MA2101S Linear Algebra II AY20/21 Semester 2

National University of Singapore

Written by : Fang Xin Yu

Audited by : Pan Jing Bin

Final Exam Suggested Solutions

Throughout:

(a) unless otherwise stated, all vector spaces are defined over a fixed field F of arbitrary characteristic, and may be infinite-dimensional;

(b) for a linear operator α on a vector space V and $v \in V$, we use the following notation:

$$\langle v \rangle_\alpha := \{\sum_{i=0}^{n} \lambda_i \alpha^i(v) \mid n \in \mathbb{Z}_{\geq 0}, \lambda_1, \ldots, \lambda_n \in F\}.$$

Question 1

Let V and W be vector spaces. Let U be a vector subspace of V, and let X be a vector subspace of W.

(a) Let $\alpha : V \to W$ be a linear transformation.

(i) Show that the function $\tilde{\alpha} : V/U \to W/X$ defined by $\tilde{\alpha}(v + U) = \alpha(v) + X$ for all $v \in V$ is well-defined if and only if $\alpha(U) \subseteq X$.

(ii) Suppose that $\alpha(U) \subseteq X$. Show that:

(A) $\tilde{\alpha}$ is linear;

(B) $v + U \in \ker(\tilde{\alpha})$ if and only if $\alpha(v) \in X$;

(C) $\tilde{\alpha}$ is injective if and only if $\alpha^{-1}(X) \subseteq U$;

(D) $\tilde{\alpha}(V/U) = (\alpha(V) + X)/X$;

(E) $\tilde{\alpha}$ is surjective if and only if $\alpha(V) + X = W$.

(b) Let $\tilde{\beta} : V/U \to W/X$ and $\gamma : U \to X$ be linear transformations. Show that there exists a linear transformation $\beta : V \to W$ such that $\beta(u) = \gamma(u)$ for all $u \in U$, and $\tilde{\beta}(v + U) = \beta(v) + X$ for all $v \in V$. (You may assume the existence of complementary vector subspaces.)

Solution

(a) (i) ($\Rightarrow$:) If $\tilde{\alpha}$ is well-defined, then $\forall u \in U$,

$$\begin{aligned} u + U = 0_V + U &\implies \tilde{\alpha}(u + U) = \tilde{\alpha}(0_V + U) \\ &\implies \alpha(u) + X = \alpha(0_V) + X \\ &\implies \alpha(u) - \alpha(0_V) \in X \\ &\implies \alpha(u) - 0_W \in X \\ &\implies \alpha(u) \in X \\ \therefore \alpha(U) \subseteq X. & \end{aligned}$$

($\Leftarrow$:) If $\alpha(U) \subseteq X$, then

$$\begin{aligned} v_1 + U = v_2 + U &\implies v_1 - v_2 \in U \\ &\implies \alpha(v_1 - v_2) \in \alpha(U) \subseteq X \\ &\implies \alpha(v_1) - \alpha(v_2) \in X \\ &\implies \alpha(v_1) + X = \alpha(v_2) + X \\ &\implies \tilde{\alpha}(v_1 + U) = \tilde{\alpha}(v_2 + U) \\ \therefore \tilde{\alpha} \text{ is well-defined.} & \end{aligned}$$

(ii) (A) $\forall v_1, v_2 \in V$ and $\lambda \in F$, we have

$$\begin{aligned}\tilde{\alpha}((v_1+U)+\lambda(v_2+U)) &= \tilde{\alpha}((v_1+\lambda v_2)+U)\\ &= \alpha(v_1+\lambda v_2)+X\\ &= \alpha(v_1)+\lambda\alpha(v_2)+X\\ &= (\alpha(v_1)+X)+(\lambda\alpha(v_2)+X)\\ &= \tilde{\alpha}(v_1+U)+\lambda\tilde{\alpha}(v_2+U)\end{aligned}$$

$\therefore \tilde{\alpha}$ is linear.

(B) $v+U \in \ker(\tilde{\alpha}) \iff \tilde{\alpha}(v+U)=\alpha(v)+X=0_W+X \iff \alpha(v)\in X$.

(C) First we notice that $\ker(\tilde{\alpha})=\{v+U \mid \alpha(v)\in X\}=\{v+U \mid v\in\alpha^{-1}(X)\}$. Since $\tilde{\alpha}$ is linear, $\tilde{\alpha}$ is injective if and only if $\ker(\tilde{\alpha})=\{0_V+U\}$. Thus we must have $v+U=0_V+U$ for all $v\in\alpha^{-1}(X)$ so $\alpha^{-1}(X)\subseteq U$.

(D)

$$\begin{aligned}(\alpha(V)+X)/X &= \{w+X|w\in(\alpha(V)+X)\}\\ &= \{(\alpha(v)+x)+X|v\in V, x\in X\}\\ &= \{\alpha(v)+X|v\in V\}\\ &= \tilde{\alpha}(V/U).\end{aligned}$$

(E) First note that $\alpha(V)+X=W \iff (\alpha(V)+X)/X=W/X$. We have

$$\begin{aligned}\alpha(V)+X=W &\iff (\alpha(V)+X)/X=W/X\\ &\iff \tilde{\alpha}(V/U)=W/X\\ &\iff \tilde{\alpha}\text{ is surjective.}\end{aligned}$$

(b) Let U' denote the complementary subspace of U, i.e. $V=U\oplus U'$. Let $\mathscr{B}_{\mathcal{U}}$ and $\mathscr{B}_{\mathcal{U}'}$ be bases for U and U' respectively. We construct a linear operator β by defining the images of the basis vectors of V as follows:

- $\forall b \in \mathscr{B}_{\mathcal{U}}$, let $\beta(b) = \gamma(b)$;
- $\forall b' \in \mathscr{B}_{\mathcal{U}'}$, choose any element $w \in W$ such that $\tilde{\beta}(b' + U) = w + X$ and let $\beta(b') = w$.

We show that such a construction yields the desired linear operator. It is clear that $\beta|_U = \gamma$. $\forall v \in V$, write $v = u + u'$ where $u \in U$ and $u' \in U'$, then $\beta(v) + X = \beta(u + u') + X = \beta(u) + \beta(u') + X = \beta(u') + X = \tilde{\beta}(u' + U)$.

Question 2

Let α be a linear operator on a vector space V, and let U and W be α-invariant subspaces of V.

(a) Recall the quotient map $q_U : V \to V/U$ defined by $q_U(v) = v + U$ for all $v \in V$, and the linear operator $\tilde{\alpha}_U$ on V/U defined by $\tilde{\alpha}_U(v + U) = \alpha(v) + U$ for all $v + U \in V/U$.

  (i) Show that $q_U(W)$ is $\tilde{\alpha}$-invariant.

  (ii) Show further that $(p(\alpha))(v) + U = (p(\tilde{\alpha}))(v + U)$ for all $p(x) \in F[x]$.

  (iii) Deduce that $q_U(\langle v \rangle_\alpha) = \langle q_U(v) \rangle_{\tilde{\alpha}}$.

(b) Prove that if $W \subseteq \langle v \rangle_\alpha$ for some $v \in V$, then $W = \langle w \rangle_\alpha$ for some $w \in W$.

(c) Suppose now that $V = \sum_{i=1}^{k} \langle v_i \rangle_\alpha$ for some $v_1, \ldots, v_k \in V$ and $k \in \mathbb{Z}^+$. Prove, by induction on k, or otherwise, that there exist $w_1, \ldots, w_k \in W$ such that

$$W = \sum_{i=1}^{k} \langle w_i \rangle_\alpha.$$

(Hint: for $k > 1$, let $U = \langle v_k \rangle_\alpha$, and consider $U \cap W$ and $q_U(W)$.)

Solution

(a) (i) For all $q_U(w) \in q_U(W)$, since W is α-invariant, we have $w' = \alpha(w) \in W$, so $\tilde{\alpha}(q_U(w)) = \tilde{\alpha}(w + U) = \alpha(w) + U = w' + U = q_U(w') \in q_U(W)$.

(ii) Let $v+U \in V+U$. First consider the case when $p(x) = x^n$ for some $n \in \mathbb{N}$. If $n = 0$, then we have $(p(\alpha))(v)+U = id_V(v)+U = v+U = id_{V/U}(v+U) = (p(\tilde{\alpha}))(v+U)$. For $n \geq 1$, by induction $\tilde{\alpha}^n(v+U) = \tilde{\alpha}^{n-1}(\tilde{\alpha}(v+U)) = \tilde{\alpha}^{n-1}(\alpha(v)+U) = \alpha^n(v)+U$. Now for any $p(x) \in F[x]$, suppose $p(x) = \sum_{i=0}^n c_i x^i$, then

$$\begin{aligned}(p(\tilde{\alpha}))(v+U) &= (\sum_{i=0}^n c_i\tilde{\alpha}^i)(v+U) = \sum_{i=0}^n c_i(\tilde{\alpha}^i(v+U)) \\ &= \sum_{i=0}^n c_i(\alpha^i(v)+U) = (\sum_{i=0}^n c_i\alpha^i(v)) + U = p(\alpha)(v) + U.\end{aligned}$$

(iii) For all $v' + U \in V/U$,

$$\begin{aligned}v' + U \in q_U(\langle v\rangle_\alpha) &\iff v' + U = p(\alpha)(v) + U \text{ for some } p(x) \in F[x] \\ &\iff v' + U = (p(\tilde{\alpha}))(v+U) \text{ for some } p(x) \in F[x] \\ &\iff v' + U \in \langle q_U(v)\rangle_{\tilde{\alpha}}.\end{aligned}$$

(b) Choose $p(x) \in F[x]$ with the least degree such that $p(\alpha)(v) \in W$. We will show that $w = p(\alpha)(v)$ generates W. Take any $w' \in W$. Then there exists $q(x) \in F[x]$ such that $q(\alpha)(v) = w'$. If $p(x) \mid q(x)$, then $w' \in \langle w\rangle_\alpha$. Suppose $p(x) \nmid q(x)$, then by the division algorithm, we can write $q(x) = p(x)a(x) + r(x)$ for some $a(x), r(x) \in F[x]$ and $\deg(r(x)) < \deg(p(x))$. But since W is α-invariant, $p(\alpha)a(\alpha)(v) \in W$, so we have $r(\alpha)(v) = q(\alpha)(v) - p(\alpha)a(\alpha)(v) \in W$, a contradiction. Therefore we must have $p(x)|q(x)$, so $W \subseteq \langle w\rangle_\alpha \implies W = \langle w\rangle_\alpha$.

(c) Proof by induction on k.

Base case: $k = 1$ is shown in (b).

Induction: Suppose that the proposition is true for some $(k-1) \in \mathbb{Z}^+$. Now suppose that $V = \sum_{i=1}^k \langle v_i\rangle_\alpha$. Let $U = \langle v_k\rangle_\alpha$. Then $V/U = \sum_{i=1}^{k-1}\langle v_i + U\rangle_{\tilde{\alpha}}$. Since $q_U(W)$ is $\tilde{\alpha}$-invariant, by the induction hypothesis, there exists $w_1, \ldots, w_{k-1} \in W$ such that $q_U(W) = \sum_{i=1}^{k-1}\langle w_i + U\rangle_{\tilde{\alpha}}$. Now $U \cap W$ is also α-invariant and $U \cap W \subseteq U$, so there exists $w_k \in W$ such that $U \cap W = \langle w_k\rangle_\alpha$. We now show that $W \subseteq \sum_{i=1}^k \langle w_i\rangle_\alpha$.

For any $w \in W$, $q_U(w) \in q_U(W)$, so there exists $w' \in \sum_{i=1}^{k-1} \langle w_i \rangle_\alpha$ such that $w + U = w' + U$. Then $w - w' \in U \cap W = \langle w_k \rangle_\alpha \implies w \in \sum_{i=1}^{k} \langle w_i \rangle_\alpha$. We conclude that $W = \sum_{i=1}^{k} \langle w_i \rangle_\alpha$ and complete the induction step.

Question 3

Let α be a linear operator on an infinite-dimensional vector space V, and suppose that α has a minimal polynomial $f(x)^k$ for some monic irreducible polynomial $f(x) \in F[x]$ and $k \in \mathbb{Z}^+$.

(a) Let $v \in V \setminus \{0_V\}$, and let $u \in \langle v \rangle_\alpha$. Prove that the following statements are equivalent:

- (i) $u = (p(\alpha))(v)$ for some $p(x) \in F[x]$ with $\gcd(p(x), f(x)) = 1$.
- (ii) $v \in \langle u \rangle_\alpha$.
- (iii) $\langle v \rangle_\alpha = \langle u \rangle_\alpha$.

(b) Let U be an α-invariant subspace of V. Show that $\langle v \rangle_\alpha \cap U = \{0_V\}$ for any $v \in \ker(f(\alpha))$ with $v \notin U$.

(c) Let $v_1, \dots, v_n \in V \setminus \ker(f(\alpha))$ such that the sum $\sum_{i=1}^{n} \langle (f(\alpha))(v_i) \rangle_\alpha$ is direct. Prove that the sum $\sum_{i=1}^{n} \langle (v_i) \rangle_\alpha$ is direct.

(d) Suppose that $(f(\alpha))(V) = \sum_{i \in I} \langle (f(\alpha))(v_i) \rangle_\alpha$ for sum indexing set I and $v_i \in V$ for all $i \in I$. Show that if $\ker(f(\alpha)) \subseteq \sum_{i \in I} \langle (v_i) \rangle_\alpha$, then $\sum_{i \in I} \langle (v_i) \rangle_\alpha = V$.
(Hint: Consider $(f(\alpha))(v)$ for $v \in V$.)

(e) Hence, or otherwise, prove that V is a direct sum of α-cyclic subspaces.
(Hint: Prove by induction on k. You may assume without proof that for every α-invariant subspace U of V, there exists an α-invariant subspace W of V such that:

- W is a direct sum of α-cyclic subspace;
- $U \cap W = \{0_V\}$;
- $\langle v \rangle_\alpha \cap (U + W) \neq \{0_V\}$ for any $v \in V \setminus (U + W)$.

Solution

(a) (i)⇒(ii): We have $\gcd(p(x), f(x)) = 1 \implies \gcd(p(x), f^k(x)) = 1$. By Bezout's Identity, there exists $a(x), b(x) \in F[x]$ such that $p(x)a(x) + f^k(x)b(x) = 1$. Then $v = p(\alpha)a(\alpha)(v) + f^k(\alpha)b(\alpha)(v) = a(\alpha)(u) \in \langle u\rangle_\alpha$.

(ii)⇒(iii):

$$\left.\begin{aligned} v \in \langle u\rangle_\alpha &\implies \langle v\rangle_\alpha \subseteq \langle u\rangle_\alpha \\ u \in \langle v\rangle_\alpha &\implies \langle u\rangle_\alpha \subseteq \langle v\rangle_\alpha \end{aligned}\right\} \implies \langle v\rangle_\alpha = \langle u\rangle_\alpha$$

(iii)⇒(i): Since $f(x)$ is irreducible, if $f(x) \nmid p(x)$, we are done. Otherwise, suppose $f(x) \mid p(x)$. Then we can write $p(x) = f(x)q(x)$ for some $q(x) \in F[x]$. Suppose the minimal polynomial of $\alpha|_{\langle v\rangle_\alpha}$ is $m_{\langle v\rangle_\alpha}(x) = f^t(x)$ where $t \in \mathbb{Z}^+$ (note that $v \neq 0_V$). Then $f^{t-1}(\alpha)(u) = f^{t-1}(\alpha)f(\alpha)q(\alpha)(v) = 0 \implies m_{\langle u\rangle_\alpha}(x) \mid f^{t-1}(x)$. But $\langle v\rangle_\alpha = \langle u\rangle_\alpha$ so $m_{\langle v\rangle_\alpha}(x) = m_{\langle u\rangle_\alpha}(x)$. Thus we have $f^t(x) \mid f^{t-1}(x)$, a contradiction.

(b) Suppose to the contrary that there exists a nonzero $v' \in \langle v\rangle_\alpha \cap U$. Then we have $v' = p(\alpha)(v)$ for some $p(x) \in F[x]$ with $\deg(p(x)) \geq 1$. Since $m_{\langle v\rangle_\alpha}(x) = f(x)$, we may assume $\deg(p(x)) < \deg(f(x))$. Then $\gcd(p(x), f(x)) = 1$. By Bezout's Identity, there exists $a(x), b(x) \in F[x]$ such that $p(x)a(x) + f(x)b(x) = 1$ Then we have

$$v = p(\alpha)a(\alpha)(v) + f(\alpha)b(\alpha)(v) = a(\alpha)(v') \in \langle v\rangle_\alpha \cap U$$

as $\langle v\rangle_\alpha \cap U$ is α-invariant. This is a contradiction to $v \notin U$.

(c) Suppose $\sum_{i=1}^n p_i(\alpha)(v_i) = 0$ for some $p_1(x), \ldots, p_n(x) \in F[x]$. We aim to show that $p_i(\alpha)(v_i) = 0 \quad \forall i \in \{1, \ldots, n\}$.

$$\begin{aligned} \sum_{i=1}^n p_i(\alpha)(v_i) = 0 &\implies f(\alpha)\left(\sum_{i=1}^n p_i(\alpha)(v_i)\right) = 0 \\ &\implies \sum_{i=1}^n p_i(\alpha)f(\alpha)(v_i) = 0 \\ &\implies p_i(\alpha)f(\alpha)(v_i) = 0 \quad \forall i \in \{1, \ldots, n\}, \end{aligned}$$

by the direct sum of $\sum_{i=1}^n \langle (f(\alpha))(v_i)\rangle_\alpha$. Note that $v_i \notin \ker(f(\alpha))$, so we must have

$f(x) \mid p_i(x) \quad \forall i \in \{1, \ldots, n\}$. Write $p_i(x) = f(x)q_i(x) \quad \forall i \in \{1, \ldots, n\}$, then

$$\begin{aligned}\sum_{i=1}^{n} p_i(\alpha)(v_i) = 0 \iff & \sum_{i=1}^{n} q_i(\alpha)f(\alpha)(v_i) = 0 \\ \implies & q_i(\alpha)f(\alpha)(v_i) = 0 \quad \forall i \in \{1, \ldots, n\} \\ \iff & p_i(\alpha)(v_i) = 0 \quad \forall i \in \{1, \ldots, n\}.\end{aligned}$$

We conclude that the sum $\sum_{i=1}^{n} \langle v_i \rangle_\alpha$ is also direct.

(d) For all $v \in V$, $f(\alpha)(v) \in (f(\alpha))(V) = \sum_{i \in I} \langle (f(\alpha))(v_i) \rangle_\alpha$, so for each $i \in I$, there exists $p_i(x) \in F[x]$ (with only finitely many of which is nonzero) such that

$$\begin{aligned} f(\alpha)(v) = \sum_{i \in I} p_i(\alpha)f(\alpha)(v_i) &= f(\alpha)\left(\sum_{i \in I} p_i(\alpha)(v_i)\right) \\ \implies f(\alpha)\left(v - \sum_{i \in I} p_i(\alpha)(v_i)\right) &= 0 \\ \implies v - \sum_{i \in I} p_i(\alpha)(v_i) &\in \ker(f(\alpha)) \subseteq \sum_{i \in I} \langle v_i \rangle_\alpha. \end{aligned}$$

Since $\sum_{i \in I} p_i(\alpha)(v_i) \in \sum_{i \in I} \langle v_i \rangle_\alpha$, we have $v \in \sum_{i \in I} \langle v_i \rangle_\alpha$. Thus $V = \sum_{i \in I} \langle v_i \rangle_\alpha$.

(e) Proof by induction on k.

Base case: If $k = 1$, then $m_\alpha(x) = f(x)$. Using the proposition in the hint with $U = \{0_V\}$, there exists an α-invariant subspace W of V such that W is a direct sum of α-cyclic subspaces, and $\langle v \rangle_\alpha \cap W \neq \{0_V\}$ for any $v \in V \setminus W$. We now show that $W = V$. Suppose not. Then there exists $v \in V \setminus W$. Since $v \in \ker(f(\alpha))$ and $v \notin W$, by (b), we have $\langle v \rangle_\alpha \cap W = \{0_V\}$, which yields a contradiction. Therefore such a v must not exist. Hence we have $V = W$ is a direct sum of α-cyclic subspaces.

Induction: Suppose now that the proposition is true for some $(k-1) \in \mathbb{Z}^+$, and consider the case when $m_\alpha(x) = f^k(x)$. By the induction hypothesis, we know that $(f(\alpha))(V) = \bigoplus_{i \in I} \langle f(\alpha)(v_i) \rangle_\alpha$ for some indexing set I and $v_i \in V \setminus \ker(f(\alpha))$ for all $i \in I$. We claim that the sum $\sum_{i \in I} \langle v_i \rangle_\alpha$ is direct. Any element in $\sum_{i \in I} \langle v_i \rangle$ is a finite sum of the form $\sum_{n=1}^{t} p_{i_n}(\alpha)v_{i_n}$ for $i_1, i_2, \cdots, i_t \in I$. Since the finite sum

$\sum_{n=1}^t \langle f(\alpha)(v_{n_i})\rangle_\alpha$ is direct, by (c) the sum $\sum_{n=1}^t \langle v_{n_i}\rangle_\alpha$ is direct. Thus

$$\sum_{n=1}^{t} p_{i_n}(\alpha)v_{i_n} = 0 \iff p_{i_n}(\alpha)v_{i_n} = 0_V \text{ for all } 1 \le n \le t.$$

Now let $U = \sum_{i\in I}\langle v_i\rangle_\alpha$, and W be an α-invariant subspace of V such that W is a direct sum of α-cyclic subspaces, $U \cap W = \{0_V\}$, and $\langle v\rangle_\alpha \cap (U+W) \neq \{0_V\}$ for any $v \in V \setminus (U+W)$. We now show that $V = U \oplus W$. Firstly, write $W = \bigoplus_{i\in I'}\langle v_i\rangle_\alpha$ for some indexing set I'(disjoint with I) and $v_i \in V$ for all $i \in I'$. Next, we will show that $\ker(f(\alpha)) \subseteq \sum_{i\in I\cup I'}\langle v_i\rangle_\alpha = U+W$. Suppose there exists $v \in \ker(f(\alpha))$ with $v \notin (U+W)$. Then by (b), $\langle v\rangle_\alpha \cap (U+W) = \{0_V\}$, contradicting our assumption about W. Therefore $\ker(f(\alpha)) \subseteq \sum_{i\in I\cup I'}\langle v_i\rangle_\alpha$. Then by (d), we have $V = \sum_{i\in I\cup I'}\langle v_i\rangle_\alpha$ and the sum is direct (since $U \cap W = \{0_V\}$).

Question 4

Let $n \in \mathbb{Z}^+$, and let ϕ be a symmetric bilinear form on an n-dimensional vector space V.

(a) Suppose that $F = \mathbb{R}$. Prove that the following statement are equivalent:

  (i) $\phi(v,v) \neq 0_F$ for all $v \in V \setminus \{0_V\}$.

  (ii) ϕ is either positive definite or negative definite.

  (iii) For any vector subspace U of V, we have $V = U \oplus U^\perp$.

(b) Show that if $|F| = 3$ and $\phi(v,v) \neq 0_F$ for all $v \in V \setminus \{0_F\}$, then $n \le 2$ and ϕ can be represented by I_n or $-I_n$.

(c) Show further that if $|F| = 5$ and $\phi(v,v) \neq 0_F$ for all $v \in V \setminus \{0_F\}$, then $n \le 2$.

Solution

(a) (i)⇒(ii): Since ϕ is non-degenerate, by the Sylvester's Law of Inertia, there exists $s, t \in \mathbb{Z}_{\geq 0}$ and an ordered basis $\mathscr{B}$ for V such that

$$[\phi]_{\mathscr{B}} = \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix}.$$

We aim to show that $st = 0$. Suppose not. Then there exists $u, v \in \mathscr{B}$ s.t. $\phi(u, u) = 1$ and $\phi(v, v) = -1$. Then $\phi(u + v, u + v) = 0$, a contradiction! So either $s = 0$ or $t = 0$, which means ϕ is either positive definite or negative definite.

(ii)⇒(iii): Since ϕ is either positive definite or negative definite, ϕ is non-degenerate. Thus for any vector subspace U of V, we have that $\phi|_{U \times U}$ is also non-degenerate so $V = U \oplus U^{\perp}$.

(iii)⇒(i): Suppose $\exists v \in V$ s.t. $\phi(v, v) = 0_F$. $U = span(\{v\})$ is a subspace of V, so we have $V = U \oplus U^{\perp}$. But $v \in U \cap U^{\perp} = \{0_V\}$, so we must have $v = 0_V$.

(b) Suppose we have $n \geq 3$. Since $char(F) > 2$, we have that there exists an orthogonal basis $\mathscr{B} = \{v_1, \ldots, v_n\}$ for V with respect to ϕ. For each $i \in \{1, \ldots, n\}$, we know that $\phi(v_i, v_i) = \pm 1_F$. If $\phi(v_1, v_1) = \phi(v_2, v_2) = \phi(v_3, v_3)$, then $\phi(v_1+v_2+v_3, v_1+v_2+v_3) = 0$, a contradiction. On the other hand, if $\phi(v_i, v_i) = -\phi(v_j, v_j)$ then $\phi(v_i + v_j, v_i + v_j) = 0$ which is another contradiction. Thus $n \leq 2$. The second part is clear for $n = 1$. When $n = 2$, assume that ϕ cannot be represented by I_n or $-I_n$. Then ϕ is represented by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

with respect to the basis $\{v_1, v_2\}$. Then $\phi(v_1 + v_2, v_1 + v_2) = 0$, a contradiction.

(c) Write $F = \{0, 1, 2, -1, -2\}$. Similar to (b), we let $\mathscr{B} = \{v_1, \ldots, v_n\}$ be an orthogonal basis for V with respect to ϕ. We consider two categories:

1) $\phi(v_i, v_i) = \pm 1$,

2) $\phi(v_i, v_i) = \pm 2$.

By the Pigeon Hole Principle, when $n \geq 3$, we would have at least 2 basis vectors that fall into the same category. WLOG, say v_1 and v_2 are in the same category. Now consider 2 cases:

1) $\phi(v_1, v_1) = -\phi(v_2, v_2)$. Then $\phi(v_1 + v_2, v_1 + v_2) = 0$, a contradiction.

2) $\phi(v_1, v_1) = \phi(v_2, v_2)$. Then $\phi(v_1 + 2v_2, v_1 + 2v_2) = \phi(v_1) + 4\phi(v_2, v_2) = 5\phi(v_1, v_1) = 0$, again, a contradiction.

Therefore $n \leq 2$. When $n = 2$, as long as the two basis vectors do not fall into the same category, ϕ is non-degenerate.