NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Johan Gunardi

MA2108 Mathematical Analysis I
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Question 1

Let $b_n = a_n - \frac{3}{2}$ for all n. So $b_1 = -\frac{1}{2}$ and

$$b_{n+1} = a_{n+1} - \frac{3}{2} = \frac{a_n + 9}{2(a_n + 2)} - \frac{3}{2} = \frac{3 - 2a_n}{2(a_n + 2)} = -\frac{b_n}{b_n + \frac{7}{2}}.$$

We prove by induction that $|b_n| \leq \frac{1}{2}$ for all n, which is readily true for $n = 1$. Assume $|b_n| \leq \frac{1}{2}$ for some n but $|b_{n+1}| > \frac{1}{2}$. Since $|b_{n+1}| = \left|\frac{b_n}{b_n+7/2}\right| = \left|1 - \frac{7/2}{b_n+7/2}\right|$, then either $1 - \frac{7/2}{b_n+7/2} > \frac{1}{2}$ or $1 - \frac{7/2}{b_n+7/2} < -\frac{1}{2}$. However, keeping in mind that $|b_n| \leq \frac{1}{2}$ and hence $b_n + 7/2 > 0$, these inequalities imply that $b_n > \frac{7}{2}$ or $b_n < -\frac{7}{6}$, both of which contradict $|b_n| \leq \frac{1}{2}$. So we have proved that $|b_n| \leq \frac{1}{2}$ for all n.

As a consequence, we have $|b_n + 7/2| \geq -\frac{1}{2} + \frac{7}{2} = 3$, and hence

$$|b_{n+1}| = \frac{|b_n|}{|b_n + 7/2|} \leq \frac{|b_n|}{3}.$$

But then $|b_n| \leq \frac{|b_1|}{3^{n-1}} \to 0$ as $n \to \infty$. So $\lim b_n = 0$ and $\lim a_n = \frac{3}{2}$.

Question 2

(i) Clearly (x_n) is bounded below by 0. Also $x_n \leq \frac{(3+1)(n^3+1)1}{n(2n+1)(3n+2)}$. The right hand side converges to $\frac{3+1}{2\cdot 3} = \frac{2}{3}$, and thus is bounded. So x_n is also bounded above.

(ii) As shown above, we must have $0 \leq \liminf x_n \leq \limsup x_n \leq \frac{2}{3}$. For $n = 6k + 3$, we have $\cos\left(\frac{n\pi}{6}\right) = \cos\left(k\pi + \frac{\pi}{2}\right) = 0$, so $x_{6k+3} = 0$. Hence the subsequence (x_{6k+3}) converges to 0, so $\liminf x_n = 0$. For $n = 12k$, we have $x_n = \frac{(3+1)(n^3+1)1}{n(2n+1)(3n+2)}$ which converges to $\frac{2}{3}$. So $\limsup x_n = \frac{2}{3}$.

(iii) No, (x_n) is not convergent because $\liminf x_n \neq \limsup x_n$.

Question 3

(a) (i) We use Root Test. Since

$$\left(\frac{n^2}{2^{n+1}}\left(1 + \frac{1}{1+4n}\right)^{2n^2}\right)^{1/n} = \frac{n^{2/n}}{2\cdot 2^{1/n}}\left(1 + \frac{1}{1+4n}\right)^{(1+4n)\cdot\frac{2n}{1+4n}} \to \frac{1}{2\cdot 1}\cdot e^{\frac{1}{2}} < 1,$$

then the series converges.

(ii) Note that $\sqrt{1+n^4} - n^2 = \frac{1}{\sqrt{1+n^4}+n^2} < \frac{1}{\sqrt{n^4}+n^2} = \frac{1}{2n^2}$ and we know that $\sum \frac{1}{2n^2}$ converges. So $\sum(\sqrt{1+n^4} - n^2)$ converges by Comparison Test.

(b) (i) The n-th partial sums of $\sum a_n$ and $\sum(a_{2n-1}+a_{2n})$ are $a_1+\cdots+a_n$ and $a_1+\cdots+a_{2n}$ respectively. So the partial sums of $\sum(a_{2n-1}+a_{2n})$ is a subsequence of the partial sums of $\sum a_n$. Since the latter converges, so does the former, i.e., $\sum(a_{2n-1}+a_{2n})$ converges.

(ii) $a_n=(-1)^n$. In this case, $\sum(a_{2n-1}+a_{2n})=\sum 0$ converges to 0 but $\sum a_n=\sum(-1)^n$ diverges.

(iii) Suppose $S=\sum(a_{2n-1}+a_{2n})$. Let $\epsilon>0$. There is N such that for all $n>N$,

$$\left|\sum_{i=1}^{n}(a_{2i-1}+a_{2i})-S\right|<\frac{\epsilon}{2}$$

and $|a_n|<\frac{\epsilon}{2}$. Take $n>2N$. If $n=2k$ is even, then $|\sum_{i=1}^n a_i-S|=\left|\sum_{i=1}^k(a_{2i-1}+a_{2i})-S\right|<\epsilon$. If $n=2k-1$ is odd, then

$$\left|\sum_{i=1}^{n}a_i-S\right|=\left|\sum_{i=1}^{k-1}(a_{2i-1}+a_{2i})+a_{2k-1}-S\right|<\left|\sum_{i=1}^{k-1}(a_{2i-1}+a_{2i})-S\right|+|a_{2k-1}|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

So $|\sum_{i=1}^n a_i-S|<\epsilon$ for all $n>2N$. Therefore $\sum a_n$ converges to S.

Question 4

(a) Let $\epsilon>0$. Choose $\delta=\min\{\frac{1}{6},\frac{\epsilon}{16}\}$ and suppose $|x-1|<\delta$. In particular, $x-1>-\frac{1}{6}$, so $3x-2>\frac{1}{2}$ Then

$$\left|\frac{x+2}{3x-2}-3\right|=\left|\frac{8(1-x)}{3x-2}\right|<\frac{8\delta}{1/2}<\epsilon.$$

(b) (i) The sequence $(a_n)=\left(\frac{1}{\sqrt{n\pi}}\right)$ converges to 0 and $\left|\sin\left(\frac{1}{a_n^2}\right)\right|=0$. On the other hand, the sequence $(b_n)=\left(\frac{1}{\sqrt{2n\pi+\frac{\pi}{2}}}\right)$ also converges to 0 and $\left|\sin\left(\frac{1}{b_n^2}\right)\right|=1$. So the limit does not exist.

(ii) For $3<x<4$, we have $[x]=3$ and $[5-x]=1$, so

$$\lim_{x\to3^+}\frac{[x]+1}{[5-x]+x^2}=\lim_{x\to3^+}\frac{3+1}{1+x^2}=\frac{4}{1+3^2}=\frac{4}{10}=\frac{2}{5}.$$

(c) Let $M>0$. There is $\delta>0$ such that for $|x-a|<\delta$, we have $f(x)>\frac{M}{2}$ and $g(x)>\frac{M}{2}$. Then for $|x-a|<\delta$, we also have $f(x)+g(x)>M$. Thus $\lim_{x\to a}(f+g)(x)=\infty$.

Question 5

(a) Suppose $f(x)=x^3$ is uniformly continuous. Let $\epsilon>0$. There is $\delta>0$ such that whenever $|x-y|<\delta$, we have $|x^3-y^3|<\epsilon$. Choose $y>\sqrt{\frac{8\epsilon}{3\delta}}$ and $x=y+\frac{\delta}{2}$. Then

$$|x^3-y^3|=|x-y||x^2+xy+y^2|=\frac{\delta}{2}((x+y/2)^2+3y^2/4)>\frac{\delta}{2}\cdot 3y^2/4>\epsilon,$$

a contradiction.

(b) (i) Yes. Keeping in mind that $\lim_{x\to0}x\cos\left(\frac{1}{x}\right)=0$, then we have

$$\lim_{x\to0+}g(x)=\lim_{x\to0+}\frac{x^{5/2}\pi}{(x+1)^2}\cdot\lim_{x\to0+}\frac{\sqrt{x}}{\pi}\cos\left(\frac{\pi}{\sqrt{x}}\right)=0.$$

(ii) Yes. Define $h(x) = g(x)$ for $x \in (0, 1]$ and $h(0) = \lim_{x\to 0+} g(x)$. Then $h(x)$ is continuous on the closed and bounded interval $[0, 1]$, and hence is uniformly continuous. So $g(x)$ is also uniformly continuous.

Question 6

(a) (i) We will prove $a_{n-1} > a_n$ for all $n > 1$, i.e.,

$$\left(\frac{n}{n-1}\right)^n > \left(\frac{n+1}{n}\right)^{n+1}$$
$$\iff n^{2n+1} > (n-1)^n(n+1)^{n+1}$$
$$\iff n^{2n+1} > (n+1)(n^2-1)^n$$
$$\iff \left(\frac{n^2}{n^2-1}\right)^n > \frac{n+1}{n}$$

The last statement is true by Binomial's Theorem:

$$\left(\frac{n^2}{n^2-1}\right)^n = \left(1+\frac{1}{n^2-1}\right)^n > 1^n + \binom{n}{1}1^{n-1}\left(\frac{1}{n^2-1}\right) = 1 + \frac{n}{n^2-1} > 1 + \frac{1}{n} = \frac{n+1}{n}.$$

(ii) Since $\lim a_n = \lim\left(1+\frac{1}{n}\right) \cdot \lim\left(1+\frac{1}{n}\right)^n = 1 \cdot e = e$ and (a_n) is decreasing, then $a_n > e$ for all n.

(b) Recall that lim sup is the largest limit among all the convergent subsequences. $\limsup y_n$ is the limit of some subsequence of y_n, which is also a subsequence of x_n. Hence $\limsup y_n \le \limsup x_n$, and analogously $\limsup z_n \le \limsup x_n$. So

$$\limsup x_n \ge \max(\limsup y_n, \limsup z_n).$$

Now take a subsequence (x_{n_k}) of x_n that converges to $\limsup x_n$. If infinitely many n_k are even, these terms form a subsequence of $(x_{2n}) = y_n$, and hence $\limsup x_n \le \limsup y_n$. Otherwise, infinitely many n_k are odd, and these terms form a subsequence of $(x_{2n-1}) = z_n$, and so $\limsup x_n \le \limsup z_n$. In any case,

$$\limsup x_n \le \max(\limsup y_n, \limsup z_n).$$

Therefore

$$\limsup x_n = \max(\limsup y_n, \limsup z_n).$$

Question 7

(a) (i) $f(x) = x$ has a supremum b, but never reaches it.

(ii) Let $g(x) = f(x)$ on $x \in [a, b)$ and $g(b) = L$. So g is continuous on $[a, b]$ and has an absolute maximum $g(x_1)$ for some $x_1 \in [a, b]$. Since $g(x_0) = f(x_0) > L$, then $g(b) = L$ is not the absolute maximum, i.e., $x_1 \in [a, b)$. Thus $f(x)$ has an absolute maximum $f(x_1)$ where $x_1 \in [a, b)$.

(iii) Yes. Define $g(x)$ as before, with absolute maximum $g(x_2)$ on $[a, b]$. If $x_2 \in [a, b)$, then $f(x_2) = g(x_2)$ is an absolute maximum of f in $[a, b)$. If $x_2 = b$, then $g(x_2) = L = f(x_1)$ is the absolute maximum of f with $x_1 \in [a, b)$. In any case, f always have an absolute maximum in $[a, b)$.

(b) Let $r \in \mathbb{R}$. Since $h(\mathbb{R}$ is not bounded above and not bounded below, there exists x_1, x_2 such that $h(x_1) < r < h(x_2)$. By the Intermediate Value Theorem, there exists x_0 between x_1 and x_2 such that $h(x_0) = r$. So $r \in h(\mathbb{R})$. Since r was an arbitrarily chosen real number, we conclude $h(\mathbb{R}) = \mathbb{R}$.

Question 8

(a) (i) We do induction on m. The base case $m = 1$ is given. Suppose $f\left(r + \frac{m-1}{n}\right) = f(r)$ is true for some natural number $m - 1$ and any rational number r, natural number n. Then

$$f\left(r + \frac{m}{n}\right) = r\left(\left(r + \frac{1}{n}\right) + \frac{m-1}{n}\right) = f\left(r + \frac{1}{n}\right) = f(r).$$

(ii) Let $a < b$ be two rational numbers. Write $b - a = \frac{m}{n}$ for some natural numbers m, n, then

$$f(b) = f\left(a + \frac{m}{n}\right) = f(b).$$

So $f(a) = f(b)$ for all rational numbers a, b. Let $c \in \mathbb{R}$ be such that $f(x) = c$ for all rational numbers x. If x is a real number, choose a sequence of rational numbers (x_n) converging to x, then $f(x) = \lim f(x_n) = \lim c = c$. Therefore f is constant.

(b) Let $\epsilon > 0$. There exists $\delta_1 > 0$ such that whenever $x, y \in (a, b]$ and $|x - y| < \delta_1$, we have $|g(x) - g(y)| < \frac{\epsilon}{2}$. Also, there exists $\delta_2 > 0$ such that whenever $x, y \in [b, c)$ and $|x - y| < \delta_2$, we have $|g(x) - g(y)| < \frac{\epsilon}{2}$.

Now choose $\delta = \min\{\delta_1, \delta_2\}$ and suppose $x, y \in (a, b)$, $|x - y| < \delta$. If x, y are both in $(a, b]$ or both in $[b, c)$, then $|g(x) - g(y)| < \frac{\epsilon}{2} < \epsilon$. Assume $x \in (a, b]$ and $y \in [b, c)$. So we have $x \leq b \leq y$ and $y - x < \delta$, then $b - x, y - b \leq \delta$, and so $|g(b) - g(x)| < \frac{\epsilon}{2}$ and $|g(y) - g(b)| < \frac{\epsilon}{2}$. Hence

$$|g(y) - g(x)| \leq |g(y) - g(b)| + |g(b) - g(x)| < \epsilon.$$

So g is uniformly continuous on (a, c).