NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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ST2131/MA2216 Probability
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Question 1

(a) We let X denote the position of the ace of spades in the deck and Y denote the position of the king of spades in the deck.

$$\begin{aligned}\mathbb{P}\{X=1\cap Y=2\} &= \mathbb{P}\{Y=2|X=1\}\times\mathbb{P}\{X=1\}\\ &= \frac{1}{52}\times\frac{1}{51}\\ &= \frac{1}{2652}\end{aligned}$$

(b) By symmetry, the probability of ace of spades being the 10th card is the same as the probability of it being the 1st card. This applies for the king of spades as well.
Therefore,

$$\begin{aligned}\mathbb{P}\{X=10\cap Y=20\} &= \mathbb{P}\{X=1\cap Y=2\}\\ &= \frac{1}{2652}\end{aligned}$$

(c) As long as the last card is a spade, it satisfies the condition that eventually only spades remain in the deck. By symmetry,

$$\mathbb{P}\{\text{Last card is a spade}\}=\mathbb{P}\{\text{First card is a spade}\}=\frac{1}{4}$$

(d) Note that if $\mathbb{P}\{Y=1|X=1\}=\frac{\mathbb{P}\{Y=1\cap X=1\}}{\mathbb{P}\{X=1\}}=\frac{\mathbb{P}\{Y=1\}\mathbb{P}\{X=1\}}{\mathbb{P}\{X=1\}}=\mathbb{P}\{Y=1\}$, then X and Y are independent. However,

$$\mathbb{P}\{Y=1|X=1\}=\frac{13}{51}\neq\frac{13}{52}=\mathbb{P}\{Y=1\}$$

Hence, X and Y are not independent.

(e)

$$\begin{aligned}E(XY) &= 0\times0\times\mathbb{P}\{X=0\cap Y=0\}+0\times1\times\mathbb{P}\{X=0\cap Y=1\}\\ &+ 1\times0\times\mathbb{P}\{X=1\cap Y=0\}+1\times1\times\mathbb{P}\{X=1\cap Y=1\}\\ &= \mathbb{P}\{X=1\cap Y=1\}\\ &= \mathbb{P}\{X=1|Y=1\}\times\mathbb{P}\{Y=1\}\\ &= \frac{13}{51}\times\frac{1}{4}=\frac{13}{204}\end{aligned}$$

(f)

$$\begin{aligned}
E(X+Y) &= E(X)+E(Y) \\
&= 1\cdot \mathbb{P}\{X=1\}+1\cdot \mathbb{P}\{Y=1\} \\
&= \frac{13}{52}+\frac{13}{52} \\
&= \frac{1}{2} \\
Var(X+Y) &= E(X+Y)^2-[E(X+Y)]^2 \\
&= E(X^2-2XY+Y^2)-(\frac{1}{2})^2 \\
&= E(X^2)-2E(XY)+E(Y^2)-\frac{1}{4} \\
&= E(X)-2(\frac{13}{204})+E(Y)-\frac{1}{4} \\
&= 0.123
\end{aligned}$$

Question 2

(a) John's father carries a Bb gene pair, because his offsprings have eyes of different colour.

(b) First of all, we know that the offsprings have eyes of different colour, so John's mother cannot have a BB gene pair, so she must have either a Bb or bb gene pair, with

$$\begin{aligned}
\mathbb{P}\{\text{John's mother has blue eyes}\} &= \mathbb{P}\{\text{having a bb gene pair} \mid \text{not having a BB gene pair}\} \\
&= \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}} \\
&= \frac{1}{3}.
\end{aligned}$$

Let M, F, J, H be the colour of the mother's, father's, John's and Hannah's eyes respectively. Also, let 1 = brown and 0 = blue.

$$\begin{aligned}
\mathbb{P}\{J=1\cap H=0\cap M=0\cap F=1\} &= \mathbb{P}\{J=1\cap H=0|M=0\cap F=1\}\times\mathbb{P}\{M=0\cap F=1\} \\
&= \left(\frac{1}{2}\times\frac{1}{2}\right)\times\frac{1}{3} \\
&= \frac{1}{12} \\
\mathbb{P}\{J=1\cap H=0\cap M=1\cap F=1\} &= \mathbb{P}\{J=1\cap H=0|M=1\cap F=1\}\times\mathbb{P}\{M=1\cap F=1\} \\
&= \left(\frac{3}{4}\times\frac{1}{4}\right)\times\left(1-\frac{1}{3}\right) \\
&= \frac{1}{8}
\end{aligned}$$

Since $\mathbb{P}\{J=1\cap H=0\cap F=1\}=\mathbb{P}\{J=1\cap H=0\cap M=0\cap F=1\}+\mathbb{P}\{J=1\cap H=0\cap M=1\cap F=1\}$,

$$\begin{aligned}
\mathbb{P}\{M=0|J=1\cap H=0\cap F=1\} &= \frac{\mathbb{P}\{M=0\cap J=1\cap H=0\cap F=1\}}{\mathbb{P}\{J=1\cap H=0\cap F=1\}} \\
&= \frac{\frac{1}{12}}{\frac{1}{12}+\frac{1}{8}} \\
&= \frac{2}{5}
\end{aligned}$$

(c) Let X be the number of babies born on 30th April.

$$X \sim Poi(100) \Rightarrow E(X) = 100$$

Since BB:Bb:bb $= 1:2:1$, expected number of blue-eyed babies $= 100 \times \frac{1}{4} = 25$.

Question 3

(a)

$$\begin{aligned}
\text{Number of unordered pairs of teams} &= \frac{8!}{(2!)^4 \times 4!} = 105 \\
\mathbb{P}\{Y=2\} &= \frac{(\frac{4!}{(2!)^2\times 2!})^2}{105} = \frac{3}{35} \\
\text{Also, } \mathbb{P}\{Y=0\} &= \frac{4!}{105} = \frac{8}{35} \\
\text{Hence, } \mathbb{P}\{Y=1\} &= 1-\mathbb{P}\{Y=0\}-\mathbb{P}\{Y=2\} = \frac{24}{35}
\end{aligned}$$

Question 4

(a) Let T be the lifetime of a lightbulb. Let D denote a defective lightbulb and N denote a normal lightbulb. Hence, $D \sim Exp(\frac{1}{500})$ and $N \sim Exp(\frac{1}{5000})$.

$$\begin{aligned}
\mathbb{P}\{D|T<1000\} &= \frac{\mathbb{P}\{D\cap T<1000\}}{\mathbb{P}\{T<1000\}} \\
&= \frac{\mathbb{P}\{D\cap T<1000\}}{\mathbb{P}\{D\cap T<1000\}+\mathbb{P}\{N\cap T<1000\}} \\
&= \frac{\frac{1}{10}(1-e^{-\frac{1000}{500}})}{\frac{1}{10}(1-e^{-\frac{1000}{500}})+\frac{9}{10}(1-e^{-\frac{1000}{5000}})} \\
&= 0.3464
\end{aligned}$$

(b) We observe from the moment generating functions that $X \sim Poi(5)$ and $Y \sim Poi(3)$. Hence, $X + Y \sim Poi(8) \Rightarrow M_{X+Y}(t) = e^{8(e^t-1)}$.

(c) Hence, from (b), $Var(X + Y) = 8$

Question 5

(a) Let I_X be the indicator function that a person alights on floor X. Then,

$$E(I_k) = \frac{1}{10} \text{ for some } 2 \le k \le 11$$

Consider floor i,

$$\mathbb{P}\{\text{A person won't alight on a floor}\} = \frac{9}{10}$$

Hence,

$$\mathbb{P}\{\text{All 10 people won't alight on a floor}\} = \left(\frac{9}{10}\right)^{10}$$

Therefore, considering 10 floors,

$$E(\text{Number of floors}) = 10 - 10\left(\frac{9}{10}\right)^{10} = 6.51$$

(b) Let X_i be the number of babies born in a city hospital in a day. i.e. $X_i \sim Poi(2)$
In 6 weeks, there are 42 days, so we denote $X = X_1 + ... + X_{42}$ as the number of babies born in these 6 weeks. i.e. $X \sim Poi(84)$
Because λ is large, we can approximate it to a normal distribution $\Rightarrow X \sim N(84, 84)$
Hence,

$$\begin{aligned}
\mathbb{P}\{X \ge 100\} &= 1 - \mathbb{P}\{X < 100\} \\
&= 1 - \Phi\left(\frac{99.5 - 84}{\sqrt{84}}\right) \\
&= 1 - \Phi(1.69) \\
&= 0.0454
\end{aligned}$$

Question 6

(a) Marginal distribution of Y:

$$\begin{aligned}
f_Y(y) &= \int_0^y \frac{e^{-y}}{y} dx = e^{-y} \\
f_{X|Y}(x|y) = \frac{f_{X,Y}(x \cap y)}{f_Y(y)} &= \frac{\frac{e^{-y}}{y}}{e^{-y}} \\
&= \frac{1}{y}
\end{aligned}$$

(b)

$$\begin{aligned} E(X^2) &= E(E(X^2|Y)) = E\left(\int_0^y x^2\left(\frac{1}{y}\right)dx\right) \\ &= E\left(\frac{Y^2}{3}\right) = \frac{1}{3}\int_0^\infty y^2 e^{-y} dy \\ &= \frac{2}{3} \end{aligned}$$

Question 7

(a) In this question, $z = \frac{x}{y}$. Since $x \in (0, \infty)$ and $y \in (0, \infty)$, $z \in (0, \infty)$.
Next, we obtain $\frac{\delta z}{\delta x} = \frac{1}{y}$, $\frac{\delta z}{\delta y} = -\frac{x}{y^2}$, $\frac{\delta y}{\delta x} = 0$ and $\frac{\delta y}{\delta y} = 1$.
Hence, $J = (\frac{1}{y})(1) - (-\frac{x}{y^2})(0) = \frac{1}{y} \Rightarrow |J|^{-1} = y$.
Therefore,

$$\begin{aligned} f_{(Z,Y)}(z, y) &= |J|^{-1} f_{(X,Y)}(x, y) \\ &= (y)\left(\frac{e^{-\frac{x}{y}} e^{-y}}{y}\right) \\ &= (y)\left(\frac{e^{-\frac{zy}{y}} e^{-y}}{y}\right) \\ &= e^{-z-y}, \qquad 0 < y < \infty, 0 < z < \infty. \end{aligned}$$

(b) Marginal distribution of Z :

$$\begin{aligned} f_Z(z) &= \int_0^\infty e^{-z-y} dy \\ &= e^{-z}\left[-e^{-y}\right]_0^\infty \\ &= e^{-z} \end{aligned}$$

Question 8

(a) By symmetry, $P[X_1 < X_2 < X_3] = P[X_1 < X_3 < X_2] = P[X_2 < X_1 < X_3] = P[X_2 < X_3 < X_1] = P[X_3 < X_1 < X_2] = P[X_3 < X_2 < X_1] = \frac{1}{6}$.
Hence, $P[X_1 < X_2 < X_3] = \frac{1}{6}$.

(b) Since the k-th order statistic of i.i.d U$(0, 1)$ X_1, ... ,X_n follows Beta$(k, n-k+1)$ density, we have

$$X_{(1)} \sim Beta(1, 3)$$
$$X_{(2)} \sim Beta(2, 2)$$

Then,

$$\begin{aligned} E(X_{(2)} - X_{(1)}) &= \frac{2}{4} - \frac{1}{4} \\ &= \frac{1}{4} \end{aligned}$$