MA2101S - Linear Algebra II(S) Suggested Solutions

(Semester 2 : AY2016/17)

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Question 1

(a) Fix $t \in \mathbb{F}$. Let $g, h \in \mathbb{F}^{\mathbb{F}}$ and $a, b \in \mathbb{F}$.

Then $\epsilon_t(ag + bh) = (ag + bh)(t) = ag(t) + bh(t) = a\epsilon_t(g) + b\epsilon_t(h)$.

Thus ϵ_t is a linear functional from $\mathbb{F}^{\mathbb{F}}$ to $\mathbb{F}$ so $\epsilon_t \in (\mathbb{F}^{\mathbb{F}})^*$.

(b) Assume, for the sake of contradiction, that $\{\epsilon_t : t \in \mathbb{F}\}$ is linearly dependent.

$\exists$ distinct $t_1, t_2, ..., t_n \in \mathbb{F}$, $a_1, a_2, ..., a_n \in \mathbb{F} \setminus \{0\}$ such that :

$$a_1\epsilon_{t_1} + a_2\epsilon_{t_2} + ... + a_n\epsilon_{t_n} = 0_V.$$

Define $f : \mathbb{F} \to \mathbb{F}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t_1. \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $f \in \mathbb{F}^{\mathbb{F}}$.

$$\begin{aligned} a_1\epsilon_{t_1}(f) + a_2\epsilon_{t_2}(f) + ... + a_n\epsilon_{t_n}(f) &= 0_V \\ a_1 f(t_1) + a_2 f(t_2) + ... + a_n f(t_n) &= 0 \\ a_1 + 0 + ... + 0 &= 0 \\ a_1 &= 0. \end{aligned}$$

This is a contradiction as $a_1 \in \mathbb{F} \setminus \{0\}$ thus the assumption is false and $\{\epsilon_t : t \in \mathbb{F}\}$ is an linearly independent set.

(c)(i) Let $a, b \in \mathbb{F}$. To prove $\epsilon_t|_{P_n} \in (P_n)^*$:

Fix $t \in \mathbb{F}$. Recall in (a) that $\epsilon_t(ag + bh) = a\epsilon_t(g) + b\epsilon_t(h) \quad \forall g, h \in \mathbb{F}^{\mathbb{F}}$.

Since $P_n \subseteq \mathbb{F}^{\mathbb{F}}$, it follows that:

$$\epsilon_t(ag + bh) = a\epsilon_t(g) + b\epsilon_t(h) \quad \forall g, h \in P_n.$$

Thus $\epsilon_t|_{P_n}$ is a linear functional from P_n to $\mathbb{F}$ so $\epsilon_t|_{P_n} \in (P_n)^*$.

To prove $\{\epsilon_t|_{P_n} : t \in T\}$ is linearly independent :

Let T $= \{t_1, t_2, ..., t_k\}$ for $k \leq n$ and consider the homogeneous equation

$$c_1\epsilon_{t_1} + c_2\epsilon_{t_2} + ... + c_k\epsilon_{t_k} = 0_V.$$

For each $t_i \in T$, define $p_i : \mathbb{F} \to \mathbb{F}$ as follows:

$$p_i(x) = (x - t_1)(x - t_2)...(x - t_{i-1})(x - t_{i+1})...(x - t_k).$$

Note that for each p_i, $\deg(p_i) \leq n-1$ and so $p_i \in P_n$.

$$\begin{aligned} c_1\epsilon_{t_1}(p_1) + c_2\epsilon_{t_2}(p_1) + ... + c_k\epsilon_{t_k}(p_1) &= 0 \\ c_1p_1(t_1) + c_2p_1(t_2) + ... + c_kp_1(t_k) &= 0 \\ c_1p_1(t_1) + 0 + ... + 0 &= 0 \\ c_1p_1(t_1) &= 0. \end{aligned}$$

Note that $p_1(t_1) \neq 0$ since each t_i is distinct. Thus we must have $c_1 = 0$. Repeating the algorithm for $t_2, t_3, ..., t_k$, we get: $c_1 = c_2 = ... = c_k = 0$. Thus only the trivial solution exists so $\{\epsilon_t|_{P_n} : t \in T\}$ is a linearly independent set.

(c)(ii) Claim: $\{\epsilon_{t_i}|_{P_n} : 1 \leq i \leq n\}$ is a basis for $(P_n)^*$.

Proof: Since P_n is finite-dimensional, $\dim((P_n)^*) = \dim(P_n) = n$. – (1)

By (c)(i), $\{\epsilon_{t_i}|_{P_n} : 1 \leq i \leq n\}$ is linearly independent and since $\{\epsilon_{t_i}|_{P_n} : 1 \leq i \leq n\}$ contains n elements, $\dim(\text{span}\{\epsilon_{t_i}|_{P_n} : 1 \leq i \leq n\}) = n$. – (2)

Also by (c)(i), each $\epsilon_{t_i}|_{P_n} \in (P_n)^*$ so $\text{span}\{\epsilon_{t_i}|_{P_n} : 1 \leq i \leq n\} \subseteq (P_n)^*$. – (3)

Combining (1), (2) and (3), we conclude that $\{\epsilon_{t_i}|_{P_n} : 1 \leq i \leq n\}$ is a basis for $(P_n)^*$.

Observe that

$$\begin{aligned} \Phi(p(x)) = \sum_{i=1}^{n} \lambda_i p(t_i) \ \ \forall p(x) \in P_n &\iff \Phi(p(x)) = \sum_{i=1}^{n} \lambda_i \epsilon_{t_i}(p(x)) \ \ \forall p(x) \in P_n \\ &\iff \Phi|_{P_n} = \sum_{i=1}^{n} \lambda_i \epsilon_{t_i}|_{P_n}. \end{aligned}$$

In other words, the two equations $\Phi(p(x)) = \sum_{i=1}^{n} \lambda_i p(t_i) \ \ \forall p(x) \in P_n$ and $\Phi|_{P_n} = \sum_{i=1}^{n} \lambda_i \epsilon_{t_i}|_{P_n}$ share the same solutions.

Since $\Phi \in (\mathbb{F}^{\mathbb{F}})^*$, $\Phi|_{P_n} \in (P_n)^*$. As $\{\epsilon_{t_i}|_{P_n} : 1 \leq i \leq n\}$ is a basis for $(P_n)^*$, $\exists$ unique $\lambda_1, \lambda_2, ..., \lambda_n$ such that:

$$\Phi|_{P_n} = \sum_{i=1}^{n} \lambda_i \epsilon_{t_i}|_{P_n}.$$

Thus $\exists$ unique $\lambda_1, \lambda_2, ..., \lambda_n$ such that $\Phi(p(x)) = \sum_{i=1}^{n} \lambda_i p(t_i) \ \ \forall p(x) \in P_n$.

Question 2

(a) To prove 'if' :

$$\begin{aligned} \alpha \circ \beta &= \alpha \circ p(\alpha) \\ &= \alpha \circ (c_n\alpha^n + c_{n-1}\alpha^{n-1} + ... + c_0 I_V) \\ &= c_n\alpha^{n+1} + c_{n-1}\alpha^n + ... + c_0\alpha \\ &= (c_n\alpha^n + c_{n-1}\alpha^{n-1} + ... + c_0 I_V) \circ \alpha \\ &= p(\alpha) \circ \alpha \\ &= \beta \circ \alpha. \end{aligned}$$

To prove 'only if' :

Since V is α-cyclic, $\exists v \in V$ such that the set $\{v, \alpha(v), \alpha^2(v), ...\}$ is a basis for V. (V may be infinite dimensional here)

Then $\beta(v) = d_0 v + d_1\alpha(v) + ... + d_k\alpha^k(v)$ for some $d_0, d_1, ..., d_k \in \mathbb{F}$.

Claim: $\beta = d_0 I_V + d_1\alpha + ... + d_k\alpha^k$.

Proof: Since $\{v, \alpha(v), \alpha^2(v), ...\}$ is a basis for V, it suffices to prove that:

$$\beta(w) = d_0 w + d_1\alpha(w) + ... + d_k\alpha^k(w) \ \ \forall w \in \{v, \alpha(v), \alpha^2(v), ...\}$$

Choose arbitrary $\alpha^j(v) \in \{v, \alpha(v), \alpha^2(v), ...\}$.

$$\begin{aligned}\beta(\alpha^j(v)) &= \alpha^j \circ \beta(v)\\ &= \alpha^j(d_0 v + d_1\alpha(v) + ... + d_k\alpha^k(v))\\ &= d_o\alpha^j(v) + d_1\alpha^{j+1}(v) + ... + d_k\alpha^{j+k}(v)\\ &= d_o\alpha^j(v) + d_1\alpha(\alpha^j(v)) + ... + d_k\alpha^k(\alpha^j(v)).\end{aligned}$$

Choose $p(x) = d_0 + d_1x + ... + d_kx^k$ and we are done.

(b) To prove 'if' :

$$\begin{gathered}\beta = q(\alpha).\\ \beta|_U = q(\alpha)|_U \;\wedge\; \beta|_W = q(\alpha)|_W.\\ \beta|_U = q(\alpha|_U) \;\wedge\; \beta|_W = q(\alpha|_W).\end{gathered}$$

Simply choose $q_U = q_W = q$ and we are done.

To prove 'only if' :

Since $\gcd(m_U(x), m_W(x)) = 1$, $\exists c(x), d(x) \in F[x]$ such that

$$c(x)m_U(x) + d(x)m_W(x) = 1.$$

Then $\forall u \in U$:

$$\begin{aligned}c(\alpha)m_U(\alpha)(u) &= c(\alpha|_U)m_U(\alpha|_U)(u).\\ &= 0_V.\\ d(\alpha)m_W(\alpha)(u) &= u - c(\alpha)m_U(\alpha)(u)\\ &= u.\end{aligned}$$

Similarly, $\forall w \in W$:

$$\begin{aligned}d(\alpha)m_W(\alpha)(w) &= d(\alpha|_W)m_W(\alpha|_W)(w)\\ &= 0_V.\\ c(\alpha)m_U(\alpha)(w) &= w - d(\alpha)m_W(\alpha)(w)\\ &= w.\end{aligned}$$

Let $v \in V$. Write $v = u + w$ for $u \in U, w \in W$.

$$\begin{aligned}&[q_W(\alpha)c(\alpha)m_U(\alpha) + q_U(\alpha)d(\alpha)m_W(\alpha)](v)\\ &= q_W(\alpha)c(\alpha)m_U(\alpha)(u + w) + q_U(\alpha)d(\alpha)m_W(\alpha)(u + w)\\ &= q_W(\alpha)(w) + q_U(\alpha)(u)\\ &= q_W(\alpha|_W)(w) + q_U(\alpha|_U)(u)\\ &= \beta|_W(w) + \beta|_U(u)\\ &= \beta(w) + \beta(u)\\ &= \beta(v).\end{aligned}$$

Since the choice of v is arbitrary, $\beta = q_W(\alpha)c(\alpha)m_U(\alpha) + q_U(\alpha)d(\alpha)m_W(\alpha)$. Choose $q(x) = q_W(x)c(x)m_U(x) + q_U(x)d(x)m_W(x)$ and the proof is complete.

Question 3

To prove (a) $\rightarrow$ (b) :

Since α is normal, α is unitarily diagonalizable. $\exists$ an orthonormal basis B with respect to ϕ such that $[\alpha]_B$ is diagonal.

$$[\alpha]_B = \begin{pmatrix}\lambda_1 & 0 & 0 & ... & 0\\ 0 & \lambda_2 & 0 & ... & 0\\ 0 & 0 & \lambda_3 & ... & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & ... & \lambda_n\end{pmatrix}$$

Each $\lambda_j \in \mathbb{C}$ so we can write $\lambda_j = a_j + ib_j$ for $a_j, b_j \in \mathbb{R}$. In other words:

$$[\alpha]_B = \begin{pmatrix} a_1 & 0 & 0 & ... & 0 \\ 0 & a_2 & 0 & ... & 0 \\ 0 & 0 & a_3 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & a_n \end{pmatrix} + i \begin{pmatrix} b_1 & 0 & 0 & ... & 0 \\ 0 & b_2 & 0 & ... & 0 \\ 0 & 0 & b_3 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & b_n \end{pmatrix}$$

Choose α_1, α_2 such that:

$$[\alpha_1]_B = \begin{pmatrix} a_1 & 0 & 0 & ... & 0 \\ 0 & a_2 & 0 & ... & 0 \\ 0 & 0 & a_3 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & a_n \end{pmatrix}, \ [\alpha_2]_B = \begin{pmatrix} b_1 & 0 & 0 & ... & 0 \\ 0 & b_2 & 0 & ... & 0 \\ 0 & 0 & b_3 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & b_n \end{pmatrix}$$

It is clear that $\alpha = \alpha_1 + i\alpha_2$.

Since $[\alpha_1]_B, [\alpha_2]_B$ only have entries in $\mathbb{R}, \alpha_1, \alpha_2$ are orthogonally diagonalisable and thus are self-adjoint(Remember that B is an orthonormal basis).

Obviously $\alpha_1 \circ \alpha_2 = \alpha_2 \circ \alpha_1$ since $[\alpha_1]_B[\alpha_2]_B = [\alpha_2]_B[\alpha_1]_B$. (Recall that multiplication of diagonal matrices are commutative)

To prove (b) $\rightarrow$ (c):

Since α_1 and α_2 are self-adjoint, α_1 and α_2 are (unitarily) diagonalisable.
Together with the fact that $\alpha_1 \circ \alpha_2 = \alpha_2 \circ \alpha_1$, we conclude that α_1 and α_2 are simultaneously diagonalisable.

$\exists$ basis B' such that $[\alpha_1]_{B'}, [\alpha_2]_{B'}$ are diagonal matrices. Since
$[\alpha]_{B'} = [\alpha_1]_{B'} + i[\alpha_2]_{B'}$, $[\alpha]_{B'}$ is also diagonal. Thus α is diagonalisable.

Write $V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus ... \oplus E_{\lambda_n}$, where each E_{λ_i} is the eigenspace of α associated with eigenvalue λ_i.

Choose arbitrary $E_{\lambda_j} \in \{E_{\lambda_1}, E_{\lambda_2}, ..., E_{\lambda_n}\}$ and let D be an orthonormal basis for E_{λ_j}

$$[\alpha|_{E_{\lambda_j}}]_D = \begin{pmatrix} \lambda_j & 0 & ... & 0 \\ 0 & \lambda_j & .. & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_j \end{pmatrix}, \ [\alpha^\star|_{E_{\lambda_j}}]_D = [\alpha|_{E_{\lambda_j}}]_D^\star = \begin{pmatrix} \overline{\lambda_j} & 0 & ... & 0 \\ 0 & \overline{\lambda_j} & .. & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \overline{\lambda_j} \end{pmatrix}$$

Then $[\alpha^\star|_{E_{\lambda_j}}]_D = \frac{\overline{\lambda_j}}{\lambda_j}[\alpha|_{E_{\lambda_j}}]_D$. By considering the polynomial $g_j(x) = \frac{\overline{\lambda_j}}{\lambda_j}x$, we conclude that $\exists g_j(x) \in \mathbb{C}[x]$ such that $\alpha^\star|_{E_{\lambda_j}} = g_j(\alpha|_{E_{\lambda_j}})$.

Since the choice of E_{λ_j} is arbitrary, $\forall E_{\lambda_j} \in \{E_{\lambda_1}, E_{\lambda_2}, ..., E_{\lambda_n}\}$, $\exists g_j(x) \in \mathbb{C}[x]$ such that
$\alpha^\star|_{E_{\lambda_j}} = g_j(\alpha|_{E_{\lambda_j}})$.

Using the fact that $V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus ... \oplus E_{\lambda_n}$ and Q2 (b), $\exists g(x) \in \mathbb{C}[x]$ such that $\alpha^\star = g(\alpha)$.

Remark: If $\lambda_k = 0$ for some $1 \leq k \leq n$, then $\alpha^\star|_{E_{\lambda_k}} = \alpha|_{E_{\lambda_k}} = 0_V$. Choosing $g(x) = 0$, the proof still holds.

To prove (c) $\rightarrow$ (a) :

$$\begin{aligned} \alpha \circ \alpha^\star &= \alpha \circ g(\alpha) \\ &= \alpha \circ (c_n\alpha^n + c_{n-1}\alpha^{n-1} + ... + c_0 I_V) \\ &= c_n\alpha^{n+1} + c_{n-1}\alpha^n + ... + c_0\alpha \\ &= (c_n\alpha^n + c_{n-1}\alpha^{n-1} + ... + c_0 I_V) \circ \alpha \\ &= g(\alpha) \circ \alpha \\ &= \alpha^\star \circ \alpha. \end{aligned}$$

Thus α is normal.

Question 4

(a)(i) Claim: $\forall$ linear operators of the form $f(\alpha|_U)$ for some $f(x) \in F[x]$,
$f(\alpha|_U)(v) = 0_V \to f(\alpha|_U) = 0_V$.

Proof: One basis of U is of the form $\{v, \alpha(v), ...\}$.

$\forall\ \alpha^j(v) \in \{v, \alpha(v), ...\}$:

$$f(\alpha|_U)(\alpha^j(v)) = f(\alpha|_U)(\alpha^j|_U(v)) = (\alpha^j|_U)f(\alpha|_U)(v) = 0_V.$$

Since $(x-\lambda)^k \in F[x]$ and $(\alpha|_U - \lambda I_V)^k(v) = 0_V, (\alpha|_U - \lambda I_V)^k = 0_V$ by our claim.

Thus $m_{\alpha|_U}(x) \mid (x-\lambda)^k$.

$m_{\alpha|_U}(x)$ is of the form : $(x-\lambda)^j$ for $1 \leq j \leq k$. We now prove that $j = k$. Assume, for the sake of contradiction, that $j < k$. Then $k-1-j \geq 0$ so $(\alpha|_U - \lambda I_V)^{k-1-j}$ exists.

$$\begin{aligned} m_{\alpha|_U}(\alpha|_U)(v) = 0_V &\to (\alpha|_U - \lambda I_V)^j(v) = 0_V \\ &\to (\alpha|_U - \lambda I_V)^{k-1-j}(\alpha|_U - \lambda I_V)^j(v) = 0_V \\ &\to (\alpha - \lambda I_V)^{k-1}(v) = 0_V. \end{aligned}$$

This is a contradiction as $v \notin \ker((\alpha - \lambda I_V)^{k-1})$. Thus the assumption is false and we conclude that $m_{\alpha|_U}(x) = (x-\lambda)^k$.

(ii) Proof that $v \in \ker((\beta - p(\lambda)(I_V))^k)$:

$$\begin{aligned} p(\lambda) - p(\lambda) = 0 &\to \lambda \text{ is a root of the polynomial: } p(x) - p(\lambda) \\ &\to (x-\lambda) \mid (p(x) - p(\lambda)) \\ &\to (x-\lambda)^k \mid (p(x) - p(\lambda))^k \\ &\to m_{\alpha|_U}(x) \mid (p(x) - p(\lambda))^k \\ &\to (p(\alpha) - p(\lambda)(I_V))^k(v) = 0_V \\ &\to v \in \ker((\beta - p(\lambda)(I_V))^k). \end{aligned}$$

Proof that $v \notin \ker((\beta - p(\lambda)(I_V))^{k-1})$:

Assume, for the sake of contradiction, that $v \in \ker((\beta - p(\lambda)(I_V))^{k-1})$.

Then $(p(\alpha) - p(\lambda)(I_V))^{k-1}(v) = 0_V$. Since $(p(x) - p(\lambda))^{k-1} \in F[x]$, by our claim in (a)(i), $(p(\alpha) - p(\lambda)(I_V))^{k-1} = 0_V$.

Thus $m_{\alpha|_U}(x) \mid (p(x) - p(\lambda))^{k-1} \to (x-\lambda)^k \mid (p(x) - p(\lambda))^{k-1}$.

By the pigeonhole principle, $(x-\lambda)^2 \mid (p(x) - p(\lambda))$. Let $g(x) = p(x) - p(\lambda)$. But then:

$$\begin{aligned} g'(x) &= p'(x) - p(\lambda)' \\ &= p'(x). \text{ (Note that here, } p(\lambda) \text{ is a constant)} \end{aligned}$$

$(x-\lambda)^2 \mid g(x) \to g'(\lambda) = 0 \to p'(\lambda) = 0$. This contradicts the given condition that $p'(\lambda) \neq 0$. Thus the assumption is false and we conclude that $v \notin \ker((\beta - p(\lambda)(I_V))^{k-1})$.

(iii) Let U' denote the β-cyclic subspace of V generated by v.

Using the same argument as (a)(i), $m_{\beta|_{U'}}(x) = (x - p(\lambda))^k$.

We first prove that $\dim(U) = \dim(U')$.

Since U is α-cyclic, $c_{\alpha|_U}(x) = m_{\alpha|_U}(x) = (x-\lambda)^k$. Similarly, $c_{\beta|_{U'}}(x) = m_{\beta|_{U'}}(x) = (x - p(\lambda))^k$. Notice that $\deg(c_{\alpha|_U}(x)) = \deg(c_{\beta|_{U'}}(x))$ so $\dim(U) = \dim(U')$.

We then prove that $U' \subseteq U$.

Choose arbitrary $w \in U'$. Since U' is β-cyclic, $\exists k(x) \in F[x]$ such that
$w = k(\beta)(v)$. Then $w = k(p(\alpha))(v)$. In other words, $\exists k'(x) \in F[x]$ such that $w = k'(\alpha)$. Thus $w \in U$ so $U' \subseteq U$.

Since $U' \subseteq U$ and $\dim(U') = \dim(U)$, we conclude that $U' = U$.

(b) Since α has a Jordan canonical form. V can be decomposed as follows:

$$V = W_1 \oplus W_2 \oplus ... \oplus W_t.$$

For each W_i, $\exists$ a basis B_i of W_i such that

$$[\alpha|_{W_i}]_{B_i} = \begin{pmatrix} \lambda_i & 1 & 0 & ... & 0 \\ 0 & \lambda_i & 1 & ... & 0 \\ 0 & 0 & \lambda_i & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \lambda_i \end{pmatrix}$$

with $B_i = \{(\alpha - \lambda_i(I_V))^{n_i-1}(v), (\alpha - \lambda_i(I_V))^{n_i-2}(v), ..., v\}$ for some $v \in W_i$.

Here, $v \in \ker((\alpha - \lambda_i(I_V))^{n_i}) \setminus \ker((\alpha - \lambda_i(I_V))^{n_i-1})$.

Let R_α and R_β denote the α-cyclic and β-cyclic subspace of V generated by v.

Claim: $R_\alpha = \text{span}(B_i)$

Proof: Since every vector in B_i is of the form $(\alpha - \lambda_i(I_V))^j(v)$ for $0 \leq j < n_i$ and $(x - \lambda_i)^j \in F[x]$, it trivially follows that $\text{span}(B_i) \subseteq R_\alpha$.

We have proven in (a)(iii) that for $v \in \ker((\alpha - \lambda_i(I_V))^{n_i}) \setminus \ker((\alpha - \lambda_i(I_V))^{n_i-1})$, the α-cyclic subspace of V generated by v, R_α, has dimension n_i. Thus $\dim(R_\alpha) = n_i = |B_i| = \dim(\text{span}(B_i))$. (Since B_i is a basis) so we conclude that $R_\alpha = \text{span}(B_i)$.

By (a)(ii), $v \in \ker((\beta - p(\lambda_i)(I_V))^{n_i}) \setminus \ker((\beta - p(\lambda_i)(I_V))^{n_i-1})$.

Define $C_i = \{(\beta - p(\lambda_i)(I_V))^{n_i-1}(v), (\beta - p(\lambda_i)(I_V))^{n_i-2}(v), ..., v\}$.

Using a similar argument as above, $\text{span}(C_i) = R_\beta$.
By (a)(iii), $R_\alpha = R_\beta$ and so we get the following equality:

$$W_i = \text{span}(B_i) = R_\alpha = R_\beta = \text{span}(C_i).$$

Thus C_i is another ordered basis for W_i. It is easy to check that:

$$[\beta|_{W_i}]_{C_i} = \begin{pmatrix} p(\lambda_i) & 1 & 0 & ... & 0 \\ 0 & p(\lambda_i) & 1 & ... & 0 \\ 0 & 0 & p(\lambda_i) & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & p(\lambda_i) \end{pmatrix}.$$

Repeat the algorithm for $W_1, W_2, ..., W_t$ to get basis $C_1, C_2, ..., C_t$ respectively.
Let $C = C_1 \cup C_2 \cup ... \cup C_t$ be an ordered basis for V. Finally we get

$$[\beta]_C = \begin{pmatrix} J_{n_1}(p(\lambda_1)) & 0 & 0 & ... & 0 \\ 0 & J_{n_2}(p(\lambda_2)) & 0 & ... & 0 \\ 0 & 0 & J_{n_3}(p(\lambda_3)) & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & J_{n_t}(p(\lambda_t)) \end{pmatrix}$$

as desired.