NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Teo Wei Hao

MA4247 Complex Analysis II
AY 2009/2010 Sem 1

Question 1

(a) False.
Assume on the contrary that such a function $f : B(0,1) \to \mathbb{C}$ exists.
Let $g : B(0,1) \to \mathbb{C}$ be the analytic function such that $g(z) = \frac{f(z)}{4}$ for all $z \in B(0,1)$.
Then, we have $g(0) = \frac{f(0)}{4} = 0$, and $|g(z)| \leq 1$ for all $z \in B(0,1)$.
Thus by Schwarz's Lemma, we have $|g(z)| \leq |z|$ for all $B(0,1)$.
However, we have $\left|g\left(\frac{1}{2}\right)\right| = \left|\frac{\pi i}{4}\right| = \frac{\pi}{4} > \frac{2}{4} = \left|\frac{1}{2}\right|$, a contradiction.

(b) True.
Let the largest domain where f and g be defined on be D_f and D_g respectively (which are domains).
Let the largest domain where $f - g$ be defined on be D.
Since f and g are meromorphic, $\mathbb{C} - D_f$ and $\mathbb{C} - D_g$ are sets of poles with no accumulation point in $\mathbb{C}$. Thus, $\mathbb{C} - D \subseteq (\mathbb{C} - D_f) \cup (\mathbb{C} - D_g)$ is a set of poles with no accumulation point in $\mathbb{C}$. So, $f - g : D \to \mathbb{C}$ is meromorphic on $\mathbb{C}$.

Let S be the set of zeroes of $f - g$ in D. Since there are infinitely many $z \in B(0,1) \subseteq \overline{B(0,1)}$ such that $f(z) = g(z)$, we have $S \cap \overline{B(0,1)}$ to be an infinite set. Since $\overline{B(0,1)}$ is compact, by Bolzano-Weierstrass Theorem, S has an accumulation point in $\overline{B(0,1)}$, say S.

Assume on the contrary that $s \notin D$. Then $f - g$ has a pole at s. This implies that there exists $r \in \mathbb{R}^+$ such that $B'(s,r) \subseteq D$, and $|(f-g)(z)| \geq 1 > 0$ for all $z \in B'(s,r)$, a contradiction to the fact that s is an accumulation point of S.

Thus $s \in D$, and so by Identity Theorem, $f - g$ is zero on D, i.e. $f(z) = g(z)$ on D.

(c) True.
Let us be given any open ball $B(a,r)$ (i.e. $a \in \mathbb{C}$, $r \in \mathbb{R}^+$), such that $r > 4$.
Then since $\pi < 4 < r$, we have $a + i\pi, a - i\pi \in B(a,r)$.
As $f(a + i\pi) = e^{a+i\pi} = e^a \cdot (-1) = e^{a-i\pi} = f(a - i\pi)$, f is not one-to-one on $B(a,r)$.

(d) True.
Let $D_1 = \{z = x + iy \in \mathbb{C} : |x| < 2,\ |y| < 2\}$. Then D_1 is a simply connected domain that is not $\mathbb{C}$. Hence, by Riemann Mapping Theorem, there exists an analytic isomorphism $f : D_1 \to B(0,1)$ such that $f(1 + i) = 0$. This implies that $f|_D$ is an analytic isomorphism from D to $B'(0,1)$.

(e) False.
Assume on the contrary there exists a harmonic function $u : D \to \mathbb{C}$ on some domain D such that

$\overline{B(0,1)} \subseteq D$ that satisfy the condition in question. Then by Mean Value Theorem, we have,

$$\begin{aligned} u(0,0) &= \frac{1}{2\pi}\int_0^{2\pi} u(\cos t, \sin t)\ dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \cos t\ dt \\ &= \frac{1}{2\pi}\left[\sin t\right]_0^{2\pi} = 0, \end{aligned}$$

a contradiction to $u(0,0) = \dfrac{1}{2}$.

Question 2

(a) Let entire functions $g, h : \mathbb{C} \to \mathbb{C}$ be such that $g(z) = 3z$, $h(z) = z^6 + 1$ for all $z \in \mathbb{C}$.
Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the simple closed contour $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$.
For all $z \in \{\gamma\}$, we have $|z| = 1$, and so $|g(z)| = 3|z| = 3 > 2 = |z|^6 + |1| \geq |h(z)|$.
Thus by Rouché Theorem, $f = g + h$ and g have same number of zeroes inside γ, i.e. in $B(0,1)$.
Since g has a zero at $0 \in B(0,1)$, f has a zero in $B(0,1)$, and so minimum of $|f|$ in $\overline{B(0,1)}$ is 0.

Since $\overline{B(0,1)}$ is a compact set whose interior $B(0,1)$ is a domain, and f is entire and non-constant in $\overline{B(0,1)}$, by Maximum Modulus Principle, the maximum of $|f|$ occurs on $\{\gamma\}$.
For all $z \in \{\gamma\}$, we have $|z| = 1$, and so $|f(z)| \leq |z|^6 + 3|z| + 1 = 5$.
At the same time, $1 \in \{\gamma\}$ is such that $|f(1)| = |1^6 + 3 + 1| = 5$.
Thus the maximum of $|f|$ on $\overline{B(0,1)}$, which is also the maximum of $|f|$ on $\{\gamma\}$, is 5.

(b) Let $z \in \mathbb{C}$ be such that $(z, \overline{z}; 0, 1) = -1$.
For the cross-ratio to be well-defined, we have $z \neq \overline{z}$, i.e. $z \notin \mathbb{R}$ (this includes $z \neq 0$ and $z \neq 1$).
Let T_z be the unique Linear Fractional Transformation such that $T_z(z) = \infty$, $T_z(\overline{z}) = 0$ and $T_z(0) = 1$. Then from the given cross-ratio, we have $T_z(1) = -1$. Notice that $\infty, 0, 1$ and -1 all lies on the extended line $\{z \in \mathbb{C} : \text{Im } z = 0\}$.
Since T_z preserves the set of circles and extended line, and $z \notin \mathbb{R}$, there exists a circle that passes through $z, \overline{z}, 0$ and 1.
From geometry, since the angle between $0, z, 1$ and $0, \overline{z}, 1$ are equal, and have to sum up to $180°$, we have z to be a point on the circle with line segment of 0 and 1 being a diameter, but not 0 or 1, i.e. $z \in \left\{ w \in \mathbb{C} : \left| w - \dfrac{1}{2} \right| = \dfrac{1}{2},\ w \neq 0,\ w \neq 1 \right\}$.

Instead, let $z \in \left\{ w \in \mathbb{C} : \left| w - \dfrac{1}{2} \right| = \dfrac{1}{2},\ w \neq 0,\ w \neq 1 \right\}$.
Then $z, \overline{z}, 0$ and 1 are distinct points on a circle in $\mathbb{C}$.
Again, let T_z be the unique Linear Fractional Transformation such that $T_z(z) = \infty$, $T_z(\overline{z}) = 0$ and $T_z(0) = 1$. Since T_z preserves the set of circles and extended line, we have $T_z(1)$ to lies on the extended line $\{z \in \mathbb{C} : \text{Im } z = 0\}$.
Next, we notice that z and $\overline{z}$ are symmetric with respect to $\{z \in \mathbb{C} : \text{Im } z = 0\}$.
Thus, by Symmetry Principle, $\{z \in \mathbb{C} : \text{Im } z = 0\}$ is mapped to the unit circle under T_z.
Since the intersection of $\{z \in \mathbb{C} : \text{Im } z = 0\}$ and the unit circle is $\{-1, 1\}$, together with the fact that $T_z(1) \neq T_z(0) = 1$, we conclude that $T_z(1) = -1$, i.e. $(z, \overline{z}; 0, 1) = -1$.

Therefore, we conclude that z is all points on the circle with the line segment between 0 and 1 as diameter, but is neither 0 nor 1, i.e. $\left\{ w \in \mathbb{C} : \left| w - \dfrac{1}{2} \right| = \dfrac{1}{2},\ w \neq 0,\ w \neq 1 \right\}$.

(c) For all $(x, y) \in \mathbb{R}^2$, we have,

$$\begin{aligned} u_x(x,y) &= 2x - e^x \sin by; \\ u_{xx}(x,y) &= 2 - e^x \sin by; \\ u_y(x,y) &= 2ay - be^x \cos by; \\ u_{yy}(x,y) &= 2a + b^2 e^x \sin by. \end{aligned}$$

Since u is harmonic on $\mathbb{R}^2$, for all $(x, y) \in \mathbb{R}^2$, we have,

$$0 = u_{xx}(x,y) + u_{yy}(x,y) = 2(1+a) + (b^2 - 1)e^x \sin by.$$

This implies that $a = -1$, and $b \in \{-1, 0, 1\}$.

For $(a, b) = (-1, -1)$, we notice that u is the real part of the analytic function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = z^2 - ie^z$. This give us Im $f(x, y) = 2xy - e^x \cos y$, and so Im $f(0, 0) = -1$.
Since harmonic conjugates differ by a constant, $v(x, y) = 2xy - e^x \cos y + 4$.

For $(a, b) = (-1, 0)$, we notice that u is the real part of the analytic function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = z^2$. This give us Im $f(x, y) = 2xy$, and so Im $f(0, 0) = 0$.
Since harmonic conjugates differ by a constant, $v(x, y) = 2xy + 3$.

For $(a, b) = (-1, 1)$, we notice that u is the real part of the analytic function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = z^2 + ie^z$. This give us Im $f(x, y) = 2xy + e^x \cos y$, and so Im $f(0, 0) = 1$.
Since harmonic conjugates differ by a constant, $v(x, y) = 2xy + e^x \cos y + 2$.

Question 3

(a) Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the simple closed contour $\gamma(t) = 2e^{it}$, $t \in [0, 2\pi]$.
$f(z)$ can be divided into 3 analytic components, $\sin z$ and $\cos z$ in the numerator, and $z^7 - z^5 + z^3 - z$ in the denominator.
We have $\sin z = 0$ iff $z = k\pi$, $k \in \mathbb{Z}$; and $\cos z = 0$ iff $z = \left(k + \frac{1}{2}\right)\pi$, $k \in \mathbb{Z}$.
So, they contributes 3 zeroes to f inside γ.

Let entire functions $g, h : \mathbb{C} \to \mathbb{C}$ be such that $g(z) = z^7$ and $h(z) = -z^5 + z^3 - z$.
For all $z \in \{\gamma\}$, we have $|z| = 2$, and so $|g(z)| = |z|^7 = 128 > 42 = |z|^5 + |z|^3 + |z| \geq |h(z)|$.
Thus by Rouché Theorem, $g + h$ and g have same number of zeroes inside γ, which is 7.
Thus, it contributes 7 poles to f inside γ.

Therefore, by Argument Principle, $\dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{f'(z)}{f(z)}\, dz = 3 - 7 = -4$.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be a function satisfying the conditions given.
Let $U = \{w \in \mathbb{C} : \text{Im } w < 0\}$ and $V = \{w \in \mathbb{C} : \text{Im } w > 0\}$.
As Im $f(z) \neq 0$ for all $z \in \mathbb{C}$, we have $f[\mathbb{C}] \subseteq U \cap V$.
Since $\mathbb{C}$ is a domain, and f is analytic, by Open Mapping Theorem, $f[\mathbb{C}]$ is a domain.
As U and V are disconnected, we must have $f[\mathbb{C}] \subseteq U$ or $f[\mathbb{C}] \subseteq V$.
Together with $f(0) = i$, we conclude that $f[\mathbb{C}] \subseteq V$.

Now since V is a simply connected domain that is not $\mathbb{C}$, by Riemann Mapping Theorem, there exists an analytic isomorphism $g : V \to B(0, 1)$. This implies that $g \circ f$ is an entire function from $\mathbb{C}$ to $B(0, 1)$, i.e. $g \circ f$ is entire bounded, and hence by Liouville's Theorem, $g \circ f$ is constant.
Since g is injective, f is a constant function, and so $f(z) = i$ for all $z \in \mathbb{C}$.

A simple check shows that this f satisfy all conditions given, and so is the only such function.

(c) Let $R \in \mathbb{R}^+$. Let $f : \mathbb{C} \to \mathbb{C}$ be such that $f(z) = e^z$.
Let entire functions $g_n : \mathbb{C} \to \mathbb{C}$ be such that $g_n = f_n - f$.
Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the simple closed contour $\gamma(t) = Re^{it}$, $t \in [0, 2\pi]$.
Since $\{\gamma\}$ is compact, and f is continuous, there exists $w \in \{\gamma\}$ such that $|f(w)| \leq |f(z)|$ for all $z \in \{\gamma\}$. Notice that $|f(w)| > 0$ since f has no zeroes.
We notice that $\left(\sum_{k=0}^{\infty} f_n\right)_{n\in\mathbb{Z}^+}$ converges to f uniformly on compact set $\{\gamma\}$. Thus, there exists $N \in \mathbb{Z}^+$ such that for all $n \in \mathbb{Z}_{\geq N}$ and $z \in \{\gamma\}$, we have $\left|\sum_{k=0}^{n} f_n(z) - f(z)\right| < |f(w)|$.
For all $z \in \{\gamma\}$, we have $|f(z)| \geq |f(w)| > |g_n(z)|$.
Thus by Rouché Theorem, $f_n = f + g_n$ and f has same number of zeroes inside γ, i.e. in $B(0, R)$.
Since f has no zeroes in $B(0, R)$, f_n has no zeroes in $B(0, R)$ for all $n \in \mathbb{Z}_{\geq N}$.

Question 4

(a) Let $U_1 = \mathbb{C} - [0, \infty)$ and $U_2 = \{w \in \mathbb{C} : \text{Im } w > 0\}$ (i.e. U_1 is the cut plane and U_2 is the upper half plane).
Let $f_1 : U_1 \to U_2$ be analytic isomorphism from U_1 to U_2 such that $f_1(z) = \text{Pr } z^{\frac{1}{2}}$.
Then $f_1(-2) = i\sqrt{2}$.
Let $f_2 : U_2 \to B(0, 1)$ be analytic isomorphism from U_2 to $B(0, 1)$ such that $f_2(z) = \dfrac{z - i\sqrt{2}}{z + i\sqrt{2}}$.
Then $f_2(i\sqrt{2}) = 0$.
Thus $f_2 \circ f_1$ is analytic isomorphism from U_1 to $B(0, 1)$ which maps -2 to 0.

(b) Let f be a linear fractional transformation that satisfy the condition given in question.
Let $L = \{w \in \mathbb{C} : \text{Im } w = 0\}$ and $C = \{w \in \mathbb{C} : |w| = 2\}$ (i.e. L is the extended real axis and C is the circle centered at 0 radius 2).
We notice that f which maps H to $B(0, 2)$, will maps L to C.
Since i and $-i$ are symmetric points with respect to L; and 1 and 4 are symmetric points with respect to C, by symmetry principle, $f(-i) = 4$. Thus, using cross-ratio, we have,

$$\begin{aligned}
(-2, 1; 4, f(z)) &= (\infty, i; -i, z) \\
\frac{f(z) - 1}{f(z) + 2}\frac{4 + 2}{4 - 1} &= \frac{z - i}{-i - i} \\
4i(f(z) - 1) &= (i - z)(f(z) + 2) \\
f(z) &= \frac{-2z + 6i}{z + 3i}.
\end{aligned}$$

A simple checks shows that this f satisfy all conditions given, and so is what we wanted.

(c) Since C_1 and C_2 intersect transversely, there are 2 intersection points. Let T be any linear fractional transformation that maps one of the intersection point to 0, and another to ∞.
Then C_1 and C_2 will be maps to L_1 and L_2 for some extended line in the extended complex plane. L_1 and L_2 are distinct, and only intersects at 0.
We see that 0 and 0 are symmetric points with respect to both L_1 and L_2, similarly for ∞ and ∞.
Let $z \in \mathbb{C} - \{0, \infty\}$. Then z does not lies on at least 1 extended line, WLOG let it be L_1.
This give us if z and z^* are symmetric with respect to L_1, then $z \neq z^*$.
Notice that any 2 distinct points in the complex plane give us a unique extended line they are symmetric to, and so it is not possible for z and z^* to be symmetric with respect to L_2.
Thus there are only 2 pairs of valid points, 0 with 0, and ∞ with ∞.

By symmetry principle, symmetry is preserved under linear fractional transformation, and so we conclude that there are only 2 pairs of valid points, which are the 2 intersection points of the circles with themselves.

Question 5

(a) Let $z_1, z_2 \in B(0,1)$, such that $f(z_1) = f(z_2)$.
This give us $z_1^2 + 2z_1 = z_2^2 + 2z_2$, i.e. $(z_1 - z_2)(z_1 + z_2 + 2) = 0$.
Since $|z_1| < 1$ and $|z_2| < 1$, by Triangle Inequality, we have $|2| \leq |z_1| + |z_2| + |z_1 + z_2 + 2|$, i.e. $|z_1 + z_2 + 2| \geq 2 - |z_1| - |z_2| > 0$. Thus $z_1 + z_2 + 2 \neq 0$, and so $z_1 - z_2 = 0$, i.e. $z_1 = z_2$.
This implies that f is injective on $B(0,1)$.

Let $r \in \mathbb{R}_{>1}$. Then, we have $-1 + \dfrac{r-1}{2}, -1 - \dfrac{r-1}{2} \in B(0,r)$.
However,

$$\begin{aligned}
f\left(-1 + \frac{r-1}{2}\right) &= \left(-1 + \frac{r-1}{2}\right)^2 + 2\left(-1 + \frac{r-1}{2}\right) \\
&= 1 - 2\left(\frac{r-1}{2}\right) + \left(\frac{r-1}{2}\right)^2 - 2 + 2\left(\frac{r-1}{2}\right) \\
&= \left(\frac{r-1}{2}\right)^2 - 1; \\
f\left(-1 - \frac{r-1}{2}\right) &= \left(-1 - \frac{r-1}{2}\right)^2 + 2\left(-1 - \frac{r-1}{2}\right) \\
&= 1 + 2\left(\frac{r-1}{2}\right) + \left(\frac{r-1}{2}\right)^2 - 2 - 2\left(\frac{r-1}{2}\right) \\
&= \left(\frac{r-1}{2}\right)^2 - 1,
\end{aligned}$$

and so f is not injective on $B(0,r)$.

Therefore, the largest possible r is 1.

(b) Firstly, let us justify that Re $f(z) \neq 0$ for all $z \in \Omega$.
Assume on the contrary that there exists $w \in \Omega$ such that Re $f(w) = 0$ (i.e. $f(w)$ is imaginary).
Then $|f(z) + 1| = \sqrt{|f(z)|^2 + 1^2} \geq 1$ and $|f(z) - 1| = \sqrt{|f(z)|^2 + 1^2} \geq 1$.
This give us $|f(w)^2 - 1| = |f(w) - 1||f(w) + 1| \geq 1$, a contradiction.

Since f is analytic on domain Ω, by Open Mapping Theorem, we have $f[\Omega]$ to be a domain.
Let $U = \{w \in \mathbb{C} : \text{Re } w > 0\}$ and $V = \{w \in \mathbb{C} : \text{Re } w < 0\}$, which are disconnected.
Since $f[\Omega] \subseteq U \cup V$, we have $f[\Omega] \subseteq U$ or $f[\Omega] \subseteq V$.

(c) Let $g : B(0,1) \to \mathbb{C}$ be such that $g(z) = \dfrac{f(z)}{z^3}$ for all $z \in B'(0,1)$; $g(0) = \dfrac{f'''(0)}{6}$.
Since f is analytic on $B(0,1)$ with a zero of order 3 at $z = 0$, we have g to be analytic on $B(0,1)$.

Assume on the contrary there exists $w \in B(0,1)$ such that $|g(w)| > 1$.
Since $|w| < 1$ and $\dfrac{1}{\sqrt[3]{|g(w)|}} < 1$, there exists $r \in \mathbb{R}^+$ such that $\max\left\{|w|, \dfrac{1}{\sqrt[3]{|g(w)|}}\right\} < r < 1$.
Then we have compact set $\overline{B(0,r)} \subseteq B(0,1)$, such that $w \in B(0,r)$ (the interior of $\overline{B(0,r)}$); and for all $z \in \partial B(0,r)$ (boundary of $B(0,r)$), we have,

$$|g(z)| = \frac{|f(z)|}{|z|^3} \leq \frac{1}{r^3} < |g(w)|,$$

a contradiction to the Maximum Modulus Principle on compact set.
Therefore, we conclude that $|g(z)| \leq 1$ for all $z \in B(0,1)$.
In particular, $|f'''(0)| = 6|g(0)| \leq 6$.

If $|f'''(0)| = 6$, then $|g(0)| = 1$, i.e. maximum modulus of g is obtained in $B(0,1)$.
Thus by Maximum Modulus Principle, g is constant in $B(0,1)$.
This implies that for all $z \in B(0,1)$, $g(z) = g(0) = e^{i\theta}$ for some $\theta \in \mathbb{R}$.
Thus $f(z) = g(z)z^3 = e^{i\theta}z^3$ for all $z \in B(0,1)$.