NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
Written by Chua Hongshen
Audited by Le Hoang Van

MA2101 Linear Algebra II
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Question 1

(a) For any $\boldsymbol{A}, \boldsymbol{A}' \in W$, $k \in \mathbb{R}$, $(\boldsymbol{A} + k\boldsymbol{A}')u = \boldsymbol{A}\boldsymbol{u} + k\boldsymbol{A}\boldsymbol{u} = \boldsymbol{0} \in W$.
$\therefore$ W is a subspace of V.

(b) For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = b, c = d$$

Clearly $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ is linearly independent.
$\therefore$ A basis for W is $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ and $\dim(W) = 2$.

(c) Extend $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ to a basis $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ for V.
Let $W' = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, then $V = W \oplus W'$.

Question 2

(a) For any $a + bx + cx^2 \in \mathrm{Ker}(T)$, we have

$$\begin{pmatrix} 1 & i & 1 \\ 0 & 1 & i \\ 1 & 2i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = t \begin{pmatrix} -2 \\ -i \\ 1 \end{pmatrix} \text{ for some } t \in \mathbb{C}$$

$\therefore$ $\{-2 - ix + x^2\}$ is a basis for $\mathrm{Ker}(T)$.
$\therefore$ $\mathrm{nullity}(T) = \dim(\mathrm{Ker}(T)) = 1$, $\mathrm{rank}(T) = \dim\ (P_2(\mathbb{C})) -$ $\mathrm{nullity}(T) = 2$.

(b) From $[T]_{E,B} = \begin{pmatrix} 1 & i & 1 \\ 0 & 1 & i \\ 1 & 2i & 0 \end{pmatrix}$, we have

$$T(1) = (1, 0, 1), T(x) = (i, 1, 2i), T(x^2) = (1, i, 0)$$

$\therefore$

$$\begin{aligned}[T]_{E,C} &= \big([(1,0,1)]_E \quad [(i,1,2i)]_E \quad [(1,i,0)]_E\big) \\ &= \begin{pmatrix} 1 & 0 & 2 \\ 0 & i & i \\ 1 & -1 & 1 \end{pmatrix}\end{aligned}$$

(c) Let $\boldsymbol{P} = [I]_{B,C} = \big([1]_B \quad [1+ix]_B \quad [1+x^2]_B\big) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then $\boldsymbol{P}$ is invertible and $[T]_{E,B}\boldsymbol{P} = [T]_{E,C}$.

Question 3

(a) Let $E = \{\boldsymbol{E}_{11}, \boldsymbol{E}_{12}, \boldsymbol{E}_{21}, \boldsymbol{E}_{22}\}$ be the standard basis for $V = M_{2\times 2}(\mathbb{R})$.

Then

$$[T]_E = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

Thus, $c_T(x) = \begin{vmatrix} x & -1 & 0 & -1 \\ -1 & x & 0 & 0 \\ 0 & 0 & x-3 & 1 \\ 0 & 0 & -2 & x \end{vmatrix} = x^4 - 3x^3 + x^2 + 3x - 2$.

(b) By Cayley-Hamilton Theorem,

$$\begin{aligned} T^4 - 3T^3 + T^2 + 3T - 2I_V &= 0_V \\ \frac{1}{2}T^4 - 3T^3 + T^2 + 3T &= I_V \\ T \circ (\frac{1}{2}(T^3 - 3T^2 + T + 3I_V)) &= I_V \end{aligned}$$

Let $p(x) = \frac{1}{2}(x^3 - 3x^2 + x + 3)$, then $T^{-1} = p(T)$.

(c) For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in E_1(T)$, we have

$$\begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{for some } t \in \mathbb{R}$$

$\therefore \dim E_1(T) = 1$.

Since $c_T(x) = (x-1)^2(x+1)(x-2)$ and $\dim(E_1(T)) = 1$, so the Jordan canonical form of T is similar to

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Question 4

(a) An example is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

By direct calculation, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^*\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(b) Since $\boldsymbol{A}$ is normal, $\boldsymbol{A}$ is unitary diagonalizable.
Hence, there exists a unitary matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}^*\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{pmatrix} = \boldsymbol{Q} \Rightarrow \boldsymbol{A} = \boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}^*$$

where $\lambda_1, ..., \lambda_n$ are the eigenvalues of $\boldsymbol{A}$.

$\therefore$

$$\boldsymbol{A}^T = \boldsymbol{A}^* = (\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}^*)^* = \boldsymbol{P}^*\boldsymbol{Q}^*\boldsymbol{P} = \boldsymbol{P}^*\boldsymbol{Q}\boldsymbol{P} = \boldsymbol{A}$$

i.e. $\boldsymbol{A}$ is symmetric.

(c) Let T be a normal operator on a finite dimensional real innner product space. If the characteristic polynomial of T can be factorized into linear factor over $\mathbb{R}$, then T is self-adjoint.

Question 5

(a) Let $\{v_1, v_2, ..., v_n\}$ be a basis for $\mathrm{Ker}(S)$ and$\{S(w_1), S(w_2), ..., S(w_n)\}$ be a basis for $R(S)$.
Define $B = \{w_1, w_2, ..., w_m, v_1, v_2, ..., v_n\}$ which is a basis for V.
Extend $\{S(w_1), S(w_2), ..., S(w_n)\}$ to a basis of $C = \{S(w_1), S(w_2), ..., S(w_n), u_1, u_2, ..., u_n\}$ for V.
Then $[S]_{C,B} = \begin{pmatrix} \boldsymbol{I}_m & \mathbf{0}_{m\times n} \\ \mathbf{0}_{n\times m} & \mathbf{0}_{n\times n} \end{pmatrix}$.

(b) $[T \circ S]_B = \begin{pmatrix} \boldsymbol{W} & \mathbf{0}_{m\times n} \\ \boldsymbol{Y} & \mathbf{0}_{n\times n} \end{pmatrix}$.

$[S \circ T]_B = \begin{pmatrix} \boldsymbol{W} & \boldsymbol{X} \\ \mathbf{0}_{n\times m} & \mathbf{0}_{n\times n} \end{pmatrix}$.

(c)

$$c_{T\circ S}(x) = \begin{vmatrix} x\boldsymbol{I}_m - \boldsymbol{W} & \boldsymbol{0}_{m\times n} \\ -\boldsymbol{Y} & x\boldsymbol{I}_n \end{vmatrix} = \det(x\boldsymbol{I}_m - \boldsymbol{W})\det(x\boldsymbol{I}_n) = x^n c_{\boldsymbol{W}}(x).$$

$$c_{S\circ T}(x) = \begin{vmatrix} x\boldsymbol{I}_m - \boldsymbol{W} & -\boldsymbol{X} \\ \boldsymbol{0}_{n\times m} & x\boldsymbol{I}_n \end{vmatrix} = \det(x\boldsymbol{I}_m - \boldsymbol{W})\det(x\boldsymbol{I}_n) = x^n c_{\boldsymbol{W}}(x).$$

$\therefore c_{T\circ S}(x) = c_{S\circ T}(x)$.

(d) No.
For example, let S and T be linear operator on $\mathbb{R}^2$ such that

$$S((x,y)) = (x,0) \text{ and } T((x,y)) = (y,0) \text{ for } (x,y) \in \mathbb{R}^2$$

Then,

$$(T\circ S)((x,y)) = T(S((x,y))) = T((x,0)) = (0,0) \text{ for } (x,y) \in \mathbb{R}^2.$$

So $T\circ S$ is the zero operator on $\mathbb{R}^2$ and hence $m_{T\circ S} = x$.

On the other hand,

$$(S\circ T)((x,y)) = S(T((x,y))) = S((0,y)) = (y,0) \text{ for } (x,y) \in \mathbb{R}^2.$$

Let $E = \{(1,0),(0,1)\}$ be a basis for $\mathbb{R}^2$.
Then $[S\circ T]_B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and hence $m_{S\circ T} = x^2$.

$T\circ S$ and $S\circ T$ do not have the same minimal polynomial.

Question 6

(a) (i) $\boldsymbol{u} \in K_i \Rightarrow Q^i(\boldsymbol{u}) = \boldsymbol{0} \Rightarrow Q^{i+1}(\boldsymbol{u}) = Q(Q^i(\boldsymbol{u})) = Q(\boldsymbol{0}) = \boldsymbol{0} \Rightarrow \boldsymbol{u} \in K_{i+1}$.
$\therefore K_i \subseteq K_{i+1}$.

(ii) We prove by induction that $K_m = K_{m+1}$, $\forall m \geq k$, which implies $K_k = K_m$, $\forall m \geq k$.
Given that $K_k = K_{k+1}$. Suppose that, $K_m = K_{m+1}$ where $m \geq k$.
For any $\boldsymbol{u} \in K_{m+2}$, we have

$$Q^{m+2}(\boldsymbol{u}) = \boldsymbol{0} \Rightarrow Q^{m+1}(Q(\boldsymbol{u})) = \boldsymbol{0} \Rightarrow Q(\boldsymbol{u}) \in K_{m+1} = K_m$$

Then $Q^{m+1}(\boldsymbol{u}) = Q^m(Q(\boldsymbol{u})) = \boldsymbol{0}$ and hence $\boldsymbol{u} \in K_{m+1}$.
So, we have shown that $K_{m+2} \subseteq K_{m+1}$ and together with $K_{m+1} \subseteq K_{m+2}$ from part (i), we have $K_{m+1} = K_{m+2}$.
Thus, by Mathematical Induction, $K_m = K_{m+1}$, $\forall m \geq k$.

(b) (i) Suppose $\boldsymbol{u} \in K \cap R$, *i.e.* $Q^s(\boldsymbol{u}) = \boldsymbol{0}$ and $\boldsymbol{u} = Q^s(\boldsymbol{v})$ for some $\boldsymbol{u} \in V$.
Then $Q^{2s}(\boldsymbol{v}) = Q^s(\boldsymbol{u}) = \boldsymbol{0}$ and hence $\boldsymbol{v}) \in K_{2s} = K_s = K$. This means $\boldsymbol{u} = Q^s(\boldsymbol{v}) = \boldsymbol{0}$.
So, we have $K \cap R = \{\boldsymbol{0}\}$ which implies that $K + R$ is a direct sum.

By the Dimension Theorem for linear transformation,

$$\dim(V) = \dim(K) + \dim(R) = \dim(K \oplus R)$$

As $K \oplus R \subseteq V$, we have $V = K \oplus R$.

(ii) For all $\boldsymbol{u} \in K$,

$$(T|_K - \lambda I_K)^s(\boldsymbol{u}) = (T - \lambda I_V)^s(\boldsymbol{u}) = Q^s(\boldsymbol{u}) = \boldsymbol{0}$$

i.e. $(T|_K - \lambda I_K)^s = 0_K$ and hence $m_{T|_K}(x)|(x-\lambda)^s \Rightarrow m_{T|_K}(x) = (x-\lambda)^t$ for some $t \leq s$.

Assume that $t < s$. Since $K_t \subsetneqq K_s$, there exists $\boldsymbol{v} \in K_s - K_t$. Then

$$(T|_K - \lambda I_K)^t(\boldsymbol{v}) = (T - \lambda I_V)^t(\boldsymbol{v}) = Q^t(\boldsymbol{v}) \neq \boldsymbol{0}$$

which contradicts with the fact that $(T - \lambda I_V)^t(\boldsymbol{v}) = 0_K$.
$\therefore m_{T|_K}(x) = (x-\lambda)^s$.

Question 7

(a) Take any $\boldsymbol{u} \in \operatorname{Ker}(T)$. Then

$$T(\boldsymbol{u}) = \boldsymbol{0} \Rightarrow (T^* \circ T)(\boldsymbol{u}) = T^*(T(\boldsymbol{u})) = T^*(\boldsymbol{0}) = \boldsymbol{0} \Rightarrow \boldsymbol{u} \in \operatorname{Ker}(T^* \circ T).$$

$\therefore \operatorname{Ker}(T) \subseteq \operatorname{Ker}(T^* \circ T)$.

Take any $\boldsymbol{v} \in \operatorname{Ker}(T^* \circ T)$. Then

$$\begin{aligned} &\langle T(\boldsymbol{v}), T(\boldsymbol{v})\rangle = \langle \boldsymbol{v}, T^*(T(\boldsymbol{v}))\rangle = \langle \boldsymbol{v}, (T^* \circ T)(\boldsymbol{v}))\rangle = \langle \boldsymbol{v}, \boldsymbol{0}\rangle = 0 \\ \Rightarrow &T(\boldsymbol{v}) = \boldsymbol{0} \\ \Rightarrow &\boldsymbol{v} \in \operatorname{Ker}(T) \end{aligned}$$

$\therefore \operatorname{Ker}(T^* \circ T) \subseteq \operatorname{Ker}(T)$.

Thus, we have $\operatorname{Ker}(T^* \circ T) = \operatorname{Ker}(T)$.

(b) No, for example, let T be a linear operator on $\mathbb{R}^2$ such that

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \text{ for } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

Then Ker $(T) = $ span $\{(1,0)^T\}$ while Ker $(T^* \circ T) = $ span $\{(0,1)^T\}$.

(c)

$$\begin{aligned}
& T(\boldsymbol{u}) \text{ is the orthogonal projection of } \boldsymbol{b} \text{ onto } R(T) \\
\Longleftrightarrow\ & \boldsymbol{b} - T(\boldsymbol{u}) \text{ is orthogonal to } R(T) \\
\Longleftrightarrow\ & \langle T(\boldsymbol{v}), \boldsymbol{b} - T(\boldsymbol{u}) \rangle = 0 \text{ for all } \boldsymbol{v} \in V \\
\Longleftrightarrow\ & \langle \boldsymbol{v}, T^*(\boldsymbol{b}) - (T^* \circ T)(\boldsymbol{u}) \rangle = 0 \text{ for all } \boldsymbol{v} \in V \\
\Longleftrightarrow\ & T^*(\boldsymbol{b}) - (T^* \circ T)(\boldsymbol{u}) = 0 \\
\Longleftrightarrow\ & T^*(\boldsymbol{b}) = (T^* \circ T)(\boldsymbol{u}) \\
\Longleftrightarrow\ & \boldsymbol{x} = \boldsymbol{u} \text{ is a solution to } T^*(\boldsymbol{b}) = (T^* \circ T)(\boldsymbol{u})
\end{aligned}$$

(d) Take any $\boldsymbol{w} \in \{\boldsymbol{u} | T(\boldsymbol{u}) = \boldsymbol{b}\}$. Then,

$$(T^* \circ T)(\boldsymbol{w}) = T^*(T(\boldsymbol{w})) = T^*(\boldsymbol{b})$$

i.e. $\boldsymbol{w} \in \{\boldsymbol{u} | (T^* \circ T)(\boldsymbol{u}) = \boldsymbol{b}\}$.
$\therefore \{\boldsymbol{u} | T(\boldsymbol{u}) = \boldsymbol{b}\} \subseteq \{\boldsymbol{u} | (T^* \circ T)(\boldsymbol{u}) = \boldsymbol{b}\}$.

Take any $\boldsymbol{w'} \in \{\boldsymbol{u} | (T^* \circ T)(\boldsymbol{u}) = \boldsymbol{b}\}$. Since $(T^* \circ T)(\boldsymbol{w}) = T^*(\boldsymbol{b})$, by part (c), $T(\boldsymbol{w'})$ is the orthogonal projection of $\boldsymbol{b}$ onto $R(T)$.
As $\boldsymbol{b} \in R(T)$, the orthogonal projection is $\boldsymbol{b}$ itself, *i.e.* $T(\boldsymbol{w'}) = \boldsymbol{b}$ and hence $\boldsymbol{w'} \in \{\boldsymbol{u} | T(\boldsymbol{u}) = \boldsymbol{b}\}$.
$\therefore \{\boldsymbol{u} | (T^* \circ T)(\boldsymbol{u}) = \boldsymbol{b}\} \subseteq \{\boldsymbol{u} | T(\boldsymbol{u}) = \boldsymbol{b}\}$.

Combining both, we have $\{\boldsymbol{u} | T(\boldsymbol{u}) = \boldsymbol{b}\} = \{\boldsymbol{u} | (T^* \circ T)(\boldsymbol{u}) = \boldsymbol{b}\}$.