NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Poh Wei Shan Charlotte

**MA3110 Mathematical Analysis II**
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## Question 1

(a) $a_0 = 1$, $a_1 = \frac{2}{5}$, $a_2 = \frac{36}{25}$, $a_3 = \frac{8}{125}$, $a_4 = \frac{1296}{625}$.

(b)

$$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{2}{5}(\frac{1}{3})^n & \text{for even } n \\ \frac{6}{5} \cdot 3^n & \text{for odd } n. \end{cases}$$

$\therefore$ lim sup $\left|\frac{a_{n+1}}{a_n}\right| = \infty$ and lim inf $\left|\frac{a_{n+1}}{a_n}\right| = 0$.

(c)

$$(a_n)^{\frac{1}{n}} = \begin{cases} \frac{6}{5} & \text{for even } n \\ \frac{2}{5} & \text{for odd } n. \end{cases}$$

$\therefore$ lim sup $(a_n)^{\frac{1}{n}} = \frac{6}{5}$ and lim inf $(a_n)^{\frac{1}{n}} = \frac{2}{5}$.

(d) Radius of Convergence $= \frac{1}{\text{lim sup}|a_n|^{\frac{1}{n}}} = \frac{5}{6}$.
When $x = -\frac{5}{6}$, the series become $\sum_{n=0}^{\infty} \left(-\frac{4+(-1)^n 2}{6}\right)^n$.
The even terms of the series are 1, hence the series does not converge by the $n$th-term divergence test.
Similarly, the series does not converge for $x = \frac{5}{6}$.
Interval of Convergence $= (-\frac{5}{6}, \frac{5}{6})$.

## Question 2

(a) For $x \in (0, \infty)$,

$$\begin{aligned} \left|\frac{x^n \cos\frac{n\pi}{x}}{(1+2x)^n}\right| &\leq \left|\frac{x^n}{(1+2x)^n}\right| \\ &\leq \left|\frac{x^n}{(2x)^n}\right| \\ &= \left(\frac{1}{2}\right)^n. \end{aligned}$$

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric series which converges. Hence, by the Weierstrass-M test, $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $(0, \infty)$.

(b) $\forall n \in \mathbb{N}$, $f_n$ is continuous on $(0, \infty)$ and from (a), $\sum_{n=1}^{\infty} f_n$ converges uniformly on $(0, \infty)$ to $f$. By the theorem on preservation of continuity on series of functions, $f$ is continuous. Therefore,

$$\begin{aligned}
\lim_{x \to 1} f(x) &= f(1) \\
&= \sum_{n=1}^{\infty} f_n(1) \\
&= \sum_{n=1}^{\infty} \frac{\cos n\pi}{3^n} \\
&= \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^n \\
&= \frac{-\frac{1}{3}}{1 - (-\frac{1}{3})} \\
&= -\frac{1}{4}.
\end{aligned}$$

(c) $\forall m \in \mathbb{N}$, $\sum_{n=1}^{m} f_n(x)$ is a finite sum of functions (hence we have the "limit of sum = sum of limits"). Therefore,

$$\begin{aligned}
\lim_{x \to \infty} \sum_{n=1}^{m} f_n(x) &= \sum_{n=1}^{m} \lim_{x \to \infty} f_n(x) \\
&= \sum_{n=1}^{m} \lim_{x \to \infty} \frac{x^n \cos \frac{n\pi}{x}}{(1+2x)^n} \\
&= \sum_{n=1}^{m} \left(\frac{1}{2}\right)^n \qquad (1) \\
&= \frac{\frac{1}{2}(1 - (\frac{1}{2})^m)}{1 - \frac{1}{2}} \\
&= 1 - \frac{1}{2^m}.
\end{aligned}$$

Note that (1) is obtained using the proof in (a) and Squeeze Theorem.

(d) Define $B_m = 1 - \frac{1}{2^m}$ $\forall m \in \mathbb{N}$.

From (a), $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to $f$ : $\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ such that $\forall x \in (0, \infty)$

$$\left| \sum_{n=1}^{m} f_n(x) - f(x) \right| < \frac{\varepsilon}{3} \text{ whenever } m > N_1 \text{ — } (*)$$

$\lim_{m \to \infty} B_m = 1$ : $\forall \varepsilon > 0$, $\exists N_2 \in \mathbb{N}$ such that

$$|B_m - 1| < \frac{\varepsilon}{3} \text{ whenever } m > N_2. \text{ — } (\triangle)$$

From (c), for each $m \in \mathbb{N}$, $\lim_{x \to \infty} \sum_{n=1}^{m} f_n(x) = B_m$ : $\forall \varepsilon > 0$, $\exists M_m > 0$ such that

$$\left| \sum_{n=1}^{m} f_n(x) - B_m \right| < \frac{\varepsilon}{3} \text{ whenever } x > M_m. \text{ — } (\blacktriangle)$$

Hence, $\forall \varepsilon > 0$, $\exists N = \max\{N_1, N_2\}$ such that $(*)$ and $(\triangle)$ are satisfied.
Next, choose an $m > N$, then $\exists M_m > 0$ such that $(\blacktriangle)$ is satisfied.

Therefore, $\forall \varepsilon > 0$, $\exists M = M_m > 0$ such that whenever $x > M$,

$$\begin{aligned} |f(x) - 1| &= |f(x) - \sum_{n=1}^{m} f_n(x) + \sum_{n=1}^{m} f_n(x) - B_m + B_m - 1| \\ &\leq |f(x) - \sum_{n=1}^{m} f_n(x)| + |\sum_{n=1}^{m} f_n(x) - B_m| + |B_m - 1| \\ &< \varepsilon. \end{aligned}$$

**Question 3**

(a) Let $F(x) = \int_a^x f(u)\ du$ for $x \in [a, b]$.
By the Fundamental Theorem of Calculus, since $f$ is continuous on $[a, b]$, $F$ is differentiable on $[a, b]$ and $F'(x) = f(x)\ \forall x \in [a, b]$.
By the Mean Value Theorem for derivatives, $\exists y_0 \in (a, b)$ such that

$$\begin{aligned} F(b) - F(a) &= F'(y_0)(b - a) \\ \therefore \int_a^b f &= f(y_o)(b - a). \end{aligned}$$

(b) What we want to show: $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that $\forall x \in [a, b]$,

$$|l_n(x) - F(x)| < \varepsilon \text{ whenever } n \geq N(\varepsilon).$$

What we have:
$F(x) = \sum_{k=0}^{n-1} \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} f(t)\ dt$.
Since $f$ is continuous on $\mathbb{R}$, $\forall n \in \mathbb{N}$, $\forall x \in \mathbb{R}$, $f$ is continuous on $[x + \frac{k}{n}, x + \frac{k+1}{n}]$.
Therefore, from (a), $\forall n \in \mathbb{N}$, $\exists\ x_n \in [x + \frac{k}{n}, x + \frac{k+1}{n}]$ such that

$$\begin{aligned} \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} f(t)\ dt &= f(x_n)\left((x + \frac{k+1}{n}) - (x + \frac{k}{n})\right) \\ &= \frac{1}{n} f(x_n). \end{aligned}$$

In addition, since $f$ is continuous on $\mathbb{R}$, $\forall \varepsilon > 0$ , $\exists\ \delta(\varepsilon) > 0$ such that $\forall x \in \mathbb{R}$,

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta(\varepsilon).$$

Using all the above, we can start our proof.

$\forall \varepsilon > 0$, we choose $N(\varepsilon) \in \mathbb{N}$ such that $N(\varepsilon) > \frac{1}{\delta(\varepsilon)}$. Then $\forall x \in [a,b]$, $\forall n \geq N(\varepsilon)$,

$$
\begin{aligned}
|l_n(x) - F(x)| &= \left| \sum_{k=0}^{n-1} \frac{1}{n} f(x + \frac{k}{n}) - \sum_{k=0}^{n-1} \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} f(t)\ dt \right| \\
&= \left| \sum_{k=0}^{n-1} \left( \frac{1}{n} f(x + \frac{k}{n}) - \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} f(t)\ dt \right) \right| \\
&= \frac{1}{n} \left| \sum_{k=0}^{n-1} \left( f(x + \frac{k}{n}) - f(x_n) \right) \right| \\
&\leq \frac{1}{n} \sum_{k=0}^{n-1} \left| \left( f(x + \frac{k}{n}) - f(x_n) \right) \right| \\
&< \frac{1}{n} \sum_{k=0}^{n-1} \varepsilon \qquad (2) \\
&= \varepsilon.
\end{aligned}
$$

The inequality (2) is due to the definition of the continuity of $f$ on $\mathbb{R}$. Since $n \geq N(\varepsilon)$, then $n > \frac{1}{\delta(\varepsilon)}$. Since $x_n \in [x + \frac{k}{n}, x + \frac{k+1}{n}]$, we have $|(x + \frac{k}{n}) - x_n| < \frac{1}{n} < \delta(\varepsilon)$. Therefore the inequality holds.

Note that $N(\varepsilon)$ is dependent on $\delta(\varepsilon)$ which is only dependent on $\varepsilon$.

Thus we have proved that $l_n(x)$ converges uniformly to $F(x)$.

**Question 4**

(a) To solve for local extrema, we will solve for $f'(x) = 0$.
$f'(x) = 3x^2 - a \implies x = \pm\sqrt{\frac{a}{3}}$.
For $x = \sqrt{\frac{a}{3}}$, $f''(x) = 6x > 0$. For $x = -\sqrt{\frac{a}{3}}$, $f''(x) < 0$.

Therefore, $f$ has a local maximum at $x = -\sqrt{\frac{a}{3}}$ where $f(-\sqrt{\frac{a}{3}}) = b + \frac{2a}{3}\sqrt{\frac{a}{3}}$ and has a local minimum at $x = \sqrt{\frac{a}{3}}$ where $f(\sqrt{\frac{a}{3}}) = b - \frac{2a}{3}\sqrt{\frac{a}{3}}$.

(b) Let $x_1 < x_2 < x_3 \in \mathbb{R}$ be the 3 real roots of $f(x) = 0$.
Then by Rolle's Theorem, $\exists\ c \in (x_1, x_2)$ and $d \in (x_2, x_3)$ such that $f'(c) = f'(d) = 0$.

Note that $a$ cannot be less than 0, otherwise from (a), $f'(x) = 3x^2 - a \neq 0\ \forall x \in \mathbb{R}$. $a$ also cannot be equal to 0, otherwise $f'(x) = 3x^2 - a = 0$ will only have one real root but $c \neq d$. Therefore, $a > 0$ and using (a), we have $c = -\sqrt{\frac{a}{3}}$ and $d = \sqrt{\frac{a}{3}}$.

Hence, we want to show that $f(-\sqrt{\frac{a}{3}}) > 0$ and $f(\sqrt{\frac{a}{3}}) < 0$. We will only prove the first inequality and the second one is similar.

Firstly, $f(-\sqrt{\frac{a}{3}}) \neq 0$ as there are only 3 real roots, $x_1, x_2, x_3$ and thus $-\sqrt{\frac{a}{3}}$ should not be another root of $f(x) = 0$.

Now, assume $f(-\sqrt{\frac{a}{3}}) < 0$, we want to obtain a result $(*)$ that $\exists\ e \in (-\sqrt{\frac{a}{3}}, x_2)$ such that $f(e) = f(-\sqrt{\frac{a}{3}})$ which is actually a contradiction.

$\because f(-\sqrt{\frac{a}{3}})$ is a local maximum, $\exists\ \delta > 0$ such that $f(-\sqrt{\frac{a}{3}}) \geq f(x)\ \forall x \in (-\sqrt{\frac{a}{3}}, -\sqrt{\frac{a}{3}} + \delta)$. (Note that we can assume $\delta$ is small enough such that $-\sqrt{\frac{a}{3}} + \delta < x_2$).

If $\forall x \in (-\sqrt{\frac{a}{3}}, -\sqrt{\frac{a}{3}} + \delta)$, $f(-\sqrt{\frac{a}{3}}) = f(x)$, then we already have the result $(*)$ that we want. If $\exists\ x' \in (-\sqrt{\frac{a}{3}}, -\sqrt{\frac{a}{3}} + \delta)$ such that $f(-\sqrt{\frac{a}{3}}) > f(x')$, then $f(x') < f(-\sqrt{\frac{a}{3}}) < 0 = f(x_2)$. By the

Intermediate Value Theorem, $\exists\ e \in (x', x_2) \subseteq (-\sqrt{\frac{a}{3}}, x_2)$ such that $f(e) = f(-\sqrt{\frac{a}{3}})$. Hence, we have proved (*).

Now we have $f(e) = f(-\sqrt{\frac{a}{3}})$, hence $\exists\ i \in (-\sqrt{\frac{a}{3}}, e)$ such that $f'(i) = 0$. However, $f'$ is a polynomial of order 2, hence $f'(x) = 0$ should only have at most 2 roots, which have already been found to be at $x = -\sqrt{\frac{a}{3}}$ and $\sqrt{\frac{a}{3}}$. Hence we obtain a contradiction. Therefore, $f(-\sqrt{\frac{a}{3}}) \not< 0$.

Therefore, $f(-\sqrt{\frac{a}{3}}) > 0$.

Using a similar argument, we will get $f(\sqrt{\frac{a}{3}}) < 0$. Hence, $f$ has a positive local maximum and a negative local minimum.

(c) From (a), we know that the local maximum is at $x = -\sqrt{\frac{a}{3}}$ and local minimum is at $x = \sqrt{\frac{a}{3}}$. Note that $a > 0$ in order to have 2 local extrema.

Therefore, we have $f(-\sqrt{\frac{a}{3}}) > 0$ and $f(\sqrt{\frac{a}{3}}) < 0$.

$\lim_{x\to-\infty} f(x) = -\infty$. Therefore $\exists\ y_1 < -\sqrt{\frac{a}{3}}$ such that $f(y_1) < 0$.

$\lim_{x\to\infty} f(x) = \infty$. Therefore $\exists\ y_2 > \sqrt{\frac{a}{3}}$ such that $f(y_2) > 0$.

$f(y_1) < 0$ and $f(-\sqrt{\frac{a}{3}}) > 0$.
By the Intermediate Value Theorem, $\exists\ x_1 \in (y_1, -\sqrt{\frac{a}{3}})$ such that $f(x_1) = 0$.

Applying the Intermediate Value theorem again on the intervals $(-\sqrt{\frac{a}{3}}, \sqrt{\frac{a}{3}})$ and $(\sqrt{\frac{a}{3}}, y_2)$, we will also find $x_2 \in (-\sqrt{\frac{a}{3}}, \sqrt{\frac{a}{3}})$ and $x_3 \in (\sqrt{\frac{a}{3}}, y_2)$ such that $f(x_2) = f(x_3) = f(x_1) = 0$ with $x_1 < x_2 < x_3$.

Therefore, $f(x) = 0$ has exactly 3 real solutions.

(d) From (b) and (c), we have the conclusion that $f(x) = 0$ has exactly 3 real solutions if and only if $f$ has a positive local maximum and negative local minimum. Therefore

$b - \frac{2a}{3}\sqrt{\frac{a}{3}} < 0$ and $b + \frac{2a}{3}\sqrt{\frac{a}{3}} > 0 \Leftrightarrow -\frac{2a}{3}\sqrt{\frac{a}{3}} < b < \frac{2a}{3}\sqrt{\frac{a}{3}} \Leftrightarrow b^2 < (\frac{2a}{3}\sqrt{\frac{a}{3}})^2 \Leftrightarrow 27b^2 - 4a^3 < 0.$

Hence, $f(x) = 0$ has exactly 3 real solutions if and only if $27b^2 - 4a^3 < 0$.