NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1104 Multivariable Calculus
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Question 1

(i) Note that $\mathbf{r}'(t) = \langle 4t^3, 3t^2, 2t\rangle$; thus one has $\mathbf{r}'(1) = \langle 4, 3, 2\rangle$. So the equation of the tangent line at $t = 1$ (i.e. at $(x, y, z) = (1, 1, 1)$) is $x = 1 + 4t$, $y = 1 + 3t$, $z = 1 + 2t$, $t \in \mathbb{R}$.

(ii) Since the plane contain the points $(2, 3, 1)$ and $(1, 1, 1)$, it follows that the plane must contain the vector $\langle 2 - 1, 3 - 1, 1 - 1\rangle = \langle 1, 2, 0\rangle$ as well.
So a vector perpendicular to the plane is $\langle 4, 3, 2\rangle \times \langle 1, 2, 0\rangle = \langle -4, 2, 5\rangle$. Then the equation of the plane must satisfy the following equation:

$$\langle x - 1, y - 1, z - 1\rangle \cdot \langle -4, 2, 5\rangle = 0.$$

This gives us the equation of the plane to be $-4x + 2y + 5z = 3$.

(iii) First of all, we note that $\mathbf{s}(0) = \langle 0, 0, 0\rangle = \mathbf{r}(0)$.
Also, since at each point X along the curve C, the speed of Q passing through X is three times of that of P, it follows that if t_1 is the amount of time needed for Q to travel from Y to Z, where Y and Z are points on the curve C, then the amount of time needed for P to travel from Y to Z is $3t_1$.

In particular, if t is the amount of time needed for Q to travel from $\mathbf{s}(0)$ to $\mathbf{s}(t)$, then the amount of time needed for P to travel from $\mathbf{r}(0) = \mathbf{s}(0)$ to $\mathbf{s}(t)$ is $3t$.
So one has $\mathbf{s}(t) = \mathbf{s}(t + 0) = \mathbf{r}(3t + 0) = \mathbf{r}(3t) = \langle (3t)^4, (3t)^3, (3t)^2\rangle = \langle 81t^4, 27t^3, 9t^2\rangle$.

Question 2

(i) Note that $f(x, y)$ is undefined if and only if the denominator is equal to zero, i.e. $x^4 + 2x^2y^4 + 3y^8 = 0$. But $x^4 + 2x^2y^4 + 3y^8 = 0$ if and only if $x = y = 0$. So the domain of f is $\mathbb{R}^2\backslash\{(0, 0)\}$.

(ii) We shall prove by the two-path test that the limit $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Along the path $x = 0$, we see that $f(x, y) = f(0, y) = 0$ for all $y \neq 0$. So as (x, y) approaches $(0, 0)$ along the path $x = 0$, we have $f(x, y) \to 0$.
Along the path $x = y^2$, we see that $f(x, y) = f(y^2, y) = \frac{(y^2)^2y^4}{(y^2)^4+2(y^2)^2y^4+3y^8} = \frac{1}{6}$ for all $y \neq 0$. So as (x, y) approaches $(0, 0)$ along the path $x = y^2$, we have $f(x, y) \to \frac{1}{6}$.

Thus, by the two-path test, we see that the limit $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Question 3

(i) Let $f(x, y, z) = x^2+2xy^2-7x^3+3y+1$. Then one has $\nabla f(x, y, z) = \langle 2x+2y^2-21x^2, 4xy+3, 0\rangle$, so $\nabla f(1, 1, 1) = \langle 2(1) + 2(1)^2 - 21(1)^2, 4(1)(1) + 3, 0\rangle = \langle -17, 7, 0\rangle$.
Hence, the equation of the tangent plane at $P(1, 1, 1)$ must satisfy the following equation:

$$\langle x - 1, y - 1, z - 1\rangle \cdot \langle -17, 7, 0\rangle = 0.$$

This gives us the equation of the tangent plane at $P(1, 1, 1)$ to be $-17x + 7y + 10 = 0$.

(ii) From part (i), we see that a normal to the tangent plane is the vector $\langle -17, 7, 0\rangle$. As the line is perpendicular to the tangent plane in part (i), it follows that the line must be parallel to the vector $\langle -17, 7, 0\rangle$.
So the equation of the line that is perpendicular to the tangent plane in part (i) and passes through $P(1, 1, 1)$ is $x = 1 - 17t$, $y = 1 + 7t$, $z = 1$, $t \in \mathbb{R}$.

Question 4

From $f(x, y) = 100 - x^2 - y^2 + 2xy$, we get $\nabla f(x, y) = \langle -2x + 2y, -2y + 2x\rangle$.
Thus $\nabla f(1, 3) = \langle -2(1) + 2(3), -2(3) + 2(1)\rangle = \langle 4, -4\rangle$.

(i) For the directional derivative $D_{\mathbf{u}}f(1, 3)$ to be maximal, we must have $\mathbf{u}$ to be in the direction of $\nabla f(1, 3)$. As $|\nabla f(1, 3)| = |\langle 4, -4\rangle| = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$, it follows that the only unit vector $\mathbf{u}$ such that the directional derivative $D_{\mathbf{u}}f(1, 3)$ is maximal is $\mathbf{u} = \frac{1}{4\sqrt{2}}\langle 4, -4\rangle = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$.

(ii) For the directional derivative $D_{\mathbf{u}}f(1, 3)$ to be equal to zero, we must have $\mathbf{u}$ to be perpendicular to $\nabla f(1, 3) = \langle 4, -4\rangle$. Let $\mathbf{u} = \langle x, y\rangle$. Then one has $x^2 + y^2 = 1$, and
$D_{\mathbf{u}}f(1, 3) = \mathbf{u} \cdot \nabla f(1, 3) = \langle x, y\rangle \cdot \langle 4, -4\rangle = 4x - 4y = 0$, so this gives us $x = y$.

This would imply that $x^2 + y^2 = x^2 + x^2 = 2x^2 = 1$, so one has $x = \pm\frac{\sqrt{2}}{2}$. Correspondingly, this gives us $y = \pm\frac{\sqrt{2}}{2}$. So the only unit vectors $\mathbf{u}$ such that the directional derivative $D_{\mathbf{u}}f(1, 3)$ is equal to zero are $\mathbf{u} = \pm\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$.

Question 5

(i) We need to find the maximum and minimum values of $f(x, y, z)$, subject to the constraint $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{144} - 1 = 0$. Let $g(x, y, z) = \frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{144} - 1$. Then one has $\nabla f(x, y, z) = \langle 1, 1, 1\rangle$ and $\nabla g(x, y, z) = \left\langle \frac{2x}{9}, \frac{y}{8}, \frac{z}{72}\right\rangle$. By the Method of Lagrange Multipliers, one has

$$\begin{aligned} & \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \\ \Rightarrow\ & \langle 1, 1, 1\rangle = \lambda \left\langle \frac{2x}{9}, \frac{y}{8}, \frac{z}{72}\right\rangle \\ \Rightarrow\ & 1 = \frac{2\lambda x}{9} = \frac{\lambda y}{8} = \frac{\lambda z}{72} \\ \Rightarrow\ & x = \frac{9}{2\lambda},\ y = \frac{8}{\lambda},\ z = \frac{72}{\lambda}. \end{aligned}$$

This implies that $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{144} - 1 = \frac{1}{9}\left(\frac{9}{2\lambda}\right)^2 + \frac{1}{16}\left(\frac{8}{\lambda}\right)^2 + \frac{1}{144}\left(\frac{72}{\lambda}\right)^2 - 1 = \frac{169}{4\lambda^2} - 1 = 0$, so one has $\lambda = \pm\frac{13}{2}$.

When $\lambda = \frac{13}{2}$, one has $x = \frac{9}{13}$, $y = \frac{16}{13}$, $z = \frac{144}{13}$. So this gives us $f(x, y, z) = f\left(\frac{9}{13}, \frac{16}{13}, \frac{144}{13}\right) = \frac{9}{13} + \frac{16}{13} + \frac{144}{13} = 13$.
When $\lambda = -\frac{13}{2}$, one has $x = -\frac{9}{13}$, $y = -\frac{16}{13}$, $z = -\frac{144}{13}$. So this gives us $f(x, y, z) = f\left(-\frac{9}{13}, -\frac{16}{13}, -\frac{144}{13}\right) = -\frac{9}{13} - \frac{16}{13} - \frac{144}{13} = -13$.
So the maximal and minimal values of $f(x, y, z)$ on S are 13 and -13 respectively.

(ii) From part (i), we see that the plane $x + y + z = -13$ is tangent to the surface S at $\left(-\frac{9}{13}, -\frac{16}{13}, -\frac{144}{13}\right)$. So the distance from S to the plane $x + y + z = -100$ is equal to the distance between the planes $x + y + z = -13$ and $x + y + z = -100$, which is then equal to $\frac{-13-(-100)}{\sqrt{1^2+1^2+1^2}} = 29\sqrt{3}$.

(iii) Let A be the point whose coordinates are $\left(-\frac{9}{13},-\frac{16}{13},-\frac{144}{13}\right)$. Notice that an unit vector perpendicular to the planes $x+y+z=-100$ and $x+y+z=-13$ is $\frac{\sqrt{3}}{3}\langle 1,1,1\rangle$. So we may let B to be the point on the plane $x+y+z=-100$ such that the length of AB is equal to the distance computed in part (ii). We have

$$\overrightarrow{OB}=\overrightarrow{OA}-29\sqrt{3}\left(\frac{\sqrt{3}}{3}\langle 1,1,1\rangle\right)=\left\langle -\frac{9}{13},-\frac{16}{13},-\frac{144}{13}\right\rangle-\langle 29,29,29\rangle=\left\langle -\frac{386}{13},-\frac{393}{13},-\frac{521}{13}\right\rangle.$$

So the coordinates of B are $\left(\frac{386}{13},-\frac{393}{13},-\frac{521}{13}\right)$.

Question 6

Notice that for all $x\neq 0$, one has $f(x,0)=\frac{x(0)(x^2-0^2)}{x^2+0^2}=0$ and $f_y(x,0)=\frac{x(x^4-4(x^2)(0^2)-0^4)}{(x^2+0^2)^2}=x$, and for all $y\neq 0$, one has $f(0,y)=\frac{(0)(y)(0^2-y^2)}{0^2+y^2}=0$ and $f_x(0,y)=\frac{y(0^4+4(0^2)(y^2)-y^4)}{(0^2+y^2)^2}=-y$. Thus

(i) $f_x(0,0)=\lim\limits_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim\limits_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim\limits_{h\to 0}\frac{0-0}{h}=0$,

(ii) $f_y(0,0)=\lim\limits_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h}=\lim\limits_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim\limits_{h\to 0}\frac{0-0}{h}=0$,

(iii) $f_{xy}(0,0)=\lim\limits_{h\to 0}\frac{f_y(0+h,0)-f_y(0,0)}{h}=\lim\limits_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h}=\lim\limits_{h\to 0}\frac{h-0}{h}=1$,

(iv) $f_{yx}(0,0)=\lim\limits_{h\to 0}\frac{f_x(0,0+h)-f_x(0,0)}{h}=\lim\limits_{h\to 0}\frac{f_x(0,h)-f_x(0,0)}{h}=\lim\limits_{h\to 0}\frac{-h-0}{h}=-1$.

Question 7

Let R denote the region enclosed by the circle $(x-1)^2+(y+2)^2=1$. Then R is the disk $(x-1)^2+(y+2)^2\leq 1$. Let $x=1+r\cos\theta$ and $y=-2+r\sin\theta$. Then it follows that on R, one has $0\leq r\leq 1$ and $0\leq\theta\leq 2\pi$. Thus, by Green's Theorem, one has

$$\begin{aligned}
\oint_C (3xy+y^2)dx+(2xy+5x^2)dy &= \iint_R \frac{\partial}{\partial x}(2xy+5x^2)-\frac{\partial}{\partial y}(3xy+y^2)\,dA\\
&= \iint_R 7x\,dA\\
&= \int_0^{2\pi}\int_0^1 7(1+r\cos\theta)r\,dr\,d\theta\\
&= \int_0^{2\pi}\left[\frac{7r^2}{2}+\frac{7r^3\cos\theta}{3}\right]_0^1 d\theta\\
&= \int_0^{2\pi}\frac{7}{2}+\frac{7\cos\theta}{3}\,d\theta\\
&= \left[\frac{7\theta}{2}+\frac{7\sin\theta}{3}\right]_0^{2\pi}=7\pi.
\end{aligned}$$

Question 8

By differentating both sides of the equation with respect to t, one has

$$\begin{aligned}
\frac{d}{dt}(f(tx,ty,tz)) &= \frac{d}{dt}(t^5f(x,y,z))\\
\Rightarrow \frac{\partial f}{\partial(tx)}\frac{d(tx)}{dt}+\frac{\partial f}{\partial(ty)}\frac{d(ty)}{dt}+\frac{\partial f}{\partial(tz)}\frac{d(tz)}{dt} &= 5t^4f(x,y,z)\\
\Rightarrow x\frac{\partial f}{\partial(tx)}+y\frac{\partial f}{\partial(ty)}+z\frac{\partial f}{\partial(tz)} &= 5t^4f(x,y,z).
\end{aligned}$$

By setting $t = 1$ in the last equation, one has $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 5f(x, y, z)$.
So $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = Cf$ for some constant C as desired, and $C = 5$.

Question 9

(i) By the given change of coordinates, we see that the image of the region E under the change of coordinates is $G = \{(u, v, w)|u^2 + v^2 + w^2 \leq 1\}$, which is the unit sphere about the origin. We have $\frac{\partial x}{\partial u} = 3u^2$, $\frac{\partial y}{\partial v} = 3v^2$, $\frac{\partial z}{\partial w} = 3w^2$, $\frac{\partial x}{\partial v} = \frac{\partial x}{\partial w} = \frac{\partial y}{\partial u} = \frac{\partial y}{\partial w} = \frac{\partial z}{\partial u} = \frac{\partial z}{\partial v} = 0$. Thus, the Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 3u^2 & 0 & 0 \\ 0 & 3v^2 & 0 \\ 0 & 0 & 3w^2 \end{vmatrix} = 27u^2v^2w^2.$$

Thus, one has

$$\text{Volume of } E = \iiint_E dx\,dy\,dz = \iiint_G \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\,dv\,dw = \iiint_G 27u^2v^2w^2\,du\,dv\,dw,$$

so $f(u, v, w) = 27u^2v^2w^2$.

(ii) By converting to spherical coordinates, i.e. $u = \rho\sin\phi\cos\theta$, $v = \rho\sin\phi\sin\theta$ and $w = \rho\cos\phi$, we see that one has $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. So one has

$$\begin{aligned}
\text{Volume of } E &= \iiint_G 27u^2v^2w^2\,du\,dv\,dw \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^1 27(\rho\sin\phi\cos\theta)^2(\rho\sin\phi\sin\theta)^2(\rho\cos\phi)^2 \cdot \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta\int_0^{\pi}\cos^2\phi\sin^5\phi\,d\phi\int_0^1 27\rho^8\,d\rho \\
&= \int_0^{2\pi}(1-\sin^2\theta)\sin^2\theta\,d\theta\int_0^{\pi}(1-\sin^2\phi)\sin^5\phi\,d\phi \cdot \left[3\rho^9\right]_0^1 \\
&= 3\left(\int_0^{2\pi}\sin^2\theta\,d\theta - \int_0^{2\pi}\sin^4\theta\,d\theta\right)\left(\int_0^{\pi}\sin^5\phi\,d\phi - \int_0^{\pi}\sin^7\phi\,d\phi\right) \\
&= 3\left(4\int_0^{\frac{\pi}{2}}\sin^2\theta\,d\theta - 4\int_0^{\frac{\pi}{2}}\sin^4\theta\,d\theta\right)\left(2\int_0^{\frac{\pi}{2}}\sin^5\phi\,d\phi - 2\int_0^{\frac{\pi}{2}}\sin^7\phi\,d\phi\right) \\
&= 24\left(\frac{\pi}{2}\cdot\frac{1}{2} - \frac{\pi}{2}\cdot\frac{1\cdot 3}{2\cdot 4}\right)\left(\frac{2\cdot 4}{1\cdot 3\cdot 5} - \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7}\right) = \frac{4\pi}{35}.
\end{aligned}$$

Question 10

(i) Let $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ where $P(x, y, z) = z^2 + Axy$, $Q(x, y, z) = x^2$ and $R(x, y, z) = Bxz$. Clearly, $\mathbf{F}$ is a field whose component functions have continuous first partial derivatives.
As $\mathbf{F}$ is conservative, one has $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, so this implies that $\frac{\partial P}{\partial y} = Ax = 2x = \frac{\partial Q}{\partial x}$, and $\frac{\partial P}{\partial z} = 2z = Bz = \frac{\partial R}{\partial x}$. By comparing coefficients, we have $A = B = 2$.

(ii) Since $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$, we have $f_x(x, y, z) = z^2 + 2xy$, $f_y(x, y, z) = x^2$, $f_z(x, y, z) = 2xz$.

By integrating f_x w.r.t. x, we get $f(x, y, z) = xz^2 + x^2y + g(y, z)$, where g is some function of y and z with continuous first partial derivatives.

By differentiating the above w.r.t. y, we get $f_y(x,y,z) = x^2 + g_y(y,z)$, so one has $g_y(y,z) = 0$. By integrating g_y w.r.t. y, we get $g(y,z) = h(z)$, where h is some continuously differentiable function of z. This implies that $f(x,y,z) = xz^2 + x^2y + h(z)$.
By differentiating the above w.r.t. z, we get $f_z(x,y,z) = 2xz + h'(z)$, so one has $h'(z) = 0$. Hence one has $h(z) = C$ for some constant C so this implies that $f(x,y,z) = xz^2 + x^2y + C$.

So a function f that satisfies the equation $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$ is $f(x,y,z) = xz^2 + x^2y$.

(iii) We have the starting point to be $\mathbf{r}(0) = (2 + 0^2 + 0^5, 1 + 3(0) - 5(0)^3, 3 - 3(0)) = (2,1,3)$, and the ending point to be $\mathbf{r}(1) = (2 + 1^2 + 1^5, 1 + 3(1) - 5(1)^3, 3 - 3(1)) = (4,-1,0)$. As $\mathbf{F}$ is conservative, it follows from the Fundamental Theorem for Line Integrals that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(4,-1,0) - f(2,1,3) = \big(4(0)^2 + 4^2(-1)\big) - \big(2(3)^2 + 2^2(1)\big) = -38.$$

Question 11

(i) curl $\mathbf{F} = \left\langle \frac{\partial}{\partial y}(ze^{x+y}) - \frac{\partial}{\partial z}(\sin(yz)), \frac{\partial}{\partial z}(2z - x^3) - \frac{\partial}{\partial x}(ze^{x+y}), \frac{\partial}{\partial x}(\sin(yz)) - \frac{\partial}{\partial y}(2z - x^3) \right\rangle = \langle ze^{x+y} - y\cos(yz), 2 - ze^{x+y}, 0\rangle$.

(ii) By Stokes' Theorem, one has

$$\iint_S (\text{curl } \mathbf{F}\cdot \mathbf{n}) d\sigma = \oint_C \mathbf{F}\cdot d\mathbf{r},$$

where C denotes the ellipse $3x^2 + 5y^2 = 15$ in the $xy-$plane with a positive orientation. Let S' denote the surface $3x^2 + 5y^2 \le 15$ in the $xy-$plane (i.e. $z = 0$). Then it follows that the boundary curve of S' is the curve C. Then on the surface S', one has

$$(\text{curl } \mathbf{F})\cdot \mathbf{k} = \langle ze^{x+y} - y\cos(yz), 2 - ze^{x+y}, 0\rangle \cdot \langle 0,0,1\rangle = 0.$$

Therefore, by Stokes' Theorem again, one has

$$\iint_S (\text{curl } \mathbf{F}\cdot \mathbf{n}) d\sigma = \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S'} (\text{curl } \mathbf{F}\cdot \mathbf{k}) d\sigma = \iint_{S'} 0\, d\sigma = 0.$$

Question 12

We have

$$\begin{aligned} 7f &= \text{div}(f\nabla f) \\ &= \text{div}(\langle ff_x, ff_y, ff_z\rangle) \\ &= (ff_x)_x + (ff_y)_y + (ff_z)_z \\ &= f_x^2 + ff_{xx} + f_y^2 + ff_{yy} + f_z^2 + ff_{zz} \\ &= f_x^2 + f_y^2 + f_z^2 + f((f_x)_x + (f_y)_y + (f_z)_z) \\ &= |\nabla f|^2 + f\text{div}(\nabla f) = 3f + f\text{div}(\nabla f). \end{aligned}$$

This implies that $f\text{div}(\nabla f) = 4f$ so one has $\text{div}(\nabla f) = 4$. Let B denote the unit sphere about the origin. Then, by Divergence Theorem, one has

$$\iint_S \nabla f\cdot \mathbf{n}\, d\sigma = \iiint_B \text{div}(\nabla f)\, dV = \iiint_B 4\, dV = 4\times \text{Volume of } B = 4\cdot \frac{4\pi}{3} = \frac{16\pi}{3}.$$