NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Zheng Shaoxuan

MA2214 Combinatorial Analysis
AY 2008/2009 Sem 2

Question 1

(i) To distribute three different tickets among twenty students such that each student gets no more than one ticket, there are 20 ways to distribute the first ticket, 19 ways remaining to distribute the second ticket, and 18 ways remaining to distribute the third ticket. Hence the number of ways is

$$20 \times 19 \times 18.$$

(ii) To distribute three different tickets among twenty students such that each student can get any number of tickets, there are 20 ways to distribute the first ticket, 20 ways to distribute the second ticket, and 20 ways to distribute the third ticket. Hence the number of ways is

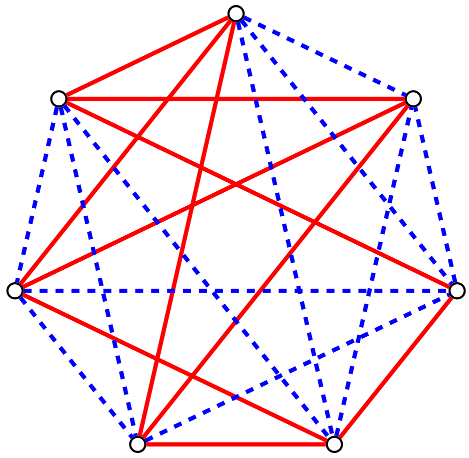
$$20^3.$$

(iii) To distribute three identical tickets among twenty students such that each student gets no more than one ticket, we choose without ordering three out of the twenty students to get the tickets. Hence the number of ways is

$$\binom{20}{3}.$$

Question 2

To show $R(3,4) > 7$, it suffices to colour the edges of a complete graph of order seven in solid red and dashed blue such that there does not exist a solid red C_3 subgraph and there does not exist a dashed blue K_4 subgraph. An example of such a colouring is as shown:



Question 3

(i) We use the Principle of Inclusion and Exclusion for (i), (ii), and (iii).

Let

- S be the set of all possible permutations of the letters of the English alphabet,
- P_1, P_2, P_3 be the properties that *dog*, *rat*, and *finch* appears in a given permutation respectively,
- $E(m)$ be the number of elements of S possessing exactly m of the 3 properties for $0 \leq m \leq 3$,
- $\omega(P_{i_1}P_{i_2}\ldots P_{i_m})$ be the number of elements of S possessing the properties P_{i_1}, P_{i_2}, $\ldots$, P_{i_m}, where $1 \leq i_m \leq 3$.
- $\omega(m) = \sum(\omega(P_{i_1}P_{i_2}\ldots P_{i_m}))$, $\omega(0) = |S|$.

By simple combinatorial observation, we have:

- $|S| = 26!$,
- $\omega(P_1) = (23+1)! = 24!$,
- $\omega(P_2) = (23+1)! = 24!$,
- $\omega(P_3) = (21+1)! = 22!$,
- $\omega(P_1P_2) = (20+2)! = 22!$,
- $\omega(P_1P_3) = (18+2)! = 20!$,
- $\omega(P_2P_3) = (18+2)! = 20!$,
- $\omega(P_1P_2P_3) = (15+3)! = 18!$.

We hence have:

- $\omega(0) = 26!$,
- $\omega(1) = 24! + 24! + 22!$,
- $\omega(2) = 22! + 20! + 20!$,
- $\omega(3) = 18!$.

The number of permutations such that none of the three sequences appear is

$$\begin{aligned} E(0) &= \omega(0) - \omega(1) + \omega(2) - \omega(3) \\ &= 26! - 2 \times 24! + 2 \times 20! - 18!. \end{aligned}$$

(ii) The number of permutations such that at least one of the three sequences appear is

$$\begin{aligned} \omega(0) - E(0) &= \omega(1) - \omega(2) + \omega(3) \\ &= 2 \times 24! - 2 \times 20! + 18!. \end{aligned}$$

(iii) The number of permutations such that exactly one of the three sequences appear is

$$\begin{aligned} E(1) &= \omega(1) - 2\omega(2) + 3\omega(3) \\ &= 2 \times 24! - 22! - 4 \times 20! + 3 \times 18!. \end{aligned}$$

Question 4

The smallest number of animals needed is 7.

If only 6 animals are chosen, we may choose 2 alpacas, 2 sheep and 2 llamas, and in this selection we do not have at least 3 of some type of animal. If 7 animals are chosen, by the Pigeonhole Principle, one of the three animals is chosen at least 3 times.

Question 5

It is not possible for all 20000 codes to be distinct.

The total number of distinct codes containing $0, 1, \ldots, 9$ of length 4 or less is $10^4 + 10^3 + 10^2 + 10^1 + 10^0 = 11111$. Hence, by the Pigeonhole Principle, if we select 20000 codes, at least 2 of them are the same code.

Question 6

Let P be the family of all partitions of n into parts of even size, and Q be the family of all partitions of n into parts such that parts of a given size occur an even number of times.

Define a function $f : P \mapsto Q$ such that for $p \in P$, let $f(p)$ be the partition of n such that the Ferrer's diagram of $f(p)$ is the transpose of the Ferrer's diagram of p. $f(p)$ is indeed an element of Q since if a part of a given size occurs an odd number of times, it would correspond to at least one part of odd size in p, a contradiction.

f is a bijection since the transposition of the Ferrer's diagram is an invertible transformation, and hence $|P| = |Q|$.

Question 7

By using generalised binomial coefficients,

$$\begin{aligned}
(1-4x)^{-\frac{1}{2}} &= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-4x)^n \\
&= \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})\ldots(-\frac{2n-1}{2})}{n!} (-4x)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^n n! (2 \cdot 4 \cdot \ldots \cdot 2n)} (-4x)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^n n! 2^n n!} (-4x)^n \\
&= \sum_{n=0}^{\infty} \frac{(2n)!}{n! n!} (x)^n \\
&= \sum_{n=0}^{\infty} \binom{2n}{n} (x)^n,
\end{aligned}$$

and hence the generating function for $a_n = \binom{2n}{n}$ is $A(x) = \dfrac{1}{\sqrt{1-4x}}$.

Question 8

Consider $A(x) = \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3} = (1+x+x^2+\ldots)(1+x^2+x^4+\ldots)(1+x^3+x^6+\ldots)$. The coefficient of x^6 in the expansion of $A(x)$ tells us how many ways there are to select one term each from each series such that the sum of the powers is 6. Since this coefficient is 7, this tells us that there are 7 partitions of the number 6 using only the numbers 1,2 and 3.

Question 9

(i) Ordering of the flags on the pole matters, so we opt to use the exponential generating function.

Since two of the colours of flags need to appear even number of times, consider the following exponential generating function:

$$\begin{aligned}
& \left(\frac{1}{2}(e^x + e^{-x})\right)\left(\frac{1}{2}(e^x + e^{-x})\right)(e^x) \\
= \; & \frac{1}{4}e^x(e^{2x} + e^{-2x} + 2) \\
= \; & \frac{1}{4}(e^{3x} + e^{-x} + 2e^x) \\
= \; & \frac{1}{4}\sum_{r=0}^{\infty}(3^r + (-1)^r + 2)\frac{x^r}{r!}.
\end{aligned}$$

The number of signals using an even number of red flags and an even number of black flags is the coefficient of $\frac{x^{12}}{12!}$ of the above generating function, which is

$$\frac{1}{4}(3^{12} + 3).$$

(ii) Since there cannot be exactly one white flag or exactly two white flags appearing on a signal this time, consider the following exponential generating function:

$$\begin{aligned}
& (e^x)(e^x)\left(e^x - \frac{x}{1!} - \frac{x^2}{2!}\right) \\
= \; & e^{3x} - xe^{2x} - \frac{1}{2}x^2e^{2x} \\
= \; & \sum_{r=0}^{\infty}(3^r)\frac{x^r}{r!} - \sum_{r=0}^{\infty}2^r(r+1)\frac{x^{r+1}}{(r+1)!} - \frac{1}{2}\sum_{r=0}^{\infty}2^r(r+2)(r+1)\frac{x^{r+2}}{(r+2)!}.
\end{aligned}$$

The number of signals using at least three white flags or no white flags at all is the coefficient of $\frac{x^{12}}{12!}$ of the above generating function, which is

$$3^{12} - 12 \cdot 2^{11} - \frac{1}{2} \cdot 12 \cdot 11 \cdot 2^{10}.$$

Question 10

(i) We solve the characteristic equation to obtain the homogenous part of a_n:

$$\begin{aligned} & x^2-4x-5=0 \\ \Rightarrow\ & (x-5)(x+1)=0 \\ \Rightarrow\ & x=5,-1. \end{aligned}$$

$$a_n^{(h)} = A\cdot 5^n + B\cdot(-1)^n,$$

where A, B are constants to be determined.

We let a particular solution of a_n, $a_n^{(p)} = Cn + D$, where C, D are constants to be determined. By substituting $a_n^{(p)}$ into the original expression, we have

$$\begin{aligned} & C(n+2)+D-4(C(n+1)+D)-5(Cn+D)=8n \\ \Rightarrow\ & \left\{\begin{array}{rcl} C-4C-5C & = & 8 \\ 2C+D-4C-4D-5D & = & 0 \end{array}\right. \\ \Rightarrow\ & C=-1,\ D=\frac{1}{4}. \end{aligned}$$

$$a_n^{(p)} = -n+\frac{1}{4}.$$

We now have $a_n = A\cdot 5^n + B\cdot(-1)^n - n + \frac{1}{4}$. By using the given initial values of a_0 and a_1,

$$\begin{aligned} & \left\{\begin{array}{rcl} A+B+\frac{1}{4} & = & 1 \\ 5A-B-1+\frac{1}{4} & = & 9 \end{array}\right. \\ \Rightarrow\ & \left\{\begin{array}{rcl} A+B & = & \frac{3}{4} \\ 5A-B & = & \frac{39}{4} \end{array}\right. \\ \Rightarrow\ & A=\frac{7}{4},\ B=-1. \end{aligned}$$

Hence, the solution is

$$a_n = \frac{7}{4}\cdot 5^n - (-1)^n - n + \frac{1}{4}.$$

(ii) We solve the characteristic equation to obtain the homogenous part of a_n:

$$\begin{aligned} & x^2+x-6=0 \\ \Rightarrow\ & (x+3)(x-2)=0 \\ \Rightarrow\ & x=2,-3. \end{aligned}$$

$$a_n^{(h)} = A\cdot 2^n + B\cdot(-3)^n,$$

where A, B are constants to be determined.

We let a particular solution of a_n, $a_n^{(p)} = Cn2^n$, where C is a constant to be determined. By substituting $a_n^{(p)}$ into the original expression, we have

$$\begin{aligned} & C(n+2)2^{n+2}-C(n+1)2^{n+1}-6Cn2^n = 10\cdot 2^n \\ \Rightarrow\ & n(4C+2C-6C)+(8C+2c)=10 \\ \Rightarrow\ & C=1. \end{aligned}$$

$$a_n^{(p)} = n2^n.$$

We now have $a_n = A \cdot 2^n + B \cdot (-3)^n + n2^n$. By using the given initial values of a_0 and a_1,

$$\begin{aligned} &\begin{cases} A + B &= 2 \\ 2A - 3B + 2 &= 1 \end{cases} \\ \Rightarrow \quad & A = 1,\ B = 1. \end{aligned}$$

Hence, the solution is

$$a_n = 2^n + (-3)^n + n2^n.$$

Question 11

Let $A(x)$ be the generating function for the sequence (a_n).

We have:

$$\begin{array}{rcccccccccccc} A(x) & = & a_0 & + & a_1x & + & a_2x^2 & + & \ldots & + & a_nx^n & + & \ldots \\ -xA(x) & = & & - & a_0x & - & a_1x^2 & - & \ldots & - & a_{n-1}x^n & + & \ldots \\ \Rightarrow (1-x)A(x) & = & a_0 & + & (a_1-a_0)x & + & (a_2-a_1)x^2 & + & \ldots & + & (a_n-a_{n-1})x^n & + & \ldots \\ & = & 5 & + & 3x & + & 3x^2 & + & \ldots & + & 3x^n & + & \ldots \\ & = & \multicolumn{11}{l}{2 + \dfrac{3}{1-x}} . \end{array}$$

$$\begin{aligned} A(x) &= \frac{2}{1-x} + \frac{3}{(1-x)^2} \\ &= 2\sum_{r=0}^{\infty} x^r + 3\sum_{r=0}^{\infty}(r+1)x^r. \end{aligned}$$

Hence, $a_n = 2 + 3(n+1) = 3n + 5$.

Question 12

(i) Each object in the line can be any one of n types, so there are n^k ways to perform the arrangement.

(ii) a_k is the number of k-permutations of the multiset $\{\infty \cdot b_1, \infty \cdot b_2, \ldots, \infty \cdot b_n\}$, where $b_1, b_2, \ldots, b_n$ are the n different objects.

Hence, the exponential generating function for a_k is

$$\begin{aligned} (e^x)^n &= e^{nx} \\ &= \sum_{r=0}^{\infty} n^r \frac{x^r}{r!}. \end{aligned}$$

a_k is the coefficient of $\dfrac{x^k}{k!}$ of the above exponential generating function. Hence, $a_k = n^k$.