NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Wang Yu

**MA2108 Mathematical Analysis I**
AY 2010/2011 Sem 1

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**Question 1**

(a) Proof: For all $n \in \mathbb{N}$, we have

$$\begin{aligned}|a_{n+2} - a_{n+1}| &= |\frac{4}{5}(a_{n+1} - 1) - \frac{4}{5}(a_n - 1)| \\ &= |\frac{4}{5}a_{n+1} - \frac{4}{5} - \frac{4}{5}a_n + \frac{4}{5}| \\ &= \frac{4}{5}|a_{n+1} - a_n| \\ &\leq \frac{4}{5}|a_{n+1} - a_n|\end{aligned}$$

Since $0 < \frac{4}{5} < 1$, $(a_n)$ is a contractive sequence, so it converges.
Let

$$a = \lim_{n\to\infty} a_n$$

Then

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{4}{5}(a_n - 1) = \frac{4}{5}(a - 1)$$

Hence $a = -4$.

(b) (i) Write $a_n = \frac{2n^3+3}{n(6n^2+5)}, n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^3+3}{n(6n^2+5)} = \lim_{n\to\infty} \frac{2 + \frac{3}{n^3}}{6 + \frac{5}{n^2}} = \frac{1}{3}.$$

Now for each $n \in \mathbb{N}, -a_n \leq x_n \leq a_n$.
So if $x_{n_k} \to x$, then

$$-\frac{1}{3} = \lim_{k\to\infty} -a_{n_k} \leq \lim_{k\to\infty} x_{n_k} = x \leq \lim_{k\to\infty} a_{n_k} = \frac{1}{3}.$$

This shows that $\frac{1}{3}$and $-\frac{1}{3}$ are upper bound and lower bound of the set of cluster points of $(x_n)$, respectively.
On the other hand,

$$x_{6k} = a_{6k}\cos(2k\pi) = a_{6k} \to \frac{1}{3}$$

and

$$x_{6k+3} = a_{6k+3}\cos((2k+1)\pi) = -a_{6k+3} \to -\frac{1}{3}.$$

Hence $\limsup x_n = \frac{1}{3}$ and $\liminf x_n = -\frac{1}{3}$.

(ii)The sequence $(x_n)$ diverges because $\limsup x_n \neq \liminf x_n$

**Question 2**

(a) (i) We use the limit comparison test:

$$\lim_{n\to\infty}\frac{\frac{n(3n^3+5)}{4n^5\sqrt{n}-3n^2+2}}{\frac{1}{n\sqrt{n}}}=\lim_{n\to\infty}\frac{3n^5\sqrt{n}+5n^2\sqrt{n}}{4n^5\sqrt{n}-3n^2+2}=\lim_{n\to\infty}\frac{3+\frac{5}{n^3}}{4-\frac{3}{n^3\sqrt{n}}+\frac{2}{n^5\sqrt{n}}}=\frac{3}{4}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$ is a p-series with $p=\frac{3}{2}>1$, so it converges.
Therefore, $\sum_{n=1}^{\infty}\frac{n(3n^3+5)}{4n^5\sqrt{n}-3n^2+2}$ converges.

(ii) We use the root test:

$$\begin{aligned}\lim_{n\to\infty}\left|n\frac{2n}{1+2n}^{n^2}\right|^{\frac{1}{n}}&=\lim_{n\to\infty}(n^{\frac{1}{n}})(\frac{2n}{1+2n})^n\\&=\lim_{n\to\infty}(n^{\frac{1}{n}})\frac{1}{(1+\frac{1}{2n})^n}\\&=\lim_{n\to\infty}(n^{\frac{1}{n}})\frac{1}{\sqrt{(1+\frac{1}{2n})^{2n}}}\\&=1\cdot\frac{1}{\sqrt{e}}=\frac{1}{\sqrt{e}}<1\end{aligned}$$

By the root test, the series converges.

(b) We observe that for every $n\in\mathbb{N}, (x_n-\frac{1}{n})^2\geq 0$, thus,

$$\frac{1}{2}(x_n^2+\frac{1}{n^2})\geq\frac{x_n}{n}.$$

Similarly, for every $n\in\mathbb{N}, (x_n+\frac{1}{n})^2\geq 0$, thus,

$$\frac{x_n}{n}\geq-\frac{1}{2}(x_n^2+\frac{1}{n^2}).$$

Hence, for every $n\in\mathbb{N}$

$$0\leq\left|\frac{x_n}{n}\right|\leq\frac{1}{2}(x_n^2+\frac{1}{n^2})$$

Since $\sum_{n=1}^{\infty}x_n^2$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converge, we can conclude that $\sum_{n=1}^{\infty}\left|\frac{x_n}{n}\right|$ converges.

(c) (i)Proof: Let $b_1=a_1+a_2+\cdots+a_{n_1}$ and $b_k=a_{n_{k-1}+1}+a_{n_{k-1}+2}+\cdots+a_{n_k}$ for all $k\geq 2$
Since $a_n\geq 0$, so $0\leq a_{n_k}\leq b_k$.
Note that $\sum_{k=1}^{\infty}b_k=\sum_{n=1}^{\infty}a_n$ which is convergent.
So $\sum_{k=1}^{\infty}a_{n_k}$ is convergent by the comparison test.

(ii) Without the assumption that $a_n\geq 0$ for all $n\in\mathbb{N}$, the series $\sum_{k=1}^{\infty}a_{n_k}$ may diverge.
Here is an example: let $a_n=\frac{(-1)^n}{n}$ and $a_{n_k}=a_{2k}$.
Then by the alternating series test, $\sum_{n=1}^{\infty}a_n$ converges,
but $\sum_{k=1}^{\infty}a_{n_k}=\sum_{k=1}^{\infty}\frac{1}{2k}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}$ diverges.

**Question 3**

(a) (i) We have

$$\left|\frac{1}{3x-4}-\frac{1}{2}\right| = \frac{1}{2}\left|\frac{2-3x+4}{3x-4}\right| = \frac{1}{2}\left|\frac{3x-6}{3x-4}\right| = \frac{3}{2}\frac{|x-2|}{|3x-4|}$$

We first restrict $x$ to $0<|x-2|<\frac{1}{2}$, then $-\frac{1}{2}<x-2<\frac{1}{2} \Rightarrow \frac{3}{2}<x<\frac{5}{2}$.
So $\frac{1}{2}<3x-4<\frac{7}{2}$.
In particular,$|3x-4|>\frac{1}{2}$, so that $0<\frac{1}{|3x-4|}<2$.
It follows that

$$0<|x-2|<\frac{1}{2} \Rightarrow \left|\frac{1}{3x-4}-\frac{1}{2}\right| < \frac{3}{2}\cdot 2|x-2| = 3|x-2|.$$

Now let $\varepsilon>0$ be given. Choose $\delta=\min(\frac{1}{2},\frac{1}{3}\varepsilon)$. Then

$$0<|x-2|<\delta \Rightarrow \left|\frac{1}{3x-4}-\frac{1}{2}\right| < 3|x-2| < 3\cdot\frac{1}{3}\varepsilon = \varepsilon.$$

(ii) Let $M>0$ be given. Choose $\delta=\frac{5}{M}$. Then

$$0<x-4<\delta \Rightarrow \frac{x+1}{x-4} = 1+\frac{5}{x-4} > \frac{5}{x-4} > \frac{5}{\delta} = M.$$

(b) (i) Write $f(x)=\cos(\frac{1}{\sqrt{x}-1})$.
For each $n\in\mathbb{N}$, let $x_n=(\frac{1}{n\pi}+1)^2$.
Then $x_n\neq 1$ for all $n\in\mathbb{N}, x_n\to 1$ and $f(x_n)=\cos n\pi=(-1)^n$, for $n\in\mathbb{N}$.
From this, we see that $f(x_{2k})\to 1$ and $f(x_{2k-1})\to -1$.
So $(f(x_n))$ diverges.
Hence the limit $\lim_{x\to 1}\cos\frac{1}{\sqrt{x}-1}$ does not exist.

(ii) For $x\in(1,1.1), [7-5x]=1$. So

$$\lim_{x\to 1^+}\frac{[7x-5]}{1+x^2} = \frac{1}{1+1^2} = \frac{1}{2}$$

## Question 4

(a) Proof: Let $\varepsilon>0$. Choose $\delta=\frac{\varepsilon}{C}$. Then

$$|x-y|<\delta \Rightarrow |f(x)-f(y)| < C|x-y| < C\delta = C\frac{\varepsilon}{C} = \varepsilon.$$

So $f$ is uniformly continuous on $I$.

(b) (i) Take

$$x_n=\frac{1}{2n\pi}, y_n=\frac{1}{(2n+1)\pi}, n\in\mathbb{N}.$$

Then

$$x_n-y_n=\frac{1}{2n\pi}-\frac{1}{(2n+1)\pi}\to 0-0=0,$$

but

$$|g(x_n)-g(y_n)| = |\cos(2n\pi)-\cos((2n+1)\pi)| = 2.$$

So $g$ is not uniformly continuous on $(0,1)$.

(ii) Define

$$H(x) = \begin{cases} 0 & \text{if } x = 0, \\ h(x) & \text{if } 0 < x < 1, \\ \sin(1) & \text{if } x = 1. \end{cases}$$

Clearly $H(x)$ is continuous on $(0, 1)$.
Since

$$-1 < \sin\frac{1}{x} < 1 \Rightarrow -x < x\sin\frac{1}{x} < x,$$

And

$$\lim_{x\to 0^+} x = \lim_{x\to 0^+} -x = 0$$

By Squeeze Theorem, we have

$$\lim_{x\to 0^+} x\sin\frac{1}{x} = 0.$$

Since

$$\lim_{x\to 0^+} H(x) = \lim_{x\to 0^+} x\sin\frac{1}{x} = 0 = H(0)$$

and

$$\lim_{x\to 1^-} H(x) = \lim_{x\to 1^-} x\sin\frac{1}{x} = \sin(1) = H(1)$$

So $H(x)$ is continuous on $[0, 1]$.
Therefore, $H(x)$ is uniformly continuous on $[0, 1]$, and so is on $(0, 1)$.
Since $h(x) = H(x)$ for $x \in (0, 1)$
So $h$ is uniformly continuous on $(0, 1)$.

**Question 5**

(a) Proof: Let $s = \limsup \frac{a_{n+1}}{a_n}$, then $0 \leq s < 1$.
There exists $r \in \mathbb{R}$ such that $s < r < 1$.
Let $\varepsilon = r - s > 0$.
Since $\limsup \frac{a_{n+1}}{a_n} < 1$, there exists $K \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} < s + \varepsilon = r, \text{ for all } n \geq K$$

So for $n \geq K$,

$$0 < a_n < ra_{n-1} < r^2 a_{n-2} < \cdots < r^{n-K} a_K = Cr^n,$$

where $C = r^{-K} a_K > 0$.
So

$$0 < \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{K-1} a_n + \sum_{n=K}^{\infty} a_n < \sum_{n=1}^{K-1} a_n + \sum_{n=K}^{\infty} Cr^n = \sum_{n=1}^{K-1} a_n + C\frac{r^K}{1-r}$$

Since $\sum_{n=1}^{\infty} a_n$ is bounded
So the series converges.

(b) Proof: Let $a_n = x_{2n-1} + x_{2n}$ for all $n \in \mathbb{N}$.
Note that

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{[(n+1)/2]} x_n &= -x_1 - x_2 + x_3 + x_4 - x_5 - x_6 + x_7 + x_8 - \cdots \\ &= -(x_1 + x_2) + (x_3 + x_4) - (x_5 + x_6) + (x_7 + x_8) - \cdots \\ &= -a_1 + a_2 - a_3 + a_4 - \cdots \\ &= \sum_{n=1}^{\infty} (-1)^n a_n. \end{aligned}$$

For all $n \in \mathbb{N}$,
Since $x_n > 0$, so $a_n = x_{2n-1} + x_{2n} > 0$.
Since $x_n \geq x_{n+1}$, so $a_n = x_{2n-1} + x_{2n} \geq x_{2n} + x_{2n+1} \geq x_{2n+1} + x_{2n+2} = a_{n+1}$.
It follows that $(a_n)$ is decreasing .
Since $\lim_{n\to\infty} x_n = 0$, so $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(x_{2n-1} + x_{2n}) = 0 + 0 = 0$.
By the alternating series test, the $\sum_{n=1}^{\infty}(-1)^n a_n$ converges, then so is $\sum_{n=1}^{\infty}(-1)^{[(n+1)/2]}x_n$.

**Question 6**

(a) Proof: Let $a > 0$. Then by putting $x = a^{\frac{1}{3}}$, we have

$$h(a^{\frac{1}{3}}) = h(x) = h(x^3) = h(a)$$

Next we put $x = a^{\frac{1}{9}} = a^{\frac{1}{3^3}}$, then

$$h(a^{\frac{1}{3^3}}) = h(x) = h(x^3) = h(a^{\frac{1}{3}})$$

Similarly,

$$h(a) = h(a^{\frac{1}{3}}) = h(a^{\frac{1}{3^3}}) = \cdots$$

By induction,$h(a) = h(a^{\frac{1}{3^n}})$, for all $n \in \mathbb{N}$.
Now $(a^{\frac{1}{3^n}})$ is a subsequence of $(a^{\frac{1}{n}})$, so

$$\lim_{n\to\infty} a^{\frac{1}{3^n}} = \lim_{n\to\infty} a^{\frac{1}{n}} = 1$$

Since $h$ is continuous at 1,

$$\lim_{n\to\infty} h(a^{\frac{1}{3^n}}) = h(1).$$

On the other hand, since $h(a) = h(a^{\frac{1}{3^n}})$, for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} h(a^{\frac{1}{3^n}}) = h(a).$$

By the uniqueness of limit, $h(a) = h(1)$.
So $h$ is a constant function on $(0, \infty)$.

(b) (i) Since $f$ is bounded on (0,1).
There exists $M > 0$, such that $|f(t)| < M$ for every $t \in (0, 1)$.
In particular, for every $x \in (0, 1), |f(t)| < M$ for every $t \in (0, x)$.
Hence, for every $x \in (0, 1), t \in (0, x)$

$$-M = \sup -M \leq \sup f(t) = g(x) \leq \sup M = M$$

So $g$ is bounded on (0,1).
Therefore, $\inf g(x)$ exists.
For $0 < x < y < 1, (0, x) \subset (0, y)$.
So $g(x) \leq g(y)$.
Therefore, $g$ is increasing.
Hence $L = \lim_{x\to 0^+} g(x) = \inf g(x)$ exists.

(ii) Proof: Given $\varepsilon > 0$.
Since $\lim_{x\to 0^+} g(x)$ exists
There exists $\delta' \in (0, 1)$ such that

$$\delta \in (0, \delta') \Rightarrow |g(\delta) - L| < \varepsilon \Rightarrow g(\delta) < L + \varepsilon$$

Now let $\delta = \frac{1}{2}\delta'$, so that $0 < \delta < \delta' < 1$.
Since $g(\delta) = \sup\{f(x) : x \in (0, \delta)\}$
So for all $x \in (0, \delta), f(x) \le g(\delta) < L + \varepsilon$.

(iii) Proof: Prove by contradiction.
Suppose that for every $\varepsilon > 0$ and for every $0 < \delta_1 < 1$,
$f(x_1) \le L - \varepsilon$ for all $x_1 \in (0, \delta_1)$.
It follows that $L - \varepsilon$ is an upper bound of $f(x_1)$ for $x_1 \in (0, \delta_1)$.
So $g(\delta_1) = \sup\{f(x) : x \in (0, \delta_1)\} \le L - \varepsilon$ for $\delta_1 \in (0, 1)$.
Therefore $g(\delta_1) < L$ for all $\delta_1 \in (0, 1)$.
This contradicts that $L$ is the infimum of $g(x)$ for $x \in (0, 1)$.

**Question 7**

(a) (i) Proof: Let

$$m = \min(f(x_i)), \text{ and } M = \max(f(x_i)),\ 1 \le i \le 4$$
$$t = \frac{1}{3}f(x_1) + \frac{1}{12}f(x_2) + \frac{5}{12}f(x_3) + \frac{1}{6}f(x_4).$$

Then

$$\begin{aligned} m &= \frac{1}{3}m + \frac{1}{12}m + \frac{5}{12}m + \frac{1}{6}m \\ &\le \frac{1}{3}f(x_1) + \frac{1}{12}f(x_2) + \frac{5}{12}f(x_3) + \frac{1}{6}f(x_4) \\ &= t \\ &\le \frac{1}{3}M + \frac{1}{12}M + \frac{5}{12}M + \frac{1}{6}M \\ &= M. \end{aligned}$$

If $m = f(x_i)$, and $M = f(x_j)$.
By the Intermediate Value Theorem,
there exists $c$ between $x_i$ and $x_j$ such that $f(c) = t$.

(b) Proof: Given $\varepsilon > 0$, since $g$ is uniformly continuous on $[0, \infty)$, there exists $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon/2$$

There exists $m \in \mathbb{N}$ such that $m \cdot \delta \ge 1$, so that $\frac{1}{m} \le \delta$.
Define $x_i = \frac{i}{m}, \quad i = 0, 1, 2, 3, \cdots, m$.
Since for any $x \ge 0$,

$$\lim_{n\to\infty} g(x + n) = 0$$

For each $x_i$, there exists a $M_i \in \mathbb{N}$, such that

$$|g(x_i + n) - 0| < \varepsilon/2, \text{ for all } n \ge M_i$$

Let $M = \max(M_i), \quad i = 0, 1, 2, 3, \cdots, m$.
For any $x \ge 0$,

$$x_0 = 0 \le x - [x] < 1 = x_m.$$

There exist a $k \in \{0, 1, 2, \cdots, m-1\}$, such that $x_k \leq x - [x] < x_{k+1}$.
So

$$\begin{aligned}|x - ([x] + x_k)| &= |(x - [x]) - x_k| \\ &= (x - [x]) - x_k \\ &< x_{k+1} - x_k \\ &= \frac{1}{m} \leq \delta\end{aligned}$$

Such that

$$|g(x) - g([x] + x_k)| < \varepsilon/2$$

Therefore for $x \geq M$, it follows $[x] \geq M$.

$$\begin{aligned}|g(x) - 0| &= |g(x) - g([x] + x_k) + g([x] + x_k) - 0| \\ &\leq |g(x) - g([x] + x_k)| + |g([x] + x_k) - 0| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Hence $\lim_{x\to\infty} g(x) = 0$