NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Ho Chin Fung

MA2101 Linear Algebra II
AY 2007/2008 Sem 1

SECTION A

Question 1

(i) $\mathcal{S} = \{1, x, x^2\}$. Then

$$\begin{aligned} T(1) &= 1 + x^2 \\ T(x) &= x + 2x^2 \\ T(x^2) &= 1 + x^2 \end{aligned}$$

So

$$[T]_{\mathcal{S}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

(ii) The characteristic polynomial of T is given by

$$\begin{aligned} c_T(x) &= \begin{vmatrix} x-1 & 0 & -1 \\ 0 & x-1 & 0 \\ -1 & 2 & x-1 \end{vmatrix} \\ &= (x-1)^3 - (x-1) \\ &= x(x-1)(x-2) \end{aligned}$$

So the eigenvalues of T are 0, 1 and 2.

(iii) Since T has 3 distinct eigenvalues, it is diagonalisable. To find eigenvectors of $[T]_S$ corresponding to $\lambda = 0$, we solve the linear system $([T]_{\mathcal{S}} - 0I)\boldsymbol{x} = \mathbf{0}$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right) \overset{R_3 - R_1 - 2R_2}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Set $x_3 = t$. Then $x_1 = -t$ and $x_2 = 0$. In particular, with $t = -1$, we obtain $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Similar computations show that the vectors $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are eigenvectors of $[T]_S$ corresponding to $\lambda = 1$ and $\lambda = 2$ respectively. So $1 - x^2$, $2 - x$, and $1 + x^2$ are eigenvectors of T corresponding to $\lambda = 0$, 1, and 2 respectively.
Let $\mathcal{B} = \{1 - x^2, 2 - x, 1 + x^2\}$.

Then $\mathcal{B}$ is a basis for $P_2(\mathbb{R})$ and $[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Question 2

(i) We compute

$$\begin{aligned}\det[T_2 \circ T_1]_{\mathcal{B}_1} &= \begin{vmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & -2 \end{vmatrix} \\ &= 0.\end{aligned}$$

So $[T_2 \circ T_1]_{\mathcal{B}_1}$ is not invertible. Since $[T_2 \circ T_1]_{\mathcal{B}_1}$ is the matrix of $T_2 \circ T_1$ with respect to $\mathcal{B}_1$, $T_2 \circ T_1$ is also not invertible.

(ii) Let

$$\begin{aligned}[T_1]_{\mathcal{B}_2,\mathcal{B}_1} &= \begin{pmatrix} a & -1 & c \\ b & 1 & d \end{pmatrix}, \\ [T_2]_{\mathcal{B}_2,\mathcal{B}_1} &= \begin{pmatrix} 1 & e \\ 0 & f \\ -1 & g \end{pmatrix}.\end{aligned}$$

Then

$$\begin{aligned}[T_2]_{\mathcal{B}_1,\mathcal{B}_2}[T_1]_{\mathcal{B}_2,\mathcal{B}_1} &= \begin{pmatrix} 1 & e \\ 0 & f \\ -1 & g \end{pmatrix}\begin{pmatrix} a & -1 & c \\ b & 1 & d \end{pmatrix} \\ \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix} = [T_2 \circ T_1]_{\mathcal{B}_1} &= \begin{pmatrix} a+eb & -1+e & c+ed \\ bf & f & fd \\ -a+bg & 1+g & -c+dg \end{pmatrix}.\end{aligned}$$

Comparing each entry, we obtain

$$a = 0, \quad b = 1, \quad c = 2, \quad d = 0, \quad e = 2, \quad f = 1, \quad g = 0.$$

So

$$\begin{aligned}[T_1]_{\mathcal{B}_2,\mathcal{B}_1} &= \begin{pmatrix} 0 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}, \\ [T_2]_{\mathcal{B}_1,\mathcal{B}_2} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}.\end{aligned}$$

(iii) Since T_1 is a linear transformation, we have

$$\begin{aligned}T_1(a+bx+cx^2) &= aT_1(1) + bT_1(x) + cT_1(x^2) \\ &= a\begin{pmatrix} 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -b+2c \\ a+b \end{pmatrix}.\end{aligned}$$

Question 3

(i) We have

$$\begin{aligned} \det(xI - A) &= c_A(x) \\ &= (x-1)^6(x-2)^5(x-3)^2. \end{aligned}$$

In particular, when $x = 0$,

$$\begin{aligned} \det(-A) &= (-1)^6(-2)^5(-3)^2 \\ (-1)^{6+5+2}\det(A) &= (1)(-32)(9) \\ (-1)\det(A) &= -288 \\ \det(A) &= 288. \end{aligned}$$

(ii) All the possible Jordan canonical forms of A are

$$\begin{pmatrix} J_3(1) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_3(1) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & J_3(2) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & J_2(2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_2(3) \end{pmatrix}, \begin{pmatrix} J_3(1) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_2(1) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & J_3(2) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_2(2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_2(3) \end{pmatrix},$$

$$\begin{pmatrix} J_3(1) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_3(2) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_2(2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_2(3) \end{pmatrix}, \begin{pmatrix} J_3(1) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_3(1) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & J_3(2) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_2(3) \end{pmatrix},$$

$$\begin{pmatrix} J_3(1) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_2(1) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & J_3(2) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_2(3) \end{pmatrix}, \begin{pmatrix} J_3(1) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_3(2) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & J_2(3) \end{pmatrix}.$$

Question 4

(i) Let

$$X = \begin{pmatrix} b & 2a+b \\ a-2b & -2b \end{pmatrix} \in W.$$

Then,

$$X = a\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} + b\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}.$$

So, $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$ spans W. Observe that $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$ are not linear multiples of each other. So $\left\{\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\right\}$ is linearly independent. Therefore, $\left\{\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\right\}$ is a basis for W.

(ii) For any $X \in W^{\perp}$,

$$\|F - \mathbf{proj}_{W^{\perp}}(F)\| \leq \|F - X\|.$$

Therefore, $\mathbf{proj}_{W^{\perp}}(F)$ can be one such matrix for P.
Now we find $\mathbf{proj}_{W^{\perp}}(F)$. Applying Gram-Schmidt process to $\left\{\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\right\}$, we obtain $\left\{\frac{1}{\sqrt{5}}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\right\}$, an orthonormal basis for W.
So,

$$\begin{aligned}
\mathbf{proj}_W(F) &= \left\langle \begin{pmatrix} 4 & 0 \\ 5 & 7 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \right\rangle \frac{1}{\sqrt{5}}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \\
&\quad + \left\langle \begin{pmatrix} 4 & 0 \\ 5 & 7 \end{pmatrix}, \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \right\rangle \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} + (-2)\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \\
&= \begin{pmatrix} -2 & 0 \\ 5 & 4 \end{pmatrix}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbf{proj}_{W^{\perp}}(F) &= F - \mathbf{proj}_W(F) \\
&= \begin{pmatrix} 4 & 0 \\ 5 & 7 \end{pmatrix} - \begin{pmatrix} -2 & 0 \\ 5 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}.
\end{aligned}$$

Therefore,

$$P = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}.$$

Question 5

(a) False.

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$. Clearly, S is a linearly dependent subset of $\mathbb{R}^3$. However $\boldsymbol{v}_1 = c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3$ has no solutions.

(b) True.

Since T_2 is an isomorphism, T_2^{-1} exists.
Let $\boldsymbol{y}_1 \in \mathcal{R}(T_1)$.
Then $\exists \boldsymbol{x}_1 \in V$ s.t. $T_1(\boldsymbol{x}_1) = \boldsymbol{y}_1$.

$$\begin{aligned} \boldsymbol{y}_1 &= T_1(T_2T_2^{-1}(\boldsymbol{x}_1)) \\ &= T_1T_2(T_2^{-1}(\boldsymbol{x}_1)) \quad \in \mathcal{R}(T_1T_2) \\ &\therefore \mathcal{R}(T_1) \subseteq \mathcal{R}(T_1T_2). \end{aligned}$$

Let $\boldsymbol{y}_2 \in \mathcal{R}(T_1T_2)$.
Then $\exists \boldsymbol{x}_2 \in V$ s.t. $T_1T_2(\boldsymbol{x}_2) = \boldsymbol{y}_2$.

$$\begin{aligned} \boldsymbol{y}_2 &= T_1T_2(\boldsymbol{x}_2) \\ &= T_1(T_2(\boldsymbol{x}_2)) \quad \in \mathcal{R}(T_1) \\ &\therefore \mathcal{R}(T_1T_2) \subseteq \mathcal{R}(T_1). \end{aligned}$$

Thus, $\mathcal{R}(T_1T_2) = \mathcal{R}(T_1)$. Moreover, since V is finite dimensional, $\mathcal{R}(T_1), \mathcal{R}(T_1T_2) \subset V$ are also finite dimensional. Therefore, $\text{rank}(T_1T_2) = \text{rank}(T_1)$.

(c) False.

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.
Both A and B have 2 eigenvalues each and are therefore diagonalizable. However, $A + B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2J_2(0)$, which is not diagonalizable.

Question 6

(a) Claim: $\text{span}\{A^m : m = 0, 1, 2, 3, \ldots\} = \text{span}\{A^n : n = 0, 1, 2, \ldots, k-1\}$.
Let the minimum polynomial of A be m_A.
By the division algorithm, $\forall m \geq 0 \in \mathbb{Z}$,

$$x^m = m_A(x)q(x) + r(x)$$

for some polynomial $p(x)$ and $r(x)$ where $\deg[r(x)] < \deg[m_A(x)] = k$.
Sub $x = A$,

$$\begin{aligned} A^m &= m_A(A)q(A) + r(A) \\ &= (0_n)q(A) + r(A) \\ &= r(A). \end{aligned}$$

A^m can be expressed as a polynomial in A of powers up to $(k-1)$.
So, $\forall m \geq 0 \in \mathbb{Z}, A^m \in \text{span}\{A^n : n = 0, 1, 2, \ldots, k-1\}$.

Thus, we have, $\operatorname{span}\{A^m : m = 0, 1, 2, 3, \dots\} \subseteq \operatorname{span}\{A^n : n = 0, 1, 2, \dots, k-1\}$.
Next, clearly, $\operatorname{span}\{A^m : m = 0, 1, 2, 3, \dots\} \supseteq \operatorname{span}\{A^n : n = 0, 1, 2, \dots, k-1\}$.
Therefore,

$$\operatorname{span}\{A^m : m = 0, 1, 2, 3, \dots\} = \operatorname{span}\{A^n : n = 0, 1, 2, \dots, k-1\}.$$

Claim: $\{A^n : n = 0, 1, 2, \dots, k-1\}$ is linearly independent.
Suppose not.
Then $\exists c_1, c_2, \dots, c_{k-1} \in \mathbb{F}$ such that

$$c_{k-1}A^{k-1} + c_{k-2}A^{k-2} + \cdots + c_1 A + a_0 I_n = 0_n.$$

The LHS is a polynomial in A that is equal to 0_n and it is of degree $(k-1) < k$. This contradict to the condition that the minimal polynomial of A is of degree k.
Therefore, $\{A^n : n = 0, 1, 2, \dots, k-1\}$ is linearly independent.
Therefore, $\dim(\operatorname{span}\{A^m : m = 0, 1, 2, 3, \dots\}) = \dim(\operatorname{span}\{A^n : n = 0, 1, 2, \dots, k-1\}) = k$.

(b) Let $c = m_A(0_{\mathbb{F}})$ be the constant term in the minimum polynomial of A.
Then, there exists $f(x) \in \mathbb{F}[x]$, with degree $(\deg[m_A(x)] - 1)$, such that

$$\begin{aligned} m_A(x) &= (f(x))x + c \\ \text{i.e.} \quad xf(x) &= m_A(x) - c. \end{aligned}$$

Claim : $c \neq 0_{\mathbb{F}}$. Suppose not, then $xf(x) = m_A(x)$.
Sub $x = A$, we have

$$\begin{aligned} Af(A) &= m_A(A) \\ f(A) &= A^{-1}0_n \quad (\because A \text{ is invertible.}) \\ &= 0_n. \end{aligned}$$

This results in a contradiction as $\deg[f(x)] < \deg[m_A(x)]$. So, $c \neq 0_{\mathbb{F}}$ and thus c^{-1} exists.
Now, let $g(x) = -(c^{-1})f(x)$. Then

$$\begin{aligned} xg(x) &= -(c^{-1})xf(x) \\ &= -(c^{-1})(m_A(x) - c) \\ &= 1_{\mathbb{F}} - (c^{-1})m_A(x) \end{aligned}$$

Sub $x = A$, we have

$$\begin{aligned} Ag(A) &= I_n - (c^{-1})m_A(A) \\ &= I_n - (c^{-1})0_n \\ &= I_n \end{aligned}$$

This give us $A^{-1} = g(A)$.

Question 7

(i) We have

$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{pmatrix} \neq 0$$

$$\Rightarrow \quad \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{pmatrix} \text{is invertible,}$$

$$\Rightarrow \quad \text{the set} \left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{r1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{r2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{rr} \end{pmatrix} \right\} \text{is linearly independent,}$$

$$\Rightarrow \quad \text{the system} \begin{cases} c_1a_{11} + c_2a_{12} + \cdots + c_ra_{1r} = 0 \\ c_1a_{21} + c_2a_{22} + \cdots + c_ra_{2r} = 0 \\ \vdots \\ c_1a_{r1} + c_2a_{r2} + \cdots + c_ra_{rr} = 0 \end{cases} \quad \text{has no non-trivial solution,}$$

$$\Rightarrow \quad \text{the system} \begin{cases} c_1a_{11} + c_2a_{12} + \cdots + c_ra_{1r} = 0 \\ c_1a_{21} + c_2a_{22} + \cdots + c_ra_{2r} = 0 \\ \vdots \\ c_1a_{n1} + c_2a_{n2} + \cdots + c_ra_{nr} = 0 \end{cases} \quad \text{has no non-trivial solution,}$$

$$\Rightarrow \quad \text{the set} \left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{nr} \end{pmatrix} \right\} \text{is linearly independent.}$$

(ii) Let $\mathrm{dr}(A)$ denotes the determinant rank of A. Then there exists S, an $r \times r$ submatrix of A, where $r = \mathrm{dr}(A)$ and $\det(S) \neq 0$.
Let A' be an $n \times n$ matrix formed by rearranging the rows of A, in such a way that S can be obtained by deleting the last $(n - r)$ rows and some $(n - r)$ columns from A'.
Since $\det S \neq 0$, then by the result from part (i), there exists a set of r linearly independent vectors in the column set of A'. Therefore, the dimension of the column space of A' is at least r. Since A and A' are row-equivalent, we have $\mathrm{Rank}A = \mathrm{Rank}A'$. Thus,

$$\mathrm{Rank}A = \mathrm{Rank}A' \geq r = \mathrm{dr}(A).$$

Let $k = (\mathrm{Rank}A)$. Certainly, we can find a set of k linearly independent columns in the column set of A. Keeping these k columns, we delete the remaining $(n - k)$ columns from A to form B. Since B has k linearly independent columns, the rank of B is again k. Therefore, we can then find a set of k linearly independent rows in the row set of B. Next, delete the remaining $(n - k)$ rows from B to form C. The rank of C is again k. C is an $k \times k$ submatrix of A. C is therefore full rank and has a nonzero determinant. By the definition of determinant rank, C cannot be larger than $\mathrm{dr}(A) \times \mathrm{dr}(A)$. So, we have

$$\mathrm{Rank}A = k \leq \mathrm{dr}(A).$$

Therefore, $\mathrm{dr}(A) = \mathrm{Rank}A$. The determinant rank of A is equal to the rank of A.

Question 8

(i) $\forall \boldsymbol{u} \in S$, we have $\boldsymbol{u} \in U$. Therefore, S is a subset of U.
Since U is a vector space, $\mathbf{0} \in U$. Since W is a subspace, $T(\mathbf{0}) = \mathbf{0} \in W$. Therefore, $\mathbf{0} \in S$.
$\forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in S, \alpha_1, \alpha_2 \in \mathbb{F}$, we have

$$\begin{array}{rl} & T(\boldsymbol{u}_1), T(\boldsymbol{u}_2) \in W \\ \Rightarrow & \alpha_1 T(\boldsymbol{u}_1) + \alpha_2 T(\boldsymbol{u}_2) \in W \\ \Rightarrow & T(\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2) \in W \\ \Rightarrow & \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 \in S. \end{array}$$

Therefore, S is a subspace of U.

(ii) Since $T : U \to V$ is a linear transformation, by the Dimension Theorem, we have

$$\begin{aligned} \dim(U) &= \operatorname{rank}(T) + \operatorname{nullity}(T) \\ &= \dim(R(T)) + \operatorname{nullity}(T). \end{aligned}$$

Similarly, $T|_S : S \to W$ is a linear transformation, and as $\ker(T) \subseteq S$, we have

$$\begin{aligned} \dim(S) &= \operatorname{rank}(T|_S) + \operatorname{nullity}(T|_S) \\ &= \dim(R(T) \cap W) + \operatorname{nullity}(T). \end{aligned}$$

We also have

$$\begin{aligned} \dim(R(T) \cap W) &= \dim(R(T)) + \dim(W) - \dim(R(T) + W) \\ \text{i.e.} \quad \dim(R(T)) - \dim(R(T) \cap W) &= \dim(R(T) + W) - \dim(W). \end{aligned}$$

Thus,

$$\begin{aligned} \dim(U) - \dim(S) &= \dim(R(T)) - \dim(R(T) \cap W) \\ &= \dim(R(T) + W) - \dim(W) \\ &\leq \dim(V) - \dim(W). \end{aligned}$$

Therefore,

$$\dim(S) \geq \dim(U) - \dim(V) + \dim(W).$$

Question 9

(i) Since T is a self-adjoint operator, the eigenvalues of T are real.
Let $\lambda \in \mathbb{R}$ be an eigenvalue of T, i.e. $T(\boldsymbol{v}) = \lambda \boldsymbol{v}$ for some $\boldsymbol{v} \in V$.
Then

$$\begin{aligned} \langle T(\boldsymbol{v}), \boldsymbol{v} \rangle &\geq 0 \\ \langle \lambda \boldsymbol{v}, \boldsymbol{v} \rangle &\geq 0 \\ \lambda \langle \boldsymbol{v}, \boldsymbol{v} \rangle &\geq 0 \\ \lambda \|\boldsymbol{v}\|^2 &\geq 0 \\ \lambda &\geq 0. \quad (\because \|\boldsymbol{v}\|^2 \geq 0) \end{aligned}$$

Therefore, all the eigenvalues of T are nonnegative.

(ii) Since T is a self-adjoint operator, T is diagonalisable. Then we may represent T by PAP^{-1} where P is a invertable matrix and A is a diagonal matrix having eigenvalues of T as diagonal entries.

$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \text{, where } \lambda_i\text{'s are eigenvalues of } T.$$

By the result of part (i), all λ_i's are nonnegative.
Let

$$B = \begin{pmatrix} \sqrt{\lambda_1} & 0 & \dots & 0 \\ 0 & \sqrt{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{\lambda_n} \end{pmatrix}.$$

Consider $(PBP^{-1})^2$.

$$\begin{aligned}
(PBP^{-1})^2 &= (PBP^{-1})(PBP^{-1}) \\
&= PB(P^{-1}P)BP^{-1} \\
&= PBIBP^{-1} \\
&= PB^2P^{-1} \\
&= P \begin{pmatrix} \sqrt{\lambda_1} & 0 & \dots & 0 \\ 0 & \sqrt{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{\lambda_n} \end{pmatrix}^2 P^{-1} \\
&= P \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} P^{-1} \\
&= PAP^{-1}.
\end{aligned}$$

Let S be a linear operator on V represented by PBP^{-1}. Then $S^2 = T$.