NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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**MA3209 Mathematical Analysis III**
AY 2007/2008 Sem 1

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Throughout this document, let $\overline{A}$ denote the closure of $A$; $N_r(x)$ be the open neighbourhood of $x$ with radius $r$. Assume that all the metric spaces stated are non-empty.

### Question 1

(a) Note that for all $x_1, x_2 \in X, y_1, y_2 \in Y$, since $d_X$ and $d_Y$ are metrics on $X$ and $Y$ respectively, so $d_X(x_1, x_2) < \infty$ and $d_Y(y_1, y_2) < \infty$. Hence, $d((x_1, y_1), (x_2, y_2)) = (d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p)^{\frac{1}{p}} < \infty$.

Since $d_X(x_1, x_2) \geq 0$ and $d_Y(y_1, y_2) \geq 0$, so $d((x_1, y_1), (x_2, y_2)) = (d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p)^{\frac{1}{p}} \geq 0$.

We have

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) = 0 &\Leftrightarrow (d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p)^{\frac{1}{p}} = 0 \\ &\Leftrightarrow d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p = 0 \\ &\Leftrightarrow d_X(x_1, x_2) = 0 \text{ and } d_Y(y_1, y_2) = 0 \\ &\Leftrightarrow x_1 = x_2 \text{ and } y_1 = y_2 \\ &\Leftrightarrow (x_1, y_1) = (x_2, y_2). \end{aligned}$$

We also have

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) &= (d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p)^{\frac{1}{p}} \\ &= (d_X(x_2, x_1)^p + d_Y(y_2, y_1)^p)^{\frac{1}{p}} \\ &= d((x_2, y_2), (x_1, y_1)) \end{aligned}$$

It suffices to show that $d$ satisfies the triangle inequality.
Take any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$.
Since $d_X$ is a metric, so $d_X(x_1, x_3) \leq d_X(x_1, x_2) + d_X(x_2, x_3)$. Hence,

$$d_X(x_1, x_3)^p \leq [d_X(x_1, x_2) + d_X(x_2, x_3)]^p$$

Similarly,

$$d_Y(y_1, y_3)^p \leq [d_Y(y_1, y_2) + d_Y(y_2, y_3)]^p$$

Recall the Minköwski's inequality:

$$\left[\sum_{i=1}^{n} |a_i + b_i|^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} |b_i|^p\right]^{\frac{1}{p}} \tag{1}$$

Putting $n = 2, a_1 = d_X(x_1, x_2),\ a_2 = d_Y(y_1, y_2),\ b_1 = d_X(x_2, x_3)$ and $b_2 = d_Y(y_2, y_3)$ into (1), we obtain:

$$\begin{aligned} d((x_1, y_1), (x_3, y_3)) &= [d_X(x_1, x_2)^p + d_Y(y_1, y_3)^p]^{\frac{1}{p}} \\ &\leq [(d_X(x_1, x_2) + d_X(x_2, x_3))^p + (d_Y(y_1, y_2) + d_Y(y_2, y_3))^p]^{\frac{1}{p}} \\ &\leq [d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p]^{\frac{1}{p}} + [d_X(x_2, x_3)^p + d_Y(y_2, y_3)^p]^{\frac{1}{p}} \qquad \text{by (1)} \\ &= d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)) \end{aligned}$$

$\therefore$ $d$ is a metric on $X \times Y$.

(b)(i) Let $f \in C[0,1]$ be a limit point of S. Then given any $\varepsilon > 0$, there exists a function $g_\varepsilon \in S$ such that $d_\infty(f, g_\varepsilon) < \varepsilon$. Hence,

$$\begin{aligned} |f(0)| &= |f(0) - g_\varepsilon(0)| \qquad \text{since } g_\varepsilon(0) = 0 \\ &\leq \sup\{|f(x) - g_\varepsilon(x)| : x \in [0,1]\} \\ &= d_\infty(f, g_\varepsilon) \\ &< \varepsilon. \end{aligned}$$

Since $\varepsilon$ is arbitrary, so $|f(0)| = 0$, which implies $f(0) = 0$. So $f \in S$.

$\therefore$ $S$ is <u>closed</u> in $(C[0,1], d_\infty)$.

(b)(ii) For any small positive $\varepsilon$, define the function $f_\varepsilon(x) : [0,1] \to \mathbb{R}$,

$$f_\varepsilon(x) = \begin{cases} \frac{2x}{\varepsilon} & \text{if } 0 \leq x < \frac{\varepsilon}{2} \\ 1 & \text{if } \frac{\varepsilon}{2} \leq x \leq 1 \end{cases}$$

Since $f_\varepsilon$ is continuous on $[0,1]$ and $f_\varepsilon(0) = 0$, so $f_\varepsilon \in S$.
Consider the function $g : [0,1] \to \mathbb{R}$, $g(x) \equiv 1$. So $g \in C[0,1]$, but $g \notin S$.

*Claim:* $g$ is a limit point of $S$.
*Proof:* Given any $\varepsilon > 0$,

$$\begin{aligned} d_1(f_\varepsilon, g) &= \int_0^1 |f_\varepsilon(x) - g(x)| \, dx \\ &= \int_0^{\frac{\varepsilon}{2}} \left| \frac{2x}{\varepsilon} - 1 \right| \, dx \\ &= \int_0^{\frac{\varepsilon}{2}} 1 - \frac{2x}{\varepsilon} \, dx \\ &= \left[ x - \frac{x^2}{\varepsilon} \right]_0^{\frac{\varepsilon}{2}} \\ &= \frac{\varepsilon}{4} \\ &< \varepsilon \end{aligned}$$

In other words, for every $\varepsilon > 0$, there exists $f_\varepsilon \in S$ such that $d_1(f_\varepsilon, g) < \varepsilon$. So $g$ is a limit point of $S$. Since $g \notin S$, so $S$ is <u>not closed</u> in $(C[0,1], d_1)$.

**Question 2**

(i) Take any $x, y \in X$.
Since $f(y) = \inf\{d(y,a) : a \in A\}$, so given any $\varepsilon > 0$, there exists $z \in A$ such that $d(y,z) \leq f(y) + \varepsilon$.
Then $f(x) \leq d(x,z) \leq d(x,y) + d(y,z) \leq d(x,y) + f(y) + \varepsilon$. So $f(x) - f(y) \leq d(x,y) + \varepsilon$.
Since $\varepsilon$ is arbitrary, so $f(x) - f(y) \leq d(x,y)$. Similarly, $f(y) - f(x) \leq d(x,y)$.
$\therefore$ $|f(x) - f(y)| \leq d(x,y)$ for all $x, y \in X$.

Now, given any $\varepsilon > 0$, we let $\delta = \varepsilon$. So whenever $x, y \in X$ and $d(x,y) < \delta$, we have $|f(x) - f(y)| \leq d(x,y) < \delta = \varepsilon$. So $f$ is uniformly continuous on $X$.

(ii)(a) Since $d(K, A) = \inf\{d(x, a) : x \in K, a \in A\}$, so given any $\varepsilon > 0$, there exists $k \in K$ and $a \in A$ such that $d(k, a) \leq d(K, A) + \varepsilon$.
Thus, $\inf\{f(x) : x \in K\} \leq f(k) = d(k, A) \leq d(k, a) \leq d(K, A) + \varepsilon$.
Since $\varepsilon$ is arbitrary, so

$$\inf\{f(x) : x \in K\} \leq d(K, A). \tag{2}$$

Furthermore, $\exists k \in K$ such that $f(k) \leq \inf\{f(x) : x \in K\} + \varepsilon$, so

$$\begin{aligned} d(K, A) &= \inf\{d(x, a) : a \in A, x \in K\} \\ &\leq \inf\{d(k, a) : a \in A\} \\ &= d(k, A) \\ &= f(k) \\ &\leq \inf\{f(x) : x \in K\} + \varepsilon \end{aligned}$$

Since $\varepsilon$ is arbitrary, so

$$d(K, A) \leq \inf\{f(x) : x \in K\} \tag{3}$$

By (2) and (3), we have $d(K, A) = \inf\{f(x) : x \in K\}$.

Since $f$ is continuous on the compact set $K$, so $f$ attains its minimum at some $k_1 \in K$, i.e. $\exists k_1 \in K$ such that $f(k_1) = \inf\{f(x) : x \in K\} = d(K, A)$.
Note that $d(K, A) \geq 0$. Suppose $d(K, A) = 0$. Then from above $f(k_1) = \inf\{d(k_1, a) : a \in A\} = 0$.
Therefore, for any $\varepsilon > 0$, there exists $a_\varepsilon \in A$ such that $d(k_1, a_\varepsilon) < \varepsilon$. So $k_1$ is a limit point of $A$.
Since $A$ is closed, so $k_1 \in A$. However, $K \cap A = \phi$ from the assumption given in the question, and this contradicts $k_1 \in K \cap A$.

$\therefore d(K, A) > 0$.

(ii)(b) From 2(ii)(a), $d(K, A) > 0$. Let $m = d(K, A)$.
Let $U := \{x \in X : d(x, K) < \frac{m}{2}\}$. Let $V := \{x \in X : d(x, A) < \frac{m}{2}\}$.
So $K \subseteq U$ and $A \subseteq V$.
*Claim:* $U \cap V = \phi$.
*Proof:* Suppose $U \cap V \neq \phi$. Take any $y \in U \cap V$. Since $y \in U$, so $d(y, K) < \frac{m}{2}$.
Since $\frac{m}{2} - d(y, K) > 0$, so there exists $k' \in K$ such that $d(y, k') < d(y, K) + [\frac{m}{2} - d(y, K)] = \frac{m}{2}$.
Similarly, there exists $a' \in A$ such that $d(y, a') < \frac{m}{2}$.
Thus, $d(k', a') \leq d(k', y) + d(y, a') < \frac{m}{2} + \frac{m}{2} = m = d(K, A) = \inf\{d(k, a) : k \in K, a \in A\}$.
This is a contradiction. Therefore, $U$ and $V$ are disjoint.$\square$

It remains to show that $U$ and $V$ are open in $X$.
Take any $x_0 \in U$.
<u>Case 1: $x_0 \in K$.</u>
For any $p$ in the neighbourhood $N_{\frac{m}{2}}(x_0)$, we have

$$\begin{aligned} d(p, K) &= \inf\{d(p, k) : k \in K\} \\ &\leq d(p, x_0) \\ &< \frac{m}{2}. \end{aligned}$$

So $p \in U$. Therefore, $N_{\frac{m}{2}}(x_0) \subseteq U$. Thus, every $x_0 \in K$ has an open neighbourhood contained in $U$.

<u>Case 2: $x_0 \notin K$.</u>

Since $K$ is closed, so $d(x_0, K) > 0$. Let $n = d(x_0, K)$.
Consider the neighbourhood $N_{\frac{m}{2}-n}(x_0) := \{x \in X : d(x_0, x) < \frac{m}{2} - n\}$.
Then for all $q \in N_{\frac{m}{2}-n}(x_0)$,

$$\begin{aligned} d(q, K) &= \inf\{d(q, k) : k \in K\} \\ &\leq \inf\{d(q, x_0) + d(x_0, k) : k \in K\} \\ &< \inf\{(\tfrac{m}{2} - n) + d(x_0, k) : k \in K\} \\ &= (\tfrac{m}{2} - n) + \inf\{(d(x_0, k) : k \in K\} \\ &= (\tfrac{m}{2} - n) + d(x_0, K) \\ &= \tfrac{m}{2} - n + n \\ &= \tfrac{m}{2} \end{aligned}$$

Thus, for all $q \in N_{\frac{m}{2}-n}(x_0)$, $q \in U$. Thus, every $x_0 \in U \setminus K$ has an open neighbourhood which is contained in $U$.

Combining the two cases, we conclude that every $x_0 \in U$ is an interior point of $U$.

$\therefore U$ is open in $X$.

Note that in the above proof that $U$ is open in $X$, we only made use of the assumption that $K$ is closed in $X$. Since $A$ being compact implies that $A$ is closed in $X$, so by the same argument as above, we can conclude that $V$ is open in $X$.

**Question 3**

(1) $\Rightarrow$ (2):
Assume $X$ is connected, and $f$ is locally constant.
Fix a point $p \in X$. Define $S_1 := \{x \in X : f(x) = f(p)\}$, $S_2 := \{x \in X : f(x) \neq f(p)\}$
Since $p \in S_1$, so $S_1$ is non-empty. Suppose $f$ is not a constant function, then there exists $x_0 \in X$ such that $f(x_0) \neq f(p)$. So $S_2$ is non-empty.

For any $x_1 \in S_1$, there exists an open neighbourhood $U_{x_1}$ containing $x_1$ such that $f(U_{x_1}) = \{p\}$ since $f$ is locally constant. Note that $U_{x_1} \subseteq S_1$. Therefore, $S_1$ is open in X. Similarly $S_2$ is open in X.

Note that $S_1 \cap S_2 = \phi$ and $S_1 \cup S_2 = X$. So $X$ can be written as the union of two non-empty disjoint sets, so $X$ is disconnected. This contradicts our assumption that $X$ is connected.$\square$

(2) $\Rightarrow$ (1):
Suppose $X$ is not connected, then $X$ is a disjoint union of two non-empty open sets $A$ and $B$. Define $f : X \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$

For any $x_0 \in X$. $x_0 \in A$ or $x_0 \in B$, but not both. Without loss of generality, assume $x_0 \in A$. Since $A$ is open, there exists an open neighbourhood $U_{x_0}$ of $x_0$ such that $U_{x_0} \subseteq A$. So $f(U_{x_0}) = \{0\}$. Thus, $f$ is locally constant. By assumption, $f$ must be a constant function. However, $f$ is not a constant function from definition. Therefore, $X$ must be connected.$\square$

**Question 4**

(i) Since $T^N$ is a contraction mapping, so there exists $c \in (0, 1)$ such that for all $x, y \in X$,

$$d(T^N(x), T^N(y)) \leq c\, d(x, y) \tag{4}$$

Since $X$ is complete, so by the contraction mapping principle, there exists a unique fixed point $x_0 \in X$ such that $T^N(x_0) = x_0$. Using (4), we obtain:

$$\begin{aligned}
d(T^N(T(x_0)), T^N(x_0)) &\leq c\ d(T(x_0), x_0) \\
d(T(T^N(x_0)), T^N(x_0)) &\leq c\ d(T(x_0), x_0) \\
d(T(x_0), x_0) &\leq c\ d(T(x_0), x_0)
\end{aligned}$$

This can only happen if $d(T(x_0), x_0) = 0$. Thus, $T(x_0) = x_0$. So $x_0$ is a fixed point of $T$. It remains to show that $T$ has at most one fixed point.

Suppose $T$ has at least two different fixed points. Denote two of these by $x_1$ and $x_2$. Since $x_1 \neq x_2$, so $d(x_1, x_2) > 0$. We have $d(x_1, x_2) = d(T^N(x_1), T^N(x_2)) \leq c\ d(x_1, x_2)$. Dividing by $d(x_1, x_2)$, we obtain $1 \leq c$, which is a contradiction.

$\therefore T$ has a unique fixed point, namely $x_0$.

(ii)(a) Suppose $\phi$ is a contraction mapping on $C[0, 1]$, then there exists $c \in (0, 1)$ such that for all $f_1, f_2 \in C[0, 1]$,

$$d(\phi(f_1), \phi(f_2)) \leq c\ d(f_1, f_2)$$

where $d$ stands for the uniform metric.
So for all $f_1, f_2 \in C[0, 1]$,

$$\begin{aligned}
d\left(\sin x + \int_0^x f_1(t)\ dt, \sin x + \int_0^x f_2(t)\ dt\right) &\leq c\ d(f_1, f_2) \\
\sup_{x\in[0,1]} \left|\left(\sin x + \int_0^x f_2(t)\ dt\right) - \left(\sin x + \int_0^x f_1(t)\ dt\right)\right| &\leq c \sup_{x\in[0,1]} |f_1(x) - f_2(x)| \\
\sup_{x\in[0,1]} \left|\int_0^x f_2(t)\ dt - \int_0^x f_1(t)\ dt\right| &\leq c \sup_{x\in[0,1]} |f_1(x) - f_2(x)|
\end{aligned}$$

In particular, let $f_1 \equiv 0$ and $f_2 \equiv 1$ on $[0, 1]$. Then

$$\begin{aligned}
\sup_{x\in[0,1]} \left|\int_0^x 1\ dt\right| &\leq c \sup_{x\in[0,1]} 1 \\
\sup_{x\in[0,1]} x &\leq c \\
1 &\leq c
\end{aligned}$$

This is a contradiction. Therefore, $\phi$ cannot be a contraction mapping.

(ii)(b) *Claim:* $\phi^2$ is a contraction mapping on $C[0, 1]$.
*Proof:* For all $f \in C[0, 1]$ and for all $x \in [0, 1]$,

$$\begin{aligned}
(\phi^2 f)(x) &= \phi\left(\sin x + \int_0^x (\phi f)(t)\ dt\right) \\
&= \sin x + \int_0^x \left(\sin t + \int_0^t f(u)\ du\right)\ dt \\
&= \sin x + \int_0^x \sin t\ dt + \int_0^x \int_0^t f(u)\ du\ dt
\end{aligned}$$

So for all $f_1, f_2 \in C[0,1]$ and for all $x \in [0,1]$,

$$\begin{aligned}
\left|(\phi^2 f_2)(x) - (\phi^2 f_1)(x)\right| &= \left|\int_0^x \int_0^t f_2(u) - f_1(u)\ du\ dt\right| \\
&\leq \int_0^x \left|\int_0^t f_2(u) - f_1(u)\ du\right| dt \\
&\leq \int_0^x t \sup_{u\in[0,1]} |f_2(u) - f_1(u)|\ dt \\
&= \left(\sup_{u\in[0,1]} |f_2(u) - f_1(u)|\right) \int_0^x t\ dt \\
&= \frac{x^2}{2}\ d(f_1, f_2) \\
&\leq \frac{1}{2}\ d(f_1, f_2)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sup_{x\in[0,1]} \left|(\phi^2 f_2)(x) - (\phi^2 f_1)(x)\right| &\leq \frac{1}{2}\ d(f_1, f_2) \\
d(\phi^2 f_1, \phi^2 f_2) &\leq \frac{1}{2}\ d(f_1, f_2)
\end{aligned}$$

So $\phi^2$ is a contraction mapping on $C[0,1]$. Since $C[0,1]$ is complete, so by part (i), $\phi$ has a unique fixed point.

**Question 5**

(a)(i) Let $(z_i)$ be a limit point of $c$. We want to show that $(z_i) \in c$.

Let $\varepsilon > 0$ be given.
Then we can choose a convergent sequence $(x_i)$ such that $d((x_i),(z_i)) = \sup_{i\in\mathbb{N}}|x_i - z_i| < \frac{\varepsilon}{3}$.
Since $(x_i)$ is Cauchy, so there exists $N_0 \in \mathbb{N}$ such that for all $m, n \geq N_0$, we have $|x_m - x_n| < \frac{\varepsilon}{3}$.
So for all $m, n \geq N_0$,

$$\begin{aligned}
|z_m - z_n| &= |(z_m - x_m) + (x_m - x_n) + (x_n - z_n)| \\
&\leq |z_m - x_m| + |x_m - x_n| + |x_n - z_n| \\
&\leq \sup_{i\in\mathbb{N}} |x_i - z_i| + \frac{\varepsilon}{3} + \sup_{i\in\mathbb{N}} |x_i - z_i| \\
&= \varepsilon
\end{aligned}$$

Thus, $(z_i)$ is a Cauchy sequence. Since $(z_i)$ is a real sequence, so it is also a convergent sequence, i.e. $(z_i) \in c$.

$\therefore c$ is closed in $(\ell^\infty, d)$.

(a)(ii) Since a closed subset of a complete metric space is complete, so $c$ being a closed subset of the complete metric space $(\ell^\infty, d)$ is complete.

(b) We need to show that $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall f \in \mathcal{F}$ and $\forall x_1, x_2 \in K$, $d(x_1, x_2) < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$.

Let $\varepsilon > 0$ be given. Since $\mathcal{F}$ is totally bounded, there exists a finite subset $S = \{f_1, f_2, ..., f_n\} \subseteq C(K)$ such that

$$\mathcal{F} \subseteq \bigcup_{i=1}^{n} N_\varepsilon(f_i) \tag{5}$$

where $N_\varepsilon(f_i) = \{f \in C(K) : \|f - f_i\| < \varepsilon\}$.

Since continuous functions on compact sets are uniformly continuous, so $f_1, f_2, ..., f_n$ are all uniformly continuous on $K$. So for all $i \in \{1, 2, ..., n\}$, $\exists \delta_i > 0$ such that for all $x_1, x_2 \in K$ with $d(x_1, x_2) < \delta_i$, we have $|f_i(x_1) - f_i(x_2)| < \varepsilon$.

Let $\delta = \min\{\delta_1, \delta_2, ..., \delta_n\}$.

Given any $f \in \mathcal{F}$, by (5), $f \in N_\varepsilon(f_i)$ for some $i \in \{1, 2, ..., n\}$, i.e. $\sup_{x \in K} |f(x) - f_i(x)| < \varepsilon$.

Then $\forall x_1, x_2 \in K$ with $d(x_1, x_2) < \delta$, we have

$$\begin{aligned} |f(x_1) - f(x_2)| &\leq |f(x_1) - f_i(x_1)| + |f_i(x_1) - f_i(x_2)| + |f_i(x_2) - f(x_2)| \\ &< \varepsilon + \varepsilon + \varepsilon \\ &= 3\varepsilon. \end{aligned}$$

$\therefore \mathcal{F}$ is equicontinuous on $K$.