NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA2214 Combinatorial Analysis
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Question 1

(a) The number of integers from 1 to N that are divisible by $a_1, a_2, ..., a_i$, is $\lfloor \frac{N}{l} \rfloor$ where $l = lcm(a_1, a_2, ..., a_i)$.

The solution of this question uses the Principle of Inclusion and Exclusion.

The numbers of integers from 1 to 8000 that are divisible by one or more of the integers in {4, 6, 14, 21} are shown in the table below.

Divisible by	Number of such integers	Divisible by	Number of such integers
4	$\lfloor \frac{8000}{4} \rfloor = 2000$	6, 21	$\lfloor \frac{8000}{42} \rfloor = 190$
6	$\lfloor \frac{8000}{6} \rfloor = 1333$	4, 6, 14	$\lfloor \frac{8000}{84} \rfloor = 95$
4, 6	$\lfloor \frac{8000}{12} \rfloor = 666$	4, 6, 21	$\lfloor \frac{8000}{84} \rfloor = 95$
4, 14	$\lfloor \frac{8000}{28} \rfloor = 285$	4, 14, 21	$\lfloor \frac{8000}{84} \rfloor = 95$
6, 14	$\lfloor \frac{8000}{42} \rfloor = 190$	6, 14, 21	$\lfloor \frac{8000}{42} \rfloor = 190$
4, 21	$\lfloor \frac{8000}{84} \rfloor = 95$	4, 6, 14, 21	$\lfloor \frac{8000}{84} \rfloor = 95$

Let

- $D(a_1, a_2, ..., a_i)$ be the set of integers from 1 to 8000 that are divisible by $a_1, a_2, ..., a_i$;
- S be the set of integers from 1 to 8000 divisible by 4 or 6;
- P_1, P_2, be the properties that are "divisible by 14" and "divisible by 21" respectively;
- $E(m)$ be the number of elements of S possessing exactly m of the 2 properties for $0 \leq m \leq 2$;
- $\omega(P_{i_1} P_{i_2} \dots P_{i_m})$ be the number of elements of S possessing the properties P_{i_1}, P_{i_2}, $\dots$, P_{i_m}, where $1 \leq m \leq 2$;
- $\omega(m) = \sum(\omega(P_{i_1} P_{i_2} \dots P_{i_m}))$, $\omega(0) = |S|$.

Hence, we have

$$\begin{aligned}
\omega(P_1) &= |D(4,14)| + |D(6,14)| - |D(4,6,14)| \\
&= 285 + 190 - 95 \\
&= 380 \\
\omega(P_2) &= |D(4,21)| + |D(6,21)| - |D(4,6,21)| \\
&= 95 + 190 - 95 \\
&= 190 \\
\omega(P_1 P_2) &= |D(4,14,21)| + |D(6,14,21)| - |D(4,6,14,21)| \\
&= 95 + 190 - 95 \\
&= 190.
\end{aligned}$$

$$\begin{aligned}
\omega(0) &= |D(4)| + |D(6)| - |D(4,6)| \\
&= 2000 + 1333 - 666 \\
&= 2667 \\
\omega(1) &= \omega(P_1) + \omega(P_2) \\
&= 380 + 190 \\
&= 570 \\
\omega(2) &= \omega(P_1 P_2) \\
&= 190.
\end{aligned}$$

Therefore, by the Principle of Inclusion and Exclusion,the number of integers from 1 to 8000 which are divisible by neither 14 nor 21 but divisible by either 4 or 6 is

$$\begin{aligned}
E(0) &= \omega(0) - \omega(1) + \omega(2) \\
&= 2667 - 570 + 190 \\
&= 2287.
\end{aligned}$$

(b) For 4 tables with at least 5 persons seated around each table,there are 6 cases to consider according to the numbers of people to be seated around the 4 tables, namely,

- $10+5+5+5$: No. of ways $= \frac{1}{3!}\binom{25}{10}\binom{15}{5}\binom{10}{5}\binom{5}{5}9!4!4!4!$;
- $9+6+5+5$: No. of ways $= \frac{1}{2!}\binom{25}{9}\binom{16}{6}\binom{10}{5}\binom{5}{5}8!5!4!4!$;
- $8+7+5+5$: No. of ways $= \frac{1}{2!}\binom{25}{8}\binom{17}{7}\binom{10}{5}\binom{5}{5}7!6!4!4!$;
- $8+6+6+5$: No. of ways $= \frac{1}{2!}\binom{25}{8}\binom{17}{6}\binom{11}{6}\binom{5}{5}7!5!5!4!$;
- $7+7+6+5$: No. of ways $= \frac{1}{2!}\binom{25}{7}\binom{18}{7}\binom{11}{6}\binom{5}{5}6!6!5!4!$;
- $7+6+6+6$: No. of ways $= \frac{1}{3!}\binom{25}{7}\binom{18}{6}\binom{12}{6}\binom{6}{6}6!5!5!5!$.

For the first calculation above, there are $\frac{1}{3!}\binom{25}{10}\binom{15}{5}\binom{10}{5}\binom{5}{5}$ ways to divide the 25 people into 4 groups, 1 group of size 10 and 3 groups of 5 each (Note that it is important to divide by 3! as the order of the 3 groups of 5 is not considered in this question since the round tables are indistinguishable.) There are 9!4!4!4! ways to arrange the people in 4 circles. The subsequent calculations are similar. Therefore, the number of arrangements is $2.572431989 \times 10^{22}$.

Question 2

(a) For each 6-element set that satisfy the conditions given, it can be converted to a sequence of binary digits. For example,

$$\{3,7,9,12,18,25\} \Leftrightarrow 00 - \mathbf{1000} - \mathbf{10} - \mathbf{100} - \mathbf{10000} - 0 - \mathbf{100000} - 0 - 1 + 75\text{`0's}$$

$$\{2,7,10,14,20,26\} \Leftrightarrow 0 - \mathbf{1000} - 0 - \mathbf{10} - 0 - \mathbf{100} - 0 - \mathbf{10000} - 0 - \mathbf{100000} - 1 + 74\text{`0's}$$

The '1's represent the 6 integers while '0's represent the integers between them. Hence there should always be at least 3'0's between 1st and 2nd '1', 1 '0' between 2nd and 3rd '1', 2 '0's between 3rd and 4th '1', 4 '0's between 4th and 5th '1' and 5 '0's between 5th and 6th '1'.

Each sequence of digits can be seen as a sequence of '1000','10','100','10000','100000','1', and 79 free '0's where the first 6 sequences of integers are slotted in between the 79 free '0's.
Hence the number of 6-element sets which satisfy the conditions given is

$$\binom{85}{6}.$$

(b) (i) Note: For an exponential generating function for a_n, A(x), we get A'(x) to be the exponential generating function for a_{n+1}. This can be seen in the following,

$$\begin{aligned} A(x) &= \sum_{i=0}^{\infty} \frac{a_i}{i!}x^i \\ &= a_0 + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \ldots \\ A'(x) &= \sum_{i=1}^{\infty} \frac{a_i}{(i-1)!}x^{i-1} \\ &= a_1 + \frac{a_2}{1!}x + \frac{a_3}{2!}x^2 + \frac{a_4}{3!}x^3 + \ldots \end{aligned}$$

Let b_n denote the number of n-digit integers (where 0 CAN be the leading digit), comprising of 0, 1, 2 or 3, in which each of the digits 0, 2 and 3 occur either zero times or at least twice and the digit 1 occurs at least once. By symmetry, the number of similar integers in which the digit 2 occurs at least once and the number of similar integers in which the digit 3 occurs at least once are both b_n as well.
Then we will have $a_n = 3b_{n-1}$ as the integers in a_n start with 1, 2 or 3, hence from the second digit onwards will be an (n-1)-digit integer where 0 can be the leading digit and the starting digit occurs at least once.
We will then have the exponential generating function of b_n to be

$$B(x) = (e^x - x)^3(e^x - 1).$$

Using the fact that $a_{n+1} = 3b_n$, we will have the exponential generating function of a_{n+1}, A'(x) to be 3B(x), where A(x) is the generating function of a_n. Therefore, we have

$$\begin{aligned} A'(x) &= 3(e^x - x)^3(e^x - 1) \\ &= 3e^{4x} - 3e^{3x} - 9xe^{3x} + 9xe^{2x} + 9x^2e^{2x} - 9x^2e^x - 3x^3e^x + 3x^3. \\ A(x) &= \int A'(x)dx \\ &= -\frac{3}{4} + \frac{3}{4}x^4 + \frac{3}{4}e^{4x} - 3xe^{3x} + \frac{9}{2}x^2e^{2x} - 3x^3e^x. \end{aligned}$$

The constant $-\frac{3}{4}$ is added to make sure we get $a_0 = 0$, even though in this question we are not very concerned with $n \leq 1$.

(ii) For $n \neq 4$, $n \geq 2$,

$$\begin{aligned} a_n &= \frac{3}{4}(4^n) - 3n(3^{n-1}) + \frac{9}{2}n(n-1)(2^{n-2}) - 3n(n-1)(n-2)(1^{n-3}) \\ &= 3(4^{n-1}) - n3^n + 9n(n-1)(2^{n-3}) - 3n(n-1)(n-2). \end{aligned}$$

For $n = 4$, we have to add $4! \times \frac{3}{4} = 18$ to the above equation.

Question 3

(a) The solution of this question uses the Principle of Inclusion and Exclusion. Let

- S be the set of permutations of all the 11 letters of the word COMMUTATORS;
- P_1, P_2, P_3, be the properties of the permutations that contain COM, MAT, TOUR respectively;
- $E(m)$ be the number of elements of S possessing exactly m of the 3 properties for $0 \le m \le 3$;
- $\omega(P_{i_1}P_{i_2}\dots P_{i_m})$ be the number of elements of S possessing the properties P_{i_1}, P_{i_2}, $\dots$, P_{i_m}, where $1 \le m \le 3$;
- $\omega(m) = \sum(\omega(P_{i_1}P_{i_2}\dots P_{i_m}))$, $\omega(0) = |S|$;

We have

$$\begin{aligned}
\omega(P_1) &= \frac{9!}{2!}. \\
\omega(P_2) &= \frac{9!}{2!}. \\
\omega(P_3) &= \frac{8!}{2!}. \\
\omega(P_1P_2) &= 7! + 7!. && (1) \\
\omega(P_1P_3) &= 6!. \\
\omega(P_2P_3) &= 6! + 6!. && (2) \\
\omega(P_1P_2P_3) &= 4! + 4! + 4! + 4!. && (3) \\
\omega(0) &= \frac{11!}{2!2!2!} \\
&= 4989600. \\
\omega(1) &= \omega(P_1) + \omega(P_2) + \omega(P_3) \\
&= 383040. \\
\omega(2) &= \omega(P_1P_2) + \omega(P_1P_3) + \omega(P_2P_3) \\
&= 12240. \\
\omega(3) &= \omega(P_1P_2P_3) \\
&= 96.
\end{aligned}$$

In (1), the first calculation is the number of permutations which contain COM and MAT separately while the second one is that which contain COMAT. (2) is similar. In (3), the first calculation is the number of permutations which contain COM, MAT, TOUR separately, the second one is that which contain COMAT and TOUR, the third one is that which contain COM and MATOUR and the last one is that which contain COMATOUR.

(i) Hence, the number of permutations which contain none of the three words is

$$\begin{aligned}
E(0) &= \omega(0) - \omega(1) + \omega(2) - \omega(3) \\
&= 4989600 - 383040 + 12240 - 96 \\
&= 4618704.
\end{aligned}$$

(ii) The number of permutations which contain exactly one of the three words is

$$\begin{aligned}
E(1) &= \omega(1) - \binom{2}{1}\omega(2) + \binom{3}{1}\omega(3) \\
&= 383040 - 2(12240) + 3(96) \\
&= 358848.
\end{aligned}$$

(iii) The number of permutations which contain exactly two of the three words is

$$\begin{aligned} E(2) &= \omega(2) - \binom{3}{2}\omega(3) \\ &= 12240 - 3(96) \\ &= 11952. \end{aligned}$$

(b) (i) If we numbered the boys from 1 to 8 and their parents respectively, we can first put the boys into group 1 to 8 according to the number they are assigned to before putting a male parent and a female parent into each group. To put the male parents into the groups with the given restriction is the same as arranging the male parents' numbers 1 to 8 where the number i should not be in the ith position, which is actually a derangement of $\{1,2,...,8\}$. Putting the female parents into the groups is similar. Therefore, the number of ways to group these 24 people without any further requirement is

$$\begin{aligned} {D_8}^2 &= (\sum_{r=0}^{8}(-1)^r\binom{8}{r}(8-r)!)^2 \\ &= (\binom{8}{0}8! - \binom{8}{1}7! + \binom{8}{2}6! - \binom{8}{3}5! + \binom{8}{4}4! - \binom{8}{5}3! + \binom{8}{6}2! - \binom{8}{7}1! + \binom{8}{8}0!)^2 \\ &= 220017889. \end{aligned}$$

(ii) WLOG, Let Dad 1 and Mum 1 be the couple that do not want to be in the same group. Hence there are 7 ways to put Dad 1, then 6 ways to put Mum 1. Observing the row of 8 Dads, there are 7 positions to put Dad 1. WLOG, Dad 1 sits on position 2, then Dad 2 will have 2 choices - to sit or not to sit on position 1. If he sits on position 1, then there are D_6 ways to sit the other 6 Dads. If he does not sit on position 1, then we will have D_7 ways to sit the 7 dads with Dad 2 not on position 1, Dad 3 not on position 3, Dad 4 not on position 4 etc. Therefore, there are $7 \times (D_7 + D_6)$ to arrange the male parents.
After arranging the male parents with Dad 1 in a particular position, there are 6 ways to put Mum 1. WLOG, if we put Dad 1 on position 2, we can put Mum 1 on position 3. Then Mum 3 can choose to sit or not to sit on position 1. Using the same argument, the number of ways to arrange the female parents will be $6 \times (D_7 + D_6)$.
Therefore, the number of ways to group the 24 people is

$$7 \times 6 \times (D_7 + D_6)^2 = 188586762.$$

Question 4

(a) (i) First let b_n denote the number of ways of paving a 2 meters by n meters floor, missing a 1 meter by 1 meter corner, using the 3 types of tiles as mentioned in the question. To count a_n, it is necessary to break a_n and b_n down into smaller quantities to form a system of recurrence relations by considering the different cases in which a tile can be placed in the top left or bottom left corner respectively.

The top half of the diagram below shows the five cases on how a_n can be broken down into smaller quantities. For instance, in the first case, if a 1 by 1 tile is placed at the top left hand corner, the question is simplified into counting b_n. For the subsequent 4 cases of placing tiles in the top left hand corner, the question is respectively broken down into a_{n-2}, a_{n-2}, a_{n-1} and b_{n-1}. Therefore the recurrence relation for a_n is

$$a_n = b_n + a_{n-1} + 2a_{n-2} + b_{n-1}. \qquad (4)$$

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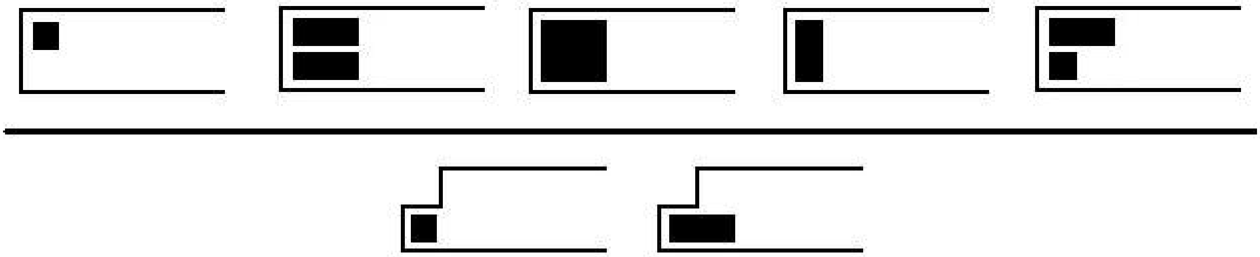

The bottom half of the diagram below shows the 2 different cases on how b_n can be counted by placing different tiles at the bottom left corner of the floor. The problem is then respectively broken down into cases of counting a_{n-1} and b_{n-1}. Hence the recurrence relation of b_n is

$$b_n = a_{n-1} + b_{n-1}. \tag{5}$$

From (4) and (5), we have

$$\begin{aligned}
b_{n-1} &= b_n - a_{n-1}. && (6)\\
a_n &= b_n + a_{n-1} + 2a_{n-2} + b_n - a_{n-1} \\
&= 2b_n + 2a_{n-2}. && (7)\\
b_n &= \frac{a_n - 2a_{n-2}}{2}. && (8)\\
b_{n-1} &= \frac{a_{n-1} - 2a_{n-3}}{2}. && (9)\\
a_n &= \frac{a_n - 2a_{n-2}}{2} + a_{n-1} + 2a_{n-2} + \frac{a_{n-1} - 2a_{n-3}}{2}. && (10)\\
a_n &= 3a_{n-1} + 2a_{n-2} - 2a_{n-3}.
\end{aligned}$$

For the above equations, we get (6) by rearranging equation (5) and thus obtain (7) by substituting (6) into (4). By rearranging (7), we obtain (8) and thus (9). (10) is obtained by substituting (8) and (9) into (4) and thus we obtain the recurrence equation that we want. By manual counting, $a_1 = 2$, $a_2 = 8$ and $b_2 = 3$. By substituting the values into (5), we get $b_3 = 11$. Hence we have $a_3 = 26$ by substituting values into (4). Therefore, the recurrence relation for a_n is

$$a_n = 3a_{n-1} + 2a_{n-2} - 2a_{n-3}.$$

with the initial conditions $a_1 = 2$, $a_2 = 8$ and $a_3 = 26$.

(ii) Using the recurrence relation in (i), we have

$$\begin{aligned}
a_4 &= 3a_3 + 2a_2 - 2a_1 \\
&= 90. \\
a_5 &= 3a_4 + 2a_3 - 2a_2 \\
&= 306. \\
a_6 &= 3a_5 + 2a_4 - 2a_3 \\
&= 1046. \\
a_7 &= 3a_6 + 2a_5 - 2a_4 \\
&= 3570. \\
a_8 &= 3a_7 + 2a_6 - 2a_5 \\
&= 12190.
\end{aligned}$$

(b) (i) Among the a_n integers, let b_n be the number of those which end with 1 and c_n be the number of those which end with 3. Note that in general $b_n \neq c_n$ as it can be shown later that $b_3 = 15$ while $c_3 = 16$. Constructing recurrence relations involving a_n, b_n and c_n, we have

$$\begin{aligned} a_n &= b_n + c_n + 3a_{n-1}. && (11)\\ b_n &= 3a_{n-2} + c_{n-1}. && (12)\\ c_n &= 2a_{n-2} + b_{n-1} + c_{n-1}. && (13) \end{aligned}$$

The RHS of (11) represents the number of integers that ends with 1 (b_n), that ends with 3 (c_n) and that ends with 0, 2, or 4 ($3a_{n-1}$). The RHS of (12) represents the number of integers that end with 01, 21 or 41 ($3a_{n-2}$) and that ends with 31 (c_{n-1}). The RHS of (13) represents the number of integers that end with 03 or 43 ($2a_{n-2}$), that end with 13 (b_{n-1}) and that end with 33 (c_{n-1}).

From (11), (12) and (13), we have

$$\begin{aligned} b_n &= a_n - c_n - 3a_{n-1} && (14)\\ &= 3a_{n-2} + c_{n-1}.\\ b_{n-1} &= a_{n-1} - c_{n-1} - 3a_{n-2}. && (15)\\ c_n &= 2a_{n-2} + a_{n-1} - c_{n-1} - 3a_{n-2} + c_{n-1} && (16)\\ &= a_{n-1} - a_{n-2}.\\ a_n - (a_{n-1} - a_{n-2}) - 3a_{n-1} &= 3a_{n-2} + a_{n-2} - a_{n-3}. && (17)\\ a_n &= 4a_{n-1} + 3a_{n-2} - a_{n-3}. \end{aligned}$$

For the above equations, we obtain (14) from (11) and (12) and thus obtain (15). By substituting (15) into (13), we obtain equation (16) and thus (17) by substituting (16) into (14). By manual counting, we get $a_1 = 4$, $a_2 = 18$ and $c_2 = 3$. Substituting the values into (12) and (16), we get $b_3 = 15$ and $c_3 = 14$ respectively. Therefore, after substituting the values into (11), we get $a_3 = 83$. Therefore, the recurrence relation for a_n is

$$a_n = 4a_{n-1} + 3a_{n-2} - a_{n-3}.$$

with the initial conditions of $a_1 = 4$, $a_2 = 18$ and $a_3 = 83$.

(ii) Using the recurrence relation in (i), we have

$$\begin{aligned} a_4 &= 4a_3 + 3a_2 - a_1\\ &= 382.\\ a_5 &= 4a_4 + 3a_3 - a_2\\ &= 1759.\\ a_6 &= 4a_5 + 3a_4 - a_3\\ &= 8099.\\ a_7 &= 4a_6 + 3a_5 - a_4\\ &= 37291.\\ a_8 &= 4a_7 + 3a_6 - a_5\\ &= 171702. \end{aligned}$$

Question 5

(a) (i) A suitable exponential generating function for a_n is

$$\begin{aligned} & \left(\frac{e^{2x}+e^{-2x}}{2}\right)\left(\frac{e^{2x}-e^{-2x}}{2}\right)(e^{3x}-1-3x) \\ = & \frac{1}{4}(e^{4x}-e^{-4x})(e^{3x}-1-3x) \\ = & \frac{1}{4}\left(e^{7x}-e^{4x}-e^{-x}+e^{-4x}-3x(e^{4x}-e^{-4x})\right). \end{aligned}$$

(ii) a_n is $n!$ times the coefficient of x^n in the above generating function. Therefore,

$$\begin{aligned} a_8 &= \frac{1}{4}\left(7^8-4^8-(-1)^8+(-4)^8-3(8)(4^7-(-4)^7)\right) \\ &= 1244592. \end{aligned}$$

(b) (i) A suitable ordinary generating function for a_n is

$$\begin{aligned} & \left(\frac{(1-x)^{-4}+(1+x)^{-4}}{2}\right)\left(\frac{(1-x)^{-4}-(1+x)^{-4}}{2}\right) \\ = & \frac{(1-x)^{-8}-(1+x)^{-8}}{4}. \end{aligned}$$

(ii) a_n is the coefficient of x^n in the above generating function. Therefore,

$$\begin{aligned} a_9 &= \frac{1}{4}\left(\binom{9+8-1}{9}-(-1)^9\binom{9+8-1}{9}\right) \\ &= 5720. \end{aligned}$$