NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to An Hoa, VU

**MA2214 Combinatorial Analysis**
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**Question 1**

The arrangement can be viewed as a two-step process: first, arrange the people to the circles and then, for each circle, arrange the seat to each person.

There are $\frac{n!}{1!\times 2!\times\cdots\times m!}$ ways to distribute the $n$ people to the circles.

For circle $T_i$, there are $(i-1)!$ ways to make seat arrangement.

Thus, the number of possible seat arrangement are:

$$\frac{n!}{1!\times 2!\times\cdots\times m!}\times(0!\times 1!\times\cdots\times(m-1)!)=\frac{n!}{m!}.$$

**Question 2**

(a) Define the following sets

$$\begin{aligned}U&=\{n\in\mathbb{N}|0<n<2010\}\\ M&=\{n\in\mathbb{N}|0<n<2010\wedge 3\mid n\}\\ N&=\{n\in\mathbb{N}|0<n<2010\wedge 4\mid n\}\\ P&=\{n\in\mathbb{N}|0<n<2010\wedge 5\mid n\}.\end{aligned}$$

The number of positive integers strictly less than 2010, are multiple of 3 or 4 and not of 5 is the cardinality of the set $(M\cup N)\cap P^c$ (where $P^c=U\backslash P$). Drawing the Venn diagram (or using algebraic manipulation), we obtain

$$|M\cup N\cap P^c|=|M|+|N|-|M\cap P|-|N\cap P|+|M\cap N\cap P|.$$

Notice that the number of positive integers less than or equal to $n$ and is divisible by $m$ is $\lfloor\frac{m}{n}\rfloor$. We can then easily get

$$\begin{aligned}|M|&=\lfloor\frac{2009}{3}\rfloor=669\\ |N|&=\lfloor\frac{2009}{4}\rfloor=502\\ |M\cap P|&=|\{n\in\mathbb{N}|0<n<2010\wedge 3\mid n\wedge 5\mid n\}|\\ &=|\{n\in\mathbb{N}|0<n<2010\wedge 15\mid n\}|\\ &=\lfloor\frac{2009}{15}\rfloor=103\end{aligned}$$

and similarly,

$$|N \cap P| = \lfloor \frac{2009}{20} \rfloor = 100$$
$$|M \cap N \cap P| = \lfloor \frac{2009}{60} \rfloor = 33.$$

Hence, the result is

$$669 + 502 - 103 - 100 + 33 = 1001.$$

(b) The number of positive integers strictly less than 3001 and is divisible by three primes in $2, 3, 7, 11$ is

$$\lfloor \frac{3000}{2 \times 3 \times 7} \rfloor + \lfloor \frac{3000}{3 \times 7 \times 11} \rfloor + \lfloor \frac{3000}{7 \times 11 \times 2} \rfloor + \lfloor \frac{3000}{11 \times 2 \times 3} \rfloor = 71 + 12 + 19 + 45 = 147$$

In the above sum, we need to subtract away four times the number of positive integers that are divisible by all the four primes $2, 3, 7, 11$. (That is because this amount is counted four times.) The final result is thus

$$147 - 4 \times \lfloor \frac{3000}{2 \times 3 \times 7 \times 11} \rfloor = 147 - 4 \times 6 = 123.$$

**Question 3**

The total number of shortest routes (i.e routes that require exactly $5 + 8 = 13$ steps) with the presence of $ABCD$ is $\binom{13}{5} = 1287$. We thus need to count the number of illegitimate routes (shortest routes that pass through either of $AB, CA, CD, DB$). Notice that a shortest route must either go upward or rightward in any step of minimum 13 steps.

Let define the following sets

$$R_1 = \{\text{routes from O to P passing through } AB\}$$
$$R_2 = \{\text{routes from O to P passing through } CA\}$$
$$R_3 = \{\text{routes from O to P passing through } CD\}$$
$$R_4 = \{\text{routes from O to P passing through } DB\}$$

Formally speaking, we need to count the number of elements of $R_1 \cup R_2 \cup R_3 \cup R_4$. According to the principle of inclusion - exclusion (PIE):

$$|R_1 \cup R_2 \cup R_3 \cup R_4| = |R_1| + |R_2| + |R_3| + |R_4| - |R_1 \cap R_2| - ... + |R_1 \cap R_2 \cap R_3| + ...$$

Note that a shortest route cannot go through three sides of $ABCD$, thus $|R_i \cap R_j \cap R_k| = 0$ for $i \neq j \neq k \neq i$. Furthermore, if it is passing through two of the edges of $ABCD$, the pair of two edges can only be $CA\&AB$ or $CD\&BD$. Hence, except for $R_1 \cap R_2$ and $R_3 \cap R_4$, $R_i \cap R_j = \emptyset$.

Thus, we only need to count $|R_1|, |R_2|, |R_3|, |R_4|, |R_1 \cap R_2|, |R_3 \cap R_4|$ in order to get the number of illegitimate routes.

For each $R_i$, the routes can be obtained by a three-step process:

(i) going from O to the first endpoint ($A$ for $R_1$, $C$ for $R_2$, etc.);

(ii) take the corresponding edge ($AB$ for $R_1$, $CA$ for $R_2$, etc.); and finally,

(iii) go from the second endpoint ($B$ for $R_1$, $A$ for $R2$, etc.) to P.

The first and the last can be obtained in the similar manner of counting the number of routes from O to P. The second step has only one choice. We have:

$$
\begin{aligned}
|R_1| &= \binom{7}{3} \times \binom{5}{2} = 350 \\
|R_2| &= \binom{6}{2} \times \binom{6}{2} = 225 \\
|R_3| &= \binom{6}{2} \times \binom{6}{3} = 300 \\
|R_4| &= \binom{7}{2} \times \binom{5}{2} = 210
\end{aligned}
$$

The rest is to count $|R_1 \cap R_2|$ and $|R_3 \cap R_4|$. But we do not need to count both of them individually, we count the sum of them instead. Notice that the route in these cases can be broken into three steps:

(i) go from $O$ to $C$: $\binom{6}{2}$ possibilities;

(ii) go from $C$ to $B$: 2 possibilities (either via $A$ or via $D$);

(iii) go from $B$ to $P$: $\binom{5}{2}$.

Thus, $|R_1 \cap R_2| + |R_3 \cap R_4| = \binom{6}{2} \times 2 \times \binom{5}{2} = 300$.

So the number of illegitimate routes are:

$$350 + 225 + 300 + 210 - 300 = 785,$$

leaving the number of shortest routes with $ABCD$ deleted being

$$1287 - 785 = 502.$$

**Question 4**

Recall that for a number $n = \prod_{i=1}^{k} p_i^{a_i}$ in standard prime factorisation ($a_i > 0$), we have

$$\phi(n) = \prod_{i=1}^{k} p_i^{a_i - 1}(p_i - 1).$$

In case $\phi(n) = 20 = 2^2 \times 5$, each prime factor must satisfy $p_i - 1 | 20$. Furthermore, if $a_i > 1$ then $p_i = 2$ or $p_i = 5$.

The divisors of 20 are 1, 2, 4, 5, 10, 20. Thus, $p_i \in \{2, 3, 5, 6, 11, 21\} \cap \mathbb{P} = \{2, 3, 5, 11\}$ (where $\mathbb{P}$ is the notation for the set of prime numbers). So $n$ can only be of form $2^a \times 3^b \times 5^c \times 11^d$. Noted that $b$ is either 0 or 1. So is $d$. $1 \le c \le 2$ (non-zero to get the factor 5) and $a \le 3$ (if $a > 3$ then $8 | \phi(n)$).

If $d = 1$ then $\phi(2^a \times 3^b \times 5^c) = 2$. In such case, we have only two choices: either $a = 2$ or $b = 1$.

If $d = 0$ then we consider the choices of $c$: If $c = 1$ then: If $b = 1$ then $a = 2$; otherwise $a = 3$. If $c = 2$ then $\phi(25) = 20$.

So all the values of $n$ such that $\phi(n) = 20$ are: 44, 33, 60, 40, 25. We have 5 such values.

**Question 5**

The ordinary generating function for the sequence $(a_n)$ can be obtained as:

$$f(x) = \underbrace{(x^{10} + x^{11} + \cdots)}_{x_1} \times \underbrace{(1 + x^2 + x^4 + \cdots)}_{x_2} \times \underbrace{(1 + x + x^2 + \cdots)}_{x_3} \times \underbrace{(x^2 + x^4 + \cdots + x^{20})}_{x_4}$$

Simplifying the sub-expressions, we have:

$$\begin{aligned}
x^{10} + x^{11} + \cdots &= x^{10}\frac{1}{1-x} \\
1 + x^2 + x^4 + \cdots &= \frac{1}{1-x^2} \\
1 + x + x^2 + \cdots &= \frac{1}{1-x} \\
x^2 + x^4 + \cdots + x^{20} &= \frac{1-x^{22}}{1-x^2} - 1 \\
&= \frac{x^2 - x^{22}}{1-x^2} \\
&= x^2\frac{1-x^{20}}{1-x^2}
\end{aligned}$$

Multiplying all the right hand side, we get

$$\begin{aligned}
f(x) &= \frac{x^{10}}{1-x}\frac{1}{1-x^2}\frac{1}{1-x}\frac{x^2-x^{22}}{1-x^2} \\
&= x^{12}\frac{1-x^{20}}{(1-x)^4(1+x)^2}.
\end{aligned}$$

This is what to be shown.

**Question 6**

Let $b_n = a_n + C_1 5^n + C_2 3^n$ i.e $b_n$ is $a_n$ plus a linear combination of two exponential functions $5^n$ and $3^n$. We shall find the constants $C_1$ and $C_2$ such that

$$b_n = 7b_{n-1} - 12b_{n-2}.$$

The sequence $b_n$ can be easily solved by standard method. And thus, $a_n$ can be obtained by the above defining equation:

$$a_n = b_n - C_1 5^n - C_2 3^n.$$

To solve for $C_1$ and $C_2$, we start from the relation

$$a_n - 7a_{n-1} + 12a_{n-2} = 5^n - 3^n,$$

and make the substitution

$$(b_n - C_1 5^n - C_2 3^n) - 7(b_{n-1} - C_1 5^{n-1} - C_2 3^{n-1}) + 12(b_{n-2} - C_1 5^{n-2} - C_2 3^{n-2}) = 5^n - 3^n,$$

or

$$\underbrace{b_n - 7b_{n-1} + 12b_{n-2}}_{0} + \underbrace{(-C_1 + 7\frac{C_1}{5} - 12\frac{C_1}{25})}_{-\frac{2}{25}C_1} 5^n - \underbrace{(C_2 - 7\frac{C_2}{3} + 12\frac{C_2}{9})}_{0} 3^n = 5^n - 3^n.$$

At this stage, we can choose $C_1 = -\frac{25}{2}$. However, we cannot make a choice of $C_2$ to fulfill our task. We thus change the defining equation to

$$a_n = b_n - C_1 5^n - C_2 n 3^n,$$

that is taking linear combination of $5^n$ and $n3^n$ instead. In such case, with a similar derivation, we obtain the requirement $nC_2 - 7(n-1)\frac{C_2}{3} + 12(n-2)\frac{C_2}{9} = 1$ or $C_2(n - \frac{7n-7}{3} + \frac{4n-8}{3}) = 1$ or $C_2(\frac{7}{3} - \frac{8}{3}) = 1$ or $C_2 = -3$.

In summary, we make substitution

$$b_n = a_n - \frac{5^{n+2}}{2} - n3^{n+1},$$

and obtain $b_n - 7b_{n-1} + 12b_{n-2} = 0$.

The first two terms of the sequence $b_n$ can now be found as

$$\begin{aligned} b_0 &= a_0 - \frac{5^2}{2} - 0 \times 3 \\ &= -\frac{23}{2} \\ b_1 &= a_1 - \frac{5^3}{2} - 1 \times 3^2 \\ &= -\frac{141}{2} \end{aligned}$$

and the sequence $b_n$ can be solved (by standard method) to be

$$b_n = \frac{49}{2} \times 3^n - 36 \times 4^n.$$

Hence, the solution to the original recurrence relation is

$$a_n = \frac{49}{2} \times 3^n - 36 \times 4^n + \frac{5^{n+2}}{2} + n3^{n+1}.$$

**Question 7**

Let $X_1, X_2, \cdots, X_6$ denote the number of die that gives the values $1, 2, \cdots, 6$ in a roll, respectively. We thus have:

$$\sum_{i=1}^{6} X_i = n,$$

as any die has only values from 1 to 6 and the total number of die is $n$. The sum of all the values (on the face of each die) is

$$S = \sum_{i=1}^{6} iX_i.$$

Notice that $S \equiv X_1 + X_3 + X_5 \pmod 2$. Thus, in order to get even sum, we thus requires the number of die that give odd value to be even. In such case, $X_1 + X_2 + X_3 \in \{0, 2, \cdots, 2N\}$. Given $X_1 + X_3 + X_5 = 2m$, the number of possibilities of $(X_1, X_3, X_5)$ is $H^3_{2m}$ (the number of non-negative integral solution of the equation $X_1 + X_3 + X_5 = 2m$). $X_2 + X_4 + X_6 = n - 2m$ and hence, there are $H^3_{n-2m}$ solutions of $(X_2, X_4, X_6)$.

So, the total number of possibilities of getting an even sum is

$$\sum_{m=0}^{N} H_{2m}^3 H_{n-2m}^3.$$

**Question 8**

(a) Let $R = \{1, 2, \cdots, n\} \backslash (S \cup T)$. Then $R, T, S$ are pairwise disjoint and $R \cup T \cup S = \{1, 2, \cdots, n\}$. That is to say: $R, T, S$ form a partition of the set of natural numbers from 1 to $n$.

The process of choosing subsets can be viewed in a different way: we first distributing the numbers from 1 to $n$ to either $R$ or $T$ or $S$ and then collect them into the corresponding set. Each number has three choices and thus, we have $3^n$ possible distribution, giving rise to exactly $3^n$ possibilities of selection of $S$ and $T$.

(b) There are $2^n$ possibilities for $S$ and so are there for $T$. Thus, there are $2^n \times 2^n = 2^{2n} = 4^n$ ways to select two subsets. Subtracting the number of disjoint pairs counted from the above part, we get $4^n - 3^n$ pairs whose intersections are not trivial.

**Question 9**

(a) Let us first assign the coordinate to the wall as depicted by the following picture:

| | | | |
|---|---|---|---|
| $(1, 0)$ | $(1, 1)$ | $\cdots$ | $(1, n-1)$ |
| $(0, 0)$ | $(0, 1)$ | $\cdots$ | $(0, n-1)$ |

Each square will be addressed by $(r, c)$ where $r \in \{0, 1\}$ indicating the row and $0 \leq c \leq n$ indicating the column.

Each tiling $T$ for case $n$ can be obtained by tiling the sub-wall of size $2 \times (n-1)$ and then pave the two new spaces i.e. pave the locations $(0, n-1)$ and $(1, n-1)$. Let $T(x, y)$ denote the type of tile used to pave the location $(x, y)$.

First, we can pave $(0, n-1)$ by any tile whose type is different from $T(0, n-2)$. Then, we need to pave the space $(1, n-1)$ with a tile whose type is different from both $T(0, n-1)$ and $T(1, n-2)$ (can be equal). This way ensures that two adjacent tiles are of different types (or colors). In the first step, there are two cases to consider:

- The type of tile to pave $(0, n-1)$ is different from $T(1, n-2)$: there are $m-2$ choices to pave $(0, n-1)$ (all except the types $T(0, n-2)$ and $T(1, n-2)$) and there are also $m-2$ possible pavement can be made for $(1, n-1)$. Thus, this case give $(m-2)^2$ possible tiling.
- The type of tile chosen to pave $(0, n-1)$ is the same as $T(1, n-2)$: (notice that $T(1, n-2) \neq T(0, n-2)$) In this case, we can tile $(1, n-1)$ using tiles of any type except for the type used to tile $(0, n-1)$ (or $(1, n-2)$). We have an additional $m-1$ possible tiling.

So to summarize, each tiling of the $2 \times (n-1)$ sub-wall give rise to exactly $(m-2)^2 + (m-1) = m^2 - 3m + 3$ possible tilings of the $2 \times n$ wall. The recurrence relation is thus obtained as

$$a(m)_n = a(m)_{n-1} \times (m^2 - 3m + 3).$$

(b) From the recurrence relation, we realize that fixing $m$, $a(m)_n$ is a geometric series and thus, we easily derive the general formula for $a(m)_n$ as a function of $m$ and $n$:

$$\begin{aligned} a(m)_n &= a(m)_0 \times (m^2 - 3m + 3)^n \\ &= \frac{m(m-1)}{m^2 - 3m + 3} \times (m^2 - 3m + 3)^n \\ &= m(m-1)(m^2 - 3m + 3)^{n-1}. \end{aligned}$$