NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Zheng Shaoxuan

MA4264 Game Theory
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Question 1

(i) This is a static game of complete information. Let $x \in [0,1]$ be the (pure) strategy of Player 1 and $s \in \{L, R\}$ be the (pure) strategy of Player 2. The payoffs to Player 1 are $u_1(x, L) = 2 - 2x$, $u_1(x, R) = x$ and the payoffs to Player 2 are $u_2(x, L) = x$, $u_2(x, R) = 1 - x$.

Consider Player 1's best response x^*. Given that $s = L$, since $u_1(x, L) = 2 - 2x$ is a decreasing function in terms of x, to maximize the payoff of Player 1, Player 1 should choose as small an x as possible. Hence the best response of Player 1, given that $s = L$, is $x^* = 0$.

On the other hand, given that $s = R$, since $u_1(x, R) = x$ is an increasing function in terms of x, to maximize the payoff of Player 1, Player 1 should choose as large an x as possible. Hence the best response of Player 1, given that $s = R$, is $x^* = 1$.

In summary the best response of Player 1 is:

$$x^* = \begin{cases} 0 & \text{if } s = L \\ 1 & \text{if } s = R. \end{cases}$$

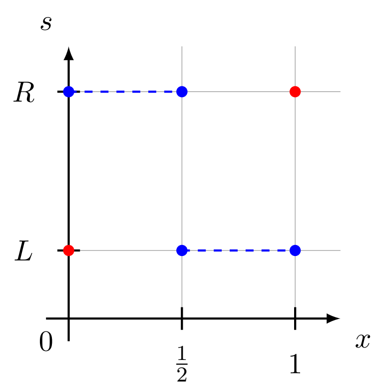
Consider Player 2's best response s^*. For any $x \in [0,1]$ that Player 1 plays,

$$\begin{aligned} s^* = L \quad &\Leftrightarrow \quad x > 1 - x \\ &\Leftrightarrow \quad x > \frac{1}{2}. \end{aligned}$$

In summary the best response of Player 2 is:

$$s^* = \begin{cases} L & \text{if } x > \frac{1}{2} \\ R & \text{if } x < \frac{1}{2} \\ \{L, R\} & \text{if } x = \frac{1}{2}. \end{cases}$$

The diagram below compares the best responses of Player 1 and Player 2, with red dots being Player 1's best response graph, and dashed blue lines being Player 2's best response graph.



Note that the graphs of the two players do not intersect. Hence there is no (pure strategy) Nash equilibrium

(ii) This is still a static game of complete information. Let $x \in [0,1]$ be the (pure) strategy of Player 1 and $q \in [0,1]$ be the mixed strategy of Player 2, where $\mathbb{P}(\text{Player 2 plays } R) = q$, i.e., $\mathbb{P}(\text{Player 2 plays } L) = 1 - q$. The payoffs to Player 1 are $u_1(x, L) = 2 - 2x$, $u_1(x, R) = x$ and the payoffs to Player 2 are $u_2(x, L) = x$, $u_2(x, R) = 1 - x$.

Consider Player 1's best response x^*. Player 1's expected payoff is

$$\begin{aligned} E(u_1(x,q)) &= \mathbb{P}(\text{Player 2 plays } L)(u_1(x,L)) + \mathbb{P}(\text{Player 2 plays } R)(u_1(x,R)) \\ &= (1-q)(2-2x) + qx \\ &= 2 - 2x - 2q + 2qx + qx \\ &= 2 - 2q + x(3q - 2). \end{aligned}$$

For maximum expected payoff of Player 1, if $3q-2 > 0$, i.e., $q > \frac{2}{3}$, then $E(u_1(x,q))$ is an increasing function, and hence $x^* = 1$ is the best response for Player 1. Similarly, if $3q - 2 < 0$, i.e., $q < \frac{2}{3}$, then $E(u_1(x,q))$ is a decreasing function, and hence $x^* = 0$ is the best response for Player 1, and if $3q - 2 = 0$, i.e., $q = \frac{2}{3}$, then $E(u_1(x,q))$ is a constant function, and hence any $x^* \in [0,1]$ is a best response for Player 1.

In summary the best response for Player 1 is:

$$x^* = \begin{cases} 1 & \text{if } q > \frac{2}{3} \\ 0 & \text{if } q < \frac{2}{3} \\ [0,1] & \text{if } q = \frac{2}{3}. \end{cases}$$

Consider Player 2's best response q^*. For any $x \in [0,1]$ that Player 1 plays,

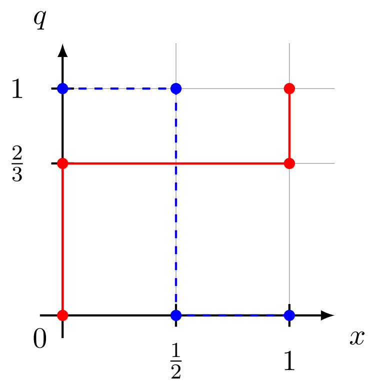
$$\begin{aligned} \text{Player 2's has higher payoff playing } L &\Leftrightarrow x > 1 - x \\ &\Leftrightarrow x > \frac{1}{2}. \end{aligned}$$

Hence Player 2 will always play L, i.e., $q = 0$ iff $x > \frac{1}{2}$, and similarly Player 2 will always play R, i.e., $q = 1$ iff $x < \frac{1}{2}$. Also, Player 2 is indifferent, i.e., $q \in [0,1]$ iff $x = \frac{1}{2}$.

In summary the best response of Player 2 is:

$$q^* = \begin{cases} 0 & \text{if } x > \frac{1}{2} \\ 1 & \text{if } x < \frac{1}{2} \\ [0,1] & \text{if } x = \frac{1}{2}. \end{cases}$$

The diagram below compares the best responses of Player 1 and Player 2, with solid red lines being Player 1's best response graph, and dashed blue lines being Player 2's best response graph.



Since the two graphs intersect at the point $(\frac{1}{2}, \frac{2}{3})$, hence the only mixed strategy Nash equilibrium is $(x^* = \frac{1}{2}, q^* = \frac{2}{3})$.

The expected payoff of Player 1 is

$$\begin{aligned}
& \mathbb{P}(\text{Player 2 plays } L)(u_1(x,L)) + \mathbb{P}(\text{Player 2 plays } R)(u_1(x,R)) \\
= \; & (1-q^*)(2-2x^*) + q^*x^* \\
= \; & \frac{1}{3}\left(2 - 2\times\frac{1}{2}\right) + \frac{2}{3}\left(\frac{1}{2}\right) \\
= \; & \frac{1}{3} + \frac{1}{3} \\
= \; & \frac{2}{3}.
\end{aligned}$$

The expected payoff of Player 2 is

$$\begin{aligned}
& \mathbb{P}(\text{Player 2 plays } L)(u_2(x,L)) + \mathbb{P}(\text{Player 2 plays } R)(u_2(x,R)) \\
= \; & (1-q^*)(x^*) + q^*(1-x^*) \\
= \; & \frac{1}{3}\left(\frac{1}{2}\right) + \frac{2}{3}\left(1 - \frac{1}{2}\right) \\
= \; & \frac{1}{6} + \frac{1}{3} \\
= \; & \frac{1}{2}.
\end{aligned}$$

Question 2

This is a two-stage game with complete information.

Stage 1: Government 1 and Government 2 simultaneously choose x_1, $x_2 \geq 0$.
Stage 2: Firm 1 and Firm 2 observes x_1, x_2 and then simultaneously choose q_1, $q_2 \geq 0$.

For $i, j = 1, 2$, $i \neq j$, the payoff for Government i is $w_i(q_1, q_2, x_1, x_2) = q_i(a-(q_i+q_j)-(c-x_i)) - \frac{x_i^2}{2}$, and the payoff for Firm i is $\pi_i(q_1, q_2, x_1, x_2) = q_i(a-(q_i+q_j)-(c-x_i))$.

We perform backward induction.

At Stage 2, given x_1 and x_2, we consider the static game between Firm 1 and Firm 2. This is similar to the Cournot model of duopoly.

For Firm i's best response q_i^*, given q_j, Firm i seeks to maximize

$$\begin{aligned}
\pi_i(q_1, q_2, x_1, x_2) &= q_i(a-(q_i+q_j)-(c-x_i)) \\
&= -q_i^2 + q_i(a - q_j - c + x_i).
\end{aligned}$$

To solve this, consider $\frac{d}{dq_i}\pi_i(q_1, q_2, x_1, x_2) = 0$:

$$\begin{aligned}
-2q_i + (a - q_j - c + x_i) &= 0 \\
q_i &= \frac{1}{2}(a - q_j - c + x_i).
\end{aligned}$$

We now solve for q_i in terms of a, c, x_i and x_j only:

$$\begin{aligned}
q_i &= \frac{1}{2}\left(a - \left(\frac{1}{2}(a - q_i - c + x_j)\right) - c + x_i\right) \\
&= \frac{1}{2}\left(a - \frac{1}{2}a + \frac{1}{2}q_i + \frac{1}{2}c - \frac{1}{2}x_j - c + x_i\right) \\
&= \frac{1}{2}\left(\frac{1}{2}a + \frac{1}{2}q_i - \frac{1}{2}c - \frac{1}{2}x_j + x_i\right) \\
&= \frac{1}{4}\left(a + q_i - c - x_j + 2x_i\right) \\
\frac{3}{4}q_i &= \frac{1}{4}\left(a - c - x_j + 2x_i\right) \\
q_i &= \frac{1}{3}\left(a - c - x_j + 2x_i\right).
\end{aligned}$$

Therefore, Firm i's best response is $q_i^* = \frac{1}{3}\left(a - c - x_j + 2x_i\right)$.

(Note that $q_i^* + q_j^* = \frac{1}{3}\left(2a - 2c + x_i + x_j\right)$.)

At Stage 1, we consider another static game between Government 1 and Government 2.

Knowing how firms will react at stage 2, to obtain the best response x_i^*, Government i seeks to maximize

$$\begin{aligned}
w_i(q_1^*, q_2^*, x_1, x_2) &= q_i^*(a - (q_i^* + q_j^*) - (c - x_i)) - \frac{x_i^2}{2} \\
&= \frac{1}{3}\left(a - c - x_j + 2x_i\right)\left(a - \frac{1}{3}\left(2a - 2c + x_i + x_j\right) - (c - x_i)\right) - \frac{1}{2}x_i^2 \\
&= \frac{1}{3}\left(a - c - x_j + 2x_i\right)\left(a - \frac{2}{3}a + \frac{2}{3}c - \frac{1}{3}x_i - \frac{1}{3}x_j - c + x_i\right) - \frac{1}{2}x_i^2 \\
&= \frac{1}{3}\left(a - c - x_j + 2x_i\right)\left(\frac{1}{3}a - \frac{1}{3}c + \frac{2}{3}x_i - \frac{1}{3}x_j\right) - \frac{1}{2}x_i^2 \\
&= x_i^2\left(\frac{4}{9} - \frac{1}{2}\right) + x_i\left(\frac{2}{3}\left(\frac{1}{3}a - \frac{1}{3}c - \frac{1}{3}x_j\right) + \frac{2}{9}(a - c - x_j)\right) + D \\
&= -\frac{1}{18}x_i^2 + x_i\left(\frac{2}{9}a - \frac{2}{9}c - \frac{2}{9}x_j + \frac{2}{9}a - \frac{2}{9}c - \frac{2}{9}x_j\right) + D \\
&= -\frac{1}{18}x_i^2 + \frac{4}{9}x_i\left(a - c - x_j\right) + D,
\end{aligned}$$

where D is the sum of the other terms in the above expansion which do not contain x_i. To solve this, consider $\frac{d}{dx_i}w_i(q_1^*, q_2^*, x_1, x_2) = 0$:

$$\begin{aligned}
-\frac{1}{9}x_i + \frac{4}{9}\left(a - c - x_j\right) &= 0 \\
x_i &= 4\left(a - c - x_j\right).
\end{aligned}$$

We now solve for x_i in terms of a and c only:

$$\begin{aligned}
x_i &= 4(a - c - 4(a - c - x_i)) \\
&= 4(a - c - 4a + 4c + 4x_i) \\
&= 4(-3a + 3c + 4x_i) \\
&= -12a + 12c + 16x_i \\
15x_i &= 12(a - c) \\
x_i &= \frac{4}{5}(a - c).
\end{aligned}$$

Hence, $x_i^* = \frac{4}{5}(a-c)$.

Going back to Stage 2, substituting x_i^* into x_i in the expression for q_i^*:

$$\begin{aligned} q_i^* &= \frac{1}{3}\left(a-c+\frac{4}{5}(a-c)\right) \\ &= \frac{1}{3}\left(\frac{9}{5}(a-c)\right) \\ &= \frac{3}{5}\left(a-c\right). \end{aligned}$$

Question 3

(i) From the extensive form tree given in the quest, we construct the normal form table for this game, where Player 1 can choose between D and U, Player 2 can choose between B and T, and player 3 can choose between L and R:

	DL	DR	UL	UR
B	2, 0, 1	1, 0, 2	3, 0, 4	0, 2, 3
T	2, 0, 1	1, 0, 2	2, 1, 0	2, 1, 0

We proceed to compare payoffs for the two strategies each Player can play, with other Players' strategies kept constant, and underline the superior payoff in each comparision. The results are as shown:

	DL	DR	UL	UR
B	2, $\underline{0}$, 1	$\underline{1}$, $\underline{0}$, $\underline{2}$	$\underline{3}$, 0, $\underline{4}$	0, $\underline{2}$, 3
T	$\underline{2}$, $\underline{0}$, 1	1, $\underline{0}$, $\underline{2}$	$\underline{2}$, $\underline{1}$, $\underline{0}$	$\underline{2}$, 1, $\underline{0}$

Nash equilibria correspond to the cells where all three numbers have been underlined. Hence in this game, the two Nash equilibria are (D, B, R) and (U, T, L).

(ii) Observe that this game has no subgames. This is because by observing the decision nodes in the extensive form of the game, the decision node corresponding to Player 1 is an initial node, the two decision nodes corresponding to Player 3 are not in singleton information sets, and the decision node corresponding to Player 2 corresponds to a subtree whichs cuts across Player 3's information set.

Hence in this game, the two subgame-perfect Nash equilibria are (D, B, R) and (U, T, L).

(iii) To obtain perfect Bayesian equilibria, we first assign a belief, $p \in [0, 1]$ to the information set of Player 3: Player 3 believes that:

$$\mathbb{P}(\text{his move was given by Player 1}) = p, \text{ i.e., } \mathbb{P}(\text{his move was given by Player 2}) = 1 - p.$$

We have two Nash equilibria obtained from (i): (D, B, R) and (U, T, L). We now only need to consider whether either of them are perfect Bayesian equlibria, subjected to restrictions on the range p can hold.

For (D, B, R): the equilibrium path passes through the decision set of Player 3. In particular, the path passes through the decision node corresponding to $p = 1$, since Player 3's move is given by Player 1. We now use $p = 1$ and check if all three players have made sequentially rational decisions.

Given that $p = 1$, Player 3's best response is to play R, as playing R gives a payoff of 2 as compared to playing L which gives a payoff of 1. Player 2's best response is to either play B or T, as there is no difference seeing that the decision node corresponding to Player 2 will not be reached given

Player 1 playing D. Player 1's best response is to play D, as playing D gives a payoff of 1 as compared to playing U, which when Player 2 plays B and Player 1 plays R, gives a payoff of 0.

We have checked that all Players have made sequentially rational decisions based on the belief that $p = 1$. Hence the corresponding perfect Bayesian equilibrium is $(D, B, R, p = 1)$.

For (U, T, L): the equilibrium path does not cut the information set of Player 3. Furthermore, there is no other information given by (U, T, L) that states that which of the two decision nodes of Player 3 is more likely to be reached. Hence we do not have a restriction of p at this point in time.

Consider Player 3's best response, given the belief $p \in [0, 1]$. Player 3's expected payoff for playing L is $p(1) + (1-p)(4) = 4 - 3p$, and for playing R is $p(2) + (1-p)(3) = 3 - p$. For L to be a rational choice for Player 3, $4 - 3p \geq 3 - p$ and hence $p \leq \frac{1}{2}$.

Player 2's best response is T, as playing T gives a payoff of 1 as compared to playing B which, since Player 3 plays L, gives a payoff of 0. Player 1's best response is either D or U, as playing U gives a payoff of 2 as compared to playing D, which when Player 3 plays L, also gives a payoff of 2.

We have checked that all Players have made sequentially rational decisions based on the belief that $p \leq \frac{1}{2}$. Hence the corresponding perfecy Bayesian equilibrium is $(U, T, L, p \leq \frac{1}{2})$.

Question 4

This is a two stage game with potentially incomplete information.

Stage 1: Plaintiff knows v and decides whether or not to reveal v to the Judge.
Stage 2: Judge, with or without the information v, decides on R.

The payoff for the Plaintiff is $R - v$ and the payoff for the Judge is $-(v - R)^2$. Furthermore, if in Stage 1, the Plaintiff does not reveal v to the Judge, the Judge adopts a belief that the v is uniformly distributed among $\{0, 1, \ldots, 9\}$.

We use backward induction.

At Stage 2, if the Judge does know v, then, in order to maximize his own payoff, his best response is $R = v$, since the payoff for the Judge would then be $-(v - v)^2 = 0$, the highest possible payoff since $-(v - R)^2 \leq 0$.

However, if the Judge does not know v, the expected payoff for the Judge is

$$-\frac{1}{10}(0 - R)^2 - \frac{1}{10}(1 - R)^2 - \ldots - \frac{1}{10}(9 - R)^2.$$

To maximize the expected payoff, we differentiate the above with respect to R and equate to 0:

$$\begin{aligned}
\frac{2}{10}((0 - R) + (1 - R) + \ldots + (9 - R)) &= 0 \\
45 &= 10R \\
R &= \frac{9}{2}.
\end{aligned}$$

Hence the best response for the Judge in this case is $R = \frac{9}{2}$.

At Stage 1, given that the Plaintiff knows how the Judge will react in Stage 2, we can calculate the Plaintiff's payoffs.

If the Plaintiff reveals v, then the Judge will play $R = v$, and hence the Plaintiff gets a payoff of $R - v = v - v = 0$.

If the Plaintiff does not reveal v, then the Judge will play $R = \frac{9}{2}$, and hence the Plaintiff gets a payoff of $R - v = \frac{9}{2} - v$. Observe that $\frac{9}{2} - v > 0 \Leftrightarrow v < \frac{9}{2}$.

Hence the best response of the Plaintiff is to reveal v if $v > \frac{9}{2}$, to not reveal v if $v < \frac{9}{2}$, and to be indifferent if $v = \frac{9}{2}$.

Hence a perfect Bayesian equilibrium for this game is:

$$\left(\text{Plaintiff} \begin{cases} \text{reveal } v & \text{if } v > \frac{9}{2}; \\ \text{not reveal } v & \text{if } v < \frac{9}{2}; \\ \text{indifferent} & \text{if } v = \frac{9}{2}. \end{cases}, R = \begin{cases} v & \text{if Plaintiff reveals } v; \\ \frac{9}{2} & \text{if Plaintiff does not reveal } v. \end{cases}, \right.$$

$$\left. \text{Judge believes that } v \text{ is uniformly distributed among } \{0, 1, \ldots, 9\}. \right)$$

Question 5

Given that $u_1 = x_1^2 y_1$ and $u_2 = x_2 y_2^2$, to compute the threat point, consider the values of x_1, x_2, y_1 and y_2 in the absence of trade. By the question, $x_1 = 3$, $y_1 = 0$, $x_2 = 0$ and $y_2 = 30$. Hence $u_1 = x_1^2 y_1 = 0$ and $u_2 = x_2 y_2^2 = 0$. Hence the threat point is $(0, 0)$.

If trade is involved, we observe that the total number of loaves of bread is still 3 and the total number of apples is still 30. Hence $x_2 = 3 - x_1$ and $y_2 = 30 - y_1$.

For Nash bargaining solution, we aim to solve

$$\begin{aligned} & \max_{(u_1, u_2)} (u_1 - 0)(u_2 - 0) \\ = \; & \max_{(x_1, y_1)} (x_1^2 y_1)(3 - x_1)(30 - y_1)^2. \end{aligned}$$

To solve the above, we perform partial differentiations on the expression above with respect to x_1 and with respect to y_1, and equate both resulting expressions with 0.

Performing partial differentiation with respect to x_1 and equating to 0:

$$\begin{aligned} y_1(30 - y_1)^2((2x_1)(3 - x_1) + (-1)(x_1^2)) &= 0 \\ y_1(30 - y_1)^2(6x_1 - 2x_1^2 - x_1^2) &= 0 \\ x_1 y_1 (30 - y_1)^2 (2 - x_1) &= 0. \end{aligned}$$

Performing partial differentiation with respect to y_1 and equating to 0:

$$\begin{aligned} x_1^2(3 - x_1)((30 - y_1)^2(1) + (y_1)(2)(30 - y_1)(-1)) &= 0 \\ x_1^2(3 - x_1)(30 - y_1)((30 - y_1) - 2y_1) &= 0 \\ 3x_1^2(3 - x_1)(30 - y_1)(10 - y_1) &= 0. \end{aligned}$$

We observe that to solve the maximization problem, $x_1 \neq 0, 3$ and $y_1 \neq 0, 30$, otherwise we obtain 0 which is definitely not the maximum possible for the above expression.

Hence the only remaining solution that yields from the two partial differentiation procedures is $x_1 = 2$, $y_1 = 10$, i.e., $x_2 = 1$, $y_2 = 20$.

Hence, $u_1 = (2)^2(10) = 30$, $u_2 = (1)(20)^2 = 400$, and thus $(40, 400)$ is the Nash bargaining solution.