MA2202 - Algebra I Suggested Solutions

(Semester 1 : AY2020/21)

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Q1

(i)

Applying the Euclidean Algorithm we will have

$$\begin{aligned}
449 &= 4(107) + 21 \\
107 &= 5(21) + 2 \\
21 &= 10(2) + 1
\end{aligned}$$

$\therefore gcd(449, 107) = 1$.

(ii)

Working backwards we will have

$$\begin{aligned}
1 &= 21 - 10(2) \\
&= 21 - 10(107 - 5(21)) \\
&= 51(21) - 10(107) \\
&= 51(449 - 4(107)) - 10(107) \\
&= 51(449) - 214(107)
\end{aligned}$$

$\therefore x = (-1)(-214) = 214$ is an integral solution.
The set of solutions hence is $214 + 449k, k \in \mathbb{Z}$

(iii)

We have $x = 4(51)(449) - 3(214)(107) = 22902$ as an integral solution.
The complete set of solutions is $22902 + k(449)(107) = 22902 + 48043k, k \in \mathbb{Z}$.

Q2

We show that $(G, \diamond)$ fulfills the group axioms.
(G1) Trivial since it is given in the question that $\diamond$ is a binary operation.
(G2) Given $x, y, z \in G$ we have that

$$\begin{aligned}(x \diamond y) \diamond z &= (x * a * y) * a * z \\ &= x * a * (y * a * z) \\ &= x * a * (y \diamond z) \\ &= x \diamond (y \diamond z).\end{aligned}$$

(G3) Given $x \in G$, we have that

$$\begin{aligned}x \diamond a^{-1} &= x * a * a^{-1} = x \\ a^{-1} \diamond x &= a^{-1} * a * x = x.\end{aligned}$$

Hence, a^{-1} is the identity element of $(G, \diamond)$

(G4) Given $x \in G$, we have that

$$\begin{aligned}x \diamond (a^{-1} * x^{-1} * a^{-1}) &= x * a * (a^{-1} * x^{-1} * a^{-1}) = a^{-1}, \\ (a^{-1} * x^{-1} * a^{-1}) \diamond x &= (a^{-1} * x^{-1} * a^{-1}) * a * x = a^{-1}.\end{aligned}$$

Hence, $a^{-1} * x^{-1} * a^{-1}$ is the inverse of x in $(G, \diamond)$
Since $(G, \diamond)$ fulfills (G1) to (G4), it is a group.

Q3

(i)

By observation we have $f = (246)(357)$.

(ii)

The order of an element is the LCM of the length of its disjoint cycles. Thus the order of f in S_8 is 3.

(iii)

Let $g = (12345)(678)$. Since g is a product of a disjoint 5-cycle and 3-cycle, the order of g is the LCM of 3 and 5, which is 15.

(iv)

False. If the order of h is 14, then h is made of disjoint cycles where 14 is the LCM of the lengths. Writing $h = c_1 c_2 ... c_r$ then we note that at least one of the c_i must have length 14 which is impossible as $h \in S_8$, or at least one c_i has length 7 and one c_j has length 2 which is impossible as $7+2=9>8$ so $h \notin S_8$.

Q4

(i)

Note that for $g \in G$, the map $\phi : H \to gHg^{-1}$ given by

$$\phi(x) = gxg^{-1}$$

is an isomorphism. In particular, ϕ is bijective.
Thus $gHg^{-1} = \phi(H) \implies |gHg^{-1}| = |\phi(H)| = |H|$

(ii)

Let $x, y \in gHg^{-1}$ and so $x = gh_1g^{-1}$ and $y = gh_2g^{-1}$ for some $h_1, h_2 \in H$. Then we have $xy^{-1} = gh_1g^{-1}gh_2^{-1}g^{-1} = gh_1h_2^{-1}g^{-1} \in gHg^{-1}$. Hence, gHg^{-1} satisfies (S) so it is a subgroup.

Remark : This question can also be done by noting that $gHg^{-1} = \phi(H)$ which is a subgroup.

(iii)

If we have $g_1H = g_2H$ then $g_1 = g_2h$ for some $h \in H$. Then we will have that $g_1Hg_1^{-1} = g_2hH(g_2h)^{-1} = g_2(hHh^{-1})g_2^{-1} = g_2Hg_2^{-1}$.

(iv)

Let n be the number of distinct subgroups of the form gHg^{-1} and then using (iii) we get that $n \leq [G : H]$. Also, note that each of the subgroups will have the identity element e. Hence, $\bigcup_{g \in G} gHg^{-1}$ has at most $1 + n(|H| - 1)$ elements. Observe that

$$1 + n(|H| - 1) \leq 1 + [G : H]|H| - [G : H] = 1 + |G| - [G : H]$$

But H is a proper subgroup of G so $[G : H] \geq 2$. Thus we conclude that $\left|\bigcup_{g \in G} gHg^{-1}\right| \leq 1 + |G| - [G : H] < |G|$ so $\bigcup_{g \in G} gHg^{-1} \neq G$.

Q5

(i)

Using the last question, if H is a subgroup then gHg^{-1} is a subgroup of G. Let K be a subgroup of G such that $gHg^{-1} \subseteq K \subseteq G$. Then we have that $g^{-1}gHg^{-1}g \subseteq g^{-1}Kg \subseteq g^{-1}Gg$, which gives $H \subseteq g^{-1}Kg \subseteq G$. By the last question we also have that $g^{-1}Kg$ is a subgroup of G.
Further given that H is a maximal subgroup of G we have that $g^{-1}Kg = H$ or $g^{-1}Kg = G$. If $g^{-1}Kg = H$ then $K = gHg^{-1}$, if $g^{-1}Kg = G$ then $K = gGg^{-1} = G$. Hence, gHg^{-1} is a maximal subgroup of G.

(ii)

If $x, y \in F = \bigcap_i H_i$ then $x, y \in H_i$ for all $i \in I$ and so $xy^{-1} \in H_i$ and $xy^{-1} \in F$ so F satisfies (S) and is a subgroup.

(iii)

Let X be the set of all maximal subgroups of G. Fix $g \in G$ and define the map $f_g : X \to X$ by

$$f_g(H) = gHg^{-1}.$$

By (i), f_g is a well-defined map.

Claim : f_g is bijective

Proof : Since G is finite, it can only have a finite number of maximal subgroups. Thus it suffices to prove that f_g is injective. Let $H_1, H_2 \in X$ such that $f_g(H_1) = f_g(H_2)$. Then

$$\begin{aligned} f_g(H_1) = f_g(H_2) &\implies gH_1g^{-1} = gH_2g^{-1} \\ &\implies g^{-1}(gH_1g^{-1})g = g^{-1}(gH_2g^{-1})g \\ &\implies H_1 = H_2. \end{aligned}$$

Thus $f_g(X) = X$ so $F = \bigcap_{i\in I} H_i = \bigcap_{i\in I} gH_ig^{-1} = g\left(\bigcap_{i\in I} H\right)g^{-1} = gFg^{-1}$. Since the choice of g is arbitrary, we conclude that F is normal.

Q6

Since the index is n we have n distinct left cosets. Consider the left cosets $H, gH, g^2H, ..., g^nH$. There are $n+1$ left cosets. By the pigeonhole principle, at least 2 left cosets are the same, i.e. $g^aH = g^aH, 0 \leq b < a \leq n$.
Then we have $g^ae \in g^bH$ and therefore $g^a = g^bh$ for some $h \in H$ hence
$g^{a-b} = h \in H$ and $1 \leq a-b \leq n$.

Q7

Let K be the Kernel of ϕ. Then we have by the First Isomorphism Theorem that G/K is isomorphic to H. So we have that $|H| = |G/K| = [G : K]$. Applying Lagrange's Theorem we will have that $|H| = [G : K] = \dfrac{|G|}{|K|}$ and so $|G| = |K| \times |H|$ and $|H|$ divides $|G|$.

Q8

(i)

We have $e_H \star \phi(e_G) = \phi(e_G) = \phi(e_G * e_G) = \phi(e_G) \star \phi(e_G)$. Applying the right cancellation law gives $e_H = \phi(e_G)$.

(ii)

Using (i) we will have $e_H = \phi(e_G) = \phi(g * g^{-1}) = \phi(g) * \phi(g^{-1})$. Multiplying $\phi(g)^{-1}$ on both sides will give $\phi(g)^{-1} = \phi(g^{-1})$.

(iii)

If we have $x, y \in A$ then we have $\phi(x), \phi(y) \in B$. We get $\phi(x * y^{-1}) =$
$\phi(x) * \phi(y^{-1}) = \phi(x) * \phi(y)^{-1} \in B$. Hence, $x * y^{-1} \in A$ and Axiom (S) is satisfied, so A is a subgroup of G.

(iv)

Suppose B is a normal subgroup and given $x \in A$ and $g \in G$. Using (i) and (ii), we have $\phi(gxg^{-1}) = \phi(g) * \phi(x) * \phi(g)^{-1} \in B$ since B is normal and $\phi(x) \in B$. Thus $gxg^{-1} \in A$ so A is a normal subgroup of H.