MA2101 - Linear Algebra II Suggested Solutions

(Semester 1 : AY2019/20)

Written by : Yip Jung Hon
Audited by : Pan Jing Bin

Question 1

(a) The inverse is $\frac{a-bi}{a^2+b^2}$. Easily checked by multiplying by $a+bi$. Note that for a non-zero element, $a^2+b^2>0$ so the inverse is well-defined.

(b) We have:

$$\left(\begin{array}{rr|r} i+1 & i+1 & i \\ 2i+1 & i & 2 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & 2i+1 \end{array}\right)$$

Thus $x=1$ and $y=1+2i$.

Question 2

(a) Define E_1, E_2, E_3 to be the 3 basis matrices given in E.

One has, $[T_A(E_1)]_E = \begin{pmatrix} 0 \\ -2b \\ 2c \end{pmatrix}$, $[T_A(E_2)]_E = \begin{pmatrix} -c \\ 2a \\ 0 \end{pmatrix}$ and $[T_A(E_3)]_E = \begin{pmatrix} b \\ 0 \\ -2a \end{pmatrix}$.

So $[T_A]_{EE} = \begin{pmatrix} 0 & -c & b \\ -2b & 2a & 0 \\ 2c & 0 & -2a \end{pmatrix}$

(b) If $A = 0_{2\times 2}$ then $\ker(T_A) = \mathfrak{sl}_2(\mathbb{C})$ which is certainly non-zero. Otherwise note that $T_{\mathbf{A}}(A) = A^2 - A^2 = 0_{2\times 2}$. Thus $A \in \ker(T_{\mathbf{A}})$ so $\ker(T_{\mathbf{A}})$ is non-trivial.

(c) Note that the characteristic polynomial for A is $c_A(x) = \det \begin{pmatrix} x-a & -b \\ -c & x+a \end{pmatrix} = x^2 - a^2 - bc$.

On the other hand, the characteristic polynomial for T is

$$c_T(x) = \det \begin{pmatrix} x & c & -b \\ 2b & x-2a & 0 \\ -2c & 0 & x+2a \end{pmatrix} = x^3 - (4a^2+4bc)x = x(x^2-4a^2-4bc).$$

Consider 2 cases :

Case 1 : $a^2 + bc \neq 0$.

Then $c_A(x)$ have 2 distinct roots and $c_T(x)$ have 3 distinct roots. Both are diagonalisable so trivially T_A is diagonalisable $\iff$ A is diagonalisable.

Case 2 : $a^2 + bc = 0$.

Then 0 is the only eigenvalue for both T_A and A. Recall that if a matrix (and linear operator) with only 1 eigenvalue is diagonalisable, it must be a diagonal matrix. Thus if T_A or A is diagonalisable, it must be the zero matrix/operator.

$$\begin{aligned} A \text{ is diagonalisable} &\iff A \text{ is the zero matrix} \\ &\iff T_{\mathbf{A}} \text{ is the zero operator} \\ &\iff T_{\mathbf{A}} \text{ is diagonalisable.} \end{aligned}$$

Question 3

(a) Let $B = \{M_{11}, M_{12}, M_{21}, M_{22}\}$ be the standard basis for $\mathcal{M}_{2\times 2}(\mathbb{C})$.

$$T(M_{11}) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, T(M_{12}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, T(M_{21}) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, T(M_{22}) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

One has: $[T]_B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $[T - \lambda I]_B = \begin{bmatrix} 1-\lambda & 0 & 1 & 1 \\ 1 & -\lambda & 1 & 0 \\ -1 & 0 & -1-\lambda & -1 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}$.

Then $\det(T - \lambda I) = \lambda^4$, and the eigenvalues are simply 0.

(b) For $\ker(T)$,

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\ker(T) = \text{span}\{[(0,1,0,0)]_B, [(-1,0,1,0)]_B\} = \text{span}\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

(c) We know $m_T(x) \mid c_T(x)$, so it is either x, x^2, x^3 or x^4. Observe that

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\ker(T^2) = \mathcal{M}_{2\times 2}(\mathbb{R})$ so $m_T(x) = x^2$.

(d) Since the minimal polynomial is of degree 2, we know the largest size of the Jordan block is of size 2.

Let $\{v_1, v_2, v_3, v_4\}$ be an ordered basis for V. We set $v_1 = [(0,1,0,0)]_B$ and $v_3 = [(-1,0,1,0)]_B$. In order to get T in Jordan Canonical Form, we need to find v_2 and v_4 such that $T(v_2) = v_1$ and $T(v_4) = v_3$. We want to solve:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Any vector that satisfies this equation will work. We can pick $v_2 = [(1,0,0-1)]_B$.

Similar, one can solve:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

One can pick $v_4 = [(0,0,0,-1)]_B$. Then the matrix $P = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$ is the one needed such that $P^{-1}TP = [T]_{B'}$.

Where B' is ordered the basis: $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\}$ and

$$[T]_{B'} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in Jordan form.

Question 4

(a) Let $T = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, and let $T^* = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then:

$$\begin{aligned} \left\langle \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} xu_1 + yu_2 \\ zu_1 + wu_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle \\ &= 4(xu_1 + yu_2)\overline{v_1} + (zu_1 + wu_2)\overline{v_2} \\ &= 4xu_1\overline{v_1} + 4yu_2\overline{v_1} + zu_1\overline{v_2} + wu_2\overline{v_2} \end{aligned}$$

$$\left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{bmatrix} \right\rangle$$
$$= 4u_1\overline{(av_1 + bv_2)} + (u_2\overline{cv_1 + dv_2})$$
$$= 4\overline{a}u_1\overline{v_1} + 4\overline{b}u_1\overline{v_2} + u_2\overline{cv_1} + u_2\overline{d}\overline{v_2}.$$

Since the adjoint is unique if it exists, one has $\overline{a} = x, 4\overline{b} = z, \overline{c} = 4y, \overline{d} = w$, and hence:

$$T^* = \begin{bmatrix} \overline{x} & \frac{1}{4}\overline{z} \\ 4\overline{y} & \overline{w} \end{bmatrix}$$

(b) For T to be self-adjoint, one needs to have.

$$T^* = \begin{bmatrix} \overline{x} & \frac{1}{4}\overline{z} \\ 4\overline{y} & \overline{w} \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Since $\overline{x} = x$, $x \in \mathbb{R}$. Similarly, $w \in \mathbb{R}$. $\frac{1}{4}\overline{z} = y$ and $4\overline{y} = z$ gives us $4y = \overline{z}$.

Question 5

(i) Let $x \in V_1 \cap V_2$. Then $T(x) \in V_1$, since V_1 is T-invariant. Similarly, $T(x) \in V_2$. So $T(x) \in V_1 \cap V_2 \implies V_1 \cap V_2$ is T-invariant.

(ii) The operator satisfies the polynomial $x^2 + 1$, so the minimal polynomial, $m_T(x) \mid x^2 + 1$. Since $x^2 + 1$ does not factor (is irreducible) in $\mathbb{R}$, we have $m_T(x) = x^2 + 1$ as well. $\deg(m_T(x)) = 2 \iff$ the dimension of the cyclic subspace generated by u is 2 for any non-zero $u \in V$.

(iii) Assume that $W_u + W_v$ is neither a direct sum nor is $W_u = W_v$. Then $\dim(W_u \cap W_v) = 1$. Since W_u and W_v are T-invariant subspaces, by (i), $W_u \cap W_v$ is also T-invariant. Thus $W_u \cap W_v$ is a one-dimensional T-invariant subspace so it must be an eigenspace of T associated with eigenvalue $\lambda \in \mathbb{R}$.

But $T^2 = -I_V \implies \lambda^2 = -1$ which is a contradiction as $\lambda \in \mathbb{R}$.

Question 6

(a) Let's first show that $\{\cos(mx), \sin(mx) \mid 0 \le m \le n\} \setminus \{0\}$ is a **orthogonal basis** first. For any m, n,

$$\langle \cos(mx), \sin(nx) \rangle = \int_{-\pi}^{\pi} \cos(mx)\sin(nx)dx = 0$$

For any n, m,

$$\langle \cos(nx), \cos(mx) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \begin{cases} \frac{1}{2} & \text{if m = n.} \\ 0 & \text{otherwise.} \end{cases}$$

If $m \neq n$, then $\langle \cos(nx), \cos(mx) \rangle = 0$. Also,

$$\langle \sin(mx), \sin(nx) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \begin{cases} \frac{1}{2} & \text{if m = n.} \\ 0 & \text{otherwise.} \end{cases}$$

If $m \neq n$, then $\langle \sin(nx), \sin(mx) \rangle = 0$. This shows that it is an orthogonal basis. When $m = n$,$\langle \sin(nx), \sin(nx) \rangle = \frac{1}{2}$, and we can choose our basis to be $\sqrt{(2)}\sin(nx)$ to normalise the inner product to 1. We may do the same for $\cos(nx)$. We have that our **orthonormal basis** is:

$$\mathscr{B} = \{1, \sqrt{2}\sin(mx), \sqrt{2}\sin(mx) \mid 0 \leq m \leq n\} \setminus \{0\} \tag{1}$$

(b) For the function $f(x) = 1 + x$, we want to 'project' it onto our $\mathscr{B}$, our orthonormal basis.

$$\begin{aligned} \text{Proj}_{\mathscr{B}}(1+x) &= \langle 1+x, 1 \rangle(1) + \langle 1+x, \sqrt{2}\sin x \rangle(\sin x) + \langle 1+x, \sqrt{2}\cos x \rangle(\cos x) + \cdots \\ &+ \langle 1+x, \sqrt{2}\sin(nx) \rangle(\sin(nx)) + \langle 1+x, \sqrt{2}\cos(nx) \rangle(\cos(nx)) \end{aligned}$$

But for each $\cos(kx)$, $\langle 1+x, \sqrt{2}\cos(kx) \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1+x)\sqrt{2}\cos(kx) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{2}\cos(kx) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{2}x\cos(kx)$. $\int_{-\pi}^{\pi}\cos(kx) = 0$, and since $\cos(kx)$ is an even function, $x\cos(kx)$ is an odd function and again, $\int_{-\pi}^{\pi} x\cos(kx) = 0$. So all the $\langle 1+x, \sqrt{2}\cos(kx) \rangle(\cos(nx))$ vanishes. The sum then reduces to:

$$\begin{aligned} &= 1 + \frac{1}{2\pi}\sqrt{2}\sin x \int_{-\pi}^{\pi}(1+x)\sqrt{2}\sin(x)dx + \cdots \\ &+ \frac{1}{2\pi}\sqrt{2}\sin(nx)\int_{-\pi}^{\pi}(1+x)\sqrt{2}\sin(nx)dx \end{aligned}$$

For each $\sin(kx)$, note that $\int_{-\pi}^{\pi}\sin(kx) = 0$, since $\sin(kx)$ is a odd function. The sum again reduces.

$$\begin{aligned} &= 1 + \frac{1}{2\pi}\sqrt{2}\sin x \int_{-\pi}^{\pi} x\sqrt{2}\sin(x)dx + \cdots + \frac{1}{2\pi}\sqrt{2}\sin(nx)\int_{-\pi}^{\pi} x\sqrt{2}\sin(nx)dx \\ &= 1 + \frac{1}{\sqrt{2}\pi}\left[\sum_{k=1}^{n}\int_{-\pi}^{\pi} x\sin(kx)dx.\sqrt{2}\sin(kx)\right] \end{aligned}$$

Since $\int_{-\pi}^{\pi} x\sin(kx)dx = \frac{2\pi(-1)^{k+1}}{k}$,

$$\begin{aligned} &= 1 + \frac{1}{\sqrt{2}\pi}\left[\sum_{k=1}^{n}\frac{2\pi(-1)^{k+1}}{k}.\sqrt{2}\sin(kx)\right] \\ &= 1 + 2\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\sin(kx) \end{aligned}$$

This is called the **Fourier Series Sawtooth Wave**. As a teaser, this is how it the above function approxmiates $f(x) = 1 + x$ on $[-\pi, \pi]$ for $n = 30$:

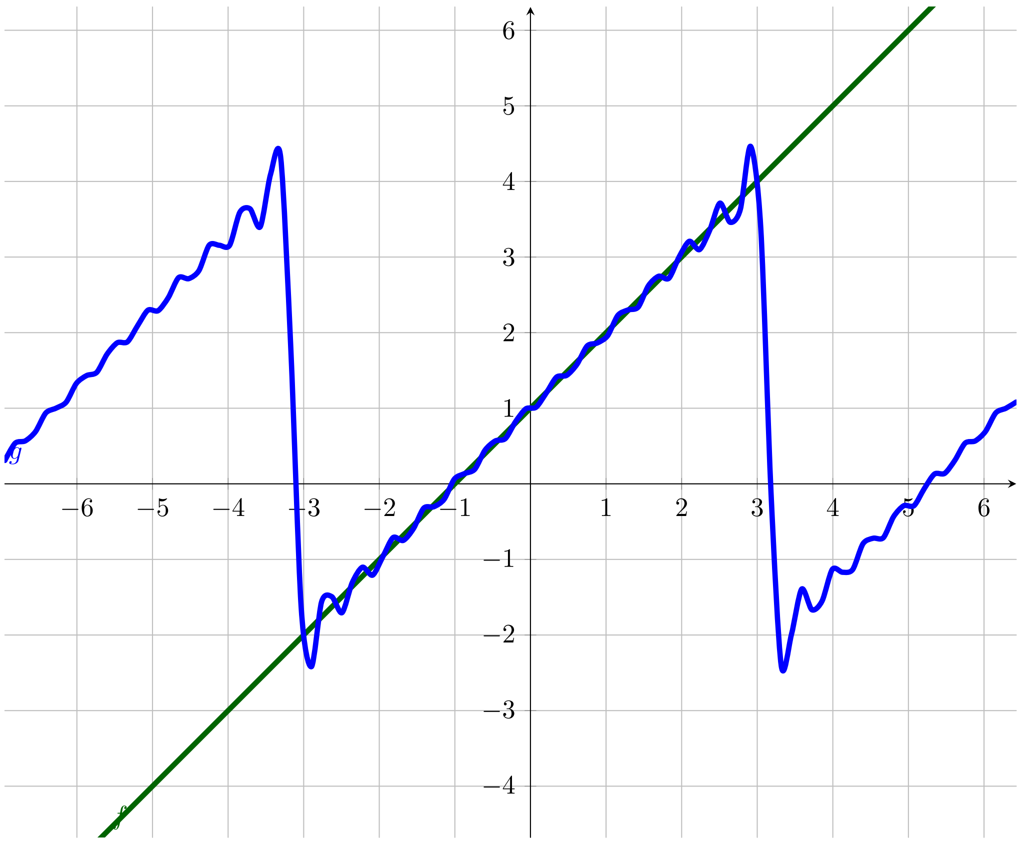

Figure 1: Fourier Series Sawtooth Wave

(c) For any f, g are functions, $d(f,g) = |f-g| = \sqrt{\langle f-g, f-g\rangle}$. Let $f = 1+x$ and set $g = 1 + 2\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\sin(kx)$.

$$\begin{aligned}
d\left(f,g\right) &= \left| x - 2\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\sin(kx) \right| \\
&= \sqrt{\frac{1}{2\pi}, \int_{-\pi}^{\pi} \left(x - 2\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\sin(kx)\right)^2} \\
&= \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 - 4x\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\sin(kx) + 4\left(\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\sin(kx))\right)^2 dx}
\end{aligned}$$

Let's take a closer took at the last term. It is impossible to integrate $\int_{-\pi}^{\pi}\left(\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\sin(kx))\right)^2$ directly. However, notice that when you do expand $\left(\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\sin(kx))\right)^2$ out, we get a lot of terms of the form $\sin(nx)\sin(mx)$. When the integral $\int_{-\pi}^{\pi}$ acts on each $\sin(nx)\sin(mx)$, if $n \neq m$, the term will go to 0. The only terms that survive are when $n = m$, that is: $\int_{-\pi}^{\pi}\left(\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\sin(kx))\right)^2 = \int_{-\pi}^{\pi}\sum_{k=1}^{n}\left(\frac{(-1)^{k+1}}{k}\sin(kx)\right)^2$.

$$= \sqrt{\frac{1}{2\pi}\frac{1}{3}[x^3]_{-\pi}^{\pi} - \frac{4}{2\pi}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\int_{-\pi}^{\pi}x\sin(kx)dx + \frac{4}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{n}\left(\frac{(-1)^{k+1}}{k}\sin(kx)\right)^2 dx}$$

$$= \sqrt{\frac{\pi^2}{3} - \frac{4}{2\pi}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\frac{(-1)^{k+1}2\pi}{k} + \frac{4}{2\pi}\sum_{k=1}^{n}\left(\frac{1}{k^2}\int_{-\pi}^{\pi}\sin^2(kx)\right)dx}$$

From (0.3) in the paper, $\int_{-\pi}^{\pi}\sin^2(kx)dx = \pi$.

$$= \sqrt{\frac{\pi^2}{3} - 4\sum_{k=1}^{n}\frac{1}{k^2} + \frac{4}{2\pi}\sum_{k=1}^{n}\left(\frac{\pi}{k^2}\right)dx}$$

$$= \sqrt{\frac{\pi^2}{3} - 2\sum_{k=1}^{n}\frac{1}{k^2}}$$

Taking $n \to \infty$ gives that $d(f,g) \to 0$, as desired.