NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA3110S Mathematical Analysis II (Version S)
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Question 1

Note that for any open set I containing a, consider $\gamma : I \to \mathbb{R}^3$ satisfying $\gamma(x) = a$ for any $x \in I$. Then, $f(\gamma(t)) = f(a) = 0 = g(a) = g(\gamma(t))$ as required.
Now, since $f(\gamma(t)) = g(\gamma(t)) = 0$, we have $D(f \circ \gamma)(t) = D(g \circ \gamma)(t) = 0$ for all $t \in I$. By chain rule, we have

$$0 = D(f \circ \gamma)(t_0) = Df(\gamma(t_0))D(\gamma(t_0)) = Df(a)D(\gamma(t_0)) = \begin{pmatrix} 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

and

$$0 = D(g \circ \gamma)(t_0) = Dg(\gamma(t_0))D(\gamma(t_0)) = Dg(a)D(\gamma(t_0)) = \begin{pmatrix} 1 & -2 & 6 \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

Solving two equations, we have $u = 0$ and $v = 3$.

Question 2

We shall use two lemmas :
Lemma 1 : $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and $h \in \mathbb{R}^n$, then $|Th| \leq |T||h|$.
Lemma 2: If $a, b \in \mathbb{R}^n$, then $||a| - |b|| \leq |a - b|$

Letting $X = Tx, H = Th$, we have :

$$\frac{\left| |T(x+h)|(x+h) - |Tx|x - \dfrac{Tx \cdot Th}{|Tx|}x - |Tx|h \right|}{|h|}$$

$$= \frac{\left| |Tx+Th|x - |Tx|x - \dfrac{Tx \cdot Th}{|Tx|}x + \left(|Tx+Th| - |Tx|\right)h \right|}{|h|}$$

$$\leq \frac{\left| |Tx+Th|x - |Tx|x - \dfrac{Tx \cdot Th}{|Tx|}x \right|}{|h|} + \frac{|\,|Tx+Th| - |Tx|\,| \cdot |h|}{|h|}$$

$$= \frac{\left| |Tx+Th| - |Tx| - \dfrac{Tx \cdot Th}{|Tx|} \right| |x|}{|T|\,|h|}|T| + |\,|Tx+Th| - |Tx|\,|$$

$$\leq \frac{\left| |Tx+Th| - |Tx| - \dfrac{Tx \cdot Th}{|Tx|} \right|}{|Th|}|T||x| + |\,Tx + Th - Tx\,|$$

$$= \frac{\left| |X+H| - |X| - \dfrac{X \cdot H}{|X|} \right|}{|H|} |T||x| + |Th|$$

Note that:

$$\frac{\left| |X+H| - |X| - \dfrac{X \cdot H}{|X|} \right|}{|H|}$$

$$= \frac{\left| \dfrac{|X+H|^2 - |X|^2}{|X+H| + |X|} - \dfrac{X \cdot H}{|X|} \right|}{|H|}$$

$$= \frac{\left|\, |X+H|^2|X| - |X|^3 - (X \cdot H)(|X+H| + |X|) \,\right|}{|H| \cdot |X| \cdot (|X+H| + |X|)}$$

$$= \frac{\left|\, \left(|X|^2 + |H|^2 + 2X \cdot H\right)|X| - |X|^3 - (X \cdot H)(|X+H| + |X|) \,\right|}{|H| \cdot |X| \cdot (|X+H| + |X|)}$$

$$= \frac{\left|\, |H|^2|X| + (X \cdot H)|X| - (X \cdot H)|X+H| \,\right|}{|H| \cdot |X| \cdot (|X+H| + |X|)}$$

$$\leq \left[\frac{|H|^2|X|}{|H|} + \frac{\left|\, (X \cdot H)|X| - (X \cdot H)|X+H| \,\right|}{|H|} \right] \frac{1}{|X| \cdot (|X+H| + |X|)}$$

$$= \left[|H||X| + \frac{|X \cdot H| \cdot \left|\, |X| - |X+H| \,\right|}{|H|} \right] \frac{1}{|X| \cdot (|X+H| + |X|)}$$

$$\leq \left[|H||X| + \frac{|X \cdot H| \cdot \left|\, X - (X+H) \,\right|}{|H|} \right] \frac{1}{|X| \cdot (|X+H| + |X|)}$$

$$= \left[|H||X| + \frac{|X \cdot H| \cdot |-1||H|}{|H|} \right] \frac{1}{|X| \cdot (|X+H| + |X|)}$$

$$= (|H||X| + |X \cdot H|) \frac{1}{|X| \cdot (|X+H| + |X|)}$$

So,

$$0 \leq \frac{\left| |T(x+h)|(x+h) - |Tx|x - \dfrac{Tx \cdot Th}{|Tx|} x - |Tx|h \right|}{|h|}$$

$$\leq \frac{\left| |X+H| - |X| - \dfrac{X \cdot H}{|X|} \right|}{|H|} |T||x| + |Th|$$

$$\leq (|H||X| + |X \cdot H|) \frac{1}{|X| \cdot (|X+H| + |X|)} |T||x| + |Th|$$

$$= (|Th||Tx| + |Tx \cdot Th|) \frac{|T||x|}{|Tx| \cdot (|Tx+Th| + |Tx|)} + |Th| - - - (1)$$

and the whole expression (1) goes to zero as $|h| \to 0$

Question 3

Since f is continuous, so is $|f|$. By extreme value theorem, there exists α such that $|f(x)| \geq |f(\alpha)|$ for all $x \in \mathbb{R}$. Since $|f(x)| > 0$ for all x, it follows that $|f(\alpha)| > 0$. Setting $M = |f(\alpha)|$, we have $f(x) \geq M$ for all $x \in \mathbb{R}^n$.
Now, consider any $\epsilon > 0$. Since (f_k) converges uniformly to f, then there exists $N_1, N_2 \in \mathbb{N}$ such that $|f_k(x) - f(x)| < \frac{M}{2}$ for all $k \geq N_1, x \in \mathbb{R}^n$, and $|f_k(x) - f(x)| < \frac{M^2\epsilon}{2}$ for all $k \geq N_2, x \in \mathbb{R}^n$. Now, if $k \geq N_1$, we have $|f(x) - f_k(x)| < \frac{M}{2}$, thus leaving us with $|f_k(x)| - |f(x)| > -\frac{M}{2}$. Thus, $|f_k(x)| > \frac{M}{2}$.
Finally, consider any $k \geq \max(N_1, N_2)$ we have $|\frac{1}{f_k(x)} - \frac{1}{f(x)}| = |\frac{f(x)-f_k(x)}{f_k(x)f(x)}| = \frac{|f(x)-f_k(x)|}{|f_k(x)||f(x)|} < \frac{\frac{M^2\epsilon}{2}}{\frac{M^2}{2}} = \epsilon$

Question 4

Since the set of discontinuity of $\cos(nx)$ is a f-null set it follows that $\int_a^b \cos nx df$. By integration by parts, we have $\int_a^b f d(\cos nx)$ and hence, $\int_a^b f \sin(nx)dx$ exists. We have, -$n \int_a^b f \sin(nx)dx = \int_a^b f d\cos(nx)$ which is bounded if and only if $\int_a^b \cos nx df$ is bounded (that happens by integration by parts). But, $f(a) - f(b) \leq \int_a^b \cos(nx)df \leq f(b) - f(a)$ since $-1 \leq \cos x \leq 1$. Thus, $\int_a^b \cos(nx)df$ is bounded. Hence, $n \int_a^b f \sin(nx)dx$ is bounded. The result follows. QED

Question 5

By Taylor's theorem, we have

$$f(x) = \sum_{k=0}^{j-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(j)}(a)}{j!} x^j$$

for some $a \in [0, x]$ which leads to

$$\frac{f(x) - P_j(x)}{x^{j-\frac{1}{2}}} = \frac{\frac{f^{(j)}(a)x^j}{j!} - \frac{f^{(j)}(0)x^j}{j!}}{x^{j-\frac{1}{2}}}$$

and hence, we have

$$\frac{f(x) - P_j(x)}{x^{j-\frac{1}{2}}} = \sqrt{x} \left(\frac{f^{(j)}(a) - f^{(j)}(0)}{j!} \right) = \frac{1}{j!\sqrt{x}} \left(x f^{(j)}(a) - x f^{(j)}(0) \right) = \frac{1}{j!\sqrt{x}} \int_0^x f^j(a) - f^j(0) dt$$

Next, clearly $\int_0^x f^{(j)}(t) - f^{(j)}(0)dt$ exists, and hence,

$$\lim_{x\to 0+} \frac{-1}{\sqrt{x}} \int_0^x |f^{(j)}(t) - f^{(j)}(0)|dt \leq \lim_{x\to 0+} \frac{1}{\sqrt{x}} \int_0^x f^{(j)}(t) - f^{(j)}(0)dt \leq \lim_{x\to 0+} \frac{1}{\sqrt{x}} \int_0^x |f^{(j)}(t) - f^{(j)}(0)|dt$$

Thus,

$$\lim_{x \to 0+} \frac{1}{\sqrt{x}} \int_0^x f^{(j)}(t) - f^{(j)}(0)dt = 0$$

which implies

$$\lim_{x \to 0+} \frac{1}{j!\sqrt{x}} \int_0^x f^{(j)}(t) - f^{(j)}(0)dt = 0 \tag{1}$$

Now, since for any function g and interval $[p, q]$, $\int_p^q g dt = g(c)(q-p)$ for some $c \in [p, q]$, we have

$$\lim_{x \to 0+} \frac{1}{j!\sqrt{x}} \int_0^x f^{(j)}(a) - f^{(j)}(t)dt = \lim_{x \to 0+} \frac{1}{j!\sqrt{x}} (f^{(j)}(a)x - f^{(j)}(b)x)$$

for some $b \in [0, x]$. Thus,

$$\lim_{x \to 0+} \frac{1}{j!\sqrt{x}} \int_0^x f^{(j)}(a) - f^{(j)}(t)dt = \lim_{x \to 0+} \frac{1}{j!} \sqrt{x}(f^{(j)}(a) - f^{(j)}(b))$$

As $f^{(j)}$ is a continuous function and since a, b both in interval $[0, x]$, we have $f^{(j)}(b) - f^{(j)}(a)$ goes to 0 as $x \to 0+$. Since $\sqrt{x}$ also goes to 0 when x goes to 0, we have

$$\lim_{x \to 0+} \frac{1}{j!\sqrt{x}} \int_0^x f^{(j)}(a) - f^{(j)}(t)dt = 0 \tag{2}$$

Adding (1) and (2), we get

$$\lim_{x \to 0+} \frac{1}{j!\sqrt{x}} \int_0^x f^{(j)}(a) - f^{(j)}(0)dt = 0$$

Thus, we get

$$\lim_{x \to 0+} \frac{f(x) - P_j(x)}{x^{j-\frac{1}{2}}} = 0$$

as desired.