MA3210 - Mathematical Analysis II Suggested Solutions

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Written by: Chow Boon Wei
Audited by: Chong Jing Quan

Question 1

True. For any $x \in [0,1]$ $f(x) < g(x)$, so $0 < g(x) - f(x)$. Define $h : [0,1] \to \mathbb{R}$ by $h(x) = g(x) - f(x)$. Since f and g are continuous, so is h. By the extreme value theorem, there exists a $x_0 \in [0,1]$ such that for any $x \in [0,1]$, we have $h(x) \geq h(x_0) = g(x_0) - f(x_0) > 0$. So, $\int_0^1 g - \int_0^1 f = \int_0^1 (g-f) = \int_0^1 h \geq h(x_0)(1-0) > 0$. Finally, we have $\int_0^1 g > \int_0^1 f$.

Question 2

i) We have

$$
\begin{aligned}
F'(h) &= \frac{d}{dh}\int_{-h}^{h} f(x+t)tdt \\
&= \frac{d}{dh}\int_{-h}^{0} f(x+t)tdt + \frac{d}{dh}\int_{0}^{h} f(x+t)tdt \\
&= -\frac{d}{dh}\int_{0}^{-h} f(x+t)tdt + \frac{d}{dh}\int_{0}^{h} f(x+t)tdt \\
&= -f(x-h)h + f(x+h)h \\
&= (f(x+h) - f(x-h))h.
\end{aligned}
$$

ii) We have

$$
\begin{aligned}
0 = \lim_{h\to 0}\frac{1}{h^3}\int_{-h}^{h} f(x+t)tdt &= \lim_{h\to 0}\frac{F(h)}{h^3} \\
&= \lim_{h\to 0}\frac{F'(h)}{3h^2} \\
&= \lim_{h\to 0}\frac{(f(x+h)-f(x-h))h}{3h^2} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{3h} \\
&= \frac{2}{3}f'(x).
\end{aligned}
$$

So, $f'(x) = 0$ for all $x \in \mathbb{R}$. Together with $f(0) = b$, we have $f(x) = b$ for all $b \in \mathbb{R}$.

Question 3

First, we have $\begin{pmatrix} f_x(a+1,b+1) & f_y(a+1,b+1) \end{pmatrix} = \begin{pmatrix} 2(a+1) & 1 \\ b+1 & a+1 \end{pmatrix}$ since $f_x(x,y) = \begin{pmatrix} 2x \\ y \end{pmatrix}$ and $f_y(x,y) = \begin{pmatrix} 1 \\ x \end{pmatrix}$. Let $A = \begin{pmatrix} 2(a+1) & 1 \\ b+1 & a+1 \end{pmatrix}$ Now, we have

$$\begin{aligned}
&\left| f(a+1+h, b+1+k) - f(a+1,b+1) - A\begin{pmatrix} h \\ k \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} (a+1+h)^2 + (b+1+k) \\ (a+1+h)(b+1+k) \end{pmatrix} - \begin{pmatrix} (a+1)^2 + (b+1) \\ (a+1)(b+1) \end{pmatrix} - \begin{pmatrix} 2(a+1) & 1 \\ b+1 & a+1 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} h(2a+2+h) + k \\ (a+1)k + (b+1)h + hk \end{pmatrix} - \begin{pmatrix} 2(a+1)h + k \\ (b+1)h + (a+1)k \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} h^2 \\ hk \end{pmatrix} \right| \\
&= \sqrt{h^4 + h^2k^2} = h\sqrt{h^2+k^2}.
\end{aligned}$$

Therefore,

$$\lim_{h\to 0, k\to 0} \frac{1}{\sqrt{h^2+k^2}} \left| f(a+1+h, b+1+k) - f(a+1,b+1) - A\begin{pmatrix} h \\ k \end{pmatrix} \right| = \lim_{h\to 0, k\to 0} h = 0$$

which shows that f is differentiable at $(a+1, b+1)$.

Question 4

i) Since f_1, f_2 and f_3 are continuously differentiable, so is f. We have

$$f'(r,\theta,z) = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So, $\det(f'(r,\theta,z)) = r$ and we have $\det\left(f'\left(b+1, \frac{\pi}{4}, ab\right)\right) = b+1 \neq 0$. Since f is continuously differentiable, by the inverse function theorem, there exists a neighborhood U of $x = (b+1, \frac{\pi}{4}, ab)$ such that f is injective from U to $f(U)$ and f^{-1} is continuously differentiable in $f(U)$.

ii) We have $Id = (Id)' = [f(f^{-1}(y))]' = f'(f^{-1}(y))(f^{-1})'(y)$. Hence,

$$(f^{-1})'(y) = (f'(f^{-1}(y)))^{-1} = (f'(x))^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{b+1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{b+1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{b+1} \begin{pmatrix} \frac{b+1}{\sqrt{2}} & \frac{b+1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Question 5

Since $|f^{(n)}(0)| \leq (a+1)|0|^{(n+1)} = 0$, we have $f^{(n)}(0) = 0$ for all $n \geq 0$. In particular, this means that $f(0) = 0$. By Taylor's theorem, for each $n \geq 0$ and $x \in (-2-a, 2+a)\backslash\{0\}$, we have

$$\begin{aligned}
|f(x)| &= \left| \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right| \\
&= \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right| \\
&\leq \frac{(a+1)|c|^{n+2}}{(n+1)!} |x|^{n+1} \\
&\leq \frac{(a+1)}{(n+1)!} (a+2)^{2n+3} \to 0 \text{ as } n \to \infty
\end{aligned}$$

for some $c \in (0,x)$ or $(x,0)$. So, $|f(x)| \leq 0$, and we have $f(x) = 0$.

Question 6

i) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = F(x) + \frac{x}{2}$. Then,

$$g(-x) = F(-x) = \frac{-x}{e^{-x}-1} + \frac{-x}{2} = \frac{-2x}{2e^{-x}-2} + \frac{-xe^{-x}+x}{2e^{-x}-2} = \frac{-x-xe^{-x}}{2e^{-x}-2} = \frac{xe^{x}+x}{2e^{x}-2}$$

and

$$g(x) = F(x) + \frac{x}{2} = \frac{x}{e^{x}-1} + \frac{x}{2} = \frac{2x}{2e^{x}-2} + \frac{xe^{x}-x}{2e^{x}-2} = \frac{xe^{x}+x}{2e^{x}-2}.$$

So, $g(x) = g(-x)$, which means that g is an even function. Therefore, $0 = g^{(2n+1)}(0) = F^{(2n+1)}(0) = B_{2n+1}$ for $n \geq 1$. (Note that the odd derivative of an even function is odd.)

ii) By writing $T(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = 1 - \frac{x}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n}$, we see that T converges when

$$\begin{aligned}
&\lim_{n\to\infty} \left| \frac{B_{2n+2}/(2n+2)!x^{2n+2}}{B_{2n}/(2n)!x^{2n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{B_{2n+2}x^2}{B_{2n}(2n+1)(2n+2)} \right| \\
&= \lim_{n\to\infty} \left| \frac{B_{2n+2}}{(-1)^n\sqrt{4\pi(n+1)}\left(\frac{n+1}{\pi e}\right)^{2n+2}} \frac{(-1)^{n-1}\sqrt{4\pi n}\left(\frac{n}{\pi e}\right)^{2n}}{B_{2n}} \frac{\sqrt{n+1}\left(\frac{n+1}{\pi e}\right)^{2n+2}}{\sqrt{n}(2n+1)(2n+2)\left(\frac{n}{\pi e}\right)^{2n}} \right| x^2 \\
&= \lim_{n\to\infty} \left| \frac{\sqrt{n+1}\left(\frac{n+1}{\pi e}\right)^{2n+2}}{\sqrt{n}(2n+1)(2n+2)\left(\frac{n}{\pi e}\right)^{2n}} \right| x^2 \\
&= \lim_{n\to\infty} \left| \sqrt{\frac{n+1}{n}} \left(\frac{n+1}{n}\right)^{2n} \frac{\left(\frac{n+1}{\pi e}\right)^2}{(2n+1)(2n+2)} \right| x^2 \\
&= \lim_{n\to\infty} e^2 \left| \frac{1}{\pi^2 e^2} \frac{(n+1)^2}{(2n+1)(2n+2)} \right| x^2 = \frac{1}{4\pi^2} x^2 < 1.
\end{aligned}$$

So, the radius of convergence is 2π.

iii) We have $F(x)(e^x - 1) = x$. So, we have

$$\begin{aligned}
x = F(x)(e^x-1) &= \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \sum_{n=1}^{\infty} \frac{1}{n!} x^n \\
&= x \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^n \\
&= x \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{B_k}{k!} \frac{1}{(n+1-k)!} x^n \\
&= x \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n+1)!} \frac{(n+1)!}{k!(n+1-k)!} B_k x^n \\
&= x \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n+1)!} \binom{n+1}{k} B_k x^n
\end{aligned}$$

So, for $n \geq 1$, we have $\sum_{k=0}^{n} \frac{1}{(n+1)!} \binom{n+1}{k} B_k = 0$ which means that $\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0$. Finally, this means that for $n \geq 2$, $\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0$