NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA2108 Mathematical Analysis I
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SECTION A

Question 1

When $x \geq 1$, we have $|2x+1| - |x-1| - 1 = (2x+1) - (x-1) - 1 = x+1$.
In this case, $|2x+1| < |x-1| - 1$ when $x+1 < 0$, which yield no answer for x.

When $\frac{-1}{2} \leq x < 1$, we have $|2x+1| - |x-1| - 1 = (2x+1) + (x-1) - 1 = 3x-1$.
Thus $|2x+1| < |x-1| - 1$ when $3x-1 < 0$, which give us $\frac{-1}{2} \leq x < \frac{1}{3}$.

When $x < \frac{-1}{2}$, we have $|2x+1| - |x-1| - 1 = -(2x+1) + (x-1) - 1 = -x-3$.
Thus $|2x+1| < |x-1| - 1$ when $-x-3 < 0$, which give us $= 3 < x < \frac{-1}{2}$.

Therefore $S = \left\{x \in \mathbb{R} \mid -3 < x < \frac{1}{3}\right\}$, and so $\inf S = -3$, $\sup S = \frac{1}{3}$.

Question 2

(a) (i) The series converges.
We have $0 \leq \dfrac{1}{n} - \dfrac{1}{n+1} = \dfrac{1}{n(n+1)} \leq \dfrac{1}{n^2}$.
Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we have $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{1+n}\right)$ to converges by Comparison Test.

(ii) The series converges.
Let $a_n = n\left(\dfrac{3n}{1+3n}\right)^{n^2}$. Then we have,

$$\begin{aligned}
\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \left(n\left(\frac{3n}{1+3n}\right)^{n^2}\right)^{\frac{1}{n}} &= \lim_{n\to\infty} n^{\frac{1}{n}} \frac{1}{\left[\left(1+\frac{1}{3n}\right)^{3n}\right]^{\frac{1}{3}}} \\
&= \left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)\left(\lim_{n\to\infty} \frac{1}{\left[\left(1+\frac{1}{3n}\right)^{3n}\right]^{\frac{1}{3}}}\right) \\
&= 1 \cdot \frac{1}{e^{\frac{1}{3}}} < 1.
\end{aligned}$$

Therefore the series is convergent by Root Test.

(b) (i) Since $c_n = \max\{a_n, b_n\}$, $a_n \geq 0$ and $b_n \geq 0$, we have $0 \leq c_n \leq a_n + b_n$ for all $n \in \mathbb{Z}^+$.
Since $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ converges, we have $\displaystyle\sum_{n=1}^{\infty} c_n$ to converges by Comparison Test.

(ii) The series $\sum_{n=1}^{\infty} c_n$ need not necessarily converge without the assumptions.

Let $a_n = (-1)^n \frac{1}{n}$ and $b_n = (-1)^{n+1}\frac{1}{n}$ for all $n \in \mathbb{Z}^+$. $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent.

However this give us $c_n = \frac{1}{n}$ for all $n \in \mathbb{Z}^+$, and so $\sum_{n=1}^{\infty} c_n$ is not convergent.

Question 3

(a) We have $a_i \geq 1$ for all $i \in \mathbb{Z}^+$. Thus we have,

$$\begin{aligned}
|a_{n+2} - a_{n+1}| = \left|\sqrt{5a_{n+1}+6} - \sqrt{5a_n+6}\right| &= \left|\frac{(5a_{n+1}+6)-(5a_n+6)}{\sqrt{5a_{n+1}+6}+\sqrt{5a_n+6}}\right| \\
&= \frac{5}{\sqrt{5a_{n+1}+6}+\sqrt{5a_n+6}}|a_{n+1}-a_n| \\
&\leq \frac{5}{2\sqrt{5+6}}|a_{n+1}-a_n|.
\end{aligned}$$

Since $\frac{5}{2\sqrt{5+6}} < 1$, (a_n) is a contractive sequence, which is a Cauchy sequence.
Therefore by Cauchy criterion, (a_n) converges.

Let $a = \lim_{n\to\infty} a_n$. Then we have $a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{5a_n+6} = \sqrt{5a+6}$.
This give us $a^2 = 5a+6$, i.e. $(a-6)(a+1) = 0$. Since $a \geq 1$, we have $a = 6$.

(b) (i) Let the set of all cluster points of (x_n) be C.
We have $\frac{-2n}{3n+1} \leq \frac{2n\cos(\frac{n\pi}{2})}{3n+1} \leq \frac{2n}{3n+1}$ for all $n \in \mathbb{Z}^+$. This give us $\lim_{n\to\infty}\frac{2n}{3n+1} = \frac{2}{3}$ to be an upper bound of C, and similarly, $\lim_{n\to\infty}\frac{-2n}{3n+1} = \frac{-2}{3}$ is a lower bound of C.
Since (x_{4k}) is a subsequence of (x_n), and $\lim_{k\to\infty} x_{4k} = \frac{2}{3}$, we have $\frac{2}{3} \in C$.
This give us $\limsup x_n = \sup C = \frac{2}{3}$.
Similarly the subsequence (x_{4k+2}) of (x_n) have $\lim_{k\to\infty} x_{4k+2} = \frac{-2}{3}$, which give us $\frac{-2}{3} \in C$.
Thus $\liminf x_n = \inf C = \frac{-2}{3}$.

(ii) Since $\liminf x_n \neq \limsup x_n$, we have (x_n) to be not convergent.

Question 4

(a) Let $\varepsilon \in \mathbb{R}^+$. Then we can let $\delta = \min\left\{\frac{1}{4}, \frac{2\varepsilon}{21}\right\}$.
Thus for all $x \in \mathbb{R}$ such that $0 < |x-1| < \delta$, we have $\frac{-1}{4} < x-1 < \frac{1}{4}$.
This give us $\frac{19}{4} < x+4 < \frac{21}{4}$ and $\frac{-3}{2} < 2x-3 < \frac{-1}{2}$, i.e. $|x+4| < \frac{21}{4}$ and $|2x-3| > \frac{1}{2}$. Thus,

$$\begin{aligned}
\left|\frac{x^2+1}{2x-3} - (-2)\right| &= \left|\frac{(x-1)(x+4)}{2x-3}\right| \\
&= \frac{1}{|2x-3|}|x+4||x-1| \\
&< (2)\left(\frac{21}{4}\right)\left(\frac{2\varepsilon}{21}\right) = \varepsilon,
\end{aligned}$$

i.e. $\lim_{x\to 1} \dfrac{x^2+1}{2x-3} = -2$.

(b) (i) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \infty$ give us $\lim_{x\to a} \dfrac{g(x)}{f(x)} = 0 \in \mathbb{R}$. If $L > 0$, then $\lim_{x\to a} \dfrac{1}{g(x)} = \dfrac{1}{L} \in \mathbb{R}$.
Thus we can obtain $\lim_{x\to a} \dfrac{1}{f(x)} = \lim_{x\to a} \dfrac{g(x)}{f(x)} \dfrac{1}{g(x)} = \left(\lim_{x\to a} \dfrac{g(x)}{f(x)}\right)\left(\lim_{x\to a} \dfrac{1}{g(x)}\right) = (0)\left(\dfrac{1}{L}\right) = 0$.
This implies that $\lim_{x\to a} \left|\dfrac{1}{f(x)}\right| = 0$, i.e. $\lim_{x\to a} |f(x)| = \infty$.
Since $f(x) > 0$ for all $x \in \mathbb{R}$, we have $\lim_{x\to a} f(x) = \lim_{x\to a} |f(x)| = \infty$.

(ii) The conclusion does not hold.
Let $f(x) = 1$ and $g(x) = x$ for all $x \in \mathbb{R} - \{0\}$, and $a = 0$.
This give us $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \infty$ and $L = 0$.
However, $\lim_{x\to a} f(x) = 1$.

Question 5

(i) Let us denote $\frac{1}{2}\mathbb{Z} = \{\frac{k}{2} \mid k \in \mathbb{Z}\}$.
x is continuous on $\mathbb{R}$, and $[x]$ is continuous on $\mathbb{R} - \mathbb{Z}$.
Therefore f is continuous on $\mathbb{R} - \frac{1}{2}\mathbb{Z}$.
When $k \in \frac{1}{2}\mathbb{Z}$, we have $\lim_{x\to k^-} x - [2x] = k - (2k-1) = -k+1$.
However, $\lim_{x\to k^+} x - [2x] = k - (2k) = -k$.
Therefore f is not continuous on $\frac{1}{2}\mathbb{Z}$.
This give us all the points where f is continuous to be $\mathbb{R} - \frac{1}{2}\mathbb{Z}$.

(ii) Assume on the contrary there exists $a \in \mathbb{R}$ such that g is continuous at a.
Then since $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are dense in $\mathbb{R}$, there exists (y_n) in $\mathbb{Q}$ and (z_n) in $\mathbb{R} - \mathbb{Q}$ such that $\lim(y_n) = \lim(z_n) = a$.
This give us $\lim_{n\to\infty} g(y_n) = g(a) = \lim_{n\to\infty} g(z_n)$.
Now $g(y_n) = y_n^2 + 1 \geq 1$ for all $n \in \mathbb{Z}^+$, and so $g(a) \geq 1$.
At the same time, $g(z_n) = -|z_n| \leq 0$ for all $n \in \mathbb{Z}^+$, and so $g(a) \leq 0$.
This is a contradiction, and hence g is not continuous anywhere in $\mathbb{R}$.

SECTION B

Question 6

(i) We shall use the established fact that for all sequence (y_n) and (z_n), we have

$$\limsup(y_n + z_n) \quad \leq \quad \limsup y_n + \limsup z_n.$$

Since $\ln x$ is continuous increasing on $\mathbb{R}$, $\limsup \ln x_n = \ln \limsup x_n$ for all sequence (x_n).
Since $a_n, b_n \geq 0$ for all $n \in \mathbb{Z}^+$, we can let $y_n = \ln a_n$ and $z_n = \ln b_n$ for all $n \in \mathbb{Z}^+$.
This give us,

$$\begin{aligned} \ln \limsup a_n b_n = \limsup \ln(a_n b_n) &= \limsup (\ln a_n + \ln b_n) \\ &\leq \limsup \ln a_n + \limsup \ln b_n \\ &= \ln \limsup a_n + \ln \limsup b_n = \ln (\limsup a_n)(\limsup b_n). \end{aligned}$$

Thus, $\limsup a_n b_n = (\limsup a_n)(\limsup b_n)$.

(ii) Let $\limsup a_n^{\frac{1}{n}} = \alpha$, $\limsup \dfrac{a_{n+1}}{a_n} = \beta$, and $\alpha_k = \sup\left\{\dfrac{a_{n+1}}{a_n} \mid n \geq k\right\}$ for all $k \in \mathbb{Z}^+$.
Then we have $\lim\limits_{k\to\infty} \alpha_k = \alpha$ and $\lim\limits_{k\to\infty} \beta_k = \beta$.
Assume on the contrary that $\alpha < \beta$. Let $\varepsilon = \dfrac{\beta - \alpha}{2}$.
Since $\varepsilon > 0$, there exists $K \in \mathbb{Z}^+$ such that $\alpha_K < \alpha + \varepsilon = \dfrac{\alpha+\beta}{2}$.
Thus for all $n \geq K$, we have $\dfrac{a_{n+1}}{a_n} \leq \alpha_K < \dfrac{\alpha+\beta}{2}$, i.e. $a_{n+1} < \left(\dfrac{\alpha+\beta}{2}\right) a_n$.
This give us $a_{K+m} < \left(\dfrac{\alpha+\beta}{2}\right)^m a_K$, for $m \in \mathbb{Z}^+$, and so $a_{K+m}^{\frac{1}{K+m}} < \left(\dfrac{\alpha+\beta}{2}\right)^{\frac{m}{K+m}} a_K^{\frac{1}{K+m}}$.
Therefore $\beta = \limsup\left(a_{K+m}^{\frac{1}{K+m}}\right) \leq \limsup\left(\left(\dfrac{\alpha+\beta}{2}\right)^{\frac{m}{K+m}} a_K^{\frac{1}{K+m}}\right)$.
Since $\lim\left(\left(\dfrac{\alpha+\beta}{2}\right)^{\frac{m}{K+m}} a_K^{\frac{1}{K+m}}\right) = \dfrac{\alpha+\beta}{2}$, we get $\beta \leq \dfrac{\alpha+\beta}{2}$, i.e. $\beta \leq \alpha$, a contradiction.

Question 7

(i) Let $\varepsilon \in \mathbb{R}^+$.
Then there exists $\delta \in \mathbb{R}^+$ such that for all $a_1, a_2 \in \mathbb{R}$ with $0 < |a_i - a| < \delta$, $i = 1, 2$, we have $|f(a_1) - f(a_2)| < \varepsilon$.

Since (x_n) converges, there exists $N \in \mathbb{Z}^+$ such that for all $k \in \mathbb{Z}_{\geq N}$, we have $|x_k - a| < \delta$.
As $x_n \in D_f$, we have $|x_k - a| \neq 0$.
Thus for all $i, j \in \mathbb{Z}_{\geq N}$, we have $0 < |x_i - a| < \delta$ and $0 < |x_j - a| < \delta$.
This give us $|f(x_i) - f(x_j)| < \varepsilon$, i.e. $(f(x_n))$ is a Cauchy sequence.

(ii) Let (y_n) and (z_n) be in domain of f with $\lim(y_n) = \lim(z_n) = a$.
Then $(f(y_n))$ and $(f(z_n))$ are Cauchy, and thus convergent by Cauchy criterion.
Let $y = \lim(f(y_n))$ and $z = \lim(f(z_n))$, and $\varepsilon \in \mathbb{R}^+$.
Then there exists $N_1 \in \mathbb{Z}^+$ such that for all $k \in \mathbb{Z}_{\geq N_1}$, we have $|f(y_k) - y| < \frac{\varepsilon}{3}$.
Similarly there exists $N_2 \in \mathbb{Z}^+$ such that for all $k \in \mathbb{Z}_{\geq N_2}$, we have $|f(z_k) - z| < \frac{\varepsilon}{3}$.
Also, we have (a_n) where $a_{2n-1} = y_n$ and $a_{2n} = z_n$ to be a sequence in domain of f with $\lim(a_n) = a$.
Thus there exists $N_3 \in \mathbb{Z}^+$ such that for all $i, j \in \mathbb{Z}_{\geq N_3}$, we have $|f(a_i) - f(a_j)| < \frac{\varepsilon}{3}$.
This implies that for all $k \in \mathbb{Z}_{\geq N_3}$, we have $|f(y_k) - f(z_k)| < \frac{\varepsilon}{3}$.
This give us by Triangle inequality,

$$|y - z| \quad \leq \quad |f(y_k) - y| + |f(z_k) - z| + |f(y_k) - f(z_k)| < \varepsilon,$$

i.e. $y = z$.
Therefore all (x_n) converging to a give the same limit for $(f(x_n))$, i.e. $\lim\limits_{x\to a} f(x)$ exists.

Question 8

(a) Since f is a continuous function on a closed bounded set, by Extreme Value Theorem, there exists $c \in [0, 1]$ such that $f(c) \leq f(x)$ for all $x \in [0, 1]$.
Since $c \in [0, 1]$, we have $f(c) > 0$, and thus we can let $\alpha = f(c)$.
This give us $\alpha > 0$ and $f(x) \geq \alpha$ for all $x \in [0, 1]$.

(b) Assume on the contrary that g is not strictly increasing.
Then there exists $a, b \in (0, 1)$, with $a < b$, such that $g(a) \geq g(b)$.

Since g is injective on $[0,1]$, we have $g(a) > g(b)$.
There are 2 cases.

If $g(a) \geq g(0)$, then $g(a) > g(0)$. Let $\beta_1 = \max\{g(0), g(b)\}$.
Since g is continuous on $[0,a]$, and $g(0) \leq \beta_1 < g(a)$, there exists $c_1 \in [0,a)$ such that $g(c_1) = \beta_1$.
Similarly g is continuous on $[a,b]$ and $g(b) \leq \beta_1 < g(b)$, thus there exists $c_2 \in (a,b]$ such that $g(c_2) = \beta_1$. This is a contradiction as $c_1, c_2 \in [0,1]$ with $c_1 \neq c_2$ but $g(c_1) = g(c_2)$.

Else $g(a) < g(0)$. This give us $g(b) < g(a) < g(0) < g(1)$. Let $\beta_2 = \min\{g(1), g(a)\}$.
Since g is continuous on $[a,b]$, and $g(b) < \beta_2 \leq g(a)$, there exists $d_1 \in [a,b)$ such that $g(d_1) = \beta_2$.
Similarly g is continuous on $[b,1]$ and $g(b) < \beta_2 \leq g(1)$, thus there exists $d_2 \in (b,1]$ such that $g(d_2) = \beta_2$. This is a contradiction as $d_1, d_2 \in [0,1]$ with $d_1 \neq d_2$ but $g(d_1) = g(d_2)$.

Therefore g is strictly increasing.