MA1100 AY15/16 Semester 1: Sample Solution

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1. Recall that if $\gcd(a, b) = 1$, then there exists $m, n \in \mathbb{Z}$ such that $am + bn = 1$.
Let P_n be the statement that $\gcd(F_n, F_{n+1}) = 1$, i.e. there exists $a_n, b_n \in \mathbb{Z}$ such that $a_n F_n + b_n F_{n+1} = 1$.
When $n = 0$, we have $F_0 = 0, F_1 = 1$. Choose any $a_0 \in \mathbb{Z}$ and $b_0 = 1$, we have $a_0(0)+1 \cdot F_1 = 1$, therefore $\gcd(F_0, F_1) = 1$ and P_0 is true.
Assume that P_k is true for some $k \in \mathbb{N} \cap \{0\}$, i.e. $\gcd(F_k, F_{k+1}) = 1$, we aim to show that P_{k+1} is true, i.e. $\gcd(F_{k+1}, F_{k+1}) = 1$.
Let $a_k, b_k \in \mathbb{Z}$. We have
$$\begin{aligned} \gcd(F_k, F_{k+1}) &= 1 \\ \Rightarrow a_k F_k + b_k F_{k+1} &= 1 \\ a_k F_k + (b_k - a_k)F_{k+1} + a_k F_{k+1} &= 1 \\ a_k(F_k + F_k + 1) + (b_k - a_k)F_{k+1} &= 1 \\ a_k F_{k+2} + (b_k - a_k)F_{k+1} &= 1 \end{aligned}$$
Since $a_k, b_k \in \mathbb{Z}$, $b_k - a_k \in \mathbb{Z}$. Therefore $\gcd(F_{k+1}, F_{k+2}) = 1$, and P_{k+1} is true.
By Principle of Mathematical Induction, P_n is true for all $n \in \mathbb{N} \cup \{0\}$.

2. (a) Consider $x \in A$. Then $f(x) \in f(A)$, so $x \in f^{-1}(f(A))$. Hence $A \subseteq f^{-1}(f(A))$.
Now assume f is injective, consider $x \in f^{-1}(f(A))$. If $x \in f^{-1}(f(A))$, $f(x) \in f(A)$, so $f(x) = f(y)$ for some $f(y)$ in $f(A)$. Since f is injective, y is in A, we have $x = y$ and therefore $x \in A$. Hence $f^{-1}(f(A)) \subseteq A$, and we conclude that $A = f^{-1}(f(A))$.
   (b) Consider $y \in f(f^{-1}(C))$. Since f is surjective, there exists an $a \in X$ in $f^{-1}(C)$ such that $f(a) = y$. Then since $a \in X$, $f(a) \in C \Rightarrow y \in C$.
Since f is surjective, for every $y \in C$ there exists an $x \in f^{-1}(C)$ such that $f(x) = y \in C$. Since $f(x) \in C$, $x \in f^{-1}(C) \Rightarrow f(x) = y \in f(f^{-1}(C))$. This gives $C \subseteq f(f^{-1}(C))$ and therefore $C = f(f^{-1}(C))$.

3. (a) Let $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $f(n) = (n, 1, 1)$ where $n \in \mathbb{N}$. If $f(x) = f(y)$, then $(x, 1, 1) = (y, 1, 1)$ gives $x = y$, therefore f is injective. Hence $|\mathbb{N}| \leq |\mathbb{N} \times \mathbb{N} \times \mathbb{N}|$.
Let $g : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $g((a, b, c)) = 2^a 3^b 5^c$. Assume that $g((a_1, b_1, c_1)) = g((a_2, b_2, c_2))$, we have $2^{a_1} 3^{b_1} 5^{c_1} = 2^{a_2} 3^{b_2} 5^{c_2}$. By Fundamental Theorem of Arithmetic, we have $a_1 = a_2, b_1 = b_2, c_1 = c_2$. This gives that g is injective, and hence $|\mathbb{N} \times \mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|$.
By Schröder-Bernstein Theorem, $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N} \times \mathbb{N}|$. Therefore there exists a bijective map from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$.
   (b) Cardinality of $\mathbb{N}$ is $|\mathbb{N}|$.
Consider $\mathrm{Maps}(\mathbb{N}, \{0, 1\})$. For each $n \in \mathbb{N}$, it can be mapped to either 0 or 1, so each number in $\mathbb{N}$ has 2 choices. Cardinality of $\mathrm{Maps}(\mathbb{N}, \{0, 1\})$ is $2^{|\mathbb{N}|}$.
Recall for any set A, $|\mathcal{P}(A)| = 2^{|A|}$. Recall also that for any set A, $\mathcal{P}(A)$ is not equivalent to A, i.e. $|A| \neq |\mathcal{P}(A)|$. Hence $|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$, and therefore $|\mathbb{N}| \neq |\mathrm{Maps}(\mathbb{N}, \{0, 1\})|$.
Hence there doesn't exist any bijective map from $\mathbb{N}$ to $\mathrm{Maps}(\mathbb{N}, \{0, 1\})$.

4. (a) Reflexive: $a - a = 0$¿ Since $n|0$ for all $n > 0$, $a \sim a$.
Symmetric: Assume $a \sim b$, then $n|(a-b)$. There exists some $k \in \mathbb{Z}$ such that $nk = a - b$. Note that $n(-k) = b - a$, so $n|(b - a)$ and therefore $b \sim a$.
Transitive: Assume $a \sim b$ and $b \sim c$. Hence $n|(a-b)$ and $n|(b-c)$, which gives $nk_1 = a-b$

and $nk_2 = b - c$ for some $k_1, k_2 \in \mathbb{Z}$. Adding the two gives $n(k_1 + k_2) = a - c$, so $n | a - c$ and $a \sim c$.
Hence $\sim$ is an equivalence relation.

(b) The question is asking for a proof that there will be n partitions for the relation $\sim$.
Define the class of an element $x \in \mathbb{Z}$ by $\overline{x} = \{y \in \mathbb{Z} | y \sim x\}$. Then $\mathbb{Z}/\sim = \{\overline{x} | x \in \mathbb{Z}\}$.
By division algorithm, any number $a \in \mathbb{Z}$ can be written as one of the following: $kn, kn + 1, kn + 2, \ldots, kn + (n-1)$ where $k \in \mathbb{Z}$. It means that we can categorise all numbers in $\mathbb{Z}$ into one of the n partitions.
Hence $|\mathbb{Z}/\sim| = n$.

5. We have that $\sum_{r=1}^{8} r = \frac{8(8+1)}{2} = 36$ and $\sum_{r=1}^{8} r^3 = \left[\frac{8(8+1)}{2}\right]^2 = 1296$. Therefore $N = 36 + 1296 = 1332$.

6. Denote C as clever and L as lazy. By (*), we have $\exists S \in C(S \in L)$, where S denotes students. Negative of (*) yields $\neg(\exists S \in C(S \in L)) = \forall S \in C(S \notin L)$. This implies C and L are disjoint sets.
Observe that (a),(b),(g),(h) are implied by the fact that C and L are disjoint. Hence $N = 4$.

7. (a) We have that $x \notin \emptyset$ for all sets x, hence it is false.
   (b) It is vacuously true that $\emptyset \subseteq A$ for all sets A, hence it is true.
   (c) true.
   (d) By property of singleton, $x \in \{x\}$ for any sets x. Hence $\emptyset \in \{\emptyset\}$ is true.
   (e) True by (b).
   (f) This is equivalent to asking whether $\emptyset \subseteq \emptyset$, by the properties of power set. From (b), since $\emptyset \subseteq \emptyset$, $\emptyset \in \mathcal{P}(\emptyset)$.
   (g) By (b), $\emptyset \subseteq \mathcal{P}(\emptyset)$.
   (h) For any set x we have $x \neq \mathcal{P}(x)$. Therefore $\emptyset \neq \mathcal{P}(\emptyset)$ and (h) is false.
   (i) We have $\emptyset \times \{\emptyset\} = \emptyset$. Since $\emptyset \notin \emptyset$, $\emptyset \notin \emptyset \times \{\emptyset\}$.
   (j) True.

8. (a) $p \cdot 0 = 0$, so $p|0$ is true.
   (b) $0|n$ only when $n = 0$. Since 0 is not a prime, $0 \not| p$. Hence (b) is false.
   (c) $p \not| 1$ for all prime numbers p, hence it is false.
   (d) Since $1 \cdot p = p$, $1|p$. Hence it is true.
   (e) $p \cdot p^{p-1} = p^p$, where p and $p-1$ are positive integers if p is a prime. Therefore $p|p^p$ is true.
   (f) Since p is a prime, $p - 2$ is nonnegative, so p^{p-2} is an integer. Since $p^2 \cdot p^{p-2} = p^p$, $p^2|p^p$ is true.
   (g) Not true when $p = 2$.
   (h) Not true.
   (i) Not true.
   (j) True, since $\gcd(5, 6) = 1$.

9. Consider the following sets and maps: $X = Z = \{0\}, Y = \{0, 1\}$, $f : X \to Y, f(0) = 1$, and $g : Y \to Z, g(0) = g(1) = 0$. While the function $g \circ f : X \to Z$, $g \circ f(0) = 0$ is bijective, g is not injective and f is not surjective. Hence (a), (d), (e), (f) are false.

   For (b), if we have $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$, then $g \circ f(x_1) = g \circ f(x_2)$ by substitution property of equality. Since $g \circ f$ is injective, $x_1 = x_2$¡ hence f is injective.

   For (c) If $g \circ f$ is surjective, then for any z in Z we have and x in X such that $g \circ f(x) = z$. Let $y := f(x) \in Y$, we will have $g(y) = z$, hence it is true.

(g) is true.

For (h) and (i), consider the following sets and maps: $X = \{0\}, Y = Z = \{0, 1\}$, $f : X \to Y, f(0) = 0, g : Y \to Z, g(0) = 0, g(1) = 1$. For (h), g is bijective, $g \circ f$ is not. For (i), f is injective and g is surjective, but $g \circ f$ is not bijective.

For (j), consider the following sets and maps: $X = Y = \{0\}, Z = \{0, 1\}, f : X \to Y, f(0) = 0, g : Y \to Z, g(0) = 0$. f is surjective and g is injective, but $g \circ f$ is not bijective.

10. (a) True; for any set A there exists a unique map $f : \emptyset \to A$.
    (b) True, $f : \emptyset \to \emptyset$ is vacuously bijective.
    (c) False, no element in A is mapped to something in $\emptyset$.
    (d) True, see (b).
    (e) False. $\{\emptyset\}$ has cardinality 1, so $f : A \to \{\emptyset\}$ is not injective if $|A| > 1$.
    (f) True. Choose any A such that $|A| = 1$.
    (g) Choose $A = \emptyset$, then there exists no mapping from $\emptyset$ to $\{\emptyset\}$.
    (h) Choose any nonempty set.
    (i) $\emptyset \times \emptyset = \emptyset$, and $f : \emptyset \to \emptyset$ is bijective.
    (j) $|\{\emptyset\}| = 1$, $|\{\emptyset\} \times \{\emptyset\}| = 1$. Since they have the same cardinality, there exists a bijective map between the two sets.