MA2216 - Probability Suggested Solutions

(Semester 2 : AY2019/20)

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Question 1

A 5-digit number is formed such that each digit is one of the nine integers 1,2...9. Each integer can be used any number of times.

i) How many 5-digit numbers can be formed?

$9^5 = 59049$.

ii) How many 5-digit numbers can be formed such that no three consecutive digits are the same?

Consider numbers with 3 or more consecutive digits first. To have 3 consecutive digits, say your repeated digit is r. You can have rrrxx, xrrrx or xxrrr. For rrrxx, choose one number to be r. The x next to the r has 8 choices for 3 consecutive digits. The x 2 steps away from the last r has 9 chocies. xxrrr has a similar reasoning. For xrrrx, both x's are beside the 3 r's, meaning that for each x, there's a total of 8 choices. In total, for 3 consecutive digits, we have: $2 * 9 * 8 * 9 + 9 * 8 * 8$.

For numbers with 4 consecutive digits, we can either have rrrrx or xrrrr. Pick one of 9 numbers to be r, then x can take on 8 numbers. Numbers with 4 consecutive digits: $2 * 8 * 9$.

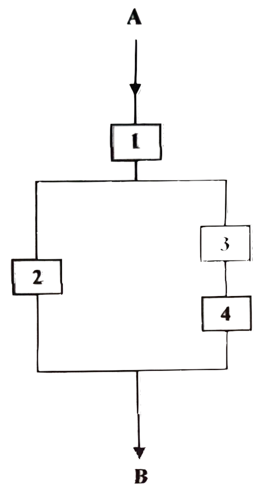
Numbers with 5 consecutive digits $= 9$.

Numbers with 3 consecutive digits $= 2 * 9 * 8 * 9 + 9 * 8 * 8 + 2 * 8 * 9 + 9 = 2025$.

Taking complements, $59049 - 2025 = 57024$.

Question 2

The system contains 4 gates and water flows from A to B. The gates work independently and the probability that a gate is open is p.



i) What is the probability that water is able to flow from A to B?

For water to flow, 1 must be open. Either 2 must be open or both 3 & 4 must be open. However, $p + p^2$ double counts the event where gates 1,2,3 are open. We have to exclude that event.

$$P(A \text{ flows to } B) = p * [p + p^2 - p^3] = p^2(p + 1 - p^2)$$

ii) What is the conditional probability that gate 2 is open given that water is able to flow from A to B?

$P(\text{gate 2 open} \cap \text{water flows from } A \text{ to } B) = p^2$. There is a p chance of gate 1 being open, and a p chance of gate 2 being open. Whether gate 3 or 4 is open or closed is of no concern. $P(\text{gate 2 open} \mid \text{water flows from } A \text{ to } B) = p^2/p^2(p + 1 - p^2) = 1/(p + 1 - p^2)$.

Question 3

The probability density funtion of X is

$$f(x) = \begin{cases} \frac{x}{100} + \frac{1}{10} & -5 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

i) Put $Y = X^2$. Find the PDF of X^2. State clearly the range of Y for which there is positive density.

Let's find the CDF of Y first.

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(X^2 \leq y) \\
&= P(-\sqrt{y} \leq x \leq \sqrt{y}) \\
&= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{x}{100} + \frac{1}{10} dx \\
&= \frac{x^2}{200} + \frac{x}{10} \bigg|_{-\sqrt{y}}^{\sqrt{y}} \\
&= \frac{\sqrt{y}}{5}
\end{aligned}$$

Since the PDF is defined to the derivative of the CDF, $f_Y(y) = F'_Y(y) = \frac{d}{dy}\frac{\sqrt{y}}{5} = \frac{1}{10\sqrt{y}}$, and it has a positive density on $0 < y < 25$.

ii) Let Z be the largest integer less than or equal to X. Find the PMF of Z. State clearly all the values that can be assumed by Z with positive probabilities.

$Z \in \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4\}$

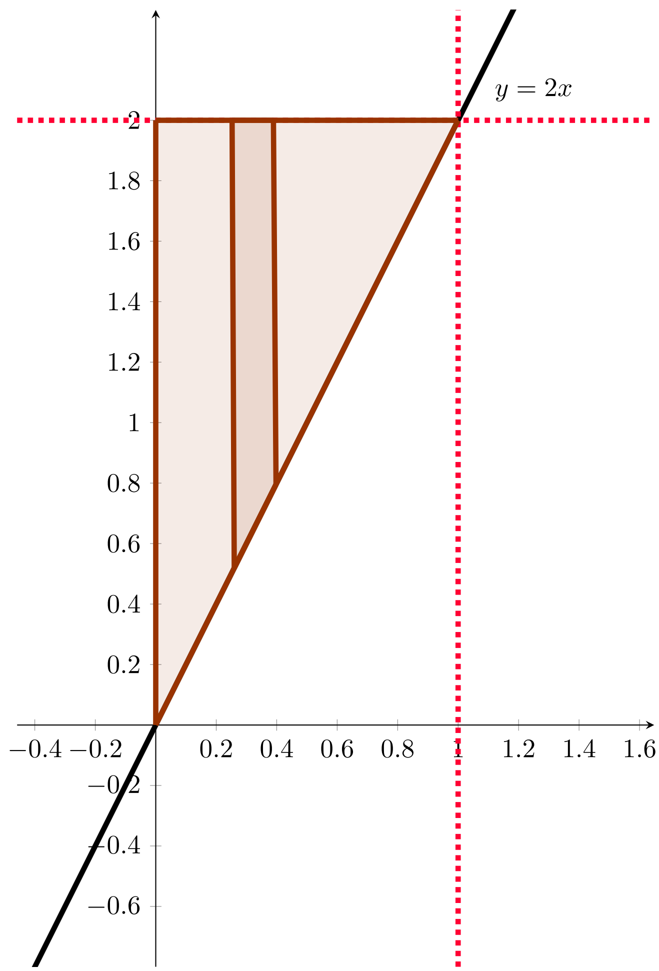
$$\begin{aligned}
P(Z = z) &= P(z < X < z + 1) \\
&= \int_{z}^{z+1} \frac{x}{100} + \frac{1}{10} dx \\
&= \frac{x^2}{200} + \frac{x}{10} \bigg|_{z}^{z+1} \\
&= \frac{2z + 21}{200}.
\end{aligned}$$

Question 4

The joint PDF for X and Y is

$$f(x, y) = \begin{cases} 3x & \text{if } x > 0, y < 2, y > 2x \\ 0 & \text{otherwise} \end{cases}$$

i) Find the PDF of X. The range for which the PDF is positive must be clearly specified.



The shaded region is the one we want. Integrating along the y-axis, our integral runs from 2 to $2x$, as seen in the vertical bar in the image above. So,

$$\begin{aligned} f_X(x) &= \int_{2x}^{2} f_{X,Y}(x,y)dy \\ &= \int_{2x}^{2} 3x\ dy \\ &= 6x - 6x^2 \end{aligned}$$

which has a positive density on $0 < x < 1$.

ii) Find the conditional PDF $f_{X|Y}(x|y)$. Are X and Y independent?

$$\begin{aligned} f_Y(y) &= \int_0^{\frac{y}{2}} f_{X,Y}(x,y)dx \\ &= \int_0^{\frac{y}{2}} 3x\ dx \\ &= \frac{3y^2}{8} \end{aligned}$$

which has a positive density on $0 < y < 2$.

$$\begin{aligned} f_{X|Y=y}(x) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{3x}{\frac{3y^2}{8}} \\ &= \frac{8x}{y^2}. \end{aligned}$$

X and Y are not independent since $f_Y(y) \times f_X(x) \neq f_{X,Y}(x,y)$.

iii) Find $\mathrm{Cov}(X,Y)$ and explain why it is positive.

$$E(XY) = \int_0^2 E(Xy \mid Y=y) f_Y(y)dy = \int_0^2 yE(X \mid Y=y)f_Y(y)dy \tag{1}$$

$$\begin{aligned} E(X \mid Y=y) &= \int_0^{\frac{y}{2}} x f_{X|Y=y}(x)dx \\ &= \int_0^{\frac{y}{2}} x\frac{8x}{y^2}dx \\ &= \left.\frac{8x^3}{3y^2}\right|_0^{\frac{y}{2}} \\ &= \frac{y}{3} \end{aligned}$$

Subbing back into (1),

$$\begin{aligned} E(XY) &= \int_0^2 y\frac{y}{3}\frac{3y^2}{8}dx \\ &= \int_0^2 \frac{y^4}{8}dy \\ &= \frac{4}{5}. \end{aligned}$$

Further, $E(X) = \int_0^1 x(6x - 6x^2)dx = \frac{1}{2}$, and $E(Y) = \int_0^2 y\frac{3y^2}{8}dy = \frac{3}{2}$.

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{4}{5} - \frac{1}{2} \times \frac{3}{2} = \frac{1}{20}.$$

$\mathrm{Cov}(X,Y)$ is positive because when the random variable takes on a larger number, the probability that Y would take on a larger number increases as well, since one must have $y > 2x$.

Question 5

Two points are selected randomly on a line of length 1. Put X as the smaller of the two points and Y as the larger of the two points.

i) Find the joint probability density function of X and Y.

   Let U and V be random variables denoting which values the first and second points land on respectively. From independence,

   $$f_{U,V}(u,v) = 1 \text{ for } 0 \leq u, v \leq 1.$$

   Put $Y = \max(U,V)$, $X = \min(U,V)$. Fix some $h > 0$. Note that:

   $$P(x < X < x+h, y < Y < y+h) = P(x < U < x+h, y < V < y+h) + P(y < U < y+h, x < V < x+h)$$

   This is because for X to take on a range of values, say in $(x, x+h)$, and Y to be in $(y, y+h)$, either U must be in $(x, x+h)$ and V must be in $(y, y+h)$, or V must be in $(x, x+h)$ and U must be in $(y, y+h)$. Taking $\lim_{h \to 0}$,

   $$f_{X,Y}(x,y) = P(x < X < x+h, y < Y < y+h) = f_{U,V}(x,y) + f_{U,V}(y,x) = 2.$$

   So $f_{X,Y}(x,y) = 2$ for $0 \leq x \leq y \leq 1$.

ii) What is the expected length of $E(Y - X)$ [1]?

   $$f_Y(y) = \int_0^y f_{X,Y}(x,y)dx = \int_0^y 2dx = 2y.$$

   $$f_X(y) = \int_x^1 f_{X,Y}(x,y)dx = \int_0^y 2dx = 2 - 2x.$$

   $$E(Y) = \int_0^1 2y^2 dy = \frac{2}{3}$$

   $$E(X) = \int_0^1 x(2-2x)dx = \frac{1}{3}$$

   Now, from linearity of expectation,

   $$E(Y - X) = E(Y) - E(X) = \frac{1}{3}.$$

[1] This is a famous problem. Generalisations include the expected distance between two points chosen randomly on the circumference of a circle and the expected distance between two random points in a square and circle.

Question 6

Jar A contains 2 tags numbered 1 and 2 respectively. Jar B also contains 2 tags numbered 1 and 2 respectively. Jar C contains 3 tags numbered 1,2 and 3 respectively. We select one tag at random from A. Record the number as X_1. We then select randomly X_1tags from B with replacement. Let X_2 be the total number on the tags drawn from B. We finally select X_2 tags randomly from C with replacement. Let X_3 be the total number on the tags drawn from C. Find $E(X_3)$.

By the LOTP, we may break $E(X_3)$ into

$$\begin{aligned} E(X_3) &= E(X_3 \mid X_2 = 1)P(X_2 = 1) + E(X_3 \mid X_2 = 2)P(X_2 = 2) \\ &\quad + E(X_3 \mid X_2 = 3)P(X_2 = 3) + E(X_3 \mid X_2 = 4)P(X_2 = 4) \end{aligned}$$

$$\begin{aligned} P(X_2 = 1) &\iff \text{you get 1 draw and you draw only 1 from A} \\ &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \\ P(X_2 = 2) &\iff \text{you get 1 draw and you draw 2, or you get 2 draws and draw both 1's} \\ &= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4} \\ &= \frac{3}{8} \\ P(X_2 = 3) &\iff \text{you get 2 draws and you draw 2 and 1, or 1 then 2} \\ &= \frac{1}{2} \times \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4} \\ P(X_2 = 4) &\iff \text{you get 2 draws and you draw both 2's} \\ &= \frac{1}{2} \times \frac{1}{4} = \frac{1}{8} \end{aligned}$$

Let's say you are only allowed to draw once from C. What is the expected value of that draw? It is,

$$E(X_3 \mid X_2 = 1) = \frac{1}{3} \times 1 + \frac{1}{3} \times 2 \times \frac{1}{3} \times 3 = 2$$

However, the draws are **with replacement**, meaning that for each draw, one returns to the urn with the values, meaning the expected value for each draw does not change. So $E(X_3 \mid X_2 = k) = 2k$.

$$E(X_3) = 2 \times \frac{1}{4} + 4 \times \frac{3}{8} + 6 \times \frac{1}{4} + 8 \times \frac{1}{8} = \frac{9}{2}.$$

Question 7

Let X and Y be independent standard normal variables. Calculate the conditional expectation $E(X^3 - Y^3 \mid X - Y = 1)$. (Hint: $X + Y$ and $X - Y$ are independent. You may assume this without proof.

$$\begin{aligned}
E(X^3 - Y^3 \mid X - Y = 1) &= E((X-Y)(X^2 + XY + Y^2 \mid X - Y = 1) \\
&= E(X^2 + XY + Y^2 \mid X - Y = 1)
\end{aligned}$$

Sub $X = 1 + Y$.

$$\begin{aligned}
&= E((1+Y)^2 + (1+Y)Y + Y^2 \mid X - Y = 1) \\
&= E(1 + 3Y + Y^2 \mid X - Y = 1)
\end{aligned}$$

Put $X - Y = U$, hence $U \sim N(0, 2)$. So $f_U(u) = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{x^2}{4}} = \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}}$.

$$\begin{aligned}
f_{Y|U=1}(y) &= \frac{f_{Y,U}(y, 1)}{f_U(1)} \\
&= \frac{f_{X,Y}(1+y, y)}{f_U(1)}
\end{aligned}$$

From independence,

$$\begin{aligned}
&= \frac{f_X(1+y) \times f_Y(y)}{f_U(1)} \\
&= \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y+1)^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y)^2}{2}}}{\frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}}} \\
&= \frac{1}{\sqrt{\pi}} \exp\left\{ -\left(\frac{(y+1)^2}{2} + \frac{y^2}{2} - \frac{1}{4} \right) \right\} \\
&= \frac{1}{\sqrt{\pi}} \exp\left\{ -\left(y^2 + y + \frac{1}{4} \right) \right\} \\
&= \frac{1}{\sqrt{\pi}} \exp\left\{ -\left(y + \frac{1}{2} \right)^2 \right\}
\end{aligned}$$

This means that $f_{Y|U=1} \sim N(-\frac{1}{2}, \frac{1}{2})$ [2]. The moment generating function of a normal distribution of mean μ and variance σ^2 is:

$$M(t) = \exp\left\{ t\mu + \frac{1}{2}\sigma^2 t^2 \right\}$$ [3]

We get that,

$$\begin{aligned}
E(X) &= M'(0) = \mu = -\frac{1}{2} \\
E(X^2) &= M''(0) = \sigma^2 + \mu^2 = \left(-\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{3}{4}.
\end{aligned}$$

[2] It is known that the distribution is uniquely specified by the CDF, PDF or MGF. The proof, however, is not easy.

[3] This is also why odd moments of a **standard normal distribution** is 0.

Lastly,

$$E(1+3Y+3Y^2 \mid X-Y=1) = 1+3\left(-\frac{1}{2}\right)+3\left(\frac{3}{4}\right) = \frac{7}{4}$$

.

Another way

Note that we may write $E(X^3 - Y^3 \mid X - Y = 1)$ as

$$\begin{aligned} E((X^2+XY+Y^2 \mid X-Y=1) &= E(X+Y)^2 - XY \mid X-Y=1) \\ &= E\left((X+Y)^2 - \left(\frac{1}{4}(X+Y)^2 - \frac{1}{4}(X-Y)^2\right) \mid X-Y=1\right) \\ &= E\left(\frac{3}{4}(X+Y)^2 + \frac{1}{4}(X-Y)^2 \mid X-Y=1\right) \end{aligned}$$

We know $X+Y \sim N(0,2)$. You can use MGF's to deduce that $E((X+Y)^2) = \mu^2 + \sigma^2 = 2$, or you can use $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$.

$$\begin{aligned} E\left(\frac{3}{4}(X+Y)^2 + \frac{1}{4}(X-Y)^2 \mid X-Y=1\right) &= \frac{3}{4}(2) + \frac{1}{4}(1) \\ &= \frac{7}{4}. \end{aligned}$$