MA2101 - Linear Algebra II Suggested Solutions 22/23

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Question 1

Note that we have

$$\begin{aligned} Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}(y_3 - y_1) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(y_1) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(y_2 - y_1) \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} y_3 - y_1 \\ y_1 \\ y_2 - y_1 \end{pmatrix} \\ &= P \begin{pmatrix} y_3 - y_1 \\ y_1 \\ y_2 - y_1 \end{pmatrix}. \end{aligned}$$

Likewise

$$Y' = P \begin{pmatrix} y_3' - y_1' \\ y_1' \\ y_2' - y_1' \end{pmatrix} = P \begin{pmatrix} (y_3 - y_1)' \\ y_1' \\ (y_2 - y_1)' \end{pmatrix}$$

by linearity of the derivative. Letting $z(x) = y_3(x) - y_1(x)$ and $w(x) = y_2(x) - y_1(x)$, our differential equation becomes

$$P \begin{pmatrix} z' \\ y_1' \\ w' \end{pmatrix} = AP \begin{pmatrix} z \\ y_1 \\ w \end{pmatrix}$$

whence multiplying both sides by P^{-1} gives

$$\begin{pmatrix} z' \\ y_1' \\ w' \end{pmatrix} = (P^{-1}AP) \begin{pmatrix} z \\ y_1 \\ w \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} z \\ y_1 \\ w \end{pmatrix} = \begin{pmatrix} z \\ 2y_1 + w \\ 2w \end{pmatrix}.$$

so that $z' = z$, $y_1' = 2y_1 + w$ and $w' = 2w$. From single-variable calculus, we know the first and third equations have solutions $z(x) = C_1 \exp x$ and $w = C_2 \exp 2x$ (where $C_1, C_2 \in \mathbb{R}$ are arbitrary), so the second equation becomes

$$y_1' = 2y_1' + w = 2y_1 + C_2 \exp 2x.$$

Using the given formula with $p(x) = -2, q(x) = \exp 2x$, we get $\mu(x) = \exp(-2x)$ and $\underline{y_1 = (C_2 x + C_3)\exp 2x}$ (with $C_3 \in \mathbb{R}$ arbitrary.) Finally, we calculate

$$\underline{y_2(x) = w(x) + y_1(x) = (C_2 x + (C_2 + C_3))\exp 2x},$$

and

$$\underline{y_3(x) = z(x) + y_1(x) = C_1 \exp x + (C_2 x + C_3)\exp 2x},$$

which is exactly what we wanted. $\square$

Question 2

Part (a)

We claim that AA^T is a real symmetric matrix, then by the spectral theorem AA^T is orthogonally diagonalizable as desired.

Indeed, write $A = (a_{i,j})_{i,j=1}^n$, then $A^T = (a_{j,i})_{i,j=1}^n$, and

$$AA^T = (a_{i,j})_{i,j=1}^n (a_{j,i})_{i,j=1}^n = \left(\sum_{k=1}^n a_{i,k} a_{j,k} \right)_{i,j=1}^n,$$

which is clearly symmetric. ■

Part (b)

We first calculate $AA^T = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$. Then AA^T clearly has eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (with eigenvalue 0) and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (with eigenvalue 4). Normalising both eigenvectors, we get $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$, so we can conclude by defining

$$\underline{P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}}.$$

□

Question 3

Part (a)

We simply look at all possible cases of elementary divisors of $m_A(x)$. Certainly we must have $(x-2)^3$ and $(x-8)$ as elementary divisors, so it suffices to "make up" elementary divisors with sum of degrees equal to 2.

- Case 1: $(x-2)$, $(x-2)$, the corresponding JCF is $\begin{bmatrix} 2 & & & & & 0 \\ & 2 & & & & \\ & & 2 & 1 & & \\ & & & 2 & 1 & \\ & & & & 2 & \\ 0 & & & & & 8 \end{bmatrix}$.
- Case 2: $(x-2)^2$, the corresponding JCF is $\begin{bmatrix} 2 & 1 & & & & 0 \\ & 2 & & & & \\ & & 2 & 1 & & \\ & & & 2 & 1 & \\ & & & & 2 & \\ 0 & & & & & 8 \end{bmatrix}$.
- Case 3: $(x-8)$, $(x-8)$, the corresponding JCF is $\begin{bmatrix} 2 & 1 & & & & 0 \\ & 2 & 1 & & & \\ & & 2 & & & \\ & & & 8 & & \\ & & & & 8 & \\ 0 & & & & & 8 \end{bmatrix}$.
- Case 4: $(x-2)$, $(x-8)$, the corresponding JCF is $\begin{bmatrix} 2 & & & & & 0 \\ & 2 & 1 & & & \\ & & 2 & 1 & & \\ & & & 2 & & \\ & & & & 8 & \\ 0 & & & & & 8 \end{bmatrix}$.

It is easily verified that we have enumerated all cases, up to permutation of the Jordan blocks. $\square$

Part (b)

Clearly the minimal polynomial $m_B(x)$ of B is either $(x-\lambda_1)^2(x-\lambda_2)$ or $(x-\lambda_1)(x-\lambda_2)^2$. Just like above, we need to "make up" a degree of 1 somewhere.

- Case 1: $m_B(x) = (x-\lambda_1)^2(x-\lambda_2)$.

- Case 1.1: Elementary divisors are $(x-\lambda_1), (x-\lambda_1)^2, (x-\lambda_2)$. The corresponding JCF is $\begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 2 \end{bmatrix}$.
- Case 1.2: Elementary divisors are $(x-\lambda_1)^2, (x-\lambda_2), (x-\lambda_2)$. The corresponding JCF is $\begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 2 & \\ & & & 2 \end{bmatrix}$.

- Case 2: $m_B(x) = (x-\lambda_1)(x-\lambda_2)^2$.
  - Case 2.1: Elementary divisors are $(x-\lambda_1), (x-\lambda_1), (x-\lambda_2)^2$. The corresponding JCF is $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix}$.
  - Case 2.2: Elementary divisors are $(x-\lambda_1), (x-\lambda_2), (x-\lambda_2)^2$. The corresponding JCF is $\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix}$.

Like above, it is easily checked that we have accounted for all possibilities up to permutation of the Jordan blocks. $\square$

Question 4

Part (a)

We note that $T|_{V_1} : V_1 \to W$ is a linear map, and that $\ker T|_{V_1}$ is clearly a subspace of $\ker T$ (because T kills every vector that $T|_{V_1}$ kills.) Furthermore, $\operatorname{im} T|_{V_1}$ is clearly a subspace of W_1. Then by the rank-nullity theorem

$$\begin{aligned}
\dim V_1 &= \dim \operatorname{im} T|_{V_1} + \dim \ker T|_{V_1} \\
&\leq \dim W_1 + \dim \ker T|_{V_1} \\
&\leq \dim W_1 + \dim \ker T.
\end{aligned}$$

■

Part (b)

Suppose that T is surjective and observe the inequality above becomes an equality if we have $\ker T|_{V_1} = \ker T$ and $W_1 = \operatorname{im} T|_{V_1}$. Since T is surjective every vector in W_1 is the target of some $\mathbf{v} \in V$ under the mapping T, so the blue condition trivially holds; on the other hand we already know that $\ker T|_{V_1} \subseteq \ker T$, so it suffices to check the reverse inclusion.

Let $\mathbf{v} \in \ker T$, then $T(\mathbf{v}) = 0_W \in W_1$ since W_1 is a subspace. By definition of V_1 we must have $\mathbf{v} \in V_1$, so $T|_{V_1}(\mathbf{v}) = T(\mathbf{v}) = 0_W$ by definition. It follows that $\mathbf{v} \in \ker T|_{V_1}$, which is exactly what we wanted. ■

Question 5

Part (a)

Let $\mathbf{v} \in W^\perp$. For any $\mathbf{w} \in W$, we have $\langle T(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, T^*(\mathbf{w}) \rangle$, but since W is T^*-invariant, we have $T^*(\mathbf{w}) \in W$. By definition of $W^\perp$, we thus have $\langle \mathbf{v}, T^*(\mathbf{w}) \rangle = 0$. Since $\mathbf{w} \in W$ is arbitrary, we have $T(\mathbf{v}) \in W^\perp$; and since $\mathbf{v} \in W^\perp$ is arbitrary, $W^\perp$ is indeed T-invariant. ■

Part (b)

Consider the following counterexample: Let $V = \mathbb{C}^2$ equipped with the standard inner product, and $U = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$. It is clear that $U^\perp = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$. Let $T : V \to V$ be defined by $T(\mathbf{v}) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{v}$, then U is clearly T-invariant since $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of T; but we have $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in U^\perp$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin U^\perp$, so $U^\perp$ is not T-invariant. ■

Question 6

Part (a)

We note that the representation of any $\mathbf{w}$ as a linear combination of $\mathbf{w}_1, \dots, \mathbf{w}_n$ is unique as $\{\mathbf{w}_n, \dots, \mathbf{w}_n\}$ is a basis; thus our function is well-defined.

We first verify that it is conjugate symmetric: indeed,

$$\begin{aligned}
\left(\sum_{j=1}^{n} y_j\mathbf{w}_j, \sum_{i=1}^{n} x_i\mathbf{w}_i\right) &= Y^tD\overline{X} \\
&= \left(Y^tD\overline{X}\right)^t && \text{since } Y^tD\overline{X} \in \mathbb{C} \\
&= \overline{X}^t D^t (Y^t)^t \\
&= \overline{X}^t D^t Y \\
&= \overline{\overline{\overline{X}^t D^t Y}} \\
&= \overline{\overline{\overline{X}^t}\,\overline{D^t Y}} \\
&= \overline{X^t\,\overline{D^t}\,\overline{Y}} \\
&= \overline{X^t D \overline{Y}} \\
&= \overline{\left(\sum_{i=1}^{n} x_i\mathbf{w}_i, \sum_{j=1}^{n} y_j\mathbf{w}_j\right)},
\end{aligned}$$

where $D = \overline{D^t}$ since positive definite matrices are Hermitian. To see that our function is linear in the second argument, let us define the vectors $\mathbf{v} = \sum_{i=1}^n x_i\mathbf{w}_i$, $\mathbf{w} = \sum_{i=1}^n y_i\mathbf{w}_i$, and $\mathbf{w} + \mathbf{w}' = \sum_{i=1}^n y_i'\mathbf{w}_i$ for complex x_i, y_i, y_i'. Then $\mathbf{w} + \mathbf{w}' = \sum_{i=1}^n (y_i + y_i')\mathbf{w}_i$, and by definition

$$\begin{aligned}
(\mathbf{v}, \mathbf{w} + \mathbf{w}') &= \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} D \begin{pmatrix} y_1 + y_1' \\ \vdots \\ y_n + y_n' \end{pmatrix} \\
&= \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} D \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} D \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} \\
&= (\mathbf{v}, \mathbf{w}) + (\mathbf{v}, \mathbf{w}').
\end{aligned}$$

Finally, the fact that our function is positive definite directly follows from the definition of D being positive definite: let $\mathbf{v} = \sum_{i=1}^n x_i\mathbf{w}_i$, then $(\mathbf{v}, \mathbf{v}) = X^tD\overline{X} \geq 0$ with equality if and only if every x_i is 0, which occurs precisely when $\mathbf{v} = 0_W$. ■

Part (b)

Subpart (i)

We calculate

$$\begin{aligned}
\left(\sum_{i=1}^n x_i\mathbf{v}_i, \sum_{j=1}^n y_j\mathbf{v}_j\right) &= \sum_{i=1}^n x_i \sum_{j=1}^n \overline{y_j}(\mathbf{v}_i, \mathbf{v}_j) \\
&= \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \begin{pmatrix} \sum_{j=1}^n \overline{y_j}(\mathbf{v}_1, \mathbf{v}_j) \\ \vdots \\ \sum_{j=1}^n \overline{y_j}(\mathbf{v}_n, \mathbf{v}_j) \end{pmatrix} \\
&= \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \begin{bmatrix} (\mathbf{v}_1, \mathbf{v}_1) & \dots & (\mathbf{v}_1, \mathbf{v}_n) \\ \vdots & \ddots & \vdots \\ (\mathbf{v}_n, \mathbf{v}_1) & \dots & (\mathbf{v}_n, \mathbf{v}_n) \end{bmatrix} \begin{pmatrix} \overline{y_1} \\ \vdots \\ \overline{y_n} \end{pmatrix} \\
&= X^t A \overline{Y}.
\end{aligned}$$

■

Subpart (ii)

We have

$$\begin{aligned}
A^* = \overline{A^t} &= \overline{\begin{bmatrix} (\mathbf{v}_1, \mathbf{v}_1) & \dots & (\mathbf{v}_1, \mathbf{v}_n) \\ \vdots & \ddots & \vdots \\ (\mathbf{v}_n, \mathbf{v}_1) & \dots & (\mathbf{v}_n, \mathbf{v}_n) \end{bmatrix}^t} \\
&= \begin{bmatrix} \overline{(\mathbf{v}_1, \mathbf{v}_1)} & \dots & \overline{(\mathbf{v}_n, \mathbf{v}_1)} \\ \vdots & \ddots & \vdots \\ \overline{(\mathbf{v}_1, \mathbf{v}_n)} & \dots & \overline{(\mathbf{v}_n, \mathbf{v}_n)} \end{bmatrix} \\
&= \begin{bmatrix} (\mathbf{v}_1, \mathbf{v}_1) & \dots & (\mathbf{v}_1, \mathbf{v}_n) \\ \vdots & \ddots & \vdots \\ (\mathbf{v}_n, \mathbf{v}_1) & \dots & (\mathbf{v}_n, \mathbf{v}_n) \end{bmatrix} = A
\end{aligned}$$

by conjugate symmetry, and likewise A^t is self-adjoint (because transpose and conjugate operators commute). ■

Subpart (ii)

Clearly if $X = 0$ then $X^t A \overline{X} = 0$. If $X \neq 0$, then from (b)(i) we know

$$X^t A \overline{X} = \left(\sum_{i=1}^n x_i\mathbf{v}_i, \sum_{j=1}^n x_j\mathbf{v}_j\right) > 0$$

since $\sum_{i=1}^n x_i\mathbf{v} \neq 0$ and inner products are positive definite. ■