NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Teo Wei Hao

MA1104 Multivariable Calculus
AY 2005/2006 Sem 2

Question 1

(a) Let $\lambda = \dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z-1}{3}$. We manipulate this to give us $\langle x, y, z\rangle = \langle 0, 0, 1\rangle + \lambda\langle 1, 2, 3\rangle$. Thus, $\langle 1, 2, 3\rangle$ is a direction vector on the plane.
Next, since $(0, 0, 1)$ and $(1, 1, 1)$ lies on the plane, we have $\langle 1, 1, 1\rangle - \langle 0, 0, 1\rangle = \langle 1, 1, 0\rangle$ to be another direction vector on the plane.
Thus we have a normal vector to the plane to be $\langle 1, 2, 3\rangle \times \langle 1, 1, 0\rangle = \langle -3, 3, -1\rangle$.
This give the equation of the plane to be,

$$\begin{aligned}
\langle x, y, z\rangle \cdot \langle -3, 3, -1\rangle &= \langle 0, 0, 1\rangle \cdot \langle -3, 3, -1\rangle \\
-3x + 3y - z &= -1 \\
-3x + 3y - z + 1 &= 0.
\end{aligned}$$

(b) g is not continuous at $(0, 0)$.
Assume on the contrary that g is continuous at $(0, 0)$. Then the limit at $(0, 0)$ exists, and we have

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} g(x, y) &= \lim_{x\to 0} g(x, x) \\
&= \lim_{x\to 0} \frac{\sin(x^2 + x^2 + x^2)}{x^2 + x^2} \\
&= \lim_{x\to 0} \frac{3}{2}\left(\frac{\sin(3x^2)}{3x^2}\right) \\
&= \frac{3}{2} \\
&\neq g(0, 0),
\end{aligned}$$

a contradiction. Thus g is not continuous at $(0, 0)$.

(c) We get $f_x(x, y) = ye^{xy}\sin y$ and $f_y(x, y) = xe^{xy}\sin y + e^{xy}\cos y$.
By integrating f_x with respect to x, we get $f(x, y) = e^{xy}\sin y + g(y)$ for some scalar function $g(y)$.
We differentiate this result with respect to y to get $f_y(x, y) = xe^{xy}\sin y + e^{xy}\cos y + g'(y)$, i.e. $g'(y) = 0$. By integrating g' with respect to y, we get $f(x, y) = e^{xy}\sin y + c$ for some arbitrary constant c. Thus we can let $c = 0$, and get $f(x, y) = e^{xy}\sin y$ to be a function that satisfy the condition.

Question 2

(a) We have $f_x = x + \dfrac{-16}{x^3}$ and $f_y = y + \dfrac{-16}{y^3}$.

$f_x = 0$ implies that $x = \pm 2$, and $f_y = 0$ implies that $y = \pm 2$.
Combining the above, we have $\nabla f = \langle 0, 0\rangle$ only when $(x, y) = (\pm 2, \pm 2)$ or $(x, y) = (\pm 2, \mp 2)$.

Next, $f_{xx} = 1 + \frac{48}{x^4}$, $f_{yy} = 1 + \frac{48}{y^4}$ and $f_{xy} = 0$. This give us $D = f_{xx}f_{yy} - (f_{xy})^2 = \left(1 + \frac{48}{x^4}\right)\left(1 + \frac{48}{y^4}\right)$. Since $D|_{(\pm 2, \pm 2)} > 0$ and $D|_{(\pm 2, \mp 2)} > 0$, there is no saddle point.

Now since $f_{xx}(\pm 2, \pm 2) > 0$ and $f_{xx}(\pm 2, \mp 2) > 0$, all 4 points are minimum points.

(b) Let D be the area enclosed by the parameters. We see that D is the area bounded by the y-axis, $y = 1$ and $y = x$. Thus we also have D to be given by $y \in [0, 1]$, $x \in [0, y]$. Therefore,

$$\begin{aligned}
\int_0^1 \int_x^1 e^{y^2}\, dy\, dx &= \int_0^1 \int_0^y e^{y^2}\, dx\, dy \\
&= \int_0^1 y e^{y^2}\, dy \\
&= \left[\frac{1}{2} e^{y^2}\right]_0^1 = \frac{1}{2}(e - 1).
\end{aligned}$$

(c) Let D be the area bounded by C. Notice that area of D is $\pi(2)^2 = 4\pi$. Thus by Green's Theorem, we have

$$\begin{aligned}
\int_C (7y - e^{\cos x})\, dx + \left[15x - \sin\left(y^3 + 8y\right)\right] dy &= \iint_D \frac{\partial}{\partial x}\left[15x - \sin\left(y^3 + 8y\right)\right] - \frac{\partial}{\partial y}(7y - e^{\cos x})\, dA \\
&= \iint_D 15 - 7\, dA \\
&= 8 \iint_D 1\, dA \\
&= 8(4\pi) = 32\pi.
\end{aligned}$$

Question 3

(a) We have $f_x = y$, $f_y = x$, $f_z = 2$. Let $g = x^2 + y^2 + z^2$. Then we have $g_x = 2x$, $g_y = 2y$ and $g_z = 2z$. We would like to find the critical points of $f(x, y, z)$ subjected to the constrain of $g(x, y, z) = 36$. Using method of Lagrange multipliers, $f_x = \lambda g_x$, $f_y = \lambda g_y$ and $f_z = \lambda g_z$ for some $\lambda \in \mathbb{R}$. Thus,

$$\begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ 2 = 2\lambda z \\ x^2 + y^2 + z^2 = 36 \end{cases}$$

We can see from the equations that $\lambda, z \neq 0$. Now if $x = 0$, then $y = 0$ and $z = \pm 6$, and so $(0, 0, \pm 6)$ are critical points. The same conclusion will be reached if $y = 0$.
We are left with the case that $x, y, z, \lambda \neq 0$. Since $y = (2\lambda)x = (2\lambda)^2 y$, we conclude that $(2\lambda)^2 = 1$, i.e. $\lambda = \pm\frac{1}{2}$.
Thus $x = 2y(\pm\frac{1}{2}) = \pm y$, and $z = \frac{1}{\lambda} = \pm 2$. This give us $36 = x^2 + x^2 + 2^2$, i.e. $x = \pm 4$. Therefore this generates 8 critical points (x, y, z), where $x \in \{-4, 4\}$, $y \in \{-4, 4\}$ and $z \in \{-2, 2\}$ (each of x, y and z has 2 choices, thus we can form $2^3 = 8$ results).

Now evaluating f at the critical points, we get $f(0, 0, \pm 6) = \pm 12$, $f(\pm 4, \pm 4, 2) = 16 + 2(2) = 20$, $f(\pm 4, \pm 4, -2) = 16 + 2(-2) = 12$, $f(\pm 4, \mp 4, 2) = -16 + 2(2) = -12$, $f(\pm 4, \mp 4, -2) = -16 + 2(-2) = -20$. Thus the maximum and minimum value of $f(x, y, z)$ subjected to the constrain of $g(x, y, z) = 36$ is 20 and -20 respectively.

(b) Let $u = x - xy$ and $v = xy$. Thus $u = x - v$, i.e. $x = u + v$, and $y = \frac{v}{x} = \frac{v}{u+v}$. This give us $\dfrac{\partial x}{\partial u} = 1$, $\dfrac{\partial x}{\partial v} = 1$, $\dfrac{\partial y}{\partial u} = \dfrac{-v}{(u+v)^2}$ and $\dfrac{\partial y}{\partial v} = \dfrac{u}{(u+v)^2}$.

Thus we have $\dfrac{\partial(x,y)}{\partial(u,v)} = (1)\left(\dfrac{u}{(u+v)^2}\right) - (1)\left(\dfrac{-v}{(u+v)^2}\right) = \dfrac{1}{u+v}$. Therefore,

$$\begin{aligned}
\iint_R \frac{1}{y}\, dA &= \int_1^2 \int_1^2 \frac{u+v}{v} \left|\frac{1}{u+v}\right| \, dv\, du \\
&= \int_1^2 \int_1^2 \frac{1}{v}\, dv\, du \\
&= \int_1^2 1\, du \int_1^2 \frac{1}{v}\, dv \\
&= [u]_1^2 [\ln v]_1^2 \\
&= \ln 2.
\end{aligned}$$

Question 4

(a) Let E be the region bounded by the parameters. We see that E is effectively the region enclosed by the surfaces $x^2 + y^2 + z^2 = 2$ and $z^2 = x^2 + y^2$ in the positive z region. Transforming these surfaces to spherical coordinates, they are $\rho^2 = 2$ and $\tan^2\phi = 1$. Since we are only working with the positive z region, we get the second surface to be $\phi = \frac{\pi}{4}$. Thus E is given by the spherical coordinates, $\rho \in [0, \sqrt{2}]$, $\theta \in [0, \pi]$, $\phi \in [0, \frac{\pi}{4}]$. So we have

$$\begin{aligned}
\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx &= \int_0^{\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} (\rho)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{\pi} 1\, d\theta \int_0^{\frac{\pi}{4}} \sin\phi\, d\phi \int_0^{\sqrt{2}} \rho^3\, d\rho \\
&= [\theta]_0^{\pi} [-\cos\phi]_0^{\frac{\pi}{4}} \left[\frac{1}{4}\rho^4\right]_0^{\sqrt{2}} \\
&= (\pi)\left(1 - \frac{\sqrt{2}}{2}\right) \\
&= \frac{\pi}{2}(2 - \sqrt{2}).
\end{aligned}$$

(b) Let S be the surface with equation $z = g(x,y) = y$ bounded by C, and D be the area of the circle centered at $(0, \frac{1}{2})$ with radius $\frac{1}{2}$. Then we see that S is a smooth surface on area D. Notice that D can be given by the polar coordinates $\theta \in [0, \pi]$, $r \in [0, \sin\theta]$. Now since

$$\begin{aligned}
\text{curl}\boldsymbol{F} &= \langle \frac{\partial}{\partial y}(e^z) - \frac{\partial}{\partial z}(x^2), \frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(e^z), \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy) \rangle \\
&= \langle 0, 0, x \rangle,
\end{aligned}$$

we have by Stoke's Theorem,

$$\begin{aligned}
\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \iint_S \text{curl}\boldsymbol{F} \cdot d\boldsymbol{S} \\
&= \iint_D -(0)(0) - (0)(1) + x \ dA \\
&= \int_0^\pi \int_0^{\sin\theta} (r\cos\theta) r \ dr \ d\theta \\
&= \int_0^\pi \frac{1}{3}\sin^3\theta\cos\theta \ d\theta \\
&= \left[\frac{1}{12}\sin^4\theta\right]_0^\pi \\
&= 0.
\end{aligned}$$

Question 5

(a) For $(x, y, z) \neq (0, 0, 0)$, we have

$$\begin{aligned}
\text{div}\boldsymbol{F} &= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) \\
&= \left(\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right) + \left(\frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right) + \left(\frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right) \\
&= 0.
\end{aligned}$$

(b) Let $D = \{(\phi, \theta) \mid 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$.
Thus the surface T is given by $\boldsymbol{r}(\phi, \theta) = a\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$, $(\phi, \theta) \in D$. This give us $\boldsymbol{r}_\phi = a\langle\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\rangle$ and $\boldsymbol{r}_\theta = a\langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0\rangle$. From this we have $\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta = a^2\langle\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\rangle$. Together with $\sqrt{x^2+y^2+z^2} = a$ on T, we have

$$\begin{aligned}
&\iint_T \boldsymbol{F} \cdot d\boldsymbol{S} \\
=& \iint_D \boldsymbol{F}(\boldsymbol{r}(\phi, \theta)) \cdot (\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta) \ dA \\
=& \iint_D \left(\frac{\sin\phi\cos\theta}{a^2}\right)(a^2\sin^2\phi\cos\theta) + \left(\frac{\sin\phi\sin\theta}{a^2}\right)(a^2\sin^2\phi\sin\theta) + \left(\frac{\cos\phi}{a^2}\right)(a^2\sin\phi\cos\phi) \ dA \\
=& \iint_D \sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta + \sin\phi\cos^2\phi \ dA \\
=& \int_0^{2\pi}\int_0^\pi \sin\phi \ d\phi \ d\theta \\
=& \int_0^{2\pi} 1 \ d\theta \int_0^\pi \sin\phi \ d\phi \\
=& [\theta]_0^{2\pi}[-\cos\phi]_0^\pi \\
=& (2\pi)(1+1) = 4\pi.
\end{aligned}$$

(c) (i) Let E be the region bounded by S_1 and S_2, E' be the region of any sphere centered at the origin that is fully contained in E, and S is the surface of of E', with outward pointing normal. Since div$\boldsymbol{F}$ is defined for all $(x, y, z) \neq (0, 0, 0)$, by Divergence Theorem on $E \cap E'^c$, we get

$$\begin{aligned}
\iint_{S_1} \boldsymbol{F} \cdot d\boldsymbol{S} + \iint_{S_2} \boldsymbol{F} \cdot d\boldsymbol{S} - \iint_{S} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iiint_{E\cap E'^c} \operatorname{div}\boldsymbol{F}\ dV \\
\iint_{S_1} \boldsymbol{F} \cdot d\boldsymbol{S} + \iint_{S_2} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iiint_{E\cap E'^c} 0\ dV + \iint_{S} \boldsymbol{F} \cdot d\boldsymbol{S} \\
&= 0 + 4\pi = 4\pi.
\end{aligned}$$

(ii) For S_1, it lies on the plane $z = 1$. Let D be the area given by the polar coordinates $r \in [0, 2]$, $\theta \in [0, 2\pi]$. We see that S_1 is a surface on D. Thus we can also have,

$$\begin{aligned}
\iint_{S_1} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iint_D \frac{1}{(x^2+y^2+1)^{\frac{3}{2}}}\ dA \\
&= \int_0^{2\pi} \int_0^2 \left(\frac{1}{r^2+1}\right) r\ dr\ d\theta \\
&= \int_0^{2\pi} 1\ d\theta \int_0^2 \left(\frac{1}{r^2+1}\right) r\ dr \\
&= [\theta]_0^{2\pi} \left[\frac{1}{2}\ln(r^2+1)\right]_0^2 \\
&= (2\pi)\left(\frac{1}{2}\ln 5\right) \\
&= \pi \ln 5.
\end{aligned}$$

Therefore we can use (5ci.) result and conclude that

$$\begin{aligned}
\iint_{S_2} \boldsymbol{F} \cdot d\boldsymbol{S} &= 4\pi - \iint_{S_1} \boldsymbol{F} \cdot d\boldsymbol{S} \\
&= 4\pi - \pi \ln 5 \\
&= \pi(4 - \ln 5).
\end{aligned}$$