NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lau Tze Siong

MA2101 Linear Algebra II
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SECTION A

Question 1

(i) For all $\begin{pmatrix} a_1 & b_1 \\ a_1 & a_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ a_2 & a_2 \end{pmatrix} \in W_1$ and $r \in \mathbb{R}$, we have,

$$\begin{aligned} \begin{pmatrix} a_1 & b_1 \\ a_1 & a_1 \end{pmatrix} + r \begin{pmatrix} a_2 & b_2 \\ a_2 & a_2 \end{pmatrix} &= \begin{pmatrix} a_1 & b_1 \\ a_1 & a_1 \end{pmatrix} + \begin{pmatrix} ra_2 & rb_2 \\ ra_2 & ra_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + ra_2 & b_1 + rb_2 \\ a_1 + ra_2 & a_1 + ra_2 \end{pmatrix} \in W_1. \end{aligned}$$

Hence W_1 is a subspace of $M_{22}(\mathbb{R})$.

(ii) Claim: $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is a basis for W_1.
Proof:
For all $w \in W_1$, $w = \begin{pmatrix} a & b \\ a & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for some $a, b \in \mathbb{R}$.
Hence $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ spans W_1.
Suppose there exist $x, y \in \mathbb{R}$ such that $x \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + y \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$.
Hence we have $\begin{pmatrix} x & y \\ x & x \end{pmatrix} = \mathbf{0}$. Therefore $x = y = 0$. This give us $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ to be linearly independent.
Therefore $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is a basis for W_1.

Claim: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ is a basis for W_2.
Proof:
For all $w \in W_2$, $w = \begin{pmatrix} a+b & b \\ c & a+c \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ for some $a, b, c \in \mathbb{R}$. Hence $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ spans W_2.
Suppose there exist $x, y, z \in \mathbb{R}$ such that $x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = 0$.
Hence we have $\begin{pmatrix} x+y & y \\ z & x+z \end{pmatrix} = \mathbf{0}$. Therefore $x = y = z = 0$.
This give us $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ to be linearly independent.
Therefore $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ is a basis for W_2.

(iii) Suppose $w \in W_1 \cap W_2$.
We have $w = \begin{pmatrix} a_1 & b_1 \\ a_1 & a_1 \end{pmatrix} = \begin{pmatrix} a_2+b_2 & b_2 \\ c_2 & a_2+c_2 \end{pmatrix}$ for some $a_1, b_1, a_2, b_2, c_2 \in \mathbb{R}$.
Hence we have $b_1 = b_2$, $a_1 = c_2$, $a_1 = a_2 + b_2$, $a_1 = a_2 + c_2$. Solving we have $a_2 = 0$, and $b_2 = c_2$.
Hence $W_1 \cap W_2 = \left\{ r \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} | r \in \mathbb{R} \right\}$. Therefore $\dim(W_1 \cap W_2) = 1$.

(iv) We have $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = 2 + 3 - 1 = 4 = \dim(M_{22}(\mathbb{R}))$.
Since $W_1 + W_2$ is a subspace of $M_{22}(\mathbb{R})$, we have $M_{22}(\mathbb{R}) = W_1 + W_2$.

(v) Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in W_1 \cup W_2$ but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \notin W_1 \cup W_2$, we have $W_1 \cup W_2$ to not be a subspace of $M_{22}(\mathbb{R})$.

Question 2

(i) Since $T(1) = 0 + x + x^2$ and $T(x) = 1 + 0x + x^2$ and $T(x^2) = 1 + x + 0x^2$. We have

$$[T]_{\mathcal{B}_1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(ii) Hence the characteristic equation of T is

$$\begin{aligned} \det(\lambda I - [T]_{\mathcal{B}_1}) &= (\lambda)(\lambda^2 - 1) + 1(-\lambda - 1) - 1(1 + \lambda) \\ &= (\lambda + 1)^2(\lambda - 2). \end{aligned}$$

Hence the eigenvalues are -1 and 2.

(iii) When $\lambda = -1$, we have

$$\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Hence $E_{-1} = \text{nullspace}\left(\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \left\{ \mu \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid \mu, \lambda \in \mathbb{R} \right\}$.
When $\lambda = 2$, we have

$$\begin{aligned} \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array} \right) &\rightarrow \left(\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ -1 & 2 & -1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right) \\ &\rightarrow \left(\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \\ &\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right). \end{aligned}$$

Hence $E_2 = \text{nullspace}\left(\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\right) = \left\{\lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} | \lambda \in \mathbb{R}\right\}$.

Hence $\mathcal{B}_2 = \{-1+x, -1+x^2, 1+x+x^2\}$ give us $[T]_{\mathcal{B}_2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ to be diagonal.

(iv) Let $\mathcal{B}_2 = \mathcal{B}_3$. Since $[\cdot]_{\mathcal{B}_3} : L(P_2(\mathbb{R}), P_2(\mathbb{R})) \to M_{33}(\mathbb{R})$ is a linear isomorphism, we have,

$$\begin{aligned} [S]_{\mathcal{B}_3} = [4T^5 + 3T^4]_{\mathcal{B}_3} &= 4[T]^5_{\mathcal{B}_3} + 3[T]^4_{\mathcal{B}_3} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 176 \end{pmatrix}. \end{aligned}$$

Therefore S is diagonalisable, with $\mathcal{B}_3 = \{-1+x, -1+x^2, 1+x+x^2\}$ and $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 176 \end{pmatrix}$.

Question 3

(a) (i) Given the characteristic equation there are 4 possible Jordan Canonical Forms.
Case 1:-

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

Case 2:-

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

Case 3:-

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

Case 4:-

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

(ii) Case 1 has minimal polynomial $= x(x-5)(x-7)$.
Case 2 has minimal polynomial $= x(x-5)^2(x-7)$.
Case 3 has minimal polynomial $= x(x-5)(x-7)^2$.
Case 4 has minimal polynomial $= x(x-5)^2(x-7)^2$.

(iii) T is not invertible, since $c_T(0) = 0$ and thus we have $\det T = 0$.

(b) Let W be the vector subspace spanned by $\mathcal{B}_1 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, ..., \boldsymbol{v}_n\}$. Since $\mathcal{B}_1$ is a linearly indepedent set, $\mathcal{B}_1$is a basis for W. Let $\langle \cdot, \cdot \rangle : W \times W \to \mathbb{R}$ be the inner product restricted to W.
Notice that $\langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle = [\boldsymbol{w}_1]_{\mathcal{B}_1}^T A [\boldsymbol{w}_2]_{\mathcal{B}_1}$ for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in W$. This give us $A = [\langle \cdot, \cdot \rangle]_{\mathcal{B}_1, \mathcal{B}_1}$.
Let $\mathcal{B}_2$ be an orthonormal basis for W. Thus we have,

$$\begin{aligned}
\langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle &= [\boldsymbol{w}_1]_{\mathcal{B}_1}^T [\langle \cdot, \cdot \rangle]_{\mathcal{B}_1, \mathcal{B}_1} [\boldsymbol{w}_2]_{\mathcal{B}_1} \\
&= \left([\mathrm{id}_W]_{\mathcal{B}_1, \mathcal{B}_2} [\boldsymbol{w}_1]_{\mathcal{B}_2}\right)^T [\langle \cdot, \cdot \rangle]_{\mathcal{B}_1, \mathcal{B}_1} \left([\mathrm{id}_W]_{\mathcal{B}_1, \mathcal{B}_2} [\boldsymbol{w}_1]_{\mathcal{B}_2}\right) \\
&= [\boldsymbol{w}_1]_{\mathcal{B}_2}^T \left([\mathrm{id}_W]_{\mathcal{B}_1, \mathcal{B}_2}^T [\langle \cdot, \cdot \rangle]_{\mathcal{B}_1, \mathcal{B}_1} [\mathrm{id}_W]_{\mathcal{B}_1, \mathcal{B}_2}\right) [\boldsymbol{w}_1]_{\mathcal{B}_2}.
\end{aligned}$$

Since $\mathcal{B}_2$ is a orthonormal basis for W, $[\langle \cdot, \cdot \rangle]_{\mathcal{B}_2, \mathcal{B}_2} = I_n$.
This give us $A = \left([\mathrm{id}_W]_{\mathcal{B}_1, \mathcal{B}_2}^T\right)^{-1} I_n \left([\mathrm{id}_W]_{\mathcal{B}_1, \mathcal{B}_2}\right)^{-1}$.
Since $[\mathrm{id}_W]_{\mathcal{B}_1, \mathcal{B}_2}$ is a linear isomorphism, $\det[\mathrm{id}_W]_{\mathcal{B}_1, \mathcal{B}_2} \neq 0$, and so,

$$\det A = \frac{1}{\det[\mathrm{id}_W]_{\mathcal{B}_1, \mathcal{B}_2}^T \det[\mathrm{id}_W]_{\mathcal{B}_1, \mathcal{B}_2}} = \frac{1}{(\det[\mathrm{id}_W]_{\mathcal{B}_1, \mathcal{B}_2})^2} > 0.$$

SECTION B

Question 4

(a) (i) Since $A_4 + 2A_3 - 2A_1 = A_2$, we have $\mathrm{span}\{A_1, A_2, A_3, A_4\} = \mathrm{span}\{A_1, A_3, A_4\}$. Also $\alpha_1 A_1 + \alpha_2 A_3 + \alpha_3 A_4 = \mathbf{0}$ if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore $\{A_1, A_3, A_4\}$ is a basis for W. Hence $\dim(W^\perp) = 1$.
Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W^\perp$. Then we have

$$0 = \mathrm{Tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \mathrm{Tr}\left(\begin{pmatrix} a & -b \\ c & -d \end{pmatrix}\right) = a - d;$$
$$0 = \mathrm{Tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix}\right) = \mathrm{Tr}\left(\begin{pmatrix} 0 & a-3b \\ 0 & c-3d \end{pmatrix}\right) = c - 3d;$$
$$0 = \mathrm{Tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}\right) = \mathrm{Tr}\left(\begin{pmatrix} b & -2b \\ d & -2d \end{pmatrix}\right) = b - 2d.$$

Hence a basis for $W^\perp$ is $\left\{\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\right\}$.

(ii) Since $\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \in M_{22}(\mathbb{R})$, and we have,

$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} = 1\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} - 1\begin{pmatrix} 0 & 0 \\ 1 & -3 \end{pmatrix} + 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 1\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix},$$

we get $F = P + Q$, with $P = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, and $Q = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$.

(iii) Since $\{\|F - X\| \mid X \in W\} = \{\|Q + Y\| \mid Y \in W\} = \left\{\sqrt{\|Q\|^2 + \|Y\|^2} \mid Y \in W\right\}$, we have $\min\{\|F - X\| : X \in W\} = \sqrt{\|Q\|^2} = \sqrt{\mathrm{Tr}\,(Q^TQ)} = \sqrt{15}$.

(b) Let $\mathcal{B}_{W_1}$ and $\mathcal{B}_{W_2}$ be ordered bases of W_1 and W_2 respectively.
Let $\varphi : W_2 \times W_1 \to \mathbb{R}$, such that $\varphi(w_2, w_1) = \langle w_2, w_1 \rangle$.
Then we have $\langle w_2, w_1 \rangle = \left([w_2]_{\mathcal{B}_{W_2}}\right)^T [\varphi]_{\mathcal{B}_{W_2},\mathcal{B}_{W_1}} [w_1]_{\mathcal{B}_{W_1}}$.
Since $[\varphi]_{\mathcal{B}_{W_2},\mathcal{B}_{W_1}}$ is a $\dim(W_2)\times\dim(W_1)$ matrix, and $\dim(W_1) < \dim(W_2)$, we have the nullspace of $[\varphi]^T_{\mathcal{B}_{W_2},\mathcal{B}_{W_1}}$ to be non-trivial, i.e. there exists $\boldsymbol{v} \in W_2\backslash\{0_V\}$, such that $[\boldsymbol{v}]_{\mathcal{B}_{W_2}} \in \text{nullspace}([\varphi]^T_{\mathcal{B}_{W_2},\mathcal{B}_{W_1}})\backslash\{0_V\}$, and for all $\boldsymbol{w} \in W_1$, we have $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \left([\boldsymbol{v}]_{\mathcal{B}_{W_2}}\right)^T [\varphi]_{\mathcal{B}_{W_2},\mathcal{B}_{W_1}} [\boldsymbol{w}]_{\mathcal{B}_{W_1}} = \mathbf{0}[\boldsymbol{w}]_{\mathcal{B}_{W_1}} = 0$, hence we are done.

Question 5

(a) (i) Since $\mathcal{R}(T) = \ker(T)$, we have $\dim(\mathcal{R}(T)) = \dim(\ker(T))$. Hence by Dimension Theorem, we have $\dim(V) = \dim(\mathcal{R}(T)) + \dim(\ker(T)) = 2\dim(\ker(T))$, i.e. n is always even.

(ii) Let $V = \mathbb{R}^2$, and T is a linear operator such that $T((1,0)) = (0,0)$ and $T((0,1)) = (1,0)$. Then we have $\ker(T) = \{r(1,0) \mid r \in \mathbb{R}\} = \mathcal{R}(T)$.

(b) Let $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$. For every given invertible $P \in M_{nn}(\mathbb{F})$, let $\mathcal{B}' = \{b_1', b_2', \ldots, b_n'\}$ such that $[b_k']_{\mathcal{B}} = k^{\text{th}}$ column of P. This give us for all $v \in V$, $P[v]_{\mathcal{B}'} = [v]_{\mathcal{B}}$, i.e, $P = [\mathrm{id}_V]_{\mathcal{B},\mathcal{B}'}$.
Hence, $P^{-1}AP = [\mathrm{id}_V]^{-1}_{\mathcal{B},\mathcal{B}'}[T]_{\mathcal{B}}[\mathrm{id}_V]_{\mathcal{B},\mathcal{B}'} = [\mathrm{id}_V^{-1} \circ T \circ \mathrm{id}_V]_{\mathcal{B}'} = [T]_{\mathcal{B}'}$.

Question 6

(a) (i) False.
Let us be given any non-singular T.
Let $S = -T$ and S is also non-singular. $S + T = (-T) + T = 0$ is the zero-map and the zero-map is singular.

(ii) True
Since A satisfy $A^3 = A$, A satisfies $A^3 - A = A(A-1)(A+1)$. Hence the minimal polynomial of A divides $A(A-1)(A+1)$. Thus the minimal polynomial of A is a product of distinct linear factors. Therefore A is diagonalisable.

(iii) True.
Applying Cauchy-Schwarz inequality we have,

$$\sum_{i=1}^{n} |\sqrt{a_i}|^2 \sum_{j=1}^{n} \left|\sqrt{\frac{1}{a_j}}\right|^2 \geq \left|\sum_{i=1}^{n} \sqrt{a_i}\sqrt{\frac{1}{a_i}}\right|^2 = n^2.$$

(b) Notice that for all polynomial $p(x) \in \mathbb{R}[x]$, we have $p(T) : V \to V$ to be a linear operator such that $p(T)(X) = p(A)X$ for all $X \in V$.
Now for all $X \in V$, $m_A(T)(X) = m_A(A)X = 0_V X = 0_V$, i.e. $m_A(T) = 0_{L(V,V)}$.
Thus $m_T(x) \mid m_A(x)$.

Also for all $X \in V$, we have $m_T(A)X = m_T(T)(X) = 0_{L(V,V)}(X) = 0_V$, i.e. $m_T(A) = 0_V$.
Thus $m_A(x) \mid m_T(x)$.

Therefore $m_T(x) = m_A(x)$.