NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Ho Chin Fung

MA3111 Complex Analysis I
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Question 1

(a) (i) Let $z = x + iy$. Then

$$\begin{aligned} f(z) &= |z|^2 + i(\mathrm{Im} z)^4 + 6i\bar{z} \\ &= x^2 + y^2 + i(y)^4 + 6i(x - iy) \\ &= x^2 + y^2 + 6y + i(y^4 + 6x). \end{aligned}$$

We have

$$u(x, y) = x^2 + y^2 + 6y, \quad v(x, y) = y^4 + 6x,$$
$$u_x = 2x, \quad u_y = 2y + 6, \quad v_x = 6, \quad v_y = 4y^3.$$

Thus, u_x, x_y, v_x, v_y are continuous at all (x, y).
To find the points where f is differentiable, we need to solve the CR equations.

$$\begin{cases} u_x = v_y, \\ u_y = -v_x. \end{cases} \Rightarrow \begin{cases} 2x = 4y^3, \\ 2y + 6 = -6. \end{cases} \Rightarrow \begin{cases} x = -432, \\ y = -6. \end{cases} \Rightarrow z = -432 - 6i.$$

Therefore, f is differentiable only at $z = -432 - 6i$.
At $z = -432 - 6i$,

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= 2(-432) + i(6) \\ &= -864 + 6i. \end{aligned}$$

(ii) First, we shall establish that f is nowhere analytic. From the result of 1(a)(i), we have f is differentiable only at $z = -432 - 6i$. Since f is differentiable only at a finite number of points, f is nowhere analytic.
Now, suppose such a domain D and function F exist. Since F is differentiable everywhere in D, F is analytic in D. Then $F'(z) = f(z)$ is also analytic in D. This is a contradiction to the fact that f is nowhere analytic. Therefore, such a domain D and function F do not exist.

(b) (i) Since g is an entire function, so is g'. Let $h(z) = \frac{g'(z)}{g(z)}$.
Consider

$$0 \le |g'(z)| < |g(z)| \quad \Rightarrow \quad g(z) \neq 0.$$

Thus, $h(z)$ is an entire function.
Next,

$$\begin{aligned} |g'(z)| &< |g(z)| \\ \left| \frac{g'(z)}{g(z)} \right| &< 1 \\ |h(z)| &< 1 \quad \forall z \in \mathbb{C} \end{aligned}$$

Therefore, $h(z)$ is a bounded function.

By Liouville's Theorem, there exists a complex constant α s.t.

$$h(z) \equiv \alpha \quad \forall z \in \mathbb{C}.$$

Since $|h(z)| < 1$, we have $|\alpha| < 1$.
We also have

$$\begin{aligned} \frac{g'(z)}{g(z)} &\equiv \alpha \\ g'(z) &= \alpha g(z) \quad \forall z \in \mathbb{C}. \end{aligned}$$

(ii) From $g'(z) = \alpha g(z)$, we have

$$g^{(n+1)}(z) = \alpha g^{(n)}(z).$$

Given that $g(0) = 1$, we have

$$\begin{aligned} g'(0) &= \alpha g(0) \\ &= \alpha \cdot 1 \\ &= \alpha. \end{aligned}$$

Next,

$$\begin{aligned} g''(z) &= \alpha g'(z) \\ g''(0) &= \alpha g'(0) \\ &= \alpha \cdot \alpha \\ &= \alpha^2. \end{aligned}$$

Proceed with induction, we have

$$g^{(n)}(0) = \alpha^n.$$

Therefore, the Maclaurin series of $g(z)$ become

$$g(z) = 1 + \alpha z + \frac{(\alpha z)^2}{2} + \frac{(\alpha z)^3}{3!} + \cdots + \frac{(\alpha z)^n}{n!} + \cdots.$$

This is also the Maclaurin series of $e^{\alpha z}$. Since Maclaurin series of an analytic function is unique, we can conclude that $g(z) = e^{\alpha z}$.

Question 2

(a) We have

$$\begin{aligned} 5\cot z &= ie^{2iz} \\ 5\cos z &= ie^{2iz}\sin z \\ 5\left(\frac{e^{iz}+e^{-iz}}{2}\right) &= ie^{2iz}\left(\frac{e^{iz}-e^{-iz}}{2i}\right) \\ 5e^{iz}+5e^{-iz} &= e^{3iz}-e^{iz} \\ e^{4iz}-6e^{2iz}-5 &= 0 \\ e^{2iz} &= \frac{6 \pm \sqrt{6^2-4(1)(-5)}}{2(1)} \\ &= 3 \pm \sqrt{14} \\ 2iz &= \ln(3+\sqrt{14}) + i(2n\pi), \quad \ln(\sqrt{14}-3) + i(2n+1)\pi, \quad n \in \mathbb{Z} \\ z &= n\pi + i\left(-\frac{\ln(3+\sqrt{14})}{2}\right), \quad \frac{(2n+1)\pi}{2} + i\left(-\frac{\ln(\sqrt{14}-3)}{2}\right), \quad n \in \mathbb{Z}. \end{aligned}$$

(b) We have $\gamma(t) = 2 + e^{it}, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.
Then

$$\begin{aligned}
\int_\gamma [\bar{z} + (z-1)^5] dz &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\overline{\gamma(t)} + (\gamma(t)-1)^5](\gamma'(t))dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [2 + e^{-it} + (2+e^{it}-1)^5](ie^{it})dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2ie^{it} + i + (e^{it}+1)^5(ie^{it})dt \\
&= 2e^{it} + it + \frac{1}{6}(e^{it}+1)^6|_{-\pi/2}^{\pi/2} \\
&= 2i + i(\frac{\pi}{2}) + \frac{1}{6}(i+1)^6 - [2(-i) + i(-\frac{\pi}{2}) + \frac{1}{6}(-i+1)^6] \\
&= 4i + i\pi + \frac{1}{6}((\sqrt{2}e^{i\pi/4})^6 - (\sqrt{2}e^{-i\pi/4})^6) \\
&= 4i + i\pi + \frac{1}{6}(-8i - 8i) \\
&= \left(\pi + \frac{4}{3}\right) i.
\end{aligned}$$

(c) We define

$$\begin{aligned}
S_1 &:= \left\{ k \;\middle|\; -\frac{\pi}{4} < \arg(z_k) \leq \frac{\pi}{4} \right\} \\
S_2 &:= \left\{ k \;\middle|\; \frac{\pi}{4} < \arg(z_k) \leq \frac{3\pi}{4} \right\} \\
S_3 &:= \left\{ k \;\middle|\; \frac{3\pi}{4} < \arg(z_k) \leq \frac{5\pi}{4} \right\} \\
S_4 &:= \left\{ k \;\middle|\; \frac{5\pi}{4} < \arg(z_k) \leq \frac{7\pi}{4} \right\}.
\end{aligned}$$

Then, $\{S_1, S_2, S_3, S_4\}$ is a partition of $\{1, 2, \ldots n\}$. By Pigeonhole Principle, there exist at least one S_i such that $|S_i| \geq \frac{n}{4}$. Consider the case where $|S_1| \geq \frac{n}{4}$.
We have

$$\begin{aligned}
\cos(\arg(z_k)) &\geq \frac{1}{\sqrt{2}} \\
\frac{\text{Re}(z_k)}{|z_k|} &\geq \frac{1}{\sqrt{2}} \\
\text{Re}(z_k) &\geq \frac{1}{\sqrt{2}}.
\end{aligned}$$

Then,

$$\begin{aligned}
\left| \sum_{k \in S_1} z_k \right| &\geq \left| \text{Re}\left(\sum_{k \in S_1} z_k \right) \right| = \left| \sum_{k \in S_1} \text{Re}(z_k) \right| \\
&\geq |S_1| \cdot \frac{1}{\sqrt{2}} \\
&\geq \frac{n}{4} \cdot \frac{1}{\sqrt{2}} = \frac{n}{4\sqrt{2}}.
\end{aligned}$$

The other cases where $|S_2| \geq \frac{n}{4}, |S_3| \geq \frac{n}{4}, |S_4| \geq \frac{n}{4}$ are similar.

Question 3

(a) Let $f(x,y) = u(x,y) + iv(x,y)$ be an entire function whose imaginary part v is given by

$$v(x,y) = 2y(x+1) + e^{-y}\sin x, \quad (x,y) \in \mathbb{R}.$$

Then

$$\begin{aligned} u_x &= v_y \\ &= 2(x+1) - e^{-y}\sin x. \end{aligned}$$

$$\begin{aligned} u_y &= -v_x \\ &= -(2y + e^{-y}\cos x) \\ &= -2y - e^{-y}\cos x. \end{aligned}$$

$$\begin{aligned} u(x,y) &= \int u_y dy \\ &= -y^2 + e^{-y}\cos x + \phi(x). \end{aligned}$$

Differentiate w.r.t x, we have

$$\begin{aligned} u_x &= -e^{-y}\sin x + \phi'(x) \\ 2(x+1) - e^{-y}\sin x &= -e^{-y}\sin x + \phi'(x) \\ \phi'(x) &= 2(x+1) \\ \phi(x) &= x^2 + 2x + c, \quad c \in \mathbb{R}. \\ &\therefore u(x,y) = x^2 - y^2 + 2x + e^{-y}\cos x + c, \quad c \in \mathbb{R}. \end{aligned}$$

Therefore, $f(x,y) = x^2 - y^2 + 2x + e^{-y}\cos x + i(2y(x+1) + e^{-y}\sin x)$ is an entire function whose imaginary part v is as given.

(b) Let $g(z) = \frac{f(z)}{(z-i)(z-2i)^2}$.
g has singular points at $z_0 = i, 2i$, which are inside γ.
By CRT, we have

$$\int_\gamma g(z)dz = 2\pi i \left(\operatorname*{Res}_{z=i} g(z) + \operatorname*{Res}_{z=2i} g(z) \right).$$

Near the point $z_0 = i$, we can write

$$g(z) = \frac{f(z)/(z-2i)^2}{z-i} = \frac{\phi(z)}{z-i},$$

where $\phi(z) = f(z)/(z-2i)^2$ is analytic at $z_0 = i$.
Thus,

$$\begin{aligned} \operatorname*{Res}_{z=i} g(z) &= \phi(i) \\ &= f(i)/(i-2i)^2 \\ &= 5/(-i)^2 \\ &= -5. \end{aligned}$$

Near the point $z_0 = 2i$, we can write

$$g(z) = \frac{f(z)/(z-i)}{(z-2i)^2} = \frac{\phi(z)}{(z-2i)^2},$$

where $\phi(z) = f(z)/(z-i)$ is analytic at $z_0 = 2i$.

Next,

$$\phi'(z) = \frac{f'(z)(z-i) - f(z)(1)}{(z-i)^2}.$$

Thus,

$$\begin{aligned}
\operatorname*{Res}_{z=2i} g(z) &= \frac{\phi'(2i)}{1!} \\
&= \frac{f'(2i)(2i-i) - f(2i)(1)}{(2i-i)^2} \\
&= \frac{(3i)(i) - 4(1)}{i^2} \\
&= \frac{-3-4}{-1} \\
&= 7.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_\gamma \frac{f(z)}{(z-i)(z-2i)^2} dz &= \int_\gamma g(z) dz \\
&= 2\pi i \left(\operatorname*{Res}_{z=i} g(z) + \operatorname*{Res}_{z=2i} g(z) \right) \\
&= 2\pi i(-5+7) \\
&= 4\pi i.
\end{aligned}$$

(c) The function g is analytic on the annulus $0 < |z| < \infty$.
Thus, by Laurent's Theorem, we have

$$g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \quad 0 < |z| < \infty.$$

For $r > 0$, let C_r be the positively oriented circle $|z| = r$.
Then for each $n \geq 1$,

$$\begin{aligned}
a_n &= \frac{1}{2\pi i} \int_{C_r} \frac{g(z)}{(z-0)^{n+1}} dz \\
&= \frac{1}{2\pi i} \int_{C_r} \frac{g(z)}{z^{n+1}} dz.
\end{aligned}$$

Next we apply ML-inequality. $L = 2\pi r$. For all $z \in C_r$,

$$\begin{aligned}
\left| \frac{g(z)}{z^{n+1}} \right| &\leq \frac{|\text{Log} z|}{|z|^{n+1}} \\
&= \frac{|\ln|z| + i\text{Arg} z|}{|z|^{n+1}} \\
&= \frac{|\ln r + i\text{Arg} z|}{r^{n+1}} \\
&\leq \frac{\sqrt{(\ln r)^2 + \pi^2}}{r^{n+1}}.
\end{aligned}$$

Thus by ML-inequality, we have

$$\begin{aligned} |a_n| &= \left|\frac{1}{2\pi i}\int_{C_r}\frac{g(z)}{z^{n+1}}dz\right| \\ &\le \frac{1}{2\pi}\cdot\frac{\sqrt{(\ln r)^2+\pi^2}}{r^{n+1}}\cdot 2\pi r \\ &= \frac{\sqrt{(\ln r)^2+\pi^2}}{r^n}. \end{aligned}$$

Letting $r \to \infty$, we have

$$\begin{aligned} |a_n| &\le \lim_{r\to\infty}\frac{\sqrt{(\ln r)^2+\pi^2}}{r^n} \\ &= 0. \end{aligned}$$

Therefore, $a_n = 0$ for each $n \ge 1$.
Similarly, for $n \ge 1$,

$$\begin{aligned} |b_n| &= \left|\frac{1}{2\pi}\int_{C_r}\frac{g(z)}{z^{-n+1}}dz\right| \\ &\le r^n\sqrt{(\ln r)^2+\pi^2} \quad \to 0 \text{ as } r \to 0. \end{aligned}$$

Therefore, $b_n = 0$ for each $n \ge 1$.
Thus, $g(z) = a_0$ for all $z \in \mathbb{C}\backslash\{0\}$. i.e., g is a constant function.
At $z = 1$, we have

$$\begin{aligned} |g(1)| &\le |\text{Log}1| \\ |a_0| &\le 0 \\ \Rightarrow a_0 &= 0. \end{aligned}$$

Thus, $g(z) \equiv 0$ for all $z \in \mathbb{C}\backslash\{0\}$. Therefore, $g(i) = 0$.

Question 4

(a) (i) Let $\frac{7z+5}{(2z+3)(z-4)} = \frac{A}{2z+3} + \frac{B}{z-4}$.
Then

$$7z+5 = A(z-4) + B(2z+3)$$
$$\Rightarrow \begin{cases} 7 = A + 2B, \\ 5 = 3B - 4A. \end{cases} \quad \Rightarrow \begin{cases} A = 1, \\ B = 3. \end{cases}$$

Thus,

$$\frac{7z+5}{(2z+3)(z-4)} = \frac{1}{2z+3} + \frac{3}{z-4}.$$

Next,

$$\begin{aligned} \frac{1}{2z+3} &= \frac{1}{2z}\cdot\frac{1}{1+\frac{3}{2z}} \\ &= \frac{1}{2z}\cdot\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{2z}\right)^n, \quad \left|\frac{3}{2z}\right| < 1 \\ &= \sum_{n=0}^{\infty}\frac{(-3)^n}{(2z)^{n+1}}, \quad \left|\frac{2z}{3}\right| > 1 \\ &= \sum_{n=0}^{\infty}\frac{(-3)^n}{2^{n+1}\cdot z^{n+1}}, \quad |z| > \frac{3}{2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{3}{z-4} &= -\frac{3}{4}\cdot\frac{1}{1-\frac{z}{4}}\\ &= -\frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n, \quad \left|\frac{z}{4}\right|<1\\ &= \sum_{n=0}^{\infty}\left(-\frac{3}{4^{n+1}}\right)z^n, \quad |z|<4.\end{aligned}$$

Therefore,

$$\frac{7z+5}{(2z+3)(z-4)} = \sum_{n=0}^{\infty}\frac{(-3)^n}{2^{n+1}\cdot z^{n+1}}+\sum_{n=0}^{\infty}\left(-\frac{3}{4^{n+1}}\right)z^n, \quad \frac{3}{2}<|z|<4.$$

(ii) Let

$$\begin{aligned}f(z) &= \frac{7z+5}{(2z+3)(z-4)}\\ &= \sum_{n=0}^{\infty}\frac{(-3)^n}{2^{n+1}\cdot z^{n+1}}+\sum_{n=0}^{\infty}\left(-\frac{3}{4^{n+1}}\right)z^n, \quad \frac{3}{2}<|z|<4.\end{aligned}$$

γ is positively orientated and lies inside the annular domain $\frac{3}{2}<|z|<4$.
So, we have

$$\begin{aligned}\int_\gamma \frac{(z^6+8)(7z+5)}{z^4(2z+3)(z-4)}dz &= \int_\gamma f(z)\left(\frac{z^6+8}{z^4}\right)dz\\ &= \int_\gamma f(z)(z-0)^{3-1}dz + 8\int_\gamma \frac{f(z)}{(z-0)^{3+1}}dz\\ &= 2\pi i(b_3)+8\cdot 2\pi i(a_3)\\ &= 2\pi i\left(\frac{(-3)^2}{2^{2+1}}\right)+16\pi i\left(-\frac{3}{4^{3+1}}\right)\\ &= \pi i\left(2\left(\frac{9}{8}\right)+16\left(-\frac{3}{256}\right)\right)\\ &= \frac{33}{16}\pi i.\end{aligned}$$

(b) Recall that

$$P.V.\int_{\infty}^{\infty}\frac{\sin(3x+1)}{x^2-6x+34}dx = \lim_{R\to+\infty}\int_{-R}^{R}\frac{\sin(3x+1)}{x^2-6x+34}dx.$$

The integrand is $\frac{\sin(3x+1)}{x^2-6x+34}$. Thus we let

$$f(z) = \frac{e^{i(3z+1)}}{z^2-6z+34}.$$

Then f has singular points at

$$z^2-6z+34=0 \quad\Leftrightarrow\quad z=3+5i, \quad 3-5i.$$

For $R > |3+5i|$, consider the semi-circular arc C_R given by $C_R(t) = Re^{it}, \quad 0 \le t \le \pi$.
By Cauchy's Residue Theorem,

$$\int_{[-R,R]} f(x)dx + \int_{C_R} f(z)dz \quad = \quad 2\pi i \operatorname*{Res}_{z=3+5i} f(z). \tag{1}$$

Write

$$f(z) = \frac{e^{i(3z+1)}}{z^2 - 6z + 34} = \frac{p(z)}{q(z)},$$

where $p(z) = e^{i(3z+1)}$ and $q(z) = z^2 - 6z + 34$ are analytic at $3+5i$ with $q'(z) = 2z - 6$.
Observe that $q(3+5i) = 0$ and $q'(3+5i) = 2(3+5i) - 6 = 10i \neq 0$. Thus

$$\begin{aligned} \operatorname*{Res}_{z=3+5i} f(z) &= \frac{p(3+5i)}{q'(3+5i)} \\ &= \frac{e^{-15+10i}}{10i}. \end{aligned}$$

Together with (1), it follows that

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \cdot \frac{e^{-15+10i}}{10i} = \frac{\pi}{5} e^{-15+10i}. \tag{2}$$

Next, we apply ML-inequality to $\int_{C_R} f(z)dz$.
First, we have $L = \frac{1}{2} \cdot (2\pi R) = \pi R$. For $z = x + iy \in C_R$,

$$\begin{aligned} |f(z)| = \left| \frac{e^{i(3z+1)}}{z^2 - 6z + 34} \right| &= \frac{|e^{i(3(x+iy)+1)}|}{|z^2 - 6z + 34|} \\ &= \frac{|e^{-3y+i(3x+1)}|}{|z^2 - 6z + 34|} \\ &= \frac{e^{-3y}}{|z^2 - (6z - 34)|} \\ &\le \frac{e^{-3 \cdot 0}}{|z^2| - |6z - 34|} \\ &\le \frac{1}{|z^2| - (|6z| + |34|)} \\ &= \frac{1}{R^2 - 6R - 34} = M. \end{aligned}$$

Thus by ML-inequality,

$$\begin{aligned} 0 \le \left| \int_{C_R} f(z)dz \right| &\le ML \\ &= \frac{1}{R^2 - 6R - 34} \cdot \pi R \longrightarrow 0 \quad \text{as } R \to +\infty. \end{aligned}$$

Thus by squeeze theorem, we have

$$\begin{aligned} \lim_{R \to +\infty} \left| \int_{C_R} f(z)dz \right| &= 0 \\ \Rightarrow \lim_{R \to +\infty} \int_{C_R} f(z)dz &= 0. \end{aligned}$$

Letting $R \to +\infty$ in (2), we have

$$
\begin{aligned}
\lim_{R\to+\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to+\infty}\int_{C_R} f(z)dz &= \frac{\pi}{5}e^{-15+10i} \\
\implies \quad \lim_{R\to+\infty}\int_{-R}^{R} \frac{e^{i(3x+1)}}{x^2-6x+34}dx + 0 &= \frac{\pi}{5}e^{-15}e^{10i} \\
\implies \quad \lim_{R\to+\infty}\int_{-R}^{R} \frac{\cos(3x+1)+i\sin(3x+1)}{x^2-6x+34}dx &= \frac{\pi}{5}e^{-15}(\cos 10 + i\sin 10).
\end{aligned}
$$

Equating the imaginary parts on both sides, we get

$$
\begin{aligned}
\lim_{R\to+\infty}\int_{-R}^{R} \frac{\sin(3x+1)}{x^2-6x+34}dx &= \frac{\pi}{5}e^{-15}\sin 10 \\
\therefore P.V.\int_{-\infty}^{\infty} \frac{\sin(3x+1)}{x^2-6x+34}dx &= \frac{\pi}{5}e^{-15}\sin 10.
\end{aligned}
$$

Question 5

(a) Let $f(z) = (z^2 - 2iz)\sin\frac{1}{z-i}$.
The integrand $f(z) = (z^2-2iz)\sin\frac{1}{z-i}$ has singular point at $z = i$. By the CRT,

$$\int_\gamma f(z)dz = 2\pi i \operatorname*{Res}_{z=i} f(z).$$

Using standard power series for $\sin z$, we have,

$$
\begin{aligned}
f(z) &= (z^2-2iz)\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{z-i})^{2n+1}}{(2n+1)!}, \quad \left|\frac{1}{z-i}\right| < \infty \\
&= ((z-i)^2+1)\left(\frac{1}{z-i} - \frac{1}{3!}\cdot\frac{1}{(z-i)^3} + \frac{1}{5!}\cdot\frac{1}{(z-i)^5} + \cdots\right) \\
&= \cdots + \left(1\cdot(-\frac{1}{6}) + 1\cdot 1\right)\frac{1}{z-i} + \cdots, \quad 0 < |z-i| < \infty.
\end{aligned}
$$

Thus, $\operatorname*{Res}_{z=i} f(z) = \frac{5}{6}$.
Therefore,

$$
\begin{aligned}
\int_\gamma (z^2-2iz)\sin\frac{1}{z-i}dz &= 2\pi i \cdot \frac{5}{6} \\
&= \frac{5\pi i}{3}.
\end{aligned}
$$

(b) Given that

$$\lim_{n\to\infty} F\left(\frac{1}{n}\right) = 1+i \quad \text{and} \quad \lim_{n\to\infty} F\left(\frac{1}{n}\right) = 1-i.$$

Observe that both sequences $\left\{\frac{1}{n}\right\}$ and $\left\{\frac{i}{n}\right\}$ approach 0 as $n \to \infty$.
However,

$$\lim_{n\to\infty} F\left(\frac{1}{n}\right) = 1+i \neq 1-i = \lim_{n\to\infty} F\left(\frac{1}{n}\right).$$

Thus, $\lim_{z\to 0} F(z)$ does not exist. Since $\lim_{z\to 0} F(z)$ is neither a finite number nor infinity, the singular point at $z = 0$ is neither a removable singular point nor a pole.
Therefore, F has an essential singular point at $z = 0$.

(c) Consider the function $f(g(z))$. We have

$$\begin{aligned} f(g(z_0)) &= f(w_0) \\ &= 0. \end{aligned}$$

$$\begin{aligned} (f(g(z_0)))' &= f'(g(z_0)) \cdot g'(z_0) \\ &= f'(w_0) \cdot g'(z_0) \\ &= 0 \cdot g'(z_0) \\ &= 0. \end{aligned}$$

$$\begin{aligned} (f(g(z_0)))'' &= f''(g(z_0)) \cdot g'(z_0) \cdot g'(z_0) + f'(g(z_0)) \cdot g''(z_0) \\ &= f''(w_0) \cdot (g'(z_0))^2 + f'(w_0) g''(z_0) \\ &= f''(w_0) \cdot (g'(z_0))^2 + 0 \cdot g''(z_0) \\ &= f''(w_0) \cdot (g'(z_0))^2 \neq 0. \end{aligned}$$

Therefore, $f(g(z))$ has a zero of order 2 at $z = z_0$. Clearly, $z - z_0$ has a zero of order 1 at $z = z_0$.
So, $h(z) = \frac{z-z_0}{f(g(z))}$ has a simple pole at $z = z_0$.
Then, $\exists R > 0$ s.t.

$$h(z) = \frac{\phi(z)}{z - z_0}, \quad 0 < |z - z_0| < R,$$

where $\phi(z)$ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$.
Thus, $\operatorname*{Res}_{z=z_0} h(z)$ is just $\phi(z_0)$.
Next,

$$\begin{aligned} \frac{z - z_0}{f(g(z))} = h(z) &= \frac{\phi(z)}{z - z_0} \\ \Rightarrow \quad \frac{1}{\phi(z)} &= \frac{f(g(z))}{(z - z_0)^2} \end{aligned}$$

Performing Taylor's expansion on $f(g(z))$ at $z = z_0$, we have

$$\begin{aligned} & f(g(z)) \\ = \; & f(g(z_0)) + \frac{1}{1!}(f(g(z_0)))'(z - z_0) + \frac{1}{2!}(f(g(z_0)))''(z - z_0)^2 + \frac{1}{3!}(f(g(z_0)))'''(z - z_0)^3 + \cdots \\ = \; & 0 + 0 \cdot (z - z_0) + \frac{1}{2} f''(w_0) \cdot (g'(z_0))^2 \cdot (z - z_0)^2 + \frac{1}{6}(f(g(z_0)))''') \cdot (z - z_0)^3 + \cdots \\ = \; & \frac{1}{2} f''(w_0) \cdot (g'(z_0))^2 \cdot (z - z_0)^2 + \frac{1}{6}(f(g(z_0)))''' \cdot (z - z_0)^3 + \cdots \end{aligned}$$

Then, we have

$$\begin{aligned} \frac{1}{\phi(z)} &= \frac{\frac{1}{2} f''(w_0) \cdot (g'(z_0))^2 \cdot (z - z_0)^2 + \frac{1}{6}(f(g(z_0)))''' \cdot (z - z_0)^3 + \cdots}{(z - z_0)^2} \\ &= \frac{1}{2} f''(w_0) \cdot (g'(z_0))^2 + \frac{1}{6}(f(g(z_0)))''' \cdot (z - z_0) + \cdots \\ \frac{1}{\phi(z_0)} &= \frac{1}{2} f''(w_0) \cdot (g'(z_0))^2 + \frac{1}{6}(f(g(z_0)))''' \cdot 0 + \cdots \\ &= \frac{1}{2} f''(w_0) \cdot (g'(z_0))^2 \\ \phi(z_0) &= \frac{2}{f''(w_0) \cdot (g'(z_0))^2} \\ \operatorname*{Res}_{z=z_0} h(z) &= \frac{2}{f''(w_0) \cdot (g'(z_0))^2}. \end{aligned}$$