NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA3111 Complex Analysis
AY 2010/2011 Sem 2

Question 1

(a) Let $z = x + iy$

$$\begin{aligned} f(z) &= 3x - 3iy + i(x^2 + y^2) - xy^2 \\ &= 3x - xy^2 + i(x^2 + y^2 - 3y) \end{aligned}$$

$$\begin{aligned} \therefore u(x,y) &= 3x - xy^2 & v(x,y) &= x^2 + y^2 - 3y \\ u_x &= 3 - y^2 & v_x &= 2x \\ u_y &= -2xy & v_y &= 2y - 3 \end{aligned}$$

since u_x, u_y, v_x, v_y are continuous at every $(x,y) \in \mathbb{R}^2$, f is differentiable at z $\Leftrightarrow$ C.R equations are satisfied.

$$\begin{aligned} u_x &= v_y & u_y &= -v_x \\ 3 - y^2 &= 2y - 3 & -2xy &= -2x \\ y^2 + 2y - 6 &= 0 & 2x(1-y) &= 0 \\ y &= \frac{-2 \pm \sqrt{4 - 4(-6)}}{2} \\ &\Rightarrow x = 0, y = -1 \pm \sqrt{7} \end{aligned}$$

$\therefore f(z)$ is differentiable only at $z = (-1 \pm \sqrt{7})i$
$f(z)$ is nowhere analytic since it is only differentiable at finite numbers of points

(b)

$$\begin{aligned} 3\tan z &= ie^{2iz} \\ 3\frac{\sin z}{\cos z} &= ie^{2iz} \\ 3\frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} &= ie^{2iz} \\ 3e^{iz} - 3e^{-iz} &= -e^{3iz} - e^{iz} \\ 4e^{iz} - 3e^{-iz} + e^{3iz} &= 0 \\ e^{4iz} + 4e^{2iz} - 3 &= 0 \\ e^{2iz} &= \frac{-4 \pm \sqrt{16 - 4(1)(-3)}}{2} \\ &= -2 \pm \sqrt{7} \\ 2iz &= log(-2 \pm \sqrt{7}) \end{aligned}$$

$$\begin{aligned} 2iz &= \ln(-2 + \sqrt{7}) + iarg(-2 + \sqrt{7}) & \text{or} \quad 2iz &= \ln(|-2 - \sqrt{7}|) + iarg(-2 - \sqrt{7}) \\ z &= \frac{-i\ln(-2 + \sqrt{7})}{2} + n\pi & \text{or} \quad z &= \frac{-i\ln(2 + \sqrt{7})}{2} + n\pi,\ n \in \mathbb{Z} \end{aligned}$$

Question 2

(a)

$$\int_\gamma \bar{z}(z+1)^2 + z^3 dz = \int_\gamma \bar{z}(z+1)^2 dz + \int_\gamma z^3 dz$$

γ is a contour joining 0 to $e^{i\frac{\pi}{2}} - 1 = i - 1$

Since $g(x) = \dfrac{x^4}{4}$ is the holomorphic primitive of $f(x) = x^3$,

$$\int_\gamma z^3 dz = g\circ\gamma(\frac{\pi}{2}) - g\circ\gamma(0) = \left.\frac{x^4}{4}\right|_0^{i-1} = -1$$

$$\begin{aligned}
\int_\gamma \bar{z}(z+1)^2 dz &= \int_0^{\frac{\pi}{2}} \overline{(e^{it}-1)}(e^{it})^2\gamma'(t)dt \\
&= \int_0^{\frac{\pi}{2}} (e^{-it}-1)(e^{it})^2(ie^{it})dt \qquad (\gamma'(t) = ie^{it}) \\
&= \int_0^{\frac{\pi}{2}} i(e^{2it} - e^{3it})dt \\
&= i\Big[(\frac{e^{2it}}{2i} - \frac{e^{3it}}{3i})\Big]_0^{\frac{\pi}{2}} \\
&= \frac{e^{i\pi}}{2} - \frac{e^{i\frac{3\pi}{2}}}{3} - \frac{1}{2} + \frac{1}{3} \\
&= -\frac{2}{3} + \frac{i}{3} \\
\therefore \int_\gamma \bar{z}(z+1)^2 + z^3 dz &= -\frac{2}{3} + \frac{i}{3} - 1 \\
&= -1\frac{2}{3} + \frac{i}{3}
\end{aligned}$$

(b) Let C be the circle $|z-4| = 1$ oriented in the counterclockwise direction.
$Logz$ is analytic on $\mathbb{C}\backslash(-\infty, 0]$
$\therefore \frac{9}{Logz}$ is analytic on $\mathbb{C}\backslash(-\infty,0]\bigcup\{z\in\mathbb{C}|Logz = 0\}$
within and on C,

$$\begin{aligned}
|4| - |z| &\leq |z-4| \text{ triangle inequality} \\
\Rightarrow 4 - |z| &\leq 1 \\
3 &\leq |z| \\
\Rightarrow Re(Logz) &= \ln|z| \\
&> 0 \\
\Rightarrow Logz &\neq 0\ \forall z \text{ within and on C}
\end{aligned}$$

Since $\forall z$ within and on C, $z \notin (-\infty, 0]$
thus $\frac{9}{Logz}$ is analytic within and on C
By C.G theorem

$$\begin{aligned}
\int_C \frac{9}{Logz} &= 0 \\
\therefore \left|\int_C \frac{\bar{z}^2+9}{Logz}dz\right| &= \left|\int_C \frac{\bar{z}^2}{Logz}dz\right| \\
\left|\frac{\bar{z}^2}{Logz}\right| &= \frac{|\bar{z}|^2}{|Logz|}
\end{aligned}$$

$$\leq \frac{5^2}{|Logz|} \qquad \text{since } |z| \leq 5 \quad \forall z \text{ on } C$$
$$\leq \frac{25}{\ln 3} = M \qquad \text{since } |Logz| = \sqrt{(\ln|z|)^2 + (Argz)^2} \geq \ln|z|$$
$$L(C) = 2\pi(1) \geq \ln 3$$

By ML inequality,

$$\begin{aligned}\left|\int_C \frac{\bar{z}^2}{Logz} dz\right| &\leq M.L \\ &= \frac{50\pi}{\ln 3} \\ &= 143 \leq 160(\text{true})\end{aligned}$$

Question 3

(a) Let $u(x,y)$ be the real part of the imaginary function

$$\begin{aligned}
u_x = v_y \quad &\Rightarrow u_x = -\frac{\cos x}{e^y} + 24xy^2 - 8x^3 \\
u_y = -v_x \quad &\Rightarrow u_y = \frac{\sin x}{e^y} - 8y^3 + 24x^2y \\
u(x,y) &= \int u_x \; dx \\
&= \int -\frac{\cos x}{e^y} + 24xy^2 - 8x^3 dx \\
&= -\frac{\sin x}{e^y} + 12x^2y^2 - 2x^4 + \phi(y) \\
u_y &= \frac{\sin x}{e^y} + 24x^2y + \phi'(y) \\
\frac{\sin x}{e^y} - 8y^3 + 24x^2y &= \frac{\sin x}{e^y} + 24x^2y + \phi'(y) \\
\phi'(y) &= -8y^3 \\
\phi(y) &= -2y^4 + c \\
\therefore u(x,y) &= -\frac{\sin x}{e^y} + 12x^2y^2 - 2x^4 - 2y^4 + c
\end{aligned}$$

(b) Let $f(z) = \frac{(g(z))^3}{(g(z))^3+1}$
since $g(z)$ is given to be entire $\Rightarrow (g(z))^3$ and $(g(z))^3 + 1$ are entire functions.
Suppose $\exists c \in \mathbb{C}$ s.t $(g(c))^3 + 1 = 0$
By the given inequality, we have,

$$\begin{aligned}
|g(c)|^3 &\leq \left|(g(c))^3 + 1\right| \\
&= 0 \\
\Rightarrow g(c) &= 0 \\
\Rightarrow (g(c))^3 + 1 &= 0^3 + 1 \\
&= 1 \neq 0(\text{contradiction}) \\
\Rightarrow (g(z))^3 + 1 &\neq 0 \quad \forall z \in \mathbb{C} \\
\Rightarrow f(z) &\text{ is entire} \\
|f(z)| &= \left|\frac{(g(z))^3}{(g(z))^3 + 1}\right|
\end{aligned}$$

$$\leq 1$$

By Liouville's theorem, $f(z)$ is a constant function

$$\begin{aligned} f(z) &= k \text{ for some } k \in \mathbb{C} \\ \frac{(g(z))^3}{(g(z))^3+1} &= k \\ (g(z))^3 &= \frac{k}{1-k} \\ \text{Differentiating, } 3(g(z))^2(g'(z)) &= 0 \end{aligned}$$

Suppose $g(z) = 0$ for some $z \in \mathbb{C}$, we will have,

$$\begin{aligned} k &= \frac{0}{1+0} \\ &= 0 \\ \Rightarrow (g(z))^3 &= 0 \\ g(z) &= 0(\text{constant}) \quad \forall z \in \mathbb{C} \end{aligned}$$

Suppose $g(z) \neq 0, \forall z \in \mathbb{C}$

$$\begin{aligned} &\Rightarrow g'(z) = 0, \forall z \in \mathbb{C} \\ &\Rightarrow g(z) \text{ is a constant function} \end{aligned}$$

$\therefore g(z)$ is a constant function

Question 4

(a) Limit does not exist
Let $z = x + iy$
C_1: vertical line $x = 0$

$$\begin{aligned} \lim_{\substack{z \to i \\ (\text{along } x=0)}} \frac{(\bar{z}+i)^4}{|z-i|^4} &= \lim_{y\to 1} \frac{(-iy+i)^4}{|iy-i|^4} \\ &= \lim_{y\to 1} \frac{(-y+1)^4}{(y-1)^4} \\ &= 1 \end{aligned}$$

C_2: line $x = y - 1$

$$\begin{aligned} \lim_{\substack{z \to i \\ (\text{along} C_2)}} \frac{(\bar{z}+i)^4}{|z-i|^4} &= \lim_{y\to 1} \frac{(y-1-iy+i)^4}{|y-1+iy-i|^4} \\ &= \lim_{y\to 1} \frac{(y-1)^4(1-i)^4}{(y-1)^4|1+i|^4} \\ &= \lim_{y\to 1} \frac{-4}{(\sqrt{2})^4} = -1 \neq 1 \end{aligned}$$

(b)

$$\begin{aligned} \frac{z+9}{(2z+3)(z+3)} &= \frac{-2}{z+3} + \frac{5}{2z+3} \\ &= \frac{-2}{(z+1)+2} + \frac{5}{2(z+1)+1} \end{aligned}$$

$$= -\frac{1}{\frac{(z+1)}{2}+1} + \frac{5}{2(z+1)}\frac{1}{1+\frac{1}{2(z+1)}}$$

$$-\frac{1}{\frac{(z+1)}{2}+1} = -\sum_{n=0}^{\infty}(-1)^n\left[\frac{(z+1)}{2}\right]^n \qquad |z+1|<2$$

$$\frac{5}{2(z+1)}\frac{1}{1+\frac{1}{2(z+1)}} = \frac{5}{2(z+1)}\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{2(z+1)}\right]^n \qquad \left|\frac{1}{2(z+1)}\right|<1$$

$$= 5\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2(z+1))^{n+1}} \qquad \frac{1}{2}<|z+1|$$

$$\therefore \frac{z+9}{(2z+3)(z+3)} = -\sum_{n=0}^{\infty}(-1)^n\left[\frac{(z+1)}{2}\right]^n + 5\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2(z+1))^{n+1}} \quad \text{for } \frac{1}{2}<|z+1|<2$$

(c) $\frac{(z+9)^2}{(z+1)^4(2z+3)(z+3)}$ has singular points at $(z+1)^4(2z+3)(z+3)=0 \Leftrightarrow z=-1,-\frac{3}{2},-3$
-1 and $-\frac{3}{2}$ are inside γ since $|-1+1|=0<1$ and $|-\frac{3}{2}+1|=\frac{1}{2}<1$ while $|-3+2|=2>1$
By Cauchy's Residue Theorem,

$$\int_\gamma \frac{(z+9)^2}{(z+1)^4(2z+3)(z+3)}dz = 2\pi i(\operatorname*{Res}_{z=-1}\frac{(z+9)^2}{(z+1)^4(2z+3)(z+3)} + \operatorname*{Res}_{z=-\frac{3}{2}}\frac{(z+9)^2}{(z+1)^4(2z+3)(z+3)})$$

at $z=\frac{3}{2}$,

$$\frac{(z+9)^2}{(z+1)^4(2z+3)(z+3)} = \frac{\frac{(z+9)^2}{(z+1)^4(z+3)}}{(2z+3)} = \frac{\phi(z)}{2z+3}$$

$$\phi(z) = \frac{(z+9)^2}{(z+1)^4(z+3)} \text{ is analytic at } z=-\frac{3}{2}$$

$$\therefore \operatorname*{Res}_{z=-\frac{3}{2}}\frac{(z+9)^2}{(z+1)^4(2z+3)(z+3)} = \phi(-\frac{3}{2}) = 600$$

at $z=-1$,

$$\frac{(z+9)^2}{(z+1)^4(2z+3)(z+3)} = \left(\frac{z+9}{(z+1)^4}\right)\left[\frac{z+9}{(2z+3)(z+3)}\right]$$

$$= (\frac{1}{(z+1)^3}+\frac{8}{(z+1)^4})\left[\begin{array}{c} -\sum_{n=0}^{\infty}(-1)^n\left[\frac{(z+1)}{2}\right]^n \\ +5\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2(z+1))^{n+1}}\end{array}\right]$$

$$= (\frac{1}{(z+1)^3}+\frac{8}{(z+1)^4})\left[\cdots-\frac{1}{4}(z+1)^2+\frac{1}{8}(z+1)^3+\cdots\right]$$

$$= \cdots + (-\frac{1}{4}+1)\frac{1}{(z+1)}+\cdots$$

$$\therefore \operatorname*{Res}_{z=-1}\frac{(z+9)^2}{(z+1)^4(2z+3)(z+3)} = \frac{3}{4}$$

$$\therefore \int_\gamma \frac{(z+9)^2}{(z+1)^4(2z+3)(z+3)}dz = 2\pi i(600+\frac{3}{4})$$

$$= 1201\frac{1}{2}\pi i$$

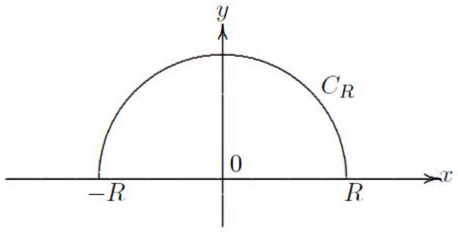
Question 5

$$P.V\int_{-\infty}^{\infty}\frac{\cos(x)\sin(x)}{4x^2-8x+5}dx=\lim_{R\to\infty}\int_{-R}^{R}\frac{\frac{1}{2}\sin(2x)}{4x^2-8x+5}dx$$

Let $f(z)=\frac{e^{i2z}}{8z^2-16z+10}$

$f(z)$ have singular points at $8z^2-16z+10=0 \Leftrightarrow z=\frac{16\pm\sqrt{(-16)^2-4(8)(10)}}{16}=1\pm\frac{1}{2}i$

for $R>|1+\frac{1}{2}i|$, consider the semi-circular arc C_R given by $C_R(t)=Re^it$, $0<t<\pi$



By Cauchy's Residue Theorem,

$$\int_{[-R,R]}f(x)dx+\int_{C_R}f(z)dz=2\pi i\operatorname*{Res}_{z=1+\frac{1}{2}i}f(z)$$

Write $f(z)=\frac{e^{i2z}}{8z^2-16z+10}=\frac{p(z)}{q(z)}$

where $p(z)=e^{2iz}$ and $q(z)=8z^2-16z+10$ are analytic at $z=1+\frac{1}{2}i$ with $q'(z)=16z-16$.

Observe that $q(1+\frac{1}{2}i)=0$ and $q'(1+\frac{1}{2}i)=8i\neq 0$

Thus,

$$\begin{aligned}\operatorname*{Res}_{z=1+\frac{1}{2}i}f(z)&=\frac{p(1+\frac{1}{2}i)}{q'(1+\frac{1}{2}i)}\\&=\frac{e^{2i-1}}{8i}\\ \therefore\int_{[-R,R]}f(x)dx+\int_{C_R}f(z)dz&=\frac{\pi(\cos 2+i\sin 2)}{4e}\end{aligned}$$

$L=\pi R$. For $z=x+iy\in C_R$,

$$\begin{aligned}|f(z)|&=\left|\frac{e^{i2z}}{8z^2-16z+10}\right|\\&=\left|\frac{e^{i2(x+iy)}}{8z^2-16z+10}\right|\\&\leq\frac{e^{-2y}\left|e^{i2x}\right|}{|8z^2|-|10-16z|}\qquad\text{since }\left|8z^2+(10-16z)\right|\geq\left|8z^2\right|-|10-16z|\\&\leq\frac{e^{-2y}}{8|z|^2-(10+16|z|)}\qquad\text{since }|10-16z|\leq|10|+|16z|\\&\leq\frac{e^{-2\cdot 0}}{8R^2-10-16R}\qquad\text{since }y\geq 0, |z|=R\text{ on }C_R\\&=M\end{aligned}$$

Thus by M.L inequality,

$$\begin{aligned}0\leq\left|\int_{C_R}f(z)dz\right|&\leq ML\\&=\frac{1}{8R^2-16R-10}\cdot\pi R\end{aligned}$$

$$\to 0 \quad \text{as } R \to +\infty$$

Thus, by squeeze theorem, we have $\lim\limits_{R\to\infty} \left|\int_{C_R} f(z)dz\right| = 0$
$\Rightarrow \lim\limits_{R\to\infty} \int_{C_R} f(z)dz = 0$
Letting $R \to \infty$ we have,

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx + \lim_{R\to\infty} \int_{C_R} f(z)dz = \frac{\pi(\cos 2 + i\sin 2)}{4e}$$
$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx + 0 = \frac{\pi(\cos 2 + i\sin 2)}{4e}$$

equating imaginary parts on both side we get,

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin(2x)}{8x^2 - 16x + 10} dx = \frac{\pi \sin 2}{4e}$$
$$P.V \int_{-\infty}^{\infty} \frac{\cos(x)\sin(x)}{4x^2 - 8x + 5} dx = \frac{\pi \sin 2}{4e}$$

Question 6

(a) $f(z)$ has singular points at $z(e^z - 1)^2 = 0 \;\Leftrightarrow z = 0, e^z - 1 = 0$
$e^z = 1 \;\Rightarrow\; z = 2n\pi i$ where $n \in \mathbb{Z}$
Let $q(z) = e^z - 1$ and $w(z) = 1 - \cos z$,

$$q'(z) = e^z$$
$$q(2n\pi i) = 0 \text{ but } q'(2n\pi i) = 1 \neq 0$$

$\Rightarrow q(z) = e^z - 1$ have zero of order 1 at $z = 2n\pi i, n \in \mathbb{Z}$

$$w'(z) = \sin z$$
$$w''(z) = \cos z$$
$$w(0) = 0$$
$$w'(0) = 0$$
$$w''(0) = 1 \neq 0$$
$$w(2n\pi i) \neq 0 \text{ for } n \neq 0, n \in \mathbb{Z}$$

$\therefore w(z)$ have zero of order 2 at $z = 0$
$\Rightarrow (1 - \cos z)^2 = (w(z))^2$ have zero of order 4 at $z = 0$
since z has zero of order 1 at $z = 0$, $z(e^z - 1)^2 = z(q(z))^2$ have zero of order $1 + 2(1) = 3 < 4$
$\Rightarrow f(z)$ have removable singular point at $z = 0$
for $z = 2in\pi$, $n \neq 0, n \in \mathbb{Z}$,
$(1 - \cos z)^2 = (w(z))^2$ have zero of order 0
$z(e^z - 1)^2 = z(q(z))^2$ have zero of order $2(1) = 2$
$\Rightarrow f(z)$ have pole of order $2 - 0 = 2$ at $z = 2in\pi$, $n \neq 0, n \in \mathbb{Z}$

(b) Let $f(z) = \frac{ze^z}{z^2+\pi^2}$
$f(z)$ have singular points at $z^2 + \pi^2 = 0 \;\Rightarrow z = \pm i\pi$

$$\frac{ze^z}{z^2 + \pi^2} = \frac{\frac{ze^z}{z - i\pi}}{z + i\pi} = \frac{\phi(z)}{z + i\pi}$$

where $\phi(z) = \frac{ze^z}{z-i\pi}$ is analytic at $z = -i\pi$

$$\therefore \operatorname*{Res}_{z=-i\pi} f(z) = \phi(-i\pi) = \frac{i\pi}{-2i\pi}$$

$$= -\frac{1}{2}$$

Similarly,

$$\frac{ze^z}{z^2+\pi^2} = \frac{\frac{ze^z}{z+i\pi}}{z-i\pi} = \frac{p(z)}{z-i\pi}$$

where $p(z) = \frac{ze^z}{z+i\pi}$ is analytic at $z = i\pi$

$$\begin{aligned}\therefore \operatorname*{Res}_{z=i\pi} f(z) = p(i\pi) &= \frac{-i\pi}{2i\pi} \\ &= -\frac{1}{2}\end{aligned}$$

Consider the contour $C : |z| = 5$ oriented in counterclockwise direction
C lies within the domain D and $z = \pm i\pi$ lies inside C since $|\pm i\pi| = \pi < 5$
By Cauchy Residue Theorem,

$$\begin{aligned}\int_C f(z)dz &= 2\pi i \left(\operatorname*{Res}_{z=i\pi} f(z) + \operatorname*{Res}_{z=-i\pi} f(z) \right) \\ &= 2\pi i(-\frac{1}{2} - \frac{1}{2}) = -2\pi i \neq 0\end{aligned}$$

$\Rightarrow f(z)$ does not have a antiderivative in $D \Rightarrow \nexists g(z)$ such that $g'(z) = f(z)$

Question 7

(a) Let $f(z) = z^2(z-2i)\cos\left(\frac{1}{z-2i}\right)$
$f(z)$ have singular point at $z = 2i$, $2i$ lies inside γ since $|2i| = 2 < 5$ Note that

$$\begin{aligned}z^2 &= (z-2i)^2 + 4iz + 4 \\ &= (z-2i)^2 + 4i(z-2i) - 4\end{aligned}$$

$$\begin{aligned}\therefore f(z) &= [(z-2i)^3 + 4i(z-2i)^2 - 4(z-2i)]\cos\left(\frac{1}{z-2i}\right) \\ &= [(z-2i)^3 + 4i(z-2i)^2 - 4(z-2i)]\sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{z-2i})^{2n}}{(2n)!} \qquad \left|\frac{1}{z-2i}\right| < \infty \\ &= [(z-2i)^3 + 4i(z-2i)^2 - 4(z-2i)]\left[\cdots + \frac{1}{24(z-2i)^4} + \frac{-1}{2(z-2i)^2} + \dots\right] \qquad |z-2i| > 0 \\ &= \left[\cdots + \left(\frac{1}{24} + 2\right)\frac{1}{z-2i} + \dots\right]\end{aligned}$$

Thus $\operatorname*{Res}_{z=2i}\left[z^2(z-2i)\cos\left(\frac{1}{z-2i}\right)\right] = \frac{49}{24}$
By Cauchy Residue theorem,

$$\begin{aligned}\int_\gamma z^2(z-2i)\cos\left(\frac{1}{z-2i}\right)dz &= 2\pi i \operatorname*{Res}_{z=2i}\left[z^2(z-2i)\cos\left(\frac{1}{z-2i}\right)\right] \\ &= \frac{49\pi i}{12}\end{aligned}$$

(b) Let $g(z) = [f(z)]^2$
Since $f(z)$ is entire $\Rightarrow g(z)$ is entire.
Let C be a positively oriented circle given by $C(t) = re^{it}, 0 \leq t \leq 2\pi$
By Cauchy Integral formula for derivative,

$$0 \leq \left|g^{(n)}(0)\right| = \left|\frac{n!}{2\pi i}\int_C \frac{g(z)}{(z-0)^{n+1}}dz\right|$$

$$
\begin{aligned}
&= \left| \frac{n!}{2\pi i} \int_0^{2\pi} \frac{[f(re^{it})]^2}{(re^{it})^{n+1}} C'(t) dz \right| \\
&\leq \frac{n!}{2\pi} \int_0^{2\pi} \left| \frac{[f(re^{it})]^2}{(re^{it})^{n+1}} rie^{it} \right| dz \\
&= \frac{n!}{2\pi} \int_0^{2\pi} \frac{\left|f(re^{it})\right|^2}{r^n} dz \quad \text{since} \left|ie^{it}\right| = 1 \\
&\leq \frac{n!}{2\pi} r^{4-n}
\end{aligned}
$$

For $n > 4$,

$$\frac{n!}{2\pi} r^{4-n} \to 0 \text{ as } r \to \infty$$

By squeeze theorem, $g^n(0) = 0$ for $n > 4$
Similarly for $n < 4$

$$\frac{n!}{2\pi} r^{4-n} \to 0 \text{ as } r \to 0$$

By squeeze theorem, $g^n(0) = 0$ for $n < 4$
$\therefore$ Maclaurin series of $g(z) = \frac{g^{(4)}(0)}{4!} z^4$

$$
\begin{aligned}
\Rightarrow f(z) &= \left(\frac{g^{(4)}(0)}{4!} \right)^{\frac{1}{2}} z^2 \\
&= cz^2 \text{ where } c \text{ is a constant}
\end{aligned}
$$