NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA4204 Group Theory
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Question 1

(a) Let $H = \text{Stab}_G(x)$, and let relation $\phi : X \to (G : H)$ be such that for all $x' \in X$ and $g \in G$ such that $g^{-1} \cdot x' = x$, we have $\phi(x') = gH$.

Since G acts transitively on X, $\phi(x')$ exists for all $x' \in X$.
Let $g_1, g_2 \in G$ such that $g_1^{-1} \cdot x' = x = g_2^{-1} \cdot x'$.
This give us $g_1 \cdot x = x' = g_2 \cdot x$, and so $g_2^{-1}g_1 \cdot x = x$.
Thus $g_2^{-1}g_1 \in \text{Stab}_G(x) = H$, i.e. $g_2H = g_1H$, and so, $\phi(x')$ is unique for all $x' \in X$.
Therefore ϕ is a function.

Let $x_1, x_2 \in X$ be such that $\phi(x_1) = \phi(x_2)$ for some $g \in G$.
Since G acts transitively on X, there exists $g_1, g_2 \in G$ such that $g_1^{-1} \cdot x_1 = x = g_2^{-1} \cdot x_2$.
This give us $g_1H = \phi(x_1) = \phi(x_2) = g_2H$, i.e. $g_2^{-1}g_1 \in H = \text{Stab}_G(x)$.
Thus $g_2^{-1}g_1 \cdot x = x$, i.e. $x_1 = g_1 \cdot x = g_2 \cdot x = x_2$, i.e. ϕ is injective.
Let $g \in G$ (equivalently, let $gH \in (G : H)$).
Since $g^{-1} \cdot (g \cdot x) = x$, we have $\phi(g \cdot x) = gH$, i.e. ϕ is surjective.
Therefore ϕ is bijective.

Notice that as $1_G^{-1} \cdot x = x$, we have $\phi(x) = 1_GH = H$, and so for all $g \in G$, $\phi(g \cdot x) = gH = g\phi(x)$.
Thus we established such a subgroup H and a bijection ϕ that satisfy our conditions.

(b) Let H and K be conjugate subgroups of G, i.e. there exists $b \in G$ such that $bHb^{-1} = K$.
Let G acts on $(G : H)$ by left composition. Since G acts transitively on $(G : H)$, with $bH \in (G : H)$, and $\text{Stab}_G(bH) = bHb^{-1} = K$, by result of (1a), there exists a bijection $\psi : (G : H) \to (G : K)$ such that $\psi(gH) = \psi(g \cdot H) = g \cdot \psi(H) = g\psi(H)$ for all $g \in G$.

Instead, let there exists a bijection $\psi : (G : H) \to (G : K)$ such that $\psi(gH) = g\psi(H)$ for all $g \in G$.
Since ψ is surjective, there exists $a \in G$ such that $\psi(aH) = K$.
Thus for all $h \in H$, we have $aha^{-1}K = aha^{-1}\psi(aH) = \psi(aha^{-1}aH) = \psi(ahH) = \psi(aH) = K$, i.e. $aha^{-1} \in K$. This implies that $aHa^{-1} \leq K$.
Also, for all $k \in K$, we have $\psi(a^{-1}kaH) = a^{-1}k\psi(aH) = a^{-1}kK = a^{-1}K = a^{-1}\psi(aH) = \psi(H)$, i.e. $a^{-1}ka \in H$, or rather $k \in aHa^{-1}$. This implies that $K \leq aHa^{-1}$.
Therefore H and K are conjugate subgroups of G.

Question 2

(a) Since G is simple, we have G to be a subgroup of A_{n+1} of index $\dfrac{k}{2}$, i.e. $|G| = \dfrac{|a_{n+1}|}{k} \geq \dfrac{|a_{n+1}|}{2n+2} = \dfrac{n!}{2}$.
If $k = 2$, then $[A_{n+1} : G] = 1$, i.e. $G = A_{n+1}$.

Else we have $1 \leq \dfrac{k}{2} - 1 \leq n$. Let $l = \dfrac{k}{2} - 1$.
Then $|(A_{n+1} : G) - \{G\}| = l$, and so $(A_{n+1} : G) - \{G\}$ is non-empty.
Let G act on $(A_{n+1} : G) - \{G\}$ by left composition (this is well-defined since $\{G\}$ is a G-orbit).
This induce a homomorphism $\varphi : G \to S_l$.

Assume on the contrary that $\ker(\varphi) = G$.
Then for all $g \in G$, $a \in A_{n+1}$, we have $gaG = aG$, i.e. $a^{-1}ga \in G$. Thus $G \lhd A_{n+1}$.
Since $n+1 \geq 5$, A_{n+1} is simple, $G = \{1_{A_{n+1}}\}$ or $G = A_{n+1}$, either way a contradiction since $[A_{n+1} : G] \neq \dfrac{(n+1)!}{2}$ and $[A_{n+1} : G] \neq 1$.

Therefore $\ker(\varphi) \neq G$, and together with G is simple, we conclude that $\ker(\varphi) = \{1_G\}$.
Therefore $G \cong \varphi[G] \leq S_l$.
Since $|\varphi[G]| = |G| \geq \dfrac{n!}{2} > 2$ and $\varphi[G]$ is simple, we have $\varphi[G] \leq A_l$.
Also $l = n$, else $l < n$ with $n \geq 5$, which give us $|G| = |\varphi[G]| \leq |S_l| = l! < \dfrac{n!}{2}$, a contradiction.
Thus $\varphi[G] \leq A_n$, and so $|\varphi[G]| \leq \dfrac{n!}{2}$.
Therefore $|\varphi[G]| = \dfrac{n!}{2} = |A_n|$, and so $G \cong \varphi[G] = A_n$.

(b) Let G be a simple group with $|G| = 60$. Let $n_5 = |\mathrm{Syl}_5(G)| \neq 1$.
Then by Sylow's Theorem, we have $n_5 \equiv 1 \mod 5$ and $n_5 \mid 12$, i.e. $n_5 = 6$.
Thus we can let G act on $\mathrm{Syl}_5(G)$ by conjugation.
Since $n_5 = 6$, this induce a homomorphism $\phi : G \to S_6$.
Since the action is transitive, we have $\ker(\phi) \neq G$, and together with G being simple, we have $\ker(\phi) = \{1_G\}$. Thus $\phi[G] \leq S_6$ with $[S_6 : \phi[G]] = 12 = 2(5) + 2 \neq 2$.
Since $\phi[G]$ is simple subgroup of S_6 of index 12, by result of (2a.), we have $G \cong \phi[G] \cong A_5$.

Question 3

(a) Let G be a group with $|G| = p^2$. Then $Z(G)$ is non-trivial, and so $|Z(G)| = p$ or $|Z(G)| = p^2$.
Either way, we have $|G/Z(G)| = 1$ or $|G/Z(G)| = p$, i.e. $G/Z(G)$ is cyclic.
Therefore G is Abelian.
Thus by classification of Abelian groups, we have $G \cong C_{p^2}$ or $G \cong C_p \times C_p$.

(b) Let G be a group with $|G| = 2p$.
Then by Sylow's Theorem, there exists $H, K \leq G$ such that $|H| = 2$, $|K| = p$.
Since $[G : K] = 2$, we have $K \lhd G$, and so $G = K \rtimes H$.

Let $\varphi : H \to \mathrm{Aut}(K)$, let $H = \{1_G, h\}$ and $K = \{1_G, k, k^2, \ldots, k^{p-1}\}$.
Then let $\varphi(h)(k) = k^l$ for some $l \in \{1, 2, \ldots, p-1\}$. This give us $\varphi(h)(k^q) = k^{ql}$ for all $q \in \mathbb{Z}$.
Since $h^2 = 1_G$, $k = \varphi(1_G)(k) = \varphi(h^2)(k) = \varphi(h)\varphi(h)(k) = k^{l^2}$, i.e. $l^2 \equiv 1 \mod p$.
By Euclid's Lemma, we have $l \equiv \pm 1 \mod p$.
If $l \equiv 1 \mod p$, then $\varphi(h)(k^q) = k^q$ for all $q \in \mathbb{Z}$, i.e. $\varphi(h) = 1_{\mathrm{Aut}(G)}$. Thus $G \cong C_2 \times C_p$.
Else $l \equiv -1 \mod p$, then $\varphi(h)(k^q) = k^{-q}$ for all $q \in \mathbb{Z}$, i.e. $G \cong D_{2p}$.
Therefore $G \cong C_2 \times C_p$ or $G \cong D_{2p}$.

Question 4

(a) Let G be a group such that $|G| = 595$. Let $n_p = |\mathrm{Syl}_p(G)|$ for $p = 5, 7, 17$.
Then by Sylow's Theorem, we have:-
$n_5 \equiv 1 \mod 5$ and $n_5 \mid 7 \cdot 17$, i.e. $n_5 = 1$.
$n_7 \equiv 1 \mod 7$ and $n_7 \mid 5 \cdot 17$, i.e. $n_7 = 1$ or $n_7 = 5 \cdot 17$.
$n_{17} \equiv 1 \mod 17$ and $n_{17} \mid 5 \cdot 7$, i.e. $n_{17} = 1$ or $n_{17} = 5 \cdot 7$.

Let $P_5 \in \mathrm{Syl}_5(G)$. Since $n_5 = 1$, we have $P_5 \lhd G$.

Assume on the contrary that $n_7 = 5 \cdot 17$.
Let $P_7 \in \mathrm{Syl}_7(G)$. Then $|N_G(P_7)| = 7$.
Since $P_5 \lhd G$, we have $P_5P_7 \leq G$.
By Sylow's Theorem, $|\mathrm{Syl}_7(P_5P_7)| \equiv 1 \mod 7$ and $|\mathrm{Syl}_7(P_5P_7)| \mid 5$.
Thus $|\mathrm{Syl}_7(P_5P_7)| = 1$, i.e. $P_7 \lhd P_5P_7$.
This give us $P_5P_7 \leq N_G(P_7)$, i.e. $35 = |P_5P_7| \leq |N_G(P_7)| = 7$, a contradiction.

Similarly, we can show that $n_{17} \neq 5 \cdot 7$, by letting $P_{17} \in \mathrm{Syl}_{17}(G)$ and show that $P_5P_{17} \leq N_G(P_{17})$.
Thus $n_7 = n_{17} = 1$, i.e. $P_7, P_{17} \lhd G$.
Since P_5, P_7 and P_{17} are cyclic and normal in G, we have G to be cyclic.

(b) Not necessarily.
Let $C_3 = \{1, p, p^2\}$ and $C_7 = \{1, q, q^2, \ldots, q^6\}$.
Let homomorphism $\varphi : C_3 \to \mathrm{Aut}(C_7)$ be such that $\varphi(p^k)(q^l) = q^{2^k l}$ for all $k, l \in \mathbb{Z}$.
Then φ is not a trivial homomorphism, and thus it can be used to form the semi-direct product $C_7 \rtimes C_3$ which is not cyclic. Thus $|(C_7 \rtimes C_3) \times C_{17}| = 3 \cdot 7 \cdot 17$, but $(C_7 \rtimes C_3) \times C_{17}$ is not cyclic.

Question 5

(a) $\to$ (b)
Let $\{1_G\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$ be a central series.
Let $H < G$, and $k \in \mathbb{Z}$ be such that $G_k \leq H$ but $G_{k+1} \not\leq H$.
This give us $[G_{k+1}, H] \leq [G_{k+1}, G] \leq G_k \leq H$.
Thus for all $g \in G_{k+1}$, $h \in H$, we have $ghg^{-1}h^{-1} \in H$, i.e. $ghg^{-1} \in hH = H$.
Therefore $G_{k+1} \leq N_G(H)$, and so $H < N_G(H)$.

(b) $\to$ (c)
Let M be a maximal subgroup of G.
Since $M < G$, we have $M < N_G(H)$. By the maximality of M, we have $N_G(H) = G$, i.e. $M \lhd G$.

(c) $\to$ (d)
Assume on the contrary that P is a Sylow subgroup which is not normal in G.
Since $N_G(P) < G$, we have $N_G(P) \leq M$, where M is a maximal subgroup of G.
Hence for all $g \in G$, we have $gPg^{-1} \leq gMg^{-1} = M$.
Since P and gPg^{-1} are Sylow subgroups of M, there exists $m \in M$ such that $gPg^{-1} = mPm^{-1}$, i.e. $m^{-1}g \in N_G(P)$. Therefore $g \in MN_G(P)$, which gives us $G = MN_G(P) = M$ (since $N_G(P) \leq M$), a contradiction.

(d) $\to$ (a)
Since every Sylow subgroup of G is normal in G, G is a direct product of its Sylow subgroups.
As every p-group is nilpotent, and finite direct products of nilpotent groups are nilpotent, we have G to be nilpotent.