NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA1101R Linear Algebra I
AY 2013/2014 Sem 2
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Question 1

(a) i)

$$\boldsymbol{A}^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ -8/5 & -1/5 & -1 \\ 6/5 & 2/5 & 1 \end{pmatrix}$$

ii)

$$\boldsymbol{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -5 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{A}^{-1} \begin{pmatrix} 2 \\ 5 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ -8/5 & -1/5 & -1 \\ 6/5 & 2/5 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ 4/5 \\ -3/5 \end{pmatrix}$$

iii)

$$\begin{pmatrix} 1 & -2 & -1 \\ 2 & 1 & 3 \\ -2 & 2 & 1 \end{pmatrix} \xrightarrow[R_3+2R_1]{R_2-2R_1} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & -2 & -1 \end{pmatrix} \xrightarrow{\frac{2}{5}R_2} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \\ 0 & -2 & -1 \end{pmatrix} \xrightarrow{R_3+R_2} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$\boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \boldsymbol{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \boldsymbol{E}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

(b) i)

$$\begin{vmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ -2 & 1 & 3 \end{vmatrix} = -1$$

The determinant is nonzero, so the columns form a basis for $\mathbb{R}^3$.

ii)

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 1 \\ 1 & 1 & 2 & 1 \\ -2 & 1 & 3 & 0 \end{array}\right) \xrightarrow{\text{Gaussian Elimination}} \left(\begin{array}{ccc|c} 2 & 0 & -1 & 1 \\ 0 & 1 & 5/2 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{array}\right)$$

Solving,

$$(\boldsymbol{w})_S = (0, 3, -1)$$

iii)

$$T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = -\begin{pmatrix}1\\1\\1\end{pmatrix} + 7\begin{pmatrix}1\\1\\0\end{pmatrix} - 3\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}3\\6\\-1\end{pmatrix}$$

$$T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\1\\1\end{pmatrix} - 4\begin{pmatrix}1\\1\\0\end{pmatrix} 2\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\-3\\1\end{pmatrix}$$

$$T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = -\begin{pmatrix}1\\1\\1\end{pmatrix} 5\begin{pmatrix}1\\1\\0\end{pmatrix} - 2\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\4\\-1\end{pmatrix}$$

Thus the standard matrix for T is

$$\begin{pmatrix}3 & -1 & 2\\6 & -3 & 4\\-1 & 1 & -1\end{pmatrix}$$

Question 2

(a) i)

$$\begin{pmatrix}1 & 2 & 3 & 4\\1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\2 & 1\\3 & 1\\4 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}4\\4\\6\\10\end{pmatrix}$$

$$\begin{pmatrix}30 & 10\\10 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}70\\24\end{pmatrix}$$

Solving: $x = 2$, $y = 1$

ii) The projection is given by

$$\begin{pmatrix}1 & 1\\2 & 1\\3 & 1\\4 & 1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\5\\7\\9\end{pmatrix}$$

iii) We prove this statement by contradiction.
Suppose not. Suppose there exists a $\boldsymbol{u}$ such that $\boldsymbol{Au} = k\boldsymbol{b}$ is consistent. Then $\frac{1}{k}\boldsymbol{Au} = \frac{1}{k}k\boldsymbol{b}$ is consistent, implying $\boldsymbol{A}\frac{\boldsymbol{u}}{k} = \boldsymbol{b}$ is consistent. Thus $\boldsymbol{x} = \frac{\boldsymbol{u}}{k}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{b}$. However, this contradicts the statement "$\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent".
If $\boldsymbol{v}$ is a least squares solution for $\boldsymbol{Ax} = \boldsymbol{b}$,
$\boldsymbol{A}^T\boldsymbol{Av} = \boldsymbol{A}^T\boldsymbol{b}$
$\implies k\boldsymbol{A}^T\boldsymbol{Av} = k\boldsymbol{A}^T\boldsymbol{b}$
$\implies \boldsymbol{A}^T\boldsymbol{A}(k\boldsymbol{v}) = \boldsymbol{A}^T(k\boldsymbol{b})$
Thus $k\boldsymbol{v}$ is a least squares solution for $\boldsymbol{Ax} = k\boldsymbol{b}$

(b) i)

$$\begin{cases} a \quad -2b \qquad\qquad\qquad\quad = 0 \\ \qquad\qquad c \quad -d \quad +2e \quad = 0 \\ a \qquad\qquad\quad +2d \quad -e \quad = 0 \end{cases}$$

W is the solution set of a homogeneous system of linear equations, so it is a subspace.

ii) Solving the above system,

$$W = \left\{ \left(-2s+t, -s+\frac{t}{2}, s-2t, s, t \right) \middle| s, t \in R \right\}$$

Thus a basis for W is given by

$$\left\{ (-2,-1,1,1,0), \left(1, \frac{1}{2}, -2, 0, 1\right) \right\}$$

and $\dim(W) = 2$.

iii) We note that a, b, c depend on d, e.
$V = \text{span}\{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)\}$

Question 3

(a)

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{array} \right) \xrightarrow{\text{Gaussian Elimination}} \left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & 0 & 0 & c-b-a \end{array} \right)$$

We must have $c - b - a = 0 \implies c = a + b$.

(b) i) Note that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal.

Get $\boldsymbol{u}_3 = \boldsymbol{v}_3 - \frac{\boldsymbol{v}_1 \cdot \boldsymbol{v}_3}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} \boldsymbol{v}_1 - \frac{\boldsymbol{v}_2 \cdot \boldsymbol{v}_3}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2} \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}$.

Orthogonal basis for span(S): $\{(1,0,1,1), (-1,1,1,0), (1,2,-1,0)\}$

ii) We must find a vector which is orthogonal to all vectors in the above basis (or S).

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 \end{array} \right)$$

Solving (there are infinite solutions, choose any one): $(-1, 0, -1, 2)$.
Add the vector $(-1, 0, -1, 2)$ to the basis found in i).

(c) Let $\boldsymbol{x}_k = \begin{pmatrix} a_k \\ a_{k+1} \end{pmatrix}$.

$$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix} = \cdots = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}^n \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

Let $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Then $\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$

Diagonalizing $\boldsymbol{A}$: the eigenvalues are $-\frac{1}{2}$ and 1. The eigenvectors are $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\boldsymbol{P} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$, $\boldsymbol{D} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

$\boldsymbol{x}_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}\boldsymbol{x}_0 = \begin{pmatrix} \left(-\frac{1}{2}\right)^n p + (1)^n q \\ \cdots \end{pmatrix}$
for some real p, q. Substituting $a_0 = 0$ and $a_1 = 1$:

$$\begin{cases} p + q = 0 \\ -\frac{1}{2}p + q = 1 \end{cases}$$

Solving, $p = -\frac{2}{3}$ and $q = \frac{2}{3}$.
Thus $a_n = -\frac{2}{3}\left(-\frac{1}{2}\right)^n + \frac{2}{3}$

Note We can verify this result using strong mathematical induction.

Question 4

(a) i)

$$\begin{pmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} \xrightarrow{\text{Gaussian Elimination}} \begin{pmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

Thus a basis for the row space of $\boldsymbol{A}$ is given by $\{(1,0,2,-1,1),(0,1,2,1,0),(0,0,1,-1,1)\}$

ii) One (the trivial solution). Refer to Question 4.24 of the textbook.
The reduced row-echelon form of $\boldsymbol{A}$ has no zero rows.
$\implies$ $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent for all $\boldsymbol{b} \in \mathbb{R}^3$.
$\implies$ The column space of $\boldsymbol{A}$ is $\mathbb{R}^3$.
$\implies$ $\text{rank}(\boldsymbol{A}) = 3$
By the dimension theorem, $\text{nullity}(\boldsymbol{A}^T) = 3 - \text{rank}(\boldsymbol{A}^T) = 3 - \text{rank}(\boldsymbol{A}) = 0$
Thus $\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{b}$ has only the trivial solution.

iii) $\text{Ker}(T) =$ nullspace of $\boldsymbol{A}$
Solving $\left(\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$: we get ($-s+t$, $-3s+2t$, $s-t$, s, t).
Thus a basis for $\text{Ker}(T)$ is given by $\{(-1,-3,1,1,0),(1,2,-1,0,1)\}$ and $\text{nullity}(T) = 2$.

(b) i) $\det(\lambda\boldsymbol{I} - \boldsymbol{B}) = \begin{vmatrix} \lambda-2 & -2 & -3 \\ -1 & \lambda-2 & -1 \\ -2 & 2 & \lambda-1 \end{vmatrix} = \lambda^3 - 5\lambda^2 + 2\lambda + 8 = (\lambda+1)(\lambda-2)(\lambda-4)$
The other eigenvalues are 2 and 4.

ii)

$$\boldsymbol{E}_{-1} : \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{E}_2 : \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}, \boldsymbol{E}_4 : \begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix}$$

The eigenspaces for each eigenvalue is the span of the corresponding eigenvector.
iii) Yes. $\boldsymbol{B}$ has 3 distinct eigenvalues.
$\boldsymbol{P} = \begin{pmatrix} -1 & -2 & 8 \\ 0 & -3 & 5 \\ 1 & 2 & 2 \end{pmatrix}$, $\boldsymbol{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$
iv) Yes. Let $\boldsymbol{C} = \begin{pmatrix} 2016 & 2 & 3 \\ 1 & 2016 & 1 \\ 2 & -2 & 2015 \end{pmatrix}$. Note that $\boldsymbol{C} = 2014\boldsymbol{I} + \boldsymbol{B}$.

Consider the equation $\det(\lambda_c \boldsymbol{I} - \boldsymbol{C}) = 0$:
$\det(\lambda_c \boldsymbol{I} - (2014\boldsymbol{I} + \boldsymbol{B})) = 0$
$\det((\lambda_c - 2014)\boldsymbol{I} - \boldsymbol{B}) = 0$
From above, $\lambda_c - 2014 = -1, 2, 4$
$\lambda_c = 2013, 2016, 2018$
$\boldsymbol{C}$ has 3 distinct eigenvalues, so it is diagonalizable.

END OF SOLUTIONS

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