NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA3111 Complex Analysis I
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Question 1

(a) Let $z = x + iy$ and $f(z = x + iy) = u(x, y) + iv(x, y)$, where x and y represent the real and imaginary parts of z respectively, and u and v are real-valued functions in x and y. Then we have

$$\begin{aligned} f(z) &= |z|^2 + i(\operatorname{Re} z)^2 + i(\operatorname{Im} z)^3 + 4\bar{z} \\ &= x^2 + y^2 + ix^2 + iy^3 + 4(x - iy) \\ &= x^2 + 4x + y^2 + i(x^2 + y^3 - 4y). \end{aligned}$$

This implies that $u(x, y) = x^2 + 4x + y^2$ and $v(x, y) = x^2 + y^3 - 4y$, from which we would get

$$u_x = 2x + 4, \quad u_y = 2y, \quad v_x = 2x \quad \text{and} \quad v_y = 3y^2 - 4.$$

Notice that the partial derivatives u_x, u_y, v_x and v_y are continuous on $\mathbb{C}$. Thus, at the points where f is differentiable, it must satisfy the Cauchy-Riemann equations. Therefore, we have

$$\begin{cases} u_x = v_y, \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} 2x + 4 = 3y^2 - 4, \\ 2y = -2x \end{cases} \Rightarrow \begin{cases} 3y^2 + 2y - 8 = 0. \end{cases}$$

This gives us $y = -2$ or $y = \frac{4}{3}$, or equivalently, $z = 2 - 2i$ or $z = -\frac{4}{3} + \frac{4}{3}i$.
At $z = 2 - 2i$, we have

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= [2(2) + 4] + i[2(2)] = 8 + 4i. \end{aligned}$$

At $z = -\frac{4}{3} + \frac{4}{3}i$, we have

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= \left[2\left(-\frac{4}{3}\right) + 4\right] + i\left[2\left(-\frac{4}{3}\right)\right] = \frac{4}{3} - \frac{8}{3}i. \end{aligned}$$

(b) We shall prove that the given inequality is true. By the Estimation Lemma, we have

$$\begin{aligned} \left| \int_\gamma \frac{(\bar{z}^2 + 5)e^{iz}}{e^{\bar{z}} - z} \, dz \right| &\leq \ell(\gamma) \cdot \sup_{z \in \gamma} \left| \frac{(\bar{z}^2 + 5)e^{iz}}{e^{\bar{z}} - z} \right| \\ &\leq \ell(\gamma) \cdot \sup_{z \in \gamma} \left| \bar{z}^2 + 5 \right| \cdot \sup_{z \in \gamma} \left| e^{iz} \right| \cdot \sup_{z \in \gamma} \left| \frac{1}{e^{\bar{z}} - z} \right| \\ &= \ell(\gamma) \cdot \sup_{z \in \gamma} \left| \bar{z}^2 + 5 \right| \cdot \sup_{z \in \gamma} \left| e^{iz} \right| \cdot \frac{1}{\inf_{z \in \gamma} \left| e^{\bar{z}} - z \right|}, \end{aligned}$$

where $\ell(\gamma)$ denotes the length of the closed contour γ. Now, we have

$$\begin{aligned}
\ell(\gamma) &= 2(2+3) = 10, \\
\sup_{z\in\gamma}\left|\bar{z}^2+5\right| &\leq \sup_{z\in\gamma}|z|^2+5 \\
&= |4+3i|^2+5 = 30, \\
\sup_{z\in\gamma}\left|e^{iz}\right| &= \sup_{z\in\gamma}\left|e^{i(\operatorname{Re} z)-\operatorname{Im} z}\right| \\
&= \sup_{z\in\gamma}\left(\left|e^{i(\operatorname{Re} z)}\right|\cdot\left|e^{-\operatorname{Im} z}\right|\right) \\
&= 1\cdot\left|e^{-0}\right| = 1, \\
\inf_{z\in\gamma}\left|e^{\bar{z}}-z\right| &\geq \inf_{z\in\gamma}\left(\left|e^{\bar{z}}\right|-|z|\right) \\
&= \inf_{z\in\gamma}\left|e^{\bar{z}}\right| - \sup_{z\in\gamma}|z| \\
&= \inf_{z\in\gamma}\left|e^{\operatorname{Re} z-i(\operatorname{Im} z)}\right| - |4+3i| \\
&= \inf_{z\in\gamma}\left(\left|e^{-i(\operatorname{Im} z)}\right|\cdot\left|e^{\operatorname{Re} z}\right|\right) - \sqrt{4^2+3^2} \\
&= 1\cdot\left|e^2\right| - 5 \geq \frac{15}{7} \\
\Rightarrow \frac{1}{\inf_{z\in\gamma}\left|e^{\bar{z}}-z\right|} &\leq \frac{7}{15}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\left|\int_\gamma \frac{(\bar{z}^2+5)e^{iz}}{e^{\bar{z}}-z}\,dz\right| &\leq \ell(\gamma)\cdot\sup_{z\in\gamma}\left|\bar{z}^2+5\right|\cdot\sup_{z\in\gamma}\left|e^{iz}\right|\cdot\frac{1}{\inf_{z\in\gamma}\left|e^{\bar{z}}-z\right|} \\
&\leq 10\cdot 30\cdot 1\cdot\frac{7}{15} = 140,
\end{aligned}$$

thereby proving the given inequality as desired.

Question 2

(a)

$$\begin{aligned}
e^{3z}+8\sinh z &= 4\cos(iz) \\
\Rightarrow e^{3z}+8\left[\frac{e^z-e^{-z}}{2}\right] &= 4\left[\frac{e^z+e^{-z}}{2}\right] \\
\Rightarrow e^{-z}\left(e^{4z}+2e^{2z}-6\right) &= 0 \\
\Rightarrow \left(e^{2z}\right)^2+2e^{2z}-6 &= 0 \quad (\because\ e^{-z}\neq 0\ \forall\ z\in\mathbb{C}) \\
\Rightarrow e^{2z} &= \frac{-2\pm\sqrt{2^2-4(1)(-6)}}{2} = -1\pm\sqrt{7}.
\end{aligned}$$

When $e^{2z} = -1+\sqrt{7}$, we have

$$\begin{aligned}
2z &= \operatorname{Log}\left(-1+\sqrt{7}\right) + i\left[\operatorname{Arg}\left(-1+\sqrt{7}\right)+2n\pi\right] \\
&= \ln\left|-1+\sqrt{7}\right| + i(0+2n\pi) \\
&= \ln\left(-1+\sqrt{7}\right) + 2n\pi i \\
\Rightarrow z &= \frac{1}{2}\ln\left(-1+\sqrt{7}\right) + n\pi i.
\end{aligned}$$

When $e^{2z} = -1 - \sqrt{7}$, we have

$$\begin{aligned} 2z &= \operatorname{Log}\left(-1-\sqrt{7}\right) + i\left[\operatorname{Arg}\left(-1-\sqrt{7}\right) + 2n\pi\right] \\ &= \ln\left|-1-\sqrt{7}\right| + i(\pi + 2n\pi) \\ &= \ln\left(1+\sqrt{7}\right) + (2n+1)\pi i \\ \Rightarrow z &= \frac{1}{2}\ln\left(1+\sqrt{7}\right) + \frac{(2n+1)\pi i}{2}. \end{aligned}$$

Thus, the solutions are $z = \frac{1}{2}\ln\left(-1+\sqrt{7}\right) + n\pi i$ or $z = \frac{1}{2}\ln\left(1+\sqrt{7}\right) + \frac{(2n+1)\pi i}{2}$.

(b) If $f = 0$, then it is clear that f is an entire function that satisfies the given inequality. Otherwise, we may assume without loss of generality that $f \neq 0$. Let g and h be functions such that $g(z) = [f(z)]^2$ and $h(z) = e^{g(z)}$ for all $z \in \mathbb{C}$. From the given inequality, we have for all $z \in \mathbb{C}$,

$$\begin{aligned} [\operatorname{Re}(f(z))]^2 &\leq [\operatorname{Im}(f(z))]^2 + 1 \\ \Rightarrow [\operatorname{Re}(f(z))]^2 &- [\operatorname{Im}(f(z))]^2 \leq 1. \end{aligned}$$

Thus, we have for all $z \in \mathbb{C}$,

$$\begin{aligned} |h(z)| &= \left|e^{g(z)}\right| \\ &= \left|e^{[f(z)]^2}\right| \\ &= \left|e^{[\operatorname{Re}(f(z))+i[\operatorname{Im}(f(z))]]^2}\right| \\ &= \left|e^{(\operatorname{Re}(f(z)))^2-(\operatorname{Im}(f(z)))^2}\right| \cdot \left|e^{2i[\operatorname{Re}(f(z))\operatorname{Im}(f(z))]}\right| \\ &\leq \left|e^1\right| \cdot 1 = e. \end{aligned}$$

This implies that h is bounded.

Also, it is clear that h is entire. Hence h is constant by the Liouville's Theorem.
Let $h(z) = c$ for all $z \in \mathbb{C}$, where $c \in \mathbb{C}$. Then it follows that

$$e^{g(z)} = h(z) = c. \tag{1}$$

By differentiating both sides of equation (1) with respect to z, we have

$$g'(z)e^{g(z)} = 0.$$

Since $e^z \neq 0$ for all $z \in \mathbb{C}$, it follows that $e^{g(z)} \neq 0$ for all $z \in \mathbb{C}$.
Thus, we must have $g'(z) = 0$ for all $z \in \mathbb{C}$, which implies that g must be constant. Let $g(z) = k$ for all $z \in \mathbb{C}$, where $k \in \mathbb{C}$. Then we have

$$[f(z)]^2 = g(z) = k. \tag{2}$$

If $k = 0$, then this would imply that $f(z) = 0$ for all $z \in \mathbb{C}$, which is a contradiction. Hence $k \neq 0$, which would imply that $f(z) \neq 0$ for all $z \in \mathbb{C}$. By differentiating both sides of equation (2) with respect to z, we have

$$2f'(z)f(z) = 0.$$

Since $f(z) \neq 0$ for all $z \in \mathbb{C}$, we must have $f'(z) = 0$ for all $z \in \mathbb{C}$. So f is constant.

Question 3

(i) Firstly, we note that $\frac{7z-5}{(2z-3)(z+4)} = \frac{1}{2z-3} + \frac{3}{z+4}$.
In the domain $|z| > \frac{3}{2}$, we have

$$\begin{aligned}\frac{1}{2z-3} &= \frac{1}{2z} \cdot \frac{1}{1-\frac{3}{2z}} \\ &= \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{3}{2z}\right)^n = \sum_{n=0}^{\infty}\left[\frac{3^n}{2^{n+1}} \cdot \frac{1}{z^{n+1}}\right].\end{aligned}$$

In the domain $|z| > 4$, we have

$$\begin{aligned}\frac{3}{z+4} &= \frac{3}{z} \cdot \frac{1}{1-(-\frac{4}{z})} \\ &= \frac{3}{z}\sum_{n=0}^{\infty}\left(-\frac{4}{z}\right)^n = \sum_{n=0}^{\infty}\left[3(-4)^n \cdot \frac{1}{z^{n+1}}\right].\end{aligned}$$

Hence, in the domain $|z| > 4$, we have

$$\begin{aligned}\frac{7z-5}{(2z-3)(z+4)} &= \frac{1}{2z-3} + \frac{3}{z+4} \\ &= \sum_{n=0}^{\infty}\left[\frac{3^n}{2^{n+1}} \cdot \frac{1}{z^{n+1}}\right] + \sum_{n=0}^{\infty}\left[3(-4)^n \cdot \frac{1}{z^{n+1}}\right] \\ &= \sum_{n=0}^{\infty}\left(\frac{3^n}{2^{n+1}} + 3(-4)^n\right)\frac{1}{z^{n+1}}.\end{aligned}$$

(ii) In the domain $|z+1|^2 > 4$ (or equivalently $|z+1| > 2$), we have

$$\begin{aligned}\frac{7z^2+14z+2}{(2z^2+4z-1)(z^2+4z+5)} &= \frac{7(z+1)^2-5}{(2(z+1)^2-3)\left((z+1)^2+4\right)} \\ &= \sum_{n=0}^{\infty}\left(\frac{3^n}{2^{n+1}} + 3(-4)^n\right)\frac{1}{[(z+1)^2]^{n+1}} \\ &= \sum_{n=0}^{\infty}\left(\frac{3^n}{2^{n+1}} + 3(-4)^n\right)\frac{1}{(z+1)^{2n+2}}.\end{aligned}$$

(iii) Define the function f on $\mathbb{C}\backslash\{1\}$ as follows:

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{3^n}{2^{n+1}} + 3(-4)^n\right)\frac{1}{(z+1)^{2n+2}}.$$

Notice that for the domain $|z+1| > 2$, we have

$$\frac{7z^2+14z+2}{(2z^2+4z-1)(z^2+4z+5)} = f(z).$$

Also, we note that $\overline{B(-1,2)} \subset \overline{B(0,4)}$, where $\overline{B(a,r)}$ denotes the closed ball centred at a with radius r. This implies that the circle $|z| = 4$ is in the domain $|z+1| > 2$.
Hence, on the circle $|z| = 4$, we must have

$$\frac{7z^2+14z+2}{(2z^2+4z-1)(z^2+4z+5)} = f(z).$$

This implies that

$$\int_\gamma \frac{\left[2(z+1)^3+(z+1)\right](7z^2+14z+2)}{(2z^2+4z-1)(z^2+4z+5)}\,dz = \int_\gamma \left[2(z+1)^3+(z+1)\right] f(z)\,dz. \qquad (3)$$

Notice that the only isolated singularity of f occurs at $z=-1$, and the singularity lies inside the open ball $B(0,4)$. Thus, it follows from Cauchy's Residue Theorem that

$$\int_\gamma \left[2(z+1)^3+(z+1)\right] f(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1}[2(z+1)^3+(z+1)]f(z).$$

To find the value of $\operatorname*{Res}_{z=-1}[2(z+1)^3+(z+1)]f(z)$, we have to find the coefficient of $\frac{1}{z+1}$ in the Laurent series expansion of $[2(z+1)^3+(z+1)]f(z)$ at $z=-1$. We have

$$\begin{aligned}
[2(z+1)^3+(z+1)]f(z) &= [2(z+1)^3+(z+1)]\sum_{n=0}^{\infty}\left(\frac{3^n}{2^{n+1}}+3(-4)^n\right)\frac{1}{(z+1)^{2n+2}} \\
&= \sum_{n=0}^{\infty}\left(\frac{3^n}{2^n}+6(-4)^n\right)\frac{1}{(z+1)^{2n-1}} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{3^n}{2^{n+1}}+3(-4)^n\right)\frac{1}{(z+1)^{2n+1}} \\
&= 7(z+1)-\frac{19}{z+1}+\sum_{n=1}^{\infty}\left(\frac{3^n}{2^{n-1}}-21(-4)^n\right)\frac{1}{(z+1)^{2n+1}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
& \operatorname*{Res}_{z=-1}[2(z+1)^3+(z+1)]f(z) \\
= \; & \text{coefficient of } \frac{1}{z+1} \text{ in Laurent series expansion of } [2(z+1)^3+(z+1)]f(z) \text{ at } z=-1 \\
= \; & -19.
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
\int_\gamma \frac{\left[2(z+1)^3+(z+1)\right](7z^2+14z+2)}{(2z^2+4z-1)(z^2+4z+5)}\,dz &= \int_\gamma \left[2(z+1)^3+(z+1)\right] f(z)\,dz \\
&= 2\pi i \operatorname*{Res}_{z=-1}[2(z+1)^3+(z+1)]f(z) \\
&= 2\pi i(-19) = -38\pi i.
\end{aligned}$$

Question 4

Firstly, we note that

$$P.V. \int_{-\infty}^{\infty} \frac{\sin(x+\alpha)}{x^2-2x+5}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{\sin(x+\alpha)}{x^2-2x+5}\,dx.$$

Next, we also note that

$$\begin{aligned}
e^{i(x+\alpha)} &= \cos(x+\alpha)+i\sin(x+\alpha) \\
\Rightarrow \int_{-R}^{R}\frac{e^{i(x+\alpha)}}{x^2-2x+5}\,dx &= \int_{-R}^{R}\frac{\cos(x+\alpha)}{x^2-2x+5}\,dx + i\int_{-R}^{R}\frac{\sin(x+\alpha)}{x^2-2x+5}\,dx.
\end{aligned}$$

Notice that the singularities of the function $\frac{e^{i(z+\alpha)}}{z^2-2z+5}$ coincide with the zeroes of the denominator z^2-2z+5, i.e. at the points z where

$$\begin{aligned} z^2-2z+5 &= 0 \\ \Rightarrow z^2-2z+1 &= -4 \\ \Rightarrow (z-1)^2 &= 4i^2 \end{aligned}$$

This gives us $z=1+2i$ or $z=1-2i$. Thus, the singularities of the function $\frac{e^{i(z+\alpha)}}{z^2-2z+5}$ occur at the points $z=1+2i$ and $z=1-2i$.
Consider the closed contour γ, consisting of the straight line from $z=-R$ to $z=R$, and the arc C_R with the parameterization $z=Re^{it}$, $0 \le t \le \pi$, where $R>4$.
Notice that the only singularity inside the contour γ is $z=1+2i$, and that the singularity at $z=1+2i$ is isolated. Thus, by Cauchy's Residue Theorem, we have

$$\int_\gamma \frac{e^{i(z+\alpha)}}{z^2-2z+5}\,dz = \int_{C_R} \frac{e^{i(z+\alpha)}}{z^2-2z+5}\,dz + \int_{-R}^{R} \frac{e^{i(z+\alpha)}}{z^2-2z+5}\,dz = 2\pi i \operatorname*{Res}_{z=1+2i}\left(\frac{e^{i(z+\alpha)}}{z^2-2z+5}\right).$$

Now, we have

$$\begin{aligned} \lim_{z\to 1+2i}[z-(1+2i)]\left(\frac{e^{i(z+\alpha)}}{z^2-2z+5}\right) &= \lim_{z\to 1+2i}\left(\frac{e^{i(z+\alpha)}}{[z-(1-2i)]}\right) \\ &= \frac{e^{i(1+2i+\alpha)}}{[1+2i-(1-2i)]} \\ &= \frac{1}{4i}e^{-2+i(1+\alpha)} \neq 0. \end{aligned}$$

This implies that the isolated singularity at $z=1+2i$ is a simple pole. Thus, we have

$$\begin{aligned} \operatorname*{Res}_{z=1+2i}\left(\frac{e^{i(z+\alpha)}}{z^2-2z+5}\right) &= \lim_{z\to 1+2i}[z-(1+2i)]\left(\frac{e^{i(z+\alpha)}}{z^2-2z+5}\right) \\ &= \frac{1}{4i}e^{-2+i(1+\alpha)} \\ \Rightarrow \int_{C_R} \frac{e^{i(z+\alpha)}}{z^2-2z+5}\,dz + \int_{-R}^{R} \frac{e^{i(z+\alpha)}}{z^2-2z+5}\,dz &= 2\pi i \operatorname*{Res}_{z=1+2i}\left(\frac{e^{i(z+\alpha)}}{z^2-2z+5}\right) \\ &= 2\pi i\left(\frac{1}{4i}e^{-2+i(1+\alpha)}\right) \\ &= \frac{\pi}{2}e^{-2+i(1+\alpha)}. \end{aligned}$$

Next, we have to estimate the value of the following integral:

$$\int_{C_R} \frac{e^{i(z+\alpha)}}{z^2-2z+5}\,dz.$$

By the Estimation Lemma, we have

$$\begin{aligned} \left|\int_{C_R} \frac{e^{i(z+\alpha)}}{z^2-2z+5}\,dz\right| &\le \ell(C_R)\cdot \sup_{z\in C_R}\left|\frac{e^{i(z+\alpha)}}{z^2-2z+5}\right| \\ &\le \ell(C_R)\cdot \sup_{z\in C_R}\left|e^{i\alpha}e^{iz}\right| \cdot \sup_{z\in C_R}\left|\frac{1}{z^2-2z+5}\right| \\ &\le \ell(C_R)\cdot \sup_{z\in C_R}\left|e^{i\alpha}\right| \cdot \sup_{z\in C_R}\left|e^{iz}\right| \cdot \frac{1}{\inf_{z\in C_R}|z^2-2z+5|} \\ &\le \ell(C_R)\cdot \sup_{z\in C_R}\left|e^{iz}\right| \cdot \frac{1}{\inf_{z\in C_R}|z^2-2z+5|}, \quad (\because \left|e^{i\alpha}\right| = 1\ \forall\ \alpha\in\mathbb{R}) \end{aligned}$$

where $\ell(C_R)$ denotes the length of the arc C_R. Now, we have

$$\begin{aligned}
\ell(C_R) &= \pi R, \\
\sup_{z\in C_R} \left|e^{iz}\right| &= \sup_{t\in[0,\pi]} \left|e^{i\left(Re^{it}\right)}\right| \\
&= \sup_{t\in[0,\pi]} \left|e^{i(R(\cos t + i\sin t))}\right| \\
&= \sup_{t\in[0,\pi]} \left(\left|e^{i(R\cos t)}\right| \cdot \left|e^{-R\sin t}\right|\right) \leq 1 \cdot \left|e^{-R(0)}\right| = 1.
\end{aligned}$$

Also, for all $z \in C_R$, we have

$$\begin{aligned}
\left|z^2\right| &= |z|^2 = R^2 > 2R = 2|z| = |2z|, \quad (\because R > 4) \\
\Rightarrow \left|z^2 - 2z\right| &\geq |z|^2 - 2|z| \quad (\because \left|z^2\right| > |2z|) \\
&= R(R-2) > 4 \cdot 2 > 5,
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \left|z^2 - 2z + 5\right| &\geq \left|\left|z^2 - 2z\right| - |-5|\right| \\
&= \left|z^2 - 2z\right| - 5 \quad (\because \left|z^2 - 2z\right| > 5) \\
&\geq |z|^2 - 2|z| - 5 = R^2 - 2R - 5 > 0 \quad (\because R(R-2) > 5) \\
\Rightarrow \inf_{z\in C_R} \left|z^2 - 2z + 5\right| &\geq R^2 - 2R - 5 > 0 \\
\Rightarrow \frac{1}{\inf_{z\in C_R} |z^2 - 2z + 5|} &\leq \frac{1}{R^2 - 2R - 5}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\left|\int_{C_R} \frac{e^{i(z+\alpha)}}{z^2 - 2z + 5}\, dz\right| &\leq \ell(C_R) \cdot \sup_{z\in C_R} \left|e^{iz}\right| \cdot \frac{1}{\inf_{z\in C_R} |z^2 - 2z + 5|} \\
&\leq \pi R \cdot 1 \cdot \frac{1}{R^2 - 2R - 5} = \frac{\pi R}{R^2 - 2R - 5}.
\end{aligned}$$

Since $\lim_{R\to\infty} \frac{\pi R}{R^2-2R-5} = 0$, it follows from the Squeeze Theorem that $\lim_{R\to\infty} \left|\int_{C_R} \frac{e^{i(z+\alpha)}}{z^2-2z+5}\, dz\right| = 0$, or equivalently, $\lim_{R\to\infty} \int_{C_R} \frac{e^{i(z+\alpha)}}{z^2-2z+5}\, dz = 0$. Thus, we get

$$\begin{aligned}
\int_{C_R} \frac{e^{i(z+\alpha)}}{z^2 - 2z + 5}\, dz + \int_{-R}^{R} \frac{e^{i(z+\alpha)}}{z^2 - 2z + 5}\, dz &= \frac{\pi}{2} e^{-2+i(1+\alpha)} \\
\Rightarrow \lim_{R\to\infty} \left(\int_{C_R} \frac{e^{i(z+\alpha)}}{z^2 - 2z + 5}\, dz + \int_{-R}^{R} \frac{e^{i(z+\alpha)}}{z^2 - 2z + 5}\, dz\right) &= \lim_{R\to\infty} \frac{\pi}{2} e^{-2+i(1+\alpha)} \\
\Rightarrow \lim_{R\to\infty} \int_{C_R} \frac{e^{i(z+\alpha)}}{z^2 - 2z + 5}\, dz + \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i(z+\alpha)}}{z^2 - 2z + 5}\, dz &= \frac{\pi}{2} e^{-2+i(1+\alpha)} \\
\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i(z+\alpha)}}{z^2 - 2z + 5}\, dz &= \frac{\pi}{2} e^{-2+i(1+\alpha)}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{i(z+\alpha)}}{z^2-2z+5}\,dz &= \frac{\pi}{2}e^{-2+i(1+\alpha)}\\
&= \frac{\pi}{2e^2}e^{i(1+\alpha)}\\
&= \frac{\pi}{2e^2}\left(\cos(1+\alpha)+i\sin(1+\alpha)\right)\\
\Rightarrow \lim_{R\to\infty}\int_{-R}^{R}\frac{\sin(z+\alpha)}{z^2-2z+5}\,dz &= \lim_{R\to\infty}\operatorname{Im}\left(\int_{-R}^{R}\frac{e^{i(z+\alpha)}}{z^2-2z+5}dz\right)\\
&= \operatorname{Im}\left(\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{i(z+\alpha)}}{z^2-2z+5}\,dz\right)\\
&= \frac{\pi}{2e^2}\sin(1+\alpha)\\
\Rightarrow P.V.\int_{-\infty}^{\infty}\frac{\sin(x+\alpha)}{x^2-2x+5}\,dx &= \frac{\pi}{2e^2}\sin(1+\alpha).
\end{aligned}$$

Question 5

(a) (i)

$$\begin{aligned}
u(x,y) &= 8x^2+kx^3y+\ell y^2(xy+1)\\
\Rightarrow u_x &= 16x+3kx^2y+\ell y^3 \quad\text{and}\quad u_y = kx^3+\ell y(3xy+2)\\
\Rightarrow u_{xx} &= 16+6kxy \quad\text{and}\quad u_{yy} = 6\ell xy+2\ell.
\end{aligned}$$

Since u is a harmonic function, it must satisfy the Laplace's equation for all $(x,y)\in\mathbb{R}^2$. Thus,

$$\begin{aligned}
u_{xx}+u_{yy} &= 0\\
\Rightarrow 16+6kxy+6\ell xy+2\ell &= 0\\
\Rightarrow 16+2\ell+6(k+\ell)xy &= 0\\
\Rightarrow 16+2\ell = 0 \quad\text{and}\quad k+\ell &= 0\\
\Rightarrow k=8 \quad\text{and}\quad \ell &= -8.
\end{aligned}$$

(ii) Let a harmonic conjugate of u be v. Then it must satisfy the following equations:

$$v_x = -u_y, \tag{4}$$

$$v_y = u_x. \tag{5}$$

From equation (4), we have

$$\begin{aligned}
v_x &= -u_y\\
&= -(kx^3+\ell y(3xy+2))\\
&= -(8x^3-8y(3xy+2)) = -8x^3+24xy^2+16y\\
\Rightarrow v(x,y) &= \int -8x^3+24xy^2+16y\,dx
\end{aligned}$$

$$\Rightarrow v(x,y) = -2x^4+12x^2y^2+16xy+g(y), \tag{6}$$

where g is a twice continuously differentiable function in y. Now, by differentiating both sides equation (6) with respect to y, we have

$$v_y = 24x^2y+16x+g'(y). \tag{7}$$

By equating both sides of equations (5) and (7), we have

$$\begin{aligned} v_y &= u_x \\ \Rightarrow 24x^2y + 16x + g'(y) &= 16x + 3kx^2y + \ell y^3 \\ &= 16x + 3(8)x^2y + (-8)y^3 \\ &= 16x + 24x^2y - 8y^3 \\ \Rightarrow g'(y) &= -8y^3 \\ \Rightarrow g(y) &= \int -8y^3\,dy \end{aligned}$$

$$\Rightarrow g(y) = -2y^4 + k. \tag{8}$$

Thus, by equating both sides of equation (6) and (8), we have

$$v(x,y) = -2x^4 + 12x^2y^2 + 16xy + g(y) = -2x^4 + 12x^2y^2 + 16xy - 2y^4 + k.$$

So a harmonic conjugate v of u is $v(x,y) = -2x^4 + 12x^2y^2 + 16xy - 2y^4$.

(b) Suppose such a function f exists.
Let g be the analytic function $g(z) = 1 + \operatorname{Log} z$ for all $z \in \mathbb{C}\backslash(-\infty, 0]$, and define the function h to be $h = f - g$ on $D\backslash(-3,-1) = D \cap \mathbb{C}\backslash(-\infty,0]$.
Then, it follows that $\operatorname{Re}[g(z)] = 1 + \ln|z| = \operatorname{Re}[f(z)]$, so we must have $\operatorname{Re}[h(z)] = \operatorname{Re}[f(z)] - \operatorname{Re}[g(z)] = 0$.
Also, since f is analytic on D and g is analytic on $\mathbb{C}\backslash(-\infty,0]$, it follows that h must be analytic on $D\backslash(-3,-1)$.
Thus, h must satisfy the Cauchy-Riemann equations for all $z \in D\backslash(-3,-1)$.
Write $h(z = x+iy) = \operatorname{Re}[h(z)] + i\operatorname{Im}[h(z)] = u(x,y) + iv(x,y)$, where u and v are functions in x and y. Then it follows that

$$\begin{gathered} u(x,y) = 0 \\ \Rightarrow u_x = u_y = 0 \\ v_x = -u_y = 0 \quad \text{and} \quad v_y = u_x = 0. \end{gathered}$$

This implies that v is a function independent of x and y, so v must be a constant, whence this forces h to be a constant as well.
So we have $f(z) = g(z) + h(z) = 1 + \operatorname{Log} z + c = k + \operatorname{Log} z$ for some constants $c \in \mathbb{C}$ and $k \in \mathbb{C}$.
Let C_C and C_A be two paths in D with the following parameterizations:

$$C_C := \{2e^{it} \mid -\pi < t \le 0\}, \quad C_A := \{2e^{it} \mid 0 \le t < \pi\}.$$

Now, along the path C_C (and also along the path C_A), we have

$$\begin{aligned} f(z) &= k + \operatorname{Log} z \\ &= k + \operatorname{Log}\left(2e^{it}\right) \\ &= k + \ln\left|2e^{it}\right| + i\operatorname{Arg}\left(2e^{it}\right) = k + \ln 2 + it. \end{aligned}$$

As we approach the point $z = -2$ along the path C_C (i.e. $t \to -\pi$), it follows that $f(z) \to k + \ln 2 - i\pi$. Likewise, as we approach the point $z = -2$ along the path C_A (i.e. $t \to \pi$), it follows that $f(z) \to k + \ln 2 + i\pi$.
This implies that $\lim_{z \to -2} f(z)$ does not exist, so f is not continuous at $z = -2$, which contradicts the fact that f is analytic (and hence continuous) on D. So such a function f does not exist.

Question 6

(a) Firstly, note that if $z = 3 + 3e^{it}$, then it follows that $dz = 3ie^{it}\,dt$. Thus

$$\begin{aligned}
\int_\gamma \frac{z}{\bar{z}-3}\,dz &= \int_0^{\frac{\pi}{2}} \frac{3+3e^{it}}{\overline{3+3e^{it}}-3}\cdot 3ie^{it}\,dt \\
&= 3i\int_0^{\frac{\pi}{2}} \frac{3+3e^{it}}{3e^{-it}}\cdot e^{it}\,dt \\
&= 3i\int_0^{\frac{\pi}{2}} \left(e^{2it}+e^{3it}\right)\,dt \\
&= 3i\left[\frac{1}{2i}e^{2it}+\frac{1}{3i}e^{3it}\right]_0^{\frac{\pi}{2}} \quad \text{(By Fundamental Theorem of Calculus for Line Integrals)} \\
&= 3i\left[\frac{1}{2i}e^{2i(\frac{\pi}{2})}+\frac{1}{3i}e^{3i(\frac{\pi}{2})}\right] - 3i\left[\frac{1}{2i}e^{2i(0)}+\frac{1}{3i}e^{3i(0)}\right] = -4-i.
\end{aligned}$$

Also, when $t = 0$, we have $z = 3 + 3e^{i(0)} = 6$, and when $t = \frac{\pi}{2}$, we have $z = 3 + 3e^{i(\frac{\pi}{2})} = 3 + 3i$. Since $\frac{1}{z}$ has an analytic anti-derivative in $\mathbb{C}\backslash(-\infty, 0]$, by the Fundamental Theorem of Calculus for Line Integrals, we have

$$\begin{aligned}
\int_\gamma \frac{4}{\pi z}\,dz &= \frac{4}{\pi}\left[\operatorname{Log} z\right]_6^{3+3i} \\
&= \frac{4}{\pi}\left[\operatorname{Log}(3+3i)\right] - \frac{4}{\pi}\left[\operatorname{Log} 6\right] \\
&= \frac{4}{\pi}\left[\ln|3+3i| + i\operatorname{Arg}(3+3i)\right] - \frac{4}{\pi}\left[\ln|6| + i\operatorname{Arg}(6)\right] \\
&= \frac{4}{\pi}\left[\ln\left(\sqrt{3^2+3^2}\right) + i\left(\frac{\pi}{4}\right)\right] - \frac{4}{\pi}\left[\ln 6 + i\,(0)\right] \\
&= \frac{2}{\pi}\ln 18 + i - \frac{4}{\pi}\ln 6 = -\frac{2}{\pi}\ln 2 + i.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\int_\gamma \left(\frac{z}{\bar{z}-3} + \frac{4}{\pi z}\right) dz &= \int_\gamma \frac{z}{\bar{z}-3}\,dz + \int_\gamma \frac{4}{\pi z}\,dz \\
&= -4 - i - \frac{2}{\pi}\ln 2 + i = -4 - \frac{2}{\pi}\ln 2.
\end{aligned}$$

(b) Since f has a simple pole at w_0, there exists some entire function ϕ_1, such that $\phi_1(w_0) \neq 0$, and for all w near w_0, but not equal to w_0, one has

$$f(w) = \frac{\phi_1(w)}{w - w_0}. \tag{9}$$

From here it follows that $\operatorname*{Res}_{w=w_0} f(w) = \phi_1(w_0) \neq 0$.
Next, since g is entire, and

$$g(z_0) - w_0 = 0, \quad g'(z_0) = 0 \quad \text{and} \quad g''(z_0) \neq 0,$$

it follows that the function $g - w_0$ has a zero of order 2.

Thus, there exists some entire function ϕ_2, such that for all z near z_0, one has

$$\begin{aligned} g(z) - w_0 &= (z - z_0)^2\phi_2(z) \\ \Rightarrow (z - z_0)^2\phi_2(z) &= \sum_{n=0}^{\infty}\left(\frac{g^{(n)}(z_0)}{n!}(z - z_0)^n\right) - w_0 \\ &= \frac{g''(z_0)}{2}(z - z_0)^2 + \frac{g'''(z_0)}{6}(z - z_0)^3 + \sum_{n=4}^{\infty}\frac{g^{(n)}(z_0)}{n!}(z - z_0)^n \end{aligned}$$

$$\Rightarrow \phi_2(z) = \frac{g''(z_0)}{2} + \frac{g'''(z_0)}{6}(z - z_0) + \sum_{n=4}^{\infty}\frac{g^{(n)}(z_0)}{n!}(z - z_0)^{n-2} \tag{10}$$

$$\Rightarrow \phi_2'(z) = \frac{g'''(z_0)}{6} + \sum_{n=4}^{\infty}\frac{g^{(n)}(z_0)(n-2)}{n!}(z - z_0)^{n-3}. \tag{11}$$

By letting $z = z_0$ in equation (10), we get $\phi_2(z_0) = \frac{g''(z_0)}{2} \neq 0$.
By letting $z = z_0$ in equation (11), we get $\phi_2'(z_0) = \frac{g'''(z_0)}{6}$.
Then it follows that for all z near z_0 (but not equal to z_0), one has

$$\begin{aligned} h(z) &= f(g(z)) \\ &= \frac{\phi_1(g(z))}{(z - z_0)^2\phi_2(z)} = \frac{\phi_1(g(z))}{\phi_2(z)} \cdot \frac{1}{(z - z_0)^2}. \end{aligned}$$

As $\phi_2(z_0) \neq 0$, this implies that the function $\frac{\phi_1 \circ g}{\phi_2}$ is analytic at $z = z_0$, so it follows that the function h has a pole of order 2 at $z = z_0$.
This implies that the function φ defined near $z = z_0$ (but not equal to z_0), where $\varphi(z) := (z - z_0)^2h(z)$, has a removable singularity at z_0.
Thus, we may define $\varphi(z_0) = \lim\limits_{z\to z_0}(z - z_0)^2h(z)$, so that φ is analytic at $z = z_0$, and φ may be analytically extended across z_0 to the analytic function $\frac{\phi_1 \circ g}{\phi_2}$ on a neighbourhood containing z_0. Therefore, we have

$$\begin{aligned} \operatorname*{Res}_{z=z_0} h(z) &= \frac{\varphi^{(2-1)}(z_0)}{(2-1)!} \\ &= \varphi'(z_0) \\ &= \left(\frac{\phi_1 \circ g}{\phi_2}\right)'(z_0) \\ &= \frac{\phi_2(z_0)\cdot(\phi_1 \circ g)'(z_0) - \phi_2'(z_0)\cdot(\phi_1 \circ g)(z_0)}{(\phi_2(z_0))^2} \\ &= \frac{\phi_2(z_0)\phi_1'(g(z_0))g'(z_0) - \phi_2'(z_0)\phi_1(g(z_0))}{(\phi_2(z_0))^2} \\ &= \frac{\left(\frac{g''(z_0)}{2}\right)\cdot\phi_1'(w_0)\cdot 0 - \left(\frac{g'''(z_0)}{6}\right)\cdot\phi_1(w_0)}{\left(\frac{g''(z_0)}{2}\right)^2} \\ &= -\frac{2g'''(z_0)}{3\left(g''(z_0)\right)^2}\,\phi_1(w_0) = -\frac{2g'''(z_0)}{3\left(g''(z_0)\right)^2}\operatorname*{Res}_{z=w_0} f(z). \end{aligned}$$

Question 7

(a) Notice that the only (and thus isolated) singularity of the integrand $(z^2+4z)\sin\left(\frac{\pi z-8}{4z}\right)$ occurs at $z=0$, and is inside the closed contour γ. Thus, by Cauchy's Residue Theorem, we have

$$\int_\gamma (z^2+4z)\sin\left(\frac{\pi z-8}{4z}\right)\,dz = 2\pi i \operatorname*{Res}_{z=0}(z^2+4z)\sin\left(\frac{\pi z-8}{4z}\right). \tag{12}$$

Now, it remains to find the coefficient of $\frac{1}{z}$ in the Laurent series expansion of $(z^2+4z)\sin\left(\frac{\pi z-8}{4z}\right)$ at $z=0$.
We have

$$\begin{aligned}
(z^2+4z)\sin\left(\frac{\pi z-8}{4z}\right) &= (z^2+4z)\sin\left(\frac{\pi}{4}-\frac{2}{z}\right)\\
&= (z^2+4z)\left(\sin\frac{\pi}{4}\cos\frac{2}{z}-\cos\frac{\pi}{4}\sin\frac{2}{z}\right)\\
&= (z^2+4z)\left(\frac{\sqrt{2}}{2}\cos\frac{2}{z}-\frac{\sqrt{2}}{2}\sin\frac{2}{z}\right)\\
&= \frac{\sqrt{2}}{2}(z^2+4z)\left(\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^{2n}}{(2n)!}\cdot\frac{1}{z^{2n}}-\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^{2n+1}}{(2n+1)!}\cdot\frac{1}{z^{2n+1}}\right)\\
&= \frac{\sqrt{2}z^2}{2}+\sqrt{2}z-5\sqrt{2}-\frac{10\sqrt{2}}{3z}\\
&+ \frac{\sqrt{2}}{2}\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(4n+5)2^{2n+2}}{(2n+3)!}\left(\frac{2n+3}{z^{2n}}+\frac{2}{z^{2n+1}}\right)\right]
\end{aligned}$$

This implies that

$$\begin{aligned}
&\operatorname*{Res}_{z=0}(z^2+4z)\sin\left(\frac{\pi z-8}{4z}\right)\\
=\ &\text{coefficient of } \frac{1}{z} \text{ in Laurent series expansion of } (z^2+4z)\sin\left(\frac{\pi z-8}{4z}\right) \text{ at } z=0\\
=\ &-\frac{10\sqrt{2}}{3}.
\end{aligned}$$

Thus, it follows from equation (12) that

$$\begin{aligned}
\int_\gamma (z^2+4z)\sin\left(\frac{\pi z-8}{4z}\right)\,dz &= 2\pi i\operatorname*{Res}_{z=0}(z^2+4z)\sin\left(\frac{\pi z-8}{4z}\right)\\
&= 2\pi i\left(-\frac{10\sqrt{2}}{3}\right) = -\frac{20\sqrt{2}\pi}{3}i.
\end{aligned}$$

(b) Since f is analytic on the ball $B(0,1)$, it follows from the definition of radius of convergence that $R\geq 1$. Suppose on the contrary that $R=1$.
By Taylor's Theorem, we may express f as a Taylor series at $z=0$ as follows:

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n, \tag{13}$$

where the series converges absolutely for all $z\in B(0,1)$, and diverges for all $|z|>1$.
By differentiating both sides of equation (13) k times, we have

$$f^{(k)}(z)=\sum_{n=k}^{\infty}\frac{f^{(n)}(0)\cdot n(n-1)\cdots(n-k+1)}{n!}z^{n-k}=\sum_{n=k}^{\infty}\frac{f^{(n)}(0)}{(n-k)!}z^{n-k}, \tag{14}$$

where the series converges absolutely for all $z \in B(0,1)$, and diverges for all $|z| > 1$.
Also, since f is analytic on the ball $B\left(\frac{1}{2},1\right)$, it follows from Taylor's Theorem that we may also express f as a Taylor series at $z = \frac{1}{2}$ as follows:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}\left(\frac{1}{2}\right)}{k!}\left(z-\frac{1}{2}\right)^k, \tag{15}$$

where the series converges absolutely for all $z \in B\left(\frac{1}{2},1\right)$.
Now, by setting $z = \frac{1}{2}$ in equation (14), we have that for all $k \geq 0$,

$$f^{(k)}\left(\frac{1}{2}\right) = \sum_{n=k}^{\infty} \frac{f^{(n)}(0)\cdot n(n-1)\cdots(n-k+1)}{n!}\cdot\left(\frac{1}{2}\right)^{n-k} = \sum_{n=k}^{\infty} \frac{f^{(n)}(0)}{(n-k)!}\cdot\frac{1}{2^{n-k}}, \tag{16}$$

where we note that the series converges absolutely for all $k \geq 0$. Now, it remains to show that the series $\sum\limits_{n=k}^{\infty} \frac{f^{(n)}(0)}{(n-k)!k!}\cdot\frac{1}{2^{n-k}}\cdot\left(z-\frac{1}{2}\right)^k$ is absolutely convergent for all $z \in B\left(\frac{1}{2},1\right)$. Notice that

$$\begin{aligned} 0 < \sum_{n=k}^{\infty}\left|\frac{f^{(n)}(0)}{(n-k)!k!}\cdot\frac{1}{2^{n-k}}\cdot\left(z-\frac{1}{2}\right)^k\right| &= \sum_{n=k}^{\infty}\left|\frac{f^{(n)}(0)}{(n-k)!}\cdot\frac{1}{2^{n-k}}\right|\cdot\left|\frac{1}{k!}\right|\cdot\left|\left(z-\frac{1}{2}\right)^k\right| \\ &< \sum_{n=k}^{\infty}\left|\frac{f^{(n)}(0)}{(n-k)!}\cdot\frac{1}{2^{n-k}}\right| = \sum_{n=k}^{\infty}\frac{f^{(n)}(0)}{(n-k)!}\cdot\frac{1}{2^{n-k}}, \end{aligned}$$

where we note that the last equality follows from the fact that $f^{(n)}(0) > 0$ for all $n \geq 0$.
Since the series on the RHS of the last inequality is absolutely convergent, it follows that the series $\sum\limits_{n=k}^{\infty} \frac{f^{(n)}(0)}{(n-k)!k!}\cdot\frac{1}{2^{n-k}}\cdot\left(z-\frac{1}{2}\right)^k$ is absolutely convergent for all $z \in B\left(\frac{1}{2},1\right)$.
Then for all $z \in B\left(\frac{1}{2},1\right)$, we may use equation (16) to rewrite equation (15) as follows:

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}\left(\frac{1}{2}\right)}{k!}\left(z-\frac{1}{2}\right)^k \\ &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{f^{(n)}(0)}{(n-k)!k!}\cdot\frac{1}{2^{n-k}}\cdot\left(z-\frac{1}{2}\right)^k \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{f^{(n)}(0)}{(n-k)!k!}\cdot\left(\frac{1}{2}\right)^{n-k}\cdot\left(z-\frac{1}{2}\right)^k \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}\cdot\left(\frac{1}{2}\right)^{n-k}\cdot\left(z-\frac{1}{2}\right)^k \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\left[\frac{1}{2}+\left(z-\frac{1}{2}\right)\right]^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n. \quad \text{(By Binomial Theorem)} \end{aligned}$$

Note: The interchanging of the summations is possible as the sums $\sum\limits_{n=k}^{\infty} \frac{f^{(n)}(0)}{(n-k)!k!}\cdot\frac{1}{2^{n-k}}\cdot\left(z-\frac{1}{2}\right)^k$ and $\sum\limits_{k=0}^{\infty} \frac{f^{(k)}\left(\frac{1}{2}\right)}{k!}\left(z-\frac{1}{2}\right)^k$ converge absolutely for all $z \in B\left(\frac{1}{2},1\right)$. This follows from the fact that any rearrangement of an absolutely convergent series converges to the same sum as the original series.

This implies that the Taylor series of f at $z = 0$ converges for all $z \in B\left(\frac{1}{2},1\right)$; and in particular for all $z \in \mathbb{R}$, $1 < z < \frac{3}{2}$, which contradicts the fact that the series diverges for all $|z| > 1$. So we must have $R > 1$ as desired.