NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA2214 Combinatorial Analysis
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Question 1

(a) (i) There are 5 vowels and 21 consonants.

A total of $2 \times 4 = 8$ consonants must be arranged between the 5 vowels in a straight line, leaving $21 - 8 = 13$ free consonants to be arranged with the 5 vowels. This can be done in $\binom{18}{5}$ ways if the vowels are identical among themselves and if the consonants are identical among themselves.

Since the vowels are distinct, there is a total of 5! ways for the vowels to be arranged among themselves. Since the consonants are distinct, there is a total of 21! ways for the consonants to be arranged among themselves.

Therefore, the total number of arrangements of the 26 letters under the given condition is

$$\binom{18}{5} \cdot 5! \cdot 21!.$$

(ii) Fix the vowel a at a position and consider the arrangements of the remaining 25 letters in a loop beginning from one side of a and ending at the other side of a. The total number of arrangements under the given condition is now equivalent to the total number of arrangements of these 25 letters (4 vowels and 21 consonants) in a straight line such that any two vowels must be separated by at least two consonants, the left-most vowel must be preceded by at least two consonants, and the right-most vowel must be succeeded by at least two consonants.

A total of $2 \times 5 = 10$ consonants must be arranged between, preceding and succeeding the 4 vowels in a straight line, leaving $21 - 10 = 11$ free consonants to be arranged with the 4 vowels. This can be done in $\binom{15}{4}$ ways if the vowels are identical among themselves and if the consonants are identical among themselves.

Since the vowels are distinct, there is a total of 4! ways for the vowels to be arranged among themselves. Since the consonants are distinct, there is a total of 21! ways for the consonants to be arranged among themselves.

Therefore, the total number of arrangements of the 26 letters under the given condition is

$$\binom{15}{4} \cdot 4! \cdot 21!.$$

(b) Let $S = \{1, 2, \ldots, n+1\}$, where $n \geq 5$, and let

$$T = \{(a, b, c, d, e, f) \in S^6 \mid a < f, b < f, c < f, d < f, e < f\}.$$

The first method to count $|T|$ is to consider the number of ways to assign values to (a, b, c, d, e) for each particular value of f, from 2 to $n + 1$ (there is no way to assign values to them when $f = 1$). For example, when f is 2, a, b, c, d, e can each hold only the value of 1, hence there is 1^5 way to assign values to (a, b, c, d, e). When f is 3, a, b, c, d, e can each hold either 1 or 2, hence there are 2^5 ways to assign values to (a, b, c, d, e). In general, when f is $r + 1$ where $1 \leq r + 1 \leq n + 1$, a, b, c, d, e can each hold integers from 1 to r, and hence there are r^5 ways to assign values to (a, b, c, d, e).

Therefore, by considering each value of f from 1 to $n+1$,

$$\begin{aligned}|T| &= 1^5 + 2^5 + \ldots + n^5 \\ &= \sum_{r=1}^{n} r^5.\end{aligned}$$

The second method to count $|T|$ is much more involved. When $n \geq 5$, there can be 2, 3, 4, 5 or 6 distinct integers amongst a, b, c, d, e and f. Considering the 5 different cases involved:

Case 1: There are 2 distinct integers amongst a, b, c, d, e and f. Let them be x_1 and x_2, where $x_1 < x_2$. There are $\binom{n+1}{2}$ ways to select them. By the definition of T, $f = x_2$. a, b, c, d and e are all x_1. Hence, the number of ways for this case is $\binom{n+1}{2}$.

Case 2: There are 3 distinct integers amongst a, b, c, d, e and f. Let them be x_1, x_2 and x_3, where $x_1 < x_2 < x_3$. There are $\binom{n+1}{3}$ ways to select them. By the definition of T, $f = x_3$. a, b, c, d and e can each be x_1 or x_2, of which there needs to be at least 1 x_1 and 1 x_2. Hence, the number of ways for this case is $\binom{n+1}{3}(2^5 - 2) = 30\binom{n+1}{3}$.

Case 3: There are 4 distinct integers amongst a, b, c, d, e and f. Let them be x_1, x_2, x_3 and x_4, where $x_1 < x_2 < x_3 < x_4$. There are $\binom{n+1}{4}$ ways to select them. By the definition of T, $f = x_4$. a, b, c, d and e can each be x_1, x_2 or x_3, of which there must be at least 1 of x_1 , 1 of x_2 and 1 of x_3. Hence the number of ways for this case is $\binom{n+1}{4}(\binom{3}{1} \times \frac{5!}{3!} + \binom{3}{2} \times \frac{5!}{2!2!}) = 150\binom{n+1}{4}$.

Case 4: There are 5 distinct integers amongst a, b, c, d, e and f. Let them be x_1, x_2, x_3, x_4 and x_5, where $x_1 < x_2 < x_3 < x_4 < x_5$. There are $\binom{n+1}{5}$ ways to select them. By the definition of T, $f = x_5$. a, b, c, d and e can each be x_1, x_2, x_3 or x_4, of which there must be at least 1 of x_1 , 1 of x_2, 1 of x_3 and 1 of x_4 (the remaining one being any one of x_1, x_2, x_3 or x_4). Hence the number of ways for this case is $\binom{n+1}{5}(\binom{4}{1} \times \frac{5!}{2!}) = 240\binom{n+1}{5}$.

Case 5: There are 6 distinct integers amongst a, b, c, d, e and f. Let them be x_1, x_2, x_3, x_4, x_5 and x_6, where $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$. There are $\binom{n+1}{6}$ ways to select them. By the definition of T, $f = x_6$. There is exactly 1 of x_1, x_2, x_3, x_4 and x_5 amongst a, b, c, d and e. Hence the number of ways for this case is $5!\binom{n+1}{6} = 120\binom{n+1}{6}$.

By considering all 5 cases above and by using the representation that $\binom{n}{k} = 0$ if $k < n$,

$$|T| = \binom{n+1}{2} + 30\binom{n+1}{3} + 150\binom{n+1}{4} + 240\binom{n+1}{5} + 120\binom{n+1}{6}, \quad n \geq 1.$$

And therefore,

$$\sum_{r=1}^{n} r^5 = \binom{n+1}{2} + 30\binom{n+1}{3} + 150\binom{n+1}{4} + 240\binom{n+1}{5} + 120\binom{n+1}{6}, \quad n \geq 1.$$

Question 2

(a) We have,

$$\begin{aligned}\frac{(5n+7)!}{(3n+5)!(2n+3)!} &= \frac{(5n+7)!}{(3n+5)!(2n+3)!}(2(3n+5) - 3(2n+3)) \\ &= 2\frac{(5n+7)!}{(3n+4)!(2n+3)!} - 3\frac{(5n+7)!}{(3n+5)!(2n+2)!} \\ &= 2\binom{5n+7}{2n+3} - 3\binom{5n+7}{3n+5}.\end{aligned}$$

$\binom{5n+7}{2n+3}$ is the number of ways to choose $2n+3$ objects from $5n+7$ distinct objects, while $\binom{5n+7}{3n+5}$ is the number of ways to choose $3n+5$ objects from $5n+7$ distinct objects. Since both of these are integers, any linear combination of them are integers too. Therefore, the given expression is an integer.

(b) *Note:* As mentioned verbally in the examination hall when I took this paper, this question is meant to be interpreted such that BOTH properties (i) and (ii) hold at the same time, and hence is not two separate question parts.

The solution of this question uses the Principle of Inclusion and Exclusion. From property (i), it is understood that the digits a_1, a_2, a_3 and a_4 can respectively hold the following values:

a_1		1	2	3	4	5	6		
a_2				3	4	5	6	7	8	9
a_3			2	3	4	5	6	7	8	
a_4	0		2		4		6		8	

Let

- S be the set of all possible combinations of 4 digit numbers $a_1a_2a_3a_4$ given property (i).
- P_1, P_2, P_3, P_4 be the properties that $a_1 = a_2$, $a_2 = a_3$, $a_3 = a_4$ and $a_4 = a_1$ respectively.
- $E(m)$ be the number of elements of S possessing exactly m of the 4 properties for $0 \le m \le 4$.
- $\omega(P_{i_1}P_{i_2}\dots P_{i_m})$ be the number of elements of S possessing the properties P_{i_1}, P_{i_2}, $\dots$, P_{i_m}, where $1 \le m \le 4$.
- $\omega(m) = \sum(\omega(P_{i_1}P_{i_2}\dots P_{i_m}))$, $\omega(0) = |S|$.

By observation of the table above and counting the number of elements of S which fits various combinations of the 4 properties above,

- $\omega(P_1) = 4 \times 7 \times 5 = 140$;
- $\omega(P_2) = 6 \times 6 \times 5 = 180$;
- $\omega(P_3) = 4 \times 6 \times 7 = 168$;
- $\omega(P_4) = 3 \times 7 \times 7 = 147$;
- $\omega(P_1P_2) = 4 \times 5 = 20$;
- $\omega(P_1P_3) = 4 \times 4 = 16$;
- $\omega(P_1P_4) = 2 \times 7 = 14$;
- $\omega(P_2P_3) = 3 \times 6 = 18$;
- $\omega(P_2P_4) = 6 \times 3 = 18$;
- $\omega(P_3P_4) = 3 \times 7 = 21$;
- $\omega(P_1P_2P_3) = 2$;
- $\omega(P_1P_2P_4) = 2$;
- $\omega(P_1P_3P_4) = 2$;
- $\omega(P_2P_3P_4) = 2$;
- $\omega(P_1P_2P_3P_4) = 2$.

Hence, $\omega(m)$ can be determined for $0 \le m \le 4$:

- $\omega(0) = |S| = 6 \times 7 \times 7 \times 5 = 1470$;
- $\omega(1) = \omega(P_1) + \omega(P_2) + \omega(P_3) + \omega(P_4) = 635$;

- $\omega(2) = \omega(P_1P_2) + \omega(P_1P_3) + \omega(P_1P_4) + \omega(P_2P_3) + \omega(P_2P_4) + \omega(P_3P_4) = 107$;
- $\omega(3) = \omega(P_1P_2P_3) + \omega(P_1P_2P_4) + \omega(P_1P_3P_4) + \omega(P_2P_3P_4) = 8$;
- $\omega(4) = \omega(P_1P_2P_3P_4) = 2$.

$E(0)$ is the desired solution since it is the number of 4-digit integers obeying property (i), which has exactly 0 of any two adjacent digits the same and first and last digit not the same too (and hence obeying property (ii)). Therefore, by the Principle of Inclusion and Exclusion:

$$\begin{aligned} E(0) &= \omega(0) - \omega(1) + \omega(2) - \omega(3) + \omega(4) \\ &= 1470 - 635 + 107 - 8 + 2 \\ &= 936. \end{aligned}$$

Question 3

(a) *Note:* This solution uses both the Principle of Inclusion and Exclusion as well as recurrence relations. The shortest pure PIE solution, in the author's perspective, will require 8 different applications, while this solution below requires just one. If there is a more elegant solution that only makes use of PIE and not recurrence relations, please contact us.

Let x_n be the number of n digit numbers comprising some or all of the six digits, namely 0, 1, 2, 3, 4 and 5, that does not contain any block of 21, and that does not begin with the number 0. The recurrence relation of x_n shall be determined. (We see that by symmetry, if the final condition is changed to that it can begin with the number 0 but does not begin with the number 1, the result will still be exactly x_n. This fact shall be used later).

Among the x_n integers, let y_n be the number of those that ends with 1. Constructing recurrence relations between x_n and y_n,

$$\begin{aligned} x_n &= 5x_{n-1} + y_n; && (1) \\ y_n &= 4x_{n-2} + y_{n-1}. && (2) \end{aligned}$$

The RHS of (1) represents the quantity of such numbers that ends with 0, 2, 3, 4 or 5 ($5x_{n-1}$) and that ends with 1 (y_n). The RHS of (2) represents the quantity of such numbers that ends with 01, 31, 41, 51 ($4x_{n-2}$) and that ends with 11 (y_{n-1}).

From (1),

$$y_n = x_n - 5x_{n-1}. \quad (3)$$

Substitute (3) into (2)

$$\begin{aligned} x_n - 5x_{n-1} &= 4x_{n-2} + x_{n-1} - 5x_{n-2} \\ x_n &= 6x_{n-1} - x_{n-2}. \end{aligned}$$

where $x_1 = 5$ and since $y_2 = 4$ and by (1), $x_2 = 5(5) + 4 = 29$.

Listing down some values of x_n of particular use for this solution:

- $x_1 = 5$;
- $x_2 = 29$;
- $x_3 = 6x_2 - x_1 = 169$;
- $x_4 = 6x_3 - x_2 = 985$;
- $x_5 = 6x_4 - x_3 = 5741$;

- $x_6 = 6x_5 - x_4 = 33461$;
- $x_7 = 6x_6 - x_5 = 195025$.

Next, let z_n be the number of n digit numbers comprising some or all of the six digits, namely 0, 1, 2, 3, 4 and 5, that does not contain any block of 21, that does not begin with the number 0, and that does not end with the number 2. (If the beginning digit condition is changed from 0 to 1, by symmetry the result will still be z_n)

It can be observed that

$$z_n = 4x_{n-1} + y_n.$$

since the RHS of the above equation represents the quantity of such numbers that ends with 0, 3, 4 or 5 ($4x_{n-1}$), and that ends with 1 (y_n). From (1), this can be further expressed as,

$$z_n = x_n - x_{n-1}.$$

where $z_1 = 4$.

Listing down some values of z_n of particular use for this solution:

- $z_1 = 4$;
- $z_2 = x_2 - x_1 = 24$;
- $z_3 = x_3 - x_2 = 140$;
- $z_4 = x_4 - x_3 = 816$;
- $z_5 = x_5 - x_4 = 4756$.

The Principle of Inclusion and Exclusion can now be implemented to find the values of a and b.

Let

- T be the set of all possible 7-digit integers in S that does not contain any block of 21;
- P_i by the property that the i-th digit of the integer in T is 1 AND the (i+1)-th digit of the integer in T is 2, for $1 \leq i \leq 6$;
- $E(m)$ be the number of elements of T possessing exactly m of the 6 properties for $0 \leq m \leq 6$;
- $\omega(P_{i_1}P_{i_2}\dots P_{i_m})$ be the number of elements of T possessing the properties P_{i_1}, P_{i_2}, $\dots$, P_{i_m}, where $1 \leq m \leq 6$;
- $\omega(m) = \sum(\omega(P_{i_1}P_{i_2}\dots P_{i_m}))$, $\omega(0) = |T|$.

(Again, integers in T cannot begin with the digit 0.)

The table below shows all possible combinations of properties from P_1 to P_6 (combinations labeled as Q) such that $\omega(Q) \neq 0$, i.e. any other combination of properties will yield the result $\omega(Q) = 0$, either because one or more of the digits is defined to be 1 and 2 at the same time which is not possible, or that properties force a block of 21 to exist, where the set T does not contain any such numbers. The resultant representation of the digits are displayed to the right of the combination of properties, where the numbers 1 and 2 are digits in the numbers which has been fixed as 1 or 2 respectively, and the underscores are positions which can contain any digit 0, 1, 2, 3, 4 or 5 as long as a block of 21 is not formed and as long as the number does not begin with 0.

P_1	1	2	_	_	_	_	_
P_2	_	1	2	_	_	_	_
P_3	_	_	1	2	_	_	_
P_4	_	_	_	1	2	_	_
P_5	_	_	_	_	1	2	_
P_6	_	_	_	_	_	1	2
P_1P_4	1	2	_	1	2	_	_
P_1P_5	1	2	_	_	1	2	_
P_1P_6	1	2	_	_	_	1	2
P_2P_5	_	1	2	_	1	2	_
P_2P_6	_	1	2	_	_	1	2
P_3P_6	_	_	1	2	_	1	2

Using the table above and by taking careful note of the definitions and the calculated values of x_n and z_n as mentioned above, we can now calculate each of the smaller cases involved in PIE.

- $\omega(P_1) = x_5 = 5741$;
- $\omega(P_2) = z_1x_4 = 3940$;
- $\omega(P_3) = z_2x_3 = 4056$;
- $\omega(P_4) = z_3x_2 = 4060$;
- $\omega(P_5) = z_4x_1 = 4080$;
- $\omega(P_6) = z_5 = 4756$;
- $\omega(P_1P_4) = z_1x_2 = 116$;
- $\omega(P_1P_5) = z_2x_1 = 120$;
- $\omega(P_1P_6) = z_3 = 140$;
- $\omega(P_2P_5) = z_1z_1x_1 = 80$;
- $\omega(P_2P_6) = z_1z_2 = 96$;
- $\omega(P_3P_6) = z_2z_1 = 96$.

Hence,

- $\omega(0) = |T| = x_7 = 195025$;
- $\omega(1) = \omega(P_1) + \omega(P_2) + \omega(P_3) + \omega(P_4) + \omega(P_5) + \omega(P_6) = 26633$;
- $\omega(2) = \omega(P_1P_4) + \omega(P_1P_5) + \omega(P_1P_6) + \omega(P_2P_5) + \omega(P_2P_6) + \omega(P_3P_6) = 648$;
- $\omega(3) = \omega(4) = \omega(5) = \omega(6) = 0$.

Note that $a = E(0)$ as $E(0)$ is the number of integers in S that do not contain a block of 21, that contains exactly 0 blocks of 12. Similarly, $b = E(1)$ as $E(1)$ is the number of integers in S that do not contain a block of 21, that contains exactly 1 block of 12. Therefore, by the Principle of Inclusion and Exclusion,

$$\begin{aligned} a &= E(0) \\ &= \omega(0) - \omega(1) + \omega(2) \\ &= 169040. \\ b &= E(1) \\ &= \omega(1) - 2\omega(2) \\ &= 25337. \end{aligned}$$

(b) (i) *Method 1:* Label the n distinct boxes Box 1 to Box n. For box 1, there are $\binom{3n}{3}$ ways to put 3 distinct objects into it. For box 2, there are $3n-3$ objects left to choose from, so there are $\binom{3n-3}{3}$ ways to put 3 distinct objects into it. This goes on until box n where there is 1 way to put the 3 distinct objects into the box since there is only 3 distinct objects left unchosen. Hence, the desired number of ways is

$$\begin{aligned} & \binom{3n}{3} \cdot \binom{3n-3}{3} \cdot \cdots \cdot \binom{3}{3} \\ = & \frac{(3n)!}{(3!)^n}. \end{aligned}$$

Method 2: Arrange the $3n$ distinct objects in the row, and make the assignment of the first 3 objects going into box 1, the next 3 going into box 2 and so on. Since the arrangement of the 3 objects in each of the n box do not matter, the desired number of ways is

$$\frac{(3n)!}{(3!)^n}.$$

(ii) The solution of this is simply the solution of (i) divided by $n!$ as the n boxes are now identical rather than distinct. Hence, the desired number of ways is

$$\frac{(3n)!}{(3!)^n n!}.$$

(iii) The desired number of ways is $F(3n, n)$, $n!$ times the Stirling number of the second kind, which is

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} (n-k)^{3n}.$$

(iv) The desired number of ways is $S(3n, n)$, the Stirling number of the second kind, which is

$$\frac{1}{n!}\sum_{k=0}^{n}(-1)^k \binom{n}{k} (n-k)^{3n}.$$

Question 4

(a) (i) *Note:* The integers as mentioned cannot begin with the number 0.

Among the a_n integers, let b_n be the number of those that ends with 1, and let c_n be the number of those that ends with 3. Note that in general, $b_n \neq c_n$. It can be shown later that, for example, $b_3 = 24$ but $c_3 = 23$.

Constructing 3 recurrence relations involving a_n, b_n and c_n,

$$\begin{aligned} a_n &= 4a_{n-1} + b_n + c_n; && (4) \\ b_n &= 4a_{n-2} + c_{n-1}; && (5) \\ c_n &= 3a_{n-2} + b_{n-1} + c_{n-1}. && (6) \end{aligned}$$

The RHS of (4) represents the quantity of relevant numbers that ends with 0, 2, 4 or 5 ($4a_{n-1}$), that ends with 1 (b_n) and that ends with 3 (c_n). The RHS of (5) represents the quantity of relevant numbers that end with 01, 21, 41 or 51 ($4a_{n-2}$) and that ends with 31 (c_{n-1}). The RHS of (6) represents the quantity of relevant numbers that ends with 03, 43 or 53 ($3a_{n-2}$), that ends with 13 (b_{n-1}) and that ends with 33 (c_{n-1}).

$a_1 = 5$ since a one digit number can be 1, 2, 3, 4 or 5. $a_2 = 28$ since for the last digit being 1 or 3, there are 4 options for the first digit and for the last digit being 0, 2, 4 or 5, there are 5 options for the first digit. By similar observation, $b_2 = 4$ and $c_2 = 4$. Hence by (5), $b_3 = 4a_1 + c_2 = 24$ and by (6), $c_3 = 3a_1 + b_2 + c_2 = 23$. Hence by (4), $a_3 = 4(28) + 24 + 23 = 159$. From (4),

$$b_n = a_n - 4a_{n-1} - c_n. \tag{7}$$

Substitute (7) into (5),

$$\begin{aligned} a_n - 4a_{n-1} - c_n &= 4a_{n-2} + c_{n-1} \\ a_n - 4a_{n-1} - 4a_{n-2} &= c_n + c_{n-1}. \end{aligned} \tag{8}$$

Substitute (7) into (6),

$$\begin{aligned} c_n &= 3a_{n-2} + a_{n-1} - 4a_{n-2} - c_{n-1} + c_{n-1} \\ c_n &= a_{n-1} - a_{n-2}. \end{aligned} \tag{9}$$

Substitute (9) into (8),

$$\begin{aligned} a_n - 4a_{n-1} - 4a_{n-2} &= a_{n-1} - a_{n-2} + a_{n-2} - a_{n-3} \\ a_n &= 5a_{n-1} + 4a_{n-2} - a_{n-3}. \end{aligned}$$

where $a_1 = 5$, $a_2 = 28$, $a_3 = 159$.

(ii) Using the above recurrence relation,

$$\begin{aligned} a_4 &= 5a_3 + 4a_2 - a_1 \\ &= 5(159) + 4(28) - 5 \\ &= 902. \\ a_5 &= 5a_4 + 4a_3 - a_2 \\ &= 5(902) + 4(159) - 28 \\ &= 5118. \\ a_6 &= 5a_5 + 4a_4 - a_3 \\ &= 5(5118) + 4(902) - 159 \\ &= 29039. \end{aligned}$$

(b) It is known that the derangement of n integers from 1 to n, D_n satisfies the following recurrence relation:

$$D_n = (n-1)(D_{n-1} + D_{n-2}) \qquad \text{, where } D_1 = 0 \text{ and } D_2 = 1.$$

From the above recurrence relation, $D_3 = (2)(1+0) = 2$. Also, $D_{n-1} = (n-2)(D_{n-2} + D_{n-3})$. By substitution,

$$\begin{aligned} D_n &= (n-1)((n-2)(D_{n-2} + D_{n-3}) + D_{n-2}) \\ &= (n-1)^2 D_{n-2} + (n-1)(n-2)D_{n-3}. \end{aligned}$$

where $D_1 = 0$, $D_2 = 1$ and $D_3 = 2$.

Question 5

(a) (i) A suitable ordinary generating function for a_n is,

$$\begin{aligned}
& \left(\frac{(1-x)^{-5}+(1+x)^{-5}}{2}\right)\left(\frac{(1-x)^{-3}-(1+x)^{-3}}{2}\right)(x^2)(1-x)^{-1} \\
= \ & \frac{1}{4}x^2(1-x)^{-1}\left((1-x)^{-8}-(1+x)^{-8}+(1-x)^{-3}(1+x)^{-5}-(1-x)^{-5}(1+x)^{-3}\right) \\
= \ & \frac{1}{4}x^2\left((1-x)^{-9}-(1-x)^{-1}(1+x)^{-8}+(1-x)^{-4}(1+x)^{-5}-(1-x)^{-6}(1+x)^{-3}\right) \\
= \ & \frac{1}{4}x^2\left(\sum_{r=0}^{\infty}\binom{r+8}{8}x^r-\left(\sum_{r=0}^{\infty}x^r\right)\left(\sum_{k=0}^{\infty}\binom{k+7}{7}(-1)^kx^k\right)\right. \\
& +\left(\sum_{r=0}^{\infty}\binom{r+3}{3}x^r\right)\left(\sum_{k=0}^{\infty}\binom{k+4}{4}(-1)^kx^k\right) \\
& \left.-\left(\sum_{r=0}^{\infty}\binom{r+5}{5}x^r\right)\left(\sum_{k=0}^{\infty}\binom{k+2}{2}(-1)^kx^k\right)\right).
\end{aligned}$$

(ii) a_n is the coefficient of x_n in the above ordinary generating function.

$$\begin{aligned}
a_n = \ & \frac{1}{4}\left(\binom{n-2+8}{8}-\sum_{k=0}^{n-2}(-1)^k\binom{k+7}{7}+\sum_{k=0}^{n-2}(-1)^k\binom{k+4}{4}\binom{n-2-k+3}{3}\right. \\
& \left.-\sum_{k=0}^{n-2}(-1)^k\binom{k+2}{2}\binom{n-2-k+5}{5}\right) \\
= \ & \frac{1}{4}\left(\binom{n+6}{8}-\sum_{k=0}^{n-2}(-1)^k\left(\binom{k+7}{7}+\binom{k+4}{4}\binom{n-k+1}{3}+\binom{k+2}{2}\binom{n-k+3}{5}\right)\right).
\end{aligned}$$

(b) (i) A suitable exponential generating function for a_n is,

$$\begin{aligned}
& (e^x-(1+x))^5 \\
= \ & e^{5x}-\binom{5}{1}e^{4x}(1+x)+\binom{5}{2}e^{3x}(1+x)^2-\binom{5}{3}e^{2x}(1+x)^3+\binom{5}{4}e^x(1+x)^4-(1+x)^5 \\
= \ & (e^{5x}-5e^{4x}+10e^{3x}-10e^{2x}+5e^x)+x(-5e^{4x}+20e^{3x}-30e^{2x}+20e^x) \\
& +x^2(10e^{3x}-30e^{2x}+30e^x)+x^3(-10e^{2x}+20e^x)+x^4(5e^x)-(1+x)^5.
\end{aligned}$$

(ii) Note that for any positive integer a, the x^n term of the expansion of e^{ax} is $\dfrac{a^n}{n!}$. Since a_n is $n!$ times the coefficient of x_n in the above exponential generating function,

$$\begin{aligned}
a_n = \ & (5^n-5\cdot 4^n+10\cdot 3^n-10\cdot 2^n+5)+n(-5\cdot 4^{n-1}+20\cdot 3^{n-1}-30\cdot 2^{n-1}+20) \\
& +(n)(n-1)(10\cdot 3^{n-2}-30\cdot 2^{n-2}+30)+(n)(n-1)(n-2)(-10\cdot 2^{n-3}+20) \\
& +(n)(n-1)(n-2)(n-3)(5).
\end{aligned}$$