NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to
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MA1102R Calculus
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Question 1

(a) By L'Hospital's Rule we have

$$\begin{aligned}\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7} &= \lim_{x\to 7}\frac{1}{2\sqrt{x+2}}\\ &= \frac{1}{2\cdot 3}=\frac{1}{6}\end{aligned}$$

(b) First observe that

$$\lim_{x\to\infty}\ln x^{\frac{1}{x^2}} = \lim_{x\to\infty}\frac{\ln x}{x^2}=0$$

since x^2 grows asymptotically faster than $\ln x$.

Hence we have

$$\begin{aligned}\lim_{x\to\infty} x^{\frac{1}{x^2}} &= \exp\left(\ln \lim_{x\to\infty} x^{\frac{1}{x^2}}\right)\\ &= \exp\left(\lim_{x\to\infty}\ln x^{\frac{1}{x^2}}\right)\\ &= e^0 = 1\end{aligned}$$

(c) Note that

$$-\left|\frac{x^2}{\sin x}\right| \le \frac{x^2\sin\frac{1}{x}}{\sin x} \le \left|\frac{x^2}{\sin x}\right|$$

Then, by L'Hospital's Rule we get

$$\lim_{x\to 0} -\left|\frac{x^2}{\sin x}\right| = \lim_{x\to 0} -\left|\frac{2x}{\cos x}\right| = 0$$

and similarly for $\lim_{x\to 0}\left|\frac{x^2}{\sin x}\right|$.

Then, by Squeeze Theorem,

$$\lim_{x\to 0}\frac{x^2\sin(\frac{1}{x})}{\sin x}=0$$

Question 2

(a) Let $u = \sqrt{x} + 1$. This gives us $dx = 2\sqrt{x}du$. Note also that $x = 1 \Rightarrow u = 2$ and $x = 4 \Rightarrow u = 3$. So applying the substitution, we have

$$\begin{aligned}
&\int_2^3 \frac{u^4}{\sqrt{x}}(2\sqrt{x})du \\
=&2\int_2^3 u^4 du \\
=&2\left(\frac{u^5}{5}\right)\Bigg|_2^3 \\
=&\frac{484}{5}
\end{aligned}$$

(b) Let $u = \sin^{-1} x$. Therefore $\sin u = x$ and $\cos u du = dx$. Note also that $x = 0 \Rightarrow u = 0$ and $x = 1 \Rightarrow u = \frac{\pi}{2}$. Then this gives us

$$\int (\sin^{-1}(x))^2 dx = \int u^2 \cos u du$$

Now let $v = u^2$ and $dw = \cos u du$. Then we have $dv = 2udu$ and $w = \sin u + C$ for some constant C. Using Integration by Parts,

$$\begin{aligned}
\int u^2 \cos u du &= u^2 \sin u - 2\int u \sin u du \\
&= u^2 \sin u - 2(-u \cos u - \int -\cos u du) \\
&= u^2 \sin u + 2u \cos u - 2 \sin u + C
\end{aligned}$$

So,

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} u^2 \cos u du \\
&= \left(u^2 \sin u + 2u \cos u - 2 \sin u\right)\Bigg|_0^{\frac{\pi}{2}} \\
&= \left(\frac{\pi}{2}\right)^2 - 2
\end{aligned}$$

Question 3

(a) Let $a_n = \dfrac{1}{\sqrt{n} + \sqrt{n+1}}$ and consider

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}} \\
&= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{1 + \sqrt{1 + \frac{1}{n}}} \\
&= \frac{0}{2} = 0
\end{aligned}$$

Note also that

$$\begin{aligned} a_{n+1} - a_n &= \frac{1}{\sqrt{n+1}+\sqrt{n+2}} - \frac{1}{\sqrt{n}+\sqrt{n+1}} \\ &= \frac{\sqrt{n}-\sqrt{n+2}}{(\sqrt{n+1}+\sqrt{n+2})(\sqrt{n}+\sqrt{n+1})} \end{aligned}$$

Now the denominator is always positive. So $a_{n+1} - a_n < 0$. Therefore $a_{n+1} < a_n$, implying that (a_n) is decreasing. Then, by Alternating Series Test, the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ converges.

(b) Let $f(x) = \frac{1}{x} - \ln\left(1+\frac{1}{x}\right)$. Then

$$f'(x) = -\frac{1}{x^2} - \frac{1}{1+\frac{1}{x}} \cdot -\frac{1}{x^2} = -\frac{1}{x^2}\left(1 - \frac{x}{x+1}\right) < 0$$

for all $x \in \mathbb{R}$. Therefore f has no turning points. Note also that $f(1) = 1 - \ln 2 > 0$ and

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \left(\frac{1}{x} - \ln\left(1+\frac{1}{x}\right)\right) \\ &= \lim_{x\to\infty} \left(\frac{1-\ln(1+(\frac{1}{x}))^x}{x}\right) \\ &= 0 \end{aligned}$$

So $f(x) > 0$ for all $x \in \mathbb{R}$. So we have $\frac{1}{n} > \ln \frac{n+1}{n}$ for all $n \in \mathbb{N}$ and hence $\frac{1}{n} > \int_n^{n+1} \frac{dx}{x}$ for all $n \in \mathbb{N}$. Furthermore, we have

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} > \int_1^2 \frac{dx}{x} + \int_2^3 \frac{dx}{x} + \cdots + \int_n^{n+1} \frac{dx}{x} = \ln(n+1)$$

Hence,

$$(\ln\ln n)^{1+\cdots+\frac{1}{n}} > (\ln\ln n)^{\ln(n+1)} > (\ln\ln n)^{\ln n} = \exp(\ln n \cdot \ln\ln\ln n) = n^{\ln\ln\ln n}$$

Note that for all $n > e^{e^{e^2}}$,

$$(\ln\ln n)^{1+\cdots+\frac{1}{n}} > n^{\ln\ln\ln n} > n^2$$

Therefore,

$$\frac{1}{(\ln\ln n)^{1+\cdots+\frac{1}{n}}} < \frac{1}{n^2} if n > e^{e^{e^2}}$$

and by Comparison Test, $\sum_{n=3}^{\infty} \frac{1}{(\ln\ln n)^{1+\cdots+\frac{1}{n}}}$ converges.

(c) Consider $f(x) = \tan x$ and note that f is continuous and differentiable on $[0, \frac{\pi}{2})$.

Also, $\left(\frac{1}{n^{2/3}+1}, \frac{1}{n^{2/3}}\right) \subseteq [0, \frac{\pi}{2})$ and so f is continuous and differentiable on the aforesaid interval.

Then, by Mean Value Theorem, there exists $x_n \in \left(\frac{1}{n^{2/3}+1}, \frac{1}{n^{2/3}}\right)$ for all n such that

$$\sec^2(x_n) = (\tan(x_n))' = \frac{\tan\frac{1}{n^{2/3}} - \tan\frac{1}{n^{2/3}+1}}{\frac{1}{n^{2/3}} - \frac{1}{n^{2/3}+1}}$$

Now we have

$$\lim_{n\to\infty} \frac{\tan\frac{1}{n^{2/3}} - \tan\frac{1}{n^{2/3}+1}}{\frac{1}{n^{2/3}} - \frac{1}{n^{2/3}+1}} = \lim_{n\to\infty} \sec^2 x_n$$

But note that $\frac{2}{3} < 1$. So $n \to \infty$ implies that $x \to 0$. Then

$$\lim_{n\to\infty} \sec^2 x_n = \lim_{x\to 0} \sec^2 x = \frac{1}{1^2} = 1$$

So, by Limit Comparison Test,

$$\sum_{n=1}^{\infty} \left(\tan\frac{1}{n^{2/3}} - \tan\frac{1}{n^{2/3}+1} \right)$$

converges if and only if

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^{2/3}} - \frac{1}{n^{2/3}+1} \right)$$

converges. But

$$\frac{1}{n^{2/3}} - \frac{1}{n^{2/3}+1} = \frac{1}{n^{2/3}(n^{2/3}+1)} < \frac{1}{n^{4/3}}$$

Note that $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ converges by the p-series test. So by Comparison Test,

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^{2/3}} - \frac{1}{n^{2/3}+1} \right)$$

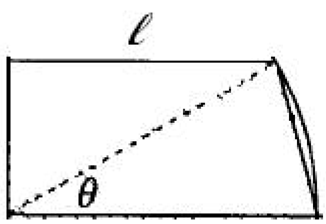
converges and so,

$$\sum_{n=1}^{\infty} \left(\tan\frac{1}{n^{2/3}} - \tan\frac{1}{n^{2/3}+1} \right)$$

converges.

Question 4

Let A be the area of 1/4 of the beam. Consider the image:



Then, $l = 15\cos\theta$. So

$$\begin{aligned} A(\theta) &= \frac{1}{2}(15)^2 \sin\theta + \frac{1}{2}(15)(15\cos\theta)\sin\theta, 0 < \theta < \frac{\pi}{2} \\ &= \frac{225}{2}\sin\theta + \frac{225}{2}\sin\theta\cos\theta \\ &= \frac{225}{4}(2\sin\theta + \sin 2\theta) \end{aligned}$$

So A is continuous and differentiable on $(0, \pi/2)$. Furthermore we have $A'(\theta) = \dfrac{225}{4}(2\cos\theta + 2\cos 2\theta)$. Now, let $A'(\theta) = 0$. Then

$$\begin{aligned}
\cos\theta + \cos 2\theta &= 0 \\
\cos\theta + 2\cos^2\theta - 1 &= 0 \\
2\cos^2\theta + \cos\theta - 1 &= 0 \\
(2\cos\theta - 1)(\cos\theta + 1) &= 0 \\
\cos\theta &= \frac{1}{2}, \cos\theta = -1
\end{aligned}$$

But $\cos\theta > 0$ for all θ. So A has only one turning point, at which $\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$. Then, by the Increasing-Decreasing Test, A attains maximum at $\theta = \frac{\pi}{3}$. So A attains maximum when the cross-section is a regular hexagon and we are done.

Question 5

Firstly we have

$$\begin{aligned}
y &= \frac{x^3}{6} + \frac{1}{2x} \\
(y')^2 &= \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2 \\
&= \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}
\end{aligned}$$

So, the arclength, s, is given by

$$\begin{aligned}
s &= \int_2^3 \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}}dx \\
&= \int_2^3 \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}}dx \\
&= \int_2^3 \sqrt{\frac{x^8 + 2x^4 + 1}{4x^4}}dx \\
&= \int_2^3 \frac{x^4 + 1}{2x^2}dx \\
&= \frac{1}{2}\int_2^3 (x^2 + x^{-2})dx \\
&= \frac{1}{2}\left(\frac{x^3}{3} - \frac{1}{x}\right)\Bigg|_2^3 \\
&= \frac{13}{4}
\end{aligned}$$

Question 6

(a) Radius of convergence, R is given by

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{1/(3n)!}{1/(3n+3)!}\right| \\ &= \lim_{n\to\infty}(3n+1)(3n+2)(3n+3) = \infty \end{aligned}$$

(b) Let $f(x) = u^3 + v^3 + w^3 - 3uvw$ and note that $f(0) = 1$. Hence, we want to show that f is constant and hence, $f(x) = 1$. To do this, it suffices to prove that $f'(x) = 0$ for all x.

Firstly, note that $u' = w$, $v' = u$, and $w' = v$. Then,

$$\begin{aligned} f'(x) &= 3u^2 \cdot u' + 3v^2 \cdot v' + 3w^2 \cdot w - 3(u'vw + uv'w + uvw') \\ &= 3u^2w + 3v^2u + 3w^2v - 3(vw^2 + u^2w + uv^2) = 0 \end{aligned}$$

Then, f is constant and therefore, $f(x) = 1$ for all x, as required.

Question 7

Suppose the circle and the parabola intersect at (a, a^2).

We have $\frac{dy}{dx} = 2x$. So, the slope of the normal at (a, a^2) is $-1/2a$, and hence, the equation of the normal is

$$y = -\frac{1}{2a}(x-a) + a^2$$

Let $O = (0, 0)$ and C be the center of the circle. Now if the circle touches the parabola at O, then $r = OC$, where OC is the distance from the origin to the center of the circle is equal to the y-coordinate of the center of the circle.

By setting $x = 0$, from the equation of the normal, we have $y = a^2 + \frac{1}{2}$ as the y-coordinate of the center and r is given by

$$r = \sqrt{\left(a^2 - a^2 - \frac{1}{2}\right) + a^2} = \sqrt{a^2 + 1/4}$$

Equating the radius and the y-coordinate of the center, we have $\sqrt{a^2 + 1/4} = a^2 + 1/2$. Therefore $a^2 + 1/4 = a^4 + a^2 + 1/4$, showing that $a = 0$. Then $r = \frac{1}{2}$.

Now we will show that if $r > \frac{1}{2}$, the circle will not touch O. Suppose $r > \frac{1}{2}$. Then $\sqrt{a^2 + \frac{1}{4}} > \frac{1}{2}$, giving $a^2 + \frac{1}{4} > \frac{1}{4}$ and so, $a^2 > 0$, implying that the circle and the parabola do not intersect at O.

So the maximum value of r is $1/2$.

Question 8

Let $g(x) = f(x) - x$ and suppose f has no fixed points. This means that $g(x) \neq 0\ \forall x \in \mathbb{R}$. By the contrapositive of the Intermediate Value Theorem, we have $g(x) > 0$ or $g(x) < 0$ for all x. Now $g(x) > 0 \Rightarrow f(x) - x > 0 \Rightarrow f(x) > x$. Evaluating f at $f(x)$, we obtain $f(x) > x \Rightarrow f(f(x)) > f(x) \Rightarrow x > f(x)$, a contradiction.

Similarly, $g(x) < 0 \Rightarrow f(x) - x < 0 \Rightarrow f(x) < x \Rightarrow f(f(x)) < f(x) \Rightarrow x < f(x)$, a contradiction.

So there must exist an $x \in \mathbb{R}$ such that $g(x) = 0$ and hence, $f(x) = x$, which means that f has at least one fixed point.