NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA3110 Mathematical Analysis II
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Question 1

(a) WLOG, suppose $f'(a) < f'(b)$ and let $k \in \mathbb{R}$ such that $f'(a) < k < f'(b)$. Now, define $h(x) := kx - f(x)$ for $x \in [a, b]$.

$$\Rightarrow h'(x) = k - f'(x) \qquad \forall x \in [a, b]$$
$$\Rightarrow h'(a) = k - f'(a) > 0 \qquad \text{and} \qquad h'(b) = k - f'(b) < 0$$

Hence $\exists \delta > 0$ such that $h(x) > h(a)$ for all $x \in (a, a + \delta)$ and $h(x) > h(b)$ for all $x \in (b - \delta, b)$. That is, a and b are not relative maximums of h. Now since h is continuous on $[a, b]$, it achieves a maximum at some $c \in (a, b)$ and $h'(c) = 0$. Therefore, $f'(c) = k$.

(b) By Taylor's Theorem, $\exists d, e \in (a, b)$ such that

$$\begin{aligned} g(b) &= g\left(\frac{a+b}{2}\right) + g'\left(\frac{a+b}{2}\right)\left(b - \frac{a+b}{2}\right) + \frac{g''(d)}{2}\left(b - \frac{a+b}{2}\right)^2 \\ &= g\left(\frac{a+b}{2}\right) + g'\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right) + \frac{g''(d)}{2}\frac{(b-a)^2}{4} \\ g(a) &= g\left(\frac{a+b}{2}\right) + g'\left(\frac{a+b}{2}\right)\left(a - \frac{a+b}{2}\right) + \frac{g''(e)}{2}\left(a - \frac{a+b}{2}\right)^2 \\ &= g\left(\frac{a+b}{2}\right) + g'\left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right) + \frac{g''(e)}{2}\frac{(a-b)^2}{4} \end{aligned}$$

Summing,

$$\begin{aligned} g(a) + g(b) &= 2g\left(\frac{a+b}{2}\right) + \frac{(b-a)^2}{4}\left(\frac{g''(d) + g''(e)}{2}\right) \\ g(a) - 2g\left(\frac{a+b}{2}\right) + g(b) &= \frac{(b-a)^2}{4}\left(\frac{g''(d) + g''(e)}{2}\right) \end{aligned}$$

If $g''(d) = g''(e)$, take $c = d$ and we are done. Otherwise, observe that $\frac{g''(d)+g''(e)}{2}$ is a real number strictly between $g''(d)$ and $g''(e)$, $\exists c \in (a, b)$ such that $g''(c) = \frac{g''(d)+g''(e)}{2}$ by applying (a) to g'.

$$\therefore g(a) - 2g\left(\frac{a+b}{2}\right) + g(b) = \frac{(b-a)^2}{4} g''(c)$$

Question 2

(a) (i) Let $M_k(h, P_n) = \sup\left\{h(x) : x \in \left[\frac{k-1}{n}, \frac{k}{n}\right]\right\}$. Since irrationals are dense in $\mathbb{R}$,

$$\Rightarrow M_k(h, P_n) = \frac{k}{n} \qquad \forall k = 1, 2, \ldots, n$$

$$\begin{aligned}\therefore U(h, P_n) &= \sum_{k=1}^{n} \frac{k}{n}\left(\frac{k+1}{n} - \frac{k}{n}\right) \\ &= \frac{1}{n^2}\sum_{k=1}^{n} k \\ &= \frac{1}{n^2}\frac{n(n+1)}{2} \\ &= \frac{1}{2} + \frac{1}{2n} \qquad \forall n \in \mathbb{N}\end{aligned}$$

$$\Rightarrow U(h) \le U(h, P_n) = \frac{1}{2} + \frac{1}{2n} \qquad \forall n \in \mathbb{N}$$

Therefore, $U(h) \le \frac{1}{2}$.

(ii) Let $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be a partition of $[0, 1]$ and let $M_k(h, P) = \sup\{h(x) : x \in [x_{k-1}, x_k]\}$. Since irrationals are dense in $\mathbb{R}$,

$$\Rightarrow M_k(h, P) = x_k \qquad \forall k = 1, 2, \ldots, n$$

$$\begin{aligned}\therefore U(h, P) &= \sum_{k=1}^{n} x_k(x_k - x_{k-1}) \\ &> \sum_{k=1}^{n} \frac{x_k + x_{k-1}}{2}(x_k - x_{k-1}) \\ &= \frac{1}{2}\sum_{k=1}^{n} x_k^2 - x_{k-1}^2 \\ &= \frac{1}{2}(x_n^2 - x_0^2) \\ &= \frac{1}{2}\end{aligned}$$

$$\Rightarrow \frac{1}{2} \le U(h)$$

Therefore, $U(h) = \frac{1}{2}$.

(iii) Let $m_k(h, P) = \inf\{h(x) : x \in [x_{k-1}, x_k]\}$. Since rationals are dense in $\mathbb{R}$,

$$\Rightarrow m_k(h, P) = -x_k$$

$$\therefore L(h, P) = \sum_{k=1}^{n} -x_k(x_k - x_{k-1}) = -U(h, P) < -\frac{1}{2}$$

(iv) Since for all partitions P of $[0, 1]$, we have $U(h, P) - L(h, P) > \frac{1}{2} + \frac{1}{2} = 1$, h is not integrable on $[0, 1]$.

(b) Suppose that f' is integrable on $[0, 1]$. Let $x \in [0, 1]$. By Fundamental Theorem of Calculus,

$$f(x) = f(0) + \int_0^x f'(t)\, dt$$

Conversely, suppose there exists an integrable function g on $[0,1]$ such that

$$f(x) = f(0) + \int_0^x g(t)\, dt, \qquad x \in [0,1].$$

Let $\varepsilon > 0$ be given. Hence there exist a partition $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ of $[0,1]$ such that $U(g,P) - L(g,P) < \frac{\varepsilon}{3}$. Let $\delta_1 = \sup\{x_k - x_{k-1} : k = 1, 2, \ldots, n\}$. Define

$$\begin{aligned} M_k(g,P) &= \sup\{g(x) : x \in [x_{k-1}, x_k]\}\,; \\ m_k(g,P) &= \inf\{g(x) : x \in [x_{k-1}, x_k]\}\,; \\ M_k(f',P) &= \sup\left\{f'(x) : x \in [x_{k-1}, x_k]\right\}; \\ m_k(f',P) &= \inf\left\{f'(x) : x \in [x_{k-1}, x_k]\right\}. \end{aligned}$$

Let $x_0 \in [x_{k-1}, x_k]$. Since f is differentiable at x_0, $\exists 0 < \delta < \delta_1$ such that for every $x \in [x_{k-1}, x_k]$,

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \frac{\varepsilon}{3} \qquad \text{whenever} \qquad 0 < |x - x_0| < \delta.$$

Now, observe that

$$\begin{aligned} m_k(g,P) \le \frac{f(x) - f(x_0)}{x - x_0} &= \frac{\int_{x_0}^x g(t)\, dt}{x - x_0} \le M_k(g,P) \\ \Rightarrow m_k(g,P) - \frac{\varepsilon}{3} &\le f'(x_0) \le M_k(g,P) + \frac{\varepsilon}{3} \end{aligned}$$

Since $x_0 \in [x_{k-1}, x_k]$ is arbitrary, we have

$$\begin{aligned} m_k(g,P) - \frac{\varepsilon}{3} \le m_k(f',P) \le M_k(f',P) &\le M_k(g,P) + \frac{\varepsilon}{3} \\ \Rightarrow M_k(f',P) - m_k(f',P) &\le M_k(g,P) - m_k(g,P) + \frac{2\varepsilon}{3} \end{aligned}$$

$$\begin{aligned} \therefore U(f',P) - L(f',P) &= \sum_{k=1}^n (M_k(f',P) - m_k(f',P))(x_k - x_{k-1}) \\ &\le \sum_{k=1}^n \left(M_k(g,P) - m_k(g,P) + \frac{2\varepsilon}{3} \right)(x_k - x_{k-1}) \\ &= U(g,P) - L(g,P) + \frac{2\varepsilon}{3} \sum_{k=1}^n x_k - x_{k-1} \\ &< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

Therefore, f' is integrable on $[0,1]$.

Question 3

(a) Let $\varepsilon > 0$ be given. Since $g_n \to g$ uniformly on $\mathbb{R}$, $\exists m \in \mathbb{N}$ such that $|g_m(x) - g(x)| < \frac{\varepsilon}{3}$ for all $x \in \mathbb{R}$. Now g_m is uniformly continuous on $\mathbb{R}$. Hence $\exists \delta > 0$ such that $\forall x, y \in \mathbb{R}$, $|g_m(x) - g_m(y)| < \frac{\varepsilon}{3}$ whenever $|x - y| < \delta$. If $x, y \in \mathbb{R}$ with $|x - y| < \delta$,

$$|g(x) - g(y)| \le |g(x) - g_m(x)| + |g_m(x) - g_m(y)| + |g_m(y) - g(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.$$

Therefore g is uniformly continuous on $\mathbb{R}$.

(b) (i) Observe that $\left|x^n(1-x^2)\right| \leq a^n$ for all $x \in [-a,a]$. Now, $\sum a^n$ converges since $0 < a < 1$. Therefore $\sum_{n=1}^{\infty} x^n(1-x)^2$ converges uniformly on $[-a,a]$.

(ii) Let $f_n(x) = x^n(1-x)^2$ for $x \in (-1,1]$. As a consequence of (3bi), $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise on $(-1,1)$. Furthermore, $\sum_{n=1}^{\infty} f_n(1) = 0$. Hence $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise on $(-1,1]$. Now, let $x_n = -1 + \frac{1}{n+1}$.

$$\begin{aligned}
\Rightarrow |f_n(x_n)| &= \left|\left(-1+\frac{1}{n+1}\right)^n\left(2-\frac{1}{n+1}\right)^2\right| \\
&= \left(1-\frac{1}{n+1}\right)^n\left(2-\frac{1}{n+1}\right)^2 \\
&\geq \left(e^{-1}\right)\left(\frac{3}{2}\right)^2
\end{aligned}$$

as $\left\{\left(1-\frac{1}{n+1}\right)^n\right\}$ is a monotone decreasing sequence converging to e^{-1} and $\left\{\left(2-\frac{1}{n+1}\right)^2\right\}$ is a monotone increasing sequence. Hence $f_n \nrightarrow 0$ uniformly on $(-1,1]$. We conclude that $\sum_{n=1}^{\infty} f_n(x)$ does not converge uniformly on $(-1,1]$.

(c) (i)

$$\left|\frac{\cos nx}{1+n^2}\right| \leq \frac{1}{1+n^2} < \frac{1}{n^2} \qquad \forall x \in \mathbb{R}, \forall n \in \mathbb{N}$$

Since $\sum \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} \frac{\cos nx}{1+n^2}$ converges uniformly on $\mathbb{R}$.

(ii) Let $f_n(x) = \frac{\cos nx}{1+n^2}$ for $x \in (0,2\pi)$.

$$\Rightarrow f_n'(x) = \frac{-n\sin nx}{1+n^2} \qquad \forall x \in (0,2\pi)$$

Let $0 < \varepsilon < \pi$ be given and consider $\sum_{n=1}^{\infty} f_n'(x)$ on $[\varepsilon, 2\pi-\varepsilon]$. Now, recall the identity $2\sin\frac{x}{2}\sin kx = \cos\left(k-\frac{1}{2}\right)x - \cos\left(k+\frac{1}{2}\right)x$ for all $k \in \mathbb{N}, \forall x \in \mathbb{R}$.

$$\begin{aligned}
\Rightarrow 2\sin\frac{x}{2}\sum_{k=1}^{n}\sin kx &= \sum_{k=1}^{n}\cos\left(k-\frac{1}{2}\right)x - \cos\left(k+\frac{1}{2}\right)x \\
&= \cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x \qquad \forall n \in \mathbb{N}, \forall x \in [\varepsilon, 2\pi-\varepsilon]
\end{aligned}$$

On $[\varepsilon, 2\pi-\varepsilon]$, we have $\sin\frac{\varepsilon}{2} \leq \sin\frac{x}{2} \leq 1$.

$$\Rightarrow \sum_{k=1}^{n}\sin kx = \frac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \qquad \forall n \in \mathbb{N}, \forall x \in [\varepsilon, 2\pi-\varepsilon]$$

$$\Rightarrow \left|\sum_{k=1}^{n}\sin kx\right| \leq \frac{\left|\cos\frac{x}{2}\right| + \left|\cos\left(n+\frac{1}{2}\right)x\right|}{2\sin\frac{\varepsilon}{2}} \leq \frac{1}{\sin\frac{\varepsilon}{2}} \qquad \forall n \in \mathbb{N}, \forall x \in [\varepsilon, 2\pi-\varepsilon]$$

Hence the sequence of partial sums of $\sum_{n=1}^{\infty}\sin nx$ is uniformly bounded on $[\varepsilon, 2\pi-\varepsilon]$. Now, since $\lim_{n\to\infty}\frac{-n}{1+n^2} = 0$ as a sequence of constants, we have $\frac{-n}{1+n^2} \to 0$ uniformly on $[\varepsilon, 2\pi-\varepsilon]$. In addition, $\frac{-n}{1+n^2}$ is monotone increasing as a sequence of constants. Hence we have $\left\{\frac{-n}{1+n^2}\right\}$ monotone increasing as a sequence of functions.

$$\Rightarrow \sum_{n=1}^{\infty} f_n'(x) \text{ converges uniformly on } [\varepsilon, 2\pi-\varepsilon]$$

Thus, f is differentiable on $[\varepsilon, 2\pi-\varepsilon]$ for all $0 < \varepsilon < \pi$. We conclude that f is differentiable on $(0,2\pi)$.

Question 4

(a) (i) Let $y = x^3$.

$$\Rightarrow \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} = \sum_{n=0}^{\infty} \frac{y^n}{(3n)!}$$

$$\lim_{n\to\infty} \left| \frac{1}{[3(n+1)]!} \middle/ \frac{1}{(3n)!} \right| = \lim_{n\to\infty} \frac{1}{(3n+3)(3n+2)(3n+1)} = 0$$

Hence the radius of convergence of $\sum_{n=0}^{\infty} \frac{y^n}{(3n)!}$ is ∞. We conclude that the radius of convergence of $\sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$ is ∞, and $I = \mathbb{R}$.

(ii)

$$f(x) + f'(x) + f''(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \sum_{n=1}^{\infty} \frac{x^{3n-1}}{(3n-1)!} + \sum_{n=1}^{\infty} \frac{x^{3n-2}}{(3n-2)!}$$

Since the radius of convergence of f is ∞, the radius of convergence of f' and f'' is ∞ which in turn implies that the radius of convergence of $f + f' + f''$ is ∞. Hence $f + f' + f''$ converges absolutely on $\mathbb{R}$ and we can rearrange

$$\sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \sum_{n=1}^{\infty} \frac{x^{3n-1}}{(3n-1)!} + \sum_{n=1}^{\infty} \frac{x^{3n-2}}{(3n-2)!}$$

to

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Therefore $f(x) + f'(x) + f''(x) = e^x$ for all $x \in \mathbb{R}$.

(b)

$$\lim_{n\to\infty} \left| \frac{1}{(n+1)(n+2)} \middle/ \frac{1}{n(n+1)} \right| = \lim_{n\to\infty} \frac{n}{n+2} = 1$$

Thus the radius of convergence is 1. Now, $\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}$.

$$\Rightarrow \frac{1}{x}\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^{n-1}}{n} = -1 - \sum_{n=1}^{\infty} \frac{x^n}{n+1} \qquad \text{for } x \neq 0$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} &= \sum_{n=1}^{\infty} \frac{x^n}{n} - \frac{x^n}{n+1} \\
&= \begin{cases} -\ln(1-x) + \frac{1}{x}\ln(1-x) + 1 & \text{if } x \in (-1,0) \cup (0,1) \\ 0 & \text{if } x = 0 \end{cases}
\end{aligned}$$

Observe that $\lim_{n\to\infty} \frac{1}{n(n+1)} = 0$ and $\frac{1}{(n+1)(n+2)} < \frac{1}{n(n+1)}$ for all $n \in \mathbb{N}$. Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$ converges.

$$\begin{aligned}
\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} &= \lim_{x\to(-1)^+} \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} \\
&= \lim_{x\to(-1)^+} -\ln(1-x) + \frac{1}{x}\ln(1-x) + 1 \\
&= 1 - 2\ln 2
\end{aligned}$$