NATIONAL UNIVERSITY OF SINGAPORE
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MA3110 Mathematical Analysis II

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Throughout this solution, we shall denote the set of positive integers by $\mathbb{N}$.

Question 1

(a) FALSE
Let $a = -1$, $b = 1$, and consider the function $f : [a, b] \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Clearly, f is differentiable on $[-1, 1] \setminus \{0\}$. Let us show that f is also differentiable at 0. Indeed, for all $x \neq 0$, we have $\frac{f(x)-f(0)}{x} = x \sin\left(\frac{1}{x}\right)$. As $\left|x \sin\left(\frac{1}{x}\right)\right| \leq |x|$, and $\lim\limits_{x\to 0} |x| = 0$, it follows from the Squeeze Theorem that $f'(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x} = 0$, and this completes the claim. So f is differentiable.
Now, let us show that f' is not continuous at 0. Indeed, for all $x \neq 0$, we have

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

Now, by a similar argument as above, we have $\lim\limits_{x\to 0} 2x \sin\left(\frac{1}{x}\right) = 0$. If f' is continuous at 0, then we must have the limit $\lim\limits_{x\to 0} f'(x) = \lim\limits_{x\to 0}[2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)] = \lim\limits_{x\to 0} \cos(\frac{1}{x}) = 0$. On the other hand, since $\cos(n\pi) = (-1)^n$, and $\lim\limits_{n\to\infty} \frac{1}{n\pi} = 0$, it follows from the Sequential Criterion for limits that $\lim\limits_{x\to 0} \cos\left(\frac{1}{x}\right) = (-1)^n$, which is a contradiction. So f' is not continuous at 0.

(b) TRUE.
This follows from the fact that the product of any two Riemann integrable functions on $[a, b]$ is again Riemann integrable.

(c) FALSE.
Let $a = 0$, $b = 1$, and for each $n \in \mathbb{N}$, let us define the function $f_n : [a, b] \to \mathbb{R}$ by $f_n(x) = \frac{x}{n}$ for all $x \in [0, 1]$. Furthermore, let us define the function $f : [a, b] \to \mathbb{N}$ by $f(x) = 0$ for all $x \in [0, 1]$. Clearly, f_n is strictly increasing for all $n \in \mathbb{N}$. Let us show that $\{f_n\}$ converges uniformly to f. To this end, let $\varepsilon > 0$ be given. Then there exists some $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. This implies that for all $n \geq N$ and $x \in [0, 1]$, we have

$$|f_n(x) - f(x)| = |f_n(x)| = \left|\frac{x}{n}\right| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon,$$

which completes the claim. However, we see that f is not strictly increasing.

(d) FALSE.
For each $n \in \mathbb{N}$, let us define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = (-1)^n$ for all $x \in [0,1]$. Then it is clear that $||f_n|| = 1$ for all $n \in \mathbb{N}$, so that $\{||f_n||\}$ is clearly a convergent sequence of real numbers. However, since $\{f_n(0)\} = \{(-1)^n\}$ is a divergent sequence of real numbers, it follows that $\{f_n\}$ is not pointwise convergent, and hence $\{f_n\}$ is not a uniformly convergent sequence of functions.

(e) TRUE.
This follows immediately from the Weierstrass M-Test.

(f) TRUE.
This follows immediately from the fact that $|(-1)^n a_n| = |a_n|$ for all $n \in \mathbb{N}$.

(g) TRUE.
This is precisely the statement of Abel's Theorem.

(h) FALSE.
For each $n \in \mathbb{N}$, let us define $a_n := \frac{1}{n}$, $a_0 := 0$ and $x_0 := 0$. As $\limsup \sqrt[n]{|a_n|} = \limsup \sqrt[n]{\frac{1}{n}} = 1$, it follows that the radius of convergence R of the series $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n = \sum\limits_{n=1}^{\infty} \frac{x^n}{n}$ is equal to $\frac{1}{1} = 1$.

Arguing by contradiction, suppose that the series $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ converges uniformly on $(x_0 - R, x_0 + R) = (-1,1)$. For each $k \in \mathbb{N}$, let us define $s_k(x) = \sum\limits_{n=1}^{k} \frac{x^n}{n}$. By Cauchy's Criterion for uniform convergence, it follows that for every $\varepsilon > 0$, there exists a positive integer N, such that for all $m > n \geq N$ and $x \in (-1,1)$, we have $|s_m(x) - s_n(x)| < \frac{\varepsilon}{2}$. In particular, since s_k is a continuous function for all $k \in \mathbb{N}$, we have

$$|s_m(1) - s_n(1)| = \lim_{x \to 1^+} |s_m(x) - s_n(x)| \leq \lim_{x \to 1^+} \frac{\varepsilon}{2} < \varepsilon$$

for all $m > n \geq N$. As $\varepsilon > 0$ is arbitrary, it follows from the Cauchy Criterion for series that the series $s_\infty(1) = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ is convergent, which is a contradiction. Therefore, the series $\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$ does not converge uniformly on $(-1,1)$.

Question 2

(a) Remark. It is implicitly assumed in this question that f is Riemann integrable, and $a = 0$ and $b = 1$.

For each $n \in \mathbb{N}$, we define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 2n, & x \in \left[\frac{1}{2n}, \frac{1}{n}\right] \\ 0, & x \in [0,1] \setminus \left[\frac{1}{2n}, \frac{1}{n}\right] \end{cases}$$

Furthermore, let us define $f : [0,1] \to \mathbb{R}$ by $f(x) = 0$ for all $x \in [0,1]$. Then it is clear that $\{f_n\}$ converges pointwise to f. Indeed, we have $f_n(0) = 0$ for all $n \in \mathbb{N}$. Moreover, for each $x \in (0,1]$, there exists some $N \in \mathbb{N}$ such that $x > \frac{1}{N}$ by the Archimedean Property, which implies that $f_n(x) = 0$ for all $n \geq N$. This completes the claim.

Now, we clearly have f to be Riemann integrable. Furthermore, for each n, we see that f_n is monotone on $\left[0, \frac{1}{2n}\right]$, $\left[\frac{1}{2n}, \frac{1}{n}\right]$ and $\left[\frac{1}{n}, 0\right]$, so that f is Riemann integrable on $\left[0, \frac{1}{2n}\right]$, $\left[\frac{1}{2n}, \frac{1}{n}\right]$ and $\left[\frac{1}{n}, 0\right]$, and hence on $[0,1]$. Since

$$\int_0^1 f_n(x)dx = \int_{\frac{1}{2n}}^{\frac{1}{n}} f_n(x)dx = \int_{\frac{1}{2n}}^{\frac{1}{n}} 2ndx = 1,$$

and $\int_0^1 f(x)dx = 0$, we see that $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = 1 \neq 0 = \int_0^1 f(x)dx$.

(b) For each $n \in \mathbb{N}$, define $a_n = b_n := \frac{(-1)^{n+1}}{\sqrt{n}}$. Then $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ converge by the Alternating Series Test, but the series $\sum\limits_{n=1}^{\infty} a_n b_n = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent.

(c) Define $f : [0,1] \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{[1/x]}, & x \in (0,1] \end{cases}$$

where $[y]$ denotes the largest integer smaller than y (Or simply, the floor function).
Then it is clear that f is monotone. Indeed, for all $x, y \in (0,1]$ such that $x < y$, we have $\left[\frac{1}{y}\right] \leq \left[\frac{1}{x}\right]$ by definition, so this implies that $f(x) = \frac{1}{[1/x]} \leq \frac{1}{[1/y]} = f(y)$ as claimed. This implies that f is Riemann integrable.

Next, let us show that for each $n \in \mathbb{N}$ greater than 1, we have f to be discontinuous at $x = \frac{1}{n}$. Indeed, for all $x \in \left(\frac{1}{n}, \frac{1}{n-1}\right]$, we have $\frac{1}{x} \in [n-1, n)$, which implies that $f(x) = \frac{1}{n-1}$. Hence, we have $\lim\limits_{x \to \frac{1}{n}^-} f(x) = \frac{1}{n-1} \neq f\left(\frac{1}{n}\right)$, which implies that f is discontinuous at $x = \frac{1}{n}$ as claimed. Therefore, f is discontinuous at infinitely many points.

(d) For each $n \in \mathbb{N}$, let us define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0, & x < n \\ x - n, & x \in [n, n+1) \\ 1, & x \geq n+1 \end{cases}$$

Furthermore, let us define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$. Then it is clear that $\{f_n\}$ converges pointwise to f. Indeed, for all $x \in \mathbb{R}$, there exists some $N \in \mathbb{N}$ such that $x < N$ by the Archimedean Property. This implies that $f_n(x) = 0$ for all $n \geq N$, and this completes the claim.

Next, let us show that for all $x \in \mathbb{R}$, and any sequence $\{x_n\}$ in $\mathbb{R}$ that converges to x, we must have $\lim\limits_{n\to\infty} f_n(x_n) = f(x)$. Indeed, let us choose any $N \in \mathbb{N}$ such that $x < N$ (which exists by the Archimedean Property), and set $\varepsilon := N - x > 0$. As $\{x_n\}$ converges to x, it follows that there exists some $K \in \mathbb{N}$, such that $|x_n - x| < \varepsilon$ for all $n \geq K$. Let $M = \max\{K, N\}$. It follows that for all $n \geq M \geq K$, we have $x_n - x < \varepsilon = N - x \leq M - x$, which implies that $x_n < M$. Hence, we have $f_n(x_n) = 0 = f(x)$ for all $n \geq M$, which implies that $\lim\limits_{n\to\infty} f_n(x_n) = f(x)$ as claimed.

Finally, let us show that $\{f_n\}$ does not converge uniformly to f. Arguing by contradiction, suppose $\{f_n\}$ converges uniformly to f. Then there exists some $N \in \mathbb{N}$, such that $|f_n(x) - f(x)| < 1$ for all $n \geq N$ and $x \in \mathbb{R}$. On the other hand, we have $|f_N(N+1) - f(N+1)| = 1$, which is a contradiction. The desired follows.

Question 3

We shall prove by induction on $n \in \mathbb{N}$ that for all C^n functions $f : [a, b] \to \mathbb{R}$ that has $n+1$ distinct zeros that there exists some $y \in [a, b]$ such that $f^{(n)}(y) = 0$. When $n = 1$, by assumption, there exist some $x_1, x_2 \in [a, b]$, such that $x_1 < x_2$, and $f(x_1) = 0 = f(x_2)$. By Mean Value Theorem, there exists some $y \in (x_1, x_2)$, such that $f'(y)(x_2 - x_1) = f(x_2) - f(x_1) = 0$. As $x_2 - x_1 \neq 0$, we must have $f^{(1)}(y) = f'(y) = 0$, and this proves the base step.

Now, suppose that the statement holds for some $n = k \in \mathbb{N}$, and suppose that $f : [a, b] \to \mathbb{R}$ is a C^{k+1} function that has $k + 2$ distinct zeros $x_1 < x_2 < \cdots < x_{k+2}$. Then f' is C^k by definition. By Mean Value Theorem, there exist $y_1, y_2, \cdots, y_{k+1} \in [a, b]$, such that $y_i \in (x_i, x_{i+1})$ for all $i = 1, 2, \cdots, k + 1$, and $f'(y_i)(x_{i+1} - x_i) = f(x_{i+1}) - f(x_i) = 0$. As $x_{i+1} - x_i \neq 0$, we must have $f'(y_i) = 0$ for all $i = 1, 2, \cdots, k + 1$. By induction hypothesis on f', there must exists some $y \in [a, b]$ such that $(f')^{(k)}(y) = 0$. As $(f')^{(k)} = f^{(k+1)}$, this completes the induction step, and we are done.

Question 4

For each $x \in [0, \infty)$, let us define $F(x) = \int_0^x f(t)dt$. Then F is differentiable on $[0, \infty)$ by the Fundamental Theorem of Calculus, and $F'(x) = f(x)$ for all $x \in [0, \infty)$. Furthermore, since f is strictly positive on $(0, \infty)$, we must have $F(x) > 0$ for all $x \in (0, \infty)$, and $f(x) = \sqrt{2F(x)}$ for all $x \in [0, \infty)$. By the chain rule for differentiation, we must have f to be differentiable on $(0, \infty)$. By applying chain rule on the left hand side of the equation $(f(x))^2 = 2F(x)$ for all $x > 0$, we have $2f(x)f'(x) = 2F'(x) = 2f(x)$. As $f(x) > 0$ for all $x > 0$, we must have $f'(x) = 1$ for all $x > 0$. This implies that there exists some $c \in \mathbb{R}$, such that $f(x) = x + c$ for all $x \in [0, \infty)$. As $f(0)^2 = 2F(0) = 0$, we must have $f(0) = 0$, so $c = 0$. So $f(x) = x$ for all $x \in [0, \infty)$ as desired.

Question 5

Let $\varepsilon > 0$ be given. As $\{f_n\}$ and $\{g_n\}$ converges to f and g respectively, there exist $N_1, N_2 \in \mathbb{N}$, such that for all $n \geq N_1$ and $x \in [a, b]$, we have $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$, and for all $n \geq N_2$ and $x \in [a, b]$, we have $|g_n(x) - g(x)| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. It follows that for all $n \geq N$ and $x \in [a, b]$, we have

$$|(f_n + g_n)(x) - (f + g)(x)| = |f_n(x) - f(x) + g_n(x) - g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows that $\{f_n + g_n\}$ converges uniformly to $f + g$ as desired.

Question 6

Let us first show that $\{\sqrt[n]{|a_n|}\}$ is bounded. Suppose not, then for each $N \in \mathbb{N}$, there exists some $k_N \in \mathbb{N}$, such that $|a_{k_N}|^{1/k_N} > N$. This implies that $|a_{k_N}| > N^{k_N} \geq N$, which implies that $\{a_n\}$ is unbounded, a contradiction.

As $\{\sqrt[n]{|a_n|}\}$ is bounded, we must have $a := \limsup \sqrt[n]{|a_n|}$ to exist in $\mathbb{R}$. Let us show that $a = 1$. If $a < 1$, then this would imply that the series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent by the Root Test, which is a contradiction. On the other hand, if $a > 1$, then let us set $\varepsilon := a - 1$. By the definition of $\limsup \sqrt[n]{|a_n|}$, there exists infinitely many n's such that $\sqrt[n]{|a_n|} > a - \frac{\varepsilon}{2} = \frac{1+a}{2}$, or equivalently, $|a_n| > \left(\frac{1+a}{2}\right)^n$. Since $\frac{1+a}{2} > 1$, it follows that the sequence $\left\{\left(\frac{1+a}{2}\right)^n\right\}$ is unbounded, and consequently, the sequence $\{a_n\}$ is unbounded, which is again a contradiction. So $a = 1$ as claimed. By definition, the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is equal to $\frac{1}{a} = 1$ as desired.

Question 7

(a) As f is differentiable (hence continuous) by definition, it follows from the sequential criterion for continuity that $f(0) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = 0$.

(b) Let us fix a $n \geq 0$, and assume without loss of generality that $x_n > 0$. By the Mean Value Theorem, there exists some $y_n \in (0, x_n)$, such that $f'(y_n)(x_n - 0) = f(x_n) - f(0) = 0$. As $x_n > 0$, we must have $f'(y_n) = 0$. Furthermore, by the choice of y_n for each $n \geq 0$, it is easy to see that $y_n \neq 0$, and $|y_n| < |x_n|$ for all $n \geq 0$. As $\lim_{n\to\infty} |x_n| = 0$, it follows from Squeeze Theorem that $\lim_{n\to\infty} y_n = 0$. Finally, since $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ is a power series with radius of convergence R, we deduce from part (a) that $f'(0) = 0$. Now, we repeat the same argument as above to deduce inductively that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. As $a_n = \frac{f^{(n)}(0)}{n!}$ by definition, we must have $f(x) = 0$ for all $x \in (-R, R)$ as desired.