MA2002 - Calculus Suggested Solutions

(Semester 1, AY2022/2023)

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Question 1

Question 1a

$$\begin{aligned}
\frac{x^2+x-2}{x^2-1}-\frac{3}{2} &= \frac{(x-1)(x+2)}{(x-1)(x+1)}-\frac{3}{2} \\
&= \frac{x+2}{x+1}-\frac{3}{2} \\
&= \frac{2(x+2)-3(x+1)}{2(x+1)} \\
&= \frac{2x+4-3x-3}{2(x+1)} \\
&= \frac{-x+1}{2(x+1)} \\
&= -\frac{x-1}{2(x+1)}.
\end{aligned}$$

Notice if we take $|x-1|<1$, then

$$\begin{aligned}
-1 < x-1 &< 1 \\
1 < x+1 &< 3 \\
1 > \frac{1}{x+1} &> \frac{1}{3}.
\end{aligned}$$

i.e., $\left|\frac{1}{x+1}\right|<1$. Hence, let $\varepsilon>0$ and $\delta=\min\{1,2\varepsilon\}$, so for $0<|x-1|<\delta$,

$$\begin{aligned}
\left|\frac{x^2+x-2}{x^2-1}-\frac{3}{2}\right| &= \left|-\frac{x-1}{2(x+1)}\right| \\
&= \left|-\frac{1}{2}\right|\cdot\frac{|x-1|}{|x+1|} \\
&< \frac{1}{2}\cdot|x-1| && \because \frac{1}{|x+1|}<1 \\
&< \frac{1}{2}\cdot 2\varepsilon \\
&= \varepsilon.
\end{aligned}$$

By the $\epsilon-\delta$ definition, we have proven that

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-1}=\frac{3}{2}.$$

Question 1b(i)

For $(-\infty, 1)$, f is a polynomial. For $(1, \infty)$, f is a rational function that has no asymptotes. This proves that f is differentiable when $x \neq 1$. It remains to prove that f is differentiable at $x = 1$. It will suffice to show that the difference quotient has both right- and left-handed limits that are equal.

For $x < 1$,

$$\begin{aligned}
\frac{f(x) - f(1)}{x - 1} &= \frac{\frac{13 - x^2}{8} - \frac{3}{2}}{x - 1} \\
&= \frac{(13 - x^2) - (3 \cdot 4)}{8(x - 1)} \\
&= \frac{1 - x^2}{8(x - 1)} \\
&= \frac{(1 - x)(1 + x)}{-8(1 - x)} \\
&= \frac{1 + x}{-8} \\
\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} &= \frac{1 + 1}{-8} \\
&= -\frac{2}{8} \\
&= -\frac{1}{4}.
\end{aligned}$$

For $x > 1$,

$$\begin{aligned}
\frac{f(x) - f(1)}{x - 1} &= \frac{\frac{x+2}{x+1} - \frac{3}{2}}{x - 1} \\
&= \frac{2(x + 2) - 3(x + 1)}{2(x + 1)(x - 1)} \\
&= \frac{2x + 4 - 3x - 3}{2(x + 1)(x - 1)} \\
&= \frac{-x + 1}{2(x + 1)(x - 1)} \\
&= -\frac{x - 1}{2(x + 1)(x - 1)} \\
&= -\frac{1}{2(x + 1)} \\
\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} &= -\frac{1}{2(1 + 1)} \\
&= -\frac{1}{4}.
\end{aligned}$$

Since

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = -\frac{1}{4},$$

$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$ exists, so $f'(1)$ exists and is equal to $-\frac{1}{4}$. Hence, the given piecewise function is differentiable everywhere.

Question 1b(ii)

$$f'(x) = \begin{cases} \frac{d}{dx}\frac{13-x^2}{8}, & x < 1 \\ \frac{-1}{4}, & x = 1 \\ \frac{d}{dx}\frac{x+2}{x+1}, & x > 1 \end{cases}$$

$$= \begin{cases} \frac{-x}{4}, & x < 1 \\ \frac{-1}{4}, & x = 1 \\ \frac{(x+1)-(x+2)}{(x+1)^2}, & x > 1 \end{cases}$$

$$= \begin{cases} \frac{-x}{4}, & x \leq 1 \\ \frac{-1}{(x+1)^2}, & x > 1 \end{cases}$$

It is clear that (i) for $0 < x \leq 1$, $f'(x) < 0$, (ii) for $x < 0$, $f'(x) > 0$, and (iii) for $x > 1$, $f'(x) < 0$. i.e., for $x < 0$, $f'(x) > 0$, and for $x > 0$, $f'(x) < 0$, so f increases for $x < 0$ and decreases for $x > 0$. Hence, the global maximum of f is at $x = 0$, so the maximum point is $(0, f(0)) = (0, \frac{13}{8})$.

Question 2

Question 2a

Let $u = \sin(x)$, so that $du = \cos(x)dx$, and when $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$.

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{\cos(x)(\sin^4(x) + 2\sin^2(x) + \sin(x) + 2)}{(\sin(x)+1)(\sin^2(x)+1)^2} dx \\
=& \int_0^1 \frac{u^4 + 2u^2 + u + 2}{(u+1)(u^2+1)^2} du \\
=& \int_0^1 \frac{(u^4 + 2u^2 + 1) + (u+1)}{(u+1)(u^2+1)^2} du \ (*) \\
=& \int_0^1 \frac{(u^2+1)^2 + (u+1)}{(u+1)(u^2+1)^2} du \\
=& \int_0^1 \frac{1}{u+1} + \frac{1}{(u^2+1)^2} du
\end{aligned}$$

$(*)$ Alternatively, one can use direct addition or partial fraction to show that

$$\frac{1}{u+1} + \frac{1}{(u^2+1)^2} = \frac{u^4 + 2u^2 + u + 2}{(u+1)(u^2+1)^2}.$$

Question 2b

$$\int_0^1 \frac{1}{u+1} du = \ln(u+1)\Big|_0^1 = \ln(2) - \ln(1) = \ln(2).$$

For the other integral, we proceed with the substitution $u = \tan\theta$. So $du = \sec^2\theta d\theta$, when $u = 0$, $\theta = 0$; when $u = 1$, $\theta = \frac{\pi}{4}$.

$$\begin{aligned}
\int_0^1 \frac{1}{(u^2+1)^2} du &= \int_0^{\frac{\pi}{4}} \frac{1}{(\tan^2(\theta)+1)^2} \cdot \sec^2(\theta) d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^4(\theta)} \cdot \sec^2(\theta) d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2(\theta)} d\theta \\
&= \int_0^{\frac{\pi}{4}} \cos^2(\theta) d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1 + \cos(2\theta)}{2} d\theta \\
&= \left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right)\Bigg|_0^{\frac{\pi}{4}} \\
&= \left(\frac{1}{2} \cdot \frac{\pi}{4} + \frac{1}{4}\sin\left(\frac{\pi}{2}\right)\right) - \left(\frac{1}{2}(0) + \frac{1}{4}\sin(0)\right) \\
&= \frac{\pi}{8} + \frac{1}{4}.
\end{aligned}$$

Alternatively, one could also use the substitution $u = \frac{1}{x}$. So, $du = -\frac{1}{x^2}dx$, and when $u = 1, x = 1$; when $u = 0, x = \infty$.

$$
\begin{aligned}
\int_0^1 \frac{1}{(u^2+1)^2} du &= \int_\infty^1 \frac{-\frac{1}{x^2}}{\left(\frac{1}{x^2}+1\right)^2} dx \\
&= \int_1^\infty \frac{x^2}{(1+x^2)^2} dx \ (*) \\
&= x \times \frac{-1}{2(1+x^2)}\bigg|_1^\infty - \int_1^\infty 1 \times \frac{-1}{2(1+x^2)} dx \\
&= \frac{1}{4} + \frac{1}{2}\arctan(x)\bigg|_1^\infty \qquad\qquad \because \int \frac{1}{1+x^2} dx = \arctan(x) \\
&= \frac{1}{4} + \frac{\pi}{8}.
\end{aligned}
$$

$(*)$ Here, we use integration by parts with

$$
\begin{aligned}
u &= x & du &= dx \\
dv &= \frac{x}{(1+x^2)^2} dx & v &= \frac{-1}{2(1+x^2)}.
\end{aligned}
$$

Then

$$\int_0^1 \frac{1}{u+1} + \frac{1}{(u^2+1)^2} du = \ln(2) + \frac{\pi}{8} + \frac{1}{4}.$$

Question 3

Question 3a

Let L denote the limit in question.

$$\begin{aligned}
L &= \lim_{x\to 2} \frac{(x-1)^{\frac{1}{x-2}}}{e^{x-1}} \\
\ln(L) &= \lim_{x\to 2} \ln\left(\frac{(x-1)^{\frac{1}{x-2}}}{e^{x-1}}\right) && \because \text{by continuity of } \ln \\
&= \lim_{x\to 2} \left(\frac{1}{x-2}\ln(x-1) - \ln\left(e^{x-1}\right)\right) \\
&= \lim_{x\to 2} \frac{\ln(x-1)}{x-2} - \lim_{x\to 2} \ln(e^{x-1}) \\
&= \lim_{x\to 2} \frac{\frac{1}{x-1}}{1} - \lim_{x\to 2}(x-1) && \because \frac{\ln(x-1)}{x-2} \to \frac{0}{0} \text{ so we apply L'H on the first limit} \\
&= \frac{1}{2-1} - (2-1) \\
&= 1-1 \\
&= 0 \\
L &= e^0 = 1.
\end{aligned}$$

Question 3b

Consider $x \in \mathbb{R}$ such that $0 < |x| < \varepsilon$. Either $f(x) > 0$ or $f(x) = 0$. Suppose x is such that $f(x) > 0$. Then

$$0 < \frac{f(x)e^{-\frac{1}{x^2}}}{1+f(x)} = \frac{e^{-\frac{1}{x^2}}}{\frac{1}{f(x)}+1} < e^{-\frac{1}{x^2}} \text{ since } 1 + \frac{1}{f(x)} > 1.$$

Suppose x is such that $f(x) = 0$. Then

$$0 \leq \frac{f(x)e^{-\frac{1}{x^2}}}{1+f(x)} = 0 < e^{-\frac{1}{x^2}}.$$

Hence, for $x \in (-\varepsilon, \varepsilon) \setminus \{0\}$,

$$0 \leq \frac{f(x)e^{-\frac{1}{x^2}}}{1+f(x)} < e^{-\frac{1}{x^2}}.$$

Since $\lim_{x\to 0} e^{-\frac{1}{x^2}} = 0$, by Squeeze Theorem,

$$\lim_{x\to 0} \frac{f(x)e^{-\frac{1}{x^2}}}{1+f(x)} = 0.$$

Question 4

Question 4a

By the disc method, we integrate along the x-axis from 0 to 2π, with the radius of each disc being $f(x) - 1 = \sin(x) + 2$.

$$
\begin{aligned}
V &= \int_0^{2\pi} \pi(\sin(x)+2)^2 dx \\
&= \pi \int_0^{2\pi} \left(\sin^2(x) + 4\sin(x) + 4\right) dx \\
&= \pi \int_0^{2\pi} \left(\frac{1-\cos(2x)}{2} + 4\sin(x) + 4\right) dx \\
&= \pi \cdot \left(\frac{x}{2} - \frac{\sin(2x)}{4} - 4\cos(x) + 4x\right)\Bigg|_0^{2\pi} \\
&= \pi\left(\left(\frac{2\pi}{2} - \frac{\sin(4\pi)}{4} - 4\cos(2\pi) + 4(2\pi)\right) - \left(\frac{0}{2} - \frac{\sin(0)}{4} - 4\cos(0) + 4(0)\right)\right) \\
&= \pi(\pi - 0 - 4 + 8\pi + 4) \\
&= \pi(9\pi) \\
&= 9\pi^2.
\end{aligned}
$$

Question 4b

By the shell method, we integrate along the x-axis from 0 to 2π, and each shell has radius x and height $f(x) = \sin(x) + 3$.

$$
\begin{aligned}
V &= \int_0^{2\pi} 2\pi x(\sin(x)+3) dx \\
&= 2\pi \int_0^{2\pi} (x\sin(x) + 3x)\, dx \\
&= 2\pi \left(\int_0^{2\pi} x\sin(x) dx + \int_0^{2\pi} 3x dx\right) \ (*) \\
&= 2\pi \left(-x\cos(x)|_0^{2\pi} + \int_0^{2\pi} (-\cos(x))\, dx + \frac{3}{2}x^2\Big|_0^{2\pi}\right) \\
&= 2\pi \left(-2\pi\cos(2\pi) + 0\cos(0) + (\sin(x))_0^{2\pi} + \left(\frac{3}{2}(2\pi)^2 - \frac{3}{2}(0)^2\right)\right) \\
&= 2\pi\left(-2\pi + (\sin(2\pi) - \sin(0)) + 6\pi^2\right) \\
&= 2\pi(6\pi^2 - 2\pi) \\
&= 4\pi^2(3\pi - 1).
\end{aligned}
$$

$(*)$ Here, we use integration by parts with

$$
\begin{aligned}
u &= x & du &= dx \\
dv &= \sin(x)dx & v &= -\cos(x).
\end{aligned}
$$

Question 4c(i)

The height is given by $f(x) - 3 = \sin(x)$. We will integrate from 0 to $\frac{\pi}{2}$ in order to have an injective substitution later on. By symmetry of $\sin(x)$, the required surface area S is 4 times of the surface area from 0 to $\frac{\pi}{2}$. By the surface area formula,

$$S = 4\int_0^{\pi/2} 2\pi \sin(x)\sqrt{1+\cos^2(x)}dx.$$

We proceed with the substitution $u = \cos(x)$, so that $du = -\sin(x)dx$, and when $x = 0, u = 1$; when $x = \frac{\pi}{2}, u = 0$.

$$S = 8\pi\int_1^0 -\sqrt{1+u^2}du = 8\pi\int_0^1 \sqrt{1+u^2}du.$$

Question 4c(ii)

We proceed with the substitution $u = \tan\theta$. So, $du = \sec^2(\theta)d\theta$, and when $u = 0, \theta = 0$; when $u = 1, \theta = \frac{\pi}{4}$.

$$\begin{aligned}
S &= 8\pi\int_0^{\pi/4} \sqrt{1+\tan^2(\theta)}\sec^2(\theta)d\theta \\
&= 8\pi\int_0^{\pi/4} \sqrt{\sec^2(\theta)}\sec^2(\theta)d\theta \\
&= 8\pi\int_0^{\pi/4} \mid\sec(\theta)\mid\sec^2(\theta)d\theta \\
&= 8\pi\int_0^{\pi/4} \sec^3(\theta)d\theta & \because \theta \in \left[0, \frac{\pi}{4}\right] \implies \sec(\theta) > 0 \\
&= 8\pi\int_0^{\pi/4} \cos^{-3}(\theta)d\theta.
\end{aligned}$$

One can proceed from here with integration by parts, but we will use the identity given, taking $n = -1$, to solve for $\int_0^{\pi/4}\cos^{-3}\theta d\theta$, and substitute back into S.

$$\begin{aligned}
\int_0^{\pi/4}\cos^{-1}(\theta)d\theta &= \frac{1}{-1}\sin(\theta)\cos^{-1-1}(\theta)\Big|_0^{\pi/4} + \frac{-1-1}{-1}\int_0^{\pi/4}\cos^{-3}(\theta)d\theta \\
\int_0^{\pi/4}\sec(\theta)d\theta &= -\sin(\theta)\cos^{-2}(\theta)|_0^{\pi/4} + 2\int_0^{\pi/4}\cos^{-3}(\theta)d\theta \\
\ln\left(\sec(\theta)+\tan(\theta)\right)|_0^{\pi/4} &= \left(-\frac{\sin(\frac{\pi}{4})}{\cos^2(\frac{\pi}{4})} + \frac{\sin(0)}{\cos^2(0)}\right) + 2\int_0^{\pi/4}\cos^{-3}(\theta)d\theta
\end{aligned}$$

$$\begin{aligned}
\ln\left(\frac{1}{\cos(\pi/4)} + \tan(\pi/4)\right) - \ln\left(\frac{1}{\cos(0)} + \tan(0)\right) &= \left(-\frac{\frac{1}{\sqrt{2}}}{\frac{1}{2}} + \frac{0}{1}\right) + 2\int_0^{\pi/4}\cos^{-3}(\theta)d\theta \\
\ln(\sqrt{2}+1) - \ln(1+0) &= -\frac{2}{\sqrt{2}} + 2\int_0^{\pi/4}\cos^{-3}(\theta)d\theta \\
\ln(\sqrt{2}+1) &= -\sqrt{2} + 2\int_0^{\pi/4}\cos^{-3}(\theta)d\theta
\end{aligned}$$

$$\int_0^{\pi/4} \cos^{-3}(\theta)d\theta = \frac{1}{2}(\ln(\sqrt{2}+1)+\sqrt{2}).$$

Substituting back into S,

$$S = 8\pi \left(\frac{1}{2}(\ln(\sqrt{2}+1)+\sqrt{2}) \right) = 4\pi \left(\ln(\sqrt{2}+1)+\sqrt{2} \right).$$

Question 5

Question 5a

Let $y = f(x)$. Then

$$\begin{aligned}
2e^x \frac{dy}{dx} &= (1 - e^{2x})e^y \\
2e^{-y} \frac{dy}{dx} &= e^{-x} - e^x \\
\int 2e^{-y} dy &= \int \left(e^{-x} - e^x\right) dx \\
-2e^{-y} &= -e^{-x} - e^x + C,\ C \in \mathbb{R} \\
e^{-y} &= \frac{1}{2}e^{-x} + \frac{1}{2}e^x - \frac{C}{2}
\end{aligned}$$

Note that we need $C < 0$ in order for the right side of the equation to be positive for all $x \in \mathbb{R}$. Assuming so, we can then take ln on both sides.

$$f(x) = -\ln\left(\frac{1}{2}e^{-x} + \frac{1}{2}e^x + C\right),\ C > 0 \qquad \because \text{Let } C' = -\frac{C}{2} > 0$$

Question 5b

Let $y = f(x)$. Then

$$\begin{aligned}
x\frac{dy}{dx} &= \ln(x) + 2y \\
\frac{dy}{dx} - \frac{2}{x}y &= \frac{\ln(x)}{x}
\end{aligned}$$

Then the integrating factor is $e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = e^{\ln(|x|^{-2})} = x^{-2}$.

$$\begin{aligned}
yx^{-2} &= \int x^{-2}\frac{\ln(x)}{x}dx \\
&= \int x^{-3}\ln(x)dx\ (*) \\
&= \frac{x^{-2}}{-2} \times \ln(x) + \int \frac{x^{-2}}{2} \times \frac{1}{x}dx \\
&= \frac{x^{-2}}{-2} \times \ln(x) + \frac{x^{-2}}{2 \times -2} \\
&= -\frac{1}{4}x^{-2}\left(2\ln(x) + 1\right) + C \\
\therefore y &= -\frac{1}{4}(2\ln(x) + 1) + Cx^2.
\end{aligned}$$

$(*)$ Here, we use integration by parts with

$$\begin{aligned}
u &= \ln(x) & du &= \frac{1}{x}dx \\
dv &= x^{-3}dx & v &= \frac{x^{-2}}{-2}
\end{aligned}$$

Substituting in $x = 1, y = 0$

$$0 = -\frac{1}{4}(2\ln(1) - 1) + C(1)^2 = -\frac{1}{4} + C$$
$$\therefore C = \frac{1}{4}.$$

Therefore, the particular solution is

$$f(x) = -\frac{1}{4}(2\ln(x) + 1 - x^2).$$

Question 6

$$\begin{aligned}
\frac{1}{n^2}\sum_{k=1}^{n}\left[kf'\left(\frac{k}{n}\right)+nf\left(\frac{k}{n}\right)\right] &= \frac{1}{n}\sum_{k=1}^{n}\left[\frac{k}{n}f'\left(\frac{k}{n}\right)+f\left(\frac{k}{n}\right)\right] \\
&= \sum_{k=1}^{n}\left[\frac{1}{n}\cdot g\left(\frac{k}{n}\right)\right],
\end{aligned}$$

where $g(x) = xf'(x) + f(x)$. Notice that g is continuous since $x, f'(x), f(x)$ are all continuous. Hence, g is integrable. To find the antiderivative of g,

$$\begin{aligned}
\int g(x)dx &= \int (xf'(x)+f(x))\,dx \\
&= \int xf'(x)dx + \int f(x)dx \ (*) \\
&= xf(x) - \int f(x)dx + \int f(x)dx \\
&= xf(x).
\end{aligned}$$

($*$) Here, we use integration by parts with

$$\begin{aligned}
u &= x & du &= dx \\
dv &= f'(x)dx & v &= f(x)
\end{aligned}$$

Note that a Riemann sum can expressed as a definite integral, i.e.,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\Delta x\cdot g(x_k)\right] = \int_a^b g(x)dx, \qquad \text{where } \Delta x = \frac{b-a}{n},\ x_k = a + k\Delta x.$$

Taking the Riemann sum of g over $[0,1]$ with even-width partition,

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n}\cdot g\left(\frac{k}{n}\right)\right] &= \int_0^1 g(x)dx \\
&= xf(x)\ \Big|_0^1 \\
&= f(1).
\end{aligned}$$