MA2108 - Mathematical Analysis I Suggested Solutions

AY19/20 Semester 1

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Question 1

(a) (i) We prove this by induction. The case for $n = 1$ is clear. Suppose the inequality holds for $n = k \geq 1$. We want to show that the inequality holds for $n = k+1$. Indeed, we have

$$x_{k+1} = \sqrt{x_k + 6} \geq \sqrt{0+6} > 0$$

and

$$x_{k+1} = \sqrt{x_k + 6} \leq \sqrt{3+6} = 3,$$

which completes the induction step.

(ii) We claim that the sequence converges to 3. Observe that

$$|x_{n+1} - 3| = |\sqrt{x_k + 6} - 3| = \left|\frac{x_k - 3}{\sqrt{x_k+6}+3}\right| < \frac{1}{3}|x_k - 3|.$$

Thus, the sequence contracts and so $\lim\limits_{k\to\infty} x_k = 3$.

(b) The answer is $\limsup y_n = 1$ and $\liminf y_n = -1$. Note that

$$\limsup y_n = \limsup\left(\frac{\cos n}{n} + \sin\frac{n\pi}{6}\right) \leq \limsup\left(\frac{\cos n}{n}\right) + \limsup\left(\sin\frac{n\pi}{6}\right) = 0 + 1 = 1.$$

On the other hand, observe that

$$\sup\left\{\frac{\cos n}{n} + \sin\left(\frac{n\pi}{6}\right), n \geq k\right\} \geq \sup\left\{\frac{\cos(12n+3)}{12n+3} + \sin\left(\frac{(12n+3)\pi}{6}\right), n \geq k\right\}.$$

Since $\lim\limits_{n\to\infty}\left(\frac{\cos(12n+3)}{12n+3} + \sin\left(\frac{(12n+3)\pi}{6}\right)\right) = 1$, it follows that $\limsup y_n \geq 1$. Thus, $\limsup y_n = 1$. The proof is similar for $\liminf y_n = -1$.

(c) For each positive integer k, and $\forall m > k$, we have $M_k := \sup\{a_n, n \geq k\} \geq a_m$ and $m_k := \inf\{b_n, n \geq k\} \leq b_m$. Then whenever $m \geq k$, we have $\frac{a_m}{b_m} \leq \frac{M_k}{m_k}$. Since the inequality works for any positive integer $m \geq k$, we get

$$\sup\left\{\frac{a_m}{b_m}, m \geq k\right\} \leq \frac{M_k}{m_k}.$$

Taking limit on both sides gives

$$\lim_{k\to\infty}\sup\left\{\frac{a_m}{b_m}, m \geq k\right\} = \limsup_{k\to\infty}\frac{a_k}{b_k} \leq \lim_{k\to\infty}\frac{M_k}{m_k} = \frac{\lim_{k\to\infty} M_k}{\lim_{k\to\infty} m_k} = \frac{\limsup a_k}{\liminf b_k}$$

since (a_n) and (b_n) are bounded sequences.

Question 2

(a) We have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2} &= \lim_{N\to\infty}\sum_{n=1}^{N}\frac{2n+1}{n^2(n+1)^2}\\ &= \lim_{N\to\infty}\sum_{n=1}^{N}\frac{(n+1)^2-n^2}{n^2(n+1)^2}\\ &= \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)\\ &= \lim_{N\to\infty}\left(1-\frac{1}{(N+1)^2}\right) = 1.\end{aligned}$$

(b) (i) We first show that $\dfrac{3n^3-2n^2+n+1}{5n^4-3n^3+2} > \dfrac{1}{5n}$ for positive integers n. Since for each positive integer n, we have $n^3 > \dfrac{2}{3}$ and $n^2 \geq 1$, this implies that

$$\frac{3n^3-2n^2+n+1}{5n^4-3n^3+2} = \frac{\frac{3}{n}-\frac{2}{n^2}+\frac{1}{n^3}+\frac{1}{n^4}}{5-\frac{3}{n}+\frac{2}{n^4}} > \frac{\frac{3}{n}-\frac{2}{n^2}}{5} \geq \frac{1}{5n}.$$

Since $\sum_{n=1}^{\infty}\dfrac{1}{n}$ diverges, the sum $\sum_{n=1}^{\infty}\dfrac{3n^3-2n^2+n+1}{5n^4-3n^3+2}$ diverges as well.

(ii) The series converges by root test. We have

$$\lim_{n\to\infty}\sqrt[n]{\frac{n^2}{10^n}\left(1+\frac{1}{2n}\right)^{4n^2}} = \lim_{n\to\infty}\frac{1}{10}n^{\frac{2}{n}}\left(1+\frac{1}{2n}\right)^{4n}.$$

Since $\lim_{n\to\infty} n^{\frac{2}{n}} = 1$ and $\lim_{n\to\infty}\left(1+\dfrac{1}{2n}\right)^{4n} = e^2 < 9$, the required limit is less than 1 and so the series converges.

(c) Write $0 \leq b_n - a_n \leq c_n - a_n$. Since $\sum_{n=1}^{\infty}(c_n-a_n)$ converges (absolutely), the series $\sum_{n=1}^{\infty}(b_n-a_n)$ converges (absolutely) by comparison test. Hence, the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}(b_n-a_n)+\sum_{n=1}^{\infty}a_n$ converges too.

Question 3

(a) Let $\varepsilon > 0$ be given. Pick $\delta = \min\left\{\dfrac{1}{4}, \dfrac{\varepsilon}{24}\right\}$ so that $0 < |x+2| < \delta \implies \left|\dfrac{2x-3}{2x+3}-7\right| < \varepsilon$. Indeed, we have

$$\left|\frac{2x-3}{2x+3}-7\right| = \left|\frac{-12x-24}{2x+3}\right| = 12|x+2|\left|\frac{1}{2x+3}\right| < 12\times 2|x+2| < 24\times\frac{\varepsilon}{24} = \varepsilon.$$

The conclusion follows.

(b) The function is only continuous at $x=2$. Let $\varepsilon > 0$ be given. Take $\delta = \dfrac{\varepsilon}{3}$ so that $0 < |x-2| < \delta \implies |f(x)-5| < \varepsilon$. Indeed, we have

$$|f(x)-5| \leq \sup\{|(3x-1)-5|, |(2x+1)-5|\} = 3|x-2| < 3\times\frac{\varepsilon}{3} = \varepsilon.$$

Thus, the function is continuous at $x=2$.

For $x \neq 2$, consider two cases. If x is rational, then $f(x) = 3x-1$. Consider a sequence of irrational numbers $(x_n)_{n=1}^{\infty}$ that converges to x. Then, $f(x_k) = 2x_k+1$ for each positive integer k. Since $x \neq 2$, the limit $\lim_{k\to\infty}(2x_k+1) = 2x+1$ does not equal to $f(x) = 3x-1$. Thus, the function is not continuous at rational values other than 2. The case for x is irrational can be handled similarly.

(c) (i) Since $\lim_{x\to\infty} \frac{g(2x)}{g(x)} = 1$, for a given ε, there exists a positive real number N so that $\left|\frac{g(2x)}{g(x)} - 1\right| < \varepsilon$ for all $x > N$. Since $2^{n-1}x \geq x$ for positive integers n, we have $\left|\frac{g(2^n x)}{g(2^{n-1}x)} - 1\right| < \varepsilon$ and we are done.

(ii) Notice that for $\alpha > 2$, we can write $\alpha = 2^k\beta$ for some positive integer k and real number $1 \leq \beta < 2$. As such, we have

$$\lim_{x\to\infty} \frac{g(\alpha x)}{g(x)} = \lim_{x\to\infty} \frac{g(2^k\beta x)}{g(x)} = \lim_{x\to\infty} \left(\frac{g(2^k\beta x)}{g(2^{k-1}\beta x)} \frac{g(2^{k-1}\beta x)}{g(2^{k-2}\beta x)} \cdots \frac{g(2\beta x)}{g(\beta x)} \frac{g(\beta x)}{g(x)} \right).$$

A modification of the proof for part (i) yields $\lim_{x\to\infty} \frac{g(2^k\beta x)}{g(2^{k-1}\beta x)} = 1$. On the other hand, since g is increasing, we have $g(x) \leq g(\beta x) < g(2x)$ and so $1 \leq \frac{g(\beta x)}{g(x)} < \frac{g(2x)}{g(x)}$. By squeeze theorem, the limit is $\lim_{x\to\infty} \frac{g(\beta x)}{g(x)} = 1$. Hence, we conclude that

$$\lim_{x\to\infty} \frac{g(\alpha x)}{g(x)} = 1.$$

Question 4

(a) By Extreme Value Theorem, f attains its supremum at $x_1 \in [0,1]$ and g attains its supremum at $x_2 \in [0,1]$. If $x_1 = x_2$, there is nothing to prove.

Suppose $f(x_1) > f(x_2)$ and $g(x_1) < g(x_2)$, i.e. f and g attains maximum at different points. Then, we see that $f(x_1) - g(x_1) = g(x_2) - g(x_1) > 0$ and $f(x_2) - g(x_2) = f(x_2) - f(x_1) < 0$. Thus, by Intermediate Value Theorem, there exists $x_0 \in [0,1]$ so that $f(x_0) = g(x_0)$.

(b) Without loss of generality, assume $x \geq 0$. Since f is uniformly continuous, there exists $\delta > 0$ so that $|x-y| < 2\delta \implies |h(x) - h(y)| < 1$. Thus, if $|x| = k\delta + r$ for some positive integer k and $0 \leq r < \delta$ by triangle inequality, we get

$$\begin{aligned} |h(x) - h(0)| &= |h(x) - h(x-\delta) + h(x-\delta) - h(x-2\delta) + \cdots + h(r) - h(0)| \\ &\leq |h(x) - h(x-\delta)| + |h(x-\delta) - h(x-2\delta)| + \cdots + |h(r) - h(0)| \\ &\leq k+1. \end{aligned}$$

Thus, $|h(x)| \leq |h(x) - h(0)| + |h(0)| \leq k + 1 + |h(0)|$. Since $|x| = k\delta + r \geq k\delta$, it follows that

$$|h(x)| \leq \frac{|x|}{\delta} + 1 + |h(0)|.$$

The proof is complete.