NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

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MA4229 Approximation Theory
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SECTION A

Question 1

(a) We know that for the inner product given, an orthogonal basis for P_2 is the Chebyshev polynomials of degree up to 2, i.e. $T_0(x) = 1$, $T_1(x) = x$ and $T_2(x) = 2x^2 - 1$. We have,

$$\begin{aligned}
\langle 1, 1\rangle &= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx \\
&= \pi; \\
\langle x, x\rangle = \langle x^2, 1\rangle &= \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\,dx \\
&= \frac{\pi}{2}; \\
\langle x^2, x^2\rangle &= \int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}}\,dx \\
&= \frac{3\pi}{8}; \\
\langle 2x^2-1, 2x^2-1\rangle &= \langle 2x^2, 2x^2\rangle - 2\langle 2x^2, 1\rangle + \langle 1, 1\rangle \\
&= 4\left(\frac{3\pi}{8}\right) - 4\left(\frac{\pi}{2}\right) + \pi \\
&= \frac{\pi}{2}.
\end{aligned}$$

Hence, after normalization, we get an orthonormal basis for P_2 to be $\left\{\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}}x, \sqrt{\frac{2}{\pi}}(2x^2-1)\right\}$.

(b) Recall that $p_2^* \in P_2$ is the least squares approximation to f iff $f - p_2^* \perp P_2$, i.e. $f - p_2^* \perp p$ for all $p \in P_2$.

Note that the Chebyshev polynomials are orthogonal to each other, hence the polynomial of degree 4, $T_4(x) = 8x^4 - 8x^2 + 1$, is orthogonal to the polynomials in the basis found in (a), which implies that $T_4 \perp P_2$. Since $\frac{1}{8}T_4(x) = \frac{1}{8}(8x^4 - 8x^2 + 1) = x^4 - \left(x^2 - \frac{1}{8}\right) = f(x) - \left(x^2 - \frac{1}{8}\right)$, we conclude that $p_2^*(x) = x^2 - \frac{1}{8}$ for all $x \in \mathbb{R}$.

(c) For all $n \in \mathbb{Z}^+$, let $q_n : \mathbb{R} \to \mathbb{R}$ be a function such that for all $x \in \mathbb{R}$, we have $q_n(x) = p_n^*(-x)$. Notice that we have $q_n \in P_n$. Also, for all $x \in \mathbb{R}$, we have $f(x) = f(-x)$. This give us,

$$\begin{aligned}
\|f - q_n\|_2^2 = \int_{-1}^{1} \frac{(f(x) - q_n(x))^2}{\sqrt{1-x^2}}\,dx &= \int_{-1}^{1} \frac{(f(-x) - q_n(-x))^2}{\sqrt{1-(-x)^2}}\,dx \\
&= \int_{-1}^{1} \frac{(f(x) - p_n^*(x))^2}{\sqrt{1-x^2}}\,dx \\
&= \|f - p_n^*\|_2^2.
\end{aligned}$$

Thus $\|f - q_n\|_2 = \min_{p \in P_n} \|f - p\|_2$, and so q_n is also a least squares approximation to f.
Since P_n is a Hilbert space, it has strictly convex norm, and so the least approximation is unique, i.e. $q_n = p_n^*$. This give us $p_n^*(-x) = p_n^*(x)$ for all $x \in \mathbb{R}$, i.e. p_n^* is an even function.

Question 2

(a) For $n \in \mathbb{Z}^+$, let us define $p_n \in P_n$ be such that $p_n(x) = \sum_{k=0}^{n} f(x_k)\ell_k(x)$ for all $x \in \mathbb{R}$.

Note that for $k, i \in \{0, 1, \ldots, n\}$, we have,

$$\ell_k(x_i) = \begin{cases} 1 & \text{if } k = i; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have $p_n(x_i) = \sum_{k=0}^{n} f(x_k)\ell_k(x_i) = f(x_i)\ell_i(x_i) = f(x_i)$, i.e. such polynomial exists.

Next assume that $g_n \in P_n$ is such that $f(x_i) = g_n(x_i)$ for all $i \in \{0, 1, \ldots, n\}$.
This give us $p_n - g_n \in P_n$ such that $p_n(x_i) - g_n(x_i) = 0$ for all $i \in \{0, 1, \ldots, n\}$.
This implies that $p_n - g_n$ is a polynomial with degree at most n but with $n + 1$ roots, and so $p_n - g_n = 0$, i.e. $p_n = g_n$. Therefore, p_n is unique.

(b) Let $f : [a, b] \to \mathbb{R}$ be such that $f(x) = x^{n+1} - \prod_{i=0}^{n}(x - x_i)$ for all $x \in [a, b]$.
We notice that f is a polynomial of degree at most $n + 1$, but at the same time, the coefficient of x^{n+1} is 0, which implies that $f \in P_n$.
Thus f is a polynomial such that $f(x_i) = x_i^{n+1}$ for all $i \in \{0, 1, \ldots, n\}$.

For all $i \in \{0, 1, \ldots, n\}$, we have $f(x_i) = x_i^{n+1} - \prod_{k=0}^{n}(x_i - x_k) = x_i^{n+1}$.

Together with what we shown in (2a.), we also have $p_n = \sum_{i=0}^{n} f(x_i)\ell_i(x) = \sum_{i=0}^{n} x_i^{n+1}\ell_i(x)$ to be a polynomial in P_n such that $f(x_i) = p_n(x_i)$ for all $i \in \{0, 1, \ldots, n\}$.
Therefore by uniqueness of the polynomial established in (2a.), we have $f = p_n$, i.e.

$$x^{n+1} - \prod_{i=0}^{n}(x - x_i) = \sum_{i=0}^{n} x_i^{n+1}\ell_i(x).$$

(c) We can extend the result of (2b.), such that for all $f \in P_n$, we have $f(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x)$ for all $x \in \mathbb{R}$, i.e. $f = \sum_{i=0}^{n} f(x_i)\ell_i$. Together with Triangle Inequality on norms, we have,

$$\begin{aligned} \|p - q\|_\infty = \left\| \sum_{i=0}^{n}(p(x_i) - q(x_i))\ell_i \right\|_\infty &\leq \sum_{i=0}^{n} |p(x_i) - q(x_i)| \|\ell_i\|_\infty \\ &\leq \sum_{i=0}^{n} \left(\max_{0 \leq i \leq n} |p(x_i) - q(x_i)| \right) \|\ell_i\|_\infty \\ &= \left(\max_{0 \leq i \leq n} |p(x_i) - q(x_i)| \right) \sum_{i=0}^{n} \|\ell_i\|_\infty. \end{aligned}$$

Question 3

(a) We have,

$$\begin{aligned}
f(x) &= \sin\left(\frac{\pi}{2}x\right); \\
f[-1] &= f(-1) = \sin\left(-\frac{\pi}{2}\right) = -1; \\
f[0] &= f(0) = \sin 0 = 0; \\
f[1] &= f(1) = \sin\frac{\pi}{2} = 1; \\
f[-1,0] &= \frac{f[0]-f[-1]}{0-(-1)} = 1; \\
f[0,1] &= \frac{f[1]-f[0]}{1-0} = 1,
\end{aligned}$$

and so we have $f[-1,0,1] = \dfrac{f[0,1]-f[-1,0]}{1-(-1)} = 0$.

(b) Let P_n be the statement "For all $x_1, x_2, \ldots, x_n \in [a,b]$, we have $f[x_1, x_2, \ldots, x_n] = f[x_n, x_{n-1}, \ldots, x_1]$" for all $n \in \mathbb{Z}^+$. The case for P_1 is trivial.

Assume that P_k is true.
Let $x_1, x_2, \ldots, x_{k+1} \in [a,b]$. Then by assumption, we have,

$$\begin{aligned}
f[x_1, x_2, \ldots, x_{k+1}] &= \frac{f[x_2, x_3, \ldots, x_{k+1}] - f[x_1, x_2, \ldots, x_k]}{x_{k+1} - x_1} \\
&= \frac{f[x_1, x_2, \ldots, x_k] - f[x_2, x_3, \ldots, x_{k+1}]}{x_1 - x_{k+1}} \\
&= \frac{f[x_k, x_{k-1}, \ldots, x_1] - f[x_{k+1}, x_k, \ldots, x_2]}{x_1 - x_{k+1}} \\
&= f[x_{k+1}, x_k, \ldots, x_1],
\end{aligned}$$

and so P_{k+1} is true.
Therefore by Mathematical Induction, we have P_n to be true for all $n \in \mathbb{Z}^+$.

(c) Let q be the polynomial of degree at most $n+1$ that interpolates f at the $n+2$ distinct nodes $x_0, x_1, \ldots, x_n, t$. Using Newton form of interpolating polynomials, we will have,

$$\begin{aligned}
p(x) &= f[x_0] + f[x_0, x_1](x-x_0) + \cdots + f[x_0, x_1, \ldots, x_n]\prod_{j=0}^{n-1}(x-x_j), \quad \forall x \in [a,b]; \\
q(x) &= f[x_0] + f[x_0, x_1](x-x_0) + \cdots \\
&\quad + f[x_0, x_1, \ldots, x_n]\prod_{j=0}^{n-1}(x-x_j) + f[x_0, x_1, \ldots, x_n, t]\prod_{j=0}^{n}(x-x_j) \\
&= p(x) + f[x_0, x_1, \ldots, x_n, t]\prod_{j=0}^{n}(x-x_j), \quad \forall x \in [a,b].
\end{aligned}$$

Since q interpolates f at t, we have $f(t) = q(t)$, and so,

$$f(t) - p(t) = q(t) - p(t) = f[x_0, x_1, \ldots, x_n, t]\prod_{j=0}^{n}(t-x_j).$$

SECTION B

Question 4

(i) Let us use the notation $\mathcal{R}[a,b]$ to be the set of Riemann integrable functions on the interval $[a,b]$, i.e. $\alpha \in \mathcal{R}[a,b]$ iff $\displaystyle\int_a^b \alpha(x)\ dx$ exists.

Since $f \in C^3_{[a,b]}$, we have $f'' \in C^1_{[a,b]}$.
Let $g \in S_1(X)$ be such that $g(x_i) = f''(x_i)$ for all $i \in \{0,1,\ldots,n\}$, i.e. g is the piecewise linear interpolation to f'' on $[a,b]$ with knots X.

Since g is continuous on $[a,b]$, we have $g \in \mathcal{R}[a,b]$.
By Fundamental Theorem of Calculus, we have $G : [a,b] \to \mathbb{R}$ such that $G(y) = \displaystyle\int_a^y g(v)\ dv$ to be a well-defined function, and $G'(y) = g(y)$ for all $y \in [a,b]$, i.e. $G \in S_2(X)$.
This again give us G to be continuous on $[a,b]$, and so $G \in \mathcal{R}[a,b]$.
Thus by Fundamental Theorem of Calculus, we have $\displaystyle\int_a^x \int_a^y g(v)\ dv\ dy$ to exists for all $x \in [a,b]$.
This allows us to have a well-defined function $s : [a,b] \to \mathbb{R}$ such that,

$$s(x) \quad = \quad f(a) + \frac{x-a}{b-a}\left(f(b) - f(a) - \int_a^b \int_a^y g(v)\ dv\ dy\right) + \int_a^x \int_a^y g(v)\ dv\ dy.$$

By substitution, we can easily obtained $s(a) = f(a)$ and $s(b) = f(b)$.
Also, s is twice differentiable, with $s''(x) = g(x)$.
Since $g \in S_1(X)$, g is piecewise linear except on X, and so s is piecewise cubic except on X, i.e. we have $s \in S_3(X)$.
Lastly, $s''(x_i) = g(x_i) = f''(x_i)$ for all $i \in \{0,1,\ldots,n\}$.
Therefore s satisfy all the conditions we wanted.

Note: The main difficulty of this problem is not to verify that the expression for s satisfy the condition (even though that is all we need to present during exams), but to obtain the expression for s.
From the setting of the question, we can deduce that s is likely obtained by performing definite integration on g twice, and so we proceed in that direction. Also, to keep the proof rigorous, we will need to keep employing Fundamental Theorem of Calculus, which is covered in the prerequisite MA3110, but is not in the scope of this module.

Integrating g, we get a function G such that $G(y) = \displaystyle\int_a^y g(v)\ dv + c_1$, for some $c_1 \in \mathbb{R}$ which we do not know the value for now.
Integrating G again, we get s to be $s(x) = \displaystyle\int_a^x \left(\int_a^y g(v)\ dv + c_1\right)\ dy + c_2$, for some $c_2 \in \mathbb{R}$.
To obtain c_1 and c_2, we make use of the remaining 2 conditions,

$$\begin{aligned}
f(a) = s(a) &= \int_a^a \left(\int_a^y g(v)\ dv + c_1\right)\ dy + c_2 = c_2 \\
&= c_2; \\
f(b) = s(b) &= \int_a^b \left(\int_a^y g(v)\ dv + c_1\right)\ dy + c_2 \\
&= \int_a^b \int_a^y g(v)\ dv\ dy + c_1(b-a) + f(a) \\
c_1 &= \frac{1}{b-a}\left(f(b) - f(a) - \int_a^b \int_a^y g(v)\ dv\ dy\right),
\end{aligned}$$

and so we obtained our expression for s.

(ii) Let $i \in \{1, 2, \ldots, n\}$.
Since s'' is linear on $[x_{i-1}, x_i]$, $s''(x_{i-1}) = f''(x_{i-1})$ and $s''(x_i) = f''(x_i)$, we have,

$$s''\left(\frac{x_{i-1}+x_i}{2}\right) = \frac{1}{2}\left(f''(x_{i-1}) + f''(x_i)\right).$$

Since $f \in C^3_{[a,b]}$, we have f''' exists. By Mean Value Theorem, there exists $\mu_1, \mu_2 \in [x_{i-1}, x_i]$ with,

$$\begin{aligned} f''\left(\frac{x_{i-1}+x_i}{2}\right) - f''(x_{i-1}) &= f'''(\mu_1)\left(\frac{x_{i-1}+x_i}{2} - x_{i-1}\right) = f'''(\mu_1)\left(\frac{x_i - x_{i-1}}{2}\right); \\ f''(x_i) - f''\left(\frac{x_{i-1}+x_i}{2}\right) &= f'''(\mu_2)\left(x_i - \frac{x_{i-1}+x_i}{2}\right) = f'''(\mu_2)\left(\frac{x_i - x_{i-1}}{2}\right). \end{aligned}$$

Since $|\mu_1 - \mu_2| \le |x_i - x_{i-1}| \le \Delta$, we have,

$$\begin{aligned} 2\left|f''\left(\frac{x_{i-1}+x_i}{2}\right) - s''\left(\frac{x_{i-1}+x_i}{2}\right)\right| &= \left|2f''\left(\frac{x_{i-1}+x_i}{2}\right) - \left(f''(x_{i-1}) + f''(x_i)\right)\right| \\ &= \left|f'''(\mu_1) - f'''(\mu_2)\right|\left|\frac{x_i - x_{i-1}}{2}\right| \\ &\le \omega(f''', [a,b], \Delta)\frac{\Delta}{2}, \end{aligned}$$

and therefore $\max\limits_{1\le i\le n}\left|f''\left(\frac{x_{i-1}+x_i}{2}\right) - s''\left(\frac{x_{i-1}+x_i}{2}\right)\right| \le \frac{1}{4}\omega(f''', [a,b], \Delta)\Delta.$

Question 5

(i) We will first show that for $g \in P_m$, $\Delta g \in P_{m-1}$ and then use recursion to prove the equation.
Firstly, let $g(x) = a_m x^m + p(x)$ where $p \in P_{m-1}$, $a_m \in \mathbb{R}$. Then,

$$\begin{aligned} \Delta g(x) &= a_m x^m + p(x) - a_m(x-1)^m - p(x-1) \\ &= a_m[x^m - (x-1)^m] + p(x) - p(x-1) \\ &= a_m\left[x^m - \sum_{i=0}^{m}\binom{m}{i}(-1)^i x^{m-i}\right] + p(x) - p(x-1) \\ &= a_m\left[-\sum_{i=1}^{m}\binom{m}{i}(-1)^i x^{m-i}\right] + p(x) - p(x-1), \end{aligned}$$

and so $\Delta g \in P_{m-1}$.
Let us define recursively, $\Delta g = g_1 \in P_{m-1}$, $\Delta g_k = g_{k+1} \in P_{m-(k+1)}$, $k = 1, 2, \ldots, m-1$.
Notice that g_m is a constant, and so $\Delta g_m = g_m - g_m = 0$.
Thus we have $\Delta^{m+1} g = \Delta^m(\Delta g) = \Delta^m g_1 = \Delta^{m-1}(\Delta g_1) = \Delta^{m-1} g_2 = \cdots = \Delta g_m = 0$.

(ii) We have obtained in lecture (or by careful manipulation) that for all function $f : \mathbb{R} \to \mathbb{R}$,

$$\Delta^{m+1} f(x) = \sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}f(x-k).$$

Now, notice that for $x \in [m+1, \infty)$ and $k \in \{0, 1, \ldots, m+1\}$, we have $(x-k)^m_+ = (x-k)^m$.
Hence together with result from (5i.), we have for $x \in [m+1, \infty)$,

$$\begin{aligned} M_m(x) = \frac{1}{m!}\Delta^{m+1} x^m_+ &= \frac{1}{m!}\sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}(x-k)^m_+ \\ &= \frac{1}{m!}\sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}(x-k)^m = \frac{1}{m!}\Delta^{m+1} x^m = 0. \end{aligned}$$

Question 6

(a) For $k, m \in \mathbb{Z}^+$, define $f_k(x) = x^k$ and $h(x) = \displaystyle\sum_{n=0}^{m} \frac{n(n-1)(n-2)\cdots(n-k)}{m^{k+1}} \binom{m}{n} x^n (1-x)^{m-n}$.

For $j \in \{1, 2, \dots, k\}$, let $a_j \in \mathbb{R}$ such that $n(n-1)(n-2)\cdots(n-k) = n^{k+1} + \displaystyle\sum_{j=1}^{k} a_j n^j$.

Now, we have,

$$\begin{aligned} h(x) &= \frac{(m-1)\cdots(m-k)}{m^k} \sum_{n=0}^{m} \frac{n(n-1)\cdots(n-k)}{m(m-1)\cdots(m-k)} \binom{m}{n} x^n (1-x)^{m-n} \\ &= \left(\frac{1}{m^k} \prod_{j=1}^{k} (m-j) \right) x^{k+1}. \end{aligned}$$

At the same time, we have,

$$\begin{aligned} h(x) &= \sum_{n=0}^{m} \frac{1}{m^{k+1}} \left(n^{k+1} + \sum_{j=1}^{k} a_j n^j \right) \binom{m}{n} x^n (1-x)^{m-n} \\ &= \sum_{n=0}^{m} \left(\left(\frac{n}{m}\right)^{k+1} \binom{m}{n} x^n (1-x)^{m-n} + \sum_{j=1}^{k} \frac{a_j}{m^{k+1-j}} \left(\frac{n}{m}\right)^j \binom{m}{n} x^n (1-x)^{m-n} \right) \\ &= B_m f_{k+1}(x) + \sum_{j=1}^{k} \frac{a_j}{m^{k+1-j}} B_m f_j(x), \end{aligned}$$

and so we have $B_m f_{k+1}(x) = \left(\displaystyle\frac{1}{m^k} \prod_{j=1}^{k} (m-j) \right) x^{k+1} - \displaystyle\sum_{j=1}^{k} \frac{a_j}{m^{k+1-j}} B_m f_j(x)$.

Therefore by performing Mathematical Induction using the above equation as inductive step, we see that $B_m f_k$ is a polynomial of degree k, with leading coefficient $\displaystyle\frac{1}{m^{k-1}} \prod_{j=1}^{k-1} (m-j)$.

(b) Firstly, we do some preparatory work to derive some important results of Bernstein Polynomials.

Let us denote $p_{m,k} = \binom{m}{k} x^k (1-x)^{m-k}$ for $m \in \mathbb{Z}^+$, $k \in \{0, 1, \dots, m\}$.

Let $x \in [0,1]$, $m \in \mathbb{Z}^+$. Using what we have in lecture and the inductive step in (6a.), we have,

$$\begin{aligned} \sum_{k=0}^{m} p_{m,k} &= 1; \\ \sum_{k=0}^{m} \frac{k}{m} p_{m,k} = B_m f_1(x) &= x; \\ \sum_{k=0}^{m} \left(\frac{k}{m}\right)^2 p_{m,k} = B_m f_2(x) &= \frac{m-1}{m} x^2 + \frac{1}{m} x; \\ \sum_{k=0}^{m} \left(\frac{k}{m}\right)^3 p_{m,k} = B_m f_3(x) &= \frac{(m-1)(m-2)}{m^2} x^3 + 3\frac{m-1}{m^2} x^2 + \frac{1}{m^2} x; \\ \sum_{k=0}^{m} \left(\frac{k}{m}\right)^4 p_{m,k} = B_m f_4(x) &= \frac{(m-1)(m-2)(m-3)}{m^3} x^4 + 6\frac{(m-1)(m-2)}{m^3} x^3 + 7\frac{m-1}{m^3} x^2 + \frac{1}{m^3} x. \end{aligned}$$

For $f \in C^2_{[0,1]}$, this give us,

$$\begin{aligned}
\sum_{k=0}^{m} mf(x)p_{m,k} &= mf(x);\\
\sum_{k=0}^{m} mf'(x)\left(\frac{k}{m}-x\right)p_{m,k} &= mf'(x)(B_m f_1(x)-x)\\
&= mf'(x)(x-x)=0;\\
\sum_{k=0}^{m} m\left(\frac{k}{m}-x\right)^2 p_{m,k} &= m\sum_{k=0}^{m}\left(\left(\frac{k}{m}\right)^2-2x\frac{k}{m}+x^2\right)p_{m,k}\\
&= m\left(B_m f_2(x)-2xB_m f_1(x)+x^2\right)\\
&= m\left(\frac{-1}{m}x^2+\frac{1}{m}x\right)=x(1-x);\\
\sum_{k=0}^{m} m\frac{f''(x)}{2}\left(\frac{k}{m}-x\right)^2 p_{m,k} &= \frac{f''(x)}{2}\sum_{k=0}^{m} m\left(\frac{k}{m}-x\right)^2 p_{m,k}=\frac{x(1-x)}{2}f''(x);\\
\sum_{k=0}^{m} m\left(\frac{k}{m}-x\right)^4 p_{m,k} &= \sum_{k=0}^{m} m\left(\left(\frac{k}{m}\right)^4-4x\left(\frac{k}{m}\right)^3+6x^2\left(\frac{k}{m}\right)^2-4x^3\frac{k}{m}+x^4\right)p_{m,k}\\
&= m(B_m f_4(x)-4xB_m f_3(x)+6x^2B_m f_2(x)-4x^3B_m f_1(x)+x^4)\\
&= \frac{1}{m}\left[\left(3-\frac{6}{m}\right)x^4+\left(-6+\frac{12}{m}\right)x^3+\left(3-\frac{7}{m}\right)x^2+\frac{1}{m}x\right].
\end{aligned}$$

Since $m \in \mathbb{Z}^+$ and $x \in [0,1]$, we have $\left|\frac{1}{m}\right| \le 1$ and $|x| \le 1$. Thus,

$$\left|\left(3-\frac{6}{m}\right)x^4+\left(-6+\frac{12}{m}\right)x^3+\left(3-\frac{7}{m}\right)x^2+\frac{1}{m}x\right| \le 3+6+6+12+3+7+1=38,$$

which give us $\sum_{k=0}^{m} m\left(\frac{k}{m}-x\right)^4 p_{m,k} = \left|\sum_{k=0}^{m} m\left(\frac{k}{m}-x\right)^4 p_{m,k}\right| \le \frac{38}{m}$.

Lastly, as a corollary of Taylor's Theorem (MA3110), there exists $\eta \in C_{[0,1]}$ such that $\eta(x)=0$, and,

$$\begin{aligned}
mB_m f(x) &= \sum_{k=0}^{m} mf\left(\frac{k}{m}\right)p_{m,k}\\
&= \sum_{k=0}^{m} m\left(f(x)+f'(x)\left(\frac{k}{m}-x\right)+\frac{f''(x)}{2}\left(\frac{k}{m}-x\right)^2+\eta\left(\frac{k}{m}\right)\left(\frac{k}{m}-x\right)^2\right)p_{m,k}\\
&= mf(x)+\frac{x(1-x)}{2}f''(x)+\sum_{k=0}^{m}\eta\left(\frac{k}{m}\right)m\left(\frac{k}{m}-x\right)^2 p_{m,k}.
\end{aligned}$$

Since η is continuous on $[0,1]$, there exists $K \in \mathbb{R}^+$ such that $|\eta(t)| \le K$ for all $t \in [0,1]$.

Finally, let us start the proof proper:-
Let $\varepsilon \in \mathbb{R}^+$. Since η is continuous on $[0,1]$, there exists $\delta \in \mathbb{R}^+$ such that for all $t \in [0,1]$ with $|t-x|<\delta$, we have $|\eta(t)| = |\eta(t)-\eta(x)| < \frac{\varepsilon}{2x(1-x)}$.

By Archimedean's Property, there exists $M \in \mathbb{Z}^+$ such that $M > \frac{76K}{\delta^2\varepsilon}$.

This implies that for all $m \ge M$, $m \in \mathbb{Z}$, we have $\frac{38K}{\delta^2 m} < \frac{\varepsilon}{2}$.

Together with Triangle Inequality, we have that for all $m \geq M$, $m \in \mathbb{Z}$,

$$
\begin{aligned}
& \left| m(B_m f(x) - f(x)) - \frac{x(1-x)}{2} f''(x) \right| \\
= & \left| \sum_{k=0}^{m} \eta\left(\frac{k}{m}\right) m \left(\frac{k}{m} - x\right)^2 p_{m,k} \right| \\
\leq & \sum_{k=0}^{m} \left| \eta\left(\frac{k}{m}\right) \right| m \left(\frac{k}{m} - x\right)^2 p_{m,k} \\
= & \sum_{\substack{k \in \{0,1,\ldots,m\} \\ |\frac{k}{m} - x| < \delta}} \left| \eta\left(\frac{k}{m}\right) \right| m \left(\frac{k}{m} - x\right)^2 p_{m,k} + \sum_{\substack{k \in \{0,1,\ldots,m\} \\ |\frac{k}{m} - x| \geq \delta}} \left| \eta\left(\frac{k}{m}\right) \right| m \left(\frac{k}{m} - x\right)^2 p_{m,k} \\
< & \frac{\varepsilon}{2x(1-x)} \sum_{\substack{k \in \{0,1,\ldots,m\} \\ |\frac{k}{m} - x| < \delta}} m \left(\frac{k}{m} - x\right)^2 p_{m,k} + \frac{K}{\delta^2} \sum_{\substack{k \in \{0,1,\ldots,m\} \\ |\frac{k}{m} - x| \geq \delta}} m \left(\frac{k}{m} - x\right)^4 p_{m,k} \\
\leq & \frac{\varepsilon}{2x(1-x)} \sum_{k=0}^{m} m \left(\frac{k}{m} - x\right)^2 p_{m,k} + \frac{K}{\delta^2} \sum_{k=0}^{m} m \left(\frac{k}{m} - x\right)^4 p_{m,k} \\
\leq & \frac{\varepsilon}{2x(1-x)} (x(1-x)) + \frac{K}{\delta^2} \left(\frac{38}{m}\right) \\
< & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}
$$

i.e. $\lim\limits_{m \to \infty} m(B_m f(x) - f(x)) = \dfrac{x(1-x)}{2} f''(x)$, for all $x \in [0,1]$.