NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA3111 Complex Analysis I
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Question 1

(a) For all $z \in \mathbb{C}$, $|z| = 1$, we have $f(z) = z + \frac{1}{z} = z + \frac{|z|^2}{z} = z + \frac{z\bar{z}}{z} = z + \bar{z} = 2\operatorname{Re} z$.
As we have $-2 = -2|z| \leq 2\operatorname{Re} z \leq 2|z| = 2$, we see that the image of the circle $|z| = 1$ under the function f is the closed interval $[-2, 2]$.

(b) If $|z| = 2$, then one has $|z + i| \leq |z| + |i| = 2 + 1 = 3$ and $|z + i| \geq ||z| - |i|| = |z| - |i| = 2 - 1 = 1$.
Also, we have $|z^3 - z - 2| \leq |z|^3 + |z| + 2 = 2^3 + 2 + 2 = 12$ and
$\left|z^3 - z - 2\right| \geq \left||z|^3 - |z| - 2\right| = |z|^3 - |z| - 2 = 2^3 - 2 - 2 = 4$.
So one has $\left|\frac{z+i}{z^3-z-2}\right| = \frac{|z+i|}{|z^3-z-2|} \leq \frac{3}{4}$ and $\left|\frac{z+i}{z^3-z-2}\right| = \frac{|z+i|}{|z^3-z-2|} \geq \frac{1}{12}$. The desired follows.

(c) We have $f(x + iy) = u(x, y) + iv(x, y) = \sqrt{|x||y|}$, so $u(x, y) = \sqrt{|x||y|}$ and $v(x, y) = 0$.
This implies that $v_x(x, y) = v_y(x, y) = 0$,
$u_x(0, 0) = \lim\limits_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim\limits_{x\to 0} \frac{0-0}{x} = 0$ and $u_y(0, 0) = \lim\limits_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim\limits_{y\to 0} \frac{0-0}{y} = 0$.
Therefore, we have $u_x(0, 0) = 0 = v_y(0, 0)$ and $u_y(0, 0) = 0 = -v_x(0, 0)$, so f satisfies the Cauchy-Riemann equations at $z = 0$.

Next, we shall show that f is not differentiable at $z = 0$.
Note that $u(x, y) = \sqrt{xy}$ for all $x, y > 0$, so one has $u_x(x, y) = \frac{1}{2}\sqrt{\frac{y}{x}}$.
Along the path $y = 0$, $x > 0$, we have $u_x(x, y) = u_x(x, 0) = 0$ for all $x > 0$. So as (x, y) approaches $(0, 0)$ along the path $y = 0$, $x > 0$, we have $u_x(x, y) \to 0$.
Along the path $y = x$, $x > 0$, we have $u_x(x, y) = u_x(x, x) = \frac{1}{2}$ for all $x > 0$. So as (x, y) approaches $(0, 0)$ along the path $y = x$, $x > 0$, we have $u_x(x, y) \to \frac{1}{2}$.

By the two-path test, we see that the limit $\lim\limits_{(x,y)\to(0,0)} u_x(x, y)$ does not exist, so u_x is not continuous at $(0, 0)$. So u does not have continuous first partial derivatives with respect to x and y, and hence f is not differentiable at $z = 0$. We are done.

Question 2

(a) Define the function $g : \mathbb{C} \to \mathbb{C}$ to be $g(z) = e^{-if(z)}$. Then it is clear that g is entire. Also, we have

$$|g(z)| = \left|e^{-if(z)}\right| = \left|e^{-i(\operatorname{Re}(f(z))+i\operatorname{Im}(f(z)))}\right| = \left|e^{\operatorname{Im}(f(z))}\right|\left|e^{-i\operatorname{Re}(f(z))}\right| \leq e^0 = 1$$

for all $z \in \mathbb{C}$. This shows that g is bounded. So g is necessarily a constant function by the Liouville's Theorem. Thus, we have $g(z) = c$ for all $z \in \mathbb{C}$ for some $c \in \mathbb{C}$.

This implies that $e^{-if(z)} = c$, so by differentiating both sides of the equation, we have
$-if'(z)e^{-if(z)} = 0$. As $e^w \neq 0$ for all $w \in \mathbb{C}$, we have $e^{-if(z)} \neq 0$, so we must have $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore f is a constant function as desired.

(b) (i) Let $z = t + it^2$. Then one has $dz = 1 + 2it\,dt$. Thus

$$\begin{aligned}\int_\gamma z\bar{z}\,dz &= \int_0^1 \left(t+it^2\right)\left(t-it^2\right)\cdot(1+2it)\,dt \\ &= \int_0^1 2it^5 + t^4 + 2it^3 + t^2\,dt \\ &= \left[\frac{it^6}{3} + \frac{t^5}{5} + \frac{it^4}{2} + \frac{t^3}{3}\right]_0^1 \quad \text{(By Fundamental Theorem for Line Integrals)} \\ &= \left[\frac{i}{3} + \frac{1}{5} + \frac{i}{2} + \frac{1}{3}\right] - [0+0+0+0] = \frac{8}{15} + \frac{5i}{6}.\end{aligned}$$

(ii) Notice that the singularities of the function $f(z) = \frac{1}{(z-a)(z-b)}$ are at $z = a$ and $z = b$.
It is easy to observe that the only (isolated) singularity inside the contour γ is at $z = a$. Moreover, we see that $\lim\limits_{z\to a}(z-a)f(z) = \lim\limits_{z\to a}\frac{1}{z-b} = \frac{1}{a-b} \neq 0$, so the singularity at $z = a$ is a simple pole.
Therefore, by the Cauchy's Residue Theorem, we have

$$\int_\gamma \frac{1}{(z-a)(z-b)}\,dz = \int_\gamma f(z)\,dz = 2\pi i \operatorname*{Res}_{z=a} f(z) = 2\pi i \lim_{z\to a}(z-a)f(z) = \frac{2\pi i}{a-b}.$$

(iii) Since $f(0) \neq 0$ and $g(0) \neq 0$, we see that the (only) singularity of the function $h(z) = \frac{f(z)g(z)}{z^3}$ is at $z = 0$. Moreover, we see that this (isolated) singularity lies inside the contour γ, and $\lim\limits_{z\to 0} z^3h(z) = \lim\limits_{z\to 0} f(z)g(z) = f(0)g(0) \neq 0$, so the singularity at $z = 0$ is a pole of order 3.

Now, define the function ϕ to be $\phi(z) = f(z)g(z)$. Then one has $\phi'(z) = f(z)g'(z) + f'(z)g(z)$ and $\phi''(z) = \frac{d}{dz}(f(z)g'(z) + f'(z)g(z)) = f(z)g''(z) + 2f'(z)g'(z) + f''(z)g(z)$.
Therefore, by the Cauchy's Residue Theorem, we have

$$\begin{aligned}\int_\gamma \frac{f(z)g(z)}{z^3}\,dz &= \int_\gamma h(z)\,dz \\ &= 2\pi i \operatorname*{Res}_{z=0} h(z) \\ &= 2\pi i\left[\frac{1}{(3-1)!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(z^3h(z)\right)\right] \\ &= \pi i \lim_{z\to 0}\frac{d^2}{dz^2}(\phi(z)) \\ &= \pi i \lim_{z\to 0}\phi''(z) \\ &= \pi i(f(0)g''(0) + 2f'(0)g'(0) + f''(0)g(0)) \\ &= \pi i(1\cdot 7 + 2\cdot(-2)\cdot 4 + 5\cdot 3) = 6\pi i.\end{aligned}$$

Question 3

(a) (i) The singularities of the function f precisely the zeroes of the function $g(z) = (1-z)\sin z$. So the singularities of f are at $z = 1$ and $z = n\pi$, $n \in \mathbb{Z}$.
Since $\lim\limits_{z\to 1}(z-1)f(z) = \lim\limits_{z\to 1}\frac{-z}{\sin z} = -\frac{1}{\sin 1} \neq 0$, we see that f has a simple pole at $z = 1$.

Next, define the function h to be $h(z) = z$. Then one has $f(z) = \frac{h(z)}{g(z)}$. Note that $g'(z) = (1-z)\cos z - \sin z$. Since $h(n\pi) = n\pi \neq 0$ and $|g'(n\pi)| = |(1-n\pi)\cos n\pi - \sin n\pi| = |1-n\pi| \neq 0$ for all $n \in \mathbb{Z}$, $n \neq 0$, we see that g has a zero of order 1 at $z = n\pi$, so we see that f has simple poles at $z = n\pi$, $n \neq 0$.
Finally, we see that $h(0) = 0$, $h'(0) = 1 \neq 0$, and $g'(0) = \cos 0 - \sin 0 = 1 \neq 0$. So both g and h have zeroes of order 1 at $z = 0$, and hence f has a removable singularity at $z = 0$.

(ii) By the definition of Laurent coefficients, one has

$$a_0 = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{1}{g(z)}\,dz \quad \text{and} \quad b_1 = \frac{1}{2\pi i}\int_\gamma f(z)\,dz,$$

where γ denotes the positively oriented circle $|z| = 4$ (Note: the contour γ lies inside the annulus $\pi < |z| < 2\pi$).
Note that the only singularities of f inside γ are at $z = 0$, $z = 1$, $z = \pi$ and $z = -\pi$. Thus, by the Cauchy's Residue Theorem, we have

$$b_1 = \frac{1}{2\pi i}\int_\gamma f(z)\,dz = \operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=\pi} f(z) + \operatorname*{Res}_{z=-\pi} f(z).$$

Now, we have

$$\begin{aligned}
\operatorname*{Res}_{z=0} f(z) &= 0 \quad \text{(because 0 is a removable singularity of } f\text{)},\\
\operatorname*{Res}_{z=1} f(z) &= \lim_{z\to 1}(z-1)f(z) = -\frac{1}{\sin 1},\\
\operatorname*{Res}_{z=\pi} f(z) &= \frac{h(\pi)}{g'(\pi)} = \frac{\pi}{(1-\pi)\cos\pi - \sin\pi} = \frac{\pi}{\pi - 1},\\
\operatorname*{Res}_{z=-\pi} f(z) &= \frac{h(-\pi)}{g'(-\pi)} = \frac{-\pi}{(1-(-\pi))\cos(-\pi) - \sin(-\pi)} = \frac{\pi}{\pi+1}.
\end{aligned}$$

Therefore, we have

$$b_1 = \operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=\pi} f(z) + \operatorname*{Res}_{z=-\pi} f(z) = -\frac{1}{\sin 1} + \frac{\pi}{\pi-1} + \frac{\pi}{\pi+1}.$$

On the other hand, we see that the singularities of $\frac{1}{g}$ coincide with the zeroes of g, i.e. at $z = 1$ and $z = n\pi$, $n \in \mathbb{Z}$.
Note that the only singularities of $\frac{1}{g}$ inside γ are at $z = 0$, $z = 1$, $z = \pi$ and $z = -\pi$. Thus, by the Cauchy's Residue Theorem, we have

$$a_0 = \frac{1}{2\pi i}\int_\gamma \frac{1}{g(z)}\,dz = \operatorname*{Res}_{z=0}\frac{1}{g(z)} + \operatorname*{Res}_{z=1}\frac{1}{g(z)} + \operatorname*{Res}_{z=\pi}\frac{1}{g(z)} + \operatorname*{Res}_{z=-\pi}\frac{1}{g(z)}.$$

Now, we see that $g'(0) = 1 \neq 0$, $g'(1) = -\sin 1 \neq 0$, $g'(\pi) = (1-\pi)\cos\pi - \sin\pi = \pi - 1 \neq 0$ and $g'(-\pi) = (1-(-\pi))\cos(-\pi) - \sin(-\pi) = -(\pi+1) \neq 0$. Therefore, we see that the poles of $\frac{1}{g}$ at these points are simple. Hence, we have

$$\begin{aligned}
\operatorname*{Res}_{z=0}\frac{1}{g(z)} &= \frac{1}{g'(0)} = 1,\\
\operatorname*{Res}_{z=1}\frac{1}{g(z)} &= \frac{1}{g'(1)} = -\frac{1}{\sin 1},\\
\operatorname*{Res}_{z=\pi}\frac{1}{g(z)} &= \frac{1}{g'(\pi)} = \frac{1}{\pi-1},\\
\operatorname*{Res}_{z=-\pi}\frac{1}{g(z)} &= \frac{1}{g'(-\pi)} = -\frac{1}{\pi+1}.
\end{aligned}$$

Therefore

$$a_0 = \operatorname*{Res}_{z=0}\frac{1}{g(z)} + \operatorname*{Res}_{z=1}\frac{1}{g(z)} + \operatorname*{Res}_{z=\pi}\frac{1}{g(z)} + \operatorname*{Res}_{z=-\pi}\frac{1}{g(z)} = 1 - \frac{1}{\sin 1} + \frac{1}{\pi-1} - \frac{1}{\pi+1}.$$

Remark. In fact, any positively oriented closed curve γ inside the annulus would work for this question as well.

(b) (i) Note that the only points for which the function $\text{Log}(z)$ fails to be analytic at are the non-positive real numbers. Moreover, we see that $z^3+2 \in \mathbb{R}$ and $z^3+2 \leq 0$ if and only if $z = ce^{\frac{2n\pi i}{3}}$, where $c \in \mathbb{R}$, $c \leq \sqrt[3]{2}$ and $n = 0, 1, 2$. Therefore, the set of points at which the function $\text{Log}\left(z^3+2\right)$ is analytic is $\mathbb{C} \backslash S$, where $S = \left\{ ce^{\frac{2n\pi i}{3}} | c \in \mathbb{R}, c \leq \sqrt[3]{2}, n = 0, 1, 2 \right\}$.

(ii) Since $\text{Log}\left(z^3+2\right)$ is analytic on and inside the circle $|z| = 1$, we have $\displaystyle\int_\gamma \text{Log}\left(z^3+2\right)\, dz = 0$ by the Cauchy-Goursat Theorem.

Question 4

(a) Let the real part of $f(z)$ be $u(x, y)$. Then we see that u must be continuously differentiable. Moreover, f is entire, so it must satisfy the Cauchy-Riemann equations, i.e.
$u_x(x, y) = v_y(x, y) = -2e^{2x} \sin 2y$ and $u_y(x, y) = -v_x(x, y) = -2e^{2x} \cos 2y - 3$.

By integrating u_x with respect to x, we have $u(x, y) = -e^{2x} \sin 2y + h(y)$, where h is some continuously differentiable function in y.
By differentiating both sides of the last equation with respect to y, we get
$u_y(x, y) = -2e^{2x} \cos 2y + h'(y) = -2e^{2x} \cos 2y - 3$, so one has $h'(y) = -3$.
Thus, we have $h(y) = -3y + c$ for some $c \in \mathbb{C}$, so one has $u(x, y) = -e^{2x} \sin 2y - 3y + c$.

Therefore, an entire function $f(z)$ whose imaginary part is $v(x, y) = e^{2x} \cos 2y + 3x$ is
$f(z) = f(x + iy) = -e^{2x} \sin 2y - 3y + i\left(e^{2x} \cos 2y + 3x\right)$.

(b) The radius of convergence R of the given series in the question is equal to

$$R = \left(\limsup_{n\to\infty} \left| \frac{2^n}{n^2} \right|^{\frac{1}{n}} \right)^{-1} = \left(\lim_{n\to\infty} \left(\frac{2^n}{n^2} \right)^{\frac{1}{n}} \right)^{-1} = \frac{1}{2} \left(\lim_{n\to\infty} n^{\frac{1}{n}} \right)^2 = \frac{1}{2}.$$

Moreover, we see that for all $|w| = \frac{1}{2}$, one has $\sum\limits_{n=1}^{\infty} \frac{2^n |w|^n}{n^2} = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent series. Therefore, the series $\sum\limits_{n=1}^{\infty} \frac{2^n z^n}{n^2}$ converges absolutely on all points of the circle $|z| = \frac{1}{2} = R$.

(c) We have

$$\begin{aligned}
\frac{1 - e^{iz}}{z^2} &= \frac{1}{z^2} \left(1 - \sum_{n=0}^{\infty} \frac{i^n z^n}{n!} \right) \\
&= -\frac{1}{z^2} \left(\sum_{n=1}^{\infty} \frac{i^n z^n}{n!} \right) \\
&= \sum_{n=1}^{\infty} \frac{i^{n-2} z^{n-2}}{n!} \\
&= \frac{1}{iz} + \sum_{n=2}^{\infty} \frac{i^{n-2} z^{n-2}}{n!} = \frac{1}{iz} + \sum_{n=0}^{\infty} \frac{i^n z^n}{(n+2)!}.
\end{aligned}$$

Now, let $f(z) = \frac{1}{iz}$ and $g(z) = \sum\limits_{n=0}^{\infty} \frac{i^n z^n}{(n+2)!}$. We see that the radius of convergence R of the series $\sum\limits_{n=0}^{\infty} \frac{i^n z^n}{(n+2)!}$ is equal to $R = \lim\limits_{n\to\infty} \left| \frac{1/(n+2)!}{1/(n+3)!} \right| = \lim\limits_{n\to\infty} n + 3 = \infty$, so g is an entire function.

By Cauchy-Goursat Theorem, we see that for any closed path γ in $\mathbb{C}$, one has $\int_\gamma g(z)\,dz = 0$, so g has an analytic anti-derivative G on $\mathbb{C}$, i.e. $G' = g$. Now, we have

$$\begin{aligned}
\int_{\gamma_\varepsilon} f(z)\,dz &= \int_{\gamma_\varepsilon} \frac{1}{iz}\,dz = \int_0^\pi \frac{1}{i\left(\varepsilon e^{it}\right)} \cdot i\varepsilon e^{it}\,dt = \int_0^\pi dt = \pi, \\
\int_{\gamma_\varepsilon} g(z)\,dz &= G\left(\varepsilon e^{i(\pi)}\right) - G\left(\varepsilon e^{i(0)}\right) \quad \text{(By Fundamental Theorem for Line Integrals)} \\
&= G(-\varepsilon) - G(\varepsilon), \\
\Rightarrow \int_{\gamma_\varepsilon} \frac{1-e^{iz}}{z^2}\,dz &= \int_{\gamma_\varepsilon} f(z) + g(z)\,dz = \int_{\gamma_\varepsilon} f(z)\,dz + \int_{\gamma_\varepsilon} g(z)\,dz = \pi + G(-\varepsilon) - G(\varepsilon), \\
\Rightarrow \lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} \frac{1-e^{iz}}{z^2}\,dz &= \lim_{\varepsilon\to 0} \left[\pi + G(-\varepsilon) - G(\varepsilon)\right] = \pi + G(0) - G(0) = \pi.
\end{aligned}$$

Question 5

(a) (i) We have

$$\begin{aligned}
|\sin z|^2 &= \sin z \overline{\sin z} \\
&= \left(\frac{e^{iz} - e^{-iz}}{2i}\right) \overline{\left(\frac{e^{iz} - e^{-iz}}{2i}\right)} \\
&= \left(\frac{e^{iz} - e^{-iz}}{2i}\right) \left(\frac{\overline{e^{iz}} - \overline{e^{-iz}}}{\overline{2i}}\right) \\
&= \frac{1}{4}\left(e^{iz+\overline{iz}} - e^{iz-\overline{iz}} - e^{-iz+\overline{iz}} + e^{-iz-\overline{iz}}\right) \\
&= \frac{1}{4}\left(e^{-2y} - e^{2ix} - e^{-2ix} + e^{2y}\right) \\
&= \frac{2 - e^{2ix} - e^{-2ix}}{4} + \frac{e^{2y} + e^{-2y} - 2}{4} \\
&= \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 + \left(\frac{e^{y} - e^{-y}}{2}\right)^2 = \sin^2 x + \sinh^2 y.
\end{aligned}$$

(ii) By the Estimation Lemma, we have

$$\left|\int_{I_R} \frac{\sin z}{z}\,dz\right| \le \ell(I_R) \cdot \sup_{z\in I_R} \left|\frac{\sin z}{z}\right| = \ell(I_R) \cdot \sup_{z\in I_R} |\sin z| \cdot \frac{1}{\inf\limits_{z\in I_R} |z|},$$

where $\ell(I_R)$ denotes the length of the line segment I_R. Now, we have

$$\begin{aligned}
\ell(I_R) &= \pi, \\
\sup_{z\in I_R} |\sin z| &= \sup_{x+iy\in I_R} \sqrt{\sin^2 x + \sinh^2 y} \\
&= \sup_{y\in[0,\pi]} \sqrt{\sin^2 R + \sinh^2 y} \\
&= \sqrt{\sin^2 R + \left(\frac{e^\pi - e^{-\pi}}{2}\right)^2} \\
&\le \sqrt{1 + 143} = 12, \\
\inf_{z\in I_R} |z| &= R.
\end{aligned}$$

Therefore, one has

$$\begin{aligned}
\left|\int_{I_R} \frac{\sin z}{z}\, dz\right| &\leq \ell(I_R)\cdot \sup_{z\in I_R} |\sin z| \cdot \frac{1}{\inf\limits_{z\in I_R} |z|} \leq \pi \cdot 12 \cdot \frac{1}{R} = \frac{12\pi}{R} \\
\Rightarrow \quad 0 \leq \lim_{R\to\infty} \left|\int_{I_R} \frac{\sin z}{z}\, dz\right| &\leq \lim_{R\to\infty} \frac{12\pi}{R} = 0 \\
\Rightarrow \quad \lim_{R\to\infty} \int_{I_R} \frac{\sin z}{z}\, dz &= 0.
\end{aligned}$$

(b) Let $f(z) = \frac{z^2+1}{z^4+1}$, $g(z) = z^2+1$ and $h(z) = z^4+1$ for all $z \in \mathbb{C}$. Then f is an even function so

$$\int_0^\infty \frac{x^2+1}{x^4+1}\, dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2+1}{x^4+1}\, dx.$$

Notice that the singularities of f coincide with the zeroes of h, i.e. at the points z where

$$\begin{aligned}
z^4 &= -1 = e^{\pi i} \\
\Rightarrow z &= e^{\frac{\pi i}{4}}, e^{-\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}, e^{-\frac{3\pi i}{4}} \\
\Rightarrow z &= \frac{\sqrt{2}}{2}(1+i), -\frac{\sqrt{2}}{2}(1+i), \frac{\sqrt{2}}{2}(-1+i), -\frac{\sqrt{2}}{2}(-1+i).
\end{aligned}$$

Consider the closed contour γ, consisting of the straight line from $z=-R$ to $z=R$, and the arc C_R with the parameterization $z = Re^{it}$, $0 \leq t \leq \pi$, where $R > 1$.
Notice that the only singularities of f inside the contour γ are at $z = \frac{\sqrt{2}}{2}(1+i)$ and $z = \frac{\sqrt{2}}{2}(-1+i)$.
Let $a = \frac{\sqrt{2}}{2}(1+i)$ and $b = \frac{\sqrt{2}}{2}(-1+i)$. Then by Cauchy's Residue Theorem, we have

$$\int_\gamma f(z)\, dz = \int_{C_R} f(z)\, dz + \int_{-R}^R f(z)\, dz = 2\pi i \operatorname*{Res}_{z=a} f(z) + 2\pi i \operatorname*{Res}_{z=b} f(z).$$

Now, we have $h'(a) = 4a^3 = 2\sqrt{2}i(1+i) \neq 0$ and $h'(b) = 4b^3 = 2\sqrt{2}i(1-i) \neq 0$, so the zeroes of h at $z=a$ and $z=b$ are of order 1.
Moreover, we have $g(a) = a^2+1 = i+1 \neq 0$ and $g(b) = b^2+1 = -i+1 \neq 0$, so this implies that the singularities at $z=a$ and $z=b$ are simple poles. Thus, we have

$$\begin{aligned}
\operatorname*{Res}_{z=a} f(z) &= \frac{g(a)}{h'(a)} = \frac{i+1}{2\sqrt{2}i(1+i)} = -\frac{i\sqrt{2}}{4}, \\
\operatorname*{Res}_{z=b} f(z) &= \frac{g(b)}{h'(b)} = \frac{-i+1}{2\sqrt{2}i(1-i)} = -\frac{i\sqrt{2}}{4}.
\end{aligned}$$

Hence, we have

$$\int_{C_R} f(z)\, dz + \int_{-R}^R f(z)\, dz = 2\pi i \operatorname*{Res}_{z=a} f(z) + 2\pi i \operatorname*{Res}_{z=b} f(z) = \pi\sqrt{2}.$$

Next, we have to estimate the value of the following integral:

$$\int_{C_R} f(z)\, dz = \int_{C_R} \frac{z^2+1}{z^4+1}\, dz.$$

By the Estimation Lemma, we have

$$\left|\int_{C_R} \frac{z^2+1}{z^4+1}\, dz\right| \leq \ell(C_R)\cdot \sup_{z\in C_R}\left|\frac{z^2+1}{z^4+1}\right| \leq \ell(C_R)\cdot \sup_{z\in C_R}\left|z^2+1\right| \cdot \frac{1}{\inf\limits_{z\in C_R}|z^4+1|},$$

where $\ell(C_R)$ denotes the length of the arc C_R. Now, we have

$$\begin{aligned}
\ell(C_R) &= \pi R, \\
\sup_{z\in C_R} \left|z^2+1\right| &\leq \sup_{z\in C_R} |z|^2+1 = R^2+1, \\
\inf_{z\in C_R} \left|z^4+1\right| &\geq \inf_{z\in C_R} \left||z|^4-1\right| = R^4-1.
\end{aligned}$$

Thus, one has

$$\begin{aligned}
& \left|\int_{C_R} \frac{z^2+1}{z^4+1}\,dz\right| \leq \ell(C_R)\cdot \sup_{z\in C_R}\left|z^2+1\right| \cdot \frac{1}{\inf\limits_{z\in C_R} |z^4+1|} \leq \frac{\pi R(R^2+1)}{R^4-1} \\
\Rightarrow\quad & 0 \leq \lim_{R\to\infty} \left|\int_{C_R} \frac{z^2+1}{z^4+1}\,dz\right| \leq \lim_{R\to\infty} \frac{\pi R(R^2+1)}{R^4-1} = 0 \\
\Rightarrow\quad & \lim_{R\to\infty} \int_{C_R} \frac{z^2+1}{z^4+1}\,dz = 0 \\
\Rightarrow\quad & \int_{-\infty}^{\infty} \frac{z^2+1}{z^4+1}\,dz = \lim_{R\to\infty}\int_{-R}^{R} \frac{z^2+1}{z^4+1}\,dz = \pi\sqrt{2} - \lim_{R\to\infty}\int_{C_R} \frac{z^2+1}{z^4+1}\,dz = \pi\sqrt{2} \\
\Rightarrow\quad & \int_0^{\infty} \frac{x^2+1}{x^4+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\,dx = \frac{\pi\sqrt{2}}{2}.
\end{aligned}$$