NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1102R Calculus
AY 2009/2010 Sem 1

Question 1

(a) By L'Hôpital's Rule,

$$\begin{aligned}
\lim_{x\to\frac{\pi}{4}} \frac{\sec^2 x - 2\tan x}{1+\cos 4x} &= \lim_{x\to\frac{\pi}{4}} \frac{2\sec^2 x\tan x - 2\sec^2 x}{-4\sin 4x} \\
&= \lim_{x\to\frac{\pi}{4}} \frac{2\sec^4 x + 4\tan^2 x\sec^2 x - 4\tan x\sec^2 x}{-16\cos 4x} \\
&= \frac{8+8-8}{-16} \\
&= -\frac{1}{2}
\end{aligned}$$

(b) $-x \le x\cos\frac{1}{\sqrt{x}} \le x$
Therefore, we have

$$\lim_{x\to 0} -x \le \lim_{x\to 0} x\cos\frac{1}{\sqrt{x}} \le \lim_{x\to 0} x$$
$$0 \le \lim_{x\to 0} x\cos\frac{1}{\sqrt{x}} \le 0$$

By Squeeze Theorem,

$$\lim_{x\to 0} x\cos\frac{1}{\sqrt{x}} = 0$$

Question 2

(a) Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{3}$, such that $|x-2| < \delta$ implies,

$$\left|\frac{3x^2-x-4}{x+1} - 2\right| = \left|\frac{3x^2-3x-6}{x+1}\right| = 3\,|x-2| < 3\delta = \epsilon$$

(b)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{((e^x+1)\sqrt{x^2+2})'(x-8)^5 - ((x-8)^5)'(e^x+1)\sqrt{x^2+2}}{((x-8)^5)^2} \\
&= \frac{\left(e^x\sqrt{x^2+2} + (e^x+1)\frac{2x}{2\sqrt{x^2+2}}\right)(x-8)^5 - 5(x-8)^4(e^x+1)\sqrt{x^2+2}}{(x-8)^{10}} \\
&= \frac{\left(e^x\sqrt{x^2+2} + (e^x+1)\frac{x}{\sqrt{x^2+2}}\right)(x-8) - 5(e^x+1)\sqrt{x^2+2}}{(x-8)^6}
\end{aligned}$$

Question 3

(a) $f'(x) = 3x^2 - 18x + 24$.
let $f'(x) > 0$, we obtain that $x > 4$ or $x < 2$.
let $f'(x) < 0$, we have $2 < x < 4$.
Therefore, $f(x)$ is increasing on $(-\infty, 2) \cup (4, \infty)$, and decreasing on $(2, 4)$.

(b) $x \in \mathbb{R}$ for $f'(x)$, let $f'(x) = 0$, we can find out all the critical points at $x = 2$, and $x = 4$.
Since $f'(x) > 0$ on $(-\infty, 2) \cup (4, \infty)$, and $f'(x) < 0$ on $(2, 4)$.
By First Derivative Test, we obtain that $f(x)$ have a local maximum at $x = 2$, and a local minimum at $x = 4$.

(c) $f''(x) = 6x - 18$.
let $f''(x) > 0$, then $x > 3$.
let $f''(x) < 0$, then $x < 3$.
Thus, $f(x)$ is concave up on $(3, \infty)$, and concave down on $(-\infty, 3)$.

(d) $x \in \mathbb{R}$ for $f''(x)$, let $f''(x) = 0$, we obtain its unique inflection point at $x = 3$.
$f(3) = 3^3 - 9 \times 3^2 + 24 \times 3 - 7 = 11$.
Hence, the coordinates of its inflection point is $(3, 11)$.

Question 4

Let x denotes the length of the side facing the main road in meters, $f(x)$ denotes the total cost.
Hence, $x > 0$,

$$f(x) = 6x + 3\left(x + \left(\frac{1200}{x}\right) \times 2\right).$$

$$f(x) = 9x + \frac{7200}{x}.$$

$$f'(x) = 9 - \frac{7200}{x^2}$$

let $f'(x) = 0$, we have $x = 20\sqrt{2}$.
In addition, $f'(x) > 0$ on $(0, 20\sqrt{2})$, and $f'(x) < 0$ on $(20\sqrt{2}, \infty)$.
Hence, $f(x)$ attains its absolute minimum at $x = 20\sqrt{2}$.
Hence, in order to minimize the cost of the fence, the length of the side facing the main road should be $20\sqrt{2}$ meters.

Question 5

(a)

$$\begin{aligned}
\int \cos x \ln(\sin x)\, dx &= \sin x \ln(\sin x) - \int \sin x \left(\frac{1}{\sin x}\right) \cos x\, dx \\
&= \sin x \ln(\sin x) - \int \cos x\, dx \\
&= \sin x \ln(\sin x) - \sin x + \mathbf{C} \quad \text{where } \mathbf{C} \text{ is a constant}
\end{aligned}$$

(b) Let $a = \sqrt{2-x}$, $x = 2 - a^2$, then $a \in (0,1)$, and $dx = -2a\ da$.

$$\begin{aligned}\int_1^2 x\sqrt{2-x}\ dx &= \int_0^1 (2-a^2)a(-2a)\ da \\ &= \int_0^1 2a^4 - 4a^2\ da \\ &= \left[\frac{2}{5}a^5 - \frac{4}{3}a^3\right]_0^1 \\ &= \frac{2}{5} - \frac{4}{3} \\ &= -\frac{14}{15}\end{aligned}$$

Question 6

(a)

$$\begin{aligned}s &= \int_1^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx \\ &= \int_1^3 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}\ dx \\ &= \int_1^3 \sqrt{1 + \frac{x^4}{4} + \frac{1}{4x^4} - \frac{1}{2}}\ dx \\ &= \int_1^3 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\ dx \\ &= \int_1^3 \frac{x^2}{2} + \frac{1}{2x^2}\ dx \\ &= \left[\frac{x^3}{6} + \frac{1}{2x}\right]_1^3 \\ &= 4\end{aligned}$$

(b)

$$\begin{aligned}V_1 &= \pi \int_a^2 (2x^2)^2\ dx \\ &= \pi \left[\frac{4}{5}x^5\right]_a^2 \\ &= \frac{128\pi}{5} - \frac{4\pi}{5}a^5\end{aligned}$$

$$\begin{aligned}V_2 &= \pi \int_0^{2a^2} a^2 - \left(\sqrt{\frac{y}{2}}\right)^2\ dy \\ &= \pi \left[a^2 y\right]_0^{2a^2} - \pi \left[\frac{y^2}{4}\right]_0^{2a^2} \\ &= 2a^4\pi - a^4\pi \\ &= a^4\pi\end{aligned}$$

thus,

$$\begin{aligned} V &= V_1 + V_2 \\ &= -\frac{4\pi}{5}a^5 + a^4\pi + \frac{128\pi}{5} \end{aligned}$$

$V' = -4a^4\pi + 4a^3\pi$, $a \in (0, 2)$, then $V' = 0$ at $a = 1$.
In addition, $V' > 0$ in $(0, 1)$, $V' < 0$ in $(1, 2)$. By First Derivative Test, for $a \in (0, 2)$, V attains its maximum value at $a = 1$.

Question 7

(a)

$$\begin{aligned} x\frac{dy}{dx} + 2y &= \frac{1}{x + x^3} \\ \frac{dy}{dx} + \frac{2}{x}y &= \frac{1}{x^2 + x^4} \end{aligned}$$

Therefore, $p(x) = \frac{2}{x}$, the integrating factor is

$$\begin{aligned} e^{\int p(x)\ dx} &= e^{\int \frac{2}{x}\ dx} \\ &= e^{2lnx} \\ &= x^2. \end{aligned}$$

$$\begin{aligned} x^2\frac{dy}{dx} + \frac{2}{x}yx^2 &= \frac{x^2}{x^2 + x^4} \\ \frac{d}{dx}x^2y &= \frac{1}{1 + x^2} \\ y &= \frac{\int \frac{1}{1+x^2}\ dx + C}{x^2}, \qquad C \in \mathbb{R} \\ y &= \frac{\arctan x + C}{x^2}, \qquad C \in \mathbb{R} \end{aligned}$$

(b)

$$\begin{aligned} \frac{dP}{dt} &= 0.0008P(100 - P) \\ \frac{1}{0.0008P(100 - P)}\ dP &=\ dt \\ \int \frac{1250}{100P - P^2}\ dP &= \int\ dt \\ 1250\int \frac{1}{50^2 - (P - 50)^2}\ dP &= t + C_1 \\ \frac{1250}{100}\ln\frac{P}{|P - 100|} = \frac{1250}{100}\ln\frac{P}{100 - P} &= t + C_1 \\ e^{0.08t + C_2} &= \frac{P}{100 - P} \\ P(t) &= \frac{100}{1 + C_3e^{-0.08t}} \end{aligned}$$

where C_1, C_2, C_3 are some real numbers. Since $P(0) = 20$, we obtain $C_3 = 4$. Hence,

$$P(t) = \frac{100}{1 + 4e^{-0.08t}}$$

Question 8

(a) Assume $g(c) = 0$ for some $c \in (a, b)$
By Mean Value Theorem, there exist two values $m \in (a, c)$, and $n \in (c, b)$ and

$$g'(m) = \frac{g(c) - g(a)}{c - a} = 0$$
$$g'(n) = \frac{g(b) - g(c)}{b - c} = 0$$

By Mean Value Theorem, there exists a value $x \in (m, n)$ such that $g''(x) = \frac{g'(n)-g'(m)}{n-m} = 0$ which contradicts with $g''(x) \neq 0$. Thus, $g(x) \neq 0$ for all $x \in (a, b)$.

(b) Let $h(x) = f(x)g'(x) - f'(x)g(x)$
Thus, we obtain that $h(a) = 0$ and $h(b) = 0$.
By Mean Value Theorem,
there exists a value $c \in (a, b)$ such that $h'(c) = 0$.
Hence,

$$h'(c) = f(c)g''(c) - f''(c)g(c) = 0$$

Since $c \in (a, b)$, we have $g(c) \neq 0$ and $g''(c) \neq 0$,
we obtain that there exists $c \in (a, b)$ such that $\frac{f(c)}{g(c)} = \frac{f''(c)}{g''(c)}$.

Question 9

Consider $xg(x)$ for $x \in \mathbb{R}$.

$$xg(x) = \int_0^1 xf(xt)\, dt = \int_0^x f(u)\, du \quad \text{where } u = xt$$
$$\Rightarrow g(x) = \frac{\int_0^x f(u)\, du}{x} \quad \text{for } x \in \mathbb{R} \setminus \{0\}$$

Hence $g(x)$ is differentiable for all $x \in \mathbb{R} \setminus \{0\}$ and

$$g'(x) = \frac{xf(x) - \int_0^x f(u)\, du}{x^2} = \frac{f(x) - g(x)}{x} \quad \text{for } x \in \mathbb{R} \setminus \{0\}.$$

Now, let $\lim_{x\to 0} \frac{f(x)}{x} = M$.

$$\Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{f(x)}{x} \cdot x = \left(\lim_{x\to 0} \frac{f(x)}{x}\right)\left(\lim_{x\to 0} x\right) = M \cdot 0 = 0$$

Hence, by the continuity of f, $f(0) = 0$. Furthermore, notice that $g(0) = \int_0^1 f(0)\,dt = f(0) = 0$.

$$\begin{aligned}
\Rightarrow g'(0) &= \lim_{x\to 0} \frac{g(x) - g(0)}{x} \\
&= \lim_{x\to 0} \frac{\int_0^x f(u)\,du}{x^2} \\
&= \lim_{x\to 0} \frac{f(x)}{2x} \quad \text{by L'Hôpital's rule} \\
&= \frac{M}{2} \\
\lim_{x\to 0} g'(x) &= \lim_{x\to 0} \frac{f(x) - g(x)}{x} \\
&= \lim_{x\to 0} \frac{f(x)}{x} - \lim_{x\to 0} \frac{g(x)}{x} \\
&= M - \frac{M}{2} = \frac{M}{2}
\end{aligned}$$

Therefore, g' is continuous at 0.

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ERRATA FOR PAST YEAR PAPER SOLUTIONS

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Question 1a The original solution evaluated the limit to be $\frac{8+8-8}{-16} = -\frac{1}{2}$. It should have been $\frac{8+8-8}{16} = \frac{1}{2}$.

Question 5b The original solution gave the definite intergal to be $\int_0^1 (2 - a^2)a(-2a)da$ after the substitution. The substitued intergal should have been $\int_1^0 (2-a^2)a(-2a)da$. This would lead the answer to be $\frac{14}{15}$ instead of the given $-\frac{14}{15}$.

Question 6a The original solution evaluated the integral as: $\int_1^3 \frac{x^2}{2} + \frac{1}{2x^2}dx = [\frac{x^3}{6} + \frac{1}{2x}]_1^3 = 4$. It should have been: $\int_1^3 \frac{x^2}{2} + \frac{1}{2x^2}dx = [\frac{x^3}{6} - \frac{1}{2x}]_1^3 = \frac{14}{3}$.

END OF ERRATA