NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lau Tze Siong

MA2108 Mathematical Analysis I
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Section A
Question 1

(a) Since for all $n \in \mathbb{N}$, we have,

$$\begin{aligned}
0 \leq & \quad \cos^2(n^3) & \leq 1 \\
0 \leq & \quad \tfrac{3}{5}\cos^2(n^3) & \leq \frac{3}{5} \\
0 \leq & \quad \left(\tfrac{3\cos^2(n^3)}{5}\right)^n & \leq \left(\frac{3}{5}\right)^n \\
0 \leq & \lim_{n\to\infty}\left(\frac{3\cos^2(n^3)}{5}\right)^n & \leq \lim_{n\to\infty}\left(\frac{3}{5}\right)^n = 0.
\end{aligned}$$

Hence we have $\displaystyle\lim_{n\to\infty}\left(\frac{3\cos^2(n^3)}{5}\right)^n = 0$.

(b) Claim: $1 \leq x_n \leq 3$ for all $n \in \mathbb{N}$.
Proof:
The case where $n = 1$ is obvious.
Suppose for some $k \in \mathbb{N}$,we have $1 \leq x_k \leq 3$.
We have $1 \leq 1 + \frac{5}{3+x_k} \leq 1 + \frac{5}{4} < 3$. Hence we have $1 \leq x_{k+1} \leq 3$.
By induction, we have $1 \leq x_n \leq 3$ for all $n \in \mathbb{N}$.

Claim: $x_{2n+2} \leq x_{2n}$ and $x_{2n+1} \geq x_{2n-1}$ for all $n \in \mathbb{N}$
Proof:
Since $x_{n+2} = 1 + \frac{5}{3+x_{n+1}} = 1 + \frac{5}{3+1+\frac{5}{3+x_n}} = \frac{9}{4} - \frac{25}{68+16x_n}$ and $x_1 = 1$, $x_2 = \frac{9}{4}$, $x_3 = \frac{41}{21}$, $x_4 = \frac{209}{104}$.
We have $x_1 \leq x_3$ and $x_2 \geq x_4$. We have the case when $n = 1$.

Suppose for some $k \in \mathbb{N}$,we have $x_{2k+2} \leq x_{2k}$ and $x_{2k+1} \geq x_{2k-1}$.
Hence we have $\frac{9}{4} - \frac{25}{68+16x_{2k+2}} \leq \frac{9}{4} - \frac{25}{68+16x_{2k}}$ and $\frac{9}{4} - \frac{25}{68+16x_{2k+1}} \geq \frac{9}{4} - \frac{25}{68+16x_{2k-1}}$.
Giving us $x_{2k+4} \leq x_{2k+2}$ and $x_{2k+3} \geq x_{2k+1}$

Hence, by induction we have $x_{2n+2} \leq x_{2n}$ and $x_{2n+1} \geq x_{2n-1}$ for all $n \in \mathbb{N}$.

By completeness of $\mathbb{R}$, $\lim\limits_{n\to\infty} x_{2n}$ and $\lim\limits_{n\to\infty} x_{2n+1}$, exist and both satisfy $x = \frac{9}{4} - \frac{25}{68+16x}$. Hence, we have $\lim\limits_{n\to\infty} x_{2n} = \lim\limits_{n\to\infty} x_{2n+1} = 2$.
Hence $\lim\limits_{n\to\infty} x_n = 2$.

Question 2

(a) (i) Since

$$\begin{aligned}\lim_{n\to\infty}\frac{\frac{(n+1)!}{1\cdot3\cdot5\cdot\dots\cdot2n+1}}{\frac{n!}{1\cdot3\cdot5\cdot\dots\cdot2n-1}} &= \lim_{n\to\infty}\frac{n+1}{2n+1}\\ &= \lim_{n\to\infty}\frac{1}{2}+\frac{1}{4n+2}\\ &= =\frac{1}{2}\end{aligned}$$

Hence, by Ratio test, the sum $\sum_{n=1}^{\infty}\frac{n!}{1\cdot3\cdot5\cdot\dots\cdot2n-1}$ exists.

(ii) Since

$$\begin{aligned}\lim_{n\to\infty}\left(n\left(1+\frac{1}{2n}\right)^{-n^2}\right)^{\frac{1}{n}} &= \lim_{n\to\infty}n^{\frac{1}{n}}\left(\left(1+\frac{1}{2n}\right)^{2n}\right)^{-\frac{1}{2}}\\ &= \lim_{n\to\infty}n^{\frac{1}{n}}\left(\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n}\right)^{-\frac{1}{2}}\\ &= 1\left(e\right)^{-\frac{1}{2}}<1\end{aligned}$$

Hence the sum $\sum_{n=1}^{\infty}n\left(1+\frac{1}{2n}\right)^{-n^2}$ exist.

(b) Since $\sum_{n=1}^{\infty}a_n,\sum_{n=1}^{\infty}b_n$ exists, there exists a $M\in\mathbb{N}$ such that for all $p,q,r,s\in\mathbb{N}_{\geq M}$ one has,

$$\left|\sum_{i=p}^{q}a_i\right|<\frac{\epsilon}{2}\text{ and }\left|\sum_{i=r}^{s}b_i\right|<\frac{\epsilon}{2}$$

We will show that $\sum_{n=1}^{\infty}c_n$ is Cauchy.

Proof:

Given any $\epsilon\in\mathbb{R}_{>0}$, let $N=2M$. Hence for any $m,n\in\mathbb{N}_{\geq N}$,

$$\begin{aligned}\left|\sum_{i=m}^{n}c_i\right| &= \left|\sum_{\lfloor\frac{m}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}b_n+\sum_{\lfloor\frac{m+1}{2}\rfloor}^{\lfloor\frac{n+1}{2}\rfloor}a_n\right|\\ &< \left|\sum_{\lfloor\frac{m}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}b_n\right|+\left|\sum_{\lfloor\frac{m+1}{2}\rfloor}^{\lfloor\frac{n+1}{2}\rfloor}a_n\right|\\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon\end{aligned}$$

Hence $\sum_{n=1}^{\infty}c_n$ is Cauchy and therefore converges.

Question 3

(a) For any given $\epsilon \in \mathbb{R}_{>0}$, choose $\delta = \min(\frac{15}{11}\epsilon, 1)$.
for $x \in \mathbb{R}$ such that $|x-3| < 1$, we have $2 < x < 4$.
Hence we have

$$3x - 1 < 12 - 1 = 11$$

.
Also

$$3x + 9 > 6 + 9 = 15$$

.
So

$$\left|\frac{3x-1}{3x+9}\right| < \frac{11}{15}$$

.

$$\begin{aligned}
\left|\frac{(x-1)(x-2)}{(x+3)} - \frac{1}{3}\right| &= \left|\frac{3x^2 - 10x + 3}{3x+9}\right| \\
&= |x-3|\left|\frac{3x-1}{3x+9}\right| \\
&< \delta\left|\frac{3x-1}{3x+9}\right| \\
&< \left(\frac{15}{11}\epsilon\right)\left(\frac{11}{15}\right) = \epsilon
\end{aligned}$$

for all $x \in \mathbb{R}$ such that $|x-3| < \delta$.

(b) (i) Suppose $\lim\limits_{x\to 1} \cos^2\left(\frac{1}{x-1}\right) = a$ Let $\epsilon = \frac{1}{2}$. For any $\delta \in \mathbb{R}_{>0}$, there exist a $n_1, n_2 \in \mathbb{N}$ such that $2n_1\pi, 2n_2\pi + \frac{\pi}{2} > \frac{1}{\delta}$. Hence we have $x_1 - 1 = \frac{1}{2n_1\pi}$ and $x_2 - 1 = \frac{1}{2n_2\pi + \frac{\pi}{2}}$. Since $|x_1 - 1| < \delta$ and $|x_2 - 1| < \delta$. We have $\left|\cos^2\left(\frac{1}{x_1-1}\right) - a\right| < \frac{1}{2}$ and $\left|\cos^2\left(\frac{1}{x_2-1}\right) - a\right| < \frac{1}{2}$. Evaluating, we have $|1-a| < \frac{1}{2}$ and $|a| < \frac{1}{2}$. Hence $1 = |1 - a + a| < |1-a| + |a| < \frac{1}{2} + \frac{1}{2} = 1$, which is a contradiction. Hence $\lim\limits_{x\to 1} \cos^2\left(\frac{1}{x-1}\right)$ does not exist.

(ii) Since $\lim\limits_{x\to 2^-} \lfloor 3x \rfloor + \lfloor 4 - x \rfloor = 5 + 2 = 7$ and
$\lim\limits_{x\to 2^+} \lfloor 3x \rfloor + \lfloor 4 - x \rfloor = 6 + 1 = 7$. One has $\lim\limits_{x\to 2} \lfloor 3x \rfloor + \lfloor 4 - x \rfloor = 7$.

Question 4

Since f and h are continuous at $x = c$ with $f(c) = h(c)$ and $f(x) \le g(x) \le h(x)$ for all $x \in \mathbb{R}$, we have $g(c) = f(c) = h(c)$. Also by Squeeze Theorem, we have $\lim_{x\to c} f(x) \le \lim_{x\to c} g(x) \le \lim_{x\to c} h(x)$. Since f and h is continuous, we have $f(c) \le \lim_{x\to c} g(x) \le h(c)$. Hence $\lim_{x\to c} g(x) = f(c) = g(c)$. Hence g is continuous at $x = c$.

Question 5

(i) We may construct (x_n) such that $x_n \in (1 - \frac{1}{2^n}, 1) \cap (\mathbb{R}\backslash\mathbb{Q})$. Hence $\lim\limits_{n\to\infty} x_n = 1$ and since f is continuous at $x = 1$, $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} ax_n + 4 = f(1) = 7$
Hence $a = 3$.

(ii) No.

We will first show that if f is continuous at $x = b$ then $b^2 + b + 5 = 3b + 4$.

Case 1) Suppose b is rational.
We may construct (x_n) such that $x_n \in (b - \frac{1}{2^n}, b) \cap (\mathbb{R}\backslash\mathbb{Q})$. Hence $\lim_{n\to\infty} x_n = b$ and since f is continuous at $x = b$, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 3x_n + 4 = f(b)$ Hence we have $3b + 4 = b^2 + b + 5$

Case 2)Suppose b is irrational.
We may construct (x_n) such that $x_n \in (b - \frac{1}{2^n}, b) \cap (\mathbb{Q})$. Hence $\lim_{n\to\infty} x_n = b$ and since f is continuous at $x = b$, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n^2 + x_n + +5 = f(b)$ Hence we have $b^2 + b + 5 = 3b + 4$

Therefore if f is continuous at $x = b$ then $b^2 + b + 5 = 3b + 4$.

Since $b = 1$ is the only solution to the equation $b^2 + b + 5 = 3b + 4$ and f is continuous at $x = 1$. f is only continuous at $x = 1$.

Section B Question 6

(a) (i) Since $\lim_{n\to\infty} y_n = L$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|y_n - L| < \frac{L}{2}$$

So $y_n > \frac{L}{2}$ for all $n \geq N$.

Let $\alpha \in \mathbb{R}$
On the other hand, since $\lim_{n\to\infty} \frac{x_n}{y_n} = \infty$,
there exists $K \in \mathbb{N}$ such that for all $n \geq K$ we have

$$\frac{x_n}{y_n} > \frac{2\alpha}{L}$$

Set $M = \max(N, K)$.
Hence we have for all $N \geq M$,

$$\begin{aligned} x_n &> \frac{2\alpha}{L} y_n \\ &> \frac{2\alpha}{L}\frac{L}{2} = \alpha \end{aligned}$$

. Since α is arbitrary, $\lim_{n\to\infty} x_n = \infty$.

(ii) No.
Let $x_n = \frac{1}{n}$, $y_n = \frac{1}{n^2}$. Hence $\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} n = \infty$ and $\lim_{n\to\infty} y_n = 0$. However $\lim_{n\to\infty} x_n = 0 \neq \infty$.

(b) Since $ca_{n+1} \leq a_{n+1}b_{n+1} - a_n b_n$ and $\sum_{n=1}^{M} ca_{n+1} \leq a_1 b_1 - a_{M+1}b_{M+1} \leq a_1 b_1$.
$\sum_{n=1}^{M} a_n \leq \frac{ca_1 + a_1 b_1}{c}$ for all $M \in \mathbb{N}$. Since $a_n > 0$ for all $n \in \mathbb{N}$, $x_M = \sum_{n=1}^{M} a_n$ is a increasing sequence. By the Completeness property of real numbers $\lim_{M\to\infty} x_M$ exists. Hence $\sum_{n=1}^{\infty} a_n$ converges.

Question 7

(a) We will first show that f is continuous.
For any $p \in \mathbb{R}$ Given any $\epsilon \in \mathbb{R}$, choose $\delta = \frac{\epsilon}{c}$, Hence we have for any $x \in \mathbb{R}$ such that $|x - p| < \delta$, we have $|f(x) - f(p)| \leq C|x - p| < C\left(\frac{\epsilon}{C}\right) = \epsilon$. Hence f is continuous.
Existence:
If $f(0) = 0$, then we are done.

Suppose $f(0) = k > 0$, for any $x \in \mathbb{R}_{>0}$ we have $k - Cx \leq f(x) \leq k + Cx$. Consider the function $h(x) = f(x) - x$.
$h(0) = k > 0$ and $h(\frac{k}{1-C} + 1) = f(\frac{k}{1-C} + 1) - \frac{k}{1-C} - 1 \leq k + c\left(\frac{k}{1-C} + 1\right) - \frac{k}{1-C} - 1 = C - 1 < 0$.
By Intermediate Value Theorem, there exist a $a \in [0, \frac{k}{1-C} + 1]$ such that $h(a) = 0$ and $f(a) = a$.

(b) For any $n \in \mathbb{N}$ Consider the function $h_n(x) = g(x) - \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x^k$. Since $g(x)$ and $\frac{1}{n}\sum_{k=1}^{n} x^k$ are continuous, $h(x)$ is continuous. Since $h_n(1) = g(1) - 1 \leq 0$ and $h_n(0) = g(0) - 0 \geq 0$. If $h_n(1) = 0$ then let $x_n = 1$ and we are done. Similarly, if $h_n(0) = 0$ let $x_n = 0$.
If both $h_n(1)$ and $h_n(0)$ are not equals 0, we have $h_n(1) < 0$ and $h_n(0) > 0$. By Intermediate Value Theorem, there exist a $c \in (0, 1)$ such that $h_n(c) = 0$. Hence by letting $x_n = c$ and we are done.

Question 8

(a) False.
Let $A = B = \{x \in \mathbb{R} | -1 \leq x \leq 0\}$. Hence $\sup A = \sup B = 0$. But $\sup C = 1 \neq (\sup A)(\sup B)$

(b) True.
Since $\displaystyle\lim_{n\to\infty} \frac{(a_n)^n}{n} = 1$, there exist a $M \in \mathbb{N}$ such that $\frac{1}{2} < \frac{(a_n)^n}{n} < \frac{3}{2}$ for all $n \in \mathbb{N}_{n \geq M}$. Hence we have $\left(\frac{n}{2}\right)^{\frac{1}{n}} < a_n < \left(\frac{3n}{2}\right)^{\frac{1}{n}}$. Therefore $\displaystyle\lim_{n\to\infty} (n)^{\frac{1}{n}} \left(\frac{1}{2}\right)^{\frac{1}{n}} \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} (n)^{\frac{1}{n}} \left(\frac{3}{2}\right)^{\frac{1}{n}}$. Hence $1 \leq \displaystyle\lim_{n\to\infty} a_n \leq 1$. Hence $\displaystyle\lim_{n\to\infty} a_n = 1$.

(c) False. Let $c \neq 0$.
Let

$$\begin{aligned} f(x) &= b \text{ for all } x \in \mathbb{R} \\ g(x) &= \begin{cases} x & \text{for all } x \neq b \\ 2c & x = b \end{cases} \end{aligned}$$

Then $g(f(x)) = g(b) = 2c$ for all $x \in \mathbb{R}$.

So $\displaystyle\lim_{x\to a} g[f(x)] = 2c \neq c$.

But $\displaystyle\lim_{x\to a} f(x) = b$. and $\displaystyle\lim_{x\to b} g(x) = \lim_{x\to b^-} g(x) = \lim_{x\to b^+} g(x) = c$

(d) Rewriting $h(2x) = h(3x)$, we have $h(y) = h(\dfrac{2}{3}y)$.
Claim:$h(p) = h\left(\left(\dfrac{2}{3}\right)^n p\right)$ for all $n \in \mathbb{N}$ and $p \in \mathbb{R}$.

Proof:
For any $p \in \mathbb{R}$. Since $h(p) = h\left(\frac{2}{3}p\right)$, we have case when $n = 1$.

Suppose for some $k \in \mathbb{N}$ we have $h(p) = h\left(\left(\frac{2}{3}\right)^k\right)$. Since we have $h\left(\left(\frac{2}{3}\right)^k\right) = h\left(\frac{2}{3}\left(\frac{2}{3}\right)^k\right) = h\left(\left(\frac{2}{3}\right)^{k+1}\right)$.
We have $h(p) = h\left(\left(\frac{2}{3}\right)^{k+1} p\right)$.

By induction, for all $n \in \mathbb{N}$ such that $h(p) = h\left(\left(\frac{2}{3}\right)^n p\right)$.

Since h is continuous, for any $p \in \mathbb{R}$, $h(p) = \lim_{n\to\infty} h(p) = \lim_{n\to\infty} h\left(\left(\dfrac{2}{3}\right)^n p\right) = h\left(\lim_{n\to\infty}\left(\dfrac{2}{3}\right)^n p\right) = h(0)$. Hence for all $p \in \mathbb{R}$, $h(p) = h(0)$. Therefore h is a constant function.