MA2101 - Linear Algebra II Suggested Solutions

(Semester 2 : AY2018/19)

Written by : Yip Jung Hon
Audited by : Pan Jing Bin

Question 1

Since B is a basis for V, $\dim(V) = n$. One has $T(u_1)\cdots T(u_n)$ are sufficient vectors to span V. We just need to show linear independence. Assume that $\exists c_1, c_2, ..., c_n$, not all zero, such that

$$\sum_{i=1}^{n} c_i T(u_i) = 0.$$

Since T is a linear transformation, we have

$$T(\sum_{i=1}^{n} c_i u_i) = 0.$$

where $\sum_{i=1}^{n} c_i u_i$ is a non-zero vector. This is a contradiction since T is injective.

Question 2

(i)

$$\begin{bmatrix} 1 & -1 & x^2 \\ 1 & -1 & x \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 0 & 0 & x^2 - x \\ 1 & -1 & x \\ 0 & 1 & 2 \end{bmatrix}.$$

We just want $x^2 - x \neq 0$. For that, we may choose $x = 2$ since $2^2 - 2 \neq 0 \pmod 3$.

(ii) Sub $x = 2$ in the above matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 2 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Hence $[v]_B = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$.

Question 3

(i) $m_T(x) = (x-1)^4(x-2)^3$ tells us that the largest Jordan block associated with eigenvalue 1 is 4, and the largest Jordan block associated with eigenvalue 2 is 3. $c_T(x) = (x-1)^7(x-2)^3$ tells us that there is **only 1** Jordan block associated with eigenvalue 2, and that all Jordan blocks comprising of 1's down the diagonal will take up a 7×7 array of the matrix. From now on, we switch to the notation used in the textbook, where $\mathbf{J}_n(\lambda)$ has λ denoting the entries down the diagonal, and n denotes the size of the Jordan block.

Here are all the non-similar forms:

$$A = \begin{pmatrix} \mathbf{J}_4(1) & & \\ & \mathbf{J}_3(1) & \\ & & \mathbf{J}_3(2) \end{pmatrix} \quad B = \begin{pmatrix} \mathbf{J}_4(1) & & & \\ & \mathbf{J}_2(1) & & \\ & & \mathbf{J}_1(1) & \\ & & & \mathbf{J}_3(2) \end{pmatrix}$$

$$C = \begin{pmatrix} \mathbf{J}_4(1) & & & & \\ & \mathbf{J}_1(1) & & & \\ & & \mathbf{J}_1(1) & & \\ & & & \mathbf{J}_1(1) & \\ & & & & \mathbf{J}_3(2) \end{pmatrix}$$

(ii) For the eigenvalue 1, the $\mathbf{J}_3(2)$ block does not 'contribute' any dimensions to the nullspace at all. $\dim(E_1)$ is **equal to the number of Jordan blocks associated to eigenvalue 1**.

For A, $\dim\ker(T-I) = 2$, since there are 2 Jordan blocks associated to eigenvalue 1. When raised to the second power, $(T-I)^2$ will have all Jordan blocks $\geq$ size 2 contributing 2 dimensions to the $\dim\ker(T-I)^2$. On the other hand, Jordan blocks < 2 cannot contribute a dimension greater than their size to $\dim\ker(T-I)^2$, so they still only contribute 1 dimension. We have that $\dim\ker(T-I)^2 = 4$.

For B, $\dim\ker(T-I) = 3$, and $\dim\ker(T-I)^2 = 5$. $\mathbf{J}_4(1)$ contributes 2 dimensions, $\mathbf{J}_2(1)$ contributes 2 dimensions, but $\mathbf{J}_1(1)$ is only able to contribute 1 dimension.

For C, $\dim\ker(T-I) = 4$. $\dim\ker(T-I)^2 = 5$. All $\mathbf{J}_1(1)$ blocks contribute 1 dimension each, and $\mathbf{J}_4(1)$ contributes 2 dimensions.

Question 4

(i) Consider $E = \{1, x, x^2\}$, the standard basis for $\mathcal{P}_2(\mathbb{R})$.

$T(1) = 1 - x - x^2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}_E$. $T(x) = 1 - x - 3x^2 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}_E$. $T(x^2) = 2x^2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}_E$.

$$T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & -3 & 2 \end{bmatrix}$$

with $c_T(x) = x^3 - 2x^2$. $\lambda = 0, 2$.

(ii)

$$T - 2I = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -3 & 0 \\ -1 & -3 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Basis for $E_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$$T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & -3 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Basis for $E_0 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(iii) $c_T(x) = x^3 - 2x^2 = x^2(x-2)$. Thus $m_T(x) = x^k(x-2)$, where $k = 1$ or 2. Since $\dim(E_0) = 1$, we must have $k = 2$ so $m_T(x) = x^2(x-2)$.

(iv) Since $m_T(x) = x^2(x-2)$, this tells us that the Jordan blocks are $J_2(0)$ & $J_1(2)$. Since the Jordan block associated with eigenvalue 0 is of size 2, this tells us we want to find a v such that $(T - 0I)(v) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & -3 & 2 & 1 \end{array}\right] \overset{RREF}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Solving, the solution is $\begin{bmatrix}-1\\0\\0\end{bmatrix} + x\begin{bmatrix}1\\1\\1\end{bmatrix}$ for $x \in \mathbb{R}$. We have that $\begin{bmatrix}-1\\0\\0\end{bmatrix}$ is such a vector. The vectors $\begin{bmatrix}-1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\1\end{bmatrix}$ will form the $J_2(0)$ Jordan block and the vector $\begin{bmatrix}0\\0\\1\end{bmatrix}$ will form the $J_1(2)$ Jordan block

Thus our desired basis is $B = \{-1 + x + x^2, -1, x^2\}$ which will give us

$$[T]_B = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 2\end{bmatrix}.$$

Question 5

(i) We want to find the second degree Lagrange polynomial, $p(x)$ such that $\int_{-1}^{1} p(x)p(x) = 1$, but $\int_{-1}^{1} p(x) = 0$ and $\int_{-1}^{1} xp(x) = 0$. Let $p(x) = ax^2 + bx + c$. Since $p(1) = 1, a + b + c = 1$. Further, $\int_{-1}^{1} ax^2 + bx + c = 0 \implies \frac{2}{3}a + 2c = 0$. Lastly, $\int_{-1}^{1} ax^3 + bx^2 + cx = 0 \implies \frac{2b}{3} = 0 \implies b = 0$.

Solving the equations, $a = \frac{3}{2}, c = -\frac{1}{2}$. The orthogonal basis is $\{1, x, \frac{3}{2}x^2 - \frac{1}{2}\}$.

(ii) However, $\int_{-1}^{1} p(x)p(x)\ dx = \frac{2}{5}, \int_{-1}^{1} 1\ dx = 2$ and $\int_{-1}^{1} x^2\ dx = \frac{2}{3}$.

Doing so, we get that the orthonormal basis $\mathscr{B} = \{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{2}}(\frac{3}{2}x^2 - \frac{1}{2})\}$.

Note that $-1 + 3x - 15x^2 \in \mathcal{P}_2(\mathbb{R})$. The projection of $5x^3$ in $\mathcal{P}_2(\mathbb{R})$ is given by

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\langle 5x^3, \frac{1}{\sqrt{2}}\rangle + \sqrt{\frac{3}{2}}x\langle 5x^3, \sqrt{\frac{3}{2}}x\rangle + \sqrt{\frac{5}{2}}(\frac{3}{2}x^2 - \frac{1}{2})\langle 5x^3, \sqrt{\frac{5}{2}}(\frac{3}{2}x^2 - \frac{1}{2})\rangle \\
&= \frac{1}{2}\int_{-1}^{1} 5x^3\ dx + \frac{3}{2}x\int_{-1}^{1} 5x^4\ dx + \frac{5}{2}(\frac{3}{2}x^2 - \frac{1}{2})\int_{-1}^{1} \frac{15}{2}x^5 - \frac{5}{2}x^3\ dx \\
&= \frac{3}{2}x(2) \\
&= 3x.
\end{aligned}$$

To perform the integrals quickly, observe that $5x^3$ and $\frac{15}{2}x^5 - \frac{5}{2}x^3$ are both odd functions. Thus those integrals evaluate to 0. The best approximation of $q(x)$ is thus given by:

$$-1 + 3x - 15x^2 + 3x = -1 + 6x - 15x^2.$$

Question 6

(i) Let $B = \{(1,0,0)^T, (0,1,0)^T, (0,0,1)^T\}$ be a basis for $\mathbb{C}^3$.

$$[T]_B = \begin{bmatrix} 1 & 0 & -i \\ 1 & -2 & 1+i \\ 0 & 1 & -i \end{bmatrix}.$$

Over the standard inner product on finite dimensional $\mathbb{C}$, $[T^*]_B$ is the conjugate transpose.

$$[T^*]_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ i & 1-i & i \end{bmatrix}$$

Thus we have

$$T^*((x,y,z)) = (x+y, -2y+z, ix+(1-i)y+iz.$$

(ii) Let $w \in W^\perp$, Then $\langle w, u\rangle = 0$ for all $u \in W$. Since W is S invariant, one has $\langle w, S(u)\rangle = 0$ for all $u \in W$, implying that $\langle S^*(w), u\rangle = 0$ for all $u \in W$. So $S^*(w) \in W^\perp \implies W^\perp$ is S^*-invariant. However, since S is self-adjoint, $W^\perp$ is also S-invariant.

Question 7

We want to show: $E_\lambda(T) = \ker(T|_{W_1} - \lambda I) \oplus \ker(T|_{W_2} - \lambda I)$.

Firstly, we show that their sum is direct. Let $x \in \ker(T|_{W_1} - \lambda I_{W_1}) \cap \ker(T|_{W_2} - \lambda I_{W_2})$. Then $x \in W_1 \cap W_2$. Since $W_1 + W_2$ is a direct sum, $W_1 \cap W_2 = \{0_V\}$ so $x = 0_V$.

Clearly, $E_\lambda(T) \supseteq \ker(T|_{W_1} - \lambda I_{W_1}) \oplus \ker(T|_{W_2} - \lambda I_{W_2})$. It suffices to prove the other set inequality. Let $v \in E_\lambda(T)$. Since $v \in V = W_1 \oplus W_2$, v can be decomposed into $v = w_1 + w_2$ uniquely, with $w_i \in W_i$ for $i = 1, 2$. Since W_1 and W_2 are T-invariant,

$$\begin{aligned} T(w_1 + w_2) &= T(w_1) + T(w_2) \\ \lambda w_1 + \lambda w_2 &= T(w_1) + T(w_2) \end{aligned}$$

By the unique expression property of direct sums, we must have $T(w_1) = \lambda w_1$ and $T(w_2) = \lambda w_2$. This gives us that $T|_{W_1}(w_1) = \lambda w_1$. w_1 is an eigenvector associated with λ for $T|_{W_1}$. The same can be said for w_2. So $w_i \in \ker(T|_{W_i} - \lambda I_{W_i})$ for $i = 1, 2$. Hence every vector $v \in E_\lambda(T)$ can be written as a direct sum of $\ker(T|_{W_1} - \lambda I)$ and $\ker(T|_{W_2} - \lambda I)$.