NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lee Yung Hei

MA1100 Basics of Mathematics
AY 2004/2005 Sem 1

Question 1

(a) We have,

$$\begin{aligned} 2057 &= 209(9) + 176 \\ 209 &= 176(1) + 33 \\ 176 &= 33(5) + 11 \\ 33 &= 11(3). \end{aligned}$$

So $\gcd(2057, 209) = 11$.

(b) We have,

$$\begin{aligned} 11 &= 176 - 33(5) \\ 176 - 33(5) &= 176 - (209 - 176)(5) = 176(6) - 209(5) \\ 176(6) - 209(5) &= [2057 - 209(9)](6) - 209(5) = 2057(6) + 209(-59). \end{aligned}$$

So a possible solution is $a = 6$ and $b = -59$.

Note: The general solution would be $a = 6 + \frac{209}{11}k = 6 + 19k$ and $b = -59 - \frac{2057}{11}k = -59 - 187k$, $k \in \mathbb{Z}$.

(c) From (b), we have $2057(6) + 209(-59) = 11$.
So,

$$\begin{aligned} 2057(12) + 209(-118) &= 22 \\ 2057(12 + \frac{209}{11}m) + 209(-118 - \frac{2057}{11}m) &= 22. \end{aligned}$$

Thus, $(x, y) = (12 + 19n, -118 - 187n)$, $n \in \mathbb{Z}$.

Question 2

(a) False.
Let $a = 3$, $b = 1$ and $c = 2$. Since $3|1 + 2$, $3 \nmid 1$ and $3 \nmid 2$, we have a counter-eg.

(b) False.
Let $R = \emptyset$. (ie. all the elements have no relation)
Then $\forall\, a, b \in A$, $(a, b) \notin R$.

(c) False.
Let $a = 4$, $b = 2$ and $c = 6$. Since $4|2 \times 6$, $4 \nmid 2$ and $4 \nmid 6$, we have a counter-eg.

(d) True.
If $f \circ g = \mathrm{id}_A$, then $\forall x_1, x_2 \in A$ such that $g(x_1) = g(x_2)$, we have $x_1 = f(g(x_1)) = f(g(x_2)) = x_2$.
This give us g to be injective. As $g : A \to A$ and A is finite, we have g to be bijective.
Therefore, g^{-1} exists, and so $g \circ f = (g \circ f) \circ (g \circ g^{-1}) = g \circ (f \circ g) \circ g^{-1} = g \circ \mathrm{id}_A \circ g^{-1} = \mathrm{id}_A$.
Thus, $g \circ f$ is also an identity mapping on A.

(e) True.
We have $A \cup B = (A \cap B^c) \cup (A^c \cap B) \cup (A \cap B)$ and $((A \cap B^c) \cup (A^c \cap B)) \cap (A \cap B) = \emptyset$.
Therefore $(A \cap B) = \emptyset$ iff $A \cup B = (A \cap B^c) \cup (A^c \cap B) \cup \emptyset = (A \cap B^c) \cup (A^c \cap B)$.

(f) True.
Let $k = \gcd(r, s)$. Then $k|r$ and $k|s$. So, $kd|rd$ and $kd|sd$.
Since $kd|a$ and $kd|b$, $kd|\gcd(a, b) \Rightarrow kd|d$. Since $k \in \mathbb{Z}^+$, $k = 1$.

(g) False.
Let $x = 2$ and $y = -2$. $\gcd(x, y) = 2$ and $\mathrm{lcm}(x, y) = 2$. $\gcd(x, y)\mathrm{lcm}(x, y) = 4 \neq 2 \times (-2)$.

(h) False.
Let $a = 2$ and $b = -2$. $\gcd(a, b) = 2 \Rightarrow d = 2$, $x = 1$ and $y = -1$. $\mathrm{lcm}(2, -2) = 2 \neq 1 \times (-1) \times 2$.

Question 3

(a) We have $x^2 + 1^2 + z^2 > 2 \Rightarrow x^2 + z^2 > 1$. Thus truth set of $P(x, 1, z)$ is

$$\{(-2,1,-2),(-2,1,-1),(-2,1,0),(-2,1,1),(-2,1,2),(-1,1,-2),$$
$$(-1,1,-1),(-1,1,1),(-1,1,2),(0,1,-2),(0,1,2),(1,1,-2),(1,1,-1),$$
$$(1,1,1),(1,1,2),(2,1,-2),(2,1,-1),(2,1,0),(2,1,1),(2,1,2)\}.$$

(b) For $P(x, 1, z)$:

$x =$	-2	-1	0	1	2
$z \in$	$\{-2,-1,0,1,2\}$	$\{-2,-1,1,2\}$	$\{-2,2\}$	$\{-2,-1,1,2\}$	$\{-2,-1,0,1,2\}$

We have truth set of $Q(z) = \{-2, 2\}$.

(c) From the table in (3b), we see that when $x = -2$, we have $P(x, 1, z)$ for all $z \in D$.
Therefore R is true.

Question 4

(a) Let $m = \sqrt{2}$ and $c = \sqrt{2}$. $(x, y) = (-1, 0)$ is an integer solution.

(b) Let $m = 2\sqrt{2}$ and $c = \sqrt{2}$. Then $y = 2\sqrt{2}x + \sqrt{2} \Rightarrow y = (2x + 1)\sqrt{2}$.
For all $x \in \mathbb{Z}$, we have $(2x + 1) \in \mathbb{Z} - \{0\}$, and so $y = (2x + 1)\sqrt{2} \notin \mathbb{Z}$.

(c) No.
Assume on the contrary a such line passes through two distinct integer points (x_1, y_1) and (x_2, y_2).
Since $m \notin \mathbb{Q}$, we have $m \neq 0$, and so $x_1 = x_2$ implies $y_1 = mx_1 + c = mx_2 + c = y_2$, a contradiction.
Therefore $x_1 \neq x_2$. This give us $y_1 - y_2 = m(x_1 - x_2)$, i.e. $m = \dfrac{y_1 - y_2}{x_1 - x_2}$.
Since $y_1, y_2, x_1, x_2 \in \mathbb{Z}$, and $x_1 - x_2 \neq 0$, we have $m \in \mathbb{Q}$, a contradiction.

Question 5

(a) Since $p \vee \neg p$ is a tautology, we have,

$$\begin{aligned}(p \vee (p \to q)) \wedge \neg p &\equiv (p \vee (\neg p \vee q)) \wedge \neg p \\ &\equiv (p \vee \neg p \vee q) \wedge \neg p \\ &\equiv (T \vee q) \wedge \neg p \\ &\equiv \neg p.\end{aligned}$$

(b) We have,

$$\begin{aligned}(p \to \neg p) \wedge p &\equiv (\neg p \vee \neg p) \wedge p \\ &\equiv \neg p \wedge p \\ &\equiv F.\end{aligned}$$

(c) We have,

$$\begin{aligned}(p \to q) \wedge (\neg p \to \neg q) &\equiv (p \to q) \wedge (q \to p) \\ &\equiv p \leftrightarrow q \equiv p \wedge q.\end{aligned}$$

Question 6

(a) $((1,2),(4,5)), ((1,2),(7,8)), ((4,5),(7,8))$.

(b) $a+b = a+b+3(0)$. So, $(a,b)R(a,b)$, i.e. R is reflexive.

If $(a,b)R(c,d)$, then there exists $k \in \mathbb{Z}$ such that $a+b = c+d+3k$. This give us $c+d = a+b+3(-k)$. Since $-k \in \mathbb{Z}$, we have $(c,d)R(a,b)$, i.e. R is symmetric.

Let $(a,b)R(c,d)$ and $(c,d)R(e,f)$.
Then $\exists k_1, k_2 \in \mathbb{Z}$ such that $a+b = c+d+3k_1$ and $c+d = e+f+3k_2$.
This give us $a+b = e+f+3k_2+3k_1 = e+f+3(k_1+k_2)$, and since $k_1, k_2 \in \mathbb{Z}$, $k_1+k_2 \in \mathbb{Z}$.
Therefore $(a,b)R(e,f)$, i.e. R is transitive.

So, R is an equivalence relation.

(c) There are 3 equivalence classes of R.
They are $\{(a,b) \mid a+b \equiv 0 \mod 3\}$, $\{(a,b) \mid a+b \equiv 1 \mod 3\}$ and $\{(a,b) \mid a+b \equiv 2 \mod 3\}$.

Question 7

(a) True.
Suppose $A, B \in \mathcal{P}(X)$ such that $F(A) = F(B)$.
Let $a \in A$, then we have $a \in F(A)$, which give us $a \in F(B)$.
Thus, there exists $b \in B$ such that $f(a) = f(b)$.
As f is injective, we have $a = b \in F(B)$, i.e. $A \subseteq B$.
Using similar argument as above, we get $B \subseteq A$, i.e. $A = B$. So, F is injective.

(b) False.
Let $X = \{0, 1\}$ and $Y = \{0\}$. $f : X \to Y$ where $f(x) = 0$ for all $x \in X$.
f is surjective. Since 0 has a pre-image.
$\mathcal{P}(Y) = \{\emptyset, \{0\}\}$ and $\mathcal{P}(X) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.
$G(\emptyset) = \emptyset$ and $G(\{0\}) = \{0, 1\}$.
Since $\{0\}$ and $\{1\}$ do not have pre-images, G is not surjective.

Question 8

(a) We have,

$$\begin{aligned}
\text{Number of integers divisible by 7} &= \left\lfloor \tfrac{500}{7} \right\rfloor = 71;\\
\text{Number of integers divisible by 11} &= \left\lfloor \tfrac{500}{11} \right\rfloor = 45;\\
\text{Number of integers divisible by both 7 and 11} &= \left\lfloor \tfrac{500}{7\times 11} \right\rfloor = 6.
\end{aligned}$$

Since 6 numbers are counted twice, answer is $71 + 45 - 6 = 110$.

(b) We have,

$$\begin{aligned}
\{2k_1 + 1 | k_1 \in \mathbb{Z}\} * \{2k_2 + 1 | k_2 \in \mathbb{Z}\} &= \{(2k_1 + 1) + (2k_2 + 1) | k_1, k_2 \in \mathbb{Z}\}\\
&= \{2(k_1 + k_2 + 1) | k_1, k_2 \in \mathbb{Z}\}\\
&= \{2(m) | m \in \mathbb{Z}\}
\end{aligned}$$

Therefore, $\{2k_1 + 1 | k_1 \in \mathbb{Z}\} * \{2k_2 + 1 | k_2 \in \mathbb{Z}\}$ is the set of all even integers.

(c) Assume on the contrary that $\sqrt[3]{2}$ is rational, i.e. $\sqrt[3]{2} = \frac{m}{n}$, where $m, n \in \mathbb{Z}$, $n \neq 0$ and $\gcd(m, n) = 1$.
Then, $2 = \frac{m^3}{n^3} \Rightarrow 2n^3 = m^3$. Therefore, m is even.
Let $m = 2p$. $2n^3 = (2p)^3 = 8p^3 \Rightarrow n^3 = 4p^3$. Since $4p^3$ is even, n^3 is even.
However, that would mean that both m and n are even.
Therefore, $2|m$ and $2|n$ implying $\gcd(m, n) \neq 1$, a contradiction.
So, $\sqrt[3]{2}$ cannot be rational.