NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Teo Wei Hao

MA3205 Set Theory
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SECTION A

Question 1

(a) Let $\beta \in \bigcup \alpha$, i.e. there exists $\gamma \in \alpha$ such that $\beta \in \gamma$. Since α is transitive, and so $\beta \in \alpha$.
This give us $\bigcup \alpha \subseteq \alpha$.

(b) Let us be given that α is not a limit ordinal.
Then $\exists \beta \in \text{Ord}$ such that $S(\beta) = \alpha$, i.e. $\beta \in \alpha$, and for all $\gamma \in \alpha$, we have $\gamma \in \beta$ or $\gamma = \beta$.
Since $(\alpha, \in)$ is a linearly ordered set, there does not exists $\gamma \in \alpha$ such that $\beta \in \gamma$.
Therefore $\beta \notin \bigcup \alpha$. This give us $\bigcup \alpha \neq \alpha$.

Instead let us be given that $\bigcup \alpha \neq \alpha$.
From (1a.), we conclude that $\alpha \nsubseteq \bigcup \alpha$, i.e. there exists $\beta \in \alpha$ such that $\beta \notin \bigcup \alpha$.
If $\gamma \in \alpha$, then we have $\beta \notin \gamma$, and since $(\alpha, \in)$ is a linearly ordered set, we have $\gamma \in \beta$ or $\gamma = \beta$.
Thus $\alpha \subseteq \beta \cup \{\beta\}$.
Instead if $\gamma \in \beta$ or $\gamma = \beta$, then since $\beta \in \alpha$, and α is transitive, we get $\gamma \in \alpha$. Thus $\beta \cup \{\beta\} \subseteq \alpha$.
Therefore $\alpha = \beta \cup \{\beta\} = S(\beta)$, i.e. α is not a limit ordinal.

(c) Let $\beta \in \mathcal{P}(\alpha)$. Since α is transitive, for all $\gamma \in \beta$, we get $\gamma \in \alpha$, and from that we get $\gamma \subseteq \alpha$.
Thus $\gamma \in \mathcal{P}(\alpha)$, and we conclude that $\mathcal{P}(\alpha)$ is transitive.

Question 2

(a) Assume on the contrary that there exists $\alpha \in \text{Ord}$ such that $\alpha \cup \{\alpha\} = S(\alpha) = \omega$.
Then since ω is infinite and $\{\alpha\}$ is finite, we have $|\omega| = |\alpha \cup \{\alpha\}| = |\alpha|$.
This give us α to be infinite, contradicting the fact that ω is the first infinite ordinal.
Thus ω is a limit ordinal.

(b) By Axiom of Foundation, there exists $\beta \in \omega$ such that for all $\gamma \in \omega$, we have $\gamma \notin \beta$.
Since ω is transitive, we have $\beta = \emptyset$, and so $\emptyset \in \omega$.

Let $\alpha \in \omega$. Since ω is a limit ordinal, we have $S(\alpha) \neq \omega$. Also, we cannot have $\omega \notin S(\alpha)$, or else it give us $\omega \in \alpha$ or $\omega = \alpha$, a contradiction to the Axiom of Foundation.
Since ordinals are comparable, we conclude that $S(\alpha) \in \omega$.

Thus ω is an inductive set.

(c) Assume on the contrary that $\omega \nsubseteq A$. Then $X = \omega - A$ is non-empty. Since $(\omega, \in)$ is a well-ordered set, there exists $\alpha \in X$ which is the least element of X. As A is inductive, $\emptyset \in A$ and so $\alpha \neq \emptyset$. Since all finite ordinals except $\emptyset$ are successor ordinals, there exists $\beta \in \omega$ such that $S(\beta) = \alpha$. By our condition on α, we have $\beta \in A$. However A is inductive, which give us $\alpha = S(\beta) \in A$, a contradiction that $\alpha \in X$. Therefore $\omega \subseteq A$.

Question 3

(a) Since A is countable, there exists a bijection $g : A \to \omega$.
Let $f : \mathcal{P}(A) \to \mathcal{P}(\omega)$ be such that $f(X) = g[X]$.
Since g is a bijection, $g[X] = g[Y]$ implies $X = Y$, i.e. f is injective.
For all $Y \subseteq \omega$, we have $f(g^{-1}[Y]) = Y$, and so f is surjective.
Therefore f is bijective, i.e. $|\mathcal{P}(\omega)| = |\mathcal{P}(A)|$.

(b) Since $\mathbb{Q}$ is countable, and $\mathbb{R}$ is uncountable, we have $\mathbb{R} - \mathbb{Q}$ to be uncountable.

(c) Let $f : (0,1) \to (1,17)$ be such that $f(x) = 16x + 1$. Since f is linear, it is injective. Also for all $y \in (1,17)$, we have $\frac{y-1}{16} \in (0,1)$ with $f(\frac{y-1}{16}) = y$, and so f is surjective.
Therefore f is bijective, i.e. $|(0,1)| = |(1,17)|$.

Let $g : [0,1] \to (0,1)$ be a well-defined injective function such that

$$g(x) = \begin{cases} \dfrac{1}{2+x}, & x = 0; \\ \dfrac{x}{1+2x}, & x = \dfrac{1}{k},\ k \in \mathbb{Z}^+; \\ x, & \text{otherwise.} \end{cases}$$

If $y = \frac{1}{k}$ for some $k \in \mathbb{N} - \{0,1,2\}$, then $\frac{y}{1-2y} \in [0,1]$ such that $g(\frac{y}{1-2y}) = y$.
If $y = \frac{1}{2}$, then $g(0) = y$.
Otherwise, $g(y) = y$. Thus g is surjective.
Therefore g is bijective, i.e. $|(0,1)| = |[0,1]|$.

Question 4

(a) If $|A| < |B|$ and $|B| \le |C|$, then there exists injective functions $f : A \to B$ and $g : B \to C$.
This give us $gf : A \to C$ to be injective, i.e. $|A| \le |C|$.
Assume on the contrary that there exists an injective function $h : C \to A$.
This give us $hg : B \to A$ to be an injective function, a contradiction to $|A| < |B|$.
Therefore $|A| < |C|$.

(b) If $|A| \le |B|$ and $|B| < |C|$, then there exists injective functions $f : A \to B$ and $g : B \to C$.
This give us $gf : A \to C$ to be injective, i.e. $|A| \le |C|$.
Assume on the contrary that there exists an injective function $h : C \to A$.
This give us $fh : C \to B$ to be an injective function, a contradiction to $|B| < |C|$.
Therefore $|A| < |C|$.

(c) Let $f : A \times B \to B \times A$ be such that $f(a,b) = (b,a)$. f is bijective, and so $|A \times B| = |B \times A|$.

SECTION B

Question 5

(a) If $\omega_1 \notin \text{Card}$, then there exists $\alpha \in \omega_1$ such that $\alpha \in \text{Card}$ and $\alpha = |\omega_1|$. However, this would implies that α is uncountable, a contradiction to the condition that ω_1 is the first uncountable ordinal.

(b) Let $f : \omega_1 \to S$ be such that $f(\alpha) = \alpha + 1$. By definition of S, f is a bijection, and so $|S| = |\omega_1|$.

Since $T \subseteq \omega_1$, $|T| \leq |\omega_1|$.
For all $\alpha \in \text{Ord}$, we have $\omega \cdot \alpha \notin S$, since it is a limit ordinal. Now if $\alpha \in \omega_1$, then α is countable.
Thus $\omega \cdot \alpha$ is a countable ordinal (since $|\omega \cdot \alpha| = |\omega \times \alpha| = \omega$), and so $\omega \cdot \alpha \in \omega_1$.
Therefore we have the well-defined function $g : \omega_1 \to T$ such that $g(\alpha) = \omega \cdot \alpha$.

Let $\alpha, \beta \in \omega_1$ such that WLOG, $\alpha < \beta$. Then there exists $\gamma \in \text{Ord}$ such that $\alpha + \gamma = \beta$.
This give us $\omega \cdot \alpha < \omega \cdot \alpha + \omega \cdot \gamma = \omega \cdot (\alpha + \gamma) = \omega \cdot \beta$, and so g is injective. Thus $|\omega_1| \leq |T|$.
Therefore by Cantor-Bernstein Theorem, $|\omega_1| = |T|$.

(c) Since we have $|\omega| < |\mathcal{P}(\omega)|$, $\mathcal{P}(\omega)$ is uncountable.
By consequence of Axiom of Choice, there exists uncountable $\alpha \in \text{Card}$ such that $\alpha = |\mathcal{P}(\omega)|$.
Since ω_1 is the first uncountable cardinal, $|\omega_1| = \omega_1 \leq \alpha = |\mathcal{P}(\omega)|$.

Question 6

Let us be given that there exists $\kappa \in \text{Card}$ such that $|\kappa| = |\mathbb{R}|$.
Then this give us a bijection $f : \mathbb{R} \to \kappa$.
Let us define the relation $\sqsubset$ on $\mathbb{R}$, such that for $r_1, r_2 \in \mathbb{R}$, we have $r_1 \sqsubset r_2$ iff $f(r_1) \in f(r_2)$.
This resulted in f being the isomorphism between $(\mathbb{R}, \sqsubset)$ and $(\kappa, \in)$.
Thus $(\mathbb{R}, \sqsubset)$ is a well-ordered set, i.e. $\mathbb{R}$ is well orderable.

Instead let us be given that $\mathbb{R}$ is well orderable, i.e. $(\mathbb{R}, \sqsubset)$ is a well-order.
Let define $g : \mathbb{R} \to V$ recursively, such that $g(x) = \bigcup\{S(g(y)) \mid y \sqsubset x\}$.
Notice that for all $x, y \in \mathbb{R}$, if $y \sqsubset x$, then $g(y) \in g(x)$, thus g is a order-preserving function.
This implies that g is injective, and so $(g[\mathbb{R}], \in)$ is isomorphic to $(\mathbb{R}, \sqsubset)$.
Let $\alpha \in g[\mathbb{R}]$, then there exists $x \in \mathbb{R}$ such that $g(x) = \alpha$.
Let $\beta \in \alpha$, i.e. $\beta \in S(g(y))$ for some $y \sqsubset x$.
Then the set $X = \{y \in \mathbb{R} \mid y \sqsubset x \wedge \beta \in S(g(y))\}$ is non-empty.
Since $(\mathbb{R}, \sqsubset)$ is a well-ordered set, there exists $b \in X$ such that b is the least element of X.
Assume on the contrary that $\beta \in g(b)$. Then $\beta \in g(c)$ for some $c \sqsubset b$, a contradiction to the condition on b. Therefore from $\beta \in S(g(b)) = g(b) \cup \{g(b)\}$, we conclude that $g(b) = \beta$, i.e. $\beta \in g[\mathbb{R}]$. Axiom of Replacement give us $g[\mathbb{R}]$ to be a set, and so we can conclude that it is transitive. Thus $g[\mathbb{R}] \in \text{Ord}$. By definition of cardinals, there exists $\kappa \in \text{Card}$ such that $\kappa = |g[\mathbb{R}]|$.
Therefore we have $|\kappa| = |\mathbb{R}|$.

Question 7

(a) Since A is countable, there exists an bijection between A and $\mathbb{N}$.
This bijection give us an enumeration of A, i.e. we can write $A = \{a_0, a_1, a_2, \ldots\}$.

Now we define a function $f : A \to \mathbb{Q} \cap (0, 1)$ recursively, such that,

$$f(a_k) \quad = \quad \frac{\sup(\{0\} \cup \{f(a_i) \mid a_i <_A a_k,\ i \in k\}) + \inf(\{1\} \cup \{f(a_i) \mid a_i >_A a_k,\ i \in k\})}{2}, \quad k \in \mathbb{N}.$$

Notice that in this definition above, $0 < a_k < 1$ for any $k \in \mathbb{N}$, also $f(a_k) > f(a_i)$ for any $a_k >_A a_i$, and $f(a_k) < f(a_i)$ for any $a_k <_A a_i$, $i \in k$. Thus f is a well-defined, order-preserving function.
Since $<_A$ is a linear order, we have f to be injective.
So by letting $B = f[A] \subseteq \mathbb{Q} \cap (0, 1)$, and $g : A \to B$ such that $g(a) = f(a)$ for all $a \in A$, we get g to be an isomorphism between $(A, <_A)$ and $(B, <)$, i.e. $(A, <_A) \cong (B, <)$.

(b) Since $\alpha < \omega_1$, we have α to be countable, and so $(\alpha, \in_\alpha)$ is a countable linearly ordered set. Thus using result of (7a.), we get a set $A_\alpha \in \mathbb{Q} \cap (0,1)$ such that $(A_\alpha, <) \cong (\alpha, \in_\alpha)$.

Let $f : A_\alpha \to \alpha$ be an isomorphism, and let $\emptyset \neq X \subseteq A_\alpha$.
Then since α is well-ordered, and $\emptyset \neq f[X] \subseteq \alpha$, there exists a least element of $f[X]$, say z. This implies that there exists $y \in X$ such that $f(y) = z$. Since f is an isomorphism, y is the least element of X, and so A_α is well-ordered by $<$.