NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Agus Leonardi

MA2216/ST2131 Probability
AY 2009/2010 Sem 2

Question 1

(a) Let $X \sim \Gamma(\alpha, \lambda)$.

$$\begin{aligned} M_X(t) &= \mathbb{E}\left(e^{tX}\right) \\ &= \int_0^\infty e^{tx} \frac{\lambda e^{-\lambda x}(\lambda x)^{t-1}}{\Gamma(t)}\, dx \\ &= \int_0^\infty \frac{\left[e^{-(\lambda-t)x}((\lambda-t)x)^{t-1}\right]\lambda^t}{(\lambda-t)^{t-1}\Gamma(t)}\, dx \\ &= \frac{\lambda^t}{(\lambda-t)^{t-1}\Gamma(t)}\left(\frac{\Gamma(t)}{\lambda-t}\right) \\ &= \left(\frac{\lambda}{\lambda-t}\right)^t \end{aligned}$$

(b) We have

$$\begin{aligned} \mathbb{E}(X) &= M_X'(t)|_{t=0} \\ &= \frac{\alpha\lambda^\alpha}{(\lambda-0)^{\alpha+1}}|_{t=0} \\ &= \frac{\alpha}{\lambda} \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}(X^2) &= M_X''(t)|_{t=0} \\ &= \frac{\alpha\lambda(\alpha+1)}{(\alpha-t)^{\alpha+2}}|_{t=0} \\ &= \frac{\alpha(\alpha+1)}{\lambda^2} \end{aligned}$$

Hence $\operatorname{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \dfrac{\alpha}{\lambda^2}$.

(c) We have $\operatorname{Exp}(\lambda) = \Gamma(1, \lambda)$. Then

$$\begin{aligned} M_{X_1+X_2}(t) &= M_{X_1}(t)M_{X_2}(t) \\ &= \frac{\lambda}{\lambda-t}\frac{\lambda}{\lambda-t} \\ &= \left(\frac{\lambda}{\lambda-t}\right)^2 \end{aligned}$$

Hence, $X_1 + X_2 \sim \Gamma(2, \lambda)$.

Question 2

(a) Either the first number is larger than the second number, or the second number is larger than the first number. By symmetry, these two probabilities are equal. Hence, the probability that the second number is larger than the first is $\dfrac{1}{2}$.

(b) Either the first number is larger than the second number, or the second number is larger than the first number, or they are equal. Let A be the event that the two numbers are equal. Let B_k be the event that the first number taken is k. Note that $\mathbb{P}(B_k) = \dfrac{1}{5}$ and $\mathbb{P}(A|B_k) = \dfrac{1}{9}$. Then

$$\begin{aligned}\mathbb{P}(A) &= \sum_{k=1}^{5} \mathbb{P}(A|B_k)\mathbb{P}(B_k) \\ &= 5 \cdot \frac{1}{9} \cdot \frac{1}{5} \\ &= \frac{1}{9}\end{aligned}$$

Hence, the probability that the second number is larger than the first is $\left(1 - \frac{1}{9}\right) \div 2 = \frac{4}{9}$.

(c) Note that $\mathbb{E}(X) = np$ and $\text{var}(X) = np(1-p)$ and the maximum value of the function $f(p) = p(1-p)$ is achieved at $p = \dfrac{1}{2}$. Now using Chebyshev's inequality,

$$\begin{aligned}\mathbb{P}\left(\left|\frac{X}{n} - p\right| \geq \varepsilon\right) &= \mathbb{P}\left(|X - np| \geq n\varepsilon\right) \\ &\leq \frac{np(1-p)}{n^2\varepsilon^2} \\ &= \frac{p(1-p)}{n\varepsilon^2} \\ &\leq \frac{1}{4n\varepsilon^2}\end{aligned}$$

Question 3

(a) Define the following events:
I: The person go to Italian restaurant.
J: The person go to Japanese restaurant.
F: The person go to French restaurant.
S: The person is satisfied with what he ate.

$$\begin{aligned}\mathbb{P}(I|S) &= \frac{\mathbb{P}(S|I)\mathbb{P}(I)}{\mathbb{P}(S|I)\mathbb{P}(I) + \mathbb{P}(S|J)\mathbb{P}(J) + \mathbb{P}(S|F)\mathbb{P}(F)} \\ &= \frac{\frac{4}{5} \cdot \frac{1}{2}}{\frac{4}{5} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{3}{10} + \frac{1}{2} \cdot \frac{1}{5}} \\ &= \frac{4}{7}\end{aligned}$$

$$\begin{aligned}\mathbb{P}(J|S) &= \frac{\mathbb{P}(S|J)\mathbb{P}(J)}{\mathbb{P}(S|I)\mathbb{P}(I)+\mathbb{P}(S|J)\mathbb{P}(J)+\mathbb{P}(S|F)\mathbb{P}(F)} \\ &= \frac{\frac{2}{3}\cdot\frac{3}{10}}{\frac{4}{5}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{3}{10}+\frac{1}{2}\cdot\frac{1}{5}} \\ &= \frac{2}{7}\end{aligned}$$

$$\mathbb{P}(F|S) = 1-\frac{4}{7}-\frac{2}{7}=\frac{1}{7}$$

(b) The required probability is equal to the probability of getting '5' in the first $n-1$ rolls, with no '6' occurring in the first $n-1$ rolls, which follows binomial distribution with parameter $n-1$ and $p=\frac{1}{5}$.
Hence $\mathbb{P}(X=k|N=n) = \begin{pmatrix} n-1 \\ k \end{pmatrix}\left(\frac{1}{5}\right)^k\left(\frac{4}{5}\right)^{n-1-k}$.

(c) Note that $N \sim \text{Geo}\left(\frac{1}{6}\right)$. We have

$$\begin{aligned}\mathbb{P}(X=k) &= \sum_{n=1}^{\infty}\mathbb{P}(X=k|N=n)\mathbb{P}(N=n) \\ &= \sum_{n=1}^{\infty}\begin{pmatrix} n-1 \\ k \end{pmatrix}\left(\frac{1}{5}\right)^k\left(\frac{4}{5}\right)^{n-1-k}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1} \\ &= \left(\frac{1}{5}\right)^k\left(\frac{1}{6}\right)\sum_{n=0}^{\infty}\begin{pmatrix} n \\ k \end{pmatrix}\left(\frac{4}{5}\right)^{n-k}\left(\frac{5}{6}\right)^{n} \\ &= \left(\frac{1}{5}\right)^k\left(\frac{1}{6}\right)\sum_{n=0}^{\infty}\begin{pmatrix} n \\ k \end{pmatrix}\left(\frac{4}{5}\right)^{n-k}\left(\frac{5}{6}\right)^{n-k}\left(\frac{5}{6}\right)^{k} \\ &= \left(\frac{1}{6}\right)^{k+1}\sum_{n=0}^{\infty}\begin{pmatrix} n \\ k \end{pmatrix}\left(\frac{2}{3}\right)^{n-k} \\ &= \left(\frac{1}{6}\right)^{k+1}\sum_{n=k}^{\infty}\begin{pmatrix} n \\ k \end{pmatrix}\left(\frac{2}{3}\right)^{n-k} \\ &= \left(\frac{1}{6}\right)^{k+1}\left(1-\frac{2}{3}\right)^{-k-1} \\ &= \frac{1}{2^{k+1}}\end{aligned}$$

Question 4

(a) We have

$$\begin{aligned}\mathbb{P}(X\le a|Y=y) &= \int_0^a \frac{1}{y}\,dx \\ &= \left[\frac{x}{y}\right]_0^a \\ &= \frac{a}{y}\end{aligned}$$

Note that from above we have $F_{X|Y}(x) = \dfrac{x}{y}$. Hence $f_{X|Y}(x) = \dfrac{d}{dx}F_{X|Y}(x) = \dfrac{1}{y}$. Then

$$\begin{aligned}\mathbb{E}(X|Y=y) &= \int_0^y \frac{x}{y}\ dx \\ &= \left[\frac{x^2}{2y}\right]_0^1 \\ &= \frac{y}{2}\end{aligned}$$

$$\begin{aligned}\mathbb{E}(X^2|Y=y) &= \int_0^y \frac{x^2}{y}\ dx \\ &= \left[\frac{x^3}{3y}\right]_0^1 \\ &= \frac{y^2}{3}\end{aligned}$$

(b) We have

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}(\mathbb{E}(X|Y)) \\ &= \mathbb{E}\left(\frac{Y}{2}\right) \\ &= \frac{1}{2}\mathbb{E}(Y) \\ &= \frac{1}{2}\int_0^1 y\ dy \\ &= \frac{1}{4}\end{aligned}$$

$$\begin{aligned}\mathbb{E}(X^2) = &= \mathbb{E}(\mathbb{E}(X^2|Y) \\ &= \mathbb{E}\left(\frac{Y^2}{3}\right) \\ &= \frac{1}{3}\mathbb{E}(Y^2) \\ &= \int_0^1 y^2\ dy \\ &= \frac{1}{9}\end{aligned}$$

Hence, $\text{var}(X) = \dfrac{1}{9} - \left(\dfrac{1}{4}\right)^2 = \dfrac{7}{144}$

(c) We have

$$\begin{aligned}f_{X,Y}(x,y) &= f_{X|Y}(x)f_Y(y) \\ &= \frac{1}{y}\cdot 1 \\ &= \frac{1}{y}\end{aligned}$$

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{1}{y}\, dy \\ &= [\ln y]_0^1 \end{aligned}$$

Question 5

(a) Let X denotes the inter-arrival time between successive occurrence. We will show that $X \sim \text{Exp}(\lambda)$.
$\mathbb{P}(X \geq t) = \mathbb{P}(N(t) = 0) = e^{-\lambda t}$.
Hence, we have $\mathbb{P}(X \leq t) = 1 - e^{-\lambda t}$. Differentiating, we get $f_X(t) = \lambda e^{-\lambda t}$, and the result follows.
We have $T_n = X_1 + X_2 + \ldots + X_n$, where X_i is the inter-arrival time between event X_{i-1} and X_i.
All X_i's are i.i.d. exponential random variables with parameter λ. Hence, $T_n \sim \text{Gamma}(n, \lambda)$.

(b) Let X denotes the number of passenger who does not turn up for the flight. Then $X \sim \text{Bin}(100, 0.06)$.
As n is large, but p is small, such that $np < 10$, then $X \sim \text{Poisson}(6)$ approximately.
The required probability is $\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) - \mathbb{P}(X = 2) = 1 - e^{-6} - 6e^{-6} - 18e^{-6} = 1 - 25e^{-6}$.

(c) $X \sim \text{Hypergeometric}(52, 13, 13)$.
Then, $\mathbb{E}(X) = \dfrac{13 \cdot 13}{52} = \dfrac{13}{4}$.

Question 6

(a) Let

$$\begin{aligned} g_1(u,v) &= \cos u \sqrt{-2\log\left(\frac{v}{2\pi}\right)} \\ g_2(u,v) &= \sin u \sqrt{-2\log\left(\frac{v}{2\pi}\right)} \end{aligned}$$

Then we have the Jacobian

$$\begin{aligned} J(u,v) &= \begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} -\sin u \sqrt{-2\log\left(\frac{v}{2\pi}\right)} & \frac{-\cos u}{v\sqrt{-2\log\left(\frac{v}{2\pi}\right)}} \\ \cos u \sqrt{-2\log\left(\frac{v}{2\pi}\right)} & \frac{-\sin u}{v\sqrt{-2\log\left(\frac{v}{2\pi}\right)}} \end{vmatrix} \\ &= \frac{\sin^2 u}{v} + \frac{\cos^2 u}{v} \\ &= \frac{1}{v} \end{aligned}$$

Therefore $|J(u,v)|^{-1} = v$.
Note that we also have

$$x^2 + y^2 = -2\log\left(\frac{v}{2\pi}\right)$$

which after rearranging and simplifying becomes

$$v = 2\pi e^{-\frac{1}{2}(x^2+y^2)}$$

Hence,

$$\begin{aligned} f_{X,Y}(x,y) &= f_{U,V}(u,v)\cdot |J(u,v)|^{-1} \\ &= \left(\frac{1}{2\pi}\right)^2 \left(2\pi e^{-\frac{1}{2}(x^2+y^2)}\right) \\ &= \left(\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right)\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\right) \end{aligned}$$

i.e. X and Y are independent standard normal random variables.

(b) We use substitution

$$\begin{aligned} x &= \cos u\sqrt{-2\log\left(\frac{v}{2\pi}\right)} \\ y &= \sin u\sqrt{-2\log\left(\frac{v}{2\pi}\right)} \end{aligned}$$

$J(u,v) = \dfrac{1}{v}$ as calculated in part (a).
Hence the integral is equivalent to

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(x^2+y^2)}{2\pi} e^{-\frac{(x^2+y^2)}{2}}\,dx\,dy &= \int_0^{2\pi}\int_0^{2\pi} \frac{1}{2\pi}\left[-2\log\left(\frac{v}{2\pi}\right)\right] e^{\frac{2\log\left(\frac{v}{2\pi}\right)}{2}} \\ &= \int_0^{2\pi}\int_0^{2\pi} -\frac{1}{2\pi^2}\log\left(\frac{v}{2\pi}\right)\,du\,dv \\ &= \int_0^{2\pi} -\frac{1}{\pi}\log\left(\frac{v}{2\pi}\right)\,dv \\ &= -\frac{1}{\pi}\left[v\log\left(\frac{v}{2\pi}\right) - v\right]_0^{2\pi} \\ &= -\frac{1}{\pi}(-2\pi) \\ &= 2 \end{aligned}$$

(c) We will show that

$$\sum_i \frac{p(i|j)}{q(j|i)} = \frac{1}{\mathbb{P}(Y=j)}$$

We have

$$\begin{aligned} \sum_i \frac{p(i|j)}{q(j|i)} &= \sum_i \frac{p(i|j)}{\left[\frac{\mathbb{P}(Y=j)p(i|j)}{\sum_j \mathbb{P}(Y=j)p(i|j)}\right]} \\ &= \sum_i \frac{\left(\sum_j \mathbb{P}(Y=j)p(i|j)\right)p(i|j)}{\mathbb{P}(Y=j)p(i|j)} \\ &= \frac{1}{\mathbb{P}(Y=j)}\sum_i\sum_j \mathbb{P}(Y=j)p(i|j) \\ &= \frac{1}{\mathbb{P}(Y=j)}\sum_i \mathbb{P}(X=i) \\ &= \frac{1}{\mathbb{P}(Y=j)} \end{aligned}$$

Then

$$\begin{aligned}\mathbb{P}(X = i, Y = j) &= p(i|j) \cdot \mathbb{P}(Y = j)\\ &= \frac{p(i|j)}{\sum_i \frac{p(i|j)}{q(j|i)}}\end{aligned}$$

as required.