NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1104 Multivariable Calculus
AY 2008/2009 Sem 1

Question 1

(a) Let $\boldsymbol{u} = a\boldsymbol{i} + b\boldsymbol{j}$ and $\boldsymbol{v} = c\boldsymbol{i} + d\boldsymbol{j}$.
$(a^2 + b^2)(c^2 + d^2) = |\boldsymbol{u}|^2|\boldsymbol{v}|^2$, $(ac + bd)^2 = (\boldsymbol{u} \cdot \boldsymbol{v})^2 = (|\boldsymbol{u}||\boldsymbol{v}|cos\theta)^2 = |\boldsymbol{u}|^2|\boldsymbol{v}|^2cos^2\theta$.
$\because cos^2\theta \leq 1$,
$\therefore (a^2 + b^2)(c^2 + d^2) \geq (a^2 + b^2)(c^2 + d^2)$.

(b) $\boldsymbol{u} \times [\boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{v})] \cdot \boldsymbol{w} = \boldsymbol{u} \cdot [\boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w}] = \boldsymbol{u} \cdot \{[(\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{u})\boldsymbol{v}] \times \boldsymbol{w}\}$
($\boldsymbol{u}$ is a unit vector, $\boldsymbol{u} \cdot \boldsymbol{u} = |\boldsymbol{u}||\boldsymbol{u}| = 1$.)
$= \boldsymbol{u} \cdot [(\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{u} \times \boldsymbol{w}] - \boldsymbol{u} \cdot [\boldsymbol{v} \times \boldsymbol{w}] = (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{w}) - \boldsymbol{u} \cdot [\boldsymbol{v} \times \boldsymbol{w}]$
$= (\boldsymbol{u} \cdot \boldsymbol{v})[(\boldsymbol{u} \times \boldsymbol{u}) \cdot \boldsymbol{w}] - \boldsymbol{u} \cdot [\boldsymbol{v} \times \boldsymbol{w}]$
$(\boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0})$
$= -\boldsymbol{u} \cdot [\boldsymbol{v} \times \boldsymbol{w}]$
$C = -1$.

Question 2

(a) The parametric equations for ellipse E can be written as,
$x = \sqrt{2}\cos t$, $y = \sqrt{2}\sin t$, $z = 4 - x = 4 - \sqrt{2}\cos t$, $0 \leqslant t \leqslant 2\pi$
The vector equation of E is $\boldsymbol{r}(t) = \langle\sqrt{2}\cos t, \sqrt{2}\sin t, 4 - \sqrt{2}\cos t\rangle$,
so, $\boldsymbol{r}'(t) = \langle-\sqrt{2}\sin t, \sqrt{2}\cos t, \sqrt{2}\sin t\rangle$
The parameter value at the point $(1, 1, 3)$ is $t = \frac{\pi}{4}$.
$\boldsymbol{r}'(\frac{\pi}{4}) = \langle-\sqrt{2}\sin\frac{\pi}{4}, \sqrt{2}\cos\frac{\pi}{4}, \sqrt{2}\sin\frac{\pi}{4}\rangle = \langle-1, 1, 1\rangle$
The parametric equation of the tangent line to E at the point $(1, 1, 3)$ is,
$x = 1 - t$, $y = 1 + t$,$z = 3 + t$.

(b) The vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ corresponding to lines $x = 2y = 2 - 2z$ and $x = y = 2z - 2$ are,
$\boldsymbol{r}_1 = \langle1, \frac{1}{2}, -\frac{1}{2}\rangle$, $\boldsymbol{r}_2 = \langle1, 1, \frac{1}{2}\rangle$
Vector $\boldsymbol{n}$ is orthogonal to the plane, and

$$\boldsymbol{n} = \boldsymbol{r}_1 \times \boldsymbol{r}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & \frac{1}{2} \end{vmatrix} = \frac{3}{4}\boldsymbol{i} - \boldsymbol{j} + \frac{1}{2}\boldsymbol{k}.$$

Point $(0, 0, 1)$ is on the plane, the scalar equation of the plane is $\frac{3}{4}x - y + \frac{1}{2}(z - 1) = 0$.

Question 3

(a) Equation of the circle is $x^2 + y^2 = 1$. Let $x = \cos(-2t)$,$y = \sin(-2t)$,$0 \leqslant t \leqslant \pi$.
At the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, the insect is moving in the direction $\boldsymbol{u} = \langle\cos\frac{-\pi}{6}, \sin\frac{-\pi}{6}\rangle = \langle\frac{\sqrt{3}}{2}, -\frac{1}{2}\rangle$
The directional derivative of $T(x, y)$ at the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ in the direction $\boldsymbol{u} = \langle\frac{\sqrt{3}}{2}, -\frac{1}{2}\rangle$ is

$D_{\boldsymbol{u}}T(\frac{1}{2},\frac{\sqrt{3}}{2}) = T_x(\frac{1}{2},\frac{\sqrt{3}}{2})\frac{\sqrt{3}}{2} - T_y(\frac{1}{2},\frac{\sqrt{3}}{2})\frac{1}{2} = \sin(2\times\frac{\sqrt{3}}{2})\frac{\sqrt{3}}{2} - 2\times\frac{1}{2}\cos(2\times\frac{\sqrt{3}}{2})\frac{1}{2} = \frac{\sqrt{3}}{2}\sin\sqrt{3} - \frac{1}{2}\cos\sqrt{3}$.

The temperature is changing at the speed of $(\frac{\sqrt{3}}{2}\sin\sqrt{3} - \frac{1}{2}\cos\sqrt{3})$ degrees Celsius per second at the point $(\frac{1}{2},\frac{\sqrt{3}}{2})$.

(b) Suppose $P(x_0, y_0, z_0)$ is on the graph of $z = x^2 + y^2 + 10$. The distance from P to the plane $x + 2y - z = 0$ is

$D = \frac{|x_0+2y_0-z_0|}{\sqrt{1^2+2^2+(-1)^2}} = \frac{|x_0+2y_o-(x_0^2+y_0^2+10)|}{\sqrt{6}} = \frac{|(x_0-\frac{1}{2})^2+(y_0-1)^2+(10-\frac{5}{4})|}{\sqrt{6}}$

$(x_0-\frac{1}{2})^2+(y_0-1)^2+(10-\frac{5}{4}) > 0, \forall x_0, y_0 \in \mathbb{R}$

$D = \frac{(x_0-\frac{1}{2})^2+(y_0-1)^2+(10-\frac{5}{4})}{\sqrt{6}}$ Let $f = (x_0-\frac{1}{2})^2+(y_0-1)^2+(10-\frac{5}{4})$

The critical points satisfy, $f_x(a,b) = 0$ and $f_y(a,b) = 0$

Which implies, $2(a-\frac{1}{2}) = 0$, and $2(b-1) = 0$. Since $f_{xx} = 2, f_{yy} = 2,$ and $f_{xy} = 0$, apply the Second Derivatives Test, $f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} > 0$. The distance is minimized when $x_0 = \frac{1}{2}$ and $y_0 = 1$.

The point $(\frac{1}{2}, 1, 11\frac{1}{4})$ is nearest to the plane.

Question 4

(a)

$$\begin{aligned}
\int_0^3\int_{\sqrt{x/3}}^1 e^{y^3}\,dy\,dx &= \int_0^1\int_0^{3y^2} e^{y^3}\,dx\,dy \\
&= \int_0^1 3y^2e^{y^3}\,dy \\
&= \Big[e^{y^3}\Big]_0^1 = e-1.
\end{aligned}$$

(b) (i) By definition of partial derivative,

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{0}{h^2}-0}{h} = 0$$
$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{0}{h^2}-0}{h} = 0.$$

(ii)

$$\begin{aligned}
\frac{\partial^2 f}{\partial x\partial y}(0,0) &= \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(0,0) \\
&= \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} \\
&= \lim_{h\to 0}\frac{h}{h} = 1. \\
\frac{\partial^2 f}{\partial y\partial x}(0,0) &= \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(0,0) \\
&= \lim_{h\to 0}\frac{f_x(0,y)-f_x(0,0)}{h} \\
&= \lim_{h\to 0}\frac{-h}{h} = -1.
\end{aligned}$$

The partial derivatives f_x and f_y exist near $(0,0)$.

$$\begin{aligned}
&\lim_{(x,y)\to(0,0)} |f_x| = \lim_{(x,y)\to(0,0)} \frac{|x^4y+4x^2y^3-y^5|}{(x^2+y^2)^2} = \lim_{(x,y)\to(0,0)} \frac{|y||x^4+4x^2y^2-y^4|}{(x^2+y^2)^2} \\
\leqslant &\lim_{(x,y)\to(0,0)} \frac{|y||x^4+4x^2y^2+y^4|}{(x^2+y^2)^2} = \lim_{(x,y)\to(0,0)} \frac{|y|(x^2+y^2)^2}{(x^2+y^2)^2} = \lim_{(x,y)\to(0,0)} |y| = 0 = f_x(0,0) \\
&\lim_{(x,y)\to(0,0)} |f_y| = \lim_{(x,y)\to(0,0)} \frac{|x^5-4x^3y^2-xy^4|}{(x^2+y^2)^2} = \lim_{(x,y)\to(0,0)} \frac{|x||x^4-4x^2y^2-y^4|}{(x^2+y^2)^2} \\
\leqslant &\lim_{(x,y)\to(0,0)} \frac{|x||x^4+4x^2y^2+y^4|}{(x^2+y^2)^2} = \lim_{(x,y)\to(0,0)} \frac{|x|(x^2+y^2)^2}{(x^2+y^2)^2} = \lim_{(x,y)\to(0,0)} |x| = 0 = f_y(0,0)
\end{aligned}$$

f_x and f_y are continuous at $(0,0)$, f is differentiable at $(0,0)$.

Question 5

(a) (i)

$$\begin{aligned}
\text{curl}\boldsymbol{F} &= [\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(\frac{x}{x^2+y^2})]\boldsymbol{i} + [\frac{\partial}{\partial z}(\frac{-y}{x^2+y^2}) - \frac{\partial}{\partial x}(z)]\boldsymbol{j} + [\frac{\partial}{\partial x}(\frac{x}{x^2+y^2}) - \frac{\partial}{\partial y}(\frac{-y}{x^2+y^2})]\boldsymbol{k} \\
&= 0\boldsymbol{i} + 0\boldsymbol{j} + (\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2})\boldsymbol{k} \\
&= \boldsymbol{0}.
\end{aligned}$$

(ii)

$$\begin{aligned}
\boldsymbol{r}(t) &= \cos t\boldsymbol{i} + \sin t\boldsymbol{j} + 0\boldsymbol{k} \\
\boldsymbol{r}'(t) &= -\sin t\boldsymbol{i} + \cos t\boldsymbol{j} + 0\boldsymbol{k} \\
\mathbf{F}(\boldsymbol{r}(t)) &= -\sin t\boldsymbol{i} + \cos t\boldsymbol{j} + 0\boldsymbol{k} \\
\oint_C \mathbf{F}\cdot d\boldsymbol{r} &= \int_0^{2\pi} 1dt = 2\pi.
\end{aligned}$$

(b) $D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)$. Since $f_{xx}(a,b)$and $f_{yy}(a,b)$ differ in sign, $f_{xx}(a,b)f_{yy}(a,b) < 0$. Also, $f_{xy}^2(a,b) \geqslant 0$. $D < 0$. We can conclude that (a,b) is a saddle point of f.

(c) The maximal value of

$$\iint_R (4-x^2-2y^2)\ dx\ dy$$

is equal to the volume of the solid that lies under the surface $z = 4-x^2-2y^2$ and the $xy-$ plane $(z=0)$. The boundary of the region R is the intersection of the two surface, $4-x^2-2y^2=0$.

Question 6

(a) D is the region enclosed by S. Let $\boldsymbol{F}(x,y,z) = \langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3}\rangle$, we have div$\boldsymbol{F} = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1$.

Thus by Divergence Theorem, we have,

$$\begin{aligned}
V &= \iiint_D 1dV \\
&= \iiint_D \text{div}\boldsymbol{F}dV \\
&= \iint_S \boldsymbol{F}\cdot d\boldsymbol{S} \\
&= \iint_S \boldsymbol{F}\cdot \boldsymbol{n}d\boldsymbol{S} \\
&= \iint_S \langle \frac{x}{3},\frac{y}{3},\frac{z}{3}\rangle \cdot \boldsymbol{n}d\boldsymbol{S} \\
&= \frac{1}{3}\iint_S \boldsymbol{r}\cdot \boldsymbol{n}d\boldsymbol{S}.
\end{aligned}$$

(b) Let $\mathbf{F} = \frac{-y}{x^2+y^2}\boldsymbol{i} + \frac{x}{x^2+y^2}\boldsymbol{j} + 1\boldsymbol{k} \neq \mathbf{0}$.
$\text{div}\boldsymbol{F} = \frac{\partial}{\partial x}(\frac{-y}{x^2+y^2}) + \frac{\partial}{\partial y}(\frac{x}{x^2+y^2}) + \frac{\partial}{\partial z}(1) = \frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} + 0 = 0.$

$$\begin{aligned}
\text{curl}\boldsymbol{F} &= [\frac{\partial}{\partial y}(1) - \frac{\partial}{\partial z}(\frac{x}{x^2+y^2})]\boldsymbol{i} + [\frac{\partial}{\partial z}(\frac{-y}{x^2+y^2}) - \frac{\partial}{\partial x}(1)]\boldsymbol{j} + [\frac{\partial}{\partial x}(\frac{x}{x^2+y^2}) - \frac{\partial}{\partial y}(\frac{-y}{x^2+y^2})]\boldsymbol{k} \\
&= 0\boldsymbol{i} + 0\boldsymbol{j} + (\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2})\boldsymbol{k} \\
&= \mathbf{0}.
\end{aligned}$$

(c) Let E be the region enclosed by S. $\text{div}\boldsymbol{F} = 3x^2+3y^2+3z^2$. Apply Divergence Theorem,

$$\begin{aligned}
\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} &= \iiint_E \text{div}\boldsymbol{F}dV \\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^2 3x^2+3y^2+3z^2\ dx\ dy\ dz \\
&= 3\int_0^{2\pi}\int_0^1\int_0^2 r^2+z^2\ dzr\ drd\theta \\
&= 3\int_0^{2\pi}\int_0^1 2r^3+\frac{8r}{3}\ drd\theta \\
&= 3\int_0^{2\pi} 2d\theta \\
&= 12\pi.
\end{aligned}$$