NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lee Yung Hei

MA3205 Set Theory
AY 2007/2008 Sem 1

Question 1

(i) $\emptyset, \{a\}, \{b\}, \{a, b\}$

(ii) $\{a, \{a, d\}\}, \{a, \{a, e\}\}, \{b, \{b, d\}\}, \{b, \{b, e\}\}, f$

(iii) a

(iv) a, b, c, d

Question 2

Any example will do, below are 2 possible solutions.

$A = \{\alpha|\alpha < \omega + 1\}$ and $B = \{\alpha|\alpha = \omega + \beta(\beta \in \omega)\}$.
$|A| = |B| = \omega$, $A \cap B = \omega$

A =set of all even numbers, B =set of all prime numbers.
$|A| = |B| = \omega$, $A \cap B = 2$

Question 3

For anyway $y \in \mathcal{P}(x)$, if $y = \emptyset$, then y is transitive.
If $y \neq \emptyset$, then $y \subseteq x$.
Since x is transitive, $\forall x \in x$, z is transitive.
So, $\forall z \in y$, z is transitive.
$\therefore$ y is transitive.

Question 4

(i) True.
Given $B \subseteq A$, $|B| \leq |A|$.

If both $|A|$ and $|B|$ are finite, then by simple addition, the statement is trivially true.

If at least $|A|$ is not finite:
Since $|A - B|, |B|$ are cardinals, $|A - B| + |B| = \max\{|A - B|, |B|\}$.
If $|B| < |A|$, then $|A - B| + |B| = \max\{|A - B|, |B|\} = |A|$.

If $|B| = |A|$, then $|A - B| \leq |A| = |B|$, so $|A - B| \leq |B|$.
$|A - B| + |B| = \max\{|A - B|, |B|\} = |B|$. $\square$

(ii) False, and this can be shown by a counter example.
$A = \omega = \{0, 1, 2, 3, \cdots\}$
$B = \omega - \{\emptyset\} = \{1, 2, 3, \cdots\}$.
$A \cap B = \{1, 2, 3, \cdots\}$, so $|A \cap B| = \omega = |A|$.
However, $\emptyset \in A$, so, $A \not\subseteq B$. $\square$

Question 5

	partially ordered	linearly ordered	well ordered
$(\mathbb{N}^*, <_{KB})$	Y	Y	N
$(V_\omega, \in)$	Y	Y	Y

Question 6

True.
By the definition of ordinals, an ordinal is transitive and well-ordered.
Since all ordinals are well-ordered and transitive, the intersection of these ordinals has to be the minimum of the set.
Therefore, the intersection is the minimum ordinal.

Question 7

(i) ω is a cardinal

(ii) ω^ω is NOT a cardinal

(iii) 5 is a cardinal

(iv) $|\omega_1 + 5| = \omega_1$, ω_1 is a cardinal, so $\omega_1 + 5$ is NOT a cardinal.

Question 8

False.
ω and $\omega + 1$ are countable and well-ordered, but they are not order isomorphic.

Question 9

(i) $\omega^6 + \omega^2 + \omega$

(ii) ω^{59}

(iii) $\omega^\omega \cdot 2 + 11$

(iv) $\omega \cdot 2 + 2$

Question 10

(i) Countable.

(ii) Countable.

(iii) Countable.

(iv) $\omega_3 > \omega_0$

Question 11

(i) $\aleph_{30}$

(ii) $\aleph_1$

Question 12

(i) $\omega^{\omega} + 1$

(ii) $\omega^{17} + \omega^2 + 5$

Question 13

(i) $f := \begin{cases} 1 & \text{if} |\alpha| = \alpha \\ 0 & \text{else} \end{cases}$

(ii)

$$\begin{aligned} g(\alpha) &= \sup\{\rho(\{\beta, \delta\}) | \beta, \delta \in \alpha\} \\ &= \sup\{S(\max\{\beta, \delta\}) | \beta, \delta \in \alpha\} \\ &= \sup\{S(\beta) | \beta \in \alpha\} \text{ (since max is a closed function)} \\ &= \alpha \text{ (by definition of ordinals)} \end{aligned}$$

Question 14

Let X be a set of sets.
$\exists$ choice function C such that $\forall x \in X$, where $x \neq \emptyset$, $C(x) \in x$.

Question 15

(i) Yes, Axiom of Choice is required.
let $g(y)$ be some $x \in X$ such that $f(x) = y$
Choice is used since it is possible that $f(x_1) = f(x_2) = y$. Thus, choice would be used to pick a particular pre-image of y.

(ii) No, Axiom of choice is NOT needed.
If $f := \mathbb{N} \to \mathbb{N}$ is surjective, we can explicitly define $g(y) = \min(\{x | f(x) = y\})$.
Since we can explicitly define g, we need not use Axiom of Choice to define g and so Axiom of Choice is not needed.

Question 16

(i) $2^{\aleph_0} = \aleph_1$

(ii) $2^{\aleph_\alpha} = \aleph_{S(\alpha)}$

(iii) No.

(iv) Yes.

Question 17

	Infinity	Power Set
$(V_{12}, \in)$	N	N
$(V_\omega, \in)$	N	Y
$(V_{\omega+1}, \in)$	Y	N

Question 18

(i) Yes. $2^{\aleph_1} > \aleph_1$

(ii) Maybe. $\aleph_5 = 2^{\aleph_4}$ or $\aleph_5 < 2^{\aleph_4}$

(iii) No. $2^{\aleph_1} < \aleph_\omega$

Section B

All questions are done. In this solution set.

Question 19

(i) We proof by induction.
Let $P(a)$ be the proposition that $f(m) \subset f(m+a)$.
$P(1) : f(m) \subset f(m+1)$ is true $\forall m \in \mathbb{N}$ since $f(m) \subset f(S(m))$
Now, suppose that $P(b)$ is true for some $b \in \mathbb{N}$.
Then consider $P(b+1)$.
Pick any $m \in \mathbb{N}$, $f(m) \subset f(m+b)$ by the induction hypothesis and $f(m+b) \subset f(S(m+b)) = f(m+b+1)$.
So, $P(b+1)$ is true.
By mathematical induction, $P(k)$ is true for all k.
If $m < n$, then since $m, n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $m + k = n$.
So, we are done.

(ii) Since there might be elements that are found in both F and A, we define $G = F \Delta A$.
We find the bijection in 2 steps.
Firstly, since A is countable, $|A| = \omega$, so there exists $f : \mathbb{N} \to A$ where f is bijective.

Since F is finite, G is also finite. ($|G| \leq |F|$)
Let $k = |G|$, so there exists $g : k \to G$ where g is bijective.

Now, we have 2 bijective functions f and g.
Define $h := \mathbb{N} \to (A \cup F)$. Note that $A \cup F = A \cup G$.
$$h(n) = \begin{cases} g(n) & n < k \\ f(n-k) & n \geq k \end{cases}$$
It is simple to check that $f^{-1} \circ h$ is a bijective map from A to $F \cup A$.

Question 20

(i) Suppose that $A \not\subseteq \alpha$, then there exists $\beta \in A$ where $\beta \notin \alpha$.
By comparability theorem, $\beta = \alpha$ or $\beta \ni \alpha$.
It is impossible for $\beta = \alpha$, since α is not in A by definition.
So, $\alpha \in \beta$ and $\beta \in A$.
However, A is transitive, so $\beta \in A$ is transitive, thus $\forall\ \kappa \in \beta$, $\kappa \in A$.
Since $\alpha \in \beta$, $\alpha \in A$. $\Rightarrow\Leftarrow$

So, $A \subseteq \alpha$.

(ii) Given $\alpha \in X$, $\alpha \in V_{card}$
Since all cardinals are ordinals, $V_{card} \subseteq V_{ord}$.
So, the supremum of ordinals is an ordinal, it is clear that $\sup(X) \in V_{ord}$.
Let $\sup(X) = \kappa$ for some ordinal κ.
If κ is a cardinal, we are done.

So, assume towards contradiction that κ is not a cardinal.
Then, $|\kappa| = \eta < \kappa$. (by definition, η is the least ordinal with the same cardinality as κ)
Since $\forall\ x \in X$, $x < \kappa$ and η is the same cardinality as κ, $x \leq \kappa$.
So, η is an upper bound of X.
This contradicts the definition that κ is the least upper bound.