NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lau Tze Siong

MA2108 Mathematical Analysis 1
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Question 1

(a) (i)

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{n+3\ln n-6n^2}{3n^2-2n+6}\right) &= \lim_{n\to\infty}\left(\frac{n}{3n^2-2n+6}+\frac{3\ln n}{3n^2-2n+6}-\frac{6n^2}{3n^2-2n+6}\right)\\
&= \lim_{n\to\infty}\left(\frac{\frac{1}{n}}{3-\frac{2}{n}+\frac{6}{n^2}}+\frac{\frac{3}{n}\ln n^{\frac{1}{n}}}{3-\frac{2}{n}+\frac{6}{n^2}}-\frac{6}{3-\frac{2}{n}+\frac{6}{n^2}}\right)\\
&= = 0+0-2\\
&= -2
\end{aligned}$$

(ii)

$$\lim_{n\to\infty}\left(\frac{n\sin(2n+1)}{n^2+1}\right) = \lim_{n\to\infty}\left(\frac{n}{n^2+1}\sin(2n+1)\right)$$

Since $-1\le\sin(2n+1)\le 1$ for all $n\in\mathbb{N}$, we have $\frac{-n}{n^2+1}\le\frac{n}{n^2+1}\sin(2n+1)\le\frac{n}{n^2+1}$. By Squeeze Theorem, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{-n}{n^2+1}\le &\quad \lim_{n\to\infty}\tfrac{n}{n^2+1}\sin(2n+1) &\le \lim_{n\to\infty}\frac{n}{n^2+1}\\
\lim_{n\to\infty}\frac{\frac{-1}{n}}{1+\frac{1}{n^2}}\le &\quad \lim_{n\to\infty}\tfrac{n}{n^2+1}\sin(2n+1) &\le \lim_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{1}{n^2}}\\
0\le &\quad \lim_{n\to\infty}\tfrac{n}{n^2+1}\sin(2n+1) &\le 0.
\end{aligned}$$

Hence we have $\lim_{n\to\infty}\frac{n}{n^2+1}\sin(2n+1)=0$.

(iii) Let m=3n+1. Hence we have,

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{1}{3n+1}\right)^n &= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{\frac{m-1}{3}}\\
&= \lim_{m\to\infty}\left(\left(\left(1+\frac{1}{m}\right)^m\right)^{\frac{1}{3}}\cdot\left(1+\frac{1}{m}\right)^{\frac{1}{3}}\right)\\
&= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^{\frac{1}{3}}\\
&= e^{\frac{1}{3}}
\end{aligned}$$

(b) $\sup S = 1$ and $\inf S = -\frac{1}{2}$
Proof:

Claim: 1 is a upper bound for S and $-\frac{1}{2}$ is a lower bound for S
For all $n, m \in \mathbb{N}$, $\frac{1}{n} \leq 1$, hence $\frac{1}{n} - \frac{1}{2m} \leq 1$.
Also, for all $m, n \in \mathbb{N}$, $-\frac{1}{2m} \geq -\frac{1}{2}$, hence $\frac{1}{n} - \frac{1}{2m} \geq -\frac{1}{2}$

Claim: $\sup S = 1$ and $\inf S = -\frac{1}{2}$
Suppose for some $\epsilon_1 \in \mathbb{R}_{>0}$, that $\sup S = 1 - \epsilon_1$.
Since there exist $p \in \mathbb{N}$ such that $\frac{1}{p} < \epsilon_1$, we have $\frac{1}{2p} < \epsilon_1$.
Hence $1 - \frac{1}{2p} > 1 - \epsilon_1$ which is a contradiction since $1 - \frac{1}{2p} \in S$.
Suppose again for some $\epsilon_2 \in \mathbb{R}_{>0}$, that $\inf S = -\frac{1}{2} + \epsilon_2$.
Since there exists a $q \in \mathbb{N}$ such that $\frac{1}{q} < \epsilon_2$.
Hence $-\frac{1}{2} + \frac{1}{q} < -\frac{1}{2} + \epsilon_2$ which is again a contradiction since $-\frac{1}{2} + \frac{1}{q} \in S$.

Question 2

(a) (i)

$$\begin{aligned}
\sum_{n=1}^{M} \frac{n^2+8n}{n^3+2n+1} &> \sum_{n=1}^{M} \frac{n^2+n}{n^3+2n+1} \\
&> \sum_{n=1}^{M} \frac{n^2+n}{n^3+3n^2+3n+1} \\
&= \sum_{n=1}^{M} \frac{n}{(n+1)^2} \\
&= \sum_{n=1}^{M} \left(\frac{1}{n+1} - \frac{1}{(n+1)^2}\right) \\
&= \sum_{n=1}^{M} \frac{1}{n+1} - \sum_{n=1}^{M} \frac{1}{(n+1)^2}
\end{aligned}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1}$ is divergent, $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+8n}{n^3+2n+1}$ is divergent by Comparison Test.

(ii)

$$\begin{aligned}
\lim_{n\to\infty} \left(\left(\frac{2n}{2n+1}\right)^{n^2}\right)^{\frac{1}{n}} &= \lim_{n\to\infty} \left(\frac{2n+1}{2n}\right)^{-n} \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^{-n} \\
&= \lim_{m\to\infty} \left(\left(1 + \frac{1}{m}\right)^{m}\right)^{-\frac{1}{2}} \\
&= (e)^{-\frac{1}{2}} \\
&< 1
\end{aligned}$$

Hence the $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{2n+1}\right)^{n^2}$ is convergent by the Root Test.

(iii)

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{\frac{(n+1)^{2n+2}}{(2n+2)!}}{\frac{n^{2n}}{(2n)!}}\right) &= \lim_{n\to\infty}\left(\frac{(n+1)^{2n+2}}{n^{2n}}\frac{(2n)!}{(2n+2)!}\right)\\ &= \lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^{2n}\frac{(n+1)^2}{(2n+2)(2n+1)}\right)\\ &= \frac{e^2}{4}\\ &> 1\end{aligned}$$

Hence the sum $\displaystyle\sum_{n=1}^{\infty}\frac{n^{2n}}{(2n)!}$ is divergent by the Ratio Test.

(iv) Since sin is a strictly increasing function from 0 to $\frac{\pi}{2}$.
We have $\sin(\frac{\pi}{n+1}) < \sin(\frac{\pi}{n})$ for all $n \in \mathbb{N}_{\geq 2}$.
Hence by Alternating Series Test, the sum $\displaystyle\sum_{n=1}^{\infty}(-1)^n\sin\left(\frac{\pi}{n}\right) = \sum_{n=2}^{\infty}(-1)^n\sin\left(\frac{\pi}{n}\right)$ converges.

(b) Since $x_{n+2} = \frac{2}{3+2x_{n+1}} = \frac{6+4x_n}{13+6x_n} = \frac{2}{3} - \frac{2}{3}\left(\frac{4}{13+6x_n}\right)$, we have $0 < x_n < \frac{2}{3}$ for all $n \in \mathbb{N}_{\geq 2}$.
Claim: For $n \in \mathbb{N}$, (x_{2n}) is either strictly increasing or strictly decreasing.
Proof:
Suppose $0 < x_0 \leq \frac{1}{2}$, then $x_0 \leq x_2$.
Suppose again for some $k \in \mathbb{N}$ we have $x_k \leq x_{k+2}$.
Then we have $\frac{2}{3} - \frac{2}{3}\left(\frac{4}{13+6x_k}\right) \leq \frac{2}{3} - \frac{2}{3}\left(\frac{4}{13+6x_{k+2}}\right)$, hence we have $x_{k+2} \leq x_{k+4}$
Suppose $\frac{1}{2} < x_0$, then $x_2 < x_0$.
Suppose again for some $k \in \mathbb{N}$ we have $x_k > x_{k+2}$.
Then we have $\frac{2}{3} - \frac{2}{3}\left(\frac{4}{13+6x_k}\right) > \frac{2}{3} - \frac{2}{3}\left(\frac{4}{13+6x_{k+2}}\right)$, hence we have $x_{k+2} > x_{k+4}$
Hence by induction, for $n \in \mathbb{N}$, (x_{2n}) is either strictly increasing or strictly decreasing.
By a similar argument, for $n \in \mathbb{N}$, (x_{2n+1}) is either strictly increasing or strictly decreasing.
Hence by Completeness of $\mathbb{R}$, $\lim\limits_{n\to\infty} x_{2n} = y$ and $\lim\limits_{n\to\infty} x_{2n+1} = y'$ exist.
Since $y = \frac{2}{3} - \frac{2}{3}\left(\frac{4}{13+6y}\right)$ and $y' = \frac{2}{3} - \frac{2}{3}\left(\frac{4}{13+6y'}\right)$, we have $y = \frac{1}{2} = y'$.
Hence, (x_n) converges and its limit is $\frac{1}{2}$.

Question 3

(a) (i) Since $-1 \leq \cos\left(\frac{1}{x^2}\right) \leq 1$, we have $\frac{x}{x+1} \leq \frac{x}{x+1}\cos\left(\frac{1}{x^2}\right) \leq -\frac{x}{x+1}$.
Hence we have $\displaystyle\lim_{x\to 0}\frac{x}{x+1} \leq \lim_{x\to 0}\frac{x}{x+1}\cos\left(\frac{1}{x^2}\right) \leq \lim_{x\to 0} -\frac{x}{x+1}$.
Therefore $\displaystyle\lim_{x\to 0}\frac{x}{x+1}\cos\left(\frac{1}{x^2}\right) = 0$.

(ii) Suppose $\lim_{x\to 0^+}\left|\sin\left(\frac{1}{x}\right)\right| = a$ exist.
Let $\epsilon = \frac{1}{4}$. For any $\delta \in \mathbb{R}_{>0}$, we can choose a $n_1 \in \mathbb{N}$ such that $x_1 = \frac{2}{\pi+4n_1\pi} < \delta$.
We can also choose $n_2 \in \mathbb{N}$ such that $x_2 = \frac{1}{2n_2\pi} < \delta$.
Hence we have $\left|\left|\sin\left(\frac{1}{x_1}\right)\right| - a\right| < \epsilon$ and $\left|\left|\sin\left(\frac{1}{x_2}\right)\right| - a\right| < \epsilon$.
Therefore we have, $|1-a| < \epsilon$ and $|0-a| < \epsilon$.
Which leads us to $|1| < 2\epsilon = \frac{1}{2}$ a contradiction.

(iii)

$$\begin{aligned}
\lim_{x\to\infty}\frac{\sqrt{x}-2x}{\sqrt{x}+2x} &= \lim_{y\to\infty}\frac{y-2y^2}{y+2y^2} \\
&= \lim_{y\to\infty}\frac{\frac{1}{y}-2}{\frac{1}{y}+2} \\
&= -1
\end{aligned}$$

(b) For any given $\epsilon \in \mathbb{R}_{>0}$, choose $\delta = \min\left(1, \frac{3\epsilon}{4}\right)$.
Then $|x-1| < \delta$ gives us,

$$|3x-2| < 4$$

and

$$\left|\frac{1}{2x+1}\right| < 1$$

Hence we have,

$$\begin{aligned}
\left|\frac{x^2-x+1}{2x+1} - \frac{1}{3}\right| &= \left|\frac{3x^2-5x+2}{3(2x+1)}\right| \\
&= \left|\frac{(x-1)(3x-2)}{3(2x+1)}\right| \\
&= |(x-1)|\left|\frac{3x-2}{3(2x+1)}\right| \\
&< \frac{4}{3}|(x-1)| \\
&< \epsilon
\end{aligned}$$

whenever $|x-1| < \delta$.
Hence $\lim_{x\to 1}\frac{x^2-x+1}{2x+1} = \frac{1}{3}$.

(c) The function $\lfloor x \rfloor$ is continuous at $x \in \mathbb{R}\backslash\mathbb{Z}$. The function $\cos x$ is continuous on $\mathbb{R}$. So $f(x)$ is continuous on $\mathbb{R}\backslash\mathbb{Z}$.
It remains to check continuity at x when $\cos x = 0, 1, -1$.

Case 1: $\cos x = 0$ when $x = \frac{pi}{2} + n\pi$, $n \in \mathbb{Z}$.

Note that if $c = \frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$. Then

$$\begin{aligned}
\lim_{x\to c^+}\lfloor \cos x \rfloor &= -1 \\
\lim_{x\to c^-}\lfloor \cos x \rfloor &= 0
\end{aligned}$$

If $c = \frac{3\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$. Then

$$\begin{aligned}\lim_{x \to c^+} \lfloor \cos x \rfloor &= 0 \\ \lim_{x \to c^-} \lfloor \cos x \rfloor &= -1\end{aligned}$$

So $f(x)$ is not continuous at $x = \frac{\pi}{2} + n\pi$ for $n \in \mathbb{Z}$.

Case 2: $\cos x = 1$

Then $x = 2n\pi$ for $n \in \mathbb{Z}$.
Let $c = 2n\pi$ for $n \in \mathbb{Z}$.

$$\begin{aligned}\lim_{x \to c^+} \lfloor \cos x \rfloor &= 0 \\ \lim_{x \to c^-} \lfloor \cos x \rfloor &= 0\end{aligned}$$

But $f(c) = 1$. So $f(x)$ is not continuous at $x = 2n\pi$, $n \in \mathbb{Z}$.

Case 3: $\cos x = -1$.

Then $x = (2n+1)\pi$ for $N \in \mathbb{Z}$.

Let $c = (2n+1)\pi$ for $n \in \mathbb{Z}$.
if $c = (2n+1)\pi$, $n \in \mathbb{Z}$. Then

$$\begin{aligned}\lim_{x \to c^+} \lfloor \cos x \rfloor &= -1 \\ \lim_{x \to c^-} \lfloor \cos x \rfloor &= -1\end{aligned}$$

So $f(x)$ is continuous at $x = (2n+1)\pi$ for $n \in \mathbb{Z}$.

In conclusion, $f(x)$ is continuous at $\mathbb{R}\backslash\{\frac{\pi}{2} + n\pi, 2n\pi : n \in \mathbb{Z}\}$. The points of continuity are $\mathbb{R}\backslash\left\{2n\pi + \frac{m\pi}{2} | n \in \mathbb{Z}, m \in \{1,3,4\}\right\}$.

Question 4

(a) Since f is continuous at $x = 0$, for $\epsilon = \frac{1}{10}$, there exists a $\delta \in \mathbb{R}_{>0}$ such that $|f(x) - f(0)| < \frac{1}{10}$ whenever $|x| < \delta$.
Hence we have $f(x) - (-1) < \frac{1}{10}$ for $|x| < \delta$.
Therefore there exist a $\delta > 0$ such that $f(x) < -\frac{9}{10}$.

(b) Since $x_n \in S$ for all $n \in \mathbb{N}$, we have $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} g(x_n)$.
Since f and g are continuous, we have $f(\lim\limits_{n\to\infty} x_n) = g(\lim\limits_{n\to\infty} x_n)$.
Hence we have $f(x) = g(x)$ and $x \in S$.

(c) Suppose $a_n < 0$ for some $n \in \mathbb{N}$.
Since (a_n) is decreasing.

$$a_m \leq a_n \text{ for all } m \geq n$$

Hence

$$\lim_{m\to\infty} a_m \leq a_n < 0.$$

Therefore $\lim_{n\to\infty} n < 0$.
Since $\sum_{n=1}^{\infty} a_n$ is convergent, $\lim_{a\to\infty} a_n = 0$ which contradicts the result. Hence $a_n \geq 0$ for all $n \in \mathbb{N}$.

Since $\sum_{n=1}^{\infty} a_n$ is convergent, the sequence of its partial sums is Cauchy.
Let $\epsilon > 0$. Then there exists N such that for all $m, n \geq N$,

$$a_n + ... + a_m < \epsilon.$$

In particular we have,

$$a_n + ... + a_{2n} < \epsilon$$
$$a_n + ... + a_{2n+1} < \epsilon.$$

Since (a_n) is decreasing,

$$\begin{aligned} \frac{1}{2}(2n)a_{2n} &= na_{2n} \leq a_n + ... + a_{2n} < \epsilon \\ \frac{1}{2}(2n+1)a_{2n+1} &\leq (n+1)a_{2n+1} \leq a_n + ... + a_{2n+1} < \epsilon \end{aligned}$$

Hence we have

$$\lim_{n\to\infty} (2n)a_{2n} = 0 = \lim_{n\to\infty} (2n+1)a_{2n+1}$$

So $\lim_{n\to\infty} na_n = 0$.

Question 5

(a) We first show that $g(x)$ is not continuous at $x \neq 1$.
Let $c \in \mathbb{R}$,$c \neq 1$.
Let $c \in \mathbb{R}$, $c \neq 1$.
Let (x_n) be a sequence of rational numbers converging to c.
Let (y_n) be a sequence of irrational numbers converging to c.
If g is continuous at c, then

$$\begin{aligned} (g(x_n)) &\to 3c \\ (g(y_n)) &\to -c+4 \end{aligned}$$

Hence we have $3c = -c+4$. Therefore we have $c = 1$ which contradicts our assumption that $c \neq 1$.

Now we shall prove that g is continuous at $x = 1$.
Let $\epsilon > 0$.
Set $\delta = \frac{\epsilon}{3}$. Let x be such that

$$|x-1| < \delta$$

If x is rational then

$$\begin{aligned} |g(x)-3| &= |3x-3| \\ &= 3|x-1| \\ &< 3\delta \\ &= \epsilon \end{aligned}$$

If x is irrational then

$$\begin{aligned} |g(x)-3| &= |-x+4-3| \\ &= |x-1| \\ &< \delta \\ &= \frac{\epsilon}{3} < \epsilon \end{aligned}$$

Therefore $\lim_{x\to 1} g(x) = 3 = g(1)$.
SO g is continuous at $x = 1$.

(b) Since (x_{3k}), (x_{3k+1}) and (x_{3k+2}) converges to the same limit a.
For any $\epsilon \in \mathbb{R}_{>0}$ there exist $M \in \mathbb{N}$ such that for all $3k, 3k+1, 3k+2 \geq M$, we have $|x_{3k} - a| < \epsilon$ and $|x_{3k+1} - a| < \epsilon$ and $|x_{3k+2} - a| < \epsilon$.
Hence, given any $\epsilon \in \mathbb{R}_{>0}$, let $N = M$ such that for all $n \in \mathbb{N}$ and $n \geq N$, we have $|x_n - a| < \epsilon$.
Hence (x_n) is convergent and converges to a.

(c) Since $\lim\limits_{n\to\infty} a_n = \infty$. We can choose a increasing subsequence (b_n) such that $\lim\limits_{n\to\infty} b_n = \infty$.
Then for any $x \in \mathbb{R}$ we can construct the sequence (c_n) such that $c_n = \max\{n \in \mathbb{Z} | n < xb_n\}$.
Therefore $xb_n - 1 \leq c_n \leq xb_n$.
Hence we have $\lim\limits_{n\to\infty} \dfrac{c_n}{b_n} \leq x$ and

$$\begin{aligned} \lim_{n\to\infty} \frac{c_n}{b_n} &\geq \lim_{n\to\infty} \frac{xb_n - 1}{b_n} \\ &= x \end{aligned}$$

Hence $\lim_{n\to\infty} \frac{c_n}{b_n} = x$.
Therefore we have $f(x) = f(\lim_{n\to\infty} \frac{c_n}{b_n})$.
Since f is continuous, we have $f(x) = \lim_{n\to\infty} f(\frac{c_n}{b_n}) = \lim_{n\to\infty} 0 = 0$.

Hence $f(x) = 0$ for all $x \in \mathbb{R}$