NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Teo Wei Hao

MA2101 Linear Algebra II
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Question 1

(a) True.
Let $X = AB$ and $Y = BA$. Then we have,

$$\begin{aligned}\operatorname{tr}(X) = \sum_{i=1}^{n} x_{ii} &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}b_{ji}\right)\\ &= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} b_{ji}a_{ij}\right)\\ &= \sum_{j=1}^{n} y_{jj} = \operatorname{tr}(Y).\end{aligned}$$

(b) True.
Let $X = BC$. From (1a.), $\operatorname{tr}(ABC) = \operatorname{tr}(AX) = \operatorname{tr}(XA) = \operatorname{tr}(BCA)$.

(c) False.
Let $A, B, C \in M_2(\mathbb{R})$ such that $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then we have

$$\operatorname{tr}(ABC) = \operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1, \qquad \operatorname{tr}(ACB) = \operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

(d) True.
Using result of (1a.), we have $\operatorname{tr}[(A)(BAB)] = \operatorname{tr}[(BAB)(A)]$, and $\operatorname{tr}[(BA)(AB)] = \operatorname{tr}[(AB)(BA)]$.
Thus

$$\begin{aligned}\operatorname{tr}[(AB - BA)(AB + BA)] &= \operatorname{tr}(ABAB - BAAB + ABBA - BABA)\\ &= \operatorname{tr}(ABAB) - \operatorname{tr}(BAAB) + \operatorname{tr}(ABBA) - \operatorname{tr}(BABA)\\ &= 0_F.\end{aligned}$$

(e) True.
Since A and B are similar, there exists $P \in M_{n\times n}(F)$ such that $A = PBP^{-1}$.
Using result of (1b.), we have $\operatorname{tr}(A) = \operatorname{tr}(PBP^{-1}) = \operatorname{tr}(BP^{-1}P) = \operatorname{tr}(B)$.

(f) the vector space structure

(g) distinction/uniqueness

(h) A^T
Notice that for all $\phi_U \in U^*$ of a vector space U, we have $[\phi_U]_{\mathcal{B}_{U^*}} = ([\phi_U]_{F,\mathcal{B}_U})^T$.

Let us be given $T \in L(V, W)$ such that $[T]_{\mathcal{B}_W,\mathcal{B}_V} = A$.
The dual map is defined to be a linear transformation $S : W^* \to V^*$, such that $S(\phi) = \phi \circ T$.
This give us for all $\phi \in W^*$, we have $[S(\phi)]_{\mathcal{B}_{V^*}} = [\phi \circ T]_{\mathcal{B}_{V^*}}$.
Thus $[S]_{\mathcal{B}_{V^*},\mathcal{B}_{W^*}}[\phi]_{\mathcal{B}_{W^*}} = ([\phi \circ T]_{F,\mathcal{B}_V})^T = ([\phi]_{F,\mathcal{B}_W}[T]_{\mathcal{B}_W,\mathcal{B}_V})^T = A^T([\phi]_{F,\mathcal{B}_W})^T = A^T[\phi]_{\mathcal{B}_{W^*}}$.
Therefore $[S]_{\mathcal{B}_{V^*},\mathcal{B}_{W^*}} = A^T$.

(i) inner product

(j) determinant

Question 2

(a) False.
Let $A, B \in M_2(\mathbb{R})$ such that $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then we have $AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}_2$. For all invertible $P \in M_2(\mathbb{R})$, we have $P(BA)P^{-1} = P\mathbf{0}_2P^{-1} = \mathbf{0}_2 \neq AB$.
Therefore AB is not similar to BA.

(b) True.
If A is invertible, then A^{-1} exists. This give us $AB = ABI = AB(AA^{-1}) = A(BA)A^{-1}$.
Therefore AB is similar to BA.

Question 3

(a) Since $f(0)g(0) = f(1)g(1)$, we have

$$\begin{aligned}
\langle D(f), g\rangle + \langle f, D(g)\rangle &= \int_0^1 f'(t)g(t)\ dt + \int_0^1 f(t)g'(t)\ dt \\
&= \int_0^1 \frac{d}{dt}(f(t)g(t))\ dt \\
&= [f(t)g(t)]_0^1 \\
&= f(1)g(1) - f(0)g(0) = 0.
\end{aligned}$$

Thus $\langle D(f), g\rangle = -\langle f, D(g)\rangle$.

(b) We have

$$\begin{aligned}
\langle f_1, f_1\rangle &= \int_0^1 1\ dt = 1, \\
\langle f_1, f_2\rangle &= \langle f_2, f_1\rangle \\
&= \int_0^1 t\ dt = \frac{1}{2}, \\
\langle f_2, f_2\rangle &= \int_0^1 t^2\ dt = \frac{1}{3}.
\end{aligned}$$

Thus $\begin{bmatrix} \langle f_1, f_1\rangle & \langle f_1, f_2\rangle \\ \langle f_2, f_1\rangle & \langle f_2, f_2\rangle \end{bmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$.

(c) By Gram-Schmidt process and results of (3b.), we create an orthogonal basis $\{u_1, u_2\}$ of V with,

$$\begin{aligned} u_1 &= f_1, \\ u_2 &= f_2 - \left(\frac{\langle u_1, f_2\rangle}{\langle u_1, u_1\rangle}\right) u_1 \\ &= f_2 - \left(\frac{\langle f_1, f_2\rangle}{\langle f_1, f_1\rangle}\right) f_1 \\ &= f_2 - \frac{1}{2} f_1. \end{aligned}$$

Then, we normalise it to get,

$$\begin{aligned} v_1 &= \left(\frac{1}{\sqrt{\langle u_1, u_1\rangle}}\right) u_1 \\ &= f_1, \\ v_2 &= \left(\frac{1}{\sqrt{\langle u_2, u_2\rangle}}\right) u_2 \\ &= \left(\frac{1}{\sqrt{\langle f_2 - \frac{1}{2}f_1, f_2 - \frac{1}{2}f_1\rangle}}\right)\left(f_2 - \frac{1}{2}f_1\right) \\ &= \left(\frac{1}{\sqrt{\langle f_2, f_2\rangle - \langle f_1, f_2\rangle + \frac{1}{4}\langle f_1, f_1\rangle}}\right)\left(f_2 - \frac{1}{2}f_1\right) \\ &= 2\sqrt{3}f_2 - \sqrt{3}f_1. \end{aligned}$$

Thus an orthonormal basis of V is $\{f_1, 2\sqrt{3}f_2 - \sqrt{3}f_1\}$.

Question 4

(a) Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\mathcal{B}_V = \{e_1, e_2, e_3, e_4\}$. Then $\mathcal{B}_V$ is a basis of V. By calculation, we get

$$L(e_1) = e_4, \qquad L(e_2) = e_3, \qquad L(e_3) = e_2, \qquad L(e_4) = e_1,$$

and thus $[L]_{\mathcal{B}_V} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Thus the characteristic polynomial of L,

$$\begin{aligned} \chi_L(x) &= \det(xI_4 - [L]_{\mathcal{B}_V}) \\ &= \begin{vmatrix} x & 0 & 0 & -1 \\ 0 & x & -1 & 0 \\ 0 & -1 & x & 0 \\ -1 & 0 & 0 & x \end{vmatrix} \\ &= x^4 - 2x^2 + 1. \qquad \text{(by cofactor expansion)} \end{aligned}$$

(b) By equating $\chi_L(\lambda) = 0$, we get $(\lambda-1)^2(\lambda+1)^2 = 0$. Thus the eigenvalues of L are ± 1.

(c) Let $v \in E_{-1}$ with $[v]_{\mathcal{B}_V} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix}^T$. Then we have $(-I_4 - [L]_{\mathcal{B}_V})[v]_{\mathcal{B}_V} = 0_{F^4}$. Thus,

$$\begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving this by Gaussian Elimination, we get $[v]_{\mathcal{B}_V} = s \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, $s, t \in F$.

Thus $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ is a basis of E_{-1}.

Next, instead let $v \in E_1$ with $[v]_{\mathcal{B}_V} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix}^T$. Then we have $(I_4 - [L]_{\mathcal{B}_V})[v]_{\mathcal{B}_V} = 0_{F^4}$. Thus,

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving this by Gaussian Elimination, we get $[v]_{\mathcal{B}_V} = s \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $s, t \in F$.

Thus $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ is a basis of E_1.

Question 5

Assume on the contrary that A is non-singular.
Then A^{-1} exists, which give us $ABA^{-1} - B = (AB - BA)A^{-1} = AA^{-1} = I_n$.
Now using result of traces established in (1b.), we have

$$\begin{aligned} \operatorname{tr}(ABA^{-1} - B) &= \operatorname{tr}(ABA^{-1}) - \operatorname{tr}(B) \\ &= \operatorname{tr}(BA^{-1}A) - \operatorname{tr}(B) \\ &= \operatorname{tr}(B) - \operatorname{tr}(B) \\ &= 0. \end{aligned}$$

However we have $\operatorname{tr}(I_n) = n$, a contradiction.

Therefore A is singular.