SUGGESTED SOLUTION FOR MA3220 ORDINARY DIFFERENTIAL EQUATIONS FINAL (AY22/23 SEM 1)

WRITTEN BY QI FULIN

AUDITED BY IKHOON EOM

Question 1.

Remark 0.1. In the actual exam, no justification is needed for Question 1. Justifications presented here are just for readers' reference.

(a) False.

Since it is a non-linear first-order equation, defining $f(t, y) = y^{\frac{1}{5}}$, we just need to verify whether both f and f_y are both continuous in some open rectangle containing $(0, 0)$. Clearly, $f_y = \frac{1}{5}y^{-\frac{4}{5}}$ is discontinuous at $x = 0$, so no such rectangle exists.

(b) True.

We can either see it from the superposition principle of linear ODEs, or

$$\frac{d^3}{dt^3}(y_1 + 2y_2) = y_1''' + 2y_3''' = e^t y_1 + 2e^t y_2 = e^t(y_1 + 2y_2).$$

(c) True.

One may verify that e^{-At} is indeed the inverse of e^{At} for all $t \in \mathbb{R}$.

(d) True.

Since $(1, 2)$ is a saddle point for the given autonomous system, we conclude that the corresponding linearised system near $(1, 2)$ has two real eigenvalues of opposite signs, and $(1, 2)$ is a critical point for the modified autonomous system.

Given that $\lambda \in \mathbb{R}$ being an eigenvalue of A implies $-\lambda \in \mathbb{R}$ being an eigenvalue of $-A$, we have the linearised system near $(1, 2)$ corresponding to the modified autonomous system also has two real eigenvalues of opposite signs, so $(1, 2)$ is indeed a saddle point for the second autonomous system.

(e) One such equation is $y'(t) = y^2$.

We have

$$\begin{aligned} y^{-2}y'(t) = 1 &\implies -y^{-1} = t - C \\ &\implies y = \frac{1}{C-t}. \end{aligned}$$

With the given initial condition, we have $C = 1$, so the (unique) solution to the equation on some open interval containing $t = 0$ is

$$y = \frac{1}{1-t},$$

which goes to infinity at $t = 1$.

(f) The suitable guess is $Y = t(At + B)(C \sin t + D \cos t)$.

Solving the characteristic equation $r^2 + 1 = 0$ gives us $r = \pm i$, so the general solution to the homogeneous solution is

$$y = A \sin t + B \cos t.$$

Hence, the suitable guess is

$$Y = t(At + B)(C \sin t + D \cos t).$$

We need to multiply $(At + B)(C \sin t + D \cos t)$ by t since otherwise, $BC \sin t + BD \cos t$ will solve the homogeneous equation.

(g) $\alpha = 3$.

Define $f(x, y) = \alpha x^2 y + xy^2$ and $g(x, y) = (x + y)x^2$. If the equation is exact, then

$$f_y = g_x \implies \alpha x^2 + 2xy = 3x^2 + 2xy \implies \alpha = 3.$$

Question 2.

(i) Rewrite the equation as $y' - t^{-1}y = t^3$, which is a first-order linear equation. Since t^3 is continuous on $\mathbb{R}$, t^{-1} is continuous on $\mathbb{R}\backslash\{0\}$, and the initial condition is $y(1) = 2$, by the existence and uniqueness theorem, the maximum interval where a solution is certain to exist is $(0, \infty)$.

(ii) The integrating factor is

$$e^{\int -t^{-1}\ dt} = t^{-1}.$$

Multiplying both sides of $y' - t^{-1}y = t^3$ by t^{-1}, we have

$$\left(t^{-1}y\right)' = t^2 \implies t^{-1}y = \frac{1}{3}t^3 + C \implies y = \frac{1}{3}t^4 + Ct.$$

From the initial condition $y(1) = 2$, we have

$$C = \frac{5}{3},$$

so the solution to the IVP is

$$y = \frac{1}{3}t^4 + \frac{5}{3}t.$$

Hence, the actual interval where the solution exists is $(-\infty, \infty)$.

Question 3.

(i) Denoting the system as $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we first try to find the eigenvalues of $\mathbf{A}$. The characteristic polynomial of $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) + 2 = \lambda^2 - 4\lambda + 5,$$

setting it to 0 giving us $\lambda = 2 \pm i$.

We now try to find an eigenvector corresponding to $\lambda = 2 + i$:

$$\begin{pmatrix} 1-i & -2 \\ 1 & -1-i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{0} \implies \begin{pmatrix} 1 & -1-i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{0}$$

$$\implies \begin{pmatrix} a \\ b \end{pmatrix} = b\begin{pmatrix} 1 \\ 1 \end{pmatrix} + bi\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ b \in \mathbb{R},$$

so an eigenvector corresponding to $\lambda = 2 + i$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

We then have a complex-valued solution

$$\begin{aligned}\mathbf{x}(t) &= e^{2t}(\cos t + i\sin t)\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \\ &= e^{2t}\left[\begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} \sin t + \cos t \\ \sin t \end{pmatrix}\right],\end{aligned}$$

so the general solution to the given system is

$$\mathbf{x}(t) = e^{2t}\left[C_1\begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + C_2\begin{pmatrix} \sin t + \cos t \\ \sin t \end{pmatrix}\right].$$

From the initial condition $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, we have

$$C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \implies C_1 = 1, C_2 = 1,$$

so the solution to the given IVP is

$$\mathbf{x}(t) = e^{2t}\left[\begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} \sin t + \cos t \\ \sin t \end{pmatrix}\right] = e^{2t}\begin{pmatrix} 2\cos t \\ \cos t + \sin t \end{pmatrix}.$$

(ii) The phase portrait is an anticlockwise outward spiral centred at $(0, 0)$. Since the real part of the eigenvalue is $2 > 0$, we conclude that the phase portrait is a spiral source and is unstable.

Question 4.

(i) We define $x_1 = x$, $x_2 = x'$. From the second order equation, we have

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_2 - \alpha \sin x_1 \end{cases} \iff \mathbf{x}' = \mathbf{f}(\mathbf{x}),$$

which is a first-order system. When $\mathbf{f}(\mathbf{x}) = 0$, we have

$$\begin{cases} x_2 = 0 \\ -x_2 - \alpha \sin x_1 = 0 \end{cases} \implies \begin{cases} x_2 = 0 \\ \sin x_1 = 0 \end{cases} \implies \begin{cases} x_2 = 0 \\ x_1 = k\pi, k \in \mathbb{Z}. \end{cases}$$

Hence, all critical points of the first order system are $(k\pi, 0)$ for $k \in \mathbb{Z}$.

(ii) We define $\mathbf{u}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

Since we have

$$\mathbf{A} = \mathbf{f}'(\mathbf{0}) = \left. \begin{pmatrix} 0 & 1 \\ -\alpha \cos x_1 & -1 \end{pmatrix} \right|_{x_1 = x_2 = 0} = \begin{pmatrix} 0 & 1 \\ -\alpha & -1 \end{pmatrix},$$

the corresponding linearised system near $(0, 0)$ is

$$\mathbf{u}'(t) = \begin{pmatrix} 0 & 1 \\ -\alpha & -1 \end{pmatrix} \mathbf{u}(t) = \mathbf{A}\mathbf{u}(t).$$

(iii) The eigenvalues of $\mathbf{A}$ are the roots of

$$p_{\mathbf{A}}(t) = \det(\mathbf{A} - \lambda \mathbf{I}) = -\lambda(-1 - \lambda) + \alpha = \lambda^2 + \lambda + \alpha,$$

which are

$$\lambda_1 = \frac{-1 + \sqrt{1 - 4\alpha}}{2}, \lambda_2 = \frac{-1 - \sqrt{1 - 4\alpha}}{2}.$$

When the critical point $(0, 0)$ is a stable node, we have $\lambda_2 < \lambda_1 < 0$, which then implies that

$$0 < \sqrt{1 - 4\alpha} < 1 \implies 0 < 1 - 4\alpha < 1 \implies 0 < \alpha < \frac{1}{4}.$$

Question 5.

(i) From the question, we have

$$y'' + \frac{1}{x} y + \left(1 - \frac{4}{x^2}\right) y = 0. \tag{1}$$

Since $\lim_{x\to 0} \frac{1}{x}$ does not exist, we conclude that $x = 0$ is a singular point.

Since $\lim_{x\to 0} x \cdot \frac{1}{x} = \lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0} x^2 \left(1 - \frac{4}{x^2}\right) = \lim_{x\to 0} x^2 - 4 = -4$, we conclude that $x = 0$ is a regular singular point.

(ii) Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ be the ansatz, so

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \wedge y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

From the question, we have

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} 4 a_n x^{n+r} = 0, \tag{2}$$

so the indicial equation is

$$r(r-1) + r - 4 = 0 \implies r^2 - 4 = 0 \implies r^2 = 4.$$

The roots of the indicial equation are thus $r = \pm 2$.

(iii) From (2), we know $\{a_n\}_{n \in \mathbb{Z}_0^+}$ satisfies

$$\begin{cases} a_1 = 0 & \text{if } n = 1; \\ (n^2 + 2nr + r^2 - 4) a_n + a_{n-2} = 0 & \text{if } n \geq 2. \end{cases} \implies \begin{cases} a_1 = 0 & \text{if } n = 1; \\ a_n = \frac{-a_{n-2}}{n^2 + 2nr + r^2 - 4} & \text{if } n \geq 2. \end{cases}$$

When $r = 2$, we have $a_i = 0$ for odd $i \in \mathbb{Z}^+$ and

$$a_2 = -\frac{1}{12} a_0,$$
$$a_4 = -\frac{1}{32} a_2 = \frac{1}{384} a_0.$$

Following this trend, we see that a_i is well-defined for all even $i \in \mathbb{Z}^+$ as well, so a series solution for the case of $r_1 = 2$ exists. The first three non-zero terms of the series solution are

$$y = a_0 x^2 - \frac{1}{12} a_0 x^4 + \frac{1}{384} a_0 x^6.$$

When $r = -2$, we have

$$a_4 = \frac{1}{16 - 16 + 4 - 4} a_2 = \frac{1}{0} a_2,$$

which is undefined. Hence, there is no series solution for the case of $r_2 = -2$.

Question 6.

(i) When $\lambda = 0$, we have

$$y'' = 0 \implies y = Cx + D$$

by integrating both sides twice with respect to t, where $C, D \in \mathbb{R}$ are arbitrary constants.

From the boundary conditions, we have $D = 0$ and $C + D - C = 0$, so all solutions to the BVP when $\lambda = 0$ must be in the form of

$$y = Cx,$$

where $C \in \mathbb{R}$. Hence, there exists non-trivial solutions to the equation when $\lambda = 0$, so $\lambda = 0$ is indeed an eigenvalue.

(ii) The characteristic equation of $y'' + \lambda y = 0$ is $m^2 + \lambda = 0$, whose roots are

$$m_1 = \sqrt{-\lambda}, m_2 = -\sqrt{-\lambda}.$$

Assume $\lambda > 0$. We have m_1, m_2 are both complex, i.e., $m_1 = i\sqrt{\lambda}, m_2 = -i\sqrt{\lambda}$. The general solution to the equation is thus

$$y(t) = A \cos \sqrt{\lambda} t + B \sin \sqrt{\lambda} t.$$

From the boundary conditions, we have

$$\begin{aligned} y(0) = 0 &\implies A = 0 \\ y(1) - y'(1) = 0 &\implies A \cos \sqrt{\lambda} + B \sin \sqrt{\lambda} + \sqrt{\lambda} A \sin \sqrt{\lambda} - \sqrt{\lambda} B \cos \sqrt{\lambda} = 0, \end{aligned}$$

which suggest

$$\begin{aligned}
B\sin\sqrt{\lambda}-\sqrt{\lambda}B\cos\sqrt{\lambda}=0 &\implies \sin\sqrt{\lambda}-\sqrt{\lambda}\cos\sqrt{\lambda}=0\\
&\implies \sqrt{1+\lambda}\sin(\sqrt{\lambda}-\arctan\sqrt{\lambda})=0\\
(*)\qquad &\implies \sqrt{\lambda_n}-\arctan\sqrt{\lambda_n}=n\pi, n\in\mathbb{Z}^+.
\end{aligned}$$

Therefore, as long as λ satisfies $(*)$, we can choose $B\in\mathbb{R}$ for free, which then implies that non-trivial solutions to the eigenvalue problem exist, i.e., λ_n is an eigenvalue.

Since by definition, $-\dfrac{\pi}{2}<\arctan x<\dfrac{\pi}{2}$, from $(*)$, we have

$$n\pi-\frac{\pi}{2}<\sqrt{\lambda_n}<n\pi+\frac{\pi}{2}\implies \left(n-\frac{1}{2}\right)^2\pi^2<\lambda_n<\left(n+\frac{1}{2}\right)^2\pi^2,$$

which then gives us the desired result.

(iii) We define $\langle f,g\rangle=\int_0^1 fg\ dt$.

Let f,g be non-trivial eigenfunctions corresponding to different eigenvalues λ_1 and $\lambda_2\in\mathbb{R}$. We have

$$\begin{aligned}
\langle -\lambda_1 f,g\rangle=\langle f'',g\rangle &=\int_0^1 f''g\ dt\\
&=[f'g]_0^1-\int_0^1 f'g'\ dt\\
&=[f'g]_0^1-\langle f',g'\rangle,\\
\langle f,-\lambda_2 g\rangle=\langle f,g'\rangle &=\int_0^1 fg''\ dt\\
&=[fg']_0^1-\int_0^1 f'g'\ dt\\
&=[fg']_0^1-\langle f',g'\rangle.
\end{aligned}$$

From the boundary conditions, we have $f(0)=g(0)=0$, $f(1)=f'(1)$, and $g(1)=g'(1)$, which then implies that

$$\begin{aligned}
\langle -\lambda_1 f,g\rangle&=f(1)g(1)-\langle f',g'\rangle,\\
\langle f,-\lambda_2 g\rangle&=f(1)g(1)-\langle f',g'\rangle.
\end{aligned}$$

This suggests that

$$\langle -\lambda_1 f, g\rangle = -\lambda_1\langle f, g\rangle = -\lambda_2\langle f, g\rangle = \langle f, -\lambda_2 g\rangle,$$

which implies $\langle f, g\rangle = 0$ given that $\lambda_1 \neq \lambda_2$. We therefore conclude that eigenfunctions corresponding to different eigenvalues must be mutually orthogonal on $[0, 1]$.

Question 7.

(i) Choose $p(t) = 0$ and $q(t) = 1$, so the equation becomes $y'' + y = 0$. We can check that

(1) our choice of $p(t)$ and $q(t)$ satisfies the question requirements;
(2) $y(t) = \sin x$ is indeed a non-constant solution to the equation which has infinitely many zeros.

(ii) Impossible.

Suppose this is possible for the sake of finding a contradiction. Let $y(t)$ be an arbitrary non-constant solution to the equation, and a, b be two consecutive zeros of $y(t)$.

Since both $p(t)$ and $g(t)$ are continuous on $\mathbb{R}$, we must have $y(t)$ is continuous on $\mathbb{R}$. Hence, by the extreme value theorem, $y(t)$ must have a local extrema in (a, b).

If $y(t)$ has a local maximum c in (a, b), we have $y(c) > 0$, $y'(c) = 0$, and $y''(c) < 0$. Since $q(c) < 0$ by assumption, we have $q(c)y(c) < 0$, which then implies that

$$0 = y''(c) + q(c)y(c) < 0,$$

a contradiction.

Similarly, if $y(t)$ has a local minimum d in (a, b), we have $y(c) < 0$, $y'(c) = 0$, and $y''(c) > 0$. Since $q(c) < 0$ by assumption, we have $q(c)y(c) > 0$, which then implies that

$$0 = y''(c) + q(c)y(c) > 0,$$

a contradiction.

Therefore, we conclude that $y(t)$ has no local extrema in (a, b), a contradiction to the extreme value theorem. We then conclude that it is impossible to have a solution with infinitely many zeros when $p(t) < 0$ for all $t \in \mathbb{R}$.