NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

**MA1102R Calculus**
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**Question 1**

$$
\begin{aligned}
f(x) &= (x^2-2)e^{x^2} \\
f'(x) &= 2xe^{x^2}(x^2-2)(2xe^{x^2}) \\
&= 2xe^{x^2}(x^2-1) \\
&= 2xe^{x^2}(x-1)(x+1) \\
f''(x) &= 2e^{x^2} + 4x^2e^{x^2} + 4x^2e^{x^2} + 2(x^2-2)e^{x^2} + (x^2-2)2xe^{x^2} \\
&= 2e^{x^2}(2x^4+x^2-1) = 2e^{x^2}(x^2+1)(2x^2-1) \\
&= 2e^{x^2}(x^2+1)(\sqrt{2}x-1)(\sqrt{2}x+1)
\end{aligned}
$$

(i)

| $x$ | $(-\infty,-1)$ | $(-1,0)$ | $(0,1)$ | $(1,\infty)$ |
|---|---|---|---|---|
| $f'(x)$ | $-$ | $+$ | $-$ | $+$ |

By increasing, decreasing test we see that $f$ is increasing in the open intervals of $(-1,0)$ and $(1,\infty)$. Additionally, f is decreasing on $f$ $(-\infty,-1)$ and $(0,1)$.

(ii) From (i) we see that $f$ has a local maximum at $(0,-2)$ and local minimums at $(-1,-e)$ and $(1,-e)$

(iii)

| $x$ | $(-\infty,-\frac{1}{\sqrt{2}})$ | $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ | $(\frac{1}{\sqrt{2}},\infty)$ |
|---|---|---|---|
| $f'(x)$ | $+$ | $-$ | $+$ |

As $f''>0 \implies f$ concaves up, $f$ concaves up at the open intervals $(-\infty,-\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}},\infty)$. Smilarly, $f$ concaves down at the open interval $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$

(iv) From (iii), the points of inflection are at $(-\frac{1}{\sqrt{2}},\frac{3}{2}\sqrt{e})$ and $(\frac{1}{\sqrt{2}},\frac{3}{2}\sqrt{e})$

**Question 2**

(a)

$$\begin{aligned}\lim_{x \implies \infty} \frac{\sqrt{2x^2+5}}{\sqrt[3]{x^3-1}} &= \lim_{x \implies \infty} \frac{\frac{1}{x}\sqrt{2x^2+5}}{\frac{1}{x}\sqrt[3]{x^3-1}} \\ &= \lim_{x \implies \infty} \frac{\sqrt{2+\frac{5}{x^2}}}{\sqrt[3]{1-\frac{1}{x^3}}} \\ &= \frac{\sqrt{2+0}}{\sqrt[3]{1-0}} = \sqrt{2}\end{aligned}$$

(b) For any $\epsilon > 0$, choose $\delta = \sqrt[4]{\epsilon}$ Now, given that $0 < |x| < \delta$

$$\begin{aligned}\left|\sqrt{x^4+1}-1\right| &= \frac{|x^4+1-1|}{\left|\sqrt{x^4+1}+1\right|} \\ &= \frac{|x^4|}{\left|\sqrt{x^4+1}+1\right|} \\ &< \frac{|x|^4}{2} \qquad (\text{as } x^4 > 0 \implies \sqrt{x^4+1}+1 > 2 \implies \tfrac{1}{|\sqrt{x^4+1}+1|} < \tfrac{1}{2}) \\ &< \delta^4 = \epsilon\end{aligned}$$

Therefore $\lim_{x \implies 0} \sqrt{x^4+1} = 1$

**Question 3**

Clearly;

$$x \leq f(x) \leq x^2 + x$$

Hence as $0 \leq f(0) \leq 0^2 + 0 = 0$ by Squeeze Theorem, we have $f(0) = 0$.
Moreover; we have:

$$1 \leq \frac{f(x)}{x} \leq x + 1$$

Therefore, as $\lim_{x \implies 0} 1 = 1$ and $\lim_{x \implies 0} x + 1 = 1$; by Squeeze Theorem;

$$\lim_{x \implies 0} \frac{f(x)}{x} = 1$$

Now;

$$\lim_{\delta x \implies 0} \frac{f(0+\delta x) - f(0)}{\delta x} = \lim_{\delta x \implies 0} \frac{f(\delta x)}{\delta x} = 1$$

So $f$ is differentiable at $x = 0$.

**Question 4**

(a) Use the substitution $u = x^3 + 1$. Hence; $du = 3x^2 dx$:

$$\begin{aligned}
\int x^5\sqrt{x^3+1}dx &= \frac{1}{3}\int x^3\sqrt{x^3+1}\ (3x^2)dx \\
&= \frac{1}{3}\int (u-1)\sqrt{u}du \\
&= \frac{1}{3}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}du \\
&= \frac{1}{3}\left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C \\
&= \frac{2}{15}(x^3+1)^{\frac{5}{2}} - \frac{2}{9}(x^3+1)^{\frac{3}{2}} + C
\end{aligned}$$

(b)

$$\begin{aligned}
\int \frac{\ln x}{(1+x)^2}dx &= -\frac{\ln x}{1+x} + \int \frac{1}{x(x+1)}dx && \text{(Using integration by parts)} \\
&= -\frac{\ln x}{1+x} + \int \frac{1}{x} - \frac{1}{x+1}dx && \text{(Using partial fractions)} \\
&= -\frac{\ln x}{1+x} + \ln|x| - \ln|x+1| + C
\end{aligned}$$

**Question 5**

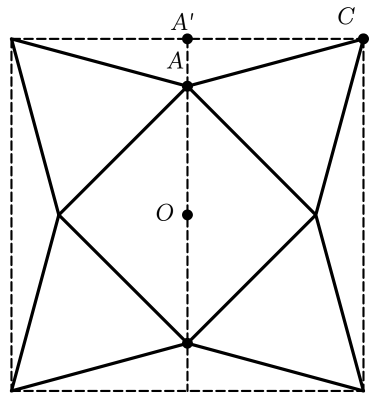


Figure 1: Unfolded Pyramid

(i) From Figure, we can use Pythagoras Theorem,to deduce that;

$$AA' = \frac{50-\sqrt{2}a}{2} = 25 - \frac{a}{\sqrt{2}}$$

$$AC = \sqrt{25^2 + \left(25 - \frac{a}{\sqrt{2}}\right)^2}$$

$$AO = \frac{a}{\sqrt{2}}$$

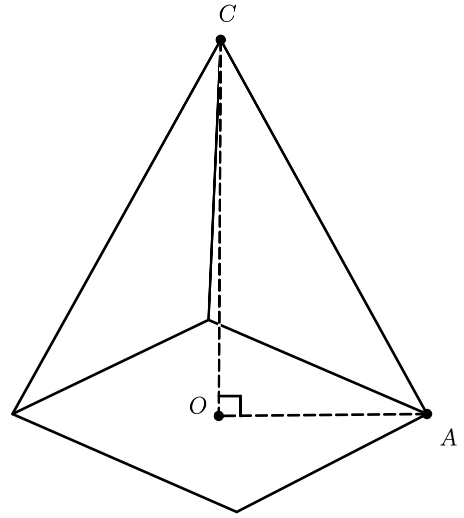


Figure 2: Pyramid

So from Figure 2, by Pythagoras Theorem;

$$\begin{aligned} OC &= \sqrt{AC^2 - AO^2} \\ &= \sqrt{25^2 + \left(25 - \frac{a}{\sqrt{2}}\right)^2 - \frac{a^2}{2}} \\ &= \sqrt{25^2 + 25^2 - \frac{50a}{\sqrt{2}} + \frac{a^2}{2} - \frac{a^2}{2}} \\ &= 5\sqrt{50 - \sqrt{2}a} \end{aligned}$$

Therefore the volume, $V$;

$$V = \frac{1}{3}a^2\left(5\sqrt{50-\sqrt{2}a}\right) = \frac{5}{3}a^2\sqrt{50-\sqrt{2}a} \text{ cm}^2$$

(ii)

$$V = \frac{5}{3}a^2\sqrt{50-\sqrt{2}a} \qquad 0 \le a \le 25\sqrt{2}$$

$$\begin{aligned} \frac{dV}{da} &= \frac{10}{3}a\sqrt{50-\sqrt{2}a} - \frac{5}{3}\frac{\sqrt{2}a^2}{2\sqrt{50-\sqrt{2}a}} \\ &= \frac{5a}{3\sqrt{50-\sqrt{2}a}}\left(100 - \frac{3}{2}\sqrt{2}a\right) \end{aligned}$$

Critical values are $a = 25\sqrt{2}$ when $\frac{dV}{da}$ is undefined and $a = 20\sqrt{2}$, when $\frac{dV}{da} = 0$.
By closed interval method;

$$V(0) = 0$$

$$V(20\sqrt{2}) = \frac{4000\sqrt{10}}{3}$$

$$V(25\sqrt{2}) = 0$$

Therefore, $V$ attains a maximum value of $\frac{4000\sqrt{10}}{3}$ cm$^3$ at $a = 20\sqrt{2}$ cm.

**Question 6**

(i) Washer Method:

$$\begin{aligned}
\text{Volume, } V = \pi \int_0^1 \left(\frac{e^x + e^{-x}}{2}\right)^2 dx &= \frac{\pi}{4}\int_0^1 e^{2x} + 2 + e^{-2x} dx \\
= \frac{\pi}{4}\left[\frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2}\right]_0^1 &= \frac{\pi}{4}\left(\frac{e^2}{2} + 2 - \frac{e^{-2}}{2} - \left(\frac{1}{2} + 0 - \frac{1}{2}\right)\right) \\
= \frac{\pi}{4}\left(\frac{e^2}{2} + 2 - \frac{1}{2e^2}\right)
\end{aligned}$$

(ii) First, using integration by parts, consider:

$$\int xe^x dx = xe^x - e^x \qquad \text{and} \qquad \int xe^{-x} dx = -xe^{-x} - e^{-x}$$

Cylindrical Shell Method:

$$\begin{aligned}
\text{Volume } V &= \int_0^1 2\pi x \left(\frac{e^x + e^{-x}}{2}\right) dx \\
&= \pi \int_0^1 xe^x + xe^{-x} dx \\
&= \pi \left[xe^x - e^x - xe^{-x} - e^{-x}\right]_0^1 \\
&= \pi \left(e - e - e^{-1} - e^{-1} - (0 - 1 - 0 - 1)\right) \\
&= \pi \left(2 - \frac{2}{e}\right)
\end{aligned}$$

(iii) It is clear that $y = \cosh x$ Therefore, $y' = \sinh x$. Hence;

$$\sqrt{1 + (y')^2} = \sqrt{1 + \sinh^2 x} = \cosh x$$

So arc length, $l$;

$$\begin{aligned}
l &= \int_0^1 \sqrt{1 + (y')^2} dx \\
&= \int_0^1 \cosh x dx \\
&= [\sinh x]_0^1 \\
&= \frac{e - e^{-1}}{2}
\end{aligned}$$

**Question 7**

Use the substitution $t = \dfrac{x+1}{x+3}$. Hence $x = \dfrac{3t-1}{1-t}$ and $dt = \dfrac{2}{(x+3)^2}dx$. Therefore;

$$\begin{aligned}
\int_0^1 \frac{1}{(x+3)^2(x+1)^3}dx &= \frac{1}{2}\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{1}{(\frac{3t-1}{1-t}+1)^3}dt \\
&= \frac{1}{2}\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{(1-t)^3}{(3t-1+1-t)^3}dt \\
&= \frac{1}{2}\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{1-3t+3t^2-t^3}{(2t)^3}dt \\
&= \frac{1}{16}\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{1}{t^3} - \frac{3}{t^2} + \frac{3}{t} - 1dt \\
&= \frac{1}{16}\left[-\frac{1}{2t^2} + \frac{3}{t} + 3\ln t - t\right]_{\frac{1}{3}}^{\frac{1}{2}} \\
&= \frac{1}{16}\left(\left[-2+6-3\ln 2-\frac{1}{2}\right] - \left[-\frac{2}{9}+9-3\ln 3-\frac{1}{3}\right]\right) \\
&= \frac{1}{16}\left(3\ln 3 - 3\ln 2 - \frac{2}{3}\right)
\end{aligned}$$

## Question 8

(a) Consider $y = (e+x)^x$

$$\begin{aligned}
& y = (e+x)^x \\
\Longrightarrow\ & \ln y = x\ln(e+x) \\
\Longrightarrow\ & \frac{1}{y}\frac{dy}{dx} = \ln(x+e) + \frac{x}{e+x} \\
\Longrightarrow\ & \frac{dy}{dx} = (e+x)^x\left(\ln(x+e) + 1 - \frac{e}{e+x}\right)
\end{aligned}$$

Applying L' Hospital's Rule twice;

$$\begin{aligned}
\lim_{x \Longrightarrow 0}\frac{(e+x)^x - e^x}{x^2} &= \lim_{x \Longrightarrow 0}\frac{(e+x)^x\left(\ln(x+e)+1-\frac{e}{e+x}\right) - e^x}{2x} \\
&= \lim_{x \Longrightarrow 0}\frac{(e+x)^x\left(\ln(x+e)+1-\frac{e}{e+x}\right)^2 + (e+x)^x\left(\frac{1}{x+e}+\frac{e}{(e+x)^2}\right) - e^x}{2} \\
&= \frac{(e+0)^0\left(\ln(0+e)+1-\frac{e}{e+0}\right)^2 + (e+0)^0\left(\frac{1}{0+e}+\frac{e}{(e+0)^2}\right) - e^0}{2} \\
&= \frac{(1+1-1)^2 + \left(\frac{1}{e}+\frac{1}{e}\right) - 1}{2} \\
&= \frac{1}{e}
\end{aligned}$$

(b) Applying FTC part 1 and chain rule we will get;

$$f'(x) = 2(\tan^{-1}x)\frac{1}{1+x^2}\int_{\sqrt{3}}^{x^2}\frac{\sqrt{t}e^{-t}}{\ln(t^2+t)}dt + (\tan^{-1}x)^2(2x)\frac{xe^{-x^2}}{\ln(x^4+x^2)}$$

Therefore;

$$\begin{aligned} f'(\sqrt{3}) &= 2(\tan^{-1}\sqrt{3})\frac{1}{1+3}\int_{\sqrt{3}}^{3}\frac{\sqrt{t}e^{-t}}{\ln(t^2+t)}dt + (\tan^{-1}\sqrt{3})^2(2\sqrt{3})\frac{\sqrt{3}e^{-3}}{\ln(9+3)} \\ &= \frac{\pi}{6}\int_{\sqrt{3}}^{3}\frac{\sqrt{t}e^{-t}}{\ln(t^2+t)}dt + \frac{2\pi^2e^{-3}}{3\ln 12} \end{aligned}$$

(Note that this is actually the answer; the remaining integral was not evaluated further.)

**Question 9**

(a) The equation is homogeneous, hence we use the substitution $z = \frac{y}{x}$. Hence, $z + x\frac{dz}{dx} = \frac{dy}{dx}$.

$$\begin{aligned} &\frac{y}{x}\cos\frac{y}{x} - \left(\frac{x}{y}\sin\frac{y}{x} + \cos\frac{y}{x}\right)\frac{dy}{dx} = 0 \\ \Longrightarrow\ & z\cos z - \left(\frac{\sin z}{z} + \cos z\right)\left(z + x\frac{dz}{dx}\right) = 0 \\ \Longrightarrow\ & z\cos z = \left(\frac{\sin z}{z} + \cos z\right)\left(z + x\frac{dz}{dx}\right) \\ \Longrightarrow\ & \frac{z\cos z}{\frac{\sin z}{z} + \cos z} = z + x\frac{dz}{dx} \\ \Longrightarrow\ & \frac{z^2\cos z}{\sin z + z\cos z} - z = x\frac{dz}{dx} \\ \Longrightarrow\ & -\frac{z\sin z}{\sin z + z\cos z} = x\frac{dz}{dx} \\ \Longrightarrow\ & \int -\frac{1}{x}dx = \int\left(\frac{1}{z} + \frac{\cos z}{\sin z}\right)dz \\ \Longrightarrow\ & -\ln|x| + C = \ln|z| + \ln|\sin z| \\ \Longrightarrow\ & C = \ln|xz\sin z| \\ \Longrightarrow\ & C_0 = zx\sin z \qquad (C_0 = \pm e^c) \\ \Longrightarrow\ & y\sin\frac{y}{x} = \pm e^c = C_0 \end{aligned}$$

(b)

$$\begin{aligned} &\frac{dS}{dt} = kS \\ \Longrightarrow\ & \int\frac{1}{S}dS = \int kdt \\ \Longrightarrow\ & \ln|S| = kt + C \\ \Longrightarrow\ & S = C_0e^{kt} \qquad (C_0 = \pm e^C) \end{aligned}$$

Given that when $t = 0, S = 50 \implies C_0 = 50$. Also, when$t = 5, S = 20 \implies \frac{\ln\frac{2}{5}}{5}$.
So when there is 10% left, $S = 5$. Therefore;$\frac{\ln\frac{2}{5}}{5}t = \ln\frac{1}{10} \implies t = 12.56... \approx 13$

**Question 10**

Let $m = \dfrac{f(b) - f(a)}{b - a}$ and $g(x) = f(x) - mx - f(b)$
It is clear that $g(a) = g(b) = 0$.
Suppose on the contrary that either $c_1$ or $c_2$ cannot be found, that is: (i)$f'(x) \geq m$ or (ii) $f'(x) \leq m$ for $x \in (a, b)$.
Case (i):
Now clearly $g'(x) = f'(x) - m \geq 0$. So $g$ is nondecreasing. Hence for all $\xi \in (a, b)$, we have $a < \xi < b \implies g(a) \leq g(\xi) \leq g(b)$, and so $g$ is identically 0 in $(a, b)$, a contradiction. Case (ii) is similar. Hence $\exists c_1, c_2 \in (a, b)$ such that

$$f'(c_1) > m \quad \text{and} \quad f'(c_2) < m$$