NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA2101 Linear Algebra II
AY 2007/2008 Semester 2

SECTION A

Question 1

(a) Since $W_2 = \text{span}\{\boldsymbol{v_1}\}$, $\dim(W_2)= 1$.

$W_1 = \text{span}\{\boldsymbol{v_1}+\boldsymbol{v_2}, \boldsymbol{v_2}+\boldsymbol{v_3}, \boldsymbol{v_3}+\boldsymbol{v_4}, \boldsymbol{v_1}+\boldsymbol{v_4}\}$. Observe that

$$\boldsymbol{v_1}+\boldsymbol{v_4} = (\boldsymbol{v_1}+\boldsymbol{v_2}) - (\boldsymbol{v_2}+\boldsymbol{v_3}) + (\boldsymbol{v_3}+\boldsymbol{v_4}).$$

Thus, $W_1 = \text{span}\{\boldsymbol{v_1}+\boldsymbol{v_2}, \boldsymbol{v_2}+\boldsymbol{v_3}, \boldsymbol{v_3}+\boldsymbol{v_4}\}$.
We suppose that for some scalars a_1, a_2 and a_3,

$$\begin{aligned} a_1(\boldsymbol{v_1}+\boldsymbol{v_2}) + a_2(\boldsymbol{v_2}+\boldsymbol{v_3}) + a_3(\boldsymbol{v_3}+\boldsymbol{v_4}) &= \boldsymbol{0} \\ \Rightarrow a_1\boldsymbol{v_1} + (a_1+a_2)\boldsymbol{v_2} + (a_2+a_3)\boldsymbol{v_3} + a_3\boldsymbol{v_4} &= \boldsymbol{0}. \end{aligned}$$

Since $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$ forms a basis for V, $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$ is linearly independent.
Solving for a_1, a_2 and a_3, we obtain $a_1 = a_2 = a_3 = 0$.
Hence, $\{\boldsymbol{v_1}+\boldsymbol{v_2}, \boldsymbol{v_2}+\boldsymbol{v_3}, \boldsymbol{v_3}+\boldsymbol{v_4}\}$ forms a basis for W_1, and $\dim(W_1)= 3$.

Let $\boldsymbol{u} \in W_1 \cap W_2$. Then,

$$\boldsymbol{u} = b_1(\boldsymbol{v_1}+\boldsymbol{v_2}) + b_2(\boldsymbol{v_2}+\boldsymbol{v_3}) + b_3(\boldsymbol{v_3}+\boldsymbol{v_4}) = b_4\boldsymbol{v_1}$$

for some scalars b_1, b_2, b_3 and b_4. We proceed to solve for the scalars.

$$(b_1 - b_4)\boldsymbol{v_1} + (b_1+b_2)\boldsymbol{v_2} + (b_2+b_3)\boldsymbol{v_3} + b_3\boldsymbol{v_4} = \boldsymbol{0}.$$

Since $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$ is linearly independent, we obtain the following system of equations:

$$\begin{cases} b_1 = b_4 \\ b_1 = -b_2 \\ b_2 = -b_3 \\ b_3 = 0 \end{cases}$$

which yields $b_1 = b_2 = b_3 = b_4 = 0$.
Hence, $\forall \boldsymbol{u} \in W_2$, $\boldsymbol{u} = \boldsymbol{0}$, which implies that $W_1 \cap W_2 = \{\boldsymbol{0}\}$.
Therefore, $\dim(W_1 \cap W_2)= 0$.

Thus,

$$\begin{aligned} \dim(W_1 + W_2) &= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \\ &= 3 + 1 - 0 \\ &= 4. \end{aligned}$$

(b) $W_1 + W_2$ is a direct sum as $\dim(W_1 \cap W_2) = 0$.
Clearly, $W_1 + W_2 \subseteq V$.
Since $\dim(W_1 + W_2) = 4 = \dim(V)$ from (a), $W_1 + W_2 = V$.

(c) A basis for V/W_2 is $\{\boldsymbol{v_2} + W_2, \boldsymbol{v_3} + W_2, \boldsymbol{v_4} + W_2\}$.

Question 2

(a) Let $\boldsymbol{A} \in \text{Ker}(T)$. Then,

$$\begin{aligned} & T(\boldsymbol{A}) = \boldsymbol{0} \\ \iff & \boldsymbol{A} - \boldsymbol{A}^T = \boldsymbol{0} \\ \iff & \boldsymbol{A} = \boldsymbol{A}^T. \end{aligned}$$

Hence, $\text{Ker}(T) = \{\boldsymbol{A} \in \mathcal{M}_{nn}(\mathbb{R}) | \boldsymbol{A} = \boldsymbol{A}^T\} = \mathcal{R}$, the set of all n by n symmetric matrices.

Let $\boldsymbol{B} \in \text{R}(T)$. Then, $\boldsymbol{B} = \boldsymbol{P} - \boldsymbol{P}^T$ for some $\boldsymbol{P} \in \mathcal{M}_{nn}(\mathbb{R})$.
Observe that

$$\begin{aligned} \boldsymbol{B}^T &= (\boldsymbol{P} - \boldsymbol{P}^T)^T \\ &= \boldsymbol{P}^T - \boldsymbol{P} \\ &= -\boldsymbol{B}. \end{aligned}$$

Thus, $\boldsymbol{B} \in \{\boldsymbol{A} \in \mathcal{M}_{nn}(\mathbb{R}) | \boldsymbol{A} = -\boldsymbol{A}^T\} = \mathcal{S}$, the set of all n by n skew symmetric matrices.
Hence, we have $\text{R}(T) \subseteq \mathcal{S}$.
Let $\boldsymbol{C} \in \mathcal{S}$. Then,

$$\begin{aligned} \boldsymbol{C} &= \frac{1}{2}\boldsymbol{C} + \frac{1}{2}\boldsymbol{C} \\ &= \frac{1}{2}\boldsymbol{C} - \frac{1}{2}\boldsymbol{C}^T. \end{aligned}$$

We thus have $\boldsymbol{C} = T(\frac{\boldsymbol{1}}{\boldsymbol{2}}\boldsymbol{C})$, which implies that $\mathcal{S} \subseteq \text{R}(T)$.
Hence, $\text{R}(T) = \mathcal{S}$.

(b) We let $\boldsymbol{E}_{ij}$ denote the matrix which has the entry 1 in its ith row and jth column, and zero everywhere else.

For all $\boldsymbol{A} = (a_{ij}) \in \mathcal{R} = \text{Ker}(T)$, since $\boldsymbol{A}$ is symmetric, $a_{ij} = a_{ji}$ for all i and j, and

$$\begin{aligned} \boldsymbol{A} &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \boldsymbol{E}_{ij} \\ &= \sum_{i=1}^{n} a_{ii} \boldsymbol{E}_{ii} + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_{ij} (\boldsymbol{E}_{ij} + \boldsymbol{E}_{ji}). \end{aligned}$$

Let $B = \{\boldsymbol{E}_{ii} | 1 \le i \le n\} \cup \{\boldsymbol{E}_{ij} + \boldsymbol{E}_{ji} | 1 \le i < j \le n\}$. Then, $\text{Ker}(T) = \text{span}(B)$.
It is easy to verify that B is linearly independent.
Hence, B is a basis for $\text{Ker}(T)$, and

$$\begin{aligned} \dim(\text{Ker}(T)) &= |B| \\ &= \sum_{k=1}^{n} k \\ &= \frac{n(n+1)}{2}. \end{aligned}$$

For all $\boldsymbol{A} = (a_{ij}) \in \mathcal{S} = \mathrm{R}(T)$, since $\boldsymbol{A}$ is anti-symmetric, $a_{ij} = -a_{ji}$ for all i and j, $a_{ii} = 0$ for all i, and

$$\begin{aligned}\boldsymbol{A} &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\boldsymbol{E}_{ij} \\ &= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_{ij}(\boldsymbol{E}_{ij} - \boldsymbol{E}_{ji}).\end{aligned}$$

Let $C = \{\boldsymbol{E}_{ij} - \boldsymbol{E}_{ji} | 1 \le i < j \le n\}$. Then, $\mathrm{R}(T) = \mathrm{span}(C)$.
It is easy to verify that C is linearly independent.
Hence, C is a basis for $\mathrm{R}(T)$, and

$$\begin{aligned}\mathrm{rank}(T) &= |C| \\ &= \frac{n(n+1)}{2} - n \\ &= \frac{n(n-1)}{2}.\end{aligned}$$

(c) We have that

$$\begin{aligned}\left[T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_B &= \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right]_B \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \\ \left[T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]_B &= \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right]_B \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \\ \left[T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]_B &= \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right]_B \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \\ \left[T\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\right]_B &= \left[\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\right]_B \\ &= \left[0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\right]_B \\ &= \begin{pmatrix} 0 \\ -2 \\ 0 \\ 2 \end{pmatrix}.\end{aligned}$$

Hence,

$$[T]_B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Question 3

(a) The characteristic polynomial is given by

$$\begin{aligned} c_{\boldsymbol{A}}(x) &= \begin{vmatrix} -x & 0 & 1 & -1 \\ 2 & 2-x & -1 & 1 \\ 0 & 0 & -1-x & 1 \\ 0 & 0 & -1 & 1-x \end{vmatrix} \\ &= (2-x)\begin{vmatrix} -x & 1 & -1 \\ 0 & -1-x & 1 \\ 0 & -1 & 1-x \end{vmatrix} \\ &= (2-x)(-x((-1-x)(1-x)+1)) \\ &= -x(2-x)(x^2) \\ &= x^3(x-2). \end{aligned}$$

To find the eigenvalues of $\boldsymbol{A}$, we solve $c_{\boldsymbol{A}}(x) = 0$, which yields $x = 0$ or $x = 2$.
Hence, the eigenvalues of $\boldsymbol{A}$ are 0 and 2.

(b) Let E_λ denote the eigenspace corresponding to eigenvalue of λ.
To solve for the eigenvectors corresponding to eigenvalue 0, we solve the system $(\boldsymbol{A} - 0\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$:

$$\left(\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right) \xrightarrow{R_4 - R_3} \left(\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 2 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_3 + R_2} \left(\begin{array}{cccc|c} 2 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, $\boldsymbol{x} = \begin{pmatrix} -s \\ s \\ t \\ t \end{pmatrix}$, where s and t are arbitrary real constants.

Thus, $E_0 = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$, and it is clear that this spanning set is linearly independent.
Hence, this spanning set forms a basis for E_0, and $\dim(E_0) = 2$.

To solve for the eigenvectors corresponding to eigenvalue 2, we solve the system $(\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$:

$$\left(\begin{array}{cccc|c} -2 & 0 & 1 & -1 & 0 \\ 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right) \xrightarrow[R_4 - \frac{1}{3}R_3]{R_2 + R_1} \left(\begin{array}{cccc|c} -2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & -\frac{4}{3} & 0 \end{array}\right)$$

Hence, we obtain $x = \begin{pmatrix} 0 \\ r \\ 0 \\ 0 \end{pmatrix}$, where r is an arbitrary real constant.

Thus, $E_2 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$.

It is then obvious that $\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ forms a basis for E_2, and hence $\dim(E_2) = 1$.

(c) Since $c_{\boldsymbol{A}}(x) = x^3(x-2)$, 0 must appear 3 times along the main diagonal of the Jordan canonical form, while 2 appears only once.
Furthermore, from (b), $\dim(E_0) = 2$ and $\dim(E_2) = 1$, which implies that there will be two Jordan blocks associated with eigenvalue 0 and one Jordan block associated with eigenvalue 2.

Hence, $\boldsymbol{A}$ must have the given Jordan canonical form $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

The minimal polynomial of $\boldsymbol{A}$ is $x^2(x-2)$.

Question 4

(a) For all $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{C}^n$,

$$\begin{aligned} \langle \boldsymbol{u}, \boldsymbol{v} \rangle &= \boldsymbol{v}^* \boldsymbol{A} \boldsymbol{u} \\ &= (\boldsymbol{v}^* \boldsymbol{A} \boldsymbol{u})^T \qquad \text{since } \boldsymbol{v}^* \boldsymbol{A} \boldsymbol{u} \text{ is a complex number} \\ &= \boldsymbol{u}^T \boldsymbol{A}^T \overline{\boldsymbol{v}} \\ &= \overline{\boldsymbol{u}^* \boldsymbol{A}^* \boldsymbol{v}} \\ &= \overline{\boldsymbol{u}^* \boldsymbol{A} \boldsymbol{v}} \qquad \text{since } \boldsymbol{A} = \boldsymbol{A}^* \\ &= \overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}. \end{aligned}$$

Hence, $\langle \ , \ \rangle$ satisfies (IP1).

For all $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbb{C}^n$,

$$\begin{aligned} \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle &= \boldsymbol{w}^* \boldsymbol{A}(\boldsymbol{u} + \boldsymbol{v}) \\ &= \boldsymbol{w}^* \boldsymbol{A} \boldsymbol{u} + \boldsymbol{w}^* \boldsymbol{A} \boldsymbol{v} \\ &= \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle. \end{aligned}$$

Hence, $\langle \ , \ \rangle$ satisfies (IP2).

For all $c \in \mathbb{C}$ and $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{C}^n$,

$$\begin{aligned} \langle c\boldsymbol{u}, \boldsymbol{v} \rangle &= \boldsymbol{v}^* \boldsymbol{A} c \boldsymbol{u} \\ &= c(\boldsymbol{v}^* \boldsymbol{A} \boldsymbol{u}) \\ &= c \langle \boldsymbol{u}, \boldsymbol{v} \rangle . \end{aligned}$$

Hence, $\langle \ , \ \rangle$ satisfies (IP3).

(b) For $\langle \ , \ \rangle$ to be an inner product, a necessary and sufficient condition is that $\langle \boldsymbol{0}, \boldsymbol{0} \rangle = 0$, and that for all non-zero $\boldsymbol{u} \in \mathbb{C}^n$, $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0$.
Now,

$$\begin{aligned} \langle \boldsymbol{0}, \boldsymbol{0} \rangle &= \boldsymbol{0}^* \boldsymbol{A} \boldsymbol{0} \\ &= \boldsymbol{0}^* \boldsymbol{0} \\ &= 0. \end{aligned}$$

Hence it is necessary and sufficient for all non-zero $\boldsymbol{u} \in \mathbb{C}^n$, $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0$ for $\langle \ , \ \rangle$ to be an inner product.
Since $\boldsymbol{A}$ is an Hermitian matrix, it is clearly normal, and is hence unitarily diagonalizable.
Let $\boldsymbol{P}$ be a unitary matrix such that

$$\boldsymbol{P}^* \boldsymbol{A} \boldsymbol{P} = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{pmatrix}$$

where $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of $\boldsymbol{A}$. For all non-zero $\boldsymbol{u} \in \mathbb{C}^n$, let $\boldsymbol{w} = \boldsymbol{P}^* \boldsymbol{u}$.

Hence, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$, where $w_1, w_2, ..., w_n \in \mathbb{C}$. Then, $\boldsymbol{u} = \boldsymbol{P}\boldsymbol{w}$, and

$$\begin{aligned} \langle \boldsymbol{u}, \boldsymbol{u} \rangle &= \boldsymbol{u}^* \boldsymbol{A} \boldsymbol{u} \\ &= \boldsymbol{w}^* \boldsymbol{P}^* \boldsymbol{A} \boldsymbol{P} \boldsymbol{w} \\ &= \boldsymbol{w}^* \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{pmatrix} \boldsymbol{w} \\ &= \boldsymbol{w}^* \begin{pmatrix} \lambda_1 w_1 \\ \lambda_2 w_2 \\ \vdots \\ \lambda_n w_n \end{pmatrix} \\ &= \lambda_1 |w_1|^2 + \lambda_2 |w_2|^2 + ... + \lambda_n |w_n|^2 \end{aligned}$$

which is greater than zero if and only if all the eigenvalues are positive real numbers.
Hence a necessary and sufficient condition on the eigenvalues of $\boldsymbol{A}$ so that $\langle \ , \ \rangle$ is an inner product is that all the eigenvalues must be positive real numbers.

SECTION B

Question 5

(a) We have that $\boldsymbol{A0} = \boldsymbol{0A} = \boldsymbol{0}$. Hence, $\boldsymbol{0} \in W$.
Let $\boldsymbol{C}, \boldsymbol{D} \in W$. Then, $\boldsymbol{AC} = \boldsymbol{CA}$ and $\boldsymbol{AD} = \boldsymbol{DA}$.

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{C}+\boldsymbol{D}) &= \boldsymbol{AC}+\boldsymbol{AD} \\ &= \boldsymbol{CA}+\boldsymbol{CD} \\ &= (\boldsymbol{C}+\boldsymbol{D})\boldsymbol{A}. \end{aligned}$$

Hence, $\boldsymbol{C}+\boldsymbol{D} \in W$.
Let $\lambda \in \mathbb{F}$. Then,

$$\begin{aligned} \boldsymbol{A}(\lambda\boldsymbol{C}) &= \lambda\boldsymbol{AC} \\ &= \lambda\boldsymbol{CA} \\ &= (\lambda\boldsymbol{C})\boldsymbol{A}. \end{aligned}$$

Hence, $\lambda\boldsymbol{C} \in W$.
Thus, W is a subspace of $\mathcal{M}_{nn}(\mathbb{F})$.

(b) Now, for $0 \le i \le n-1$, we have

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{A}^i) &= \boldsymbol{A}^{i+1} \\ &= (\boldsymbol{A}^i)A. \end{aligned}$$

Hence, $\boldsymbol{A}^i \in W$ for $0 \le i \le n-1$.
Suppose that $\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^{n-1}$ are not linearly independent, i.e. there exists scalars $\lambda_0, \lambda_1, ..., \lambda_{n-1} \in \mathbb{F}$, not all zeros, such that

$$\lambda_0\boldsymbol{I} + \lambda_1\boldsymbol{A} + ... + \lambda_{n-1}\boldsymbol{A}^{n-1} = \boldsymbol{0}.$$

Multiplying both sides of the equation by $\boldsymbol{v}$, we get the conclusion that there exists scalars $\lambda_0, \lambda_1, ..., \lambda_{n-1} \in \mathbb{F}$, not all zeros, such that

$$\lambda_0\boldsymbol{Iv} + \lambda_1\boldsymbol{Av} + ... + \lambda_{n-1}\boldsymbol{A}^{n-1}\boldsymbol{v} = \boldsymbol{0}$$

which implies that the set of vectors $\{\boldsymbol{v}, \boldsymbol{Av}, \boldsymbol{A}^2\boldsymbol{v}, ..., \boldsymbol{A}^{n-1}\boldsymbol{v}\}$ is linearly dependent, which contradicts the fact that $\{\boldsymbol{v}, \boldsymbol{Av}, \boldsymbol{A}^2\boldsymbol{v}, ..., \boldsymbol{A}^{n-1}\boldsymbol{v}\}$ is a basis for $\mathbb{F}^n$.
Hence, $\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^{n-1}$ are linearly independent vectors contained in W.

(c) Let $\text{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^{n-1}\} = V$.
Let $\boldsymbol{X} \in V$. Then,

$$\begin{aligned} \boldsymbol{X} &= = a_0\boldsymbol{I} + a_1\boldsymbol{A} + ... + a_{n-1}\boldsymbol{A}^{n-1} \qquad \text{for some } a_0, a_1, ..., a_{n-1} \in \mathbb{F} \\ \Rightarrow \boldsymbol{AX} &= \boldsymbol{A}(a_0\boldsymbol{I} + a_1\boldsymbol{A} + ... + a_{n-1}\boldsymbol{A}^{n-1}) \\ &= a_0\boldsymbol{A} + a_1\boldsymbol{A}^2 + ... + a_{n-1}\boldsymbol{A}^n \\ &= (a_0\boldsymbol{I} + a_1\boldsymbol{A} + ... + a_{n-1}\boldsymbol{A}^{n-1})\boldsymbol{A}. \\ &= \boldsymbol{XA} \end{aligned}$$

Thus, we have $\boldsymbol{X} \in W$. Hence, $V \subseteq W$.
Let $\boldsymbol{B} \in W$.
Now, $\boldsymbol{Bv} \in \mathbb{F}^n$, and thus,

$$\boldsymbol{Bv} = b_0\boldsymbol{v} + b_1\boldsymbol{Av} + b_2\boldsymbol{A}^2\boldsymbol{v} + ... + b_{n-1}\boldsymbol{A}^{n-1}\boldsymbol{v} \tag{1}$$

for some $b_0, b_1, ..., b_{n-1} \in \mathbb{F}$.

Since $\boldsymbol{AB} = \boldsymbol{BA}$, we have $\boldsymbol{A}^i\boldsymbol{B} = \boldsymbol{BA}^i$ for any $i \in \mathbb{N}$ by induction.
Hence, we have $\boldsymbol{A}^i\boldsymbol{Bv} = \boldsymbol{BA}^i\boldsymbol{v}$. From (1), we have

$$\begin{aligned} \boldsymbol{BA}^i\boldsymbol{v} &= \boldsymbol{A}^i\boldsymbol{Bv} \\ &= (b_0\boldsymbol{A}^i + b_1\boldsymbol{A}^{i+1} + b_2\boldsymbol{A}^2 + ... + b_{n-1}\boldsymbol{A}^{n+i-1})\boldsymbol{v} \\ &= (b_0\boldsymbol{I} + b_1\boldsymbol{A} + ... + b_{n-1}\boldsymbol{A}^{n-1})\boldsymbol{A}^i\boldsymbol{v}. \end{aligned}$$

Thus,

$$\boldsymbol{BA}^i\boldsymbol{v} = (b_0\boldsymbol{I} + b_1\boldsymbol{A} + ... + b_{n-1}\boldsymbol{A}^{n-1})\boldsymbol{A}^i\boldsymbol{v} \qquad \text{for all } i \in \mathbb{N}. \tag{2}$$

Now, suppose that $\boldsymbol{u} \in \mathbb{F}^n$. Then,

$$\boldsymbol{u} = c_0\boldsymbol{v} + c_1\boldsymbol{Av} + ... + c_{n-1}\boldsymbol{A}^{n-1}\boldsymbol{v}$$

for some $c_0, c_1, ..., c_{n-1} \in \mathbb{F}$. Then,

$$\begin{aligned} \boldsymbol{Bu} &= \boldsymbol{B}(c_0\boldsymbol{v} + c_1\boldsymbol{Av} + ... + c_{n-1}\boldsymbol{A}^{n-1}\boldsymbol{v}) \\ &= c_0\boldsymbol{Bv} + c_1\boldsymbol{BAv} + ... + c_{n-1}\boldsymbol{BA}^{n-1}\boldsymbol{v} \\ &= c_0(\boldsymbol{BA}^0\boldsymbol{v}) + c_1(\boldsymbol{BAv}) + ... + c_{n-1}(\boldsymbol{BA}^{n-1}\boldsymbol{v}) \\ &= \sum_{i=0}^{n-1} c_i\boldsymbol{BA}^i\boldsymbol{v} \\ &= \sum_{i=0}^{n-1} c_i(b_0\boldsymbol{I} + b_1\boldsymbol{A} + ... + b_{n-1}\boldsymbol{A}^{n-1})\boldsymbol{A}^i\boldsymbol{v} \qquad \text{by (2)} \\ &= (b_0\boldsymbol{I} + b_1\boldsymbol{A} + ... + b_{n-1}\boldsymbol{A}^{n-1})\sum_{i=0}^{n-1} c_i\boldsymbol{A}^i\boldsymbol{v} \\ &= (b_0\boldsymbol{I} + b_1\boldsymbol{A} + ... + b_{n-1}\boldsymbol{A}^{n-1})(c_0\boldsymbol{v} + c_1\boldsymbol{Av} + ... + c_{n-1}\boldsymbol{A}^{n-1}\boldsymbol{v}) \\ &= (b_0\boldsymbol{I} + b_1\boldsymbol{A} + ... + b_{n-1}\boldsymbol{A}^{n-1})\boldsymbol{u}. \end{aligned}$$

Hence, for all $\boldsymbol{u} \in \mathbb{F}^n$, we have $\boldsymbol{Bu} = (b_0\boldsymbol{I} + b_1\boldsymbol{A} + ... + b_{n-1}\boldsymbol{A}^{n-1})\boldsymbol{u}$.
This implies that $\boldsymbol{B} = b_0\boldsymbol{I} + b_1\boldsymbol{A} + ... + b_{n-1}\boldsymbol{A}^{n-1}$, i.e. $\boldsymbol{B} \in V$.
Thus we have $W \subseteq V$. Combined with the fact that $V \subseteq W$, we have $W = V$.

In conclusion, we have $W =$span$\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^{n-1}\}$. Combined with the result in 5(b), we have proven that $\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^{n-1}\}$ is a basis for W.

Question 6

(a) Let dim(V)= n, and let $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_k\}$ be a basis for Ker(T).
Since Ker$(T)$$\in V$, we can extend B to a basis B_2 for V, where

$$B_2 = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_k, \boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_l\}$$

and $k + l = n$. Then,

$$\begin{aligned} \text{Range}(T) &= \text{span}\{T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), ..., T(\boldsymbol{u}_k), T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), ..., T(\boldsymbol{v}_l)\} \\ &= \text{span}\{T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), ..., T(\boldsymbol{v}_l)\}. \end{aligned}$$

Now, by the Dimension Theorem, $\dim(V)$=Nullity(T)+Rank(T).
Hence, Rank(T)= $n-k=l$. Thus, we have that $\{T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), ..., T(\boldsymbol{v}_l)\}$ forms a basis for Range(T).

Let $B_3 = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_k, T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), ..., T(\boldsymbol{v}_l)\}$. Then, we have that $|B_3| = k + l = n$.
We now proceed to show that B_3 is a set of linearly independent vectors in V.
Suppose that

$$a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + ... + a_k\boldsymbol{u}_k + b_1T(\boldsymbol{v}_1) + b_2T(\boldsymbol{v}_2) + b_lT(\boldsymbol{v}_l) = \boldsymbol{0} \quad (3)$$

for some $a_1, a_2, ..., a_k, b_1, b_2, ..., b_l \in \mathbb{F}$. Rearranging, we obtain

$$a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + ... + a_k\boldsymbol{u}_k = -(b_1T(\boldsymbol{v}_1) + b_2T(\boldsymbol{v}_2) + b_lT(\boldsymbol{v}_l)).$$

Hence, we obtain that $(a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + ... + a_k\boldsymbol{u}_k) \in$Ker($T$)$\cap$Range($T$). Since we are given that Ker(T)$\cap$Range(T)$=\{\boldsymbol{0}\}$, we obtain

$$a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + ... + a_k\boldsymbol{u}_k = \boldsymbol{0}.$$

Since $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_k\}$, we must have $a_1 = a_2 = ... = a_k = 0$.
From (3), we have

$$b_1T(\boldsymbol{v}_1) + b_2T(\boldsymbol{v}_2) + b_lT(\boldsymbol{v}_l) = \boldsymbol{0}.$$

Since $\{T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), ..., T(\boldsymbol{v}_l)\}$ forms a basis for Range(T), we have $b_1 = b_2 = ... = b_l = 0$.
Hence, we conclude that B_3 is a set of linearly independent vectors in V.
Combined with the fact that $|B_3| = k + l = n$, B_3 forms a basis for V.

Thus, for any $\boldsymbol{v} \in V$, we can express v as

$$\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + ... + c_k\boldsymbol{u}_k + d_1T(\boldsymbol{v}_1) + d_2T(\boldsymbol{v}_2) + d_lT(\boldsymbol{v}_l)$$

for some unique $c_1, c_2, ..., c_k, d_1, d_2, ..., d_l \in \mathbb{F}$.
Hence,

$$\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{y}$$

for some unique $\boldsymbol{x} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + ... + c_k\boldsymbol{u}_k \in$Ker($T$) and some unique $\boldsymbol{y} = d_1T(\boldsymbol{v}_1) + d_2T(\boldsymbol{v}_2) + d_lT(\boldsymbol{v}_l) \in$Range($T$).
Together with the fact that Ker(T)$\cap$Range(T)$=\{\boldsymbol{0}\}$, we have proven that V =Ker(T)$\oplus$Range(T).

(b) Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ for all $(x, y) \in \mathbb{R}^2$.

Now, Range(T)=Ker(T)$= \{(x, 0)|x \in \mathbb{R})\}$, and hence we have $\mathbb{R}^2 \neq \text{Ker}(T) + \text{R}(T)$.

(c) For any $i = 1, 2, 3, ...$, suppose we have $\boldsymbol{u} \in \text{Ker}(T^i)$. Then,

$$\begin{aligned} T^i(\boldsymbol{u}) &= \boldsymbol{0} \\ \Rightarrow T(T^i(\boldsymbol{u})) &= T(\boldsymbol{0}) \\ \Rightarrow T^{i+1}(\boldsymbol{u}) &= \boldsymbol{0}. \end{aligned}$$

Hence, $\boldsymbol{u} \in \text{Ker}(T^{i+1})$, and thus we have $\text{Ker}(T^i) \subseteq \text{Ker}(T^{i+1})$ for $i = 1, 2, 3...$

(d) From (c), we have Ker(T^i)$\subseteq$Ker(T^{i+1}), which implies that Nullity(T^i)$\leq$Nullity(T^{i+1}) for all $i \in \mathbb{N}$.
We also have that for all $i \in \mathbb{N}$, we have Nullity(T^i)$\leq$dim(V).
Since V is finite dimensional, there must exist a positive integer m such that:
Nullity(T^m)=Nullity(T^{m+1})=Nullity(T^{m+2})=..., otherwise V cannot be finite dimensional.

Combined with the fact that Ker(T^m)$\subseteq$Ker(T^{m+1})$\subseteq$Ker(T^{m+2})$\subseteq$..., we have the result that there exists a positive integer m such that Ker(T^m)=Ker(T^n) for all positive integers $n \geq m$.
In particular, we have Ker(T^m)=Ker(T^{2m}).

Let $\boldsymbol{u} \in$Ker(T^m)$\cap$R(T^m).
Then, we have $T^m(\boldsymbol{u}) = \mathbf{0}$, and $\boldsymbol{u} = T^m(\boldsymbol{v})$ for some $\boldsymbol{v} \in V$.
Now,

$$\begin{aligned} \boldsymbol{u} &= T^m(\boldsymbol{v}) \\ \Rightarrow T^m(\boldsymbol{u}) &= T^{2m}(\boldsymbol{v}) \\ \Rightarrow T^{2m}(\boldsymbol{v}) &= \mathbf{0}. \end{aligned}$$

Thus, $\boldsymbol{v} \in \mathrm{Ker}(T^{2m}) = \mathrm{Ker}(T^m)$, which implies that $\mathbf{0} = \boldsymbol{u} = T^m(\boldsymbol{v})$.
Hence, $\mathrm{Ker}(T^m) \cap \mathrm{R}(T^m) = \{\mathbf{0}\}$.
By (a), we conclude that $V = \mathrm{Ker}(T^m) \oplus \mathrm{R}(T^m)$, and we are done.

Question 7

(a) Since T is a self-adjoint operator on V, we have $T = T^*$. Hence, for all $\boldsymbol{u} \in V$,

$$\begin{aligned} \langle T(\boldsymbol{u}), \boldsymbol{u} \rangle &= \langle \boldsymbol{u}, T^*(\boldsymbol{u}) \rangle \\ &= \langle \boldsymbol{u}, T(\boldsymbol{u}) \rangle \\ &= \overline{\langle T(\boldsymbol{u}), \boldsymbol{u} \rangle}. \end{aligned}$$

Hence, $\langle T(\boldsymbol{u}), \boldsymbol{u} \rangle$ is a real number for all $\boldsymbol{u} \in V$.

(b) (i) P is self-adjoint since $P^* = (S^* \circ S)^* = S^* \circ S^{**} = S^* \circ S = P$.
Let $\boldsymbol{u} \in V$. Then,

$$\begin{aligned} \langle P(\boldsymbol{u}), \boldsymbol{u} \rangle &= \langle S^* \circ S(\boldsymbol{u}), \boldsymbol{u} \rangle \\ &= \langle S(\boldsymbol{u}), S(\boldsymbol{u}) \rangle \geq 0 \end{aligned}$$

by the axioms of inner product spaces.

(ii) Now, $\langle \boldsymbol{u}, \boldsymbol{u} \rangle$ is a real number greater than or equal to zero for all $\boldsymbol{u} \in V$.
Let $\boldsymbol{v}$ be an eigenvector of P associated with eigenvalue λ. Then,

$$\begin{aligned} \langle P(\boldsymbol{v}), \boldsymbol{v} \rangle &= \langle \lambda \boldsymbol{v}, \boldsymbol{v} \rangle \\ &= \lambda \langle \boldsymbol{v}, \boldsymbol{v} \rangle \end{aligned}$$

From (a) and (b), we know that $\langle P(\boldsymbol{v}), \boldsymbol{v} \rangle$ is a positive real number for all $\boldsymbol{v} \in V$.
Since $\langle \boldsymbol{v}, \boldsymbol{v} \rangle$ is a real number greater than or equal to zero, we must have λ to be a real number greater than or equal to zero.
Thus, we conclude that all eigenvalues of P are non-negative real numbers.

Since P is also self-adjoint and V is a finite dimensional inner product space over $\mathbb{C}$, it is normal, and hence is unitarily diagonalizable. We thus can find an orthnormal basis B such that

$$[P]_B = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{pmatrix}$$

where $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of P, and all of them are non-negative real numbers. We now construct a linear operator S, such that

$$\begin{aligned} [S]_B &= \begin{pmatrix} \sqrt{\lambda_1} & 0 & \cdots & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \sqrt{\lambda_n} \end{pmatrix} \\ &= ([S]_B)^* \\ &= [S^*]_B \end{aligned}$$

Then,

$$\begin{aligned} [S^* \circ S]_B &= [S^*]_B [S]_B \\ &= \begin{pmatrix} \sqrt{\lambda_1} & 0 & \cdots & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \sqrt{\lambda_n} \end{pmatrix}^2 \\ &= \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{pmatrix} \\ &= [P]_B \end{aligned}$$

which implies that $P = S^* \circ S$, and we are done.