MA2104 Multivariable Calculus Suggested Solutions

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Question 1

Let $U \subseteq \mathbb{R}^2$ be an open set in the plane, and let $\mathbf{F} : U \to \mathbb{R}^2$ be a vector-valued function, with components $\mathbf{F} = \begin{pmatrix} u \\ v \end{pmatrix}$. Suppose $\mathbf{F}$ is twice continuously differentiable, and that for all $p \in U$, one has

$$\mathbf{F}'(p)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{F}'(p) \quad \textit{as a } 2 \times 2\textit{-matrix.}$$

(a) *Show that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \quad \textit{on } U.$$

Write

$$\mathbf{F}'(p) = \begin{pmatrix} \frac{\partial u}{\partial x}(p) & \frac{\partial u}{\partial y}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial v}{\partial y}(p) \end{pmatrix}.$$

Then, we have

$$\mathbf{F}'(p)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x}(p) & \frac{\partial u}{\partial y}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial v}{\partial y}(p) \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial y}(p) & -\frac{\partial u}{\partial x}(p) \\ \frac{\partial v}{\partial y}(p) & -\frac{\partial v}{\partial x}(p) \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{F}'(p) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x}(p) & \frac{\partial u}{\partial y}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial v}{\partial y}(p) \end{pmatrix} = \begin{pmatrix} -\frac{\partial v}{\partial x}(p) & -\frac{\partial v}{\partial y}(p) \\ \frac{\partial u}{\partial x}(p) & \frac{\partial u}{\partial y}(p) \end{pmatrix}.$$

This yields

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

Hence, we get

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) \\
&= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} \\
&= 0 \quad \text{by Clairaut's Theorem.}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \\
&= -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} \\
&= 0.
\end{aligned}$$

(b) *Suppose further that for all $p \in U$, one has*

$$\mathbf{F}(p) \neq \mathbf{0} \quad \text{in } \mathbb{R}^2.$$

Show that the $\mathbb{R}$-valued function $\varphi : U \to \mathbb{R}$ defined by

$$\varphi(p) = \log|\mathbf{F}(p)| \quad \text{for } p \in U$$

satisfies the equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad \text{on } U.$$

(Here, $\log$ denotes the natural logarithm function on positive real numbers.)

Note that we have $|\mathbf{F}(p)| = \sqrt{(u(p))^2 + (v(p))^2}$, so $\log|\mathbf{F}(p)| = \frac{1}{2}\log((u(p))^2 + (v(p))^2)$. For convenience, we will simply write u for $u(p)$ and v for $v(p)$.

By differentiating $\frac{1}{2}\log(u^2 + v^2)$ twice with respect to x, one has

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\left(\frac{1}{2}\log(u^2+v^2)\right) &= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\log(u^2+v^2)\right)\right) \\
&= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{2uu_x + 2vv_x}{u^2+v^2}\right) \\
&= \frac{1}{2(u^2+v^2)^2}((2u_x^2 + 2uu_{xx} + 2v_x^2 + 2vv_{xx})(u^2+v^2) - (2uu_x + 2vv_x)^2) \\
&= \frac{1}{(u^2+v^2)^2}((u_x^2 + uu_{xx} + v_x^2 + vv_{xx})(u^2+v^2) - 2(uu_x + vv_x)^2). \qquad (1)
\end{aligned}$$

Similarly, by differentiating $\log(u^2 + v^2)$ twice with respect to y, one has

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{2}\log(u^2+v^2)\right) = \frac{1}{(u^2+v^2)^2}((u_y^2 + uu_{yy} + v_y^2 + vv_{yy})(u^2+v^2) - 2(uu_y + vv_y)^2). \qquad (2)$$

Using part (a), adding (1) and (2) together yields

$$\begin{aligned}
&\frac{1}{(u^2+v^2)^2}((u_x^2 + uu_{xx} + v_x^2 + vv_{xx})(u^2+v^2) - 2(uu_x + vv_x)^2) \\
&+ \frac{1}{(u^2+v^2)^2}((u_y^2 + uu_{yy} + v_y^2 + vv_{yy})(u^2+v^2) - 2(uu_y + vv_y)^2) \\
&= \frac{1}{u^2+v^2}((u_x^2 + u_y^2 + v_x^2 + v_y^2)(u^2+v^2) - 2(uu_x + vv_x)^2 - 2(uu_y + vv_y)^2) \\
&= \frac{1}{u^2+v^2}(2(u_x^2 + u_y^2)(u^2+v^2) - 2(uu_x + vv_x)^2 - 2(uu_y + vv_y)^2). \qquad (3)
\end{aligned}$$

We now observe that

$$\begin{aligned}
(uu_x + vv_x)^2 + (uu_y + vv_y)^2 &= (uu_x)^2 + 2uu_xvv_x + (vv_x)^2 + (uu_y)^2 + 2uu_yvv_y + (vv_y)^2 \\
&= (uu_x)^2 - 2uv_yvu_y + (vv_x)^2 + (uu_y)^2 + 2uu_yvv_y + (vv_y)^2 \\
&= (uu_x)^2 + (vv_x)^2 + (uu_y)^2 + (vv_y)^2 \\
&= (u_x^2 + u_y^2)(u^2+v^2). \qquad (4)
\end{aligned}$$

Substituting (4) into (3) yields the desired result.

Question 2

(a) *Compute the value of the integral*

$$\int_R (x^2+y^2)d(x,y)$$

where R is the region in the first quadrant $x \geq 0, y \geq 0$ of $\mathbb{R}^2$ bounded by the curves

$$x^2-y^2=1, x^2-y^2=4, xy=1, xy=3.$$

Consider the following C^1 transformation

$$F(x,y)=(x^2-y^2,xy).$$

The Jacobian matrix of the transformation is

$$J_F = \begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix}.$$

It is also easy to see that this is a continuous bijection with continuous inverse from R to the rectangular domain $S=[1,4]\times[1,3]$. This is because we can write the inverse function for F by solving quadratic equations:

$$F^{-1}(a,b)=\left(\sqrt{\frac{a+\sqrt{a^2+4b^2}}{2}}, b\sqrt{\frac{2}{a+\sqrt{a^2+4b^2}}}\right).$$

and we see that the function F^{-1} is also continuously differentiable on the (open) first quadrant.

Thus, by denoting $F(x,y)=(a,b)$, we have $\det J_F = 2x^2+2y^2>0$, so $\det J_F^{-1}=(2x^2+2y^2)^{-1}$. We see that

$$\int_R (x^2+y^2)d(x,y)=\int_S (x^2+y^2)(2x^2+2y^2)^{-1}d(a,b)=\int_S \frac{1}{2}d(a,b)=\frac{1}{2}\times 6=3.$$

(b) *Compute the volume of the following subset of $\mathbb{R}^3$:*

$$\{(x,y,z)\in\mathbb{R}^3 : 2\sqrt{x^2+y^2}+|z|\leq 1\}.$$

We first see that by changing to cylindrical coordinates we have

$$-(1-2r)\leq z\leq 1-2r.$$

We also see that $0\leq r\leq \frac{1}{2}$, and the height $h=2-4r$. Therefore the volume can be computed using cylindrical coordinates:

$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^{\frac{1}{2}}(2-4r)rdrd\theta \\
&= \int_0^{2\pi}\int_0^{\frac{1}{2}} 2r-4r^2drd\theta \\
&= 2\pi\times[r^2-\frac{4r^3}{3}]_0^{\frac{1}{2}} \\
&= 2\pi\times\frac{1}{12}=\frac{\pi}{6}.
\end{aligned}$$

Question 3

(a) *Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(\rho) = \int_0^{2\pi} e^{\rho\cos\theta}\cos(\rho\sin\theta)d\theta$$

is a constant function.

First we use the differentiation under the integral sign:

$$\frac{df}{d\rho} = \int_0^{2\pi} \cos\theta e^{\rho\cos\theta}\cos(\rho\sin\theta) - e^{\rho\cos\theta}\sin\theta\sin(\rho\sin\theta)d\theta. \tag{5}$$

We will show that this expression evaluates to 0. Assume $\rho \neq 0$.

Let $\vec{F} = (e^x \sin y, e^x \cos y) = (P, Q)$ and let $\vec{r}(\theta) = (\rho\cos\theta, \rho\sin\theta)$. Let C denote the circle traced by $\vec{r}(\theta)$ and R be the region bounded by the circle. Then we see that by the Green's theorem, we have

$$\int_C \vec{F}\cdot d\vec{r} = \int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}dA = \int_R e^x\cos y - e^x\cos y dA = 0$$

On the other hand, we see that by noting $x = \rho\cos\theta$ and $y = \rho\sin\theta$

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} \vec{F}\cdot\vec{r}'(\theta)\ d\theta \\
&= \int_0^{2\pi} \begin{pmatrix} e^x\sin y \\ e^x\cos y \end{pmatrix} \cdot \begin{pmatrix} -\rho\sin\theta \\ \rho\cos\theta \end{pmatrix}\ d\theta \\
&= \int_0^{2\pi} \rho\cos\theta e^{\rho\cos\theta}\cos(\rho\sin\theta) - \rho e^{\rho\cos\theta}\sin\theta\sin(\rho\sin\theta)\ d\theta \\
&= \rho\frac{df}{d\rho}.
\end{aligned}$$

Since $\rho \neq 0$, we have $\frac{df}{d\rho} = 0$. If $\rho = 0$, we can also show that $\frac{df}{d\rho} = 0$ since

$$\frac{df}{d\rho} = \int_0^{2\pi}\cos\theta d\theta = 0.$$

So $\frac{df}{d\rho} = 0$ on $\mathbb{R}$. Therefore f is a constant function of ρ.

Remark. Observe that the following identity

$$\frac{d}{d\theta}\left(e^{\rho\cos\theta}\sin(\rho\sin\theta)\right) = e^{\rho\cos\theta}\rho\cos\theta\cos(\rho\sin\theta) - \rho\sin\theta e^{\rho\cos\theta}\sin(\rho\sin\theta)$$

can be used to compute the integral on (5). Indeed, for $\rho \neq 0$, one has

$$\begin{aligned}
\frac{df}{d\rho} &= \int_0^{2\pi} \cos\theta e^{\rho\cos\theta}\cos(\rho\sin\theta) - e^{\rho\cos\theta}\sin\theta\sin(\rho\sin\theta)d\theta \\
&= \int_0^{2\pi} \frac{1}{\rho}\frac{d}{d\theta}\left(e^{\rho\cos\theta}\sin(\rho\sin\theta)\right)\ d\theta \\
&= \frac{1}{\rho}\left[e^{\rho\cos\theta}\sin(\rho\sin\theta)\right]_0^{2\pi}\ =\ 0.
\end{aligned}$$

Then, one may continue to handle the case for $\rho = 0$ and finish the proof as above.

(b) *Compute the value of the integral*

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\ d\theta.$$

The required integral is $f(1)$. Instead, we can compute $f(0)$ since f is constant. So we see that

$$f(1) = f(0) = \int_0^{2\pi} 1\ d\theta = 2\pi.$$

Question 4

Let B be the closed unit ball in $\mathbb{R}^3$

$$B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$$

and let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x, y, z) = xyz.$$

(a) *Determine the global maximum of f on B, as well as the points of B at which f attains this maximum volume.*

First assume that $x, y, z \geq 0$. Then we can use the AM-GM inequality:

$$\frac{x^2 + y^2 + z^2}{3} \geq (xyz)^{\frac{2}{3}}$$

Since $x^2 + y^2 + z^2 \leq 1$, we have $xyz \leq \frac{1}{\sqrt{27}}$. By letting $x = y = z = \frac{1}{\sqrt{3}}$, we see that the inequality is tight and the maximum value is indeed $\frac{1}{\sqrt{27}}$. Using the symmetry of the function, we see that the points of B where f attains maximum are

$$\{(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})\}$$

(b) *Compute the surface integral*

$$\int_{\partial B} \nabla f \cdot \mathbf{n} \, d\sigma$$

of the vector field ∇f over the boundary sphere ∂B oriented with the outward pointing unit normal vectors $\mathbf{n}$.

We use Gauss's Theorem:

$$\begin{aligned}
\int_{\partial B} \nabla f \cdot \mathbf{n} \, d\sigma &= \int_B \Delta f \, dV \\
&= \int_B \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \, dV \\
&= \int_B 0 \, dV \\
&= 0.
\end{aligned}$$

Question 5

(a) *Compute the line integral*

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

of the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *over the curve* C *paramaterized by* $\mathbf{r} : [0,1] \to \mathbb{R}^3$, *where*

$$\mathbf{F}(x,y,z) := \begin{pmatrix} e^{-y} - ze^{-x} \\ e^{-z} - xe^{-y} \\ e^{-x} - ye^{-z} \end{pmatrix} \quad \text{and} \quad \mathbf{r}(t) := \left(\frac{\log(1+t)}{\log(2)}, \sin\frac{\pi t}{2}, \frac{1-e^{-t}}{1-e} \right).$$

We show that $\mathbf{F}$ is conservative by finding a potential function f. Note that one has

$$\begin{aligned} \frac{\partial f}{\partial x} &= e^{-y} - ze^{-x} \implies f = xe^{-y} + ze^{-x} + h_1(y,z) \\ \frac{\partial f}{\partial y} &= e^{-z} - xe^{-y} \implies f = ye^{-z} + xe^{-y} + h_2(x,z) \\ \frac{\partial f}{\partial z} &= e^{-x} - ye^{-z} \implies f = ze^{-x} + ye^{-z} + h_3(x,z) \end{aligned}$$

for some functions h_1, h_2, h_3. By inspection, one such f is given by

$$f(x,y,z) = ze^{-x} + ye^{-z} + xe^{-y}.$$

Since $\mathbf{r}(0) = (0,0,0)$ and $\mathbf{r}(1) = (1,1,1)$, it follows from gradient theorem that the integral is $e^{-1} + e^{-1} + e^{-1} - 0 = 3e^{-1}$.

(b) *Suppose* $f : \mathbb{R}^3 \to \mathbb{R}$ *is a scalar-valued function and* $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$ *is a vector field, both assumed to be twice continuously differentiable on* $\mathbb{R}^3$. *Suppose also that both* f *and* $\mathbf{G}$ *are of compact support, which is to say there exists a (solid) ball* B *in* $\mathbb{R}^3$ *centered at the origin such that* f *and* $\mathbf{G}$ *are zero outside* B. *Show that the dot-product of the gradient vector field* ∇f *with curl vector field* $\nabla \times \mathbf{G}$ *has a zero volume integral over* B, *i.e. show that*

$$\int_B (\nabla f) \cdot (\nabla \times \mathbf{G}) \, dV = 0.$$

Note that

$$\nabla \cdot (f \nabla \times \mathbf{G}) = (\nabla f) \cdot (\nabla \times \mathbf{G}) + f \nabla \cdot (\nabla \times \mathbf{G}) = (\nabla f) \cdot (\nabla \times \mathbf{G})$$

since $\nabla \cdot (\nabla \times \mathbf{G}) = 0$. By Gauss's Theorem, one has

$$\int_B \nabla \cdot (f \nabla \times \mathbf{G}) \, dV = \int_{\partial B} (f \nabla \times \mathbf{G}) \cdot \mathbf{n} \, dS = 0$$

since f and $\mathbf{G}$ have compact support and are identically zero on the boundary of B by continuity. This completes the proof.