MA2108S - Mathematical Analysis I(S) Suggested Solutions

(Semester 2 : AY2017/18)

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Question 1

(a) $\sum_{n=4}^{\infty} \frac{1}{n\log(\sqrt{n}+\cos n)}$ diverges. Note that

$$\sum_{n=4}^{\infty} \frac{1}{n\log(\sqrt{n}+\cos n)} \geq \sum_{n=4}^{\infty} \frac{1}{n\log(\sqrt{n}+1)}.$$

By Cauchy Condensation Test, set $a_n = \frac{1}{n\log(\sqrt{n}+1)}$, then $2^n a_{2^n} = \frac{2^n}{2^n\log(\sqrt{2^n}+1)} = \frac{1}{\log(\sqrt{2^n}+1)}$. The sum $\sum_{n=4}^{\infty} \frac{1}{n\log(\sqrt{n}+1)}$ will diverge if $\sum_{n=4}^{\infty} \frac{1}{\log(\sqrt{2^n}+1)}$ diverges. Now observe

$$\sum_{n=4}^{\infty} \frac{1}{\log(\sqrt{2^n}+1)} \geq \sum_{n=4}^{\infty} \frac{1}{\log(\sqrt{2^n}+\sqrt{2^n})} = \sum_{n=4}^{\infty} \frac{1}{\log(2^{\frac{n}{2}+1})} = \frac{2}{\log(2)} \sum_{n=4}^{\infty} \frac{1}{(n+2)}$$

which diverges by comparison with p-series.

(b) $\sum_{k=2}^{\infty} \frac{\sin k}{\log k}$ converges. Obviously $\{\frac{1}{\log k}\}_{k=1}^{\infty}$ is monotone and converges to 0. The key point is that $\sum \sin k$ is bounded. Once you have $\sum \sin k$ bounded, then $\sum_{k=2}^{\infty} \frac{\sin k}{\log k}$ converges by Dirichlet's test. To show $\sum \sin k$ is bounded, note that by factor formula,

$$2(\sin k)(\sin(0.5)) = \cos(k-0.5) - \cos(k+0.5).$$

Summing up from $1 \cdots n$, we get a cancellation between adjacent terms. This implies that:

$$\sum_{k=1}^{n} \sin k = \frac{\cos(0.5) - \cos(n+0.5)}{2\sin(0.5)}$$

which is clearly bounded.

<u>You should read this part after you read through part (c) and (d).</u>

Our solution for (c) and (d) uses 'regrouping' of terms, so before we tackle that, we first prove a theorem first.

Let $\{a_n\}$ be a sequence converging to 0, let $s_n = \sum_{k=1}^{n} a_k$, let $\{b_n\}$ be such that $b_n = a_{2n} + a_{2n-1}$ and let $t_n = \sum_{k=1}^{n} b_k$.

Claim : $\{s_n\}$ converges $\iff$ $\{t_n\}$ converges. The intuition is that (t_n) is a subsequence of (s_n), and since the a_n's goes to 0, it is not big enough for (t_n) to "wriggle" out of that ϵ radius.

Proof : Obviously $\{s_n\}$ converges $\implies$ $\{t_n\}$ converges since $\{t_n\}$ is a subsequence of $\{s_n\}$. Thus we only prove that $\{t_n\}$ converges $\implies$ $\{s_n\}$ converges.

Suppose $\{t_n\} \to \alpha$. Then there exists some N_1 such that $n > N_1 \implies |t_n - \alpha| < \frac{\epsilon}{2}$. Since $\{a_n\} \to 0$, again there exists N_2 such that $n > N_2 \implies |a_n| < \frac{\epsilon}{2}$. Since for every two a terms there is a b term (ie. $b_1 = a_2 + a_1$, $b_2 = a_4 + a_3 + \cdots$), we pick $N = \max\{2N_1, N_2\}$.

For all $n > N$, one has $|s_n - \alpha| = |a_n + t_{\text{floor(n/2)}} - \alpha| \leq |a_n| + |t_{\text{floor(n/2)}} - \alpha| < \epsilon$ (where $a_n = 0$ if n is even since $t_{\text{floor(n/2)}} = s_n$). So $\{t_n\} \to \alpha$ implies $\{s_n\} \to \alpha$.

We have thus shown that provided $a_n \to 0$, there is no difference between $a_1+a_2+a_3+a_4+\cdots$ and $(a_1 + a_2) + (a_3 + a_4) + c \ldots$. The reason why this is important is because there are some instances where you cannot "regroup" the terms, such as in the case of $1 - 1 + 1 - 1 + 1 - 1$ as this series is divergent, but regrouping them $(1 - 1) + (1 - 1) + \ldots$ causes it to sum to 0.

In (c) and (d), the sequences go to zero, thus what we have proved above applies and we may regroup the terms.

(c) Let's write the series out explicitly.

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n} + (-1)^n} = \frac{1}{\sqrt{3}-1} - \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{5}-1} - \frac{1}{\sqrt{4}+1} + \ldots$$

Our series can be rewritten as:

$$\frac{1}{\sqrt{3}-1} - \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{5}-1} - \frac{1}{\sqrt{4}+1} + \ldots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}-1} - \frac{1}{\sqrt{2n}+1}$$

Now note that $\sqrt{2n}+1 \geq \sqrt{2n+1}$ for $n \in \mathbb{N}$. This can be easily seen by squaring both sides.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}-1} - \frac{1}{\sqrt{2n}+1} &\geq \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}-1} - \frac{1}{\sqrt{2n+1}} \\
&\geq \sum_{u \in \{3,5,7\cdots\}}^{\infty} \frac{1}{\sqrt{u}-1} - \frac{1}{\sqrt{u}} \\
&= \sum_{u \in \{3,5,7\cdots\}}^{\infty} \frac{1}{u-\sqrt{u}} \\
&\geq \sum_{u \in \{3,5,7\cdots\}}^{\infty} \frac{1}{u}
\end{aligned}$$

which diverges.

(d) $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^\alpha + (-1)^n}$ converges if $\frac{1}{2} < \alpha$. As before, we may rewrite our sum as such,

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^\alpha + (-1)^n} = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^\alpha - 1} - \frac{1}{(2n)^\alpha + 1}.$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(2n+1)^\alpha - 1} - \frac{1}{(2n)^\alpha + 1} &\leq \sum_{n=1}^{\infty} \frac{1}{(2n)^\alpha - 1} - \frac{1}{(2n)^\alpha + 1} \\ &\leq \sum_{n=1}^{\infty} \frac{2}{(2n)^{2\alpha} - 1}\end{aligned}$$

which converges based on our knowledge of the p-series.

Question 2

Observe that

$$\begin{aligned}\gamma &= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{1}{i} - \log n\right] \\ &= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{n-1} \log\left(\frac{i+1}{i}\right)\right] \\ &= \lim_{n\to\infty} \left[\log(1+\frac{1}{n}) + \sum_{i=1}^{n} \frac{1}{i} - \log\left(1 + \frac{1}{i}\right)\right] \\ &= \lim_{n\to\infty} \log(1+\frac{1}{n}) + \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{1}{i} - \log\left(1 + \frac{1}{i}\right)\right] \\ &= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{1}{i} - \log\left(1 + \frac{1}{i}\right)\right]\end{aligned}$$

However, $\frac{1}{i} - \log\left(1 + \frac{1}{i}\right) > 0$, which means that the sequence of partial sums is monotone increasing. The reason why this is so is from **Bernolli's inequality**, which says $e^x \geq 1 + x$ for all $x \in \mathbb{R}$. Taking log on both sides and replacing x with $\frac{1}{i}$ gives the desired result. Hence, we just need to show the sequence of partial sums is bounded above. Let $x = \frac{1}{i}$ and apply taylor expansion on $\log(1+x)$. The Taylor expansion of $\log(1+x)$ is absolutely convergent on $-1 < x < 1$. **Note that** $x = \frac{1}{i}$ **has** $x \leq 1$, which means that this is a valid operation to do and it will not alter our infinite series.

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}[x-\log(1+x)] &= \lim_{n\to\infty}\sum_{i=1}^{n} x-\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}-\cdots\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{2i^2}-\frac{1}{3i^3}+\frac{1}{4i^4}-\cdots\right)
\end{aligned}$$

We can subtract off smaller numbers by making the denominators bigger for the negative numbers.

$$\begin{aligned}
&\le \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{2i^2}-\frac{1}{4i^4}+\frac{1}{4i^4}-\cdots\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{2i^2}\right)\\
&= \sum_{i=1}^{\infty}\left(\frac{1}{2i^2}\right)\\
&= \frac{\pi^2}{12} \text{ and we are done.}
\end{aligned}$$

Question 3

Given

$$f(1)+\frac{x-1}{2}\le f(x)\le f(1)+\frac{x-1}{3} \qquad : x\in[0,1].$$

For any $x\in[0,1]$, one has:

$$\frac{1-x}{3}\le f(1)-f(x)\le\frac{1-x}{2}.$$

Note that both $\frac{1-x}{3}$ and $\frac{1-x}{2}$ are positive so $f(1)-f(x)$ is positive. In other words

$$x\in[0,1]\to f(x)\in[0,1]. \tag{1}$$

Then

$$|f(x)-f(1)|\le|\frac{x-1}{2}|.$$

Since $x_0\in[0,1]$, and using (1), we deduce by mathematical induction that $x_j\in[0,1]\ \forall j\in\mathbb{N}$. Now by using $f(x_{n-1})=x_n$ and $f(1)=1$

$$|x_n-1|=|f(x_{n-1})-f(1)|\le\frac{1}{2}|x_{n-1}-1|$$

Then $\{x_n\}_{n=1}^{\infty}$ is a contractive sequence so $\lim_{n\to} x_n=1$.

Question 4

We want to show: If there is no $x \in X$ such that $f(x) = x \implies \exists \epsilon > 0$ such that $d(f(x), x) \geq \epsilon$ for all $x \in X$. We can do it by contradiction.

Suppose that for all ϵ, there exists x such that $d(f(x), x) < \epsilon$. Then we can take our successive ϵ's to be $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3} \cdots$ and we have an x_n fulfilling this condition for all $n \in \mathbb{N}$,

$$d(f(x_n), x_n) < \frac{1}{2^n}. \tag{2}$$

Consider the sequence of x_n's. We shall denote the sequence $\{x_n\}_{n=1}^{\infty}$. By compactness, $\{x_n\}_{n=1}^{\infty}$ contains a Cauchy subsequence that converges to a point $x \in M := (X, d)$. Let this subsequence be $\{x'_n\}_{n=1}^{\infty}$.

By continuity of f, the sequence $\{f(x'_n)\}_{n=1}^{\infty}$ converges to $f(x)$. If we can show that $\{f(x'_n)\}_{n=1}^{\infty}$ converges to x, then we automatically have $x = f(x)$ since the limit that a sequence converges to is unique if it exists.

Fix $\epsilon > 0$. Since $\{x'_n\}_{n=1}^{\infty}$ converges to x, $\exists N_1 \in \mathbb{N}$ such that $n_1 \geq N_1 \implies d(x'_{n_1}, x) \leq \frac{\epsilon}{2}$. From (2), $\exists N_2 \in \mathbb{N}$ such that $n_2 \geq N_2 \implies d(f(x_{n_2}), x_{n_2}) < \frac{1}{2^{n_2}} < \frac{\epsilon}{2}$. Take $N = \max\{N_1, N_2\}$. Then $n \geq N$ implies:

$$d(f(x'_n), x) \leq d(f(x'_n), x'_n) + d(x, x'_n) < \epsilon.$$

Thus the sequence $\{f(x'_n)\}_{n=1}^{\infty}$ converges to x so $x = f(x)$, a contradiction.

Question 5

Suppose we have points $s < t < u$. Now,

$$\begin{aligned} t &= \frac{t(u-s)}{u-s} \\ &= \frac{tu - ts + su - su}{u-s} \\ &= \left(\frac{t-s}{u-s}\right) u + \left(\frac{u-t}{u-s}\right) s. \end{aligned} \tag{3}$$

It can be checked that

$$1 - \frac{t-s}{u-s} = \frac{u-t}{u-s}.$$

So choose $\lambda = \frac{t-s}{u-s}$ and we have $t = \lambda u + (1-\lambda)s$. By the convexity of f,

$$\begin{aligned} f(t) &\leq \lambda f(u) + (1-\lambda) f(s) \\ f(t) &\leq \lambda f(u) + f(s) - \lambda f(s) \\ f(t) - f(s) &\leq \lambda (f(u) - f(s)) \\ f(t) - f(s) &\leq \left(\frac{t-s}{u-s} \right) (f(u) - f(s)) \\ \frac{f(t) - f(s)}{t-s} &\leq \frac{f(u) - f(s)}{u-s}. \end{aligned}$$

Now let $\mu = 1 - \lambda = \frac{u-t}{u-s}$ and $1 - \mu = \frac{t-s}{u-s}$. We rewrite (3) in μ as follows:

$$t = \mu s + (1-\mu) u.$$

The equality gives us,

$$\begin{aligned} f(t) &\leq \mu f(s) + (1-\mu) f(u) \\ f(t) &\leq \mu f(s) + f(u) - \mu f(u) \\ f(t) - f(u) &\leq \mu (f(s) - f(u)) \\ f(t) - f(u) &\leq \left(\frac{u-t}{u-s} \right) (f(s) - f(u)) \\ \frac{f(t) - f(u)}{u-t} &\leq \frac{f(s) - f(u)}{u-s}. \end{aligned}$$

Multiplying -1 to both sides gives us

$$\frac{f(u) - f(s)}{u-s} \leq \frac{f(u) - f(t)}{u-t}.$$

We get the following inequality. If $s < t < u$,

$$\frac{f(t) - f(s)}{t-s} \leq \frac{f(u) - f(s)}{u-s} \leq \frac{f(u) - f(t)}{u-t}. \tag{4}$$

Let $x \in (0,1)$ and we will prove that f is continuous at x. Since $(0,1)$ is an open interval, there exists some $\delta > 0$ such that $(x-\delta, x+\delta) \subset (a,b)$. To show continuity from the left hand side, let z be an arbitrary point $z \in (x-\delta, x)$. Applying (4) to $x - \delta < z < x$,

$$\frac{f(z) - f(x-\delta)}{z - (x-\delta)} \leq \frac{f(x) - f(x-\delta)}{\delta} \leq \frac{f(x) - f(z)}{x-z}.$$

Ignore the left inequality. We may also apply (4) on $z < x < x + \delta$,

$$\frac{f(x) - f(z)}{x-z} \leq \frac{f(x+\delta) - f(z)}{x+\delta-z} \leq \frac{f(x+\delta) - f(x)}{\delta}.$$

Ignore the middle inequality. We get,

$$\frac{f(x) - f(x-\delta)}{\delta} \leq \frac{f(x) - f(z)}{x-z} \leq \frac{f(x+\delta) - f(x)}{\delta}.$$

Now, δ is fixed, so $\frac{f(x)-f(x-\delta)}{\delta}$ and $\frac{f(x+\delta)-f(x)}{\delta}$ are constants. Let $\frac{f(x)-f(x-\delta)}{\delta} = K_1$ and $\frac{f(x+\delta)-f(x)}{\delta} = K_2$.

$$K_1(x-z) \leq f(x) - f(z) \leq K_2(x-z).$$

Since $z \in (x-\delta, x)$, we can only consider $z \to x$ from the left:

$$\lim_{z\to x^-} K_1(x-z) \leq \lim_{z\to x^-} f(x) - f(z) \leq \lim_{z\to x^-} K_2(x-z).$$

By squeeze theorem:

$$\lim_{z\to x^-} f(z) = f(x).$$

Similarly, let $y \in (x, x+\delta)$ and apply (4) to $x < y < x+\delta$,

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(x+\delta)-f(x)}{\delta} \leq \frac{f(x+\delta)-f(y)}{x+\delta-y}.$$

Ignore the right inequality. Apply (4) to $x-\delta < x < y$,

$$\frac{f(x)-f(x-\delta)}{\delta} \leq \frac{f(y)-f(x-\delta)}{y-(x-\delta)} \leq \frac{f(y)-f(x)}{y-x}.$$

Ignore the middle inequality. Again, we see that,

$$\frac{f(x)-f(x-\delta)}{\delta} \leq \frac{f(y)-f(x)}{y-x} \leq \frac{f(x+\delta)-f(x)}{\delta}.$$

Again, $\frac{f(x)-f(x-\delta)}{\delta}$ and $\frac{f(x+\delta)-f(x)}{\delta}$ are constants, so put them as K_3 and K_4.

$$K_3(y-x) \leq f(y) - f(x) \leq K_4(y-x).$$

Since $y \in (x, x+\delta)$, we now consider $y \to x$ from the right:

$$\lim_{y\to x^+} K_3(y-x) \leq \lim_{y\to x^+} f(y) - f(x) \leq \lim_{y\to x^+} K_4(y-x).$$

Using squeeze theorem again,

$$\lim_{y\to x^+} f(y) = f(x).$$

Finally

$$\lim_{z\to x^-} f(z) = f(x) \text{ and } \lim_{y\to x^+} f(y) = f(x) \implies \lim_{y\to x} f(y) = f(x).$$

so f is continuous at x. Since the choice of x is arbitrary, f is continuous on $(0,1)$.

One final thing to note is that when choosing $(x-\delta, x+\delta) \subset (a,b)$, we relied on (a,b) being an open interval. In fact, this result would fail at the end points, ie. at $x=a$ or $x=b$.