NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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ST2131/MA2216 Probability
AY 2006/2007 Sem 2

Question 1

(i) Notice that $X \sim H(13, 52, 4)$. Thus we have,

$$f_X(x) = \begin{cases} \frac{\binom{4}{x}\binom{48}{13-x}}{\binom{52}{13}}, & x = 0, 1, 2, 3, 4; \\ 0, & \text{otherwise.} \end{cases}$$

Also $E(X) = \dfrac{13 \times 4}{52} = 1$.

(ii) We notice that $Y|(X = 0) \sim H(13, 39, 4)$ Thus we have,

$$f_{Y|X}(y|0) = \begin{cases} \frac{\binom{4}{y}\binom{35}{13-y}}{\binom{39}{13}}, & y = 0, 1, 2, 3, 4; \\ 0, & \text{otherwise.} \end{cases}$$

Also $E(Y|X = 0) = \dfrac{13 \times 4}{39} = \dfrac{4}{3}$.

(iii) We have,

$$\begin{aligned} \mathbb{P}\{X = 0, Y = 1\} &= \mathbb{P}\{Y = 1 \mid X = 0\}\mathbb{P}\{X = 0\} \\ &= f_{Y|X}(1|0)f_X(0) \\ &= \left(\frac{\binom{4}{1}\binom{35}{12}}{\binom{39}{13}}\right)\left(\frac{\binom{4}{0}\binom{48}{13}}{\binom{52}{13}}\right). \end{aligned}$$

(iv) X and Y are not independent.

Assume on the contrary that they are independent. Then we have $f_Y(1) = \mathbb{P}\{Y = 1 \mid X = 0\} \neq 0$. However, notice that if all 4 aces are obtained in the first 13 cards, it is not possible to get an ace in the next 13 cards. This give us $f_Y(1) = \mathbb{P}\{Y = 1 \mid X = 4\} = 0$, a contradiction.

Question 2

(i) Since $f(x)$ is given to be the p.d.f. of r.v. X, we have,

$$\begin{aligned} 1 = \int_{\mathbb{R}} f(x)\, dx &= \int_0^1 Cx^2(1-x)^2\, dx \\ &= C\int_0^1 x^4 - 2x^3 + x^2\, dx \\ &= C\left[\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3\right]_0^1 \\ &= C\left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right) = \frac{C}{30}. \end{aligned}$$

Therefore $C = 30$.

(ii) We have,

$$\begin{aligned}
\mathbb{P}\left\{\left|X-\frac{1}{2}\right|<\frac{1}{4}\right\} &= \mathbb{P}\left\{\frac{1}{4}<X<\frac{3}{4}\right\} \\
&= \int_{\frac{1}{4}}^{\frac{3}{4}} 30x^2(1-x)^2\ dx \\
&= 30\left[\frac{1}{5}x^5-\frac{1}{2}x^4+\frac{1}{3}x^3\right]_{\frac{1}{4}}^{\frac{3}{4}} \\
&= \frac{203}{256}.
\end{aligned}$$

(iii) We have,

$$\begin{aligned}
E\left(\frac{1}{X}\right)=\int_{\mathbb{R}}\frac{1}{x}f(x)\ dx &= \int_0^1 30(x^3-2x^2+x)\ dx \\
&= 30\left[\frac{1}{4}x^4-\frac{2}{3}x^3+\frac{1}{2}x^2\right]_0^1 = 5/2.
\end{aligned}$$

(iv) Let $Y = X^2$. We see that when $y \le 0$, $F_Y(y) = 0$.

For $0 \ge y$, since $\mathbb{P}\{X \le -\sqrt{y}\} = 0$, we have

$$\begin{aligned}
F_Y(y)=\mathbb{P}\{Y\le y\} &= \mathbb{P}\{X^2\le y\} \\
&= \mathbb{P}\{-\sqrt{y}\le X\le\sqrt{y}\} \\
&= \mathbb{P}\{X\le\sqrt{y}\}=F_X(\sqrt{y}).
\end{aligned}$$

This give us $F_Y(y) = 1$ when $y \ge 1$.

For $0 < y < 1$, we have

$$\begin{aligned}
f_Y(y)=\frac{d}{dy}F_Y(y) &= \frac{d}{dy}F_X(\sqrt{y}) \\
&= f_X(\sqrt{y})\frac{d}{dy}(\sqrt{y}) \\
&= [30y(1-\sqrt{y})^2]\left(\frac{1}{2\sqrt{y}}\right) \\
&= 15\sqrt{y}(1-\sqrt{y})^2.
\end{aligned}$$

Therefore the p.d.f of X^2 is:

$$f_{X^2}(y) = \begin{cases} 15\sqrt{y}(1-\sqrt{y})^2, & 0<y<1; \\ 0, & \text{otherwise.} \end{cases}$$

Question 3

(i) Notice that $\{0 < x < 1,\ 0 < y < 1,\ x > y\} = \{x \in (0,1),\ y \in (0,x)\}$.
Since $f(x,y)$ is given to be the joint p.d.f. of X and Y, we have

$$\begin{aligned} 1 = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\ dy\ dx &= \int_0^1\int_0^x Cx\ dy\ dx \\ &= \int_0^1 Cx[y]_0^x\ dx \\ &= \int_0^1 Cx^2\ dx \\ &= C\left[\frac{1}{3}x^3\right]_0^1 \\ &= C/3. \end{aligned}$$

Therefore $C = 3$.
As a by-product, we obtain $f_X(x) = 3x^2$ for $0 < x < 1$, and $f_X(x) = 0$ otherwise.

(ii) When $0 < y < 1$, we have

$$\begin{aligned} f_Y(y) = \int_{\mathbb{R}} f(x,y)\ dx &= \int_y^1 3x\ dx \\ &= \left[\frac{3}{2}x^2\right]_y^1 \\ &= \frac{3(1-y^2)}{2}, \qquad 0 < y < 1. \end{aligned}$$

Thus the marginal p.d.f. of Y is given by,

$$f_Y(y) = \begin{cases} \frac{3(1-y^2)}{2}, & 0 < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

(iii) Given that $Y = y$, we have for $1 > x > y$,

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{2x}{1-y^2}. \end{aligned}$$

Thus the conditional p.d.f. of X given that $Y = y$ is,

$$f_{X|Y}(x|y) = \begin{cases} \frac{2x}{1-y^2}, & y < x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

(iv) We have,

$$\begin{aligned} \mathbb{P}\left\{X < \frac{1}{2} \mid Y = \frac{1}{4}\right\} &= \int_{-\infty}^{\frac{1}{2}} f_{X|Y}\left(x \mid \frac{1}{4}\right)\ dx \\ &= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{32x}{15}\ dx \\ &= \left(\frac{16}{15}\right)\left[x^2\right]_{\frac{1}{4}}^{\frac{1}{2}} = \frac{1}{5}. \end{aligned}$$

(v) To find $\mathrm{Cov}(X, Y)$, we firstly obtain $E(X)$, $E(Y)$ and $E(XY)$.

$$\begin{aligned}
E(X) = \int_{\mathbb{R}} x f_X(x)\ dx &= \int_0^1 3x^3\ dx \\
&= \left[\frac{3}{4}x^4\right]_0^1 = \frac{3}{4}, \\
E(Y) = \int_{\mathbb{R}} y f_Y(y)\ dy &= \int_0^1 \frac{3}{2}y - \frac{3}{2}y^3\ dy \\
&= \left[\frac{3}{4}y^2 - \frac{3}{8}y^4\right]_0^1 = \frac{3}{8}, \\
E(XY) = \int_{\mathbb{R}}\int_{\mathbb{R}} xy f(x, y)\ dy\ dx &= \int_0^1 \int_0^x 3x^2 y\ dy\ dx \\
&= \int_0^1 3x^2 \left[\frac{1}{2}y^2\right]_0^x\ dx \\
&= \int_0^1 \frac{3}{2}x^4\ dx \\
&= \left[\frac{3}{10}x^5\right]_0^1 = \frac{3}{10}.
\end{aligned}$$

Thus we get $\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{3}{10} - (\frac{3}{4})(\frac{3}{8}) = \frac{3}{160}$.
Since $\mathrm{Cov}(X, Y) \neq 0$, X and Y are not independent.

Question 4

Let $a = E\left(\frac{X_1}{\sum_{i=1}^n X_i}\right)$. Since $X_1, X_2, \ldots, X_n$ are i.i.d. r.v., we have

$$E\left(\frac{X_k}{\sum_{i=1}^n X_i}\right) = a, \quad \forall\ k = 1, 2, \ldots, n.$$

Now

$$\begin{aligned}
1 = E\left(\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i}\right) &= \sum_{i=1}^n E\left(\frac{X_i}{\sum_{i=1}^n X_i}\right) \\
&= \sum_{i=1}^n a = na.
\end{aligned}$$

This give us $a = \frac{1}{n}$, and so

$$\begin{aligned}
E\left(\frac{\sum_{i=1}^k X_i}{\sum_{i=1}^n X_i}\right) &= \sum_{i=1}^k E\left(\frac{X_i}{\sum_{i=1}^n X_i}\right) \\
&= \frac{k}{n}.
\end{aligned}$$

Question 5

(i) Let P_t be the statement $Q_t = (1-\lambda)^t Q_0 + \sum_{i=0}^{t-1}(1-\lambda)^i \lambda X_{t-i}$ for all $t \in \mathbb{Z}^+$.
Since $Q_1 = (1-\lambda)Q_0 + \lambda X_1$, we have P_1 to be true.
Assume that P_k is true for some $k \in \mathbb{Z}^+$, we consider P_{k+1}.

$$\begin{aligned} Q_{k+1} = (1-\lambda)Q_k + X_{k+1} &= (1-\lambda)\left((1-\lambda)^k Q_0 + \sum_{i=0}^{k-1}(1-\lambda)^i \lambda X_{k-i}\right) + X_{k+1} \\ &= (1-\lambda)^{k+1}Q_0 + \sum_{i=0}^{(k+1)-1}(1-\lambda)^i \lambda X_{(k+1)-i}. \end{aligned}$$

Thus P_{k+1} is true.
Therefore by Mathematical Induction, P_t is true for all $t \in \mathbb{Z}^+$.
Since the X_i's are i.i.d., we have $E(Q_t) = (1-\lambda)^t Q_0 + \sum_{i=0}^{t-1}(1-\lambda)^i \lambda \mu_0 = (1-\lambda)^t Q_0 + (1-(1-\lambda)^t)\mu_0$.
Thus $E(Q_{100}) = (1-\lambda)^{100}Q_0 + (1-(1-\lambda)^{100})\mu_0$. Since $0 < \lambda < 1$, we have $\lim_{t\to\infty}(1-\lambda)^t = 0$, and so $\lim_{t\to\infty} E(Q_t) = 0 + (1-0)\mu_0 = \mu_0$.

(ii) Similarly as the X_i's are i.i.d. r.v., we have,

$$\begin{aligned} \mathrm{Var}(Q_{100}) = \mathrm{Var}\left((1-\lambda)^{100}Q_0 + \sum_{i=0}^{99}(1-\lambda)^i \lambda X_{100-i}\right) &= \sum_{i=0}^{99}(1-\lambda)^{2i}\lambda^2\sigma_0^2 \\ &= \frac{1-(1-\lambda)^{200}}{1-(1-\lambda)^2}(\lambda\sigma_0)^2. \end{aligned}$$

(iii) We have

$$\begin{aligned} \mathbb{P}\{Q_1 = y \mid Q_0 = u\} = \frac{d}{dy}\mathbb{P}\{Q_1 \le y \mid Q_0 = u\} &= \frac{d}{dy}\mathbb{P}\{(1-\lambda)u + \lambda X_1 \le y\} \\ &= \frac{d}{dy}\mathbb{P}\left\{X_1 \le \frac{y-(1-\lambda)u}{\lambda}\right\} \\ &= \left(\frac{1}{\lambda}\right) f\left(\frac{y-(1-\lambda)u}{\lambda}\right). \end{aligned}$$

Notice that when $-h < y < h$, we have $E(R \mid Q_1 = y) - 1 = E(R \mid Q_0 = y)$.
When $|y| > h$, we have $E(R \mid Q_1 = y) - 1 = 0$. Thus,

$$\begin{aligned} E(R \mid Q_0 = u) &= \int_{\mathbb{R}} E(R \mid Q_0 = u, Q_1 = y)\mathbb{P}\{Q_1 = y \mid Q_0 = u\}\ dy \\ &= \int_{\mathbb{R}} \mathbb{P}\{Q_1 = y \mid Q_0 = u\}\ dy + \int_{\mathbb{R}} (E(R \mid Q_0 = u, Q_1 = y) - 1)\,\mathbb{P}\{Q_1 = y \mid Q_0 = u\}\ dy \\ &= 1 + \int_{\mathbb{R}} (E(R \mid Q_0 = u, Q_1 = y) - 1)\left(\frac{1}{\lambda}\right) f\left(\frac{y-(1-\lambda)u}{\lambda}\right)\ dy \\ &= 1 + \frac{1}{\lambda}\int_{\mathbb{R}} (E(R \mid Q_1 = y) - 1)\, f\left(\frac{y-(1-\lambda)u}{\lambda}\right)\ dy \\ &= 1 + \frac{1}{\lambda}\int_{-h}^{h} E(R|Q_0 = y) f\left(\frac{y-(1-\lambda)u}{\lambda}\right)\ dy. \end{aligned}$$