NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1100 Basics of Mathematics
AY 2005/2006 Sem 1

Question 1

(a) If $r = 0$, then for any $x \in \mathbb{R} - \mathbb{Q}$, we have $rx = 0 \in \mathbb{Q}$.

If $r \neq 0$, we claim that rx is not rational.
Assume on the contrary that we have $r \in \mathbb{Q}$ and $x \in \mathbb{R} - \mathbb{Q}$, such that $rx \in \mathbb{Q}$.
Then, $rx = \frac{a}{b}$ and $r = \frac{c}{d}$ for some $a, b, c, d \in \mathbb{Z}^+ \backslash \{0\}$.
So, $x = rx \div r = \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$, but $\frac{ad}{bc} \in \mathbb{Q}$. We have a contradiction.
Therefore, $rx \notin \mathbb{Q}$.

Question 2

Assume on the contrary that there exists an injective function $f : \mathbb{R} \to \mathbb{Q}$.
Since $\mathbb{Q}$ is countable, there exists a bijective function $g : \mathbb{Q} \to \mathbb{N}$.
This give us $gf : \mathbb{R} \to \mathbb{N}$ to be an injective function, thus $|\mathbb{R}| \leq |\mathbb{N}|$.
This give us $\mathbb{R}$ to be countable, a contradiction.
Therefore such an injective function from $\mathbb{R}$ to $\mathbb{Q}$ does not exists.

Question 3

(i) $1, 2, 5, 7, 10, 14, 35, 70$.

(ii) 3 of them are prime, i.e. $2, 5$ and 7.

(iii) There are 6 positive divisors of 18, i.e. $1, 2, 3, 6, 9$ and 18.

(iv) 2 of them are prime, i.e. 2 and 3.

(v) Yes.
We can see from the list in (i) that 70 has no divisors of the form k^2, $k \in \mathbb{Z}^+ \backslash \{1\}$.

(vi) No.
Since $9|18$ and $9 = 3^2$, 18 is not square free.

(vii) Let n be a positive square-free integer.
This give us n to be a multiple of distinct prime numbers, i.e. $n = p_1 p_2 \dots p_m$, (where $m \in \mathbb{Z}^+$).
Let the set of positive divisors of n be D and the set of prime factors of n be $S = \{p_1, p_2, \dots, p_m\}$.
Consider the well-defined function $f : D \to S$, such that $f(d) = \{p \text{ is prime} \mid p|d\ \}$.
If $f(d_1) = f(d_2)$, then $p|d_1$ iff $p|d_2$ for all prime p of n. Since n is positive square-free integer, so is d_1 and d_2. This give us $d_1 = d_2$. Thus f is injective.

Let $s = \{p_{i_1}, p_{i_2}, \ldots p_{i_k}\}$ for some $1 \le i_1 < i_2 < \cdots < i_k \le m$. Then we have $d = p_{i_1}p_{i_2}\cdots p_{i_k} \in D$ such that $f(d) = s$. Thus f is surjective.
Therefore f is bijective, and so $|D| = |\mathcal{P}(S)| = 2^k$ for some $k \in \mathbb{Z}^+$.

Question 4

(i) We notice that $\gcd(31349, -35351) = \gcd(31349, 35351)$. Using Euclidean algorithm, we have

$$\begin{aligned} 35351 &= (31349)(1) + 4002 \\ 31349 &= (4002)(7) + 3335 \\ 4002 &= (3335)(1) + 667 \\ 3335 &= (667)(5). \end{aligned}$$

Therefore, $d = 667$.

(ii) Using the equations generated in (4i.), we have,

$$\begin{aligned} d &= 4002 - (3335)(1) = 4002 - [31349 - (4002)(7)](1) \\ &= (31349)(-1) + (4002)(8) = (31349)(-1) + [35351 - (31349)(1)](8) \\ &= (35351)(8) + (31349)(-9) \\ &= (31349)(-9) - (35351)(-8). \end{aligned}$$

$\therefore s = -9$ and $t = -8$ satisfy the condition.

(iii) Since

$$\begin{aligned} (31349)(-9) - (35351)(-8) &= (31349)(-9) - (35351)(-8) + (31349)(35351) - (35351)(31349) \\ &= (31349)(-9 + 35351) - (35351)(-8 + 31349) \\ &= (31349)(35342) - (35351)(31341). \end{aligned}$$

$\therefore x = 35342$ and $y = 31341$ is another pair that satisfy the condition.

Question 5

(i) $\pm 1, \pm 2, \pm 3$ and ± 6.

(ii) 6.

(iii) Since there exists primes p_i and p_j such that $p_i \neq p_j$, we have $\gcd(p_i, p_j) = 1$.
Notice that $\gcd(p_1, p_2, ..., p_n) \mid p_i$ and $\gcd(p_1, p_2, ..., p_n) \mid p_j$, thus $\gcd(p_1, p_2, ..., p_n) \mid \gcd(p_i, p_j) = 1$.
This give us $\gcd(p_1, p_2, ..., p_n) = 1$.

(iv) Let $\gcd(c_1, c_2, \ldots, c_k, c_{k+1}) = d_1$, $\gcd(c_1, c_2, \ldots, c_k) = d_2$, and $\gcd(d_2, c_{k+1}) = d_3$.

Since $d_1 \mid c_i$ for all $i = 1, 2, \ldots, k$, we have $d_1 \mid \gcd(c_1, c_2, \ldots, c_k) = d_2$.
Also $d_1 \mid c_{k+1}$. Thus $d_1 \mid \gcd(d_2, c_{k+1}) = d_3$.

On the other hand, we also have $d_3 \mid d_2$.
Since $d_2 \mid c_i$ for all $i = 1, 2, \ldots, k$, we have $d_3 \mid c_i$ for all $i = 1, 2, \ldots, k$.
Together with the fact that $d_3 \mid c_{k+1}$, we have $d_3 \mid \gcd(c_1, c_2, \ldots, c_k, c_{k+1}) = d_1$.

(v) We shall prove a stronger claim.
Let P_n be the proposition that there exists integers $s_1, s_2, \ldots, s_n$ such that

$$\gcd(c_1, c_2, \ldots, c_n) = s_1c_1 + s_2c_2 + \cdots + s_nc_n,$$

for $n \geq 2$.

The existence of $s_1, s_2 \in \mathbb{Z}^+$ for $\gcd(c_1, c_2) = s_1c_1 + s_2c_2$ is guaranteed by Euclidean Algorithm. Thus P_2 is true.

Assume P_k is true for some $k \geq 2$, i.e. there exists integers $a_1, a_2, \ldots, a_k$ such that

$$\gcd(c_1, c_2, \ldots, c_k) = a_1c_1 + a_2c_2 + \cdots + a_nc_k.$$

Consider P_{k+1}.
We have from result of (5iv.),

$$\begin{aligned}
\gcd(c_1, c_2, \ldots, c_k, c_{k+1}) &= \gcd(\gcd(c_1, c_2, \ldots, c_k), c_{k+1}) \\
&= \gcd(a_1c_1 + a_2c_2 + \cdots + a_kc_k, c_{k+1}) \qquad \text{(from induction hypothesis)} \\
&= b_1(a_1c_1 + a_2c_2 + \cdots + a_kc_k) + b_2c_{k+1} \qquad \text{(result of Euclidean Algorithm)} \\
&= (a_1b_1)c_1 + (a_2b_1)c_2 + \cdots + (a_kb_1)c_k + b_2c_{k+1} \\
&= s_1c_1 + s_2c_2 + \cdots + s_kc_k + s_{k+1}c_{k+1}, \quad s_i \in \mathbb{Z},\ i = 1, 2, \ldots, k+1.
\end{aligned}$$

Thus P_{k+1} is true.

Since P_2 is true, and P_k is true implies that P_{k+1} is true, by Mathematical Induction, P_n is true for all $n \geq 2$. In particular, it is true for all $n \geq 3$, which is what we wanted.

(vi) Notice that $\gcd(24, -36, 60, 90) = 6 = \gcd(24, 90)$. By trial and error, or the Euclidean Algorithm on 24 and 90, we can get $\gcd(24, -36, 60, 90) = 24(4) + 36(0) + 60(0) + 90(-1)$.
Thus we have $\{s_1, s_2, s_3, s_4\} = \{4, 0, 0, -1\}$ to be one possible solution.