MA2002 - Calculus Suggested Solutions
(Semester 2: AY2022/23)

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Question 1

(a)

$$y = \left(\frac{1}{x}\right)^{\ln x}$$

Taking natural log both sides,

$$\begin{aligned}\ln y &= \ln x \ln \frac{1}{x} \\ &= -(\ln x)^2\end{aligned}$$

Then by taking derivatives both sides,

$$\begin{aligned}\frac{y'}{y} &= \frac{-2\ln x}{x} \\ \implies \frac{dy}{dx} &= \frac{-2y\ln x}{x} \\ &= -2\left(\frac{1}{x}\right)^{\ln x}\frac{\ln x}{x}\end{aligned}$$

Equation of tangent line at $x = e$:

$$\begin{aligned}\text{slope } m &= -2\left(\frac{1}{e}\right)^{\ln e}\frac{\ln e}{e} \\ &= -2\cdot\frac{1}{e}\cdot\frac{1}{e} \\ &= \frac{-2}{e^2}\end{aligned}$$

So $y = \frac{-2}{e^2}(x-e) + \frac{1}{e}$.
Hence, at $x = 0$:

$$\begin{aligned} y &= \frac{-2}{e^2}(-e) + \frac{1}{e} \\ &= \frac{2}{e} + \frac{1}{e} \\ &= \frac{3}{e} \end{aligned}$$

(b) By definition, define volume $V = a^2b = 128$, i.e., $b = \frac{128}{a^2}$. Similarly, define cost $C = 2a^2 + \frac{1}{2} \cdot 4ab = 2a^2 + 2ab$. Then

$$\begin{aligned} C(a) &= 2a^2 + 2a\frac{128}{a^2} \\ &= 2\left(a^2 + \frac{128}{a}\right) \end{aligned}$$

By taking derivative with respect to a,

$$\begin{aligned} \frac{dC(a)}{da} &= 0 \\ \implies 2\left(2a - \frac{128}{a^2}\right) &= 0 \\ \implies a &= 4 \end{aligned}$$

Now, apply Second Derivative Test,

$$\frac{d^2C(a)}{da^2} = 2\left(2 + \frac{2 \cdot 128}{a^3}\right) > 0$$

So when $a = 4$, we indeed minimise the cost.
Hence, the dimensions are as followed:

$$\begin{aligned} &(\text{length, width, height}) \\ =&(a, a, b) \\ =&(4, 4, \frac{128}{16}) \\ =&(4, 4, 8) \end{aligned}$$

Question 2

(a)

$$\begin{aligned}&\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)\\&=\lim_{x\to 0}\left(\frac{\sin^2 x - x^2}{x^2\sin^2 x}\right)\end{aligned}$$

Notice that $\lim_{x\to 0}(\sin^2 x - x^2) = 0$ and $\lim_{x\to 0} x^2\sin^2 x = 0$.
We can apply LH rule,

$$\begin{aligned}&=\lim_{x\to 0}\frac{\frac{d}{dx}(\sin^2 x - x^2)}{\frac{d}{dx}(x^2\sin^2 x)}\\&=\lim_{x\to 0}\frac{2\sin x\cos x - 2x}{2\sin x\cos x\cdot x^2 + 2x\sin^2 x}\\&=\lim_{x\to 0}\frac{\frac{1}{2}\sin 2x - x}{\frac{1}{2}\sin 2x\cdot x + x\sin x}\end{aligned}$$

Again, $\lim_{x\to 0}\frac{1}{2}\sin 2x - x = 0$ and $\lim_{x\to 0}\frac{1}{2}\sin 2x\cdot x + x\sin x = 0$.
By LH rule,

$$\begin{aligned}&\lim_{x\to 0}\frac{\cos 2x - 1}{x\sin 2x + x^2\cos 2x + \sin^2 x + x\sin 2x}\\&=\lim_{x\to 0}\frac{\cos 2x - 1}{x^2\cos 2x + \sin^2 x + 2x\sin 2x}\end{aligned}$$

Again, $\lim_{x\to 0}\cos 2x - 1 = 0$ and $\lim_{x\to 0} x^2\cos 2x + \sin^2 x + 2x\sin 2x = 0$.
By LH rule,

$$\begin{aligned}&\lim_{x\to 0}\frac{-2\sin 2x}{-2x^2\sin 2x + 2x\cos 2x + \sin 2x + 2\sin 2x + 4x\cos 2x}\\&=\lim_{x\to 0}\frac{-2\sin 2x}{6x\cos 2x + 3\sin 2x - 2x^2\sin 2x}\end{aligned}$$

Again, $\lim_{x\to 0} -2\sin 2x = 0$ and $\lim_{x\to 0} 6x\cos 2x + 3\sin 2x - 2x^2\sin 2x = 0$.
By LH rule,

$$\begin{aligned}&\lim_{x\to 0}\frac{-4\cos 2x}{-4x^2\cos 2x - 4x\sin 2x + 6\cos 2x - 12x\sin 2x + 6\cos 2x}\\&=\lim_{x\to 0}\frac{-4\cos 2x}{-4x^2\cos 2x - 16x\sin 2x + 12\cos 2x}\end{aligned}$$

Taking limit, we get: $\frac{-4}{0-0+12} = -\frac{1}{3}$.
Hence, $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right) = -\frac{1}{3}$.

(b) Let $f(x) = 2\sin x - 3x + 5$. Notice that f is continuous and differentiable in $\mathbb{R}$.
Now, consider $x = 0$ and $x = \pi$, $f(0) = 5$ and $f(\pi) = 5 - 3\pi < 0$.
Apply IVT, $\exists c_1 \in (0, \pi)$ such that $f(c_1) = 0$.
So the equation must have at least 1 solution. Now, assume there are 2 solutions: c_1 and c_2, i.e., $f(c_1) = 0$ and $f(c_2) = 0$.
WLOG, suppose $c_1 < c_2$. By MVT, $\exists c_3 \in (c_1, c_2)$ such that $f'(c_3) = 0$. Then $2\cos x - 3 = 0 \implies \cos x = \frac{3}{2}$. Contradiction, since $|\cos x| \leq 1$.
Hence, the equation $2\sin x = 3x + 5$ has exactly one solution.

Question 3

(a)

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2}$$

By Riemann Sum: Let $\triangle x = \frac{b-a}{n}$, then

$$\begin{aligned}\int_a^b f(x)dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\cdot\frac{b-a}{n}\right)\frac{b-a}{n} \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} f(a+k\triangle x)\triangle x\end{aligned}$$

We know:

$$\frac{n}{n^2+k^2} = \left(\frac{1}{1+(\frac{k}{n})^2}\right)\cdot\frac{1}{n}$$

Now let $a = 0, b = 1$, and $f(x) = \frac{1}{1+x^2}$, then $\triangle x = \frac{1}{n}$.
Hence,

$$\begin{aligned}S &= \int_0^1 \frac{1}{1+x^2}dx \\ &= \left[\tan^{-1}(x)\right]_0^1 \\ &= \tan^{-1}(1) \\ &= \frac{\pi}{4}\end{aligned}$$

(b)

$$\begin{aligned}&\lim_{x\to 0} \frac{1}{ax-\sin x}\int_0^x \frac{t^2}{\sqrt{b+t^2}}dt \\ &= \lim_{n\to\infty} \frac{\int_0^x \frac{t^2}{\sqrt{b+t^2}}dt}{ax-\sin x}\end{aligned}$$

Notice that $\lim_{x\to 0}\int_0^x \frac{t^2}{\sqrt{b+t^2}}dt = \lim_{x\to 0}(ax-\sin x) = 0$.

By LH rule,

$$\begin{aligned}
\lim_{x\to 0} \frac{\frac{d}{dx}(\int_0^x \frac{t^2}{\sqrt{b+t^2}}dt)}{\frac{d}{dx}(ax-\sin x)} &= 5 \\
\implies \lim_{x\to 0} \frac{\frac{x^2}{\sqrt{b+x^2}}}{a-\cos x} &= 5 \\
\implies \lim_{x\to 0} \frac{x^2}{(a-\cos x)\sqrt{b+x^2}} &= 5
\end{aligned}$$

Consider two cases:

- Case 1: $a = 1$

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2}{(1-\cos x)\sqrt{b+x^2}} &= 5 \\
\implies \lim_{x\to 0} \frac{x^2}{\sin^2 x} \frac{(1+\cos x)}{\sqrt{b+x^2}} &= 5
\end{aligned}$$

  We know $\lim_{x\to 0} \frac{\sin x}{x} = 1$, then $\lim_{x\to 0} \frac{x^2}{\sin^2 x} = 1$.
  So $\frac{2}{\sqrt{b}} = 5$, then $b = \frac{4}{25}$.

- Case 2: $a \neq 1$
  Now consider two cases:
  - Case 1: $b = 0$

$$\begin{aligned}
\lim_{x\to 0} \frac{x}{a-\cos x} &= 5 \\
\implies \frac{0}{a-1} &= 5
\end{aligned}$$

    Contradiction, since $0 \neq 5$.
  - Case 2: $b \neq 0$

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2}{(a-\cos x)\sqrt{b+x^2}} &= 5 \\
\implies \frac{0}{(a-1)\sqrt{b}} &= 5
\end{aligned}$$

    Contradiction, since $0 \neq 5$.

Hence, $a = 1, b = \frac{4}{25}$.

Question 4

(a)

$$I = \int_0^\infty \frac{1}{1+e^x} dx$$

Let $e^x = t$, then $e^x dx = dt$, so $dx = \dfrac{dt}{e^x} = \dfrac{dt}{t}$.
When $x = 0$, $t = e^0 = 1$. Similarly, when $x \to \infty$, $t \to \infty$.
By substitution,

$$\begin{aligned} I &= \int_1^\infty \frac{1}{t(1+t)} dt \\ &= \int_1^\infty \left[\frac{1}{t} - \frac{1}{1+t}\right] dt \\ &= [\ln t - \ln(1+t)]_1^\infty \\ &= \left[\ln \frac{t}{1+t}\right]_1^\infty \\ &= \lim_{t\to\infty} \ln \frac{t}{1+t} - \ln \frac{1}{2} \end{aligned}$$

Since $\ln x$ is continuous, then

$$\begin{aligned} I &= \ln(\lim_{t\to\infty} \frac{t}{1+t}) + \ln 2 \\ &= \ln 2 + \ln \lim_{t\to\infty} \frac{1}{\frac{1}{t}+1} \\ &= \ln 2 + ln 1 \\ &= \ln 2 \end{aligned}$$

(b)

$$\begin{aligned} f(x) &= x^c(1-x)^d \\ \implies f'(x) &= cx^{c-1}(1-x)^d - d(1-x)^{d-1}x^c \\ &= x^{c-1}(1-x)^{d-1}[(1-x)c - dx] \end{aligned}$$

Because the local maxima of f must be a critical point, i.e., when $f'(x) = 0$ or $f'(x)$ is undefined. Since $f'(x)$ is always well-defined, we solve for $f'(x) = 0$. So $x = 0$, $x = 1$, $c - cx - dx = 0$ or $x = \frac{c}{c+d}$.
Now, $f(0) = f(1) = 0$, $f(\frac{c}{c+d}) = (\frac{c}{c+d})^c(\frac{d}{c+d})^d > 0$.
It is clear that $f(\frac{c}{c+d}) > f(0) = f(1) = 0$.
Having compared the values of critical points, given $0 \le x \le 1$, we also need to check the boundaries which has already been done in this case.
Hence, the global maximum value is $f(\frac{c}{c+d}) = (\frac{c}{c+d})^c(\frac{d}{c+d})^d$.

Question 5

(a)

$$I = \int_{-2}^{3} |x^2 - 1| dx$$

Note that

$$|x^2 - 1| = \begin{cases} x^2 - 1, \ x \geq 1 \\ 1 - x^2, \ x \leq 1 \end{cases}$$

Then

$$\begin{aligned} I &= \int_{-2}^{-1} x^2 - 1dx + \int_{-1}^{1} 1 - x^2 dx + \int_{1}^{3} x^2 - 1dx \\ &= \left[\frac{x^3}{3} - x\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{x^3}{3} - x\right]_{1}^{3} \\ &= \frac{-1}{3} + 1 + \frac{8}{3} - 2 + 1 - \frac{1}{3} - \frac{1}{3} + 9 - 3 - \frac{1}{3} + 1 \\ &= \frac{28}{3} \end{aligned}$$

(b) Define radius $r(y) = y + 2$ and height $h(y) = 1 + y^2 - 2y^2 = 1 - y^2$. Given the curves, we know they intersect at $y = 1$ and $y = -1$, then

$$\begin{aligned} \text{volume } V &= \int_{-1}^{1} 2\pi r(y) h(y) dy \\ &= \int_{-1}^{1} 2\pi (y + 2)(1 - y^2) dy \\ &= 2\pi \int_{-1}^{1} y + 2 - y^3 - 2y^2 dy \\ &= 2\pi \left[\frac{y^2}{2} + 2y - \frac{y^4}{4} - \frac{2y^3}{3}\right]_{-1}^{1} \\ &= \frac{16\pi}{3} \end{aligned}$$

Question 6

(a) Apply Squeeze Theorem, it is clear that $1 - x^4 \leq f(x) \leq x^2 + 1$.
Now, calculate the limits of upper bound and lower bound

$$\lim_{x \to 0} x^2 + 1 = 0 + 1 = 1$$
$$\lim_{x \to 0} 1 - x^4 = 1 - 0 = 1$$

Hence, $\lim_{x \to 0} f(x) = 1$.

(b) We know

$$\int_0^1 |f(x)| = \int_0^{\frac{1}{2}} |f(x)| dx + \int_{\frac{1}{2}}^1 |f(x)| dx$$

Calculate the integrals separately by hint:

- $\int_0^{\frac{1}{2}} |f(x)| dx$: Take $x \in [0, \frac{1}{2}]$ and f is continuous and differentiable on $[0, x]$. By MVT, $\exists c_x \in (0, x)$ such that

$$f'(c_x) = \frac{f(x) - f(0)}{x - 0}$$

Then

$$\begin{aligned} \frac{f(x)}{x} &= f'(x) \\ \implies |\frac{f(x)}{x}| &= |f'(c_x)| \leq M \\ \implies |f(x)| &\leq Mx \\ \implies \int_0^{\frac{1}{2}} |f(x)| dx &\leq \int_0^{\frac{1}{2}} Mx dx = \frac{M}{8} \end{aligned}$$

- $\int_{\frac{1}{2}}^1 |f(x)| dx$: Similarly, take $x \in [\frac{1}{2}, 1]$ and f is continuous and differentiable on $[x, 1]$. By MVT, $\exists d_x \in (x, 1)$ such that

$$f'(d_x) = \frac{f(x) - f(1)}{x - 1}$$

Then

$$\begin{aligned} \frac{f(x)}{x-1} &= f'(d_x) \\ \implies \frac{|f(x)|}{1-x} &= |f'(d_x)| \leq M \\ \implies |f(x)| &\leq M(1-x) \\ \implies \int_{\frac{1}{2}}^{1} |f(x)| dx &\leq \int_{\frac{1}{2}}^{1} M(1-x) dx = \frac{M}{8} \end{aligned}$$

Hence,

$$\int_0^1 |f(x)| dx \leq \frac{M}{8} + \frac{M}{8} = \frac{M}{4}$$

Question 7

(a)

$$y = \frac{\ln x}{x + 4x(\ln x)^2}$$

For $e \leq x \leq e^2$,

$$\begin{aligned} \ln x &\geq 1 \\ \implies y &\geq 0 \end{aligned}$$

Then

$$\text{area } A = \int_e^{e^2} \frac{\ln x}{x[1 + 4(\ln x)^2]} dx$$

Let $\ln x = t$, then $\frac{dx}{x} = dt$.

$$\begin{aligned} A &= \int_{\ln e}^{\ln(e^2)} \frac{t}{1 + 4t^2} dt \\ &= \int_1^2 \frac{t}{1 + 4t^2} dt \end{aligned}$$

Let $t^2 = v$, then $tdt = \frac{dv}{2}$.

$$\begin{aligned} A &= \int_{1^2}^{2^2} \frac{1}{2(1 + 4v)} dv \\ &= \left[\frac{1}{8} \ln |1 + 4v|_1^4 \right] \\ &= \frac{1}{8} [\ln 17 - \ln 5] \\ &= \frac{1}{8} \ln \frac{17}{5} \end{aligned}$$

(b) Apply formula, then

$$
\begin{aligned}
\text{area } S &= \int_{\frac{3}{4}}^{\frac{15}{4}} 2\pi \cdot \sqrt{y}\sqrt{1+(\frac{1}{2\sqrt{y}})^2}dy \\
&= 2\pi \int_{\frac{3}{4}}^{\frac{15}{4}} \sqrt{y+\frac{1}{4}}dy \\
&= 2\pi \left[\frac{2}{3}(y+\frac{1}{4})^{\frac{3}{2}}\right]_{\frac{3}{4}}^{\frac{15}{4}} \\
&= \frac{4\pi}{3}\left[4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] \\
&= \frac{28\pi}{3}
\end{aligned}
$$

Question 8

(a)

$$\frac{dy}{dx} = -(1 + \frac{y}{x})$$
$$\implies \frac{dy}{dx} + \frac{y}{x} = -1$$

Define $P(x) = \dfrac{1}{x},\ Q(x) = -1$, then let $v(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$.
We know

$$\begin{aligned} y &= \frac{1}{v(x)} \int v(x)Q(x)dx \\ &= \frac{1}{x} \int -xdx \\ &= \frac{1}{x}(-\frac{x^2}{2} + C) \\ &= -\frac{x}{2} + \frac{C}{x} \end{aligned}$$

Since $(1, 3)$ is a solution, then $C = \dfrac{7}{2}$.

Hence, the equation is $y = -\dfrac{x}{2} + \dfrac{7}{2x}$.

(b) Let S be amount of salt at time t.
Define the rate of change of salt:

$$\frac{dS}{dt} = 3r - \frac{Sr}{100}$$
$$\implies \frac{dS}{dt} + \frac{Sr}{100} = 3r$$

Solve the first order differential equation:
Define $P(t) = \dfrac{r}{100},\ Q(t) = 3r$, then let $v(t) = e^{\int P(t)dt} = e^{\int \frac{r}{100}dt} = e^{\frac{rt}{100}}$.
We know

$$\begin{aligned} S &= \frac{1}{v(t)} \int v(t)Q(t)d \\ &= e^{-\frac{rt}{100}} \int 3re^{\frac{rt}{100}}dt \\ &= e^{-\frac{rt}{100}} \cdot (300e^{\frac{rt}{100}} + C) \\ &= 300 + Ce^{-\frac{rt}{100}} \end{aligned}$$

When $t = 0$, it is given that $S = 100$, then $C = 100 - 300 = -200$.
Now, when $t = 45$ and $S = 200$, then

$$\begin{aligned} 200 &= 300 - 200e^{\frac{-45r}{100}} \\ \implies e^{\frac{-45r}{100}} &= \frac{1}{2} \\ \implies \frac{45r}{100} &= \ln 2 \\ \implies r &= \frac{20\ln 2}{9} \end{aligned}$$