NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to An Hoa, VU

MA4207 Mathematical Logic
AY 2009/2010 Sem 2

Question 1

(a) Let $\varphi = Px$, $\psi = Qx$ and $\Sigma = \{\exists x\varphi, \exists x\psi\}$. It is clear that $\Sigma \models \exists x\varphi$ and $\Sigma \models \exists x\psi$. But $\Sigma \not\models \exists x(\varphi \wedge \psi)$ as we can take the structure $\mathfrak{A}$ with $|\mathfrak{A}| = \{0, 1\}$, $P^{\mathfrak{A}} = \{0\}$ and $Q^{\mathfrak{A}} = \{1\}$.

(b) Let $\varphi = \forall yx \not\approx y$ and $t = y$. Then $\varphi^x_t = \forall yy \not\approx y$. Clearly $\forall x\varphi \to \varphi^x_t$ is not valid.

(c) Let the language consist only of the binary relation $<$. Take $\mathfrak{A} = (\mathbb{Q}, <_{\mathbb{Q}})$ and $\mathfrak{B} = (\mathbb{R}, <_{\mathbb{R}})$. From the lecture, we have $\mathfrak{A} \equiv \mathfrak{B}$ but $\mathfrak{A} \not\cong \mathfrak{B}$.

(d) Let the language consist of 2010 predicates $P_1, P_2, ..., P_{2010}$. Take a structure $\mathfrak{A}$ with $|\mathfrak{A}| = \{1, 2, ..., 2010\}$ and $P_i^{\mathfrak{A}} = \{i\}$. Then $\mathfrak{A}$ has 2010 elements and each of them is definable: $\{i\}$ is defined by the formula $P_i x$.

(e) Consider the structure $\mathfrak{A} = (\mathbb{Z}, <)$ over the language with only the binary relation $<$. All the automorphisms over this structures are "translations" i.e mapping of form

$$\begin{aligned} \phi_z : \mathbb{Z} &\to \mathbb{Z} \\ x &\mapsto x + z \end{aligned}$$

where $z \in \mathbb{Z}$. So we have a countably infinitely many automorphisms over this structure.

Question 2

(a) We have

$$\begin{aligned} &\exists x\varphi \vdash \exists x\psi \\ \iff &\neg\forall x\neg\varphi \vdash \neg\forall x\neg\psi \\ \iff &\forall x\neg\psi \vdash \forall x\neg\varphi \\ \iff &\vdash \forall x\neg\psi \to \forall x\neg\varphi \end{aligned}$$

Note that $\varphi \vdash \psi$ implies $\vdash \varphi \to \psi$. So $\vdash \neg\psi \to \neg\varphi$ (rule T) and hence $\vdash \forall x(\neg\psi \to \neg\varphi)$ by generalization theorem. Then from axiom group 3, one has $\vdash \forall x(\neg\psi \to \neg\varphi) \to \forall x\neg\psi \to \forall x\neg\varphi$ and then we can use MP to deduce $\vdash \forall x\neg\psi \to \forall x\neg\varphi$. Now, reversing the iff of the above, we get what we want.

(b) Let $\delta_k = \bigwedge_{1\leq i<j\leq k} x_i \not\approx x_j$. Now, notice that

$$\delta_{k+1} = \delta_k \wedge \bigwedge_{1\leq i\leq k} x_i \not\approx x_{k+1}$$

and that we have $\alpha \wedge \beta \to \alpha$, so we can deduce

$$\delta_{k+1} \vdash \delta_k$$

Applying the above result, we have:

$$\exists x_{k+1}\delta_{k+1} \vdash \exists x_{k+1}\delta_k.$$

Since x_{k+1} does not appear in δ_k, we also have

$$\exists x_{k+1}\delta_k \vdash\dashv \delta_k.$$

(The above is equivalent to $\neg\forall x_{k+1}\neg\delta_k \vdash\dashv \delta_k$. By contrapositive, we get its equivalence $\forall x_{k+1}\neg\delta_k \vdash\dashv \neg\delta_k$. This is true by generalization theorem and the fact that x does not occur free in δ_k.) From this we get

$$\exists x_{k+1}\delta_{k+1} \vdash \delta_k.$$

Now, applying the above with φ and ψ being the LHS and RHS of the above k times, we get

$$\lambda_{k+1} \vdash \lambda_k.$$

Question 3

Suppose that $|\mathfrak{A}| = \{a_1, a_2, ..., a_n\}$. Let

$$\begin{aligned} \tau := (&\bigwedge_{1\leq i<j\leq k} x_i \not\approx x_j \wedge \forall y \bigwedge_{i=1}^{n} y \approx x_i \\ &\wedge \bigwedge_{(a_i,a_j)\in P^{\mathfrak{A}}} Px_ix_j \wedge \bigwedge_{(a_i,a_j)\notin P^{\mathfrak{A}}} \neg Px_ix_j \\ &\wedge \bigwedge_{f^{\mathfrak{A}}(a_i)=a_j} fx_i \approx x_j \wedge \bigwedge_{f^{\mathfrak{A}}(a_i)\neq a_j} fx_i \not\approx x_j) \end{aligned}$$

Consider the following sentence:

$$\sigma = \exists x_1 \exists x_2 ... \exists x_n \tau$$

This sentence describes fully the structure $\mathfrak{A}$ and it is true in $\mathfrak{A}$ if one assign $x_i \mapsto a_i$. Since $\mathfrak{A} \equiv \mathfrak{B}$, it must be the case that $\models_{\mathfrak{B}} \sigma$. Notice that this validity will be unravelled to $\models_{\mathfrak{B}} \tau[s]$ for some assignment s.

Consider the mapping: $h : |\mathfrak{A}| \to |\mathfrak{B}|, a_i \mapsto x_i \mapsto s(x_i)$. We claim that this map is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

Question 4

(a) We call a formula whose connectives are all from C to be C-formula. We prove this by induction on the C-formula α.

- Base case: If α is a sentential symbol then $G_\alpha := \alpha$ and so, $G_\alpha(F) = F \leq G_\alpha(T) = T$. If $\alpha = \top$ or $\alpha = \bot$ then G_α is constant and hence, clearly monotonic.

- Induction: Suppose that α, β are C-formula and that G_α, G_β are monotonic. We need to show that $G_{\alpha\wedge\beta}$ and $G_{\alpha\vee\beta}$ are also monotonic. First, we extend G_α and G_β to $\bar{G}_\alpha$ and $\bar{G}_\beta$ which are the same Boolean function but cover all the variables that appear in both α and β. These functions are still monotonic.
  Then $G_{\alpha\wedge\beta} = \bar{G}_\alpha \wedge \bar{G}_\beta$ and $G_{\alpha\vee\beta} = \bar{G}_\alpha \vee \bar{G}_\beta$. We can clearly check these cases.

So by principle of induction, we conclude that if α is a C-formula then G_α is monotonic.

(b) We will prove this by induction again. In this problem, I shall use the notation $\bar{A}$ where A is a sentential symbol to denote the values assigned to A instead of $v(A)$. Also we will abuse the notation by using $\alpha \wedge \beta$ instead of $v(\alpha \wedge \beta)$.

- Base case $n = 0$: If $n = 0$ then f is constant function and it can be realize by either $\top$ or $\bot$ depending on its values.
- Induction: Suppose that the property holds for all monotonic Boolean function of $n = k$ variables. Consider the case of a monotonic function f with $k + 1$ variables $A_1, A_2, ..., A_{k+1}$. Note that $f(A_1, ..., A_k, T)$ and $f(A_1, A_2, ..., A_k, F)$ are monotonic functions with k variables and hence, by induction hypothesis, should be realizable by C-formulae α and β respectively. We claim that the formula $\gamma = (\alpha \wedge A_{k+1}) \vee \beta$ realizes f i.e $f \equiv G_\gamma$ as functions. Note that $G_\gamma(\bar{A}_1, \bar{A}_2, ..., \bar{A}_{k+1}) = (f(\bar{A}_1, \bar{A}_2, ...\bar{A}_k, T) \wedge \bar{A}_{k+1}) \vee f(\bar{A}_1, \bar{A}_2, ...\bar{A}_k, F)$.
  We have two cases. If $\bar{A}_{k+1} = F$ then $f(\bar{A}_1, \bar{A}_2, ...\bar{A}_k, T) \wedge \bar{A}_{k+1} = F$ and hence, G_γ and f agree with each other. If $\bar{A}_{k+1} = T$ then if $f(\bar{A}_1, \bar{A}_2, ...\bar{A}_k, F) = F$, we have what we want ($G_\gamma$ and f agrees). If not, then due to monotonicity, we must have $f(\bar{A}_1, \bar{A}_2, ...\bar{A}_k, F) = f(\bar{A}_1, \bar{A}_2, ...\bar{A}_k, T) = T$ and so G_γ and f also agree. Hence, in any case, G_γ and f are identical (as boolean functions). This proves that γ realizes f.

So by principle of mathematical induction, any monotonic function is realizable by a C-formula.

(c) First, C is incomplete. The reason is that $\neg$ is not monotonic and hence, is not expressible by a C-formula due to the earlier part. To prove its maximal incompleteness, we need to prove that given any g not realizable, $C \cup \{g\}$ can realize the negation.

Suppose that g is not realizable. Then g is not monotonic. That is to say there are truth-values (i.e T or F) $x_1, x_2, ..., x_{i-1}, x_i, ..., x_n$ such that $g(x_1, ..., x_{i-1}, F, x_i, ..., x_n) = T$ and $g(x_1, ..., x_{i-1}, T, x_i, ..., x_n) = F$. Hence, $g\,\overline{x_1}\,\overline{x_2}...\alpha\overline{x_{i+1}}...\overline{x_n}$ realizes $\neg\alpha$ where $\overline{x_j} = \top$ if $x_j = T$ and $\overline{x_j} = \bot$ if $x_j = F$. So $C \cup \{g\}$ is complete.

Question 5

First, we add to the language the new constants symbols which are labelled by the rational numbers $\{c_r : r \in \mathbb{Q}\}$. Then consider the set of sentences

$$\Sigma = \mathrm{Th}\mathfrak{A} \cup \{c_r < c_s : r, s \in \mathbb{Q} \,\&\, r < s\}.$$

Σ is finitely satisfiable: if Σ_0 is a finite subset of Σ then it contains finitely many sentences of form $c_r < c_s$. Let all the rational labels appearing in Σ_0 to be $r_1 < r_2 < ... < r_k$. Then $\mathfrak{A}$ together with the interpretation $c_{r_i} \mapsto i$ for $i = 1..k$ and $c_q \mapsto 0$ if $q \notin \{r_1, ..., r_k\}$ satisfy Σ_0.

By Compactness Theorem, Σ is satisfiable. Let $\mathfrak{B}$ be a model for Σ in which we ignore all the interpretations of constants. Then $\mathfrak{B} \equiv \mathfrak{A}$ because $\mathfrak{B}$ is a model for Th$\mathfrak{A}$. Also, we can embed $\mathbb{Q}$ into $\mathfrak{B}$ by the injection such that:

$$r \mapsto c_r^{\mathfrak{B}}.$$

This embedding preserves ordering of $\mathbb{Q}$ due to the requirement $c_r < c_s$ for $r < s$. We proved the assertion.