NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1101R Linear Algebra I
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Question 1

(i) $\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \overset{GJE}{\rightarrow} \left(\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

A basis for the row space of A is: {(1 0 -1 0 0),(0 1 1 0 0),(0 0 0 0 1)}.

A basis for the column space of A is: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

(ii) $\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \overset{GJE}{\rightarrow} \left(\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Solve the equation system $\begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \\ x_5 = 0 \end{cases}$, we get $\begin{cases} x_1 = s \\ x_2 = -s \\ x_3 = s \\ x_4 = t \\ x_5 = 0 \end{cases}$ where s,t are arbitrary parameters.

Hence, the homogeneous system Ax=0 has a general solution x=s $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ + t $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ where s,t are arbitrary parameters.

Thus a basis for the nullspace of A is $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(iii) rank(A)=3, nullity(A)=2,
nullity(A^T)=no of columns of A^T- rank(A^T)=no of columns of A^T- rank(A)=$4 - 3 = 1$.

(iv) We form a matrix K=$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ using the vectors in the basis for the nullspace of A. We choose (0 1 0 0 0),(0 0 1 0 0),(0 0 0 0 1) to extend the basis. These vectors are linearly independent, hence {(1 -1 1 0 0),(0 0 0 1 0),(0 1 0 0 0),(0 0 1 0 0),(0 0 0 0 1)} is a basis for $\mathbb{R}^5$.

(v) Suppose there is a non-zero vector u such that u belongs to the nullspace of A and the row space of A.

Hence Au=$\mathbf{0}$, $a_i \cdot u$=0, where a_i is the rowvector of A, for i=1,2 ,..,n.

And u=$c_1 \cdot a_1 + c_2 \cdot a_2 + ... + c_n \cdot a_n$, where c_i is constant for i=1,2 ,..,n.

Hence $u \cdot u$=$(\sum c_i \cdot a_i) \cdot u$=$\sum c_i \cdot (a_i \cdot u)$=$\sum c_i \cdot 0 = 0$. But u is a non-zero vector, therefore $u \cdot u$ >0.

So there is no non-zero vector u such that u belongs to both the nullspace of A and the row space of A.

(vi) Suppose there is a matrix B such that AB is invertible. We write $A = \begin{pmatrix} a_1^T \\ a_2^T \\ a_3^T \\ a_4^T \end{pmatrix}$,

then $A^T = (a_1 \ a_2 \ a_3 \ a_4)$, $B^T A^T = (B^T a_1 \ B^T a_2 \ B^T a_3 \ B^T a_4)$. Since A has two identical rows, $a_3^T = a_4^T$, $B^T A^T$ has two identical columns, $B^T a_3 = B^T a_4$. Hence the determinant of $B^T A^T$ is zero. Hence $\det(AB) = \det((AB)^T) = \det(B^T A^T) = 0$. AB is singular. So it is impossible to find a matrix B such that AB is an invertible matrix.

Question 2

(a)

(i) Since A is a triangular matrix, λ_1=1, λ_2=2.

(ii) For λ_1=1,

$(\lambda I - A)x = 0$

$$\Leftrightarrow \left(\begin{array}{cccc|c} 1-1 & -1 & 0 & 0 & 0 \\ 0 & 1-1 & 0 & 0 & 0 \\ 0 & 0 & 1-2 & -1 & 0 \\ 0 & 0 & 0 & 1-2 & 0 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{cccc|c} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right)$$

$\Leftrightarrow x = t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ where t is an arbitrary parameter.

Hence, a basis for the eigenspace of A associated with λ_1=1 is $\{(1\ 0\ 0\ 0)\}$.

For λ_2=2,

$(\lambda I - A)x = 0$

$$\Leftrightarrow \left(\begin{array}{cccc|c} 2-1 & -1 & 0 & 0 & 0 \\ 0 & 2-1 & 0 & 0 & 0 \\ 0 & 0 & 2-2 & -1 & 0 \\ 0 & 0 & 0 & 2-2 & 0 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Leftrightarrow x = t\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ where t is an arbitrary parameter.

Hence, a basis for the eigenspace of A associated with λ_2=2 is $\{(0\ 0\ 1\ 0)\}$.

(iii) Since we only have two linearly independent eigenvectors, A is not diagonalizable.

(b)

(i) Since B$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and B$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, B has two linearly independent eigenvectors, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$,$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so B is diagonalizable. So we can write B=PDP^{-1}, where P=$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, D=$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$. Since det(P)=0, P is invertible, P^{-1}=$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$.

(ii) B^n=PD^nP^{-1}

$=\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2^n & 0 \\ 0 & -1^n \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$=\begin{pmatrix} \frac{2^n+(-1)^n}{2} & \frac{2^n+(-1)^{n+1}}{2} \\ \frac{2^n+(-1)^{n+1}}{2} & \frac{2^n+(-1)^n}{2} \end{pmatrix}$.

(iii) From (ii) B^1=$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$, suppose $\boldsymbol{v}$=$\begin{pmatrix} x \\ y \end{pmatrix}$

B$\boldsymbol{v} = \boldsymbol{v}$

$\Leftrightarrow \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$

$\Leftrightarrow \left(\begin{array}{cc|c} -\frac{1}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 \end{array}\right)$

$\Leftrightarrow \boldsymbol{v} = \mathbf{0}$

Hence, it is impossible to find a non-zero column vector $\boldsymbol{v}$ such that B$\boldsymbol{v} = \boldsymbol{v}$.

Question 3

(a)

(i) The equation

$c_1(1,1,0) + c_2(0,1,1) + c_3(0,0,1) = (0,0,0)$

give us a linear system

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

we find this linear system has only the trivial solution, so S is linearly independent. And $|S|$=dim($\mathbb{R}^3$)=3, so S is a basis for $\mathbb{R}^3$.

The equation

$c_1(1,0,1)+c_2(0,1,1)+c_3(0,1,0)=(0,0,0)$

give us a linear system

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \overset{GJE}{\rightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

we find this linear system has only the trivial solution, so T is linearly independent. And $|T|$=dim($\mathbb{R}^3$)=3, so T is a basis for $\mathbb{R}^3$.

(ii) By $\left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right) \overset{GJE}{\rightarrow} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right)$

so P=$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$ is the transition matrix from T to S.

(iii) Since $(w)_T=(1,2,-1)$, $w=1(1,0,1)+2(0,1,1)+(-1)(0,1,0)=(1,1,3)$.

Since $[w]_S=P[w]_T=\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$, $(w)_S=(1,0,3)$.

(iv) Yes, let $v=(1,1,1)$, $(v)_T=(1,0,1)=(v)_S$.

(b)

(i) True

Suppose $\dim(U\cap V)=0$, let S=$\{s_1,s_2\}, T=\{t_1,t_2,t_3\}$ be bases of U and V. Since $U\cap V=0$, s_1,s_2 are not linear combinations of t_1,t_2,t_3. Hence $K=\{s_1,s_2,t_1,t_2,t_3\}$ is linearly independent. However $|K|=5>\dim(\mathbb{R}^4)$, so K is linearly dependent. This is a contradiction.

(ii) True

Since $U\bigcap V$ is a subspace of U, then $\dim(U\cap V)\leq\dim(U)=2$, and since U is not a subset of V, $U\cap V\neq U$, dim($U\cap V$)<dim(U)=2, and dim($U\cap V$)≥1, so dim($U\cap V$)=1.

Question 4

(a)

(i) $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right) \overset{GJE}{\rightarrow} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right)$

the last column of a row-echelon form of the augmented matrix is a pivot column, so Ax=b is an inconsistent system.

(ii) $A^TAx=A^Tb$

$$\Rightarrow\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}\text{x}=\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$\Rightarrow$x=$\begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$.

(iii) Let $T = \{(1,1,0),(-1,1,1)\}$, since T is linearly independent and (1 1 0)$\cdot$ (-1 1 1)=0, so T is an orthogonal basis for the column space of A, the projection p of b onto the column space of A is p=$\frac{b\cdot t_1}{\|t_1\|^2}t_1+\frac{b\cdot t_2}{\|t_2\|^2}t_2$=(0.5,0.5,0).

(iv) Let $n = b - p = \begin{pmatrix} -0.5 \\ 0.5 \\ -1 \end{pmatrix}$, then n is orthogonal to column space of A,

and T$\cup\{n\}$=$\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -0.5 \\ 0.5 \\ -1 \end{pmatrix}\right\}$ is an orthogonal basis for $\mathbb{R}^3$. And by normalizing the vectors, an orthonormal basis S for $\mathbb{R}^3$ is S=$\left\{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \end{pmatrix}\right\}$.

(v) $(b)_S = (b\cdot s_1, b\cdot s_2, b\cdot s_3)$=($\frac{1}{\sqrt{2}}$,0,$\frac{3}{\sqrt{6}}$).

(b) Let A=(a_1 a_2 .. a_n), $n = b - p$ is orthogonal to the column space of A, so $a_1\cdot n$=0, $a_2\cdot n$=0,.., $a_n\cdot n$=0. That is

$$\begin{pmatrix} a_1^T \\ a_2^T \\ .. \\ a_n^T \end{pmatrix}\text{n=0}$$

Hence A^Tn=0, $n = b - p$ is a solution of $A^Tx = 0$.

Question 5

(i) T$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$=T$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$-T$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$=$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$-$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$=$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

T$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$=T$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$-T$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$=$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$-$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$=$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

(ii) The standard matrix A for T is

A=(T(e_1) T(e_2) T(e_3))=$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$.

(iii) The kernel of T is the nullspace of A,

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right) \overset{GJE}{\rightarrow} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right)$$

Hence, ker(T)=$\left\{\begin{pmatrix} t \\ 2t \\ t \end{pmatrix} \mid t\in\mathbb{R}\right\}$ =span$\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right\}$.

(iv) True

We reduce A to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix}$

so the dimension of R(T), rank(T)=rank(A)=2.

Since R(T) is a subspace of $\mathbb{R}^2$ and dim(R(T))=dim($\mathbb{R}^2$)=2, R(T)=$\mathbb{R}^2$. Hence, every vector in $\mathbb{R}^2$ is an image under T.

(v) The formula of S $\circ$ T is

$$\mathrm{S} \circ \mathrm{T}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y+2z \\ -y+2z \\ -y+2z \end{pmatrix}$$

(vi) Since S $\circ$ T(v)=v

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y+2z \\ -y+2z \\ -y+2z \end{pmatrix} \Rightarrow \begin{pmatrix} x+y-2z \\ 2y-2z \\ y-z \end{pmatrix} = \mathbf{0}$$

$$\mathrm{v} \in \left\{ \begin{pmatrix} t \\ t \\ t \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

(vii) Since $\mathrm{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y+z \\ -x+z \end{pmatrix}$

Let $x-y+z=-x+z$, we get $-2x+y+0z=0$, hence $-2x+y+0z=0$ is the equation of the plane in $\mathbb{R}^3$ that is transformed to the line $x-y=0$ in $\mathbb{R}^2$ under T.