NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Ku Cheng Yeaw

solutions prepared by Tay Jun Jie

MA2108 Mathematical Analysis I
AY 2009/2010 Sem 1

Question 1

(a) Let $M > 0$ be given and let $\delta = \min\left\{\sqrt{\frac{2}{M}}, 2\right\}$. If $|x+2| < \sqrt{\frac{2}{M}}$, then

$$\frac{1}{(x+2)^2} = \frac{1}{|x+2|^2} > \frac{M}{2}.$$

On the other hand, if $|x+2| < 2$, then

$$x^2 - 3x + 2 > 2.$$

Therefore, if $|x+2| < \delta$,

$$\Rightarrow \frac{1}{(x+2)^2} > \frac{M}{2} \quad \text{and} \quad x^2 - 3x + 2 > 2$$
$$\Rightarrow \frac{x^2-3x+2}{(x+2)^2} > M$$

(b) Firstly we claim that if S is a set of positive real numbers such that $\sup S = K$, then $\inf \frac{1}{S} = \frac{1}{K}$ where the set $\frac{1}{S} := \left\{\frac{1}{s} \ : \ s \in S\right\}$.

$$\Rightarrow K > 0 \qquad \text{and} \qquad s \le K \ \forall s \in S$$
$$\Rightarrow \frac{1}{s} \ge \frac{1}{K} \qquad \forall s \in S$$

Hence $\frac{1}{K}$ is an lower bound of $\frac{1}{S}$. Now, let $\varepsilon > 0$ be given.

$$\Rightarrow 1 + \varepsilon K > 1$$
$$\Rightarrow \frac{K}{1+\varepsilon K} < K$$
$$\Rightarrow K - \frac{K}{1+\varepsilon K} > 0$$
$$\Rightarrow \exists s_0 \in S \quad \text{such that} \quad s_0 > K - \left(K - \frac{K}{1+\varepsilon K}\right)$$
$$\Rightarrow \frac{1}{s_0} < \frac{1+\varepsilon K}{K} = \frac{1}{K} + \varepsilon$$

Thus proving the claim. Let $y_m := \sup\{x_n \ : \ 1 \le n \le m\}$ for each positive integer m.

$$\Rightarrow \lim_{m\to\infty} y_m = L$$
$$\Rightarrow \frac{1}{y_m} = \inf\left\{\frac{1}{x_n} \ : \ 1 \le n \le m\right\} \quad \text{by the claim}$$
$$\Rightarrow \liminf \frac{1}{x_n} = \lim_{m\to\infty} \frac{1}{y_m} = \frac{1}{L}$$

Question 2

(a) (i) Since factorials grow faster then exponentials, $\exists N \in \mathbb{N}$ such that $2^{2n} \leq n! \; \forall n \geq N$.

$$\Rightarrow \sum_{n=N}^{\infty} \frac{n! + 2^{2n}}{n^n} \leq 2 \sum_{n=N}^{\infty} \frac{n!}{n^n} \leq 2 \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Now, we shall perform the ratio test on $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

$$\lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1)^{n+1}} \div \frac{n!}{n^n} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n = \frac{1}{e} < 1$$

Hence $\sum_{n=N}^{\infty} \frac{n!+2^{2n}}{n^n}$ converges. We conclude that $\sum_{n=1}^{\infty} \frac{n!+2^{2n}}{n^n}$ converges absolutely.

(ii)

$$\sum_{n=1}^{\infty} \frac{n-1}{n\sqrt{n+1}-1} = \sum_{m=1}^{\infty} \frac{m}{(m+1)\sqrt{m+2}-1}$$

Note that $\frac{m}{(m+1)\sqrt{m+2}-1} > 0 \; \forall m \in \mathbb{N}$. We shall perform the limit comparison test with $\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}}$.

$$\lim_{m\to\infty} \frac{m}{(m+1)\sqrt{m+2}-1} \div \frac{1}{\sqrt{m}} = \lim_{m\to\infty} \frac{1}{\left(1+\frac{1}{m}\right)\sqrt{1+\frac{2}{m}} - \frac{1}{m\sqrt{m}}} = 1 > 0$$

Since $\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}}$ diverges, we conclude that $\sum_{n=1}^{\infty} \frac{n-1}{n\sqrt{n+1}-1}$ diverges.

(iii) Firstly, observe that $\cos \frac{4}{\pi} \leq \cos \frac{4}{n\pi} < 1 \; \forall n \in \mathbb{N}$.

$$\Rightarrow \frac{\cos \frac{4}{\pi} \sin \frac{n\pi}{4}}{\sqrt{n}} \leq \frac{\cos \frac{4}{n\pi} \sin \frac{n\pi}{4}}{\sqrt{n}} < \frac{\sin \frac{n\pi}{4}}{\sqrt{n}}$$

Now $\left(\frac{1}{\sqrt{n}}\right)$ is a decreasing sequence with $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and the partial sums of $\sum_{n=1}^{\infty} \sin \frac{n\pi}{4}$ is bounded. By Dirichlet's Test, $\sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{4}}{\sqrt{n}}$ is convergent. Hence $\sum_{n=1}^{\infty} \frac{\cos \frac{4}{n\pi} \sin \frac{n\pi}{4}}{\sqrt{n}}$ converges. On the other hand, suppose $\sum_{n=1}^{\infty} \left| \frac{\cos \frac{4}{n\pi} \sin \frac{n\pi}{4}}{\sqrt{n}} \right|$ converges. Observe that

$$\left| \frac{\cos \frac{4}{n\pi} \sin \frac{n\pi}{4}}{\sqrt{n}} \right| = \frac{\cos \frac{4}{n\pi} \left|\sin \frac{n\pi}{4}\right|}{\sqrt{n}} \geq \frac{\cos \frac{4}{\pi} \left|\sin \frac{n\pi}{4}\right|}{\sqrt{n}} \geq 0$$

Since $\sum_{n=1}^{\infty} \left| \frac{\cos \frac{4}{n\pi} \sin \frac{n\pi}{4}}{\sqrt{n}} \right|$ converges, we have the absolute convergence of $\sum_{n=1}^{\infty} \frac{\left|\sin \frac{n\pi}{4}\right|}{\sqrt{n}}$. By rearranging $\sum_{n=1}^{\infty} \frac{\left|\sin \frac{n\pi}{4}\right|}{\sqrt{n}}$, we have the convergence of

$$\frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{1}{\sqrt{4k+1}} + \sum_{k=0}^{\infty} \frac{1}{\sqrt{4k+2}} + \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{1}{\sqrt{4k+3}}.$$

However,

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{4k+2}} = \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{1}{\sqrt{2k+1}} \geq \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{1}{k+1}.$$

Since $\sum_{k=0}^{\infty} \frac{1}{k+1}$ diverges, we arrived at a contradiction. In conclusion, $\sum_{n=1}^{\infty} \frac{\cos \frac{4}{n\pi} \sin \frac{n\pi}{4}}{\sqrt{n}}$ converges conditionally.

(iv) Firstly, observe that $\frac{1}{4^3} < 2\left(\frac{2}{3}\right)$ and $\frac{1}{4^2} < \left(\frac{2}{3}\right)^2$.

$$\Rightarrow \frac{1}{4^{2k+3}} < 2\left(\frac{2}{3}\right)^{2k+1} \qquad \forall k \in \mathbb{Z}_{\geq 0}$$

Now, consider the absolute convergence of the series.

$$\begin{aligned}
&\frac{1}{4^1} + \frac{2^1}{3^0} + \frac{1}{4^3} + \frac{2^3}{3^2} + \frac{1}{4^5} + \frac{2^5}{3^4} + \frac{1}{4^7} + \frac{2^7}{3^6} + \frac{1}{4^9} + \frac{2^9}{3^8} + \cdots \\
<&\frac{1}{4} + 2\left(\frac{2}{3}\right)^0 + 2\left(\frac{2}{3}\right)^1 + 2\left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right)^3 + 2\left(\frac{2}{3}\right)^4 + \cdots \\
=&\frac{1}{4} + 2\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n
\end{aligned}$$

Since $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$ converges, we conclude that the series converges absolutely.

(b) Consider the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. By the Alternating series test, it is convergent. Now consider the following rearrangement.

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{41} - \frac{1}{4} + \cdots$$

This rearrangement is achieved by adding consecutive odd terms till the sum is at least the next positive integer, then the next even term is added. For example, 1 is the first positive integer and the first odd term is also 1, then the first even term $-\frac{1}{2}$ is added. Continuing, the next positive integer is 2 and we add consecutive odd terms till the sum is at least 2, then the next even term $-\frac{1}{4}$ is added. Hence, the partial sums of the rearranged series will be unbounded above. In conclusion, the rearranged series diverges.

Question 3

(a) (i) In particular, $\frac{m^2}{1+2m} \in S \ \forall m \in \mathbb{N}$. Since $\lim_{m\to\infty}\frac{m^2}{1+2m} = \infty$, we cannot have S bounded. Hence $\sup S$ does not exist.

(ii) Let $m_0 \in \mathbb{N}$ be fixed.

$$\begin{aligned}
\frac{m_0 n}{m_0 + n + 1} &= m_0 - \frac{m_0(m_0+1)}{m_0+n+1} \\
&\geq m_0 - \frac{m_0(m_0+1)}{m_0+1+1} \\
&= \frac{m_0}{m_0+2} \\
&= 1 - \frac{2}{m_0+2} \\
&\geq 1 - \frac{2}{1+2} \\
&= \frac{1}{3}
\end{aligned}$$

Since m_0 is arbitrary, we have $s \geq \frac{1}{3} \ \forall s \in S$. Furthermore, by choosing $m = n = 1$, we have $\frac{1}{3} \in S$. Therefore $\inf S = \frac{1}{3}$.

(b) Let $x \in \mathbb{Q}\setminus\{0\}$, $y \in \mathbb{R}\setminus\mathbb{Q}$.

$$\Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1}{x} \text{ does not exists}$$

Hence f is not continuous at 0. Let $a \in \mathbb{R}$ such that $a \neq 0$ and $a \neq \pm 1$.

$$\Rightarrow \lim_{x\to a} f(x) = \lim_{x\to a}\frac{1}{x} = \frac{1}{a} \neq a = \lim_{y\to a} y = \lim_{y\to a} f(y).$$

Thus f is not continuous on $\mathbb{R}\setminus\{\pm 1\}$. Now, let $\varepsilon > 0$ be given and let $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\}$. If $|x-1| < \delta$,

$$\Rightarrow \frac{2}{3} < \frac{1}{x} < 2.$$

If $x \in \mathbb{Q}$,

$$\Rightarrow |f(x) - f(1)| = \left|\frac{1}{x} - 1\right| = \frac{|x-1|}{|x|} = \frac{|x-1|}{x} < 2\,|x-1| < 2\delta \leq 2\frac{\varepsilon}{2} = \varepsilon.$$

On the other hand, if $x \in \mathbb{R}\setminus\mathbb{Q}$,

$$\Rightarrow |f(x) - f(1)| = |x-1| < \delta \leq \frac{\varepsilon}{2} < \varepsilon.$$

That is, $\forall x \in \mathbb{R}, |x-1| < \delta \Rightarrow |f(x) - f(1)| < \varepsilon$. Therefore f is continuous at 1. Similarly, if $|x-(-1)| < \delta$,

$$\Rightarrow \frac{1}{|x|} < 2$$

If $x \in \mathbb{Q}$,

$$\Rightarrow |f(x) - f(-1)| = \left|\frac{1}{x} - (-1)\right| = \frac{|x+1|}{|x|} < 2\,|x+1| < 2\delta \leq 2\frac{\varepsilon}{2} = \varepsilon.$$

On the other hand, if $x \in \mathbb{R}\setminus\mathbb{Q}$,

$$\Rightarrow |f(x) - f(-1)| = |x-(-1)| < \delta \leq \frac{\varepsilon}{2} < \varepsilon.$$

That is, $\forall x \in \mathbb{R}, |x-1| < \delta \Rightarrow |f(x) - f(-1)| < \varepsilon$. Therefore f is continuous at -1. In conclusion, f is continuous at ± 1 only.

(c) Let $\lambda \in \mathbb{R}$ and (x_n) be a sequence on $\mathbb{R}$ such that $x_n \to 1$. Hence (λx_n) is a sequence in $\mathbb{R}$ such that $\lambda x_n \to \lambda$.

$$\Rightarrow \lim_{n\to\infty} f(\lambda x_n) = \lim_{n\to\infty} x_n f(\lambda) = f(\lambda)\lim_{n\to\infty} x_n = f(\lambda)$$

Hence, f is continuous on $\mathbb{R}$.

Question 4

(a) (i) Let $x_n = \frac{1}{n+1}$ and $y_n = \frac{1}{(n+1)^2}$ where $n \in \mathbb{N}$. Now, (x_n) and (y_n) are sequences in $(0,\infty)$ with

$$\lim_{n\to\infty}(x_n - y_n) = 0.$$

However,

$$|f(x_n) - f(y_n)| = \left|\ln\frac{1}{n+1} - \ln\frac{1}{(n+1)^2}\right| = |\ln(n+1)| \geq \ln 2$$

Hence f is not uniformly continuous on $(0,\infty)$.

(ii) For $x \in [0,1]$, $x \leq 1$ and $x^2+1 \geq 1$. Hence $|f(x)| \leq 1\ \forall x \in [0,1]$. On the other hand, for $x \in (1,\infty)$, $x < x^2 < x^2+1$. Thus $|f(x)| < 1\ \forall x \in (1,\infty)$. Henceforth, $|f(x)| \leq 1\ \forall x \in [0,\infty)$. Now, let $\varepsilon > 0$ be given and let $\delta = \frac{\varepsilon}{2}$. Suppose $|x-y| < \delta$ where $x, y \in [0,\infty)$.

$$\begin{aligned}
|f(x)-f(y)| &= \left|\frac{x}{x^2+1} - \frac{y}{y^2+1}\right| \\
&= |x-y| \frac{|1-xy|}{(x^2+1)(y^2+1)} \\
&\leq |x-y| \frac{1+|xy|}{(x^2+1)(y^2+1)} \quad \text{by triangle inequality} \\
&\leq |x-y| \left(1 + \left|\frac{x}{x^2+1}\right| \left|\frac{y}{y^2+1}\right|\right) \\
&\leq |x-y|\,(1+1) \\
&< \varepsilon
\end{aligned}$$

Therefore f is uniformly continuous on $[0,\infty)$.

(b) Recall that if g is continuous on $[b,\infty)$ with $\lim_{x\to\infty} f(x)$ existing, then g is uniformly continuous on $[b,\infty)$. As a consequence of the definition of f, we have

$$\lim_{x\to a^+} f(x) = \inf\{f(x) : x \in I\} = L \in \mathbb{R} \quad \text{and} \quad \lim_{x\to\infty} f(x) = \sup\{f(x) : x \in I\} \in \mathbb{R}$$

Let $\varepsilon > 0$ be given. Hence $\exists K > a$ such that $x \in (a,K) \Rightarrow |f(x)-L| < \varepsilon$. Firstly, since $K \in I$, f fulfils the above condition and thus f is uniformly continuous on $[K,\infty)$. So $\exists \delta_1 > 0$ such that $\forall x, y \in [K,\infty)$,

$$|x-y| < \delta_1 \Rightarrow |f(x)-f(y)| < \varepsilon. \tag{1}$$

Secondly, since f is continuous at K, $\lim_{x\to K^-} f(x)$ exist. As a result,

$$f \text{ is continuous on } (a,K) \text{ with } \lim_{x\to a+} f(x) \text{ and } \lim_{x\to K^-} f(x) \text{ existing.}$$

Hence f is uniformly continuous on (a,K). So $\exists \delta_2 > 0$ such that $\forall x, y \in (a,K)$,

$$|x-y| < \delta_2 \Rightarrow |f(x)-f(y)| < \varepsilon. \tag{2}$$

Lastly, again by the continuity at K, $\exists \delta_3 > 0$ such that

$$x \in (K-\delta_3, K+\delta_3) \Rightarrow |f(x)-f(K)| < \frac{\varepsilon}{2}. \tag{3}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Let $x, y \in I$ with $|x-y| < \delta$.

Case 1 $x, y \in [K,\infty)$
By (1), $|f(x)-f(y)| < \varepsilon$.

Case 2 $x, y \in (a,K)$
By (2), $|f(x)-f(y)| < \varepsilon$.

Case 3 $x \in (a,K)$ and $y \in [K,\infty)$

$$\begin{aligned}
&\Rightarrow x, y \in (K-\delta_3, K+\delta_3) \\
&\Rightarrow |f(x)-f(K)| < \frac{\varepsilon}{2} \quad \text{and} \quad |f(y)-f(K)| < \frac{\varepsilon}{2} \qquad \text{by (3)} \\
&\Rightarrow |f(x)-f(y)| \leq |f(x)-f(K)| + |f(y)-f(K)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

Case 4 $y \in (a, K)$ and $x \in [K, \infty)$
Similar to **Case 3**.

Therefore, f is uniformly continuous on I.

Question 5

(a) Suppose f is continuous on I. By Extreme Value Theorem, $\exists m_1 \in I$ such that $f(m_1) \leq f(x)\ \forall x \in I$. By definition, $\exists m_2, m_3 \in I, m_2 \neq m_1, m_2 \neq m_3, m_1 \neq m_3$ such that $f(m_1) = f(m_2) = f(m_3)$. WLOG, assume $m_1 < m_2 < m_3$. Now let $f(m_1) = M$ and let $x_1, x_2 \in I$ such that $m_1 < x_1 < m_2 < x_2 < m_3$.

$$\Rightarrow M < f(x_1) \qquad \text{and} \qquad M < f(x_2)$$

Let $L = \min\{f(x_1), f(x_2)\}$. Then $M < \frac{M+L}{2} < L$. By Intermediate Value Theorem, $\exists y_1, y_2, y_3, y_4 \in I$ such that $m_1 < y_1 < x_1 < y_2 < m_2 < y_3 < x_2 < y_4 < m_3$ with

$$f(y_1) = f(y_2) = f(y_3) = f(y_4) = \frac{M+L}{2}$$

Hence there is a contradiction and we conclude that f is not continuous on I.

(b) Let $\varepsilon > 0$ be given.

$$\Rightarrow \exists N \in \mathbb{N} \text{ such that } |x_k - c| < \frac{\varepsilon}{2}\ \forall k \geq N$$

By Archimedean Principle,

$$\exists M \in \mathbb{N} \text{ such that } \frac{\sum_{k=1}^{N} |x_k - c|}{n} < \frac{\varepsilon}{2}\ \forall n \geq M$$

Now, let $K = \max\{M, N\}$. If $n \geq K$,

$$\begin{aligned}
|a_n - c| &= \left| \frac{\sum_{k=1}^{n} x_k}{n} - c \right| \\
&= \left| \frac{\sum_{k=1}^{n} (x_k - c)}{n} \right| \\
&\leq \sum_{k=1}^{n} \frac{|x_k - c|}{n} \quad \text{by triangle inequality} \\
&= \sum_{k=1}^{N} \frac{|x_k - c|}{n} + \sum_{k=N+1}^{n} \frac{|x_k - c|}{n} \\
&< \frac{\varepsilon}{2} + \frac{n-N}{n} \cdot \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

Therefore, $\lim_{n\to\infty} a_n = c$.