2$, so we must have $u_1 = u_2, v_1 = v_2$.

So $T : G \to R$ is an injection as desired.

(iii) We have $\frac{\partial x}{\partial u} = \frac{\cos u}{\cos v}$, $\frac{\partial x}{\partial v} = \frac{\sin u \sin v}{\cos^2 v}$, $\frac{\partial y}{\partial u} = \frac{\sin u \sin v}{\cos^2 u}$ and $\frac{\partial y}{\partial v} = \frac{\cos v}{\cos u}$. Thus, the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{\cos u}{\cos v}\cdot\frac{\cos v}{\cos u} - \frac{\sin u \sin v}{\cos^2 v}\cdot\frac{\sin u\sin v}{\cos^2 u} = 1 - \left(\frac{\sin u \sin v}{\cos u \cos v}\right)^2.$$

Thus, one has

$$\begin{aligned}\iint_R \frac{1}{1-(xy)^2}dx\,dy &= \iint_G \frac{1}{1-\left(\left(\frac{\sin u}{\cos v}\right)\left(\frac{\sin v}{\cos u}\right)\right)^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \\ &= \iint_G \frac{1}{1-\left(\frac{\sin u \sin v}{\cos u\cos v}\right)^2}\cdot\left(1-\left(\frac{\sin u\sin v}{\cos u\cos v}\right)^2\right)du\,dv \\ &= \iint_G du\,dv \\ &= \text{Area of } G \\ &= \frac{1}{2}\cdot\frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}.\end{aligned}$$

(iv) Firstly, we note that for all $n \in \mathbb{Z}$, $n \geq 0$, one has

$$\iint_R (xy)^{2n}\,dx\,dy = \int_0^1 x^{2n}\,dx \int_0^1 y^{2n}\,dy = \frac{1}{(2n+1)^2}.$$

Secondly, we note that since $0 < x < 1$ and $0 < y < 1$, one has $0 < |xy| < 1$, so the series $\sum\limits_{n=0}^{\infty} (xy)^{2n}$ converges, and is equal to $\frac{1}{1-(xy)^2}$.
Using these two facts above, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\ &= \sum_{n=0}^{\infty} \iint_R (xy)^{2n}\, dx\, dy \\ &= \iint_R \sum_{n=0}^{\infty} (xy)^{2n}\, dx\, dy \\ &= \iint_R \frac{1}{1-(xy)^2}\, dx\, dy = \frac{\pi^2}{8}.\end{aligned}$$

At the same time, we also have

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Therefore, we have $S = \sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{6}$.

Remark: Strictly speaking, it is necessary to check that the equality $\sum\limits_{n=0}^{\infty} \iint_R (xy)^{2n}\, dx\, dy$ $= \iint_R \sum\limits_{n=0}^{\infty} (xy)^{2n}\, dx\, dy$ does hold, before interchanging the summation and the integration signs. Indeed, equality does hold by the Lebesgue's Monotone Convergence Theorem (also known as Beppo Levi Theorem), but the proof of which shall be omitted here as it is beyond the scope of the MA1104 course.

Question 9

(i) Let $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ where $P(x,y,z) = e^{xy}(xy+k)\sin z$, $Q(x,y,z) = x^2e^{xy}\sin z$ and $R(x,y,z) = xe^{xy}(\cos z) + 2z$. Clearly, $\mathbf{F}$ is a field whose component functions have continuous first partial derivatives.
Note that $\frac{\partial P}{\partial z} = e^{xy}(xy+k)\cos z$ and $\frac{\partial R}{\partial x} = (x(ye^{xy}) + e^{xy}(1))\cos z = e^{xy}(xy+1)\cos z$. As $\mathbf{F}$ is conservative, one has $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, so this implies that $e^{xy}(xy+k)\cos z = e^{xy}(xy+1)\cos z$. Then it is easy to see that $k = 1$.

(ii) Since $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$, we have $f_x(x,y,z) = e^{xy}(xy+1)\sin z$, $f_y(x,y,z) = x^2e^{xy}\sin z$, $f_z(x,y,z) = xe^{xy}(\cos z) + 2z$.

By integrating f_z w.r.t. z, we get $f(x,y,z) = xe^{xy}(\sin z) + z^2 + g(x,y)$, where g is some function of x and y with continuous first partial derivatives.
By differentiating the above w.r.t. y, we get $f_y(x,y,z) = x^2e^{xy}(\sin z) + g_y(x,y)$, so one has $g_y(x,y) = 0$.
By integrating g_y w.r.t. y, we get $g(x,y) = h(x)$, where h is some continuously differentiable function of x. This implies that $f(x,y,z) = xe^{xy}(\sin z) + z^2 + h(x)$.
By differentiating the above w.r.t. x, we get $f_x(x,y,z) = e^{xy}(xy+1)\sin z + h'(x)$, so one has $h'(x) = 0$.
Hence one has $h(x) = C$ for some constant C so one has $f(x,y,z) = xe^{xy}(\sin z) + z^2 + C$.

So a function f that satisfies $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$ is $f(x,y,z) = xe^{xy}(\sin z) + z^2$.

(iii) We have the start point to be $\mathbf{r}(0) = (\sin 0, \cos^2 0, 0) = (0,1,0)$, and the end point to be $\mathbf{r}\left(\frac{\pi}{2}\right) = \left(\sin\frac{\pi}{2}, \cos^2\frac{\pi}{2}, \frac{\pi}{2}\right) = \left(1,0,\frac{\pi}{2}\right)$. As $\mathbf{F}$ is conservative, it follows from the Fundamental Theorem for Line Integrals that

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= f(\mathbf{r}(1)) - f(\mathbf{r}(0)) \\ &= f\left(1,0,\frac{\pi}{2}\right) - f(0,1,0) \\ &= \left[(1)e^{1(0)}\left(\sin\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right)^2\right] - [(0)e^{0(1)}(\sin 0) + (0)^2] = 1 + \frac{\pi^2}{4}.\end{aligned}$$

Question 10

We have
$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}\left(z^2 x\right) + \frac{\partial}{\partial y}\left(\frac{1}{3}y^3 + \sin z\right) + \frac{\partial}{\partial z}\left(x^2 z + y^2\right) = z^2 + y^2 + x^2.$$
By converting into spherical coordinates, i.e. $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$, we see that S is given by $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \frac{\pi}{2}$. Therefore, by Divergence Theorem, one has

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_H \text{div } \mathbf{F}\, dV \\ &= \iiint_H z^2 + y^2 + x^2\, dV \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^2\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\phi\, d\phi \int_0^1 \rho^4\, d\rho \\ &= [\theta]_0^{2\pi}[-\cos\phi]_0^{\pi/2}\left[\frac{\rho^5}{5}\right]_0^1 \\ &= 2\pi\cdot 1\cdot\frac{1}{5} = \frac{2\pi}{5},\end{aligned}$$

where H denotes the unit hemisphere above the $xy-$plane.