NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Chang Hai Bin

MA1102R Calculus
AY 2009/2010 Sem 1

Question 1

(a)

$$\begin{aligned}\lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2} &= \lim_{x\to 4}\frac{(\sqrt{1+2x}-3)(\sqrt{1+2x}+3)}{\sqrt{1+2x}+3}\cdot\frac{\sqrt{x}+2}{(\sqrt{x}-2)(\sqrt{x}+2)}\\ &= \lim_{x\to 4}\frac{1+2x-9}{\sqrt{1+2x}+3}\cdot\frac{\sqrt{x}+2}{x-4}\\ &= \lim_{x\to 4}\frac{2(x-4)}{\sqrt{1+2x}+3}\cdot\frac{\sqrt{x}+2}{x-4}\\ &= \lim_{x\to 4}\frac{2(\sqrt{x}+2)}{\sqrt{1+2x}+3} = \frac{2(\sqrt{4}+2)}{\sqrt{1+2\cdot 4}+3} = \frac{4}{3}\end{aligned}$$

(b) For $x > 0$, $-1 \le \sin(\frac{1}{x}) \le 1$
$-\frac{1}{x} \le \frac{1}{x}\sin(\frac{1}{x}) \le \frac{1}{x}$
As $\lim_{x\to\infty}\frac{-1}{x} = \lim_{x\to\infty}\frac{1}{x} = 0$,
By Squeeze Theorem, $\lim_{x\to\infty}\frac{1}{x}\sin(\frac{1}{x}) = 0$

Question 2

(a) let g(x) = f^{-1}(x)
$f(g(x)) = x \quad \forall x \in D_{f^{-1}}$
$\frac{2}{e^{g(x)}-e^{-g(x)}} = x \quad \Rightarrow \frac{2}{x} = e^{g(x)} - \frac{1}{e^{g(x)}}$
$\Rightarrow \left[e^{g(x)}\right]^2 - \frac{2}{x}\left[e^{g(x)}\right] - 1 = 0$

$$\begin{aligned}e^{g(x)} &= \frac{\frac{2}{x}\pm\sqrt{\frac{4}{x^2}+4}}{2} = \frac{1}{x}\pm\sqrt{1+\frac{1}{x^2}} = \frac{1}{x}\pm\frac{\sqrt{x^2+1}}{\mid x\mid}\\ &= \begin{cases}\frac{1\pm\sqrt{1+x^2}}{x}, & x>0;\\ \frac{1}{x}\pm\frac{\sqrt{1+x^2}}{-x}, & x<0.\end{cases}\\ &= \begin{cases}\frac{1+\sqrt{1+x^2}}{x}, & x>0;\\ \frac{1-\sqrt{1+x^2}}{x}, & x<0.\end{cases}\end{aligned}$$

Note that for $g(x) \in \mathbb{R}$, $e^{g(x)} > 0$, and so we reject answers with negative value.
For example, when $x > 0$, $\frac{1-\sqrt{1+x^2}}{x} < 0$; and when $x < 0$, $\frac{1+\sqrt{1+x^2}}{x} < 0$.
So, g(x) = $\begin{cases}\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right), & x>0;\\ \ln\left(\frac{1-\sqrt{1+x^2}}{x}\right), & x<0.\end{cases}$

(b)

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \begin{cases} \frac{1}{\frac{1+\sqrt{1+x^2}}{x}} \quad \frac{d}{dx}\left[\frac{1+\sqrt{1+x^2}}{x}\right], & x > 0; \\ \frac{1}{\frac{1-\sqrt{1+x^2}}{x}} \quad \frac{d}{dx}\left[\frac{1-\sqrt{1+x^2}}{x}\right], & x < 0. \end{cases}$$
$$= \begin{cases} \frac{x}{1+\sqrt{1+x^2}} \times \quad \frac{-(1+\sqrt{1+x^2})}{x^2\sqrt{1+x^2}}, & x > 0; \\ \frac{x}{1-\sqrt{1+x^2}} \times \quad \frac{(1-\sqrt{1+x^2})}{x^2\sqrt{1+x^2}}, & x < 0. \end{cases}$$
$$= \begin{cases} \frac{-1}{x\sqrt{1+x^2}}, & x > 0; \\ \frac{1}{x\sqrt{1+x^2}}, & x < 0. \end{cases}$$

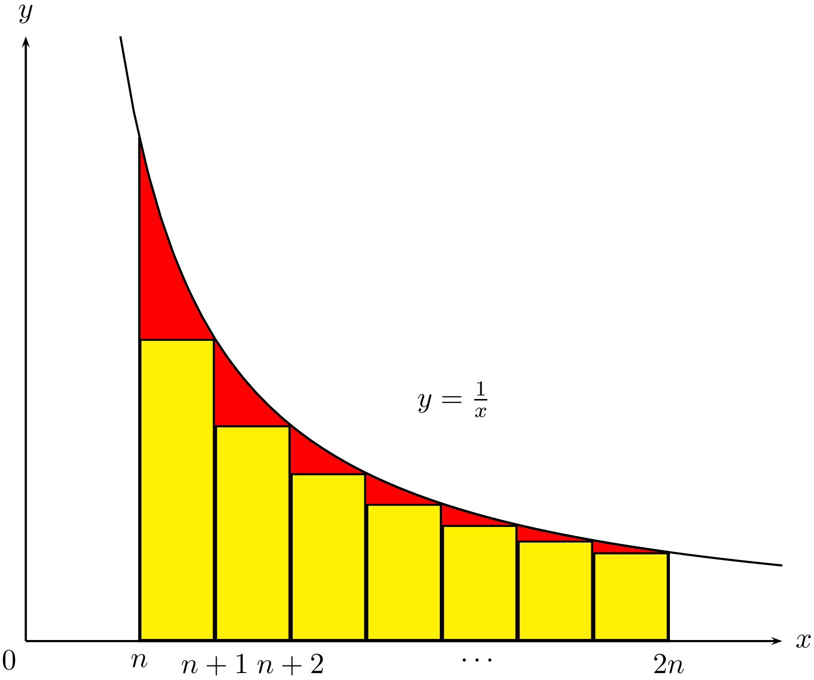
Question 3

(a) Using L' Hopital's Rule,

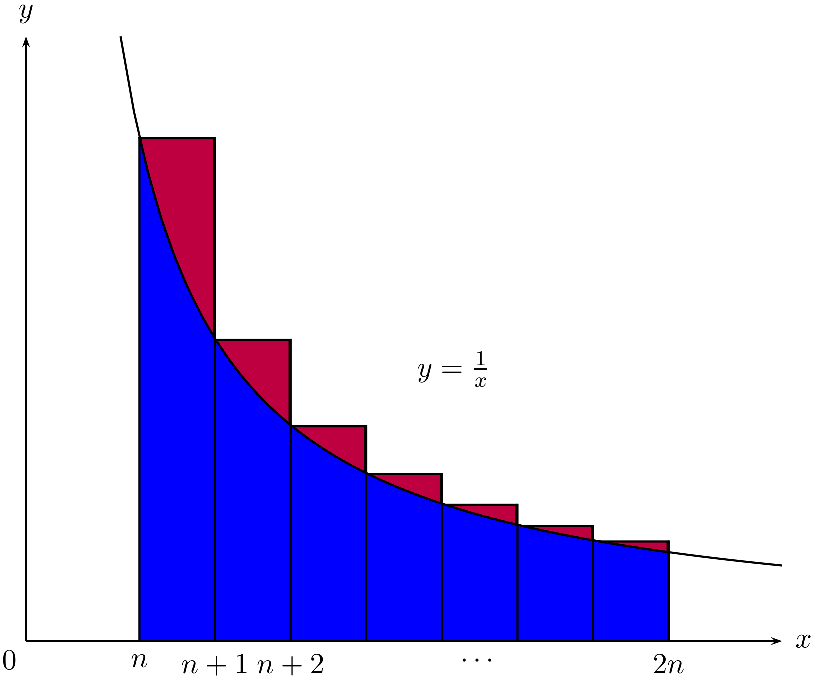
$$\begin{aligned} \lim_{x\to 0} \ln\left(\frac{{a_1}^x + \cdots + {a_n}^x}{n}\right)^{\frac{1}{x}} &= \lim_{x\to 0} \frac{\ln\left(\frac{{a_1}^x + \cdots + {a_n}^x}{n}\right)}{x} \\ &= \lim_{x\to 0} \frac{\frac{1}{\frac{{a_1}^x + \ldots + {a_n}^x}{n}} \cdot \frac{1}{n} \cdot [(\ln a_1){a_1}^x + \cdots + (\ln a_n){a_n}^x]}{1} \\ &= \lim_{x\to 0} \frac{(\ln a_1){a_1}^x + \cdots + (\ln a_n){a_n}^x}{{a_1}^x + \cdots + {a_n}^x} \\ &= \frac{(\ln a_1)\cdot 1 + \cdots + (\ln a_n)\cdot 1}{1 + \cdots + 1} \\ &= \frac{1}{n}\left[\ln(a_1 \cdot a_2 \cdot \cdots \cdot a_n)\right] \\ &= \ln\left(a_1 a_2 \cdots a_n\right)^{\frac{1}{n}} \end{aligned}$$

$$\lim_{x\to 0}\left(\frac{{a_1}^x + \cdots + {a_n}^x}{n}\right)^{\frac{1}{x}} = (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$$

(b) From the picture below, $\frac{1}{n+1} + \cdots + \frac{1}{2n} \leq \int_n^{2n} \frac{1}{x}dx$ (Area of yellow region $\leq$ Area of yellow and red Region)



From the picture below, $\int_{n+1}^{2n+1} \frac{1}{x} dx \leq \frac{1}{n+1} + \cdots + \frac{1}{2n}$ (Area of blue region $\leq$ Area of blue and purple Region)



$$\int_{n+1}^{2n+1} \frac{1}{x} dx \leq \frac{1}{n+1} + \cdots + \frac{1}{2n} \leq \int_{n}^{2n} \frac{1}{x} dx$$
$$\ln \frac{2n+1}{n+1} \leq \frac{1}{n+1} + \cdots + \frac{1}{2n} \leq \ln 2.$$

As $\lim_{n\to\infty} \frac{2n+1}{n+1} = \lim_{n\to\infty} \left(2 - \frac{1}{n+1}\right) = \lim_{n\to\infty} \ln 2 = \ln 2$,

By Squeeze Theorem, $\lim_{n\to\infty} \left(\frac{1}{n+1} + \cdots + \frac{1}{2n}\right) = \ln 2$

Question 4

(a) Note that the function is defined at $x \in \mathbb{R}\backslash(-1,1)$.

Using Integration by Parts,

$$\begin{aligned}\int \frac{1}{x^3} \cdot \sqrt{x^2-1}dx &= \left[-\frac{1}{2x^2}\right]\sqrt{x^2-1} - \int\left[-\frac{1}{2x^2}\right]\left[\frac{2x}{2\sqrt{x^2-1}}\right]dx \\ &= -\frac{\sqrt{x^2-1}}{2x^2} + \int \frac{1}{2x\sqrt{x^2-1}}dx \\ &= -\frac{\sqrt{x^2-1}}{2x^2} + \frac{1}{2}\text{csgn}(x)\sec^{-1}x + C\end{aligned}$$

Where $\text{csgn}(x) = \begin{cases} 1, & x > 0; \\ -1, & x < 0. \end{cases}$.

If $\int \frac{1}{x\sqrt{x^2-1}}dx$ is not given in formula sheet, let $x = \sec u, u \in [0,\pi]\backslash\{\frac{\pi}{2}\}$, $\frac{dx}{du} = \sec(u)\cdot\tan(u)$,

$$\begin{aligned}\int \frac{1}{x\sqrt{x^2-1}}dx &= \int \frac{1}{\sec(u)|\tan(u)|}\cdot\sec(u)\tan(u)du \\ &= \begin{cases}\int 1du, & \tan u > 0; \\ \int -1du, & \tan u < 0.\end{cases} \\ &= \begin{cases} u, & \tan u > 0 \Leftrightarrow \sec u > 0; \\ -u, & \tan u < 0 \Leftrightarrow \sec u < 0.\end{cases} \\ &= \begin{cases}\sec^{-1}x, & \sec u > 0 \Leftrightarrow x > 0; \\ -\sec^{-1}x, & \sec u < 0 \Leftrightarrow x > 0.\end{cases} \\ &= \text{csgn}(x)\sec^{-1}(x)\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{2}{(x-1)(x^2+1)}dx &= \int \frac{1}{x-1} - \frac{x}{x^2+1} - \frac{1}{x^2+1}dx \\ &= \ln(x-1) - \frac{1}{2}\ln(x^2+1) - \tan^{-1}(x) + C\end{aligned}$$

Question 5

(a) $h^2 + r^2 = 100$.
If θ is in radians, $10\theta = 2\pi r$.
Hence $r = \frac{5}{\pi}\theta, h = \sqrt{100 - \frac{25}{\pi^2}\theta^2}$

(b) So $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\frac{25}{\pi^2}\theta^2\sqrt{100-\frac{25}{\pi^2}\theta^2}$
Note that it is possible to consider V as a function of θ, with $\theta \in [0, 2\pi]$, and still get the Volume maximizing answer (although it might be messy).

Alternatively, we can consider V as a function of r, with $r \in [0, 10]$, or consider V as a function of $\alpha = r^2, \alpha \in [0, 100]$.
eg. $V(\alpha) = \frac{1}{3}\pi\alpha\sqrt{100-\alpha}$,

$\frac{dV}{d\alpha} = ... = \frac{\pi}{6}\left[\frac{200-3\alpha}{\sqrt{100-\alpha}}\right]$

So for $\alpha \in [0, 100]$, $\frac{dV}{d\alpha} = 0$ when $\alpha = \frac{200}{3}$

Using the Closed Interval Method,

$\alpha =$	type	Volume $V(\alpha)$
0	End-Point	0
$\frac{200}{3}$	Critical Point	$\frac{2000\sqrt{3}}{27}\pi$
100	End-Point	0

Hence, $r^2 = \frac{200}{3}$ will correspond to $\theta = \sqrt{\frac{8}{3}}\pi$.

$\theta = \sqrt{\frac{8}{3}}\pi$ will result in cone of largest volume $\frac{2000\sqrt{3}}{27}\pi$.

Question 6

(a)

$$\begin{aligned}
V &= \int_0^{\frac{1}{3}} \pi y^2 \, dx \\
&= \pi \int_0^{\frac{1}{3}} \frac{1}{9}x - \frac{2}{3}x^2 + x^3 \, dx \\
&= \frac{\pi}{9} \int_0^{\frac{1}{3}} x - 6x^2 + 9x^3 \, dx \\
&= \frac{\pi}{9}\left[\frac{x^2}{2} - 2x^3 + \frac{9}{4}x^4\right]_0^{\frac{1}{3}} \\
&= \frac{7}{972}\pi
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{6}x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} \\
\left(\frac{dy}{dx}\right)^2 + 1 &= \frac{1}{36x} - \frac{1}{2} + \frac{9}{4}x + 1 \\
\sqrt{\left(\frac{dy}{dx}\right)^2 + 1} &= \sqrt{\frac{1}{36x} + \frac{1}{2} + \frac{9}{4}x} = \frac{1}{6\sqrt{x}} + \frac{3}{2}\sqrt{x}
\end{aligned}$$

$$\begin{aligned}
S &= \int_0^{\frac{1}{3}} 2\pi y \sqrt{\left(\frac{dy}{dx}\right)^2 + 1} \, dx \\
&= 2\pi \int_0^{\frac{1}{3}} \left(\frac{1}{3}\sqrt{x} - x\sqrt{x}\right)\left(\frac{1}{6\sqrt{x}} + \frac{3}{2}\sqrt{x}\right) \, dx \\
&= 2\pi \int_0^{\frac{1}{3}} \frac{1}{18} - \frac{x}{6} + \frac{x}{2} - \frac{3}{2}x^2 \, dx \\
&= 2\pi \left[\frac{1}{18}x - \frac{x^2}{12} + \frac{x^2}{4} - \frac{1}{2}x^3\right]_0^{\frac{1}{3}} \\
&= \frac{1}{27}\pi
\end{aligned}$$

Question 7

(a) $f(0+0) = \frac{f(0)+f(0)}{1-f(0)\cdot f(0)}$
So either: $f(0) = 0$, or $1 - [f(0)]^2 = 2 \Rightarrow [f(0)]^2 = -1$ (Reject, since imaginary)
So $f(0) = 0$
As f is differentiable on $x = 0$, $\lim_{\delta x \to 0} \frac{f(0+\delta x)-f(0)}{\delta x} = \lim_{\delta x \to 0} \frac{f(\delta x)}{\delta x}$ exists, let it be L.
Hence $\lim_{\delta x \to 0} f(\delta x) = \left[\lim_{\delta x \to 0} \frac{f(\delta x)}{\delta x}\right] \cdot [\lim_{\delta x \to 0} \delta x] = L \cdot 0 = 0$
(Alternatively, differentiability implies continuity at $x = 0$, and $f(0) = 0$.)
Hence, for $\beta \in (-1, 1)$,

$$\begin{aligned}
\lim_{\delta x \to 0} \left[\frac{f(\beta+\delta x) - f(\beta)}{\delta x}\right] &= \lim_{\delta x \to 0} \frac{\frac{f(\beta)+f(\delta x)}{1-f(\beta)f(\delta x)} - f(\beta)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{f(\delta x) + (f(\beta))^2 f(\delta x)}{(\delta x)(1 - f(\beta)f(\delta x))} \\
&= \lim_{\delta x \to 0} \frac{f(\delta x)}{\delta x} \cdot \frac{1 + (f(\beta))^2}{1 - f(\beta)f(\delta x)} \\
&= L \cdot \frac{1 + (f(\beta))^2}{1 - f(\beta) \cdot 0} = L \cdot (1 + f(\beta)^2)
\end{aligned}$$

Since the limit exists for arbitrary $\beta \in (-1, 1)$, f is differentiable on $(-1, 1)$.

(b) If $L = \frac{\pi}{2}$,
$f'(\beta) = \frac{\pi}{2}(1 + (f(\beta))^2)$. Using a change of notation,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\pi}{2}(1 + y^2) \\
\int \frac{1}{1+y^2} dy &= \int \frac{\pi}{2} dx \\
\tan^{-1} y &= \frac{\pi x}{2} + C
\end{aligned}$$

Since it passes through $(x, y) = (0, 0), \tan^{-1} 0 = \frac{\pi \cdot 0}{2} + C$, hence $C = 0$.
So $f(x) = \tan\left(\frac{\pi \cdot x}{2}\right)$

Question 8

(a)

$$\frac{dy}{dx} + \left(\frac{1}{x} - 1\right) y = \frac{1}{x} e^{2x}$$

Note that $\frac{d}{dx}(\ln x - x) = \frac{1}{x} - 1$ and $e^{\ln x - x} = e^{\ln x} \cdot e^{-x} = xe^{-x}$.

$$\begin{aligned}
\left[xe^{-x}\right] \frac{dy}{dx} + \left[xe^{-x}\right] \left(\frac{1}{x} - 1\right) y &= \left[xe^{-x}\right] \frac{1}{x} e^{2x} \\
\left(xe^{-x}\right) \frac{dy}{dx} + \left(e^{-x} - xe^{-x}\right) y &= e^x \\
\frac{d}{dx}\left[\left(xe^{-x}\right) y\right] &= e^x \\
xe^{-x} y &= e^x + C, \qquad C \in \mathbb{R} \\
y &= \frac{e^{2x} + Ce^x}{x}, \qquad C \in \mathbb{R}
\end{aligned}$$

Since $1 = \lim_{x\to 0^+} y(x) = \lim_{x\to 0^+} \frac{e^{2x}+Ce^x}{x}$, we have $\lim_{x\to 0^+} e^{2x} + Ce^x = 0$. (Or else $\lim_{x\to 0^+} y(x)$ is undefined). Hence, $e^{2\times 0} + Ce^0 = 0, C = -1$. So $y(x) = \frac{e^{2x}-e^x}{x}, x > 0$.

(b) If P is constant, let $12P = Y$,

$$\begin{aligned}
\frac{dA}{dt} &= 0.05A - Y \\
\int \frac{1}{0.05A - Y} &= \int dt \\
\frac{1}{0.05} \ln|0.05A - Y| &= t + K \\
|0.05A - Y| &= e^{0.05t+0.05K} = e^{0.05t+L} \\
A &= 20Y - \left(20e^L\right) e^{0.05t} = 20Y - Me^{0.05t}
\end{aligned}$$

for suitable constants K, L, and M.

(Note: Yearly payment must be greater than the interest payment, hence $0.05A - Y$ is negative, $|0.05A - Y| = -(0.05A - Y)$)

So, when $t = 0, A = \$1$ mil. When $t = 20$ years, $A = \$0$.

$$\begin{cases} 1\text{mil} & = 20Y - Me^0 \quad = 20Y - M, \\ 0 & = 20Y - Me^{0.05\cdot 20} \quad = 20Y - Me, \end{cases}$$

By solving, $Y = \frac{e\cdot \$1\text{mil}}{(e-1)(20)} \approx \$79098.84$
And P = Y/12 = \$6591.57

(Note: A 5% interest per year is not the same as a 2.5 % compound interest every half-year, or two times per year. The 5% interest in this question should be considered as a $\frac{5}{n}$% interest compounded n times per year, with $n \to \infty$.)

Question 9

Lemma: If $g(x)$ is continuous over $[a, b]$, and $g(x) \neq 0\,\forall x \in (a, b)$, then

(i) $g(x) > 0\,\forall x \in (a, b)$, $g(a), g(b) \geq 0$ and $\int_a^b g(x)dx > 0$; OR

(ii) $g(x) < 0\,\forall x \in (a, b)$, $g(a), g(b) \leq 0$ and $\int_a^b g(x)dx < 0$

Proof: Note that $g(x)$ must be either: (i) all strictly positive, or (ii) all strictly negative for $x \in (a, b)$ (or else by the Intermediate Value Theorem, there exists $\alpha \in (a, b)$, such that $g(\alpha) = 0$)

$g(a), g(b)$ can be 0, but cannot be negative in Case (i), and cannot be positive in Case (ii).

(i) Case 1: $g(x) > 0\,\forall x \in (a, b)$
By choosing a small positive δ (for example, let $\delta = \frac{b-a}{1000}$), let $u = \text{minimum}\{g(x) : x \in [a + \delta, b - \delta]\}$ (u exists because of Extreme Value Theorem).

$$\begin{aligned}
\int_a^b g(x)dx &= \int_a^{a+\delta} g(x)dx + \int_{a+\delta}^{b-\delta} g(x)dx + \int_{b-\delta}^{b} g(x)dx \\
&\geq 0 + u\int_{a+\delta}^{b-\delta} dx + 0 = u \times (b - a - 2\delta) > 0
\end{aligned}$$

(ii) Case 2: $g(x) < 0\,\forall x \in (a, b)$
Similarly, by letting $u = \text{maximum}\{g(x) : x \in [a + \delta, b - \delta]\}$, $\int_a^b g(x)dx < 0$

Proof of Question 9:

Assume (for a contradiction) that $f(x)$ has at most one real root in $(0, \pi)$.

Case 1: $f(x)$ has no real root in $(0, \pi)$,

Note that $\sin(x) > 0\, \forall x \in (0, \pi)$,
So, using the Lemma above,

(i) if $f(x) > 0\, \forall x \in (0, \pi)$, then $f(x)\sin(x) > 0\, \forall x \in (0, \pi)$, and hence $\int_0^\pi f(x)\sin(x)dx > 0$
(ii) if $f(x) < 0\, \forall x \in (0, \pi)$, then $f(x)\sin(x) < 0\, \forall x \in (0, \pi)$, and hence $\int_0^\pi f(x)\sin(x)dx < 0$

Either subcase, a contradiction.

Case 2: $f(x) = 0$ has a real root at $\beta, 0 < \beta < \pi$, and no other real root in $(0, \pi)$.

(i) Subcase 1: $f(x) > 0\, \forall x \in (0, \pi)\backslash\{\beta\}$
Then $\int_0^\pi f(x)\sin(x)dx = \int_0^\beta f(x)\sin(x)dx + \int_\beta^\pi f(x)\sin(x)dx > 0 + 0 = 0$
(ii) Subcase 2: $f(x) < 0\, \forall x \in (0, \pi)\backslash\{\beta\}$
Then $\int_0^\pi f(x)\sin(x)dx = \int_0^\beta f(x)\sin(x)dx + \int_\beta^\pi f(x)\sin(x)dx < 0 + 0 = 0$
(iii) Subcase 3: $f(x) > 0\, \forall x \in (0, \beta), f(x) < 0\, \forall x \in (\beta, \pi)$
Note that $\sin(x - \beta) < 0\, \forall x \in (0, \beta), \sin(x - \beta) > 0\, \forall x \in (\beta, \pi)$
Hence $f(x)\sin(x - \beta) < 0\, \forall x \in (0, \beta), f(x)\sin(x - \beta) < 0\, \forall x \in (\beta, \pi)$
$\int_0^\pi f(x)\sin(x - \beta)dx = \int_0^\beta f(x)\sin(x - \beta)dx + \int_\beta^\pi f(x)\sin(x - \beta)dx < 0 + 0 = 0$
But $\int_0^\pi f(x)\sin(x - \beta)dx = \int_0^\pi f(x)\left[\sin(x)\cos(\beta) - \cos(x)\sin(\beta)\right] dx$
$= \cos(\beta)\int_0^\pi f(x)\sin(x)dx - \sin(\beta)\int_0^\pi f(x)\cos(x)dx = \cos(\beta)\cdot 0 - \sin(\beta)\cdot 0 = 0$
(iv) Subcase 4: $f(x) < 0\, \forall x \in (0, \beta), f(x) > 0\, \forall x \in (\beta, \pi)$
Note that $\sin(x - \beta) < 0\, \forall x \in (0, \beta), \sin(x - \beta) > 0\, \forall x \in (\beta, \pi)$
Hence $f(x)\sin(x - \beta) > 0\, \forall x \in (0, \beta), f(x)\sin(x - \beta) > 0\, \forall x \in (\beta, \pi)$
$\int_0^\pi f(x)\sin(x - \beta)dx = \int_0^\beta f(x)\sin(x - \beta)dx + \int_\beta^\pi f(x)\sin(x - \beta)dx > 0 + 0 = 0$
But $\int_0^\pi f(x)\sin(x - \beta)dx = 0$, as shown before.

Either Subcase, a contradiction.

Conclusion: $f(x) = 0$ has at least two real roots in $(0, \pi)$, as it is shown not possible to have none, or to have only one root.