NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA4267 Discrete Time Finance
AY 2008/2009 Sem 1

Question 1

(a) WLOG, we assume $S_0^0 = 1$. Suppose there exists a free lunch in the market, i.e. $\exists \phi = (\phi^0, \phi^1, \cdots, \phi^d) \in \mathbb{R}^{d+1}$ s.t. $\phi \cdot S_0 < 0$ and $\phi \cdot S_1 \geq 0$ almost surely. Let $\phi' = (\phi^0 - \phi \cdot S_0, \phi^1, \cdots, \phi^d)$. Then we have $\phi' \cdot S_0 = \phi \cdot S_0 - \phi \cdot S_0 = 0$ and $\phi' \cdot S_1 = \phi \cdot S_1 - (1+r)\phi \cdot S_0 > 0$. Thus, ϕ' is an arbitrage opportunity.

(b) (i) Suppose (ϕ^0, ϕ^1) is a free lunch in the market. Thus we have

$$\begin{cases} \phi^0 + 2\phi^1 & < 0 \\ \phi^0 + 2\phi^1 & \geq 0 \\ \phi^0 + 3\phi^1 & \geq 0. \end{cases}$$

Obviously, there is no solution to the above linear system of equations. Thus, there is no free lunch in the market.

(ii)

$$\begin{cases} \mathbb{Q}(w_1) + \mathbb{Q}(w_2) & = 1 \\ 2\mathbb{Q}(w_1) + 3\mathbb{Q}(w_2) & = 2 \end{cases} \Longrightarrow \begin{cases} \mathbb{Q}(w_1) & = 1 \\ \mathbb{Q}(w_2) & = 0 \end{cases}$$

(iii) Since $\mathbb{Q}(w_2) = 0$, the risk-neutral measure $\mathbb{P}^\star$ found in (ii) is not equivalent to $\mathbb{P}$.

(c) Portfolio $(-2, 1)$ is an arbitrage opportunity in part (b), while we have shown in part (b) that there is no free lunch in that market model. Thus we have found a counter example showing that the reverse to part (a) is not true.

Question 2

(a) Because (U_t) is the smallest super-martingale that dominates the sequence (Z_t), which is a super-martingale itself. Thus, we have $U_t = Z_t$. $\tau_0 = \inf\{t \geq 0 : U_t = Z_t\} \equiv 0$ is the smallest optimal stopping time.

(b) Since (Z_t) is a martingale, which is also a supermartingale, we have $U_t = Z_t$ by following the same argument in part (a). By the criteria of optimal stopping times in notes, we deduce that all stopping times in $\mathcal{T}_{0,T}$ are optimal.

(c) By mathematical induction, we can show that $U_t = \mathbb{E}[U_{t+1}|\mathcal{F}_t]$, which implies that (U_t) is a martingale. By the criteria of optimal stopping times in notes, we can deduce that the largest optimal stopping time for (Z_t) is T. Remark: The above is a brief answer to Question 2. We can also discuss something related with Doob's Decompostion.

Question 3

(a) $\tau \sim \text{Geo}(1/2)$

$$\mathbb{P}(\tau = n) = \mathbb{P}(\xi_i = -1, \text{ for } 1 \leq i \leq n-1, \ \xi_n = 1) = \frac{1}{2^n}$$

(b) Since H_n depends on ξ_{n-1}, $H_n \in \mathcal{F}_{t-1}$, which means that $\{H_n\}$ is previsible relative to $\mathcal{F}_t$.

(c)

$$\begin{aligned}
\mathbb{E}\left[\sum_{j=1}^{\tau-1} H_j \triangle W_j\right] &= \sum_{n=1}^{\infty}\left[\sum_{j=1}^{n-1} 2^{j-1}(-1)\right]\mathbb{P}(\tau = n) \\
&= -\sum_{n=1}^{\infty}[2^{n-1} - 1]\frac{1}{2^n} \\
&= -\sum_{n=1}^{\infty}\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{2^n} \\
&= -\infty
\end{aligned}$$

Question 4

(a) Denote $M_T^+ = \max(M_T, 0)$, $M_T^- = \max(-M_T, 0)$. Then we can write $M_T = M_T^+ - M_T^-$. Since $M_T^+ \geq 0$ and $M_T^- \geq 0$, we can regard M_T^+ and M_T^- as two contingent claims. Because there is only one equivalent martingale measure $\mathbb{P}^\star$, the market is viable and complete. Thus trading strategies can be found to replicate M_T^+ and M_T^-, i.e. $\exists$ trading strategies ϕ_1 and ϕ_2 s.t. $M_T^+ = V_T(\phi_1)$ and $M_T^- = V_T(\phi_2)$. Denote $\phi = \phi_1 - \phi_2$. Then we have $V_T(\phi) = V_T(\phi_1) - V_T(\phi_2) = M_T^+ - M_T^- = M_T$. Since (M_t) is a martingale, $V_t(\phi) = \mathbb{E}^\star(M_T|\mathcal{F}_t) = M_t$. Moreover $V_t(\phi) = V_0 + \sum_{k=1}^t \phi_k \cdot (S_k - S_{k-1})$. Thus $M_t = M_0 + \sum_{k=1}^t \phi_k \cdot (S_k - S_{k-1})$.

(b) Suppose $\mathbb{P}^\star \in \mathcal{P}$. $M_t = \mathbb{E}^\star(I_A|\mathcal{F}_t)$ is a $\mathbb{P}^\star$-martingale. Since the martingale representation property of the $\mathbb{P}^\star$-martingale holds, we can find predictable process ϕ s.t. $M_t = V_t(\phi)$. Thus $M_T = I_A = V_T(\phi)$, i.e. I_A is attainable. Suppose both $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are in $\mathcal{P}$, then $\mathbb{E}_{\mathbb{Q}_1}[I_A|\mathcal{F}_0] = \mathbb{E}_{\mathbb{Q}_1}[V_T(\phi)|\mathcal{F}_0] = V_0(\phi) = \mathbb{E}_{\mathbb{Q}_2}[V_T(\phi)|\mathcal{F}_0] = \mathbb{E}_{\mathbb{Q}_2}[I_A|\mathcal{F}_0]$. Thus we have $\mathbb{E}_{\mathbb{Q}_1}[I_A] = \mathbb{E}[\mathbb{E}_{\mathbb{Q}_1}[I_A|\mathcal{F}_0]] = \mathbb{E}[\mathbb{E}_{\mathbb{Q}_2}[I_A|\mathcal{F}_0]] = \mathbb{E}_{\mathbb{Q}_2}[I_A]$.