MA2108S - Mathematical Analysis I(S) Suggested Solutions

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Question 1

The module coordinator have acknowledged that this question is fundamentally flawed. Counterexample:

$$\text{Let } a_n = \begin{cases} 1 & \text{if } n \text{ is odd.} \\ -1 & \text{otherwise.} \end{cases}$$

Then $|s_k| \leq 1 \ \forall k \in \mathbb{N}$. By choosing $M = 2$ and $r = 0, |s_k| < 2 = (2)(k)^0 \ \forall k \in \mathbb{N}$.

But $\sum_{k=1}^{\infty} \frac{|a_k|}{k}$ is just the p-series:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$$

which obviously diverges.

Question 2

Remark: There is a small typo as the series should be $\sum_{n=1}^{\infty} a_n$ instead of $\sum_{n=0}^{\infty} a_n$.

(a) Yes, the series must converge to L.

Proof: Let $\{b_n\}_{n=1}^{\infty}$ denote the rearranged series. Let $s_k = \sum_{n=1}^{k} a_n$ and $p_k = \sum_{n=1}^{k} b_n$. Our aim is to prove that $\lim_{n\to\infty} p_n = L$.

$$\text{First note that } |s_k - p_k| = \begin{cases} 0 & \text{if k} \equiv 0 \pmod 3. \\ |a_{k+2} - a_k| & \text{if k} \equiv 1 \pmod 3. \\ |a_{k+1} - a_{k-1}| & \text{if k} \equiv 2 \pmod 3. \end{cases}$$

Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} a_n$ is a convergent series, $\lim_{n\to\infty} a_n = 0$. $\exists N_1 \in \mathbb{N}$ such that

$$n_1 \geq N_1 \to |a_{n_1 - 1}| < \frac{\epsilon}{3}.$$

Then $|a_{n_1+2} - a_{n_1}| \leq |a_{n_1+2}| + |a_{n_1}| < \frac{2\epsilon}{3}$. Similarly, $|a_{n_1+1} - a_{n_1-1}| < \frac{2\epsilon}{3}$ so we have:

$$n_1 \geq N_1 \to |s_{n_1} - p_{n_1}| < \frac{2\epsilon}{3}.$$

Since $\lim_{n\to\infty} s_n = L$, $\exists N_2 \in \mathbb{N}$ such that

$$n_2 \geq N_2 \to |s_n - L| < \frac{\epsilon}{3}.$$

Choose $N = \max\{N_1, N_2\}$. Then:

$$n \geq N \to |p_n - L| \leq |p_n - s_n| + |s_n - L| < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus $\lim_{n\to\infty} p_n = L$ as desired.

(b) No, the series does not necessarily converge.

Counterexample: Let a_n be a series such that each group $\{a_{2^n}, a_{2^n+1}, ..., a_{2^{n+1}-1}\}$ is of the following form.

$$a_k = \begin{cases} \frac{1}{2^n} & \text{if k is odd.} \\ -\frac{1}{2^n} & \text{otherwise.} \end{cases}$$

The original series is of the form:

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{8} + ...$$

which converges by alternating series test.

Let $\sum_{n=1}^{\infty} b_n$ be the rearranged series. Then $\sum_{n=1}^{\infty} b_n$ is of the form:

$$\frac{1}{1} + \frac{1}{2} - \frac{1}{2} + \frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{8} + ...$$

In particular, note that $\forall i \in \mathbb{Z}_{\geq 0}$:

$$\sum_{n=1}^{3(2^i)-1} b_n = \frac{3}{2} \;, \quad \sum_{n=1}^{2^{i+1}-1} b_n = 1.$$

Thus is it clear that $\sum_{n=1}^{\infty} b_n$ cannot be convergent.

Question 3

Assume that the sequence $\{x_n\}_{n=1}^{\infty}$ does not converge to x_0. Then $\exists \epsilon > 0$ such that $\forall N \in \mathbb{N}, \exists n \in \mathbb{N}$ such that:

$$n \geq N \;\wedge\; \rho(x_n, x_0) \geq \epsilon.$$

Choose $N = 1$. Then $\exists n_1 \geq 1$ such that $\rho(x_{n_1}, x_0) \geq \epsilon$.

Assume n_k has been defined. Choose $N = n_k + 1$. Then $\exists n_{k+1} \geq N$ such that $\rho(x_{n_k+1}, x_0) \geq \epsilon$. Inductively, we have defined a sequence $\{x_{n_k}\}_{k=1}^{\infty}$, which is a subsequence of $\{x_n\}_{n=1}^{\infty}$.

$\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of a totally bounded sequence so it is also totally bounded. As $\langle M, \rho \rangle$ is a complete metric space, $\exists$ subsequence of $\{x_{n_k}\}_{k=1}^{\infty}, \{y_i\}_{i=1}^{\infty}$ such that $\{y_i\}_{i=1}^{\infty}$ converges to some $L \in M$. Obviously $L \neq x_0$ since $\rho(y_i, x_0) \geq \epsilon \; \forall i \in \mathbb{N}$.

Since $\{x_n\}_{n=1}^{\infty}$ has the property $x_m \neq x_n$ if $m \neq n$, $\{y_i\}_{n=1}^{\infty}$ also have the same property as it is a subsequence of $\{x_n\}_{n=1}^{\infty}$. But this means that $\exists$ a sequence, $\{y_i\}_{i=1}^{\infty}$, such that $y_m \neq y_n$ if $m \neq n$ and $\lim_{n\to\infty} y_n = L$. This means that L is also a cluster point of M. This is a contradiction as $L \neq x_0$ and x_0 is the only cluster point of M. Thus the assumption is false and the sequence $\{x_n\}_{n=1}^{\infty}$ must converge to x_0.

Question 4

Remark: If M is empty then trivially $U = M$. Thus we will only focus on the case where M is non-empty.

Since M is non-empty, $\exists x \in M$. Let U be the set of all reachable points from x. We will prove that $U = M$.

Claim 1 : U is open in M.

Let $y \in U$. Then $\exists$ finitely many sets $G_1, ..., G_n$ such that $x \in G_1, y \in G_n$ and $G_i \cap G_{i+1} \neq \emptyset,\ 1 \leq i < n$. Since G_n is open, $\exists \epsilon_1 > 0$ such that $B[y, \epsilon_1] \subseteq G_n$.

Obviously $G_n \subseteq U$. Thus $B[y, \epsilon_1] \subseteq G_n \to B[y, \epsilon_1] \subseteq U$ so U is open.

Claim 2 : U is closed in M.

Let $z \in \overline{U}$. Then $z \in M$. Since $\mathcal{G}$ is an open cover for M, $\exists G_k \in \mathcal{G}$ such that $z \in G_k$. G_k is open so $\exists \epsilon_2 > 0$ such that $B[z, \epsilon_2] \subseteq G_k$.

On the other hand, $z \in \overline{U} \to \exists z' \in U$ such that $\rho(z, z') < \epsilon_2$. This means that $z' \in G_k$ as $B[z, \epsilon_2] \subseteq G_k$.

Since $z' \in U, \exists$ finitely many sets $G_1, ..., G_j$ such that $x \in G_1, z \in G_j$ and $G_i \cap G_{i+1} \neq \emptyset,\ 1 \leq i < j$. Thus we have:

$$z' \in G_j \ \wedge \ z' \in G_k \to G_j \cap G_k \neq \emptyset.$$

By defining $G_{j+1} = G_k$, the finite collection of sets $G_1, ..., G_j, G_{j+1}$ still retains the property $G_i \cap G_{i+1} \neq \emptyset,\ 1 \leq i < j+1$. Since $x \in G_1 \ \wedge \ z \in G_{j+1}$, we conclude that $z \in U$. Thus U is closed in M.

Since $x \in U,\ U \neq \emptyset$. But U is both open and closed in M, a connected metric space. Hence we finally conclude that $U = M$.

Question 5

Obviously $\overline{E}$ is closed. Since $\overline{E}$ is a closed set in M, a complete metric space, $\overline{E}$ is complete. Thus to prove that $\overline{E}$ is compact, it suffices to prove that $\overline{E}$ is totally bounded.

Let $\epsilon > 0$. Choose $r = \frac{\epsilon}{4}$. Then $\exists$ compact set $A \subseteq M$ such that $E \subseteq (A)_{\frac{\epsilon}{4}}$. Since A is compact, $\exists$ finitely many $x_1, x_2, ..., x_n$ such that $A \subseteq \bigcup_{i=1}^n B[x_i, \frac{\epsilon}{4}]$.

Let $z \in E$. Since $E \subseteq (A)_{\frac{\epsilon}{4}}, z \in (A)_{\frac{\epsilon}{4}}$. Then $\exists y \in A$ such that $\rho(z, y) < \frac{\epsilon}{4}$. Since $y \in A$, $\exists x_j \in \{x_1, x_2, ..., x_n\}$ such that $y \in B[x_j, \frac{\epsilon}{4}]$. Then:

$$\rho(x_j, z) \leq \rho(x_j, y) + \rho(y, z) < \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} = \tfrac{\epsilon}{2}.$$

Thus $z \in B[x_j, \frac{\epsilon}{2}]$ so $E \subseteq \bigcup_{i=1}^n B[x_i, \frac{\epsilon}{2}]$.

Finally, let $v \in \overline{E}$. Then $\exists v' \in E$ such that $\rho(v, v') < \frac{\epsilon}{2}$. Since $E \subseteq \bigcup_{i=1}^n B[x_i, \frac{\epsilon}{2}]$, $\exists x_k \in \{x_1, x_2, ..., x_n\}$ such that $v' \in B[x_k, \frac{\epsilon}{2}]$. Then similarly:

$$\rho(v, x_k) \leq \rho(v, v') + \rho(v', x_k) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

To conclude, $v \in B[x_k, \epsilon]$ so $\overline{E} \subseteq \bigcup_{i=1}^n B[x_i, \epsilon]$. Thus $\overline{E}$ is totally bounded.

Question 6

It suffices to prove that $\forall f \in M,\ \forall \epsilon > 0,\ \exists f' \in \bigcup_{n=1}^\infty F_n$ such that $\rho(f, f') < \epsilon$.

Obviously $[0, 1]$ is compact under the Euclidean metric. Thus any continuous function with $[0, 1]$ as its domain will be uniformly continuous. In other words, M is the set of all **uniformly** continuous functions $f : [0, 1] \to [0, 1]$.

Let $f \in M$ and let $\epsilon > 0$. By uniform continuity of f, $\exists \delta > 0$ such that:

$$\forall x, y \in [0, 1],\ |x - y| < \delta \to |f(x) - f(y)| < \frac{\epsilon}{2}.$$

Let $i = \lceil \frac{1}{\delta} \rceil + 1$. Then $\frac{1}{i} < \delta$. We will now construct the function f' as follows:

Divide the interval [0,1] into

$$[0, \frac{1}{i}], [\frac{1}{i}, \frac{2}{i}], [\frac{2}{i}, \frac{3}{i}], ..., [\frac{i-1}{i}, 1].$$

Within each segment $[\frac{k-1}{i}, \frac{k}{i}]$:

$$f'(t) = [(1-k)f(\frac{k}{i}) + kf(\frac{k-1}{i})] + t[if(\frac{k}{i}) - if(\frac{k-1}{i})].$$

Since $a_k = [(1-k)f(\frac{k}{i}) + kf(\frac{k-1}{i})]$ and $b_k = [if(\frac{k}{i}) - if(\frac{k-1}{i})]$, f' is linear on each segment. Thus $f' \in F_i$ so $f' \in \bigcup_{n=1}^{\infty} F_n$.

Note that $f'(\frac{k-1}{i}) = f(\frac{k-1}{i})$ and $f'(\frac{k}{i}) = f(\frac{k}{i})$. By linearity of f' on $[\frac{k-1}{i}, \frac{k}{i}]$:

$$\begin{aligned}\forall x \in [\frac{k-1}{i}, \frac{k}{i}],\ |f'(\frac{k}{i}) - f'(x)| &\leq |f'(\frac{k}{i}) - f'(\frac{k-1}{i})| \\ &= |f(\frac{k}{i}) - f(\frac{k-1}{i})|.\end{aligned}$$

Finally, we will prove that $\rho(f, f') < \epsilon$.

Let $\lambda \in [0,1]$. Then $\exists j \in \mathbb{N}, 1 \leq j \leq i$, such that $\lambda \in [\frac{j-1}{i}, \frac{j}{i}]$.

$$\begin{aligned}|f(\lambda) - f'(\lambda)| &\leq |f(\lambda) - f(\frac{j}{i})| + |f(\frac{j}{i}) - f'(\lambda)| \\ &\leq |f(\lambda) - f(\frac{j}{i})| + |f(\frac{j}{i}) - f(\frac{j-1}{i})| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Since $|f(\lambda) - f'(\lambda)| < \epsilon\ \forall \lambda \in [0,1]$, $\max\{|f(x) - f'(x)| : x \in [0,1]\} < \epsilon$. Thus $\rho(f, f') < \epsilon$.