NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA2108 Mathematical Analysis I
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Question 1

(a) Firstly, it is easy to see that (x_n) is bounded below by 1.
This is because for $n = 1$, we have $x_1 = 2 \geq 1$, and for all $n \in \mathbb{N}$, $n > 1$, one has

$$x_{n+1} = \frac{1}{2}\left(x_n^2 - 2x_n + 3\right) = \frac{1}{2}(x_n - 1)^2 + 1 \geq 1.$$

This would imply that $x_n \geq 1$ for all $n \in \mathbb{N}$, so (x_n) is bounded below by 1 as desired.

Next, we shall prove by induction that (x_n) is decreasing.
Let $P(n)$ be the statement $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. $P(1)$ is clearly true since $x_1 = 2 \geq \frac{3}{2} = x_2$.
Suppose $P(n)$ is true for some $k \in \mathbb{N}$. By induction hypothesis, we have $x_k \geq x_{k+1}$.
Then, it follows that

$$\begin{aligned}
x_k &\geq x_{k+1} \geq 1 \\
\Rightarrow x_k - 1 &\geq x_{k+1} - 1 \geq 0 \\
\Rightarrow (x_k - 1)^2 &\geq (x_{k+1} - 1)^2 \\
\Rightarrow x_{k+1} &= \frac{1}{2}(x_k - 1)^2 + 1 \\
&\geq \frac{1}{2}(x_{k+1} - 1)^2 + 1 = x_{k+2}.
\end{aligned}$$

So $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Since (x_n) is decreasing and bounded below, (x_n) converges by the Monotone Convergence Theorem. Let x be the limit of (x_n). Then we have

$$\begin{aligned}
& \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n^2 - 2x_n + 3\right) \\
\Rightarrow \quad & 2\lim_{n\to\infty} x_{n+1} = \left(\lim_{n\to\infty} x_n\right)^2 - 2\left(\lim_{n\to\infty} x_n\right) + 3 \\
\Rightarrow \quad & 2x = x^2 - 2x + 3 \\
\Rightarrow \quad & x^2 - 4x + 3 = 0 \\
\Rightarrow \quad & x = 1 \quad \text{and} \quad x = 3.
\end{aligned}$$

As $x_1 \geq x_n$ for all $n \in \mathbb{N}$, we have

$$2 = x_1 = \lim_{n\to\infty} x_1 \geq \lim_{n\to\infty} x_n = x.$$

So we must have $x = 1$.

(b) Let the limit of (y_{n_ℓ}) be y. Fix a $m \in \mathbb{N}$. Then for all $\ell \geq m$, $\ell \in \mathbb{N}$, one has

$$\begin{aligned}
& m \leq n_m < n_\ell \\
\Rightarrow \quad & y_m \leq y_{n_m} \leq y_{n_\ell} \quad (\because (y_n) \text{ is increasing})
\end{aligned}$$

$$\Rightarrow y_m = \lim_{\ell\to\infty} y_m \le \lim_{\ell\to\infty} y_{n_\ell} = y$$

As m is arbitrary, this implies that $y_n \le y$ for all $n \in \mathbb{N}$, so (y_n) is bounded above by y.
Since (y_n) is bounded above and increasing, (y_n) converges by the Monotone Convergence Theorem.

Question 2

(a) (i) Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^{1+\frac{1}{n}}}$. Then it follows that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n^{\frac{1}{n}} = 1 > 0.$$

Since $\sum\limits_{n=1}^{\infty} a_n$ diverges, we have $\sum\limits_{n=1}^{\infty} b_n$ to diverge by the Limit Comparison Test.

(ii) Let $a_n = \frac{3^n}{5n}\left(1+\frac{1}{2n^2}\right)^{-4n^3}$. Then it follows that

$$\begin{aligned}
a_n^{\frac{1}{n}} &= \frac{3}{(5n)^{\frac{1}{n}}} \cdot \left(1+\frac{1}{2n^2}\right)^{-4n^2} \\
&= \frac{3}{5^{\frac{1}{n}} \cdot n^{\frac{1}{n}}} \cdot \frac{1}{\left(1+\frac{1}{2n^2}\right)^{4n^2}} \\
\Rightarrow \lim_{n\to\infty} a_n^{\frac{1}{n}} &= \lim_{n\to\infty} \left(\frac{3}{5^{\frac{1}{n}} \cdot n^{\frac{1}{n}}} \cdot \frac{1}{\left(\left(1+\frac{1}{2n^2}\right)^{2n^2}\right)^2} \right) \\
&= \frac{3}{\lim\limits_{n\to\infty} 5^{\frac{1}{n}} \cdot \lim\limits_{n\to\infty} n^{\frac{1}{n}}} \cdot \frac{1}{\left(\lim\limits_{n\to\infty}\left(1+\frac{1}{2n^2}\right)^{2n^2}\right)^2} \\
&= \frac{3}{1\cdot 1} \cdot \frac{1}{e^2} \\
&= \frac{3}{e^2} < 1.
\end{aligned}$$

So $\sum\limits_{n=1}^{\infty} a_n$ converges by the Root Test.

(b) Let $a_n = \frac{1}{n!}$. Then it follows that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

So $\sum\limits_{n=3}^{\infty} \frac{1}{n!}$ converges by the Ratio Test. Therefore, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n^2-n-1}{n!} &= \frac{1^2-1-1}{1!} + \frac{2^2-2-1}{2!} + \sum_{n=3}^{\infty}\left(\frac{n(n-1)}{n!} - \frac{1}{n!}\right) \\
&= \sum_{n=3}^{\infty} \frac{1}{(n-2)!} - \sum_{n=3}^{\infty} \frac{1}{n!} - \frac{1}{2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} - \sum_{n=3}^{\infty} \frac{1}{n!} - \frac{1}{2} \\
&= \frac{1}{1!} + \frac{1}{2!} + \sum_{n=3}^{\infty} \frac{1}{n!} - \sum_{n=3}^{\infty} \frac{1}{n!} - \frac{1}{2} \\
&= \frac{3}{2} - \frac{1}{2} = 1.
\end{aligned}$$

Remark: It is necessary to check that the sum $\sum\limits_{n=3}^{\infty} \frac{1}{n!}$ is convergent, so that the steps above actually make sense.

(c) Since (a_n) and (a_{n+1}) are convergent sequences, it follows that $(a_n a_{n+1})$ is a convergent as well. Hence, $(a_n a_{n+1})$ is bounded, which implies that there exists some $m > 0$, such that for all $n \in \mathbb{N}$, one has

$$|a_n a_{n+1}| \leq \frac{1}{m}$$
$$\Rightarrow \left|\frac{1}{a_n a_{n+1}}\right| \geq m. \tag{1}$$

Next, since $(a_n a_{n+1})$ is convergent, $a_n a_{n+1} \neq 0$ for all $n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} a_n a_{n+1} = \left(\lim\limits_{n\to\infty} a_n\right)^2 \neq 0$, it follows that $\left(\frac{1}{a_n a_{n+1}}\right)$ is convergent as well.
Thus, there exists some $M > 0$, such that for all $n \in \mathbb{N}$, one has

$$\left|\frac{1}{a_n a_{n+1}}\right| \leq M. \tag{2}$$

By combining inequalities (1) and (2), it follows that for all $n \in \mathbb{N}$, one has

$$m \leq \left|\frac{1}{a_n a_{n+1}}\right| \leq M$$

$$\Rightarrow m \sum_{n=1}^{\infty} |a_{n+1} - a_n| \leq \sum_{n=1}^{\infty} \frac{|a_{n+1} - a_n|}{|a_n a_{n+1}|} = \sum_{n=1}^{\infty} \left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right| \leq M \sum_{n=1}^{\infty} |a_{n+1} - a_n|$$

Using the Comparison Test, we have:

$$\sum_{n=1}^{\infty} \left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right| \text{converges} \Rightarrow m \sum_{n=1}^{\infty} |a_{n+1} - a_n| \text{ converges} \Rightarrow \sum_{n=1}^{\infty} |a_{n+1} - a_n| \text{ converges},$$

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| \text{ converges} \Rightarrow M \sum_{n=1}^{\infty} |a_{n+1} - a_n| \text{ converges} \Rightarrow \sum_{n=1}^{\infty} \left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right| \text{converges.}$$

Thus, we have $\sum\limits_{n=1}^{\infty} \left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right|$ converges $\Leftrightarrow \sum\limits_{n=1}^{\infty} |a_{n+1} - a_n|$ converges.
Likewise, by a similar argument as above, we can also conclude that $\sum\limits_{n=1}^{\infty} \left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right|$ diverges $\Leftrightarrow$ $\sum\limits_{n=1}^{\infty} |a_{n+1} - a_n|$ diverges.
So we conclude that the two series either both converge or both diverge.

Question 3

(a) Let $\varepsilon > 0$ be given. Choose $\delta = \min\left\{\frac{1}{4}, \frac{\varepsilon}{24}\right\}$.
Then it follows that if $\left|x - \frac{1}{2}\right| < \delta$, then we must have

$$\left|x - \frac{1}{2}\right| < \delta \leq \frac{1}{4}, \tag{3}$$

$$\left|x - \frac{1}{2}\right| < \delta \leq \frac{\varepsilon}{24}. \tag{4}$$

Using the triangle inequality and inequality (3), we get

$$|x| \leq \left|x - \frac{1}{2}\right| + \left|\frac{1}{2}\right| < \frac{1}{4} + \frac{1}{2} = \frac{3}{4}, \tag{5}$$

$$\begin{aligned} |1-x| &= \left|\frac{1}{2} - \left(x - \frac{1}{2}\right)\right| \\ &\geq \left|\left|\frac{1}{2}\right| - \left|x - \frac{1}{2}\right|\right| \\ &= \frac{1}{2} - \left|x - \frac{1}{2}\right| \quad \left(\because \frac{1}{2} > \frac{1}{4} > \left|x - \frac{1}{2}\right|\right) \\ &> \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \\ &\Rightarrow \frac{1}{|1-x|} < 4. \end{aligned} \tag{6}$$

Therefore, using inequalities (4), (5) and (6), one has

$$\begin{aligned} \left|\frac{2-3x}{(x-1)^2} - 2\right| &= \left|\frac{x(1-2x)}{(1-x)^2}\right| \\ &= 2|x|\left|x - \frac{1}{2}\right|\left(\frac{1}{|1-x|}\right)^2 \\ &< 2 \cdot \frac{3}{4} \cdot \frac{\varepsilon}{24} \cdot 4^2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows from the $\varepsilon - \delta$ definition that $\lim\limits_{x \to \frac{1}{2}} \frac{2-3x}{(x-1)^2} = 2$.

(b) (i) Write $f(x) = \sin^2\left(\frac{1}{x+1}\right)$, and let $x_n = -1 + \frac{2}{n\pi}$ for all $n \in \mathbb{N}$.
Then it is clear that $x_n \neq -1$ for all $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} x_n = -1$ and $f(x_n) = \sin^2\left(\frac{n\pi}{2}\right)$. From here, we get that for all $k \in \mathbb{N}$,

$$\begin{aligned} f(x_{2k}) &= \sin^2(k\pi) = 0 \\ \Rightarrow \lim_{k\to\infty} f(x_{2k}) &= 0, \\ f(x_{2k-1}) &= \sin^2\left(\frac{(2k-1)\pi}{2}\right) = 1 \\ \Rightarrow \lim_{k\to\infty} f(x_{2k-1}) &= 1. \end{aligned}$$

Consequently, $(f(x_n))$ diverges so the limit $\lim\limits_{x\to-1} f(x)$ does not exist.

(ii) Without loss of generality, we may assume $x < 1$. Clearly, we have $f(x) > 0$ for all $x \in (0, 1)$. Also, we note that if $f(x) = \frac{1}{n^2}$, where $n \in \mathbb{N}$, then we must have

$$\begin{aligned} &\frac{1}{n+1} < x \leq \frac{1}{n} \\ \Rightarrow\ & 1 \leq n \leq \frac{1}{x} < n+1 \\ \Rightarrow\ & 0 < \frac{1}{x} - 1 < n \\ \Rightarrow\ & \frac{1-x}{x} < n \\ \Rightarrow\ & \frac{x^2}{(1-x)^2} > \frac{1}{n^2} = f(x). \end{aligned}$$

Thus, for all $x \in (0,1)$, we have $0 < f(x) < \frac{x^2}{(1-x)^2}$.
Since $\lim_{x\to 0^+} 0 = 0$ and $\lim_{x\to 0^+} \frac{x^2}{(1-x)^2} = 0$, it follows from Squeeze Theorem that $\lim_{x\to 0^+} f(x) = 0$.

(iii) Without loss of generality, we may assume that $\sin x < 1$.
Let $\left[\frac{1}{\sin x}\right] = n$, where $n \in \mathbb{N}$. Then, it follows that

$$n \le \frac{1}{\sin x} < n+1 \tag{7}$$

$$\Rightarrow \frac{1}{\sin x} - 1 < n$$

$$\Rightarrow \frac{\sin x}{1-\sin x} > \frac{1}{n} \tag{8}$$

From inequality (7), we get

$$\frac{1}{n+1} < \sin x \le \frac{1}{n}, \tag{9}$$

$$n \le \left[\frac{1}{\sin x}\right] < n+1, \tag{10}$$

$$2n \le \frac{2}{\sin x} < 2n+2$$

$$\Rightarrow 2n \le \left[\frac{2}{\sin x}\right] < 2n+2. \tag{11}$$

From inequalities (10) and (11), we get

$$3n \le \left[\frac{1}{\sin x}\right] + \left[\frac{2}{\sin x}\right] \le 3n+3 \tag{12}$$

Combining inequalities (9) and (12), we get

$$\begin{aligned}
&\frac{3n}{n+1} < (\sin x)\left(\left[\frac{1}{\sin x}\right] + \left[\frac{2}{\sin x}\right]\right) < \frac{3n+3}{n} \\
\Rightarrow\quad & 3 - \frac{3}{n+1} < (\sin x)\left(\left[\frac{1}{\sin x}\right] + \left[\frac{2}{\sin x}\right]\right) < 3 + \frac{3}{n} \\
\Rightarrow\quad & 3 - 3\sin x < (\sin x)\left(\left[\frac{1}{\sin x}\right] + \left[\frac{2}{\sin x}\right]\right) < 3 + \frac{3\sin x}{1-\sin x}
\end{aligned}$$

(The last inequality follows from inequalities (8) and (9).)
Since $\lim_{x\to 0^+} (3 - 3\sin x) = 3$ and $\lim_{x\to 0^+} \left(3 + \frac{3\sin x}{1-\sin x}\right) = 3$, it follows from Squeeze Theorem that $\lim_{x\to 0^+} (\sin x)\left(\left[\frac{1}{\sin x}\right] + \left[\frac{2}{\sin x}\right]\right) = 3$.

Question 4

Firstly, we note that $c = c^2$ if and only if $c = 0$ or $c = 1$. Take a rational sequence (x_n) and an irrational sequence (y_n) such that $\lim_{n\to\infty} x_n = c = \lim_{n\to\infty} y_n$.
Then it follows that $\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} x_n = c$, and $\lim_{n\to\infty} h(y_n) = \lim_{n\to\infty} y_n^2 = c^2$.
Thus, if $c \ne 0$ and $c \ne 1$, then $c \ne c^2$, so $\lim_{n\to\infty} h(x_n) \ne \lim_{n\to\infty} h(y_n)$, which would imply that $\lim_{x\to c} h(x)$ does not exist.
Consequently, h is not continuous at $x = c$ with $c \ne 0$ and $c \ne 1$, so it remains to show that h is continuous at $x = 0$ and $x = 1$.

For $x = 0$, let $\varepsilon > 0$ be given. Choose $\delta = \min\{\varepsilon, \sqrt{\varepsilon}\}$.
Then, it follows that if $|x - 0| = |x| < \delta$, then we must have $|x| < \delta \le \varepsilon$ and $|x| < \delta \le \sqrt{\varepsilon}$.
For $x \in \mathbb{Q}$, we have $|h(x) - h(0)| = |x| < \varepsilon$.
For $x \notin \mathbb{Q}$, we have $|h(x) - h(0)| = \left|x^2\right| = |x|^2 < (\sqrt{\varepsilon})^2 = \varepsilon$.
So we have $|h(x) - h(0)| < \varepsilon$ for all $x \in \mathbb{R}$. This shows that $\lim_{x\to 0} h(x) = h(0) = 0$ so h is continuous at $x = 0$.

For $x = 1$, let $\varepsilon > 0$ be given. Choose $\delta = \min\left\{1, \frac{\varepsilon}{3}\right\}$.
Then, it follows that if $|x - 1| < \delta$, then we must have $|x - 1| < \delta \le 1$, $|x - 1| < \delta \le \frac{\varepsilon}{3}$ and

$$|x + 1| = |x - 1 + 2| \le |x - 1| + 2 < 3.$$

For $x \in \mathbb{Q}$, we have $|h(x) - h(1)| = |x - 1| < \frac{\varepsilon}{3} < \varepsilon$.
For $x \notin \mathbb{Q}$, we have $|h(x) - h(1)| = \left|x^2 - 1\right| = |x + 1||x - 1| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$.
So we have $|h(x) - h(1)| < \varepsilon$ for all $x \in \mathbb{R}$. This shows that $\lim_{x\to 1} h(1) = h(1) = 1$ so h is continuous at $x = 1$.

So we conclude that h is continuous only at the points $x = 0$ and $x = 1$.

Question 5

(i) Take $f(x) = \frac{1}{x-1}$. Then it is clear that f is continuous on $(1, 2)$, but $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x-1} = \infty$, so f is not bounded on $(1, 2)$.

(ii) Since g is uniformly continuous on $(1, 2)$, it follows that $\lim_{x\to 1^+} g(x)$ and $\lim_{x\to 2^-} g(x)$ exist.
Thus, we may define $g(1) = \lim_{x\to 1^+} g(x)$ and $g(2) = \lim_{x\to 2^-} g(x)$ so that g is continuous on the closed interval $[1, 2]$. Hence it must be bounded on $[1, 2]$ so g is bounded on $(1, 2)$ as required.

Question 6

(a) Firstly, let $\varepsilon > 0$ be given, and let $\lim_{k\to\infty} a_{n_k} = a = \lim_{k\to\infty} a_{m_k}$.
Then, there exist N_1, $N_2 \in \mathbb{N}$, such that for all $n_k \ge n_{N_1}$ and $m_k \ge m_{N_2}$, one has $|a_{n_k} - a| < \varepsilon$ and $|a_{m_k} - a| < \varepsilon$. Let $N = \max\{n_{N_1}, m_{N_2}\}$.

Next, from the condition $\{n_k : k \in \mathbb{N}\} \cup \{m_k : k \in \mathbb{N}\} = \mathbb{N}$, we get $\mathbb{N} \subseteq \{n_k : k \in \mathbb{N}\} \cup \{m_k : k \in \mathbb{N}\}$.
This implies that for all $n \in \mathbb{N}$, we must have either $n \in \{n_k : k \in \mathbb{N}\}$ or $n \in \{m_k : k \in \mathbb{N}\}$.

Consider all $n \ge N$.
If $n \in \{n_k : k \in \mathbb{N}\}$, then we must have $n \ge N \ge n_{N_1}$, so we have $|a_n - a| < \varepsilon$, by virtue of the fact that $|a_{n_k} - a| < \varepsilon$ for all $n_k \ge n_{N_1}$.
Otherwise, if $n \in \{m_k : k \in \mathbb{N}\}$, then we must have $n \ge N \ge m_{N_2}$, so we have $|a_n - a| < \varepsilon$ as well, by virtue of the fact that $|a_{m_k} - a| < \varepsilon$ for all $m_k \ge m_{N_2}$.
Thus, we have $|a_n - a| < \varepsilon$ for all $n \ge N$.
Since $\varepsilon > 0$ is arbitrary, this shows that $\lim_{n\to\infty} a_n = a$ so (a_n) converges as desired.

(b) We shall prove that f is continuous at $x = c$. Pick any sequence (c_n) in $\mathbb{R}$ such that $\lim_{n\to\infty} c_n = c$.

Consider two subsequences of (c_n), (c_{n_k}) and (c_{m_k}), where $c_{n_k} \in \mathbb{Q}$ for all $k \in \mathbb{N}$, $c_{m_k} \notin \mathbb{Q}$ for all $k \in \mathbb{N}$, and each term c_n in (c_n) appears as a term in (c_{n_k}) or in (c_{m_k}), i.e. either $n = n_p$ for some $p \in \mathbb{N}$ or $n = m_q$ for some $q \in \mathbb{N}$. (This is possible because for each $n \in \mathbb{N}$, we have either

$c_n \in \mathbb{Q}$ or $c_n \notin \mathbb{Q}$.)
Note that $\lim_{k\to\infty} c_{n_k} = \lim_{k\to\infty} c_{m_k} = c$. Thus, from the condition given in part (b), we have $\lim_{k\to\infty} f(c_{n_k}) = \lim_{k\to\infty} f(c_{m_k}) = f(c)$.

Also, we note that since each term c_n in (c_n) appears as a term in (c_{n_k}) or in (c_{m_k}), it follows that for all $n \in \mathbb{N}$, we must have either $n \in \{n_k : k \in \mathbb{N}\}$, or $n \in \{m_k : k \in \mathbb{N}\}$.

Thus, by a similar argument in part (a), we have $\lim_{n\to\infty} f(c_n) = f(c)$.
So f is continuous at $x = c$ by the Sequential Criterion for Continuity.

Question 7

(i) Since (x_n) is bounded, there exists some $M > 0$, such that $|x_n| \leq M$ for all $n \in \mathbb{N}$, or equivalently, $-M \leq x_n \leq M$ for all $n \in \mathbb{N}$. As f is continuous on $\mathbb{R}$, it must be continuous on $[-M, M]$, so f is bounded on $[-M, M]$. Consequently, the sequence $(f(x_n))$ must be bounded.

(ii) To prove that $f(M) = \limsup f(x_n)$, we need to show that for a given $\varepsilon > 0$,
(1) There exists some $N \in \mathbb{N}$, such that for all $n \geq N$, one has $f(x_n) < f(M) + \varepsilon$, and
(2) There are infinitely many n's such that $f(x_n) > f(M) - \varepsilon$.

For (1), let $\varepsilon > 0$ be given. By the continuity of f, there exists some $\delta > 0$, such that if $|x - M| < \delta$, then $|f(x) - f(M)| < \varepsilon$.
Equivalently, if $M - \delta < x < M + \delta$, then necessarily one has $f(M) - \varepsilon < f(x) < f(M) + \varepsilon$.
Since f is increasing, the above equations would further imply that if $x < M + \delta$, then $f(x) < f(M) + \varepsilon$.

Since $M = \limsup x_n$, by definition it follows that there exists some $N_\delta \in \mathbb{N}$, such that for all $n \geq N_\delta$, one has $x_n < M + \delta$. This would then imply that $f(x_n) < f(M) + \varepsilon$ for all $n \geq N_\delta$. Thus, condition (1) is proven as desired.

By a similar argument as above, we may also prove condition (2) as well. So we conclude that $f(M) = \limsup f(x_n)$.

(iii) Without the assumption that f is increasing, the desired conclusion may not necessarily hold.
Take $f(x) = x^2$, and construct a sequence (x_n) with $x_{2k-1} = -2$ and $x_{2k} = 1$ for all $k \in \mathbb{N}$.
Then it is clear that $M = \limsup x_n = 1$ and $f(M) = 1$.
However, we have $f(x_{2k-1}) = 4$ and $f(x_{2k}) = 1$ for all $k \in \mathbb{N}$, so we have $\limsup f(x_n) = 4$.
Thus in this case, we have $f(M) \neq \limsup f(x_n)$, so the conclusion does not hold in general.

Question 8

(a) Let $f(x_i) = \min\{f(x_1), f(x_2), \cdots, f(x_n)\}$, $f(x_j) = \max\{f(x_1), f(x_2), \cdots, f(x_n)\}$, and without loss of generality let's assume that $x_i \leq x_j$.
Then it is clear that for $1 \leq k \leq n$, $k \in \mathbb{N}$, one has

$$\begin{aligned} & f(x_i) \leq f(x_k) \leq f(x_j) \\ \Rightarrow\ & a_k f(x_i) \leq a_k f(x_k) \leq a_k f(x_j) \\ \Rightarrow\ & f(x_i) \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} a_k f(x_k) \leq f(x_j) \sum_{k=1}^{n} a_k \\ \Rightarrow\ & f(x_i) \leq \frac{\sum_{k=1}^{n} a_k f(x_k)}{\sum_{k=1}^{n} a_k} \leq f(x_j). \end{aligned}$$

As f is continuous on $[x_i, x_j]$, it follows from the Intermediate Value Theorem that there exists some $c \in [x_i, x_j] \subseteq \mathbb{R}$, such that $f(c) = \frac{\sum_{k=1}^{n} a_k f(x_k)}{\sum_{k=1}^{n} a_k}$, and we are done.

(b) Note that a sequence is convergent if and only if it is Cauchy. We shall first show that g is continuous on $(0,1)$.
Fix a $c \in (0,1)$. We shall show that for any sequence (x_n) in $(0,1)$ such that $\lim_{n\to\infty} x_n = c$, one has $\lim_{n\to\infty} g(x_n) = g(c)$.

Suppose that this is not the case. Then there exists some Cauchy sequence (x_n) in $(0,1)$ such that $\lim_{n\to\infty} g(x_n) \neq g(c)$. Let $\lim_{n\to\infty} g(x_n) = k$, where we note that $g(c) \neq k$.
Construct a new sequence (y_n) where $y_{2n} = x_n$ and $y_{2n-1} = c$ for all $n \in \mathbb{N}$.
Since $\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} x_n = c$ and $\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} c = c$, it follows that (y_n) must converge, and $\lim_{n\to\infty} y_n = c$. It follows then that $(g(y_n))$ must be Cauchy as well.

However, we note that $\lim_{n\to\infty} g(y_{2n}) = \lim_{n\to\infty} g(x_n) = k$, and $\lim_{n\to\infty} g(y_{2n-1}) = \lim_{n\to\infty} g(c) = g(c)$.
This implies that $\lim_{n\to\infty} g(y_{2n}) \neq \lim_{n\to\infty} g(y_{2n-1})$, so the sequence $(g(y_n))$ does not converge, which contradicts the fact that $(g(y_n))$ is Cauchy.
Thus, for any sequence (x_n) in $(0,1)$ such that $\lim_{n\to\infty} x_n = c$, one has $\lim_{n\to\infty} g(x_n) = g(c)$. Hence, by the Sequential Criterion for Continuity, g is continuous at $x = c$. As c is arbitrary, g must be continuous on $(0,1)$.

By a similar argument as above, we can also show that $\lim_{x\to 0^+} g(x)$ and $\lim_{x\to 1^-} g(x)$ exist.

Since $\lim_{x\to 0^+} g(x)$ and $\lim_{x\to 1^-} g(x)$ exist, we may define $g(0) = \lim_{x\to 0^+} g(x)$ and $g(1) = \lim_{x\to 1^-} g(x)$ so that the extended function g is continuous on the closed interval $[0,1]$. Thus, it follows that g must be uniformly continuous on $[0,1]$, so it must be uniformly continuous on $(0,1)$ as desired.