NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA2202 Algebra I
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Question 1

(i) $|G| = 9$. By Lagrange theorem, any subgroup of G has either 1, 3 or 9 elements.

Case 1: Subgroup has 1 element. As all subgroups must contain the identity element, the only subgroup is $\{(\overline{0}, \overline{0})\}$.

Case 2: Subgroup has 3 elements. As 3 is prime, the subgroup must be cyclic, i.e. it must be generated by some $g \in G$. We can proceed by listing all the generators of g.
(a) $\langle(\overline{0}, \overline{1})\rangle = \langle(\overline{0}, \overline{2})\rangle = \{(\overline{0}, \overline{0}), (\overline{0}, \overline{1}), (\overline{0}, \overline{2})\}$
(b) $\langle(\overline{1}, \overline{0})\rangle = \langle(\overline{2}, \overline{0})\rangle = \{(\overline{0}, \overline{0}), (\overline{1}, \overline{0}), (\overline{2}, \overline{0})\}$
(c) $\langle(\overline{1}, \overline{1})\rangle = \langle(\overline{2}, \overline{2})\rangle = \{(\overline{0}, \overline{0}), (\overline{1}, \overline{1}), (\overline{2}, \overline{2})\}$
(d) $\langle(\overline{1}, \overline{2})\rangle = \langle(\overline{2}, \overline{1})\rangle = \{(\overline{0}, \overline{0}), (\overline{1}, \overline{2}), (\overline{2}, \overline{1})\}$
The last generator, $\langle(\overline{0}, \overline{0})\rangle$ is not of order 3.

Case 3: Subgroup has 9 elements. Then the subgroup is G itself.

(ii) G is not a cyclic group. By part (i), we have exhausted all generators of $g \in G$. None of them is equal to G.

Question 2

(i) $e_{G_1} \in G_1$, therefore $\varphi(e_{G_1}) \in \varphi(G_1)$, so $\varphi(G_1) \neq \emptyset$.

Let $h_1, h_2 \in \varphi(G_1)$. Then $\exists g_1, g_2 \in G_1$ such that $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$. So,

$$h_1 * h_2^{-1} = \varphi(g_1) * \varphi(g_2)^{-1} = \varphi(g_1) * \varphi(g_2^{-1}) = \varphi(g_1 * g_2^{-1}) \in \varphi(G_1)$$

as $g_1 * g_2^{-1} \in G_1$.

This shows that $\varphi(G_1) \leq G_2$.

(ii) No. We can construct a counter-example.

Let $\varphi : S_2 \to S_3$ be the identity map. Let $N_1 = S_2$. Then $\varphi(N_1) = \{(1), (12)\}$. But $(13)(12)(13)^{-1} = (13)(12)(13) = (23) \notin \varphi(N_1)$. This shows that $\varphi(N_1)$ is not a normal subgroup of S_3.

Question 3

(i) $(1) \in H_n$, therefore $H_n \neq \emptyset$.

Let $h_1, h_2 \in H_n$, so $h_1(n) = h_2(n) = n$, and $h_2^{-1}(n) = n$. Then

$$(h_1 \circ h_2^{-1})(n) = h_1(h_2^{-1}(n)) = h_1(n) = n \Rightarrow h_1 \circ h_2^{-1} \in H_n$$

Therefore $H_n \leq S_n$.

(ii) $|H_n| = |S_{n-1}| = (n-1)!$.

(iii) K is not a subgroup of S_n as the identity element, (1), is not in K.

(iv) We can define a function $\phi : K \to H_n$ such that $\phi(k) = (n-1, n) \circ k \ \ \forall k \in K$. Then ϕ is a bijection. Therefore $|K| = |H_n| = (n-1)!$.

Question 4

$A_4 = \{(1), (123), (124), (134), (234), (321), (421), (431), (432), (12)(34), (13)(24), (14)(23)\}$ has 12 elements. The first element has order 1, the next 8 elements have order 3, and the last 3 elements have order 2.

Suppose it has a subgroup of order 6. We shall attempt to construct it. As there are only 2 types of groups of order 6, namely, a cyclic group of order 6 ($\cong (\mathbb{Z}_6, +)$) and a non-cyclic non-Abelian group of order 6 ($\cong (S_3, \circ)$), the subgroup in question must be isomorphic to one of them. As A_4 has no element of order 6, the subgroup must be of the second type (isomorphic to S_3).

$S_3 = \langle (12), (123) \rangle$ is generated by an element of order 2 and an element of order 3. Therefore, the subgroup of order 6 can also be generated by an element of order 2 and an element of order 3. Without loss of generality, assume that the element of order 3 is $(123) \in A_4$. There are 3 choices for the element of order 2, namely $(12)(34), (13)(24)$ and $(14)(23)$. However, it can be shown that $\langle (123), (12)(34) \rangle = \langle (123), (13)(24) \rangle = \langle (123), (14)(23) \rangle = A_4$, i.e. it is impossible to construct a subgroup of order 6.

Question 5

(i) Let $G = \{e_G, a_1, a_2, a_3\}$.

$e_G g = g e_G \ \ \forall g \in G$.

For $i = j$, $a_i a_j = a_j a_i \ \ \forall i, j$, since $a_i = a_j$.

For $i \neq j$, without loss of generality, consider the product $a_1 a_2$. Clearly the product cannot be $= a_1, a_2$ as it will make the other element the identity element. So either $a_1 a_2 = e_G$ or a_3. If it is the former, then the 2 elements are inverses of each other and $a_1 a_2 = e_G = a_2 a_1$. If it is the latter, then consider $a_2 a_1$. This product now cannot be $= a_2, a_1$ by the same reason. It also cannot be $= e_G$ as they will then be inverses of each other and this will make $a_1 a_2 = e_G$. So $a_2 a_1 = a_3 = a_1 a_2$.

By exhausting all the cases, we have proven that G is Abelian.

(ii) All subgroups of G must have order 1, 2 or 4 by the Lagrange theorem. Suppose G is not a cyclic group. Then $o(a_1) = o(a_2) = o(a_3) = 2$ (cannot be 1, otherwise $a_i = e_G$; and cannot be 4, otherwise G is cyclic). So $a_1^2 = a_2^2 = a_3^2 = e_G$. Consider a_1a_2. By part (i)'s reasoning, the product cannot be equal to a_1 or a_2. Neither can it be equal to e_G, as this will make $a_2 = a_1^{-1} = a_1$. So $a_1a_2 = a_3$. Similarly, $a_2a_3 = a_1$ and $a_3a_1 = a_2$.

$*$	e_G	a_1	a_2	a_3
e_G	e_G	a_1	a_2	a_3
a_1	a_1	e_G	a_3	a_2
a_2	a_2	a_3	e_G	a_1
a_3	a_3	a_2	a_1	e_G

$+$	$(\overline{0},\overline{0})$	$(\overline{0},\overline{1})$	$(\overline{1},\overline{0})$	$(\overline{1},\overline{1})$
$(\overline{0},\overline{0})$	$(\overline{0},\overline{0})$	$(\overline{0},\overline{1})$	$(\overline{1},\overline{0})$	$(\overline{1},\overline{1})$
$(\overline{0},\overline{1})$	$(\overline{0},\overline{1})$	$(\overline{0},\overline{0})$	$(\overline{1},\overline{1})$	$(\overline{1},\overline{0})$
$(\overline{1},\overline{0})$	$(\overline{1},\overline{0})$	$(\overline{1},\overline{1})$	$(\overline{0},\overline{0})$	$(\overline{0},\overline{1})$
$(\overline{1},\overline{1})$	$(\overline{1},\overline{1})$	$(\overline{1},\overline{0})$	$(\overline{0},\overline{1})$	$(\overline{0},\overline{0})$

We see that the multiplication table for G and the addition table for $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ are similar. Hence the 2 groups are isomorphic.

Question 6

Let $g \in G$ and $h \in H$. Use $ghH = gH$ and $g^{-1}H = g^{-1}H$. Then, by the property given in the question,

$$ghg^{-1}H = gg^{-1}H = e_GH = H \Rightarrow ghg^{-1} \in H$$

This proves that H is a normal subgroup of G.

Question 7

The centre $Z(G)$ of G is a subgroup of G. By the Lagrange theorem, $|Z(G)| = 1, 3, 9$ or 27.

If $|Z(G)| = 1$, then by the class equation, there exists a subset of G, $\{x_j\}$, such that the centralizers $C_G(x_j) \neq G$ and $|G| = |Z(G)| + \sum_j |G : C_G(x_j)|$. But $C_G(x_j) \leq G$ and $C_G(x_j) \neq G$, so $|C_G(x_j)| = 1$, 3, or 9 by the Lagrange theorem, and $|G : C_G(x_j)| = 27$, 9 or 3. Then $\sum_j |G : C_G(x_j)|$ is divisible by 3, and hence

$$1 = |Z(G)| = |G| - \sum_j |G : C_G(x_j)| = 27 - \sum_j |G : C_G(x_j)|$$

is divisible by 3. This is a contradiction.

If $|Z(G)| = 3$, then we are done.

If $|Z(G)| = 9$, then $G/(Z(G))$ forms a group with order $|G : Z(G)| = 3$, implying that such a group is cyclic. So we have $G/Z(G) = \{e_G * Z(G), a * Z(G), a^2 * Z(G)\}$ for some $a \in G$. Let $g_1, g_2 \in G$. Then $g_1 = a^iz_1$ and $g_2 = a^jz_2$ for some $i, j \in \mathbb{N}, z_1, z_2 \in Z(G)$. Then

$$g_1g_2 = a^iz_1 * a^jz_2 = a^ia^j * z_1z_2 = a^{i+j} * z_2z_1 = a^ja^i * z_2z_1 = a^jz_2 * a^iz_1 = g_2g_1$$

This proves that G is Abelian.

If $|Z(G)| = 27 = |G|$, then it means that $Z(G) = G$ and $gz = zg\ \forall g, z \in G$, implying that G is Abelian.

Question 8

It suffices to prove that $S \subseteq H$. This is because if we have proven it, then $\langle S \rangle$ is generated by a subset of H, so it must be closed in H (i.e. $\langle S \rangle \subseteq H$). Subsequently, since $\langle S \rangle \leq G$ and $\langle S \rangle \subseteq H$, we will have $\langle S \rangle \leq H$.

Since $|G : H| = 2$, we must have H' as a coset of H in G such that $H \neq H'$ and $H \cup H' = G$. We must also have $(G/H, \times)$ as a group, where H is the identity element and $H' \times H' = H$. For any $g \in G$, since $gg^{-1} = e_G \in H$, we must have $g \in H \Leftrightarrow g^{-1} \in H$.

(i) If $h_1, h_2 \in H$, then $h_1h_2 \in H$ by closure of subgroup H.
(ii) If $h_1 \in H, h_2 \in H'$, then $h_1h_2 \in H'$, because if not, then $h_1h_2 \in H \Rightarrow h_2 = (h_1^{-1}h_1)h_2 = h_1^{-1}(h_1h_2) \in H$ since $h_1^{-1}, (h_1h_2) \in H$, which leads to a contradiction.
(iii) By the same logic, if $h_1 \in H', h_2 \in H$, then $h_1h_2 \in H'$.
(iv) Finally, if $h_1, h_2 \in H'$, then $H' = h_1H = h_2H$ and $(h_1h_2)H = h_1H \times h_2H = H' \times H' = H \Rightarrow h_1h_2 \in H$.

Let $s \in S$. Then $\exists g_1, g_2 \in G$ such that $s = g_1^{-1}g_2^{-1}g_1g_2$. We split into 4 cases:
(i) $g_1, g_2 \in H$. Then $g_1^{-1}, g_2^{-1} \in H$ and by closure of the subgroup H, $s = g_1^{-1}g_2^{-1}g_1g_2 \in H$.
(ii) $g_1 \in H, g_2 \in H'$. Then $g_1^{-1} \in H, g_2^{-1} \in H'$. By the previous paragraph, we have $g_1^{-1}g_2^{-1} \in H', g_1^{-1}g_2^{-1}g_1 \in H'$ and $s = g_1^{-1}g_2^{-1}g_1g_2 \in H$.
(ii) $g_1 \in H', g_2 \in H$. Then $g_1^{-1} \in H', g_2^{-1} \in H$. By the previous paragraph, we have $g_1^{-1}g_2^{-1} \in H', g_1^{-1}g_2^{-1}g_1 \in H$ and $s = g_1^{-1}g_2^{-1}g_1g_2 \in H$.
(ii) $g_1, g_2 \in H'$. Then $g_1^{-1}, g_2^{-1} \in H'$. By the previous paragraph, we have $g_1^{-1}g_2^{-1} \in H, g_1^{-1}g_2^{-1}g_1 \in H'$ and $s = g_1^{-1}g_2^{-1}g_1g_2 \in H$.

Therefore we have shown that $s \in S \Rightarrow s \in H$. This shows that $S \subseteq H$.

END OF SOLUTIONS

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