NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Joseph Nah

MA2213 Numerical Analysis I
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Question 1

The matrix

$$\begin{pmatrix} 0.003000 & 59.14 & 0 & 0 & 0 \\ 5.291 & 6.130 & 0 & 0 & 0 \\ 0 & 0 & 0.600 & 0 & 0 \\ 0 & 0 & 0 & 0.002110 & 0.08204 \\ 0 & 0 & 0 & 0.3370 & 12.84 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 59.17 \\ 46.78 \\ 1.00 \\ 0.04313 \\ 6.757 \end{pmatrix}$$

can be separated into 3 different systems of equations

$$\begin{pmatrix} 0.003000 & 59.14 \\ 5.291 & 6.130 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 59.17 \\ 46.78 \end{pmatrix}, 0.600x_3 = 1.00, \begin{pmatrix} 0.002110 & 0.08204 \\ 0.3370 & 12.84 \end{pmatrix} \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0.04313 \\ 6.757 \end{pmatrix}$$

For the 1st matrix, note that $a_{21} > a_{11}$, so row exchange is needed. Hence, we have

$$\begin{pmatrix} 5.291 & -6.130 \\ 0.003000 & 59.14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 46.78 \\ 59.17 \end{pmatrix}$$

Now, to perform row reduction,

$$\begin{aligned} a_{22} &= 59.14 - \frac{0.003000}{5.291} \times -6.130 \\ &= 59.14 - 0.0005670 \times -6.130 \\ &= 59.14 + 0.003476 \\ &= 59.14 \end{aligned}$$

$$\begin{aligned} b_2 &= 59.17 - \frac{0.003000}{5.291} \times 46.78 \\ &= 59.17 - 0.0005670 \times 46.78 \\ &= 59.17 - 0.02652 \\ &= 59.14 \end{aligned}$$

We obtain

$$\begin{pmatrix} 5.291 & -6.130 \\ 0 & 59.14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 46.78 \\ 59.14 \end{pmatrix}$$

Hence,

$$\begin{aligned} a_2 &= \frac{59.14}{59.14} \\ &= 1.000 \end{aligned}$$

$$\begin{aligned} a_1 &= \frac{46.78 + 6.130 \times 1.000}{5.291} \\ &= \frac{46.78 + 6.130}{5.291} \\ &= \frac{52.91}{5.291} \\ &= 10.00 \end{aligned}$$

For the 2nd equation, we can solve it to get $a_3 = 1.667$.
For the 3rd matrix, note that $a_{21} > a_{11}$, so row exchange is needed. Hence, we have

$$\begin{pmatrix} 0.3370 & 12.84 \\ 0.002110 & 0.08204 \end{pmatrix} \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 6.757 \\ 0.04313 \end{pmatrix}$$

Now, to perform row reduction,

$$\begin{aligned} a_{22} &= 0.08204 - \frac{0.002110}{0.3370} \times 12.84 \\ &= 0.08204 - 0.006261 \times 12.84 \\ &= 0.08204 - 0.08039 \\ &= 0.001650 \end{aligned}$$

$$\begin{aligned} b_2 &= 0.04313 - \frac{0.002110}{0.3370} \times 6.757 \\ &= 0.04313 - 0.006261 \times 6.757 \\ &= 0.04313 - 0.04231 \\ &= 0.0008200 \end{aligned}$$

We obtain

$$\begin{pmatrix} 0.3370 & 12.84 \\ 0 & 0.001650 \end{pmatrix} \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 6.757 \\ 0.0008200 \end{pmatrix}$$

Hence,

$$\begin{aligned} a_5 &= \frac{0.0008200}{0.001650} \\ &= 0.4970 \end{aligned}$$

$$\begin{aligned} a_4 &= \frac{6.757 - 12.84 \times 0.4970}{0.3370} \\ &= \frac{6.757 - 6.381}{0.3370} \\ &= \frac{0.3760}{0.3370} \\ &= 1.116 \end{aligned}$$

Therefore, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 10.00 \\ 1.000 \\ 1.667 \\ 1.116 \\ 0.4970 \end{pmatrix}$$

Question 2

(a) We construct the divided difference table:

$$
\begin{aligned}
f(-2)=2, f(-1)=8, f(0)=22, f(1)=32, f(2)=26, f(3)=-8 \\
f[-2,-1]=6, f[-1,0]=14, f[0,1]=10, f[1,2]=-6, f[2,3]=-34 \\
f[-2,-1,0]=4, f[-1,0,1]=-2, f[0,1,2]=-8, f[1,2,3]=-14 \\
f[-2,-1,0,1]=-2, f[-1,0,1,2]=-2, f[0,1,2,3]=-2 \\
f[-2,-1,0,1,2]=0, f[-1,0,1,2,3]=0 \\
f[-2,-1,0,1,2,3]=0
\end{aligned}
$$

From the divided difference table, we conclude that

$$f(x)=2+6(x+2)+4(x+1)(x+2)-2x(x+1)(x+2)$$

Hence, the degree of the minimal degree polynomial is 3.

(b) First we observe that

$$f(x)=f(x_0)+(x-x_0)f[x_0,x_1]+\cdots+(x-x_0)(x-x_1)\cdots(x-x_{n-1})f[x_0,x_1,\ldots,x_n]$$

Hence, if $f(x)$ has a degree of $m<n$, the coefficient of $f[x_0,x_1,\ldots,x_n]=$ the coefficient of $x^n=0$. If $f(x)$ has a degree of $m=n$, the coefficient of $f[x_0,x_1,\ldots,x_n]=$ the coefficient of $x^n=x^m=1$.

Question 3

(a) The first step is to apply Trapezoidal rule with step size $h=\frac{b-a}{2}$. This produces:

$$\int_a^b f(x)dx=T(a,b)-\frac{h^3}{12}f^{(2)}(\mu),\ \text{for some } \mu\in(a,b)$$

The next step is to determine an accuracy approximation that does not require $f^{(2)}(\mu)$. To do this, we apply the Composite Trapezoidal rule with step size $\frac{b-a}{4}=\frac{h}{2}$, giving

$$
\begin{aligned}
\int_a^b f(x)dx &= T(a,\frac{a+b}{2})+T(\frac{a+b}{2},b)-(\frac{h}{2})^2\frac{b-a}{12}f^{(2)}(\mu),\ \text{for some } \mu\in(a,b) \\
&= T(a,b)-\frac{h^3}{12}f^{(2)}(\mu)
\end{aligned}
$$

This gives us

$$
\begin{aligned}
||T(a,b)-T(a,\frac{a+b}{2})-T(\frac{a+b}{2},b)|| &= \frac{3h^3}{48}f^{(2)}(\mu) \\
||\int_a^b f(x)dx-T(a,\frac{a+b}{2})-T(\frac{a+b}{2},b)|| &= \frac{h^3}{48}f^{(2)}(\mu)
\end{aligned}
$$

Hence,

$$||T(a,b)-T(a,\frac{a+b}{2})-T(\frac{a+b}{2},b)||=3\times||\int_a^b f(x)dx-T(a,\frac{a+b}{2})-T(\frac{a+b}{2},b)||$$

(b) This is a special case of the Higher Order Newton-Cotes Formula with n = 4 and h = 1. Therefore, the degree of precision = 5.

Question 4

The degree of precision of an interpolation polynomial is always odd, so if it is exact of all polynomials of degree at most 6, it must be exact for a polynomial of degree 7. Therefore, $E(x^7) = 0$.

Question 5

Since $q(x)$ is only a polynomial of degree 4, it needs only 5 data points to extrapolate the polynomial. We will take the 1st 5 data points. Hence,

$$p(x) = q(x) + a_5(x+2)(x+1)(x)(x-1)(x-2)$$

Substitute $x = 3$, we get

$$\begin{aligned} p(3) &= q(3) + a_5(3+2)(3+1)(3)(3-1)(3-2) \\ 30 &= 61 + a_5(120) \\ a_5 &= -\frac{31}{120} \end{aligned}$$

Hence,

$$p(x) = x^4 - x^3 + x^2 - x + 1 - \frac{31}{120}(x+2)(x+1)(x)(x-1)(x-2)$$