NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA1102R Calculus
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Question 1

(a) Let $f(x) = x^3$, we get the following identity:

$$\lim_{y\to 0} \frac{(x+y)^3 - x^3}{y} = \lim_{y\to 0} \frac{f(x+y) - f(x)}{y} \tag{1}$$

From the definition of derivative, the right hand side of (1) also equals to $f'(x)$. Hence we can choose $a(x) = f'(x) = 3x^2$ as the value for the given limit.

(b) For every $\epsilon > 0$, choose $\delta = \min\left\{\frac{\epsilon}{3|x|+1}; 1\right\}$. For $0 < |y| < \delta$, we have:

$$\begin{aligned}\left|\frac{(x+y)^3 - x^3}{y} - 3x^2\right| &= \left|\frac{3x^2y + 3xy^2 + y^3}{y} - 3x^2\right| \\ &= \left|3x^2 + 3xy + y^2 - 3x^2\right| \\ &= |y||3x+y| < \delta(3|x| + \delta) \le \frac{\epsilon}{3|x|+1}(3|x|+1) = \epsilon.\end{aligned}$$

Therefore, from the definition of limit, we have:

$$\lim_{y\to 0} \frac{(x+y)^3 - x^3}{y} = 3x^2$$

.

(c) From a), the limit is the value of the derivative of x^3 at the point x. Hence, from the geometric mean of derivative, the limit is the slope of the tangent line of the curve $y = x^3$ at the point x.

Question 2

(a)

$$\lim_{x\to 2} \frac{x^m - 2^m}{\sin(x^n - 2^n)} = \lim_{x\to 2} \frac{x^m - 2^m}{x^n - 2^n} \cdot \lim_{x\to 2} \frac{x^n - 2^n}{\sin(x^n - 2^n)}$$

- $\lim_{x\to 2} \frac{x^m - 2^m}{x^n - 2^n} = \lim_{x\to 2} \frac{mx^{m-1}}{nx^{n-1}} = \frac{m}{n} \cdot 2^{m-n}$ (L'Hospital's Rule)

- $\lim_{x\to 2} \frac{x^n - 2^n}{\sin(x^n - 2^n)} = \lim_{t\to 0} \frac{t}{\sin t} = 1$ (Let $t = x^n - 2^n \Rightarrow t \to 0$ when $x \to 2$)

  Hence, by multiplying two sub-limits together, we get:

$$\lim_{x\to 2} \frac{x^m - 2^m}{\sin(x^n - 2^n)} = \frac{m}{n} \cdot 2^{m-n}$$

(b)

$$\begin{aligned} \lim_{y\to 0} \frac{\sin(y) - \sin(y)\cos(y)}{y - \sin(y)} &= \lim_{y\to 0} \frac{\cos(y) - \cos(2y)}{1 - \cos(y)} \quad \text{(by L'Hospital's Rule)} \\ &= \lim_{y\to 0} (1 + 2\cos(y)) = 3. \end{aligned}$$

(c)

$$\lim_{x\to 3} \frac{x^2 - 3x + 1}{x^3 + 2x - 27} = \frac{3^2 - 3\cdot 3 + 1}{3^3 + 2\cdot 3 - 27} = \frac{1}{6}.$$

Question 3

(a) We apply l'Hospital's rule:

$$\begin{aligned} \lim_{x\to 2} \frac{x^m - 2^m}{\sin(x^n - 2^n)} &= \lim_{x\to 2} \frac{mx^{m-1}}{\cos(x^n - 2^n)(nx^{n-1})} \\ &= \frac{m}{n} \lim_{x\to 2} \left(\frac{1}{\cos(x^n - 2^n)} \cdot x^{m-n} \right) \\ &= \frac{m2^{m-n}}{n} \end{aligned}$$

(b) Again, we apply l'Hospital's rule:

$$\begin{aligned} \lim_{y\to 0} \frac{\sin y - \sin y \cos y}{y - \sin y} &= \lim_{y\to 0} \frac{\cos y - \cos^2 y + \sin^2 y}{1 - \cos y} \\ &= \lim_{y\to 0} \left(\cos y + \frac{\sin^2 y}{1 - \cos y} \right) \\ &= 1 \end{aligned}$$

(c) The function $f(x) = \frac{x^2-3x+1}{x^3+2x-27}$ is rational and defined at $x = 3$. Hence $\lim_{x\to 3} f(x) = f(3) = \frac{1}{6}$.

Question 4

(a) We substitute $u = t^3 - x^3$ so that $du = 3t^2 dt$:

$$\begin{aligned}\frac{d}{dx}\int_0^x t^2\cos\left(x^3-t^3\right)dt &= \frac{d}{dx}\int_0^x t^2\cos\left(t^3-x^3\right)dt\\ &= \frac{1}{3}\frac{d}{dx}\int_{-x^3}^0 \cos(u)\,du\\ &= \frac{-1}{3}\cos\left(x^3\right)\end{aligned}$$

by the Fundamental Theorem of Calculus.

(b) We substitute $u = 1 - x$; $du = -dx$:

$$\begin{aligned}\int_0^1 \frac{x^2}{x^2+(1-x)^2}dx &= \int_1^0 \frac{-(1-u)^2}{(1-u)^2+u^2}du\\ &= \int_0^1 \frac{(1-u)^2}{(1-u)^2+u^2}du\\ &= \int_0^1 \frac{(1-x)^2}{x^2+(1-x)^2}dx\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^1 \frac{x^2}{x^2+(1-x)^2}dx &= \frac{1}{2}\left(\int_0^1 \frac{x^2+(1-x)^2}{x^2+(1-x)^2}dx\right)\\ &= \frac{1}{2}\end{aligned}$$

(c)

$$\begin{aligned}\frac{dy}{dx} &= e^{\cos\left(x^2\right)}\cdot\left(-\sin\left(x^2\right)\right)(2x)\\ &= -2xe^{\cos\left(x^2\right)}\sin\left(x^2\right)\\ \frac{d^2y}{dx^2} &= -2e^{\cos\left(x^2\right)}\sin\left(x^2\right) - 2x\sin\left(x^2\right)\cdot\left(-2xe^{\cos\left(x^2\right)}\sin\left(x^2\right)\right) - 2xe^{\cos\left(x^2\right)}\left(2x\cos x^2\right)\\ &= -2e^{\cos\left(x^2\right)}\sin\left(x^2\right) + 4x^2e^{\cos\left(x^2\right)}\sin^2\left(x^2\right) - 4x^2e^{\cos\left(x^2\right)}\cos\left(x^2\right)\end{aligned}$$

(d) Differentiate implicitly:

$$\begin{aligned}\frac{x^2}{9}+\frac{y^2}{16} &= 1\\ \frac{2x}{9}+\frac{y}{8}\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= \frac{-2x}{9}\cdot\frac{8}{y}\\ &= \frac{-16x}{9y}\end{aligned}$$

Question 5

(a) We substitute $u = t^3 - x^3$ so that $du = 3t^2 dt$:

$$\begin{aligned} \frac{d}{dx}\int_0^x t^2 \cos\left(x^3 - t^3\right) dt &= \frac{d}{dx}\int_0^x t^2 \cos\left(t^3 - x^3\right) dt \\ &= \frac{1}{3}\frac{d}{dx}\int_{-x^3}^{0} \cos\left(u\right) du \\ &= \frac{-1}{3}\cos\left(x^3\right) \end{aligned}$$

by the Fundamental Theorem of Calculus.

(b) We substitute $u = 1 - x$; $du = -dx$:

$$\begin{aligned} \int_0^1 \frac{x^2}{x^2 + (1-x)^2} dx &= \int_1^0 \frac{-(1-u)^2}{(1-u)^2 + u^2} du \\ &= \int_0^1 \frac{(1-u)^2}{(1-u)^2 + u^2} du \\ &= \int_0^1 \frac{(1-x)^2}{x^2 + (1-x)^2} dx \end{aligned}$$

It follows that

$$\begin{aligned} \int_0^1 \frac{x^2}{x^2 + (1-x)^2} dx &= \frac{1}{2}\left(\int_0^1 \frac{x^2 + (1-x)^2}{x^2 + (1-x)^2} dx\right) \\ &= \frac{1}{2} \end{aligned}$$

(c)

$$\begin{aligned} \frac{dy}{dx} &= e^{\cos\left(x^2\right)} \cdot \left(-\sin\left(x^2\right)\right)(2x) \\ &= -2xe^{\cos\left(x^2\right)}\sin\left(x^2\right) \\ \frac{d^2y}{dx^2} &= -2e^{\cos\left(x^2\right)}\sin\left(x^2\right) - 2x\sin\left(x^2\right)\cdot\left(-2xe^{\cos\left(x^2\right)}\sin\left(x^2\right)\right) - 2xe^{\cos\left(x^2\right)}\left(2x\cos x^2\right) \\ &= -2e^{\cos\left(x^2\right)}\sin\left(x^2\right) + 4x^2e^{\cos\left(x^2\right)}\sin^2\left(x^2\right) - 4x^2e^{\cos\left(x^2\right)}\cos\left(x^2\right) \end{aligned}$$

(d) Differentiate implicitly:

$$\begin{aligned} \frac{x^2}{9} + \frac{y^2}{16} &= 1 \\ \frac{2x}{9} + \frac{y}{8}\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-2x}{9}\cdot\frac{8}{y} \\ &= \frac{-16x}{9y} \end{aligned}$$

Question 6

(a)

$$\begin{aligned}16x^4 - 40x^2 + 9 = 0 &\Leftrightarrow (4x^2 - 1)(4x^2 - 9) = 0\\ &\Leftrightarrow x = \pm 0.5 \text{ or } x = \pm 1.5\end{aligned}$$

Hence the equation has exactly 2 solutions (0.5 and -0.5) in the interval $[-1, 1]$.

(b) We have:

$$\begin{aligned}9 - x^2 = 3 - x &\Leftrightarrow (3 - x)(2 + x) = 0\\ &\Leftrightarrow x = -2 \text{ or } x = 3\end{aligned}$$

Therefore, using the Disk/Washer method, we get the volume of the solid:

$$\begin{aligned}\int_{-2}^{3} \pi|(9 - x^2)^2 - (3 - x)^2|dx &= \pi \int_{-2}^{3} ((9 - x^2)^2 - (3 - x)^2)dx\\ &= \pi \int_{-2}^{3} (x^4 - 19x^2 + 6x + 72)dx\\ &= \pi \left(\frac{x^5}{5} - \frac{19x^3}{3} + 3x^2 + 72x \right)\Bigg|_{-2}^{3}\\ &= \frac{625\pi}{3}\end{aligned}$$

(Since $(9 - x^2)^2 - (3 - x)^2 = (x - 3)^2(x + 2)(x + 4) \geq 0$ on $[-2, 3]$)

Question 7

(a) We have:

$$|g(1 + x) - 2| \leq x^2 \qquad \forall x : -1 \leq x \leq 1 \tag{2}$$

i)
Let $x = 0$ in (2), we get: $|g(1) - 2| \leq 0 \Rightarrow g(1) = 2$.

ii) and iii)
We compute the limit: $\lim\limits_{\Delta x \to 0} \dfrac{g(1 + \Delta x) - g(1)}{\Delta x}$.

We have, from (**??**) and i):

$$0 \leq \left| \frac{g(1 + \Delta x) - g(1)}{\Delta x} \right| = \left| \frac{g(1 + \Delta x) - 2}{\Delta x} \right| \leq |\Delta x|$$

Note that $|\Delta x| \to 0$, by Squeeze Theorem, we conclude:

$$\lim_{\Delta x \to 0} \frac{g(1 + \Delta x) - g(1)}{\Delta x} = 0$$

Therefore, g is differentiable at $x = 1$ and $g'(1) = 0$.

(b) Let h(m) be the height of the tank and a(m) be the side length of the bottom. Then we have the following relation:

$$a^2h = 64 \quad (a, h > 0)$$

The cost for the bottom: $8a^2$.
The cost for the 4 sides: $4 \cdot 4ah = 16ah$.
Hence the total cost: $8a^2 + 16ah = 8a^2 + \dfrac{1024}{a}$ (Since $ah = \dfrac{64}{a}$).
tyle
Let $f(x) = 8x^2 + \dfrac{1024}{x}$. Then $f'(x) = 16x - \dfrac{1024}{x^2}$.
Thus:

$$f'(x) = 0 \Leftrightarrow 16x - \frac{1024}{x^2} = 0 \Leftrightarrow 16x^3 - 1024 = 0 \Leftrightarrow x = 4$$

On $(0, 4) : f'(x) < 0 \Rightarrow f$ is decreasing.
On $(4, \infty) : f'(x) > 0 \Rightarrow f$ is increasing.

Thus $x = 4$ is the global minimum of f on $(0, \infty)$.

Hence, the cost is minimized when $a = 4$, at which $h = 4$.

Conclusion: the tank with the least cost has the height of $4m$ and the bottom side length of $4m$.

Question 8

We have $g'_{i,n}(x) = -n(1-x)^{n-1}(1+ix) + i(1+x)^n = (1+x)^{n-1}(i - n - ix - inx) < 0$. Since the function is decreasing, it is one-to-one. Furthermore, $g(1) = 0$ and $g(0) = 1$. Since the function is continuous, we conclude from the Intermediate Value Theorem that the range of $g_{i,n}$ is indeed $[0, 1]$.

Question 9

(a)

$$f_n(x) = (n+1)x^n - nx^{n+1}$$

(i)
We have:

$$f'_n(x) = n(n+1)(x^{n-1} - x^n)$$

$$f'_n(x) = 0 \Leftrightarrow x^{n-1} - x^n = 0 \tag{3}$$

Now we consider 3 cases:
Case 1: $n = 1$: (**??**) $\Leftrightarrow x = 1$

x	$(-\infty, 1)$	$(1, \infty)$
$f'_n(x)$	$+$	$-$

Hence, $f_n(x)$ is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. (6.a)
Case 2: $n > 1$ and n is odd: (**??**) $\Leftrightarrow x = 1$ or $x = 0$

x	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f'_n(x)$	$+$	$+$	$-$

Hence, f_n is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. (6.b)
Case 3: $n > 1$ and n is even: (**??**) $\Leftrightarrow x = 1$ or $x = 0$

x	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f_n'(x)$	$-$	$+$	$-$

Hence, f_n is increasing on $(0, 1)$ and decreasing on $(-\infty, 0)$ and $(1, \infty)$. (7)

(ii)
From (i), we conclude that:
If n is odd, f_n has only 1 local maximum at $x = 1$ or the point $(1, 1)$.
If n is even, f_n has 1 local minimum at $(0, 0)$ and 1 local maximum at $(1, 1)$.

(iii)
We consider 4 cases:
Case 1: $n = 1$. Here we have $f_n''(x) = -2 < 0\ \forall x$. Hence f is concave down on $\mathbb{R}$.

Case 2: $n = 2$. Here we have $f_n''(x) = 6(1 - 2x)$ hence $f_n''(x) = 0 \Leftrightarrow x = \dfrac{1}{2}$.

x	$(-\infty, \frac{1}{2})$	$(\frac{1}{2}, \infty)$
$f_n''(x)$	$+$	$-$

Hence f_n is concave up on $\left(-\infty, \dfrac{1}{2}\right)$ and concave down on $\left(\dfrac{1}{2}, \infty\right)$.

Case 3: $n > 2$ and n is even. Here we have $f_n''(x) = n(n+1)x^{n-2}[(n-1) - nx]$ hence $f_n''(x) = 0 \Leftrightarrow x = \dfrac{n-1}{n}$ or $x = 0$.

x	$(-\infty, 0)$	$(0, \frac{n-1}{n})$	$(\frac{n-1}{n}, \infty)$
$f_n''(x)$	$+$	$+$	$-$

Hence f_n is concave up on $\left(-\infty, \dfrac{n-1}{n}\right)$ and concave down on $\left(\dfrac{n-1}{n}, \infty\right)$.

Case 4: $n > 2$ and n is odd. Here we have $f_n''(x) = n(n+1)x^{n-2}[(n-1) - nx]$ hence $f_n''(x) = 0 \Leftrightarrow x = \dfrac{n-1}{n}$ or $x = 0$.

x	$(-\infty, 0)$	$(0, \frac{n-1}{n})$	$(\frac{n-1}{n}, \infty)$
$f_n''(x)$	$-$	$+$	$-$

Hence f_n is concave up on $\left(0, \dfrac{n-1}{n}\right)$ and concave down on $(-\infty, 0)$ and $\left(\dfrac{n-1}{n}, \infty\right)$.

(iv)
From (iii), we conclude:

For $n = 1$: f_n has no inflection points.

For $n > 1$ and n is even: f_n has 1 inflection point at $\left(\dfrac{n-1}{n}, 2\left(\dfrac{n-1}{n}\right)^n\right)$ (Since $f_n\left(\dfrac{n-1}{n}\right) = 2\left(\dfrac{n-1}{n}\right)^n$).

For $n > 1$ and n is odd, f_n has the above and 1 additional inflection point at $(0, 0)$ (Since $f(0) = 0$).

(b) (i)
From (6.a) and (6.b), we conclude that f_n attains its only global maximum value at $x = 1$ if n is odd.

Therefore, f_n has an absolute maximum value and it attains that value at exactly 1 point.

(ii)
From (7), we conclude that f_n attains its maximum value on $(0, \infty)$ at and only at $x = 1$ if n is even.
Therefore, f_n has an absolute maximum value on $(0, \infty)$ and it attains that value at exactly 1 point.

END OF SOLUTIONS

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