NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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ST2131/MA2216 Probability
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Question 1

(a) (i) Since (X, Y) have a bivariate standard normal distribution, we get $X, Y \sim N(0, 1)$. This give us $X^2 \sim \chi^2 = \Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$. Thus the p.d.f. of X^2 is,

$$f_{X^2}(y) = \begin{cases} \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} y^{-\frac{1}{2}} e^{-\frac{1}{2}y}, & y > 0; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Let $W = \dfrac{X^2 + Y^2}{2}$.
Since (X, Y) have a bivariate standard normal distribution, we also get $Y^2 \sim \Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$, also X and Y are independent r.v.. This give us $X^2 + Y^2 \sim \Gamma\left(\frac{1}{2} + \frac{1}{2}, \frac{1}{2}\right) = \Gamma\left(1, \frac{1}{2}\right) = \text{Exp}\left(\frac{1}{2}\right)$.
Thus $W = \dfrac{X^2 + Y^2}{2} \sim \text{Exp}(1)$.

(b) (i) We have $X \sim \text{Exp}(2)$, and for $x > 0$, we have $Y|(X = x) \sim U(0, x)$.
Thus when $x > y > 0$, we get $f_{(X,Y)}(x, y) = f_{Y|X}(y|x) f_X(x) = \left(\frac{1}{x}\right)\left(2e^{-2x}\right) = \frac{2}{x}e^{-2x}$.
Therefore the joint p.d.f. of X and Y is,

$$f_{(X,Y)}(x, y) = \begin{cases} \frac{2}{x}e^{-2x}, & 0 < y < x; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Since $Y|(X = x) \sim U(0, x)$, we have $E(Y \mid X = x) = \frac{0+x}{2} = \frac{x}{2}$, and so $E(Y \mid X) = \frac{X}{2}$.

(iii) Since $X \sim \text{Exp}(2)$, we have $E(Y) = E(E(Y \mid X)) = E\left(\frac{X}{2}\right) = \frac{1}{2}E(X) = \frac{1}{4}$.

(iv) We have,

$$\begin{aligned} E(XY) = E(E(XY \mid X)) &= E(X\ E(Y \mid X)) \\ &= E\left(\frac{X^2}{2}\right) \\ &= \frac{1}{2}\left[\text{Var}(X) + E(X)^2\right] \\ &= \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4}. \end{aligned}$$

Thus $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{1}{4} - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{8}$.

Question 2

(a) Let X be the r.v. of the number of cars driven to workshop in a day, i.e. $X \sim P(40)$.
Thus by referring to the statistical table, we have,

$$\begin{aligned} \mathbb{P}\{X \leq k\} &> 0.75 \\ \mathbb{P}\{X \geq k + 1\} &< 0.25 \\ k + 1 &\geq 45 \\ k &\geq 44. \end{aligned}$$

(b) M1: Let X be the r.v. of the number of heads appeared in 225 tosses.
Let X_i be the indicator r.v. that the i-th throw is a head, $i = 1, 2, \ldots, 225$.
Notice that $X_i \sim B(1, 0.5)$, $X = \sum_{i=1}^{225} X_i$, and the X_i's are mutually independent r.v..
Thus we can apply Central Limit Theorem to get,

$$\begin{aligned}\mathbb{P}\{X \leq 99\} = \mathbb{P}\{X < 99.5\} &= \mathbb{P}\left\{\frac{X - (225)(0.5)}{\sqrt{(0.25)(225)}} < \frac{99.5 - (225)(0.5)}{\sqrt{(0.25)(225)}}\right\} \\ &= \mathbb{P}\left\{Z < -1.73333\right\} = 4.15182 \times 10^{-2}.\end{aligned}$$

M2: Let X be the r.v. of the number of tosses to get 100 heads. Let X_i be the r.v. of the number of tosses from the $i-1$-th head to get the i-th head, $i = 1, 2, \ldots, 100$.
Notice that $X_i \sim \text{Geom}(0.5)$, $X = \sum_{i=1}^{100} X_i$, and the X_i's are mutually independent r.v..
Thus we can apply Central Limit Theorem to get,

$$\begin{aligned}\mathbb{P}\{X \geq 226\} = \mathbb{P}\{X > 225.5\} &= \mathbb{P}\left\{\frac{X - (100)(2)}{\sqrt{(2)(100)}} > \frac{225.5 - (100)(2)}{\sqrt{(2)(100)}}\right\} \\ &= \mathbb{P}\left\{Z > 1.80312\right\} = 3.56845 \times 10^{-2}.\end{aligned}$$

Note: Notice that both Method 1 and 2 are acceptable usage of the C.L.T., however the difference in answer occurred due to difference in the term approximated, and the degree of continuity correction. This is alright due to the fact that this question only asked for an estimation.

(c) (i) Since X and Y are independent r.v. with $X, Y \sim U(0, 1)$, for $0 < z < 1$, we have,

$$\begin{aligned}\mathbb{P}\{X \geq zY\} = \int_{\mathbb{R}} \mathbb{P}\{X \geq zy\}\, dy &= \int_0^1 \int_{zy}^1 1\, dx\, dy \\ &= \int_0^1 1 - zy\, dy \\ &= \left[y - \frac{zy^2}{2}\right]_0^1 = 1 - \frac{z}{2}.\end{aligned}$$

(ii) Since X, Y and Z are mutually independent r.v., using result of (2ci.), we get,

$$\begin{aligned}\mathbb{P}\{X \geq YZ\} = \int_{\mathbb{R}} \mathbb{P}\{X \geq zY\}\, dz &= \int_0^1 1 - \frac{z}{2}\, dz \\ &= \left[z - \frac{z^2}{4}\right]_0^1 = \frac{3}{4}.\end{aligned}$$

Question 3

(i) Let $w = x - y$, $z = \frac{x}{x-y}$. This give us $x = wz$, $y = w(z-1)$.
Given $0 < y < x < 1$, we have $w \in (0, 1)$, $z \in (1, \infty)$.
Thus together with $0 < w(z-1) < wz < 1$, we have $z < \frac{1}{w}$.
Now, $\dfrac{\partial w}{\partial x} = 1$, $\dfrac{\partial w}{\partial y} = -1$, $\dfrac{\partial z}{\partial x} = -\dfrac{y}{(x-y)^2}$, $\dfrac{\partial w}{\partial x} = \dfrac{x}{(x-y)^2}$, which give us $J(x, y) = \dfrac{1}{x-y}$.

Since $x-y>0$, $\dfrac{1}{|J(x,y)|}=x-y$. Thus,

$$\begin{aligned} f_{(W,Z)}(w,z) &= \frac{1}{|J(x,y)|} f_{(X,Y)}(x,y) \\ &= (x-y)(2) \\ &= 2w, \quad 0<w<1,\ 1<z<\frac{1}{w}. \end{aligned}$$

Therefore the joint p.d.f. of W and Z is,

$$f_{(W,Z)}(w,z) = \begin{cases} 2w, & 0<w<1,\ 1<z<\frac{1}{w}; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) For $0<w<1$, we have,

$$\begin{aligned} f_W(w) = \int_{\mathbb{R}} f_{(W,Z)}(w,z)\ dz &= \int_1^{\frac{1}{w}} 2w\ dz \\ &= 2w\left(\frac{1}{w}-1\right) = 2-2w. \end{aligned}$$

Thus the marginal p.d.f. of W is,

$$f_W(w) = \begin{cases} 2-2w, & 0<w<1; \\ 0, & \text{otherwise.} \end{cases}$$

(iii) For $1<z$, we have,

$$\begin{aligned} f_Z(z) = \int_{\mathbb{R}} f_{(W,Z)}(w,z)\ dw &= \int_0^{\frac{1}{z}} 2w\ dw \\ &= \left[w^2\right]_0^{\frac{1}{z}} = \frac{1}{z^2}. \end{aligned}$$

Thus the marginal p.d.f. of Z is,

$$f_Z(z) = \begin{cases} \frac{1}{z^2}, & 1<z; \\ 0, & \text{otherwise.} \end{cases}$$

(iv) W and Z are not independent, since $f_W(w)f_Z(z) = (2-2w)\left(\frac{1}{z^2}\right) \neq 2w = f_{(W,Z)}(w,z)$.

Question 4

(a) Let A and B be the events that the women is pregnant and the test is positive respectively. Thus we have,

$$\begin{aligned} \mathbb{P}(A \mid B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} &= \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B \mid A)\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\mathbb{P}(A^c)} \\ &= \frac{(0.99)(0.75)}{(0.99)(0.75)+(0.02)(0.25)} = 0.993311. \end{aligned}$$

(b) (i) Let X be the r.v. of the number of empty urns, and $Y_{(i,j)}$ be the r.v. that the i ball goes into the j urns. Then we have $X = \sum_{k=1}^{n} I_k$. Now for $k = 1, 2, \ldots, n$, we have,

$$\begin{aligned} E(I_k) = \mathbb{P}\left(\bigcap_{i=1}^{n} Y^c_{(i,k)}\right) &= \prod_{i=1}^{n} \mathbb{P}\left(Y^c_{(i,k)}\right) \\ &= \prod_{i=1}^{k-1} \mathbb{P}\left(Y^c_{(i,k)}\right) \cdot \prod_{i=k}^{n} \mathbb{P}\left(Y^c_{(i,k)}\right) \\ &= 1 \cdot \prod_{i=k}^{n} \frac{i-1}{i} = \frac{k-1}{n}. \end{aligned}$$

Thus $E(X) = \sum_{k=1}^{n} E(I_k) = \frac{1}{n}\sum_{k=1}^{n}(k-1) = \frac{1}{n}\left[\frac{n(n-1)}{2}\right] = \frac{n-1}{2}$.

(ii) For $2 \le j < k \le n$, we have,

$$\begin{aligned} \mathbb{P}\{I_j = 1,\ I_k = 1\} &= \mathbb{P}\left(\bigcap_{i=1}^{n} Y^c_{(i,j)} Y^c_{(i,k)}\right) \\ &= \prod_{i=1}^{n} \mathbb{P}\left(Y^c_{(i,j)} Y^c_{(i,k)}\right) \\ &= \prod_{i=1}^{j-1} \mathbb{P}\left(Y^c_{(i,j)} Y^c_{(i,k)}\right) \cdot \prod_{i=j}^{k-1} \mathbb{P}\left(Y^c_{(i,j)} Y^c_{(i,k)}\right) \cdot \prod_{i=k}^{n} \mathbb{P}\left(Y^c_{(i,j)} Y^c_{(i,k)}\right) \\ &= (1) \cdot \left(\prod_{i=j}^{k-1} \frac{i-1}{i}\right) \cdot \left(\prod_{i=k}^{n} \frac{i-2}{i}\right) \\ &= \frac{(j-1)(k-2)}{n(n-1)}. \end{aligned}$$

(iii) For $2 \le k \le 5$, we have $\mathrm{Var}(I_k) = \left(\frac{k-1}{5}\right)\left(\frac{5-k+1}{5}\right) = \frac{1}{25}(k-1)(6-k)$.
For $2 \le j < k \le 5$, using result of (4bii.), we have,

$$\begin{aligned} \mathrm{Cov}(I_j, I_k) = E(I_j I_k) - E(I_j)E(I_k) &= \frac{(j-1)(k-2)}{5(5-1)} - \left(\frac{j-1}{5}\right)\left(\frac{k-1}{5}\right) \\ &= \frac{-1}{100}(j-1)(6-k). \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{Var}(X) &= \sum_{k=1}^{5} \mathrm{Var}(I_k) + 2\sum_{1\le j<k\le 5} \mathrm{Cov}(I_j, I_k) \\ &= \sum_{k=2}^{5}\left(\mathrm{Var}(I_k) + 2\sum_{1\le j<k} \mathrm{Cov}(I_j, I_k)\right) \\ &= \sum_{k=2}^{5}(6-k)\left[\frac{1}{25}(k-1) + \frac{-2}{100}\sum_{1\le j<k}(j-1)\right] \\ &= \sum_{k=2}^{5}(6-k)\left[\frac{1}{25}(k-1) - \frac{(k-1)(k-2)}{100}\right] \\ &= \frac{1}{100}\sum_{k=2}^{5}(6-k)^2(k-1) \\ &= \frac{1}{100}\left((4)^2(1) + (3)^2(2) + (2)^2(3) + (1)^2(4)\right) = \frac{1}{2}. \end{aligned}$$

Question 5

(i) Let W_i be the r.v. of the number of success in i trials, and I_i be the indicator r.v. of success on the i-th trial, $i \in \mathbb{Z}^+$. This give us $W_i \sim B\left(i, \dfrac{1}{2}\right)$ and $I_i \sim B\left(1, \dfrac{1}{2}\right)$.
Thus from the condition on Y given, we see that

$$\begin{aligned}
\mathbb{P}\{Y = k\} &= \mathbb{P}\{W_{k-1} = r\} \cdot \mathbb{P}\{I_k = 1\} \\
&= \binom{k-1}{r}\left(\frac{1}{2}\right)^{k-1} \cdot \left(\frac{1}{2}\right) = \binom{k-1}{r}\left(\frac{1}{2}\right)^{k}.
\end{aligned}$$

(ii) Since the I_i's are independent r.v., we have for $r \leq k \leq n$,

$$\begin{aligned}
\mathbb{P}\{X = n+2 \mid Y = k+1\} &= \mathbb{P}\{I_{n+2} = 1\} \prod_{i=k+2}^{n+1} \mathbb{P}\{I_i = 0\} \\
&= \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)^{n-k} = \left(\frac{1}{2}\right)^{n-k+1}.
\end{aligned}$$

(iii) Since $X \sim NB\left(r+2, \frac{1}{2}\right)$, we have,

$$\begin{aligned}
\mathbb{P}\{X = n+2\} &= \sum_{k=r}^{n} \mathbb{P}\{X = n+2,\ Y = k+1\} \\
&= \sum_{k=r}^{n} \mathbb{P}\{X = n+2 \mid Y = k+1\} \cdot \mathbb{P}\{Y = k+1\} \\
\binom{n+1}{r+1}\left(\frac{1}{2}\right)^{n+2} &= \sum_{k=r}^{n} \left(\frac{1}{2}\right)^{n-k+1} \cdot \binom{k}{r}\left(\frac{1}{2}\right)^{k+1} \\
&= \sum_{k=r}^{n} \binom{k}{r}\left(\frac{1}{2}\right)^{n+2} \\
\binom{n+1}{r+1} &= \sum_{k=r}^{n} \binom{k}{r},
\end{aligned}$$

which is what we wanted.