MA3209 - Metric and Topological Spaces Suggested Solutions

(Semester 1, AY2021/2022)

Written by: Chow Boon Wei
Audited by: Chong Jing Quan

Question 1

i) For any $x, y \in X$, we have

$$\begin{aligned}
\rho(x,y) = 0 &\iff \sqrt{d(x,y)+4} - 2 = 0 \\
&\iff \sqrt{d(x,y)+4} = 2 \\
&\iff d(x,y) + 4 = 4 \\
&\iff d(x,y) = 0 \\
&\iff x = y.
\end{aligned}$$

Also, $\rho(x,y) = \sqrt{d(x,y)+4} - 2 = \sqrt{d(y,x)+4} - 2 = \rho(y,x)$.

Finally, we have

$$\begin{aligned}
\rho(x,y) + \rho(y,z) &= \sqrt{d(x,y)+4} - 2 + \sqrt{d(y,z)+4} - 2 \\
&= \sqrt{(\sqrt{d(x,y)+4} + \sqrt{d(y,z)+4})^2} - 4 \\
&= \sqrt{d(x,y) + 4 + 2\sqrt{d(x,y)+4}\sqrt{d(y,z)+4} + d(y,z) + 4} - 4 \\
&= \sqrt{d(x,y) + d(y,z) + 2\sqrt{d(x,y)d(y,z) + 4d(x,y) + 4d(y,z) + 16} + 8} - 4 \\
&\geq \sqrt{d(x,z) + 2\sqrt{d(x,y)d(y,z) + 4d(x,z) + 16} + 8} - 4 \\
&\geq \sqrt{d(x,z) + 2\sqrt{4d(x,z) + 16} + 8} - 4 \\
&\geq \sqrt{d(x,z) + 4 + 4\sqrt{d(x,z)+4} + 4} - 4 \\
&= \sqrt{d(x,z)+4} + 2 - 4 \\
&= \sqrt{d(x,z)+4} - 2.
\end{aligned}$$

ii) Let $(a_n)_{n \in \mathbb{N}}$ be a sequence that is Cauchy convergent with respect to ρ. Then, I claim that this sequence is also Cauchy convergent with respect to d. Let $\epsilon > 0$ be given. We can find an $N \in \mathbb{N}$ such that

$$n, m \geq N \implies \rho(a_n, a_m) < \sqrt{\epsilon + 4} - 2.$$

Then, whenever $n, m \geq N$, we have $d(a_n, a_m) = (\rho(a_n, a_m) + 2)^2 - 4 < \epsilon$. So, this sequence is also Cauchy convergent with respect to d. Since $(X < d)$ is complete, we can find an $a \in X$ such that $a_n \to a$ with respect to d. That is, for every $\epsilon > 0$, we can fine a $K \in \mathbb{N}$ such that

$$n \geq K \implies d(a_n, a) < (\epsilon + 2)^2 - 4.$$

Then, whenever $n \geq K$, we have $\rho(a_n, a) = \sqrt{d(a_n, a) + 4} - 2 < \epsilon$. Hence, $a_n \to a$ with respect to ρ. This means that the sequence is convergent. Finally, we see that every sequence that is Cauchy convergent with respect to ρ is also convergent with respect to ρ. So, (X, ρ) is complete.

iii) Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover for (X,ρ). Then, for each $\lambda \in \Lambda$, U_λ is open in (X,ρ). So, for each $x \in U_\lambda$, we can find a $\epsilon > 0$ such that $B_\rho(x,\epsilon) \subset U_\lambda$. Now, I claim that $B_d\left(x,(\epsilon+2)^2-4\right) \subset U_\lambda$. Indeed, we have

$$\begin{aligned} y \in B_d\left(x,(\epsilon+2)^2-4\right) &\iff d(y,x) < (\epsilon+2)^2-4 \\ &\iff \rho(x,y) < \epsilon \\ &\iff y \in B_\rho(x,\epsilon) \\ &\implies y \in U_\lambda. \end{aligned}$$

So, U_λ is also open in (X,d). In particular, $\{U_\lambda\}_{\lambda\in\Lambda}$ is also an open cover for (X,d). Since, (X,d) is compact, we can find a finite subcover $\{U_{\lambda_1},\dots,U_{\lambda_n}\}$. This shows that (X,ρ) is compact.

Remark: In fact, (X,d) and (X,ρ) have the same topology.

Question 2

i) Take $f : [0,2] \to \mathbb{R}$ given by $f(x) = x$ for each $x \in [0,2]$. Let $\epsilon > 0$ be given. Take $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. For any $n \geq N$, we have

$$\begin{aligned} d_2(f,f_n) &= \sqrt{\int_0^2 |f(x)-f_n(x)|dx} \\ &= \sqrt{\int_0^{1/n^3} |f(x)-f_n(x)|dx + \int_{1/n^3}^2 |x-f_n(x)|dx} \\ &= \sqrt{\int_0^{1/n^3} |x-n-(1-n^4)x|dx + \int_{1/n^3}^2 |x-x|dx} \\ &= \sqrt{\int_0^{1/n^3} |-n+n^4x|dx} \\ &= \sqrt{\int_0^{1/n^3} n-n^4xdx} \\ &= \sqrt{\frac{1}{2n^2}} \\ &< \frac{1}{N} \\ &< \epsilon. \end{aligned}$$

Therefore, this sequence converges in $(C[0,2],d_2)$.

ii) Let $K = \mathbb{Z}_{\geq 0}$. Then, K is compact because give any open cover $C = \{U_\lambda\}_{\lambda\in\Lambda}$, we can choose a finite subcover as follows. Fix $U_{\lambda_0} \in C$. Since U_{λ_0} is open in $(\mathbb{Z},\tau_{cofinite})$, $\mathbb{Z}\backslash U_{\lambda_0}$ is finite. So, $K\backslash U_{\lambda_0}$ is also finite. If $K\backslash U_{\lambda_0} = \emptyset$, we are done. Otherwise, write $K\backslash U_{\lambda_0} = \{x_1,\dots,x_n\}$. For each $1 \leq i \leq n$, we can find a $U_{\lambda_i} \in C$ such that $x_i \in U_{\lambda_i}$. Finally, $\{U_{\lambda_0},U_{\lambda_1},\dots,U_{\lambda_n}\}$ is the finite subcover we need. Now, the closed sets of $(\mathbb{Z},\tau_{cofinite})$ are finite subsets and $\mathbb{Z}$. Since K is infinite and $K \subset \overline{K}$, we see that $\overline{K} = \mathbb{Z}$. But, K is not open in $\overline{K} = \mathbb{Z}$ because $\mathbb{Z}\backslash K = \mathbb{Z}_{<0}$ is infinite.

Question 3

a) For reference, $Z = \{f : [2,4] \to \mathbb{R} : \forall x \in [2,4], 4 \le f(x) \le 8\}$ We can define $F : Z \to Z$ by $F(f) = \left(x \mapsto \sqrt{f\left(\frac{x+2}{2}\right) + x + 10}\right)$ for any $f \in Z$. This map is well-defined because $\sqrt{f\left(\frac{x+2}{2}\right) + x + 10} \le \sqrt{8+4+10} < 8$ and $\sqrt{f\left(\frac{x+2}{2}\right) + x + 10} \ge \sqrt{4+2+10} > 4$ for any $x \in [2,4]$. Now, with respect to the supremum norm, we have

$$\begin{aligned}
\sup_{x\in[2,4]} |F(f)(x) - F(g)(x)| &= \sup_{x\in[2,4]} \left| \sqrt{f\left(\frac{x+2}{2}\right) + x + 10} - \sqrt{g\left(\frac{x+2}{2}\right) + x + 10} \right| \\
&= \sup_{x\in[2,4]} \left| \frac{f\left(\frac{x+2}{2}\right) + x + 10 - g\left(\frac{x+2}{2}\right) - x - 10}{\sqrt{f\left(\frac{x+2}{2}\right) + x + 10} + \sqrt{g\left(\frac{x+2}{2}\right) + x + 10}} \right| \\
&= \sup_{x\in[2,4]} \left| \frac{f\left(\frac{x+2}{2}\right) - g\left(\frac{x+2}{2}\right)}{\sqrt{4+2+10} + \sqrt{4+2+10}} \right| \\
&= \sup_{x\in[2,3]} \left| \frac{f(x) - g(x)}{8} \right| \\
&\le \frac{1}{8} \sup_{x\in[2,4]} |f(x) - g(x)|.
\end{aligned}$$

So, F is a contraction map. Let $B([2,4])$ be the set of real valued bounded functions on $[2,4]$. Since $Z \subset B([2,4])$ and we know that $B([2,4])$ with the supremum norm is complete, all we have to show is that Z is closed to deduce that $B([2,4])$ is also complete. Let $(f_n)_{n\in\mathbb{N}} \subset Z$ be a sequence in Z that is convergent in $B([2,4])$. Denote its limit by f. Then, $4 \le f_n(x) \le 8$ for each $x \in [2,4]$. As $n \to \infty$, we have $4 \le f(x) \le 8$. So, $f \in Z$ which shows that Z is closed in $B([2,4])$. Hence, Z is complete. Finally, by Banach's fixed point theorem, there exists a unique $f \in Z$ such that $F(f) = f$. So, there is a unique $f \in Z$ such that

$$f(x)^2 = f\left(\frac{x+2}{2}\right) + x + 10$$

for all $x \in [2,4]$.

b) The result is clear if $X = \emptyset$ or $Y = \emptyset$. So we shall assume that we are not in these cases.

Fix $(z_1, z_2) \in (X \times Y)^a$. Then, for every open neighborhood U of (z_1, z_2), we have

$$U \cap (X \times Y)\backslash\{(z_1, z_2)\} \ne \emptyset.$$

In particular, we can choose $U = A \times B$ where A and B are neighborhoods of z_1 and z_2 respectively. But,

$$\emptyset \ne (A \times B) \cap (X \times Y)\backslash\{(z_1, z_2)\} = (A \cap X) \times (B \cap Y)\backslash\{(z_1, z_2)\} = (A\backslash\{z_1\} \times Y) \cup (X \times B\backslash\{z_2\}).$$

So, $A\backslash\{z_1\} \times Y \ne \emptyset$ or $X \times B\backslash\{z_2\} \ne \emptyset$. Hence, $A\backslash\{z_1\} \ne \emptyset$ or $B\backslash\{z_2\} \ne \emptyset$. Suppose that $A\backslash\{z_1\} = A \cap (X\backslash\{z_1\}) \ne \emptyset$ for all open neighborhood A of z_1. Then, we have $z_1 \in X^a$. Otherwise, if $A\backslash\{z_1\} = A \cap (X\backslash\{z_1\}) = \emptyset$ for some A open neighborhood A of z_1, then we can let B be arbitrary. So, $B\backslash\{z_2\} \ne \emptyset$ for any open neighborhood B of z_2. Therefore, $z_2 \in Y^a$. We have $z_1 \in X^a$ or $z_2 \in Y^a$. Therefore, $(z_1, z_2) \in (X^a \times Y) \cup (X \times Y^a)$. Hence, we have $(X \times Y)^a \subset (X^a \times Y) \cup (X \times Y^a)$.

Fix $(z_1, z_2) \in (X^a \times Y) \cup (X \times Y^a)$. So, we have $(z_1, z_2) \in X^a \times Y$ or $(z_1, z_2) \in X \times Y^a$. Suppose that $(z_1, z_2) \in X^a \times Y$. Then, $z_1 \in X^a$. So, for every open neighborhood A of z_1, we have $A\backslash\{z_1\} = A \cap (X\backslash\{z_1\}) \ne \emptyset$. Now, let U be an arbitrary open neighborhood of (z_1, z_2). Then, there exists open neighborhoods A of z_1 and B of z_2 such that $(z_1, z_2) \in A \times B \subset U$. Therefore,

$$\begin{aligned} U \cap (X \times Y)\backslash\{(z_1, z_2)\} &\supset (A \times B) \cap ((X \times Y)\backslash\{(z_1, z_2)\}) \\ &= (A \cap X) \times (B \cap Y)\backslash\{(z_1, z_2)\} \\ &= (A \times B)\backslash\{(z_1, z_2)\} \\ &= (A\backslash\{z_1\} \times Y) \cup (X \times B\backslash\{z_2\}) \\ &\neq \emptyset. \end{aligned}$$

Since U is arbitrary, we have $(z_1, z_2) \in (X \times Y)^a$. A similar argument holds for $(z_1, z_2) \in X \times Y^a$. Finally, we have $(X^a \times Y) \cup (X \times Y^a) \subset (X \times Y)^a$.

Question 4

i) Denote $Z = \{x \in X : f(x) \neq g(x)\}$.Fix $x \in X\backslash Z$. Then, $f(x) \neq g(x)$. So, there exists open sets $U, V \subset Y$ such that $f(x) \in U$, $g(x) \in V$ but $U \cap V = \emptyset$. Since, f and g are continuous, $f^{-1}(U)$ and $g^{-1}(V)$ are both open in X. Now, $x \in f^{-1}(U) \cap g^{-1}(V)$ means that $f^{-1}(U) \cap g^{-1}(V) \neq \emptyset$. Furthermore, $f^{-1}(U) \cap g^{-1}(V)$ is open in X. Finally, $f^{-1}(U) \cap g^{-1}(V) \subset X\backslash Z$ because

$$\begin{aligned} x \in f^{-1}(U) \cap g^{-1}(V) &\implies f(x) \in U, g(x) \in V \\ &\implies f(x) \neq g(x) \\ &\implies x \notin Z. \end{aligned}$$

So, for every $x \in X\backslash Z$, we can find an open set $W = f^{-1}(U) \cap g^{-1}(V)$ such that $x \in W \subset (X\backslash Z)$. Hence, $X\backslash Z$ is open in X and so Z is closed in X.

ii) Take $X = \mathbb{R}$ with the usual topology and $Y = \mathbb{R}$ with the indiscrete topology. Since $\tau_Y = \{\emptyset, \mathbb{R}\}$, it is clearly non-Hausdorff. Then, take $F, G : X \to Y$ to be given by $F(x) = 1$ and

$$G(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

for every $x \in \mathbb{R}$. Clearly, F and G are continuous (in fact any map $K : X \to Y$ is), and

$$\{x \in X : F(x) = G(x)\} = \mathbb{R}^+$$

is not closed in X.

Question 5

a) Note that $[0,2]$ is compact. For each $x \in [0,2]$ and for any $n \in \mathbb{N}$, we have

$$0 \leq g_n(x) = \frac{2\sqrt{x}}{n} + \int_0^x (f_n(t))^2 dt \leq \frac{2\sqrt{2}}{n} + 9\int_0^x t^2 dt \leq 2\sqrt{2} + 9\frac{x^3}{3} \leq 2\sqrt{2} + 24.$$

Therefore, $\{g_n\}_{n\in\mathbb{N}}$ is a point-wise bounded (in fact uniformly bounded) family of functions. Furthermore, let $x \in [0,2]$ be arbitrary and let $\epsilon > 0$ be given. Since $(x \mapsto \sqrt{x})$ is uniformly continuous on $[0,2]$, there is a $\delta_1 > 0$ such that whenever $|x-y| < \delta$, we have $|\sqrt{x} - \sqrt{y}| \leq \frac{\epsilon}{4}$. Take $\delta = \min(\delta_1, \frac{\epsilon}{72})$. Then, for any $n \in \mathbb{N}$ and $y \in [0,2]$, whenever $|x-y| < \delta$,

$$\begin{aligned}
|g_n(x) - g_n(y)| &= \left| \frac{2\sqrt{x}}{n} + \int_0^x (f_n(t))^2 dt - \frac{2\sqrt{y}}{n} - \int_0^y (f_n(t))^2 dt \right| \\
&\leq \left| \frac{2\sqrt{x}}{n} - \frac{2\sqrt{y}}{n} \right| + \left| \int_0^x (f_n(t))^2 dt - \int_0^y (f_n(t))^2 dt \right| \\
&\leq \frac{2}{n} \left| \sqrt{x} - \sqrt{y} \right| + \left| \int_x^y (f_n(t))^2 dt \right| \\
&\leq \frac{2}{1} \times \frac{\epsilon}{4} + \left| \int_x^y 9t^2 dt \right| \\
&\leq \frac{\epsilon}{2} + 3\left| y^3 - x^3 \right| \\
&\leq \frac{\epsilon}{2} + 3\left| (y-x)(x^2 + xy + y^2) \right| \\
&< \frac{\epsilon}{2} + 36\delta \\
&< \epsilon.
\end{aligned}$$

Remark: To show that $(x \mapsto \sqrt{x})$ is uniformly continuous on $[0,2]$, take $\delta = \epsilon^2$. Therefore, $\{g_n\}_{n\in\mathbb{N}}$ is a point-wise equicontinuous (in fact uniformly equicontinuous) family of functions. By the Arzela-Ascoli theorem, $(g_n)_{n\in\mathbb{N}}$ has a subsequence which converges uniformly to some continuous function on $[0,2]$.

b) Denote $K := \bigcap_{n\in\mathbb{N}} K_n$. Suppose that K is not connected. Then, there exists open sets $G, H \subset X$ such that $G \cap H \cap K = \emptyset$, $K \subset G \cup H$, $G \cap K \neq \emptyset$ and $H \cap K \neq \emptyset$. Since X is metrizable, we can choose G and H to be disjoint. But, for each $n \in \mathbb{N}$, K_n is connected. So, we have either $G \cap H \cap K_n \neq \emptyset$, $K_n \not\subset G \cup H$, $G \cap K_n = \emptyset$, or $H \cap K_n = \emptyset$. But, $G \cap K_n \supset G \cap K \neq \emptyset$ and $H \cap K_n \supset H \cap K \neq \emptyset$ so $G \cap K_n = \emptyset$ and $H \cap K_n = \emptyset$ are not possible. This leaves us with $G \cap H \cap K_n \neq \emptyset$ or $K_n \not\subset G \cup H$. Since G and H are chosen to be disjoint, we only have $K_n \not\subset G \cup H$. For each $n \in \mathbb{N}$, take $F_n = K_n \backslash (G \cup H)$. Since X is a metric space, it is Hausdorff. In a Hausdorff space, compact sets are closed. Thus, K_n is closed. Therefore, F_n is a closed subset of a compact set and is therefore also compact. Also, we have $F_{n+1} = K_{n+1} \backslash (G \cup H) \subset K_n \backslash (G \cup H) = F_n$ Finally, $\bigcap_{n\in\mathbb{N}} F_n = \bigcap_{n\in\mathbb{N}} K_n \backslash (G \cup H) = K \backslash (G \cup H) = \emptyset$ which is a contradiction.