NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Guan Yizheng, Shi Xiaojie

MA1104 Multivariable Calculus
AY 2008/2009 Sem 2

Question 1

(a) find the intersection of the two surfaces:

$$z = x^2 - y^2$$

$$z = x^2 + xy - 1.$$

hence,

$$x^2 - y^2 = x^2 + xy - 1$$

$$x = \frac{1 - y^2}{y}$$

.
let y = t,

$$x = \frac{1 - t^2}{t}, z = \frac{1 - 2t^2}{t^2}$$

where $t \neq 0$.
the parameterization could be $\langle \frac{1-t^2}{t}, t, \frac{1-2t^2}{t^2} \rangle$.

(b) reform the equation $(\boldsymbol{x} - \boldsymbol{a})(\boldsymbol{x} - \boldsymbol{b}) = c^2$

$$\boldsymbol{x}^2 - (\boldsymbol{a} + \boldsymbol{b})\boldsymbol{x} + \boldsymbol{a}\boldsymbol{b} - c^2 = 0$$

complete the square:

$$(\boldsymbol{x} - \frac{\boldsymbol{a} + \boldsymbol{b}}{2})^2 = c^2 + (\frac{\boldsymbol{a} - \boldsymbol{b}}{2})^2.$$

the equation represents a sphere centered at $\frac{\boldsymbol{a}+\boldsymbol{b}}{2}$ where the radius is $\sqrt{c^2 + (\frac{\boldsymbol{a}-\boldsymbol{b}}{2})^2}$.
$R^2 = c^2 + |\frac{\boldsymbol{a}-\boldsymbol{b}}{2}|^2$.

Question 2

(a) for every point where $x \neq y$

$$g(x, y) = \frac{x^2 - y^2}{x - y} = x + y$$

.
since $x + y$ is a continuous function

$g(x, y)$ is continuous at all points where $x \neq y$.
for all points where $x = y$ and $x \neq 0$,

$$g(x, y) = 3x.$$

$$\lim_{y \to x} g(x, y) = \lim_{y \to x} \frac{x^2 - y^2}{x - y} = \lim_{y \to x} x + y = 2x.$$

$\because x \neq 0$,
$\therefore g(x, y) \neq \lim_{y \to x} g(x, y)$
so, $g(x, y)$ is discontinuous at points where $x = y$ and $x \neq 0$ for the point $(0, 0)$.
$g(0, 0) = 0$.

$$\lim_{y \to 0 x \to 0} g(x, y) = x + y = 0$$

if $x \neq y$.

$$\lim_{y \to 0 x \to 0} g(x, y) = 3x = 0$$

if $x = y$.
$g(x, y)$ is continuous at the point (0,0). In conclusion $g(x, y)$ is continuous at $\{(x, y) \| x \neq y\} \cup \{(0, 0)\}$.

(b) let

$$u = (cos\theta, sin\theta)$$

$$\begin{aligned} D_{\boldsymbol{u}} f(0, 0) &= \lim_{h \to 0} \frac{f((0, 0) + h\boldsymbol{u}) - f(0, 0)}{h - 0} \\ &= \lim_{h \to 0} \frac{cos^2\theta sin\theta h^3}{h^7 cos^6\theta + 2h^3 sin^2\theta} \\ &= \lim_{h \to 0} \frac{sin\theta cos^2\theta}{h^4 cos^6\theta + 2 sin^2\theta} \\ &= \frac{sin\theta cos^2\theta}{2 sin^2\theta} \end{aligned}$$

if $sin\theta \neq 0$ the $D_{\boldsymbol{u}} f(0, 0)$ exists. Otherwise not.

No.
let (x, y) goes to zero along the curve $y = x^4$

$$\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{x \to 0} \frac{x^6}{x^6 + 2x^8} = \lim_{x \to 0} \frac{1}{1 + 2x^2} = 1.$$

let (x,y) goes to zero along y=x.

$$\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{x \to 0} \frac{x^3}{x^6 + 2x^2} = \lim_{x \to 0} \frac{x}{x^4 + 2} = 0.$$

the two are not equal, thus we can conclude that f is not continous at (0,0).

Question 3

(a) find the curve first.

$$x^3 + 2xy + yz = 73x^2 - yz = 1 \tag{1}$$

we can get:

$$x^3 + 2xy + 3x^2 = 8$$

let $x = t$,then, $y = \frac{8-t^3-3t^2}{2t}$, $z = \frac{6t^3-2t}{8-t^3-3t^2}$
so, the curve C can be represented by $\boldsymbol{r}(t) = \langle t, \frac{8-t^3-3t^2}{2t}, \frac{6t^3-2t}{8-t^3-3t^2}\rangle$
point $(1, 2, 1)$ is $\boldsymbol{r}(1)$.

$$\boldsymbol{r}'(t) = \langle 1, \frac{-4t^3 - 6t^2 - 16}{4t^2}, \frac{-18t^4 - 4t^3 + 138t^2 - 16}{(8 - t^3 - 3t^2)^2}\rangle.$$

$$\boldsymbol{r}'(1) = \langle 1, -\frac{13}{2}, \frac{25}{4}\rangle.$$

so, the parameterizations of the equation could be represented

$$\langle 1, 2, 1\rangle + s\langle 1, -\frac{13}{2}, \frac{25}{4}\rangle$$

the parameterizations is given by

$$\langle t + 1, -\frac{13}{2}t + 2, \frac{25}{4}t + 1, \rangle.$$

(b) to find all local extreme points, take the partial derivetives and make them to zero.

$$\begin{aligned} \frac{\partial f}{\partial x} &= -24 - 6xy = 0 \\ \frac{\partial f}{\partial y} &= 3y^2 - 3x^2 = 0 \end{aligned}$$

we can get $xy = -4$ and $x^2 = y^2$.
all the local extreme points are$(2, -2)$ and $(-2, 2)$

$$f_{xx} = -6y, f_{yy} = 6y, f_{xy} = -6x.$$

hence, $\boldsymbol{D} = -36(x^2 + y^2) < 0$ for the two points.
Thus, those two points are saddle points of f.

(c) as one of the vertex of the rectangular box is on the ellipsoid and the coordinate would be (x, y, z) where $x, y, z > 0$.
so, the volumn $V = 8xyz$. use the lagrange multiplier method to do the find the maximum value of V.
let $F(x, y, z) = (\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2$

$$\begin{aligned} \frac{\partial V}{\partial x} &= \lambda\frac{\partial F}{\partial x} \\ \frac{\partial V}{\partial y} &= \lambda\frac{\partial F}{\partial y} \\ \frac{\partial V}{\partial z} &= \lambda\frac{\partial F}{\partial z} \end{aligned}$$

we can get :

$$\begin{aligned} 4yz &= \lambda\frac{x}{a^2} \\ 4xz &= \lambda\frac{y}{b^2} \\ 4xy &= \lambda\frac{x}{c^2} \end{aligned}$$

solve the equations:

$$\begin{aligned} x &= \frac{\lambda}{4bc} \\ y &= \frac{\lambda}{4ac} \\ z &= \frac{\lambda}{4ab} \\ (\lambda)^2 &= \frac{16a^2 * b^2 * c^2}{3} \end{aligned}$$

sub in the value:

$$V_{max} = 8xyz = \frac{\lambda^3}{8a^2b^2c^2} = \frac{8}{3}\sqrt{\frac{a^2b^2c^2}{3}} = \frac{8abc}{3\sqrt{3}}$$

So, the maximum Volume is

$$\frac{8abc}{3\sqrt{3}}$$

Question 4

(a) $\boldsymbol{J} \times \boldsymbol{r} = -\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{r'}) = -(\boldsymbol{r} \cdot \boldsymbol{r'})\boldsymbol{r} + \boldsymbol{r}^2\boldsymbol{r'} = \boldsymbol{r}^2\boldsymbol{r'}$
therefore, $\boldsymbol{r'} = \frac{\boldsymbol{J}\times\boldsymbol{r}}{\boldsymbol{r}^2}$

(b) let $x = rcos\theta$, $y = rsin\theta$.

$$\begin{aligned} \int_0^{\frac{1}{2}} \int_{\sqrt{3}} x^{\sqrt{1-x^2}} e^{x^2+y^2}\ dy\ dx &= \int_0^1 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} e^{r^2} r\ d\theta\ dr \\ &= (\frac{\pi}{2} - \frac{\pi}{3}) \int_0^1 re^{r^2}\ dr \\ &= \frac{\pi}{12}\left[e^{r^2}\right]_0^1 = \frac{\pi}{12}(e-1). \end{aligned}$$

(c) By observing the planes, we find that it is a Parallelepiped.
We can caculate the volume by the vector triple product.
first we find the four points PQRS
let plane SPR be $x + y + z = 1$, plane SPQ be $y + z = 2$, plane PQR be $x + 2y = 0$

find point P by the equations

$$\begin{aligned} x+y+z&=1\\ y+z&=2\\ x+2y&=0 \end{aligned}$$

$P : \langle -1, \frac{1}{2}, \frac{3}{2} \rangle$.
find Q:

$$\begin{aligned} x+y+z&=2\\ y+z&=2\\ x+2y&=0 \end{aligned}$$

$Q : \langle 0, 0, 2 \rangle$.
find R:

$$\begin{aligned} x+y+z&=1\\ y+z&=4\\ x+2y&=0 \end{aligned} \tag{2}$$

$R : \langle -3, \frac{3}{2}, \frac{5}{2} \rangle$.
Find S

$$\begin{aligned} x+y+z&=1\\ y+z&=2\\ x+2y&=1 \end{aligned} \tag{3}$$

$S : \langle -1, 1, 1 \rangle$.

therefore,

$$\begin{aligned} PS &= \langle 0, \frac{1}{2}, -\frac{1}{2} \rangle\\ PR &= \langle -2, 1, -1 \rangle\\ PQ &= \langle 1, -\frac{1}{2}, \frac{1}{2} \rangle \end{aligned}$$

the volume would be: $V = (PS \times PR) \cdot PQ = 1 - \frac{1}{2} + \frac{1}{2} = 1$

Question 5

(a) suppose $\boldsymbol{F}$ is conservative.

$$f = \int 2x + y \; dx = x^2 + xy + C_1$$

where C_1 is a function of y and z;

$$\frac{\partial f}{\partial x} = x + \frac{\partial C_1}{\partial y} = x + 2yz^2$$

therefore, $C_1 = y^2 * z^2 + C_2$

$$f = x^2 + xy + y^2 * z^2 + C_2$$

$$\frac{\partial f}{\partial z} = 2y^2 * z + \frac{\partial C_2}{\partial z} = 2y^2 * z$$

hence C_2 is a constant.
therefore, $f = x^2 + xy + y^2 * z^2 + C$ where C is a constant.

(b) let $\boldsymbol{F} = \langle A, B, C\rangle$

$$\oint_C \mathbf{F} \cdot d\boldsymbol{r} = \iint_D \frac{\partial A}{\partial z} - \frac{\partial C}{\partial x} dxdz = \iint_D (-2x) dxdz = 0.$$

remark: by stoke's theorem you can get the above result. the last integral is because D is symmetric of x axis

(c) let $\boldsymbol{F} = \langle bz - cy, cx - az, ay - bz\rangle$
then $curlF = \langle 2a, 2b, 2c\rangle = 2\boldsymbol{n}$ let u be a unit vector with the same direction with n. therefore :

$$\frac{1}{2|\boldsymbol{n}|} \oint_C (bz - cy)dx + (cx - az)dy + (ay - bx)dz = \frac{1}{2|\boldsymbol{n}|} \int\int_R 2\boldsymbol{n} \cdot \boldsymbol{u} dS = \int\int_R 1dS$$

$\iint_R 1dS$ is the area of R.
Remark: using the stokes theorem.

Question 6

(a) let $\boldsymbol{F} = (F_1, F_2, F_3), \boldsymbol{G} = (G_1, G_2, G_3)$

$$\boldsymbol{F} * \boldsymbol{G} = (F_2G_3 - F_3G_2, F_3G_1 - F_1G_3, F_1G_2 - F_2G_1)$$

$$lefthandside = div(\boldsymbol{F} * \boldsymbol{G}) = \frac{\partial(F_2G_3 - F_3G_2)}{\partial x} + \frac{\partial(F_3G_1 - F_1G_3)}{\partial y} + \frac{\partial(F_1G_2 - F_2G_1)}{\partial z}$$

$$curl\boldsymbol{F} = (\frac{(\partial(F_3))}{\partial y} - \frac{\partial(F_2)}{\partial z}, \frac{\partial(F_1)}{\partial z} - \frac{\partial(F_3)}{\partial x}, \frac{\partial(F_2)}{\partial x} - \frac{\partial(F_1)}{\partial y})$$

$$curl\boldsymbol{G} = (\frac{(\partial(G_3))}{\partial y} - \frac{\partial(G_2)}{\partial z}, \frac{\partial(G_1)}{\partial z} - \frac{\partial(G_3)}{\partial x}, \frac{\partial(G_2)}{\partial x} - \frac{\partial(G_1)}{\partial y})$$

$$curl\boldsymbol{F} \cdot \boldsymbol{G} = G_1\frac{\partial(F_3)}{\partial y} - G_1\frac{\partial(F_2)}{\partial z} + G_2\frac{\partial(F_1)}{\partial z} - G_2\frac{\partial(F_3)}{\partial x} + G_3\frac{\partial(F_2)}{\partial x} - G_3\frac{\partial(F_1)}{\partial y}$$

similarly,

$$curl\boldsymbol{G} \cdot \boldsymbol{F} = F_1\frac{\partial(G_3)}{\partial y} - F_1\frac{\partial(G_2)}{\partial z} + F_2\frac{\partial(G_1)}{\partial z} - F_2\frac{\partial(G_3)}{\partial x} + F_3\frac{\partial(G_2)}{\partial x} - F_3\frac{\partial(G_1)}{\partial y}$$

thus,

$$\begin{aligned} \text{RHS} &= curl\boldsymbol{F} \cdot \boldsymbol{G} - curl\boldsymbol{G} \cdot \boldsymbol{F} \\ &= \frac{\partial(F_2G_3 - F_3G_2)}{\partial x} + \frac{\partial(F_3G_1 - F_1G_3)}{\partial y} + \frac{\partial(F_1G_2 - F_2G_1)}{\partial z} \\ &= \text{LHS} \end{aligned}$$

(b) Let $g(x, y) = \sqrt{1 - x^2}$, then we have

$$\frac{\partial g}{\partial x} = -\frac{x}{\sqrt{1 - x^2}} \quad \& \quad \frac{\partial g}{\partial y} = 0$$

First we use the reduction formula,

$$\begin{aligned} \int \cos^3(x)\ dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\ dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C, \quad C \in \mathbb{R} \end{aligned}$$

$$\begin{aligned}
\iint\limits_S y^2\,dS &= 2\int_{-1}^{1}\int_{0}^{3-x} y^2\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dy\,dx \\
&= 2\int_{-1}^{1}\int_{0}^{3-x} y^2\sqrt{1+\frac{x^2}{1-x^2}}\,dy\,dx \\
&= \frac{2}{3}\int_{-1}^{1}\left[y^3\right]_{y=0}^{y=3-x}\left(\sqrt{\frac{1}{1-x^2}}\right)\,dx \\
&= \frac{2}{3}\int_{-1}^{1}\frac{(3-x)^3}{\sqrt{1-x^2}}\,dx \\
(\text{substitute } x=\cos\theta) &= \frac{2}{3}\int_{\pi}^{2\pi}\frac{(3-\cos\theta)^3}{\sqrt{1-\cos^2\theta}}\,\frac{dx}{d\theta}\,d\theta \\
&= \frac{2}{3}\int_{\pi}^{2\pi}\frac{(3-\cos\theta)^3}{\sin\theta}\,\sin\theta\,d\theta \\
&= \frac{2}{3}\int_{\pi}^{2\pi}(3-\cos\theta)^3\,d\theta \\
&= \frac{2}{3}\int_{\pi}^{2\pi}27-27\cos\theta+9\cos^2\theta-\cos^3\theta\,d\theta \\
(\cos^2\theta=\frac{1}{2}(\cos 2\theta+1)) &= \frac{2}{3}\left[27\pi-0+\frac{9}{2}\int_{\pi}^{2\pi}\cos 2\theta+1\,d\theta-\int_{\pi}^{2\pi}\cos^3\theta\,d\theta\right] \\
&= \frac{2}{3}\left[27\pi+\frac{9}{2}(0+\pi)-(0+0)\right] \\
&= \frac{2}{3}\left[\frac{63}{2}\pi\right] \\
&= 21\pi
\end{aligned}$$

(c) The flux E across the field is the value of $div\boldsymbol{E}$.

$$\boldsymbol{E}=\nabla(\frac{z}{\rho^3})$$

$$div\boldsymbol{E}=\nabla\cdot\boldsymbol{E}=\nabla^2(\frac{z}{\rho^3})=\frac{\partial^2(\frac{z}{\rho^3})}{\partial x^2}+\frac{\partial^2(\frac{z}{\rho^3})}{\partial y^2}+\frac{\partial^2(\frac{z}{\rho^3})}{\partial z^2}$$

$$\frac{\partial(\frac{1}{\rho^3})}{\partial x}=-\frac{3x}{(x^2+y^2+z^2)^{(\frac{5}{2})}}\frac{\partial^2(\frac{1}{\rho^3})}{\partial(x^2)}=-\frac{3}{(x^2+y^2+z^2)^{(\frac{5}{2})}}+\frac{15x^2}{(x^2+y^2+z^2)^{(\frac{7}{2})}}$$

by symmetry,

$$\begin{aligned}
\frac{\partial^2(\frac{1}{\rho^3})}{\partial(y^2)} &= -\frac{3}{(x^2+y^2+z^2)^{(\frac{5}{2})}}+\frac{15y^2}{(x^2+y^2+z^2)^{(\frac{7}{2})}} \\
\frac{\partial^2(\frac{1}{\rho^3})}{\partial(z^2)} &= -\frac{3}{(x^2+y^2+z^2)^{(\frac{5}{2})}}+\frac{15z^2}{(x^2+y^2+z^2)^{(\frac{7}{2})}} \\
\frac{\partial^2(\frac{z}{\rho^3})}{\partial(z^2)} &= z(-\frac{3}{(x^2+y^2+z^2)^{(\frac{5}{2})}}+\frac{15x^2}{(x^2+y^2+z^2)^{(\frac{7}{2})}})-\frac{6x}{(x^2+y^2+z^2)^{(\frac{5}{2})}}
\end{aligned}$$

hence,

$$div\boldsymbol{E}=\nabla\cdot\boldsymbol{E}=z(-\frac{9}{(x^2+y^2+z^2)^{(\frac{5}{2})}}+\frac{15(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{(\frac{7}{2})}})-\frac{6x}{(x^2+y^2+z^2)^{(\frac{5}{2})}}=0$$

Thus, the flux accros the field is zero.