NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

Written by Lin Mingyan, Simon
Audited by Chua Hongshen

MA1101R Linear Algebra I
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Question 1

(i) We have

$$\boldsymbol{Av} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0\cdot 1 + 1\cdot 1 + 1\cdot 2 + 0\cdot 2 \\ (-1)\cdot 1 + 0\cdot 1 + 0\cdot 2 + (-1)\cdot 2 \\ 1\cdot 1 + 0\cdot 1 + 0\cdot 2 + 1\cdot 2 \\ 0\cdot 1 + (-1)\cdot 1 + (-1)\cdot 2 + 0\cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \\ -3 \end{pmatrix} = \boldsymbol{b_1}.$$

(ii) By Gaussian elimination on the augmented matrix $(\boldsymbol{A}|\boldsymbol{0})$, we have

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{array}\right) \xrightarrow{R_2+R_3\to R_2} \left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{array}\right)$$
$$\xrightarrow{R_1+R_4\to R_4} \left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
$$\xrightarrow{R_1\leftrightarrow R_2} \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
$$\xrightarrow{R_1\leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

So a basis for the nullspace of $\boldsymbol{A}$ is $\left\{(-1,0,0,1)^T, (0,-1,1,0)^T\right\}$.

(iii) The solution set for the matrix equation $\boldsymbol{Ax} = \boldsymbol{b_1}$ is $\left\{(1-s, 1-t, 2+t, 2+s)^T \mid s,t \in \mathbb{R}\right\}$.

(iv) By Gaussian elimination on the augmented matrix $(\boldsymbol{A}|\boldsymbol{b_2})$, we have

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 2 \\ -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & -2 \\ 0 & -1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_2+R_3\to R_2} \left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 1 & -2 \\ 0 & -1 & -1 & 0 & 1 \end{array}\right)$$
$$\xrightarrow{R_1+R_4\to R_4} \left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right)$$

$$\xrightarrow{R_2+\frac{2}{3}R_4\to R_2} \left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right)$$

$$\xrightarrow{R_1\leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right)$$

$$\xrightarrow{R_2\leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right)$$

$$\xrightarrow{R_3\leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Since $(\boldsymbol{A}|\boldsymbol{b_2})$ has a row echelon form where the last column is a pivot column, we see that the matrix equation $\boldsymbol{Ax} = \boldsymbol{b_2}$ is inconsistent.

(v) By part (ii), we have $B = \left\{(0,-1,1,0)^T, (1,0,0,-1)^T\right\}$ to be a basis for the column space of $\boldsymbol{A}$. Furthermore, it is clear that B is orthogonal. Let $\boldsymbol{u_1} = (0,-1,1,0)^T$ and $\boldsymbol{u_2} = (1,0,0,-1)^T$. Then the projection $\boldsymbol{p}$ of $\boldsymbol{b_2}$ onto the column space of $\boldsymbol{A}$ is given by

$$\begin{aligned} \boldsymbol{p} &= \frac{\boldsymbol{u_1}\cdot\boldsymbol{b_2}}{||\boldsymbol{u_1}||^2}\boldsymbol{u_1} + \frac{\boldsymbol{u_2}\cdot\boldsymbol{b_2}}{||\boldsymbol{u_2}||^2}\boldsymbol{u_2} \\ &= \frac{(0,-1,1,0)^T\cdot(2,0,-2,1)^T}{2}(0,-1,1,0)^T + \frac{(1,0,0,-1)^T\cdot(2,0,-2,1)^T}{2}(1,0,0,-1)^T \\ &= \left(\frac{1}{2},1,-1,-\frac{1}{2}\right)^T. \end{aligned}$$

By Gaussian elimination on the augmented matrix $(\boldsymbol{A}|\boldsymbol{p})$, we have

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & \frac{1}{2} \\ -1 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & 0 & -\frac{1}{2} \end{array}\right) \xrightarrow{R_2+R_3\to R_2} \left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & 0 & -\frac{1}{2} \end{array}\right)$$

$$\xrightarrow{R_1+R_4\to R_4} \left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1\leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2\leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Since a least square solution to the equation $\boldsymbol{Ax} = \boldsymbol{b_2}$ is a solution to the equation $\boldsymbol{Ax} = \boldsymbol{p}$, and vice versa, we see that two different least square solutions to the equation $\boldsymbol{Ax} = \boldsymbol{b_2}$ are

$\boldsymbol{x_1} = \left(-1, \frac{1}{2}, 0, 0\right)^T$ and $\boldsymbol{x_2} = \left(0, 0, \frac{1}{2}, -1\right)^T$. Lastly, we check that

$$\begin{aligned}
\boldsymbol{Ax_1} &= \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \cdot (-1) + 1 \cdot \frac{1}{2} + 1 \cdot 0 + 0 \cdot 0 \\ (-1) \cdot (-1) + 0 \cdot \frac{1}{2} + 0 \cdot 0 + (-1) \cdot 0 \\ 1 \cdot (-1) + 0 \cdot \frac{1}{2} + 0 \cdot 0 + 1 \cdot 0 \\ 0 \cdot (-1) + (-1) \cdot \frac{1}{2} + (-1) \cdot 0 + 0 \cdot 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ 1 \\ -1 \\ -\frac{1}{2} \end{pmatrix}, \text{ and} \\
\boldsymbol{Ax_2} &= \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \cdot 0 + 1 \cdot 0 + 1 \cdot \frac{1}{2} + 0 \cdot (-1) \\ (-1) \cdot 0 + 0 \cdot 0 + 0 \cdot \frac{1}{2} + (-1) \cdot (-1) \\ 1 \cdot 0 + 0 \cdot 0 + 0 \cdot \frac{1}{2} + 1 \cdot (-1) \\ 0 \cdot 0 + (-1) \cdot 0 + (-1) \cdot \frac{1}{2} + 0 \cdot (-1) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ 1 \\ -1 \\ -\frac{1}{2} \end{pmatrix},
\end{aligned}$$

so we have $\boldsymbol{Ax_1} = \boldsymbol{Ax_2}$ as desired.

(vi) It is not possible for the matrix equation $\boldsymbol{Ax} = \boldsymbol{b_1} + \boldsymbol{b_3}$ to be consistent. Note that for a given $\boldsymbol{a} \in \mathbb{R}^4$, we have the equation $\boldsymbol{Ax} = \boldsymbol{a}$ to be consistent if and only if $\boldsymbol{a}$ belongs to the column space of $\boldsymbol{A}$. Arguing by contradiction, suppose that the equation $\boldsymbol{Ax} = \boldsymbol{b_1} + \boldsymbol{b_3}$ is consistent. Then $\boldsymbol{b_1} + \boldsymbol{b_3}$ must belong to the column space of $\boldsymbol{A}$. As the column space of $\boldsymbol{A}$ is a vector subspace of $\mathbb{R}^4$, and $\boldsymbol{b_1}$ belongs to the column space of $\mathbf{A}$ by part (a), we must have $\boldsymbol{b_3} = (\boldsymbol{b_1} + \boldsymbol{b_3}) - \boldsymbol{b_1}$ to belong to the column space of $\boldsymbol{A}$, a contradiction.

Question 2

By Gaussian elimination on the augmented matrix $(\boldsymbol{X}|\boldsymbol{I})$, we have

$$\begin{aligned}
\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 1 & 3 & -2 & 0 & 0 & 1 \end{array}\right) &\xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 1 & 3 & -2 & 0 & 0 & 1 \end{array}\right) \\
&\xrightarrow{R_2 - R_1 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 2 & -1 & -\frac{1}{2} & 1 & 0 \\ 1 & 3 & -2 & 0 & 0 & 1 \end{array}\right) \\
&\xrightarrow{R_3 - R_1 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 2 & -1 & -\frac{1}{2} & 1 & 0 \\ 0 & 3 & -2 & -\frac{1}{2} & 0 & 1 \end{array}\right) \\
&\xrightarrow{R_2 - \frac{1}{2}R_3 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{4} & 1 & -\frac{1}{2} \\ 0 & 3 & -2 & -\frac{1}{2} & 0 & 1 \end{array}\right)
\end{aligned}$$

$$\xrightarrow{2R_2\to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 2 & -1 \\ 0 & 3 & -2 & -\frac{1}{2} & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3-3R_2\to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 2 & -1 \\ 0 & 0 & -2 & 1 & -6 & 4 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}R_3\to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 2 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} & 3 & -2 \end{array}\right).$$

Hence, we have $\boldsymbol{X}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 2 & -1 \\ -\frac{1}{2} & 3 & -2 \end{pmatrix}$.

(ii) We have $\det(\boldsymbol{X}) = 2(2)(-2) + 0(-1)(1) + 0(1)(3) - 0(2)(1) - 0(1)(-2) - 2(-1)(3) = -2$. Thus, we have

$$\mathbf{adj}(\boldsymbol{X}) = \det(\boldsymbol{X})\boldsymbol{X}^{-1} = -2\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 2 & -1 \\ -\frac{1}{2} & 3 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -4 & 2 \\ 1 & -6 & 4 \end{pmatrix}.$$

(iii) Let us denote the standard matrix for the linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ by $\boldsymbol{Z}$. Since we have $(T \circ S)(\boldsymbol{x}) = \boldsymbol{x}$ for all non-zero $\boldsymbol{x} \in \mathbb{R}^3$, it follows that $T \circ S = I$, where I denotes the identity transformation on $\mathbb{R}^3$. Hence, the standard matrix for the linear transformation $T \circ S : \mathbb{R}^3 \to \mathbb{R}^3$ is the identity matrix $\boldsymbol{I}$, which implies that $\boldsymbol{ZX} = \boldsymbol{I}$. Hence, we have $\boldsymbol{Z} = \boldsymbol{X}^{-1}$, so the formula for the linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$S\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 2 & -1 \\ -\frac{1}{2} & 3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{x_1}{2} \\ -\frac{x_1}{2} + 2x_2 - x_3 \\ -\frac{x_1}{2} + 3x_2 - 2x_3 \end{pmatrix}$$

for all $(x_1, x_2, x_3)^T \in \mathbb{R}^3$.

(iv) Note that

$$\lambda\boldsymbol{I} - \boldsymbol{X} = \begin{pmatrix} \lambda - 2 & 0 & 0 \\ -1 & \lambda - 2 & 1 \\ -1 & -3 & \lambda + 2 \end{pmatrix}$$

for all $\lambda \in \mathbb{R}$. This implies that

$$\begin{aligned} \det(\lambda\boldsymbol{I} - \boldsymbol{X}) &= (\lambda - 2)(\lambda - 2)(\lambda + 2) - (\lambda - 2)(1)(-3) \\ &= (\lambda - 2)^2(\lambda + 2) + 3(\lambda - 2) \\ &= (\lambda - 2)((\lambda - 2)(\lambda + 2) + 3) \\ &= (\lambda - 2)(\lambda^2 - 4 + 3) \\ &= (\lambda - 2)(\lambda^2 - 1) \\ &= (\lambda - 2)(\lambda - 1)(\lambda + 1). \end{aligned}$$

Note that $\det(\lambda\boldsymbol{I} - \boldsymbol{X}) = 0$ if and only if $\lambda = 2, 1, -1$. So the eigenvalues λ of $\boldsymbol{X}$ are $\lambda = 2, 1, -1$.

When $\lambda = 2$, we see from Gaussian elimination on the augmented matrix $(\lambda\boldsymbol{I} - \boldsymbol{X}|\boldsymbol{0})$ that

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -3 & 4 & 0 \end{array}\right) \xrightarrow{-R_2+R_3\to R_3} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So a basis for E_2 is $\{(1,1,1)^T\}$.

When $\lambda = 1$, we see from Gaussian elimination on the augmented matrix $(\lambda \boldsymbol{I} - \boldsymbol{X}|\boldsymbol{0})$ that

$$\left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -3 & 3 & 0 \end{array}\right) \xrightarrow{-R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & -3 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{-R_1+R_3 \to R_3} \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{-3R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So a basis for E_1 is $\{(0,1,1)^T\}$.

When $\lambda = -1$, we see from Gaussian elimination on the augmented matrix $(\lambda \boldsymbol{I} - \boldsymbol{X}|\boldsymbol{0})$ that

$$\left(\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ -1 & -3 & 1 & 0 \\ -1 & -3 & 1 & 0 \end{array}\right) \xrightarrow{-R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ -1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So a basis for E_{-1} is $\{(0,1,3)^T\}$.

(v) Since $\boldsymbol{X}$ has 3 distinct eigenvalues, $\boldsymbol{X}$ is diagonalizable, and a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{X}$ is $\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}$.

(vi) Clearly, we have $E_{\lambda_1} + E_{\lambda_2} + \cdots + E_{\lambda_k} \subseteq \mathbb{R}^n$, so it remains to show that $\mathbb{R}^n \subseteq E_{\lambda_1} + E_{\lambda_2} + \cdots + E_{\lambda_k}$. To this end, write $S_{\lambda_i} = \{\boldsymbol{u}_{i,1}, \boldsymbol{u}_{i,2}, \cdots, \boldsymbol{u}_{i,n_i}\}$ for all $i = 1, 2, \cdots, k$, where n_i is a positive integer for all $i = 1, 2, \cdots, k$. Furthermore, let $S = \bigcup_{i=1}^{k} S_{\lambda_i}$. Then S is linearly independent, since the S_{λ_i}'s are chosen to be bases for the E_{λ_i}'s for all $i = 1, 2, \cdots, k$. As $\boldsymbol{Y}$ is a diagonalizable matrix of order n, we must have $|S| = n$. Hence, we have S to be a basis for $\mathbb{R}^n$.

Now, let us take any $\boldsymbol{x} \in \mathbb{R}^n$. As S is a basis for $\mathbb{R}^n$, it follows that for all $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, n_i$, there exist (unique) $a_{i,j} \in \mathbb{R}$, such that $\boldsymbol{x} = \sum_{i=1}^{k} \left(\sum_{j=1}^{n_i} a_{i,j} \boldsymbol{u}_{i,j} \right)$. As E_{λ_i} is a vector subspace of $\mathbb{R}^n$ for all $i = 1, 2, \cdots, k$, we must have $\boldsymbol{v}_i := \sum_{j=1}^{n_i} a_{i,j} \boldsymbol{u}_{i,j} \in E_{\lambda_i}$ for all $i = 1, 2, \cdots, k$.

This implies that $\boldsymbol{x} = \sum_{i=1}^{k} \left(\sum_{j=1}^{n_i} a_{i,j} \boldsymbol{u}_{i,j} \right) = \sum_{i=1}^{k} \boldsymbol{v}_i \in E_{\lambda_1} + E_{\lambda_2} + \cdots + E_{\lambda_k}$. As $\boldsymbol{x} \in \mathbb{R}^n$ is arbitrary, this shows that $\mathbb{R}^n \subseteq E_{\lambda_1} + E_{\lambda_2} + \cdots + E_{\lambda_k}$, and we are done.

Question 3

(i) Let $\boldsymbol{A}$ denote the matrix $(\boldsymbol{u_1}\ \boldsymbol{u_2}\ \boldsymbol{u_3})^T$. By Gaussian elimination on the augmented matrix $(\boldsymbol{A}|\boldsymbol{0})$, we have

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_2+R_3\to R_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & \frac{3}{2} & \frac{5}{2} & 0 \end{array}\right).$$

Since the augmented matrix $(\boldsymbol{A}|\boldsymbol{0})$ has a row echelon form where all rows are non-zero, we see that S is linearly independent, and hence S is a basis for V.

(ii) Let $\boldsymbol{B}$ denote the matrix $\left(\boldsymbol{u_1^T}\ \boldsymbol{u_2^T}\ \boldsymbol{u_3^T}\right)$. By Gaussian elimination on the augmented matrix $(\boldsymbol{B}|\boldsymbol{w}^T)$, we have

$$\begin{aligned}
\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 1 & -2 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 3 \end{array}\right) &\xrightarrow{-R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -2 & 1 & -4 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 3 \end{array}\right) \\
&\xrightarrow{-2R_1+R_4\to R_4} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -2 & 1 & -4 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & -3 \end{array}\right) \\
&\xrightarrow{2R_3+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 3 & -6 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & -3 \end{array}\right) \\
&\xrightarrow{-R_3+R_4\to R_4} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 3 & -6 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right) \\
&\xrightarrow{-3R_4+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right) \\
&\xrightarrow{-R_4+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right) \\
&\xrightarrow{R_2\leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right) \\
&\xrightarrow{R_3\leftrightarrow R_4} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right).
\end{aligned}$$

Since the last row of the last augmented matrix in the above equation is zero, it follows that $\boldsymbol{w} \in V$, and $(\boldsymbol{w})_S = (3, 1, -2)$.

(iii) By Gram-Schmidt process on $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$, we have

$$\boldsymbol{v_1} := \boldsymbol{u_1}$$

$$
\begin{aligned}
&= (1,1,0,2),\\
\boldsymbol{v_2} &:= \boldsymbol{u_2} - \frac{\boldsymbol{u_2}\cdot\boldsymbol{v_1}}{||\boldsymbol{v_1}||^2}\boldsymbol{v_1}\\
&= (0,-2,1,1) - \frac{(0,-2,1,1)\cdot(1,1,0,2)}{1^2+1^2+2^2}(1,1,0,2)\\
&= (0,-2,1,1),\\
\boldsymbol{v_3} &:= \boldsymbol{u_3} - \frac{\boldsymbol{u_3}\cdot\boldsymbol{v_1}}{||\boldsymbol{v_1}||^2}\boldsymbol{v_1} - \frac{\boldsymbol{u_3}\cdot\boldsymbol{v_2}}{||\boldsymbol{v_2}||^2}\boldsymbol{v_2}\\
&= (0,1,1,2) - \frac{(0,1,1,2)\cdot(1,1,0,2)}{1^2+1^2+2^2}(1,1,0,2) - \frac{(0,1,1,2)\cdot(0,-2,1,1)}{(-2)^2+1^2+1^2}(0,-2,1,1)\\
&= \left(-\frac{5}{6},\frac{1}{2},\frac{5}{6},\frac{1}{6}\right).
\end{aligned}
$$

So an orthogonal basis T for V is $\left\{(1,1,0,2),(0,-2,1,1),\left(-\frac{5}{6},\frac{1}{2},\frac{5}{6},\frac{1}{6}\right)\right\}$.

(iv) By assumption, we have $(0,1,1,2) = \frac{5}{6}(1,1,0,2)+\frac{1}{6}(0,-2,1,1)+\left(-\frac{5}{6},\frac{1}{2},\frac{5}{6},\frac{1}{6}\right)$, so this implies that

$$
\begin{aligned}
&(3,-1,-1,3)\\
=\ & 3(1,1,0,2)+(0,-2,1,1)-2(0,1,1,2)\\
=\ & 3(1,1,0,2)+(0,-2,1,1)-2\left(\frac{5}{6}(1,1,0,2)+\frac{1}{6}(0,-2,1,1)+\left(-\frac{5}{6},\frac{1}{2},\frac{5}{6},\frac{1}{6}\right)\right)\\
=\ & \frac{4}{3}(1,1,0,2)+\frac{2}{3}(0,-2,1,1)-2\left(-\frac{5}{6},\frac{1}{2},\frac{5}{6},\frac{1}{6}\right).
\end{aligned}
$$

Hence, we have $(\boldsymbol{w})_T = \left(\frac{4}{3},\frac{2}{3},-2\right)$.

(v) Note that a row echelon form for the matrix $\boldsymbol{A}$ is $\begin{pmatrix}1&1&0&2\\0&-2&1&1\\0&0&\frac{3}{2}&\frac{5}{2}\end{pmatrix}$. Since the last column in the above row echelon form of $\boldsymbol{A}$ is a non-pivot column, we see that a basis for $\mathbb{R}^4$ that contains S is $\{(1,1,0,2),(0,-2,1,1),(0,1,1,2),(0,0,0,1)\}$.

(vi) Remark. We note that if the kernel of a linear transformation $T:\mathbb{R}^4\to\mathbb{R}^2$ is V, then by the Rank-Nullity Theorem, we must have $\text{rank}(T)+\text{nullity}(T)=4$, which implies that $\text{rank}(T)=1$, since $\text{nullity}(T)=\dim(\text{Ker}(T))=\dim(V)=3$. As

$$\{(1,1,0,2),(0,-2,1,1),(0,1,1,2),(0,0,0,1)\}$$

is a basis for $\mathbb{R}^4$ that contains S, and S is a basis for V, the strategy to constructing a linear transformation $T:\mathbb{R}^4\to\mathbb{R}^2$ that satisfies the given conditions is to first set the images of each of the elements in S under T to $\mathbf{0}$, and set the image of $(0,0,0,1)$ under T to any non-zero element of $\mathbb{R}^2$. The formula for T could be found thereafter using Gaussian elimination.

Let $\boldsymbol{u_4}=\boldsymbol{e_4}=(0,0,0,1)$, and $\boldsymbol{C}$ denote the matrix $\left(\boldsymbol{u_1^T}\ \boldsymbol{u_2^T}\ \boldsymbol{u_3^T}\ \boldsymbol{u_4^T}\right)$. By Gaussian elimination on the augmented matrix $(\boldsymbol{C}|\boldsymbol{e_1}|\boldsymbol{e_2}|\boldsymbol{e_3})$, we have

$$
\left(\begin{array}{cccc|c|c|c}1&0&0&0&1&0&0\\1&-2&1&0&0&1&0\\0&1&1&0&0&0&1\\2&1&2&1&0&0&0\end{array}\right)
\xrightarrow{-R_1+R_2\to R_2}
\left(\begin{array}{cccc|c|c|c}1&0&0&0&1&0&0\\0&-2&1&0&-1&1&0\\0&1&1&0&0&0&1\\2&1&2&1&0&0&0\end{array}\right)
$$

$$
\xrightarrow{-2R_1+R_4\to R_4}
\left(\begin{array}{cccc|c|c|c}1&0&0&0&1&0&0\\0&-2&1&0&-1&1&0\\0&1&1&0&0&0&1\\0&1&2&1&-2&0&0\end{array}\right)
$$

$$\xrightarrow{R_2-R_3\to R_2}\left(\begin{array}{cccc|c|c|c}1&0&0&0&1&0&0\\0&-3&0&0&-1&1&-1\\0&1&1&0&0&0&1\\0&1&2&1&-2&0&0\end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R_2\to R_2}\left(\begin{array}{cccc|c|c|c}1&0&0&0&1&0&0\\0&1&0&0&\frac{1}{3}&-\frac{1}{3}&\frac{1}{3}\\0&1&1&0&0&0&1\\0&1&2&1&-2&0&0\end{array}\right)$$

$$\xrightarrow{-R_2+R_3\to R_3}\left(\begin{array}{cccc|c|c|c}1&0&0&0&1&0&0\\0&1&0&0&\frac{1}{3}&-\frac{1}{3}&\frac{1}{3}\\0&0&1&0&-\frac{1}{3}&\frac{1}{3}&\frac{2}{3}\\0&1&2&1&-2&0&0\end{array}\right)$$

$$\xrightarrow{-R_2+R_4\to R_4}\left(\begin{array}{cccc|c|c|c}1&0&0&0&1&0&0\\0&1&0&0&\frac{1}{3}&-\frac{1}{3}&\frac{1}{3}\\0&0&1&0&-\frac{1}{3}&\frac{1}{3}&\frac{2}{3}\\0&0&2&1&-\frac{7}{3}&\frac{1}{3}&-\frac{1}{3}\end{array}\right)$$

$$\xrightarrow{-2R_3+R_4\to R_4}\left(\begin{array}{cccc|c|c|c}1&0&0&0&1&0&0\\0&1&0&0&\frac{1}{3}&-\frac{1}{3}&\frac{1}{3}\\0&0&1&0&-\frac{1}{3}&\frac{1}{3}&\frac{2}{3}\\0&0&0&1&-\frac{5}{3}&-\frac{1}{3}&-\frac{5}{3}\end{array}\right).$$

Now, let us define a linear transformation $T:\mathbb{R}^4\to\mathbb{R}^2$ by

$$T\left(x_1,x_2,x_3,x_4\right)=\left(-\frac{5x_1}{3}-\frac{x_2}{3}-\frac{5x_3}{3}+x_4,0\right)$$

for all $(x_1,x_2,x_3,x_4)\in\mathbb{R}^4$. Let us show that $\mathrm{Ker}(T)=V$. Since

$$\begin{aligned}T\left(1,1,0,2\right)&=\left(-\frac{5}{3}-\frac{1}{3}+0+2,0\right)=(0,0)\,,\\T\left(0,-2,1,1\right)&=\left(0+\frac{2}{3}-\frac{5}{3}+1,0\right)=(0,0)\,,\text{ and}\\T\left(0,1,1,2\right)&=\left(0-\frac{1}{3}-\frac{5}{3}+2,0\right)=(0,0)\,,\end{aligned}$$

we see that $S\subseteq\mathrm{Ker}(T)$. As $\mathrm{Ker}(T)$ is a vector subspace of $\mathbb{R}^4$, we must have $V=\mathrm{span}(S)\subseteq\mathrm{Ker}(T)$, so that $\mathrm{nullity}(T)=\dim(\mathrm{Ker}(T))\geq\dim(V)=3$. Next, since $(0,0,0,1)\notin\mathrm{Ker}(T)$, we must have $\mathrm{Ker}(T)\neq\mathbb{R}^4$, so we must have $\mathrm{nullity}(T)<4$. So $\mathrm{nullity}(T)=3$, and hence we must have $\mathrm{Ker}(T)=V$ as desired.

Question 4

(i) By Gaussian elimination on $\boldsymbol{Y}$, we have

$$\begin{pmatrix}2&-4&6&8\\2&-1&3&2\\4&-5&9&10\\0&-1&1&2\end{pmatrix}\xrightarrow{-R_1+R_2\to R_2}\begin{pmatrix}2&-4&6&8\\0&3&-3&-6\\4&-5&9&10\\0&-1&1&2\end{pmatrix}$$

$$\xrightarrow{-2R_1+R_3\to R_3}\begin{pmatrix}2&-4&6&8\\0&3&-3&-6\\0&3&-3&-6\\0&-1&1&2\end{pmatrix}$$

$$\xrightarrow{3R_4+R_2\to R_2} \begin{pmatrix} 2 & -4 & 6 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & -6 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow{3R_4+R_3\to R_3} \begin{pmatrix} 2 & -4 & 6 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow{R_2\leftrightarrow R_4} \begin{pmatrix} 2 & -4 & 6 & 8 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, we have rank$(\boldsymbol{Y}) = 2$ as required.

(ii) By Rank-Nullity Theorem, we have nullity$(\boldsymbol{Y}) = 4 - \text{rank}(\boldsymbol{Y}) = 2$. Moreover, we have

$$\boldsymbol{Yv_1} = \begin{pmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\cdot 2+(-4)\cdot 0+6\cdot(-2)+8\cdot 1 \\ 2\cdot 2+(-1)\cdot 0+3\cdot(-2)+2\cdot 1 \\ 4\cdot 2+(-5)\cdot 0+9\cdot(-2)+10\cdot 1 \\ 0\cdot 2+(-1)\cdot 0+1\cdot(-2)+2\cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ and}$$

$$\boldsymbol{Yv_2} = \begin{pmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2\cdot 1+(-4)\cdot 5+6\cdot(-1)+8\cdot 3 \\ 2\cdot 1+(-1)\cdot 5+3\cdot(-1)+2\cdot 3 \\ 4\cdot 1+(-5)\cdot 5+9\cdot(-1)+10\cdot 3 \\ 0\cdot 1+(-1)\cdot 5+1\cdot(-1)+2\cdot 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This implies that $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ is contained in the nullspace of $\boldsymbol{Y}$. Finally, since $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ is clearly linearly independent, and nullity$(\boldsymbol{Y}) = 2$, we must have $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ to be a basis for the nullspace of $\boldsymbol{Y}$ as desired.

(iii) By Gaussian elimination on the augmented matrix $(\boldsymbol{Z}|\boldsymbol{v_1})$, we have

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-R_1+R_3\to R_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & -4 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_3+R_4\to R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 & -3 \end{array}\right)$$

$$\xrightarrow{-R_2+R_4\to R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & -3 \end{array}\right)$$

$$\xrightarrow{R_2\leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{array}\right).$$

Since $(\boldsymbol{Z}|\boldsymbol{v_1})$ has a row echelon form where the last column is a pivot column, we see that the matrix equation $\boldsymbol{Zx} = \boldsymbol{v_1}$ is inconsistent.

Now, we note that $B = \left\{(1,0,1,0)^T, (1,0,0,1)^T, (0,1,1,0)^T\right\}$ to be a basis for the column space of $\boldsymbol{Z}$. Let $\boldsymbol{u_1} = (0,1,1,0)^T$, $\boldsymbol{u_2} = (1,0,0,1)^T$ and $\boldsymbol{u_3} = (1,0,1,0)^T$. Then it is clear that $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are orthogonal to each other. By Gram-Schmidt process on $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$, we have

$$\begin{aligned} & \boldsymbol{u_3} - \frac{\boldsymbol{u_3} \cdot \boldsymbol{u_1}}{||\boldsymbol{u_1}||^2}\boldsymbol{u_1} - \frac{\boldsymbol{u_3} \cdot \boldsymbol{u_2}}{||\boldsymbol{u_2}||^2}\boldsymbol{u_2} \\ = \; & (1,0,1,0)^T - \frac{(1,0,1,0)^T \cdot (0,1,1,0)^T}{1^2+1^2}(0,1,1,0)^T - \frac{(1,0,1,0)^T \cdot (1,0,0,1)^T}{1^2+1^2}(1,0,0,1)^T \\ = \; & \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T. \end{aligned}$$

So an orthogonal basis B' for the column space of $\boldsymbol{Z}$ is

$$\left\{(0,1,1,0)^T, (1,0,0,1)^T, \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T\right\}.$$

Now, let $\boldsymbol{p}$ be the projection of $\boldsymbol{v_1}$ onto the column space of $\boldsymbol{Z}$, and $\boldsymbol{w_1} = (0,1,1,0)^T$, $\boldsymbol{w_2} = (1,0,0,1)^T$ and $\boldsymbol{w_3} = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T$. Now, we have

$$\begin{aligned} w_{\boldsymbol{1}} \cdot \boldsymbol{v_1} &= (0,1,1,0)^T \cdot (2,0,-2,1)^T = -2, \\ w_{\boldsymbol{2}} \cdot \boldsymbol{v_1} &= (1,0,0,1)^T \cdot (2,0,-2,1)^T = 3, \\ w_{\boldsymbol{3}} \cdot \boldsymbol{v_1} &= \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T \cdot (2,0,-2,1)^T = -\frac{1}{2}, \\ \boldsymbol{p} &= \frac{\boldsymbol{w_1} \cdot \boldsymbol{v_1}}{||\boldsymbol{w_1}||^2}\boldsymbol{w_1} + \frac{\boldsymbol{w_2} \cdot \boldsymbol{v_1}}{||\boldsymbol{w_2}||^2}\boldsymbol{w_2} + \frac{\boldsymbol{w_3} \cdot \boldsymbol{v_1}}{||\boldsymbol{w_3}||^2}\boldsymbol{w_3} \\ &= \frac{-2}{2}(0,1,1,0)^T + \frac{3}{2}(1,0,0,1)^T - \frac{1}{2}\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T \\ &= \left(\frac{5}{4}, -\frac{3}{4}, -\frac{5}{4}, \frac{7}{4}\right)^T. \end{aligned}$$

Since $\boldsymbol{p}$ belongs to the column space of $\boldsymbol{Z}$, we must have the matrix equation $\boldsymbol{Zx} = \boldsymbol{p}$ to be consistent. Furthermore, we have $d(\boldsymbol{p}, \boldsymbol{v_1}) \leq d(\boldsymbol{Zx}, \boldsymbol{v_1})$ for all $\boldsymbol{x} \in \mathbb{R}^4$. So a vector $\boldsymbol{w}$ such that $\boldsymbol{Zx} = \boldsymbol{w}$ is consistent and $d(\boldsymbol{w}, \boldsymbol{v}_1)$ is as small as possible is $\boldsymbol{p} = \left(\frac{5}{4}, -\frac{3}{4}, -\frac{5}{4}, \frac{7}{4}\right)^T$.

(iv) Arguing by contradiction, suppose there exist elementary matrices $\boldsymbol{E_1}, \boldsymbol{E_2}, \cdots, \boldsymbol{E_k}$, such that $\boldsymbol{E_k} \cdots \boldsymbol{E_1}\boldsymbol{Y} = \boldsymbol{Z}$. Then this would imply that $\boldsymbol{Y}$ is row-equivalent to $\boldsymbol{Z}$. In particular, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ must have the same row space, and hence the same rank. But from part (iii), we see that $\text{rank}(\boldsymbol{Z}) = 3 \neq 2 = \text{rank}(\boldsymbol{Y})$, which is a contradiction. So there do not exist elementary matrices $\boldsymbol{E_1}, \boldsymbol{E_2}, \cdots, \boldsymbol{E_k}$, such that $\boldsymbol{E_k} \cdots \boldsymbol{E_1}\boldsymbol{Y} = \boldsymbol{Z}$.

(v) Since $\boldsymbol{Yu_1}$, $\boldsymbol{Yu_2}$, $\boldsymbol{Yu_3}$, $\boldsymbol{Yu_4}$ all belong to the column space of $\boldsymbol{Y}$, it follows that U is a vector subspace of the column space of $\boldsymbol{Y}$, and hence $\dim(\text{span}(U)) \leq \text{rank}(\boldsymbol{Y}) = 2$. As $\{(2,2,4,0)^T, (-4,-1,-5,-1)^T\}$ is clearly linearly independent, and $\boldsymbol{Ye_1} = (2,2,4,0)^T$, $\boldsymbol{Ye_2} = (-4,-1,-5,-1)^T$, we see that by setting $\boldsymbol{u_1} = \boldsymbol{e_1}$, $\boldsymbol{u_2} = \boldsymbol{e_2}$, and $\boldsymbol{u_3} = \boldsymbol{u_4} = \boldsymbol{0}$, we have $\{\boldsymbol{Yu_1}, \boldsymbol{Yu_2}\}$ to form a basis for $\text{span}(U)$. So the largest possible value of $\dim(\text{span}(U))$ is 2.