NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA2101 Linear Algebra II
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Question 1

We now rewrite W_1 in terms of vector so that it is easier for us to check the linear independency of the basis of W_1.

$$W_1 = \text{span}\{1+x^2, 1+2x^2, 1+3x^2\} = \text{span}\{v_1, v_2, v_3\} = \text{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}\}$$

$$v_2 = \frac{1}{2}v_1 + \frac{1}{3}v_3 \quad \& \quad \text{rank}\begin{pmatrix} v_1 & v_3 \end{pmatrix} = 2$$

Then we have $\dim W_1 = 2$.

Next, note that $W_2 \neq \emptyset$ and

$$W_2 = \text{span}\{1+ix^3\} = \text{span}\{u_1\} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}\right\}$$

Hence we have $\dim W_2 = 1$.

In order to find $\dim W_1 \cap W_2$, we want to study the vectors which are both in W_1 and W_2, notice that u_1 is a linear combinations of v_1 and v_3, ie.

$$\left(\frac{3}{2} - \frac{1}{2}i\right)v_1 + \left(-\frac{1}{2} + \frac{1}{2}i\right)v_3 = u_1$$

Therefore we know that $W_2 \subset W_1$, hence $W_1 \cap W_2 = W_2$, then $\dim W_1 \cap W_2 = \dim W_2 = 1$.

Next, we have, by a theorem in the lecture notes,

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$$

Therefore by substituting all known arguments, we have

$$2 + 1 = 1 + \dim(W_1 + W_2) \Rightarrow \dim(W_1 + W_2) = 2$$

Let $V = \text{span}\{v_1,\ v_3,\ w_1,\ w_2\}$.

$$\frac{V}{W_1} = \{x + y \mid x \in V,\ y \in W_1\} = \text{span}\{w_1 + y,\ w_2 + y,\} \quad \text{where } y \in W_1$$

Question 2

(a) Since $X \in M_{22}(\mathbb{R})$, we treat X as a vector in $\mathbb{R}^4$, with

$$X = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \mapsto \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = X$$

Therefore B is now

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\} = \{b_1,\ b_2,\ b_3,\ b4\}$$

$$T(X) = \begin{pmatrix} p-r & q-s \\ p-r & q-s \end{pmatrix} \mapsto \begin{pmatrix} p-r \\ q-s \\ p-r \\ q-s \end{pmatrix}$$

We now can find $[T]_{\{e_1,e_2,e_3,e_4\}}$

$$[T]_{\{e_1,e_2,e_3,e_4\}} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$Tb_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = 0b_1 + b_2 + 0b_3 - b_4 \quad Tb_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = b_1 + 0b_2 + b_3 + 0b_4$$

$$Tb_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = 0b_1 + b_2 + 0b_3 - b_4 \quad Tb_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = b_1 + 0b_2 + b_3 + 0b_4$$

Then

$$[T]_B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

(b)

$$\text{rank } T = \text{rank } [T]_B = 2$$

$$\text{null } T = \dim(M_{22}(\mathbb{R})) - \text{rank } T = 4 - 2 = 2$$

(c) No. This is because null $T \neq 0$, hence T is not an injective linear transformation, and thus not an isomorphism.

Question 3

(a) *Proof.* To show that W is indeed a subspace of $\mathbb{C}^4$, we need to verify that $\forall w_1,\ w_2 \in W$, $\alpha w_1 \in W$ and $w_1 + w_2 \in W$ for all $\alpha \in \mathbb{C}$.

Consider

$$\begin{aligned} w_1 &= (a_1, a_1 - b_1 i, a_1 + 2b_1 i, a_1 + 3b_1 i), \quad \text{and} \\ w_2 &= (a_2, a_2 - b_2 i, a_2 + 2b_2 i, a_2 + 3b_2 i) \end{aligned}$$

Then for all $\alpha_1, \alpha_2 \in \mathbb{C}$,

$$\begin{aligned} &\alpha_1 w_1 + \alpha_2 w_2 \\ &= (a_3, a_3 - b_3 i, a_3 + 2b_3 i, a_3 + 3b_3 i) \\ &\in W \end{aligned}$$

where $a_3 = \alpha_1 a_1 + \alpha_2 a_2 \in \mathbb{C}$ and $b_3 = \alpha_1 b_1 + \alpha_2 b_2 \in \mathbb{C}$. Therefore W is a subspace of $\mathbb{C}^4$.

$\square$

(b) Note that $W = \text{span}\{(1,1,1,1), (0,-i,2i,3i)\}$. Hence, applying Gram-Schmidt process to $S = \{(1,1,1,1), (0,-i,2i,3i)\}$ gives us

$$\begin{aligned} w_1 &= (1,1,1,1) \\ w_2 &= (0,-i,2i,3i) - \frac{4i}{4}(1,1,1,1) \\ &= (-i,-2i,i,2i) \end{aligned}$$

Normalizing the vectors gives us

$$\begin{aligned} w_1' &= \frac{1}{2}(1,1,1,1) \\ w_2' &= \frac{1}{\sqrt{10}}(-i,-2i,i,2i) \end{aligned}$$

Then $S' = \{w_1', w_2'\}$ is an orthonormal basis for W.

(c) Since W is a proper subspace of $\mathbb{C}^4$, there exists a proper subspace of $\mathbb{C}^4$, V such that

$$\mathbb{C}^4 = V \oplus W$$

We know that there exists $v_1, v_2 \in V$ such that

$$V = \text{span}\{v_1,\ v_2\}$$

since V is a vector space.
Now consider any $x \in \mathbb{C}^4$, x could be linearly expressed as

$$x = c_1 v_1 + c_2 v_2 + c_3 w_1' + c_4 w_2' \quad c_1, c_2, c_3, c_4 \in \mathbb{C}$$

then we have the projection formula

$$\text{Proj}_W(x) = \langle w_1', x \rangle\, w_1' + \langle w_2', x \rangle\, w_2' = c_3 w_1' + c_4 w_2'$$

Question 4

(a) Let $\mathbf{u} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}$ and $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Now evaluating $\mathbf{u}A\mathbf{u}^T$ gives us a 1×1 matrix with its entry as

$$a_{11}u_1^2 + a_{22}u_2^2 + a_{33}u_3^2 + (a_{12} + a_{21})u_1u_2 + (a_{13} + a_{31})u_1u_3 + (a_{23} + a_{32})u_2u_3$$

Now comparing this with the expanded expression

$$(u_1 + u_2 + u_3)^2 = u_1^2 + 2u_1u_2 + u_2^2 + 2u_1u_3 + 2u_2u_3 + u_3^2$$

Solving a_{ij} for $i, j = 1, 2, 3$,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(b) We need to find the eigenvalues and their corresponding eigenvectors. There are two methods in finding eigenvalues and their corresponding eigenvectors for this question.

Method I: Computing the eigenvalues

Let λ be the eigenvalues of A, then we have

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = \lambda^2(3 - \lambda) = 0$$

Hence we have $\lambda = 0, 3$. We now find eigenvectors, we have

$$\begin{pmatrix} x+y+z \\ x+y+z \\ x+y+z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow x = y = z = 1$$

$$\begin{pmatrix} x+y+z \\ x+y+z \\ x+y+z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The RREF form of A will be

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x = -y, x = -z$$

Then eigenvectors of A corresponding to 0 is

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \& \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Method II: By Observations

By observations, we know that each entries of A are the same, ie.

$$\begin{pmatrix} x+y+z \\ x+y+z \\ x+y+z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

We try some guesses by substituting $x = 1$, $y = -1$ and $z = 0$

$$A \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

and $x = 1$, $y = 0$ and $z = -1$

$$A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

We can now obtain the third orthogonal vector by considering the cross product of the first two vectors that we obtained

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Hence we obtain a set of orthogonal vectors for P. Now to find an orthogonal matrix P, we want to normalize the three eigenvectors. So normalizing the vectors gives us

$$\begin{aligned} \mathbf{w}_1 &= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^T \\ \mathbf{w}_2 &= \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)^T \\ \mathbf{w}_3 &= \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)^T \end{aligned}$$

Now we just take $P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$ and we have our result.

(c) By direct computation, we have

$$\begin{aligned} Q(\mathbf{u}) &= Q(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3) \qquad \text{(since } Q \text{ is a linear transformation)} \\ &= Q(a_1\mathbf{v}_1) + Q(a_2\mathbf{v}_2) + Q(a_3\mathbf{v}_3) \\ &= (a_1\mathbf{v}_1)^T A(a_1\mathbf{v}_1) + (a_2\mathbf{v}_2)^T A(a_2\mathbf{v}_1) + (a_3\mathbf{v}_3)^T A(a_1\mathbf{v}_3) \\ &= a_1^2(\mathbf{v}_1^T A\mathbf{v}_1) + a_2^2(\mathbf{v}_2^T A\mathbf{v}_2) + a_3^2(\mathbf{v}_3^T A\mathbf{v}_3) \end{aligned}$$

Now let $\mathbf{v}_1 = \mathbf{w}_1$, $\mathbf{v}_2 = \mathbf{w}_2$ and $\mathbf{v}_3 = \mathbf{w}_3$ as defined in (b). Then

$$\begin{aligned} Q(\mathbf{u}) &= a_1^2(\mathbf{v}_1^T A\mathbf{v}_1) + a_2^2(\mathbf{v}_2^T A\mathbf{v}_2) + a_3^2(\mathbf{v}_3^T A\mathbf{v}_3) \\ &= \lambda_1 a_1^2 + \lambda_2 a_2^2 + \lambda_3 a_3^2 \end{aligned}$$

where $\lambda_1 = 3$ and $\lambda_2 = 0 = \lambda_3$.

Question 5

(a) Since $V = W_1 \oplus W_2$, then $\forall v \in V$, v can be uniquely expressed as $v = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$.

In order to prove that P is a linear transformation, we might want to ask ourselves what does P send v to. From $v - P(v) \in W_2$, we know that $v - P(v) = w_2' \in W_2$. We claim that w_2' is indeed w_2. Since $P(v) \in W_1$, we have $P(v) = w_1'$. Then

$$v - P(v) = w_1 + w_2 - w_1' = (w_1 - w_1') + w_2 = w_2'$$

In order for the above to be true, $w_1 - w_1'$ must be in W_2. However w_1, w_1' are both in W_1, which does not contain any element of W_2. Thus w_1 must be equal to w_1' and then $w_2' = w_2$. Then we have a rough idea of how P transforms v

$$P(v) = P(w_1 + w_2) = w_1$$

In order to prove P is a linear transformations, we would like to show that $\forall v_1, v_2 \in V$, $P(\alpha v_1) = \alpha P(v_1)$ and $P(v_1 + v_2) = P(v_1) + P(v_2)$ for $\alpha \in F$. Let $v_1 = w_{11} + w_{21}$ and $v_2 = w_{12} + w_{22}$,

$$P(\alpha v_1) = P(\alpha(w_{11} + w_{21})) = \alpha w_{11} = \alpha P(v_1)$$

$$P(v_1+v_2) = P(w_{11}+w_{12}+w_{21}+w_{22}) = P((w_{11}+w_{12})+(w_{21}++w_{22})) = w_{11}+w_{21} = P(v_1)+P(v_2)$$

Hence P is linear transformations.

(b) As deduced from part (a), for $v \in W_1$ and $w \in W_2$, expressed $v = v + 0$ and $w = 0 + w$,

$$P(v) = P(v + 0) = v \quad \& P(w) = P(0 + w) = 0$$

(c) (i) Given W_1 and W_2 are both T-invariant, this implies that

$$T(W_1) \subseteq W_1 \quad \& \quad T(W_2) \subseteq W_2$$

To prove that $T \circ P = P \circ T$, we need to make sure they both map $v \in V$ into the same element in V. Note that the domain and codomain of both functions are the same, which is V. Let $v = w_1 + w_2$, then by invariant properties of T, for $w_1 \in W_1$ and $w_2 \in W_2$,

$$T(w_1) = w_1' \in W_1 \quad \& \quad T(w_2) = w_2' \in W_2$$

Then we have

$$(P \circ T)(v) = (P \circ T)(w_1 + w_2) = P(w_1' + w_2') = w_1' = T(w_1) = (T \circ P)(w_1 + w_2) = (T \circ P)(v)$$

as desired.

(ii) No. For example, consider T which has the following transformation with $w_1 \in W_1$ and $w_2 \in W_2$

$$T(w_1) = w_1' \in W_1 \quad \& \quad T(w_2) = w_2' \notin W_2 \quad \text{ie. } w_2' \in W_1$$

Then

$$(P \circ T)(v) = (P \circ T)(w_1 + w_2) = P(w_1' + w_2') = w_1' + w_2' \neq w_1' = T(w_1) = (T \circ P)(w_1 + w_2) = (T \circ P)(v)$$

Question 6

(a) (i) S is not surjective. Let

$$S(p(x)) = 1$$

Then $a_0 = 1$. This forces $a_i = 1$ for all i. But this gives

$$S(p(x)) = 1 - x^{n+1}$$

Therefore there are no elements in $\mathcal{P}(\mathbb{R})$ which maps to 1.

T is injective. Consider $\ker T = \{v \in \mathcal{P}(\mathbb{R}) | T(v) = 0\}$
For $v \in \mathcal{P}$,

$$\begin{aligned} T(v) &= 0 \\ T(a_0 + a_1 x + \cdots + a_n x^n) &= 0 \\ (a_0 - a_1) + (a_1 - a_2) + \cdots + (a_{n-1} - a_n)x^{n-1} + a_n x^n &= 0 \end{aligned}$$

$$\Rightarrow a_0 = a_1 = \cdots = a_n$$

But $a_n = 0$ implies $a_i = 0$ for all i. Hence $\ker T = \{0\} \Rightarrow T$ is injective.

(ii) *Proof.* We have

$$< q, S(p) > = < \sum_{i=1}^{m} b_i x^i, a_0 + \sum_{i=1}^{n} (a_i - a_{i-1}) x^i - a_n x^{n+1} >$$

$$< T(q), p > = < \sum_{i=0}^{m-1} (b_i - b_{i+1}) x^i + b_m x^m, \sum_{i=0}^{n} a_i x^i >$$

If $n \leq m$, then

$$\begin{aligned} & a_0 b_0 + (a_1 - a_0) b_1 + \cdots + (a_n - a_{n-1}) b_n \\ =& a_0 b_0 + a_1 b_1 - a_0 b_1 + \cdots + a_n b_n - a_{n-1} b_n \\ =& a_0 (b_0 - b_1) + a_1 (b_1 - b_2) + \cdots + a_{n-i} (b_{n-1} - b_n) + a_n b_n \\ =& < T(q), p > \end{aligned}$$

If $n > m$, then

$$\begin{aligned}
&a_0b_0 - (a_1 - a_0)b_1 + \cdots + (a_m - a_{m-1})b_m \\
=&a_0(b_0 - b_1) + a_1(b_1 - b_2) + \cdots + a_{m-1}(b_{m-1} - b_m) + a_m b_m \\
=& < T(q), p >
\end{aligned}$$

Therefore in both cases, $< S(p), q > = < p, T(q) >$ which shows that S is the adjoint of T. □

(b) (i) Considering an inner product operator ϕ, let T^* be the adjoint of T, ie.

$$\forall x, y \in V, \quad < Tx, y >=< x, T^*y > \Leftrightarrow \phi(T(x), y) = \phi(x, T^*(y))$$

Let $y_1, y_2 \in V$. To prove the injectivity of T^*, we consider $T^*(y_1) = T^*(y_2)$ and we want to show that $y_1 = y_2$. Suppose $< x, T^*(y_1) >=< x, T^*(y_2) >$, then we have

$$< x, T^*(y_1) > - < x, T^*(y_2) >= 0 \Rightarrow < x, T^*(y_1) - T^*(y_2) > = 0$$

Since T^* is a linear operator, we have

$$< x, T^*(y_1) - T^*(y_2) > = 0 \Rightarrow < x, T^*(y_1 - y_2) > = 0$$

Now we have to make use of the surjectivity of T. Since T is surjective, $\forall T(x) \in V$, there exists a corresponding $x \in V$, then we have

$$< x, T^*(y_1 - y_2) > = 0 = < Tx, y_1 - y_2 >$$

Since the above is true for all $x \in V$, and $< Tx, y_1 - y_2 >= 0$, then we have $y_1 - y_2 = 0$, which gives us $y_1 = y_2$ as desired.

(ii) No. For example, consider S and T for this question. S is the adjoint of T and T is injective by part (a)(i), however S is not surjective.

Question 7

(a) (i) Let

$$(\boldsymbol{J}_t(\lambda) - \lambda I_t)^s \boldsymbol{e}_k = \begin{pmatrix} 0 & 1 & & 0 & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & 1 \\ & & & & 0 \end{pmatrix}^s \boldsymbol{e}_k = B^s \boldsymbol{e}_k$$

Essentially, $(\boldsymbol{J}_t(\lambda) - \lambda I_t)^s \boldsymbol{e}_k$ is the k-th column of the matrix of $(\boldsymbol{J}_t(\lambda) - \lambda I_t)^s$. Notice that when s is increase by 1, then the column of the matrix are shifted right by 1 unit. To illustrate this, we use the following example with $t = 4$, $s = 1, 2, 3, 4$.

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In general,

$$B^s \boldsymbol{e}_k = \boldsymbol{e}_{k-s} \quad \text{where } \mathbf{e}_j = \mathbf{0} \text{ if } j \leq 0$$

(ii) By observing on B^s, $\text{rank}(B^s) = t - s$. By Rank-Nullity Theorem, $\text{null}(B^s) = t - (t - s) = s$.

(b) (i) For $s \geq 1$, the nullity of $(\boldsymbol{J}-\lambda I_n)^s$ is equal to the sum of the nullity of each of $(\boldsymbol{J}_{t_i}(\lambda)-\lambda I_{t_i})^s$. If $s \geq t_i$, then nullity of $(\boldsymbol{J}_{t_i}(\lambda)-\lambda I_{t_i})^s = s$. If $s < t_i$, then nullity of $(\boldsymbol{J}_{t_i}(\lambda)-\lambda I_{t_i})^s = t_i$. Hence we have

$$\text{null}(\boldsymbol{J}-\lambda I_n)^s = \sum_{j=1}^{m} \text{null } \left(\boldsymbol{J}_{t_j}(\lambda)-\lambda I_{t_j}\right)^s = \sum_{j=1}^{m} \min\{s, t_j\}$$

(ii) Given $n = t_1 + \cdots + t_m$, we now rearrange $t_1, \ldots, t_m$ into $d_1, \ldots, d_m$ such that $d_1 \leq d_2 \leq \cdots \leq d_m$. Let $d_i = d_{i+1} = \cdots = d_j = s-1$ and $d_k \geq s$ for all $k \geq j+1$. Then we have

$$\mathit{nullity}((\boldsymbol{A}-\lambda I_n)^s) - \mathit{nullity}((\boldsymbol{A}-\lambda I_n)^{s-1}) = r$$

$$\begin{aligned}
& (d_1 + d_2 + \cdots + d_{i-1} + d_i + d_{i+1} + \cdots + d_j + s + \cdots + s) \\
& - (d_1 + d_2 + \cdots + d_{i-1} + d_i + d_{i+1} + \cdots + d_j + (s-1) + \cdots + (s-1)) \\
= \; & (d_1 + d_2 + \cdots + d_{i-1} + (s-1) + (s-1) + \cdots + (s-1) + s + \cdots + s) \\
& - \; (d_1 + d_2 + \cdots + d_{i-1} + (s-1) + (s-1) + \cdots + (s-1) + (s-1) + \cdots + (s-1)) \\
= \; & (m-j) \\
= \; & r
\end{aligned}$$

Hence we know that there are r elements in $\{t_1, t_2, \ldots, t_m\}$ such that their size is at least s.