NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1101R Linear Algebra 1

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Question 1

(a)

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix} \xrightarrow{-R_2+R_3} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

So $\det(\mathbf{A}) = 4$.

(b) As can be seen from the row-reduced form of $\mathbf{A}$ in part (a), $\mathbf{A}$ is of full rank. So S is linearly independent, and since $\dim(\mathbb{R}^3) = 3$, S is also a basis for $\mathbb{R}^3$. To find $(\mathbf{w})_S = (w_1, w_2, w_3)$, we solve the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & x \\ 0 & 1 & 0 & y \\ 0 & 1 & 4 & z \end{array}\right) \xrightarrow{-R_2+R_3} \left(\begin{array}{ccc|c} 1 & -1 & 3 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 4 & -y+z \end{array}\right)$$

We thus have $w_2 = y$, $w_3 = \frac{1}{4}(-y+z)$, $w_1 = x + w_2 - 3w_3 = x + y - \frac{3}{4}(-y+z) = x + \frac{7}{4}y - \frac{3}{4}z$. So $(\mathbf{w})_S = (x + \frac{7}{4}y - \frac{3}{4}z, y, \frac{1}{4}(-y+z))$.

(c) Using part (b), we have

$$\mathbf{v}_1 = 2\mathbf{u}_1 + \mathbf{u}_2 + 0\mathbf{u}_3$$

and

$$\mathbf{v}_2 = \frac{17}{4}\mathbf{u}_1 + 3\mathbf{u}_2 + \frac{1}{4}\mathbf{u}_3$$

(d) Any such vector $\mathbf{v} = (x, y, z)^T$ satisfies $\mathbf{v} \cdot \mathbf{u}_3 = 0$ and $\mathbf{v} \cdot \mathbf{v}_2 = 0$. We thus solve the system

$$3x + 0y + 4z = 0$$
$$2x + 3y + 4z = 0$$

Or equivalently, to find the null-space of: $\begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Or, using Row2 -Row1: $\begin{bmatrix} 3 & 0 & 4 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ We thus have:

$$\mathbf{v} = \begin{pmatrix} 1 \\ \frac{1}{3} \\ -\frac{3}{4} \end{pmatrix} s$$

for $s \in \mathbb{R}$.

(e) We note that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \mathbb{R}^3$. We thus require to find all $\mathbf{y} = (y_1, y_2, y_3)^T \in \mathbb{R}^3$ such that $\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{y}\} = \mathbb{R}^3$. Note that this vector $\mathbf{y}$ could be found by finding a vector $\mathbf{y}$ which is linearly independent with both $\mathbf{v_1}$ and $\mathbf{v_2}$, i.e

$$\mathbf{0} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + c_3 \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

has only solutions $c_1 = c_2 = c_3 = 0$. Geometrically speaking, imagine $\text{span}\{\mathbf{v_1}, \mathbf{v_2}\}$ is a plane, we want to find $\mathbf{y}$ that is not in the plane generated by $\mathbf{v_1}$ and $\mathbf{v_2}$. Hence we need to find a vector normal to the plane first. We let this normal vector be $\mathbf{n}$, then $\mathbf{v_1}.\mathbf{n} = 0$ and $\mathbf{v_2}.\mathbf{n} = 0$, i.e

$$\begin{aligned} n_1 + n_2 + n_3 &= 0 \\ 2n_1 + 3n_2 + 4n_3 &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 4 & 0 \end{array}\right) \xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \xrightarrow{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$n_1 = n_3, \quad n_2 = -2n_3, \quad \mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Therefore all $\mathbf{y}$ such that $\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{y}\} = \mathbb{R}^3$ are of the form

$$\mathbf{y} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{n}, \quad \forall\, c_1, c_2, c_3 \in \mathbb{R}$$

(f) Let $(\alpha, 0, 0)^T$ be an eigenvector of B corresponding to eigenvalue λ. Then we have

$$B\begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha b_{11} \\ \alpha b_{21} \\ \alpha b_{31} \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$$

Therefore $b_{11} = \lambda$ and $b_{21} = b_{31} = 0$. For simplicity reason, we let $\lambda = 1$ and let $\mathbf{B}$ has only one eigenvalue and one eigenvector. Then $\mathbf{B}(1, 0, 0)^T = (1, 0, 0)^T$. So $\mathbf{B}$ may take the form

$$\begin{pmatrix} 1 & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{pmatrix}$$

For simplicity reason so that we can have an invertible matrix, we might want to let $b_{32} = 0$ and $b_{22} = 1$ and $b_{33} = 1$ both non zero to have an invertible matrix, $\mathbf{B}$, giving

$$\begin{pmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{pmatrix} \quad \& \quad \lambda\mathbf{I} - \mathbf{B} = \begin{pmatrix} 0 & -b_{12} & -b_{13} \\ 0 & 0 & -b_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

Consider the null space of $\lambda\mathbf{I} - \mathbf{B}$, we perform row reduced operation. As long as $b_{12} \neq 0$ and $b_{23} \neq 0$, then it is $\lambda\mathbf{I} - \mathbf{B}$ row-reducible to

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Implying that the only eigenvector is $\mathbf{v_1}$. Thus an invertible $\mathbf{B}$ satisfying the conditions is given by taking $a = b = c = 1$, i.e.

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Question 2

(a) We row-reduce the augmented system $(\mathbf{A}|\mathbf{B})$ viz.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & a & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right) \xrightarrow[R_1+R_3]{R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & a+1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

Let $\mathbf{X} = (x_{ij})$. If $a = -1$, we have $x_{3j} = 1$ and $x_{1j} = \frac{1}{2} - x_{2j}$ for $j = 1, 2, 3$. We thus have

$$\mathbf{X} = \begin{pmatrix} \frac{1}{2} - r & \frac{1}{2} - s & \frac{1}{2} - t \\ r & s & t \\ 1 & 1 & 1 \end{pmatrix}$$

for $r, s, t \in \mathbb{R}$. Taking $r = s = t = 0$ and $r = s = t = \frac{1}{2}$ gives us

$$\mathbf{X} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$\mathbf{X} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix}$$

as two different solutions to $\mathbf{AX} = \mathbf{B}$.

(b) Note that

$$\mathbf{b} = 2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Using the answer in (a), we have $\mathbf{x} = 2(0, \frac{1}{2}, 1)^T = (0, 1, 2)^T$ as a solution to $\mathbf{Ax} = \mathbf{b}$ (i.e. $\|\mathbf{Ax} - \mathbf{b}\| = 0$ in this case).

(c) We can view l as a vector space spanned by $\{(1/\sqrt{5}, 0, 2/\sqrt{5})^T\}$. We note that

$$\begin{aligned} (1,0,0)^T \cdot (1/\sqrt{5}, 0, 2/\sqrt{5})^T &= 1/\sqrt{5} \\ (0,1,0)^T \cdot (1/\sqrt{5}, 0, 2/\sqrt{5})^T &= 0 \\ (0,0,1)^T \cdot (1/\sqrt{5}, 0, 2/\sqrt{5})^T &= 2/\sqrt{5} \end{aligned}$$

Thus the standard matrix for T_2 is

$$\mathbf{F} = \begin{pmatrix} \frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 0 \\ \frac{2}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

The standard matrix for $T_1 \circ T_2$ is therefore

$$\mathbf{BF} = \frac{3}{10}\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

(d) (i) We have in this case,

$$x\mathbf{I} - \mathbf{A} = \begin{pmatrix} x-1 & -1 & 0 \\ 1 & x-5 & -1 \\ 1 & 1 & x-1 \end{pmatrix}$$

The characteristic equation $\det(x\mathbf{I} - \mathbf{A})$ is therefore

$$\begin{aligned}((x-1)^2(x-5)+1) - (-(x-1)-(x-1)) &= (x-1)[(x-1)(x-5)+2]+1 \\ &= (x-1)(x^2-6x+7)+1\end{aligned}$$

Taking $x = 2$ gives us 0, and we conclude that 2 is an eigenvalue of A.

(ii) Consider

$$2\mathbf{I} - \mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -3 & -1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_2+R_3} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, if $\mathbf{x} = (x, y, z)^T$ satisfies $(2\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$,

$$\begin{aligned} x - y + 0z &= 0 \\ 0x - 2y - z &= 0 \end{aligned}$$

We then have $z = -2y \Rightarrow x = -\frac{1}{2}z$. So a basis for the eigenspace E_2 is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$$

(e) From working in (a), range of $T_3 \neq \mathbb{R}^3 \Leftrightarrow \text{rank}(\mathbf{A}) < 3 \Leftrightarrow a = -1$.

(f) We have

$$x\mathbf{I} - (\mathbf{B} - \mathbf{C}) = x\mathbf{I} - \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{3}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{5}{2} \end{pmatrix} = \begin{pmatrix} x+\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & x+\frac{3}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & x+\frac{5}{2} \end{pmatrix}$$

The characteristic equation is $(x+\frac{1}{2})(x+\frac{3}{2})(x+\frac{5}{2}) = 0$. Thus $\mathbf{B} - \mathbf{C}$ has 3 distinct eigenvalues, and we conclude that it is diagonalizable.

Question 3

(a) All $\mathbf{w}_i \in S$ are vectors from $\mathbb{R}^3$. Since $|S| = 4 > 3 = \mathbb{R}^3$, it follows that S is linearly dependent.

(b) It suffices to show that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent. This follows by noting that the matrix

$$\begin{pmatrix} 3 & 1 & 3 \\ 6 & 2 & 0 \\ 0 & 2 & -1 \end{pmatrix} \xrightarrow{-2R_1+R_2} \begin{pmatrix} 3 & 1 & 3 \\ 0 & 0 & -6 \\ 0 & 2 & -1 \end{pmatrix}$$

is of full rank.

(c) This follows from (b) and the fact that $|S| = 3 = \dim(\mathbb{R}^3)$.

(d) We need to find $a_{11}, a_{22}, \ldots, a_{33}$ such that

$$\mathbf{e_1} = a_{11}\mathbf{w_1} + +a_{21}\mathbf{w_2} + a_{31}\mathbf{w_3}$$

$$\mathbf{e_2} = a_{12}\mathbf{w_1} + +a_{22}\mathbf{w_2} + a_{32}\mathbf{w_3}$$

$$\mathbf{e_3} = a_{13}\mathbf{w_1} + +a_{23}\mathbf{w_2} + a_{33}\mathbf{w_3}$$

By

$$\left(\begin{array}{ccc|c|c|c} 3 & 1 & 3 & 1 & 0 & 0 \\ 6 & 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -\frac{1}{18} & \frac{7}{36} & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{12} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{6} & 0 \end{array}\right)$$

Therefore the transition matrix from U to V is

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{18} & \frac{7}{36} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{12} & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{6} & 0 \end{pmatrix}$$

(e) We note that $\{\frac{1}{\sqrt{5}}(1,2,0)^T, \frac{1}{2}(0,0,2)^T\}$ is an orthonormal basis for $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$. Then the projection of $\mathbf{w}_3$ onto W is given by

$$\left(\frac{1}{\sqrt{5}}\begin{pmatrix}3\\0\\-1\end{pmatrix} \cdot \begin{pmatrix}1\\2\\0\end{pmatrix}\right)\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\\0\end{pmatrix} + \left(\frac{1}{2}\begin{pmatrix}3\\0\\-1\end{pmatrix} \cdot \begin{pmatrix}0\\0\\2\end{pmatrix}\right)\frac{1}{2}\begin{pmatrix}0\\0\\2\end{pmatrix}$$

$$= \frac{3}{5}\begin{pmatrix}1\\2\\0\end{pmatrix} - \frac{2}{4}\begin{pmatrix}0\\0\\2\end{pmatrix} = \begin{pmatrix}3/5\\6/5\\-1\end{pmatrix}$$

(f) We seek to find $\alpha, \beta, \gamma, \delta$ such that

$$\alpha\begin{pmatrix}3\\6\\0\end{pmatrix} + \beta\begin{pmatrix}1\\2\\2\end{pmatrix} = \gamma\begin{pmatrix}3\\0\\-1\end{pmatrix} + \delta\begin{pmatrix}8\\5\\-6\end{pmatrix}$$

We thus consider the matrix

$$\begin{pmatrix} 3 & 1 & -3 & -8 \\ 6 & 2 & 0 & -5 \\ 0 & 2 & 1 & 6 \end{pmatrix} \xrightarrow{-2R_1+R_2} \begin{pmatrix} 3 & 1 & -3 & -8 \\ 0 & 0 & 6 & 11 \\ 0 & 2 & 1 & 6 \end{pmatrix}$$

We thus have $\gamma = -\frac{11}{6}\delta$. Note that

$$-11\begin{pmatrix}3\\0\\-1\end{pmatrix} + 6\begin{pmatrix}8\\5\\-6\end{pmatrix} = \begin{pmatrix}15\\30\\-25\end{pmatrix} = 5\begin{pmatrix}3\\6\\-5\end{pmatrix}$$

A basis for $\text{span}\{\mathbf{w}_1, \mathbf{w}_2\} \cap \text{span}\{\mathbf{w}_3, \mathbf{w}_4\}$ is therefore

$$\left\{\begin{pmatrix}3\\6\\-5\end{pmatrix}\right\}$$

Question 4

(a) $\mathbf{A}+\mathbf{B}$ is positive semidefinite since

$$\mathbf{x}^T(\mathbf{A}+\mathbf{B})\mathbf{x} = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{B}\mathbf{x} \geq 0$$

if both $\mathbf{A}$ and $\mathbf{B}$ are positive semidefinite.

(b) For any $\mathbf{x} = (x, y, z)^T$, we have

$$\begin{aligned}\mathbf{x}^T\begin{pmatrix}4 & 1 & 0\\ 1 & 5 & 1\\ 0 & 1 & 1\end{pmatrix}\mathbf{x} &= x(4x+y) + y(x+5y+z) + z(y+z)\\ &= 4x^2 + 5y^2 + z^2 + 2xy + 2yz\\ &= 3x^2 + 3y^2 + (x+y)^2 + (y+z)^2 \geq 0\end{aligned}$$

for all $x, y, z \in \mathbb{R}$. This proves the claim.

(c) Take $\mathbf{x} = \mathbf{e}_i$ for each i. The claim follows since $\mathbf{e}_i^T\mathbf{A}\mathbf{e}_i = a_{ii}$.

(d) Consider $\mathbf{x} = (x_k)^T$ with

$$x_k = \begin{cases} a_{jj} & k = i\\ -a_{ij} & k = j\\ 0 & \text{otherwise}\end{cases}$$

Now let $\mathbf{c}_k$ denote the kth column of $\mathbf{A}$. Then

$$\begin{aligned}\mathbf{x}^T\mathbf{A}\mathbf{x} &= \mathbf{x}(a_{jj}\mathbf{c}_i - a_{ij}\mathbf{c}_j)\\ &= a_{jj}(a_{jj}a_{ii} - a_{ij}^2) - a_{ij}(a_{jj}a_{ij} - a_{ij}a_{jj})\\ &= a_{jj}^2 a_{ii} - a_{jj}a_{ij}^2 \geq 0\end{aligned}$$

From (c), $a_{jj} \geq 0$, so $a_{jj}a_{ii} - a_{ij}^2 \geq 0 \Rightarrow a_{ij}^2 \leq a_{ii}a_{jj}$.