MA2104 Suggested Solutions

Written by: Nicholas Russell Saerang
Audited by: Chow Yong Lam

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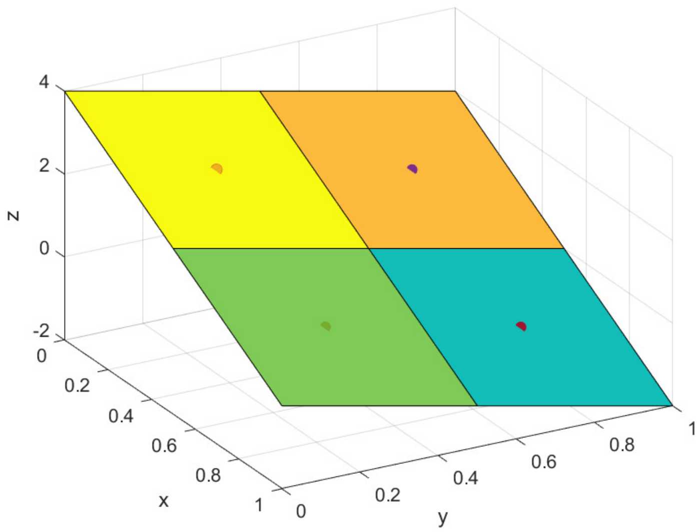
1. Note that S is a graph of a function $g(x,y) = 4 - 4x - 2y$, where $(x,y) \in D = [0,1]^2$. Therefore,

$$\begin{aligned}\iint_S f(x,y,z)\,dS &= \iint_D f(x,y,g(x,y))\sqrt{g_x^2+g_y^2+1}\,dA \\ &= \iint_D f(x,y,g(x,y))\sqrt{(-4)^2+(-2)^2+1}\,dA \\ &= \sqrt{21}\iint_D f(x,y,g(x,y))\,dA \\ &\approx \sqrt{21}\sum_{(x^*,y^*)} f(x^*,y^*,g(x^*,y^*))\Delta x\Delta y \\ &= \sqrt{21}(3-1+2+7)\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{11\sqrt{21}}{4}.\end{aligned}$$

Here, $\sum_{(x^*,y^*)}$ is basically summing over the four points provided in the question.



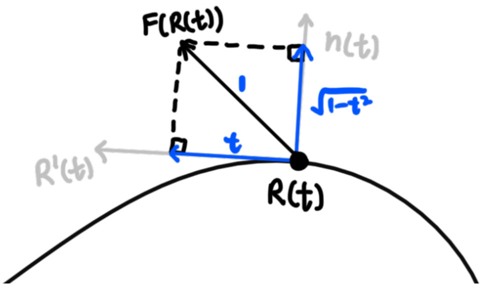
2. Suppose $\mathbf{F}(x,y) = \begin{pmatrix} X(x,y) \\ Y(x,y) \end{pmatrix}$. Since $X_y = Y_x = 3x^2 + 2y$, $\mathbf{F}$ is conservative.
Also, a possible potential function is $f(x,y) = x^3y + xy^2 + \frac{1}{3}y^3 - \frac{1}{4}\cos 2x$. This is because we have

$$\begin{aligned} f(x,y) &= \int 3x^2y + \sin x \cos x + y^2 \; dx \\ &= x^3y - \frac{1}{4}\cos 2x + xy^2 + p(y) \\ f(x,y) &= \int x^3 + 2xy + y^2 \; dy \\ &= x^3y + xy^2 + \frac{1}{3}y^3 + q(x) \end{aligned}$$

for some function $p(y)$ and $q(x)$, to which we can set them to be $\frac{1}{3}y^3$ and $-\frac{1}{4}\cos 2x$, respectively. Therefore,

$$\begin{aligned} \int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} &= f(0, -\pi^2) - f(-\pi, 0) \\ &= \left(-\frac{1}{3}\pi^6 - \frac{1}{4}\right) - \left(-\frac{1}{4}\right) = -\frac{1}{3}\pi^6. \end{aligned}$$

3. Since $\text{comp}_{\mathbf{n}(t)}\mathbf{F}(R(t)) = \sqrt{1-t^2}$ and $\|\mathbf{F}\| = 1$, using the fact that $\mathbf{n}(t)$ are perpendicular to $R'(t)$, we must have $\text{comp}_{R'(t)}\mathbf{F}(R(t)) = t$. You can take a look at the diagram below for the visualization.



Thus,

$$\begin{aligned} \int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(R(t)) \cdot R'(t) \; dt \\ &= \int_0^1 \text{comp}_{R'(t)}\mathbf{F}(R(t)) \cdot \|R'(t)\| \; dt \\ &= \int_0^1 t\sqrt{1^2 + (2t)^2 + (6t)^2} \; dt \\ &= \int_0^1 t\sqrt{40t^2 + 1} \; dt \\ &= \frac{(40+1)^{\frac{3}{2}}}{120} - \frac{(1)^{\frac{3}{2}}}{120} = \frac{41\sqrt{41} - 1}{120}. \end{aligned}$$

4. Since $f(x,y) = f(x,-y)$, then

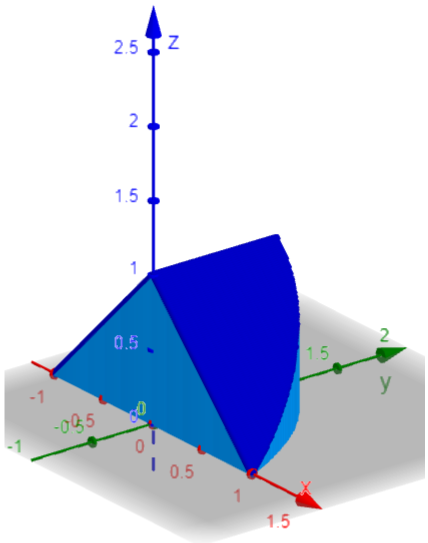
$$\int_0^1 f(x,y)dy = \int_{-1}^0 f(x,y)dy = \frac{1}{2}\int_{-1}^1 f(x,y)dy$$

Therefore,

$$\begin{aligned}
&\int_0^2 \int_{-2}^2 f(x,y)dxdy \\
&= \int_1^2 \int_{-2}^2 f(x,y)dxdy + \int_0^1 \int_{-2}^1 f(x,y)dxdy + \int_0^1 \int_1^2 f(x,y)dxdy \\
&= \int_1^2 \int_{-2}^2 f(x,y)dxdy + \int_{-2}^1 \int_0^1 f(x,y)dydx + \int_1^2 \int_0^1 f(x,y)dydx \\
&= \int_1^2 \int_{-2}^2 f(x,y)dxdy + \frac{1}{2}\int_{-2}^1 \int_{-1}^1 f(x,y)dydx + \int_1^2 \int_{-1}^0 f(x,y)dydx \\
&= 1 + \frac{1}{2}\int_{-1}^1 \int_{-2}^1 f(x,y)dxdy + \int_{-1}^0 \int_1^2 f(x,y)dxdy \\
&= 1 + \frac{1}{2}\cdot 6 + \frac{1}{3} = \frac{13}{3}.
\end{aligned}$$

5. Since $0 \le z \le 1 - |x|$, we have $|x| \le 1 - z \to z - 1 \le x \le 1 - z$ while $0 \le z \le 1$. Thus,

$$\int_{-1}^1 \int_0^{1-x^2} \int_0^{1-|x|} f(x,y,z)\ dzdydx = \int_0^1 \int_{z-1}^{1-z} \int_0^{1-x^2} f(x,y,z)\ dydxdz.$$



6. Using the spherical coordinate transformation, we have $(x, y, z) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$ and

$$f(x, y, z) = \begin{cases} \frac{1}{10(1+\rho^3)}, & \text{if } \rho \leq k \\ 0, & \text{otherwise} \end{cases}$$

(a) We must have

$$\begin{aligned} 1 &= \int_0^{\pi}\int_0^{2\pi}\int_0^{k} \frac{1}{10(1+\rho^3)}\rho^2 \sin\phi \; d\rho d\theta d\phi \\ &= \int_0^{\pi}\int_0^{2\pi} \frac{\ln(1+k^3)}{30} \sin\phi \; d\theta d\phi \\ &= \frac{\pi}{15}\ln(1+k^3)\int_0^{\pi} \sin\phi \; d\phi \\ &= \frac{2\pi}{15}\ln(1+k^3). \end{aligned}$$

Therefore, $k = \sqrt[3]{e^{\frac{15}{2\pi}} - 1}$.

(b) Note that

$$z = \sqrt{x^2+y^2} \Rightarrow \phi = \frac{\pi}{4}$$

$$\sqrt{(x^2+y^2+z^2)^3} + 1 = e^{\frac{z}{\sqrt{x^2+y^2+z^2}}} \Rightarrow \rho^3 + 1 = e^{\cos\phi} \Rightarrow \rho = \sqrt[3]{e^{\cos\phi} - 1}$$

Also, since $\rho \leq \sqrt[3]{e^{\cos\phi} - 1} \leq \sqrt[3]{e-1} < k$, all the points inside the region will have a nonzero f-value. So this question boils down to evaluating the following integral.

$$\begin{aligned} &\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\sqrt[3]{e^{\cos\phi}-1}} \frac{1}{10(1+\rho^3)}\rho^2 \sin\phi \; d\rho d\theta d\phi \\ &= \int_0^{\frac{\pi}{4}}\int_0^{2\pi} \frac{\sin\phi\cos\phi}{30} \; d\theta d\phi \\ &= \int_0^{\frac{\pi}{4}} \frac{2\pi\sin\phi\cos\phi}{30} \; d\phi \\ &= \int_0^{\frac{\pi}{4}} \frac{\pi\sin 2\phi}{30} \; d\phi \\ &= \frac{\pi}{30}\cdot\frac{-1}{2}\cdot\left(\cos\frac{\pi}{2} - \cos 0\right) = \frac{\pi}{60}. \end{aligned}$$

7. (a) Let $(u,v) = (x-2y, 2x+y)$. Thus $(x,y) = \left(\frac{u+2v}{5}, \frac{v-2u}{5}\right)$ and $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{vmatrix} = \frac{1}{5}$.

Note that $5x^2+5y^2+1 = u^2+v^2+1$. Suppose $(u,v) \in S = [-1,2]\times[0,3]$, then by the 2D-Jacobian we have

$$\begin{aligned}
\iint_R f(x,y)\, dA &= \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\, dudv \\
&= \frac{1}{5}\iint_S (u^2+v^2+1)\, dudv \\
&= \frac{1}{5}\int_0^3\int_{-1}^2 (u^2+v^2+1)\, dudv \\
&= \frac{1}{5}\int_0^3\int_{-1}^2 (u^2+v^2)\, dudv + \frac{1}{5}(3-0)(2+1) \qquad \text{(take 1 out and it becomes the area)} \\
&= \frac{1}{5}\int_0^3 \left(\frac{2^3+1^3}{3} + (2+1)v^2\right) dv + \frac{9}{5} \\
&= \frac{3}{5}\int_0^3 (v^2+1)\, dv + \frac{9}{5} \\
&= \frac{3}{5}\left(\frac{3^3-0^3}{3} + (3-0)\right) + \frac{9}{5} = 9.
\end{aligned}$$

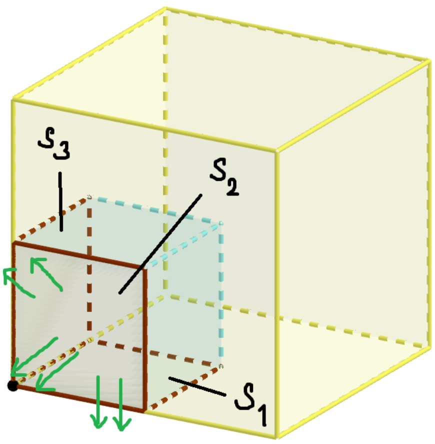
(b) Note that along the path, $x^2+y^2=4$ is constant, so the average depth is just the constant value of $f(x,y)$ which is 21.

(c) We can parameterize the path as $R(t) = (-2\cos t, 2\sin t), 0 \le t \le \pi \Rightarrow R'(t) = (2\sin t, 2\cos t)$.
The work done by the ocean is

$$\begin{aligned}
\int_0^\pi \mathbf{F}(x(t),y(t)) \cdot R'(t)\, dt &= \int_0^\pi \begin{pmatrix} 4\cos^2 t \\ 4\sin^2 t \end{pmatrix} \cdot \begin{pmatrix} 2\sin t \\ 2\cos t \end{pmatrix} dt \\
&= \int_0^\pi 8\sin t\cos^2 t + 8\cos t\sin^2 t\, dt \\
&= \left[\frac{8}{3}(\sin^3 t - \cos^3 t)\right]_0^\pi \\
&= \frac{8}{3} + \frac{8}{3} = \frac{16}{3}.
\end{aligned}$$

8. (a) Suppose $\mathbf{F}(x,y) = \begin{pmatrix} X(x,y,z) \\ Y(x,y,z) \\ Z(x,y,z) \end{pmatrix}$. Since $xz = X_y \neq Y_x = -2xy$, $\mathbf{F}$ is not conservative.

(b) In the diagram below, the black point is the origin, while the yellow and cyan cubes represent E and E' respectively.



The flux across $\mathbf{S}$ is just the overall flux across the surfaces of E, call this $\mathbf{T}$, minus the total flux on the three sides of E', highlighted in brown. Suppose we denote the sides $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$, where each side lies on the xy-plane, xz-plane, and the yz-plane, respectively. Note that by Gauss' Theorem,

$$\begin{aligned}
\iint_{\mathbf{S}} \mathbf{F} \cdot \, d\mathbf{S} &= \iint_{\mathbf{T}} \mathbf{F} \cdot \, d\mathbf{T} - \left(\iint_{\mathbf{S}_1} \mathbf{F} \cdot \, d\mathbf{S}_1 + \iint_{\mathbf{S}_2} \mathbf{F} \cdot \, d\mathbf{S}_2 + \iint_{\mathbf{S}_3} \mathbf{F} \cdot \, d\mathbf{S}_3 \right) \\
&= \iiint_E \operatorname{div} \mathbf{F} \, dV - \left(\iint_{\mathbf{S}_1} \mathbf{F} \cdot \, d\mathbf{S}_1 + \iint_{\mathbf{S}_2} \mathbf{F} \cdot \, d\mathbf{S}_2 + \iint_{\mathbf{S}_3} \mathbf{F} \cdot \, d\mathbf{S}_3 \right) \\
&= \iiint_E (yz - x^2 + 1 + x^2 - yz) \, dV - \left(\iint_{\mathbf{S}_1} \mathbf{F} \cdot \, d\mathbf{S}_1 + \iint_{\mathbf{S}_2} \mathbf{F} \cdot \, d\mathbf{S}_2 + \iint_{\mathbf{S}_3} \mathbf{F} \cdot \, d\mathbf{S}_3 \right) \\
&= \iiint_E dV - \left(\iint_{\mathbf{S}_1} \mathbf{F} \cdot \, d\mathbf{S}_1 + \iint_{\mathbf{S}_2} \mathbf{F} \cdot \, d\mathbf{S}_2 + \iint_{\mathbf{S}_3} \mathbf{F} \cdot \, d\mathbf{S}_3 \right) \\
&= 1 - \left(\iint_{\mathbf{S}_1} \mathbf{F} \cdot \, d\mathbf{S}_1 + \iint_{\mathbf{S}_2} \mathbf{F} \cdot \, d\mathbf{S}_2 + \iint_{\mathbf{S}_3} \mathbf{F} \cdot \, d\mathbf{S}_3 \right).
\end{aligned}$$

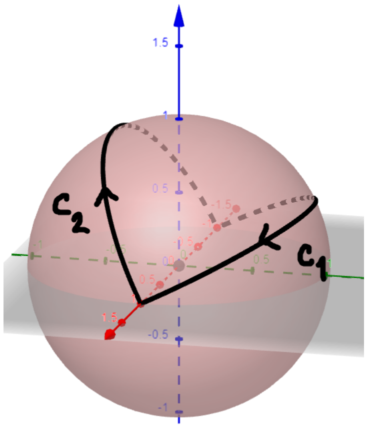
For $\mathbf{S}_1$, the normal vector is $-\mathbf{k}$ but $z = 0$, so $\iint_{\mathbf{S}_1} \mathbf{F} \cdot \, d\mathbf{S}_1 = 0$. Similarly, for $\mathbf{S}_2$, the normal vector is $-\mathbf{j}$ but $y = 0$, so $\iint_{\mathbf{S}_2} \mathbf{F} \cdot \, d\mathbf{S}_2 = 0$. For $\mathbf{S}_3$, the normal vector is $-\mathbf{i}$ and $x = 0$. However, the dot product is non-zero, which is

$$\begin{aligned}
\iint_{\mathbf{S}_3} \mathbf{F} \cdot \, d\mathbf{S}_3 &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \mathbf{F} \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} dydz \\
&= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} (-xyz - 1) \, dydz \\
&= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} (-1) \, dydz = -\frac{1}{4}.
\end{aligned}$$

Therefore, the flux of $\mathbf{F}$ across $\mathbf{S}$ is $1 - (0 + 0 - \frac{1}{4}) = \frac{5}{4}$.

(c) First, we have

$$\text{curl } \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz+1 & -x^2y+y & x^2z-\frac{1}{2}yz^2 \end{vmatrix} = \begin{pmatrix} -\frac{1}{2}z^2 \\ x(y-2z) \\ -x(2y+z) \end{pmatrix}.$$



A possible surface that has the concatenated curve as the boundary is the quarter sphere shown in the picture above. Note that

$$\int_{\mathbf{C}_1 \cdot \mathbf{C}_2} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathbf{S}} \text{curl } \mathbf{F} \cdot d\mathbf{S}.$$

The value of curl $\mathbf{F} \cdot d\mathbf{S}$ depends on the curl and the normal vector on each point. Since the surface is symmetric to the plane $x = 0$, we can see that the **j**-component and the **k**-component of the two curls at (x, y, z) and $(-x, y, z)$ will just cancel each other where the corresponding components of the normal vectors at both points are the same.

On the other hand, the **i**-component of the normal vector will also cancel each other for the two points (x, y, z) and $(-x, y, z)$ while that component of the curl is the same for both points, which is $-\frac{1}{2}z^2$, so this concludes that the sum of the dot product curl $\mathbf{F} \cdot d\mathbf{S}$ of both points will be 0 and therefore the resulting integral will also be 0.

Appendix

Question 8c

Alternatively, if we do the long way, using the following parametrization.

$$R(\theta,\phi) = (\cos\phi, \cos\theta\sin\phi, \sin\theta\sin\phi), 0 \le \phi \le \pi, \tan^{-1}\left(\frac{1}{2}\right) \le \theta \le \tan^{-1}\left(\frac{1}{2}\right) + \frac{\pi}{2}$$

we have

$$\begin{aligned} R_\theta \times R_\phi &= \begin{pmatrix} 0 \\ -\sin\theta\sin\phi \\ \cos\theta\sin\phi \end{pmatrix} \times \begin{pmatrix} -\sin\phi \\ \cos\theta\cos\phi \\ \sin\theta\cos\phi \end{pmatrix} \\ &= \begin{pmatrix} -\sin\phi\cos\phi \\ -\cos\theta\sin^2\phi \\ -\sin\theta\sin^2\phi \end{pmatrix} \end{aligned}$$

which is the correct direction of the induced orientation since the **k**-component is negative. Finally, using Stokes' Theorem, we have

$$\begin{aligned} \int_{\mathbf{C_1}\cdot\mathbf{C_2}} \mathbf{F}\cdot d\mathbf{r} &= \iint_{\mathbf{S}} \text{curl } \mathbf{F}\cdot d\mathbf{S} \\ &= \iint_{\mathbf{S}} \begin{pmatrix} -\frac{1}{2}z^2 \\ x(y-2z) \\ -x(2y+z) \end{pmatrix} \cdot d\mathbf{S} \\ &= \int_0^\pi \int_{\tan^{-1}(\frac{1}{2})}^{\tan^{-1}(\frac{1}{2})+\frac{\pi}{2}} \begin{pmatrix} -\frac{1}{2}(\sin\theta\sin\phi)^2 \\ \cos\phi((\cos\theta\sin\phi) - 2(\sin\theta\sin\phi)) \\ -\cos\phi(2(\cos\theta\sin\phi) + (\sin\theta\sin\phi)) \end{pmatrix} \cdot \begin{pmatrix} -\sin\phi\cos\phi \\ -\cos\theta\sin^2\phi \\ -\sin\theta\sin^2\phi \end{pmatrix} d\theta d\phi \\ &= \int_0^\pi \int_{\tan^{-1}(\frac{1}{2})}^{\tan^{-1}(\frac{1}{2})+\frac{\pi}{2}} \sin^3\phi\cos\phi \begin{pmatrix} -\frac{1}{2}(\sin\theta)^2 \\ \cos\theta - 2\sin\theta \\ -2\cos\theta - \sin\theta \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -\cos\theta \\ -\sin\theta \end{pmatrix} d\theta d\phi \\ &= \int_0^\pi \sin^3\phi\cos\phi \, d\phi \cdot \int_{\tan^{-1}(\frac{1}{2})}^{\tan^{-1}(\frac{1}{2})+\frac{\pi}{2}} \begin{pmatrix} -\frac{1}{2}(\sin\theta)^2 \\ \cos\theta - 2\sin\theta \\ -2\cos\theta - \sin\theta \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -\cos\theta \\ -\sin\theta \end{pmatrix} d\theta \\ &= 0 \end{aligned}$$

since

$$\int_0^\pi \sin^3\phi\cos\phi \, d\phi = \frac{\sin^4\pi - \sin^4 0}{4} = 0.$$