NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Teo Wei Hao

MA2108S Mathematical Analysis I (version S)
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Question 1

(a) For all $x \in \mathbb{R}$, we have $|\sin x| \le |x|$ and $|\cos x| \le 1$. Thus,

$$\begin{aligned} 0 \le \left|\sin\sqrt{n+1} - \sin\sqrt{n}\right| &= \left|2\cos\frac{\sqrt{n+1}+\sqrt{n}}{2}\sin\frac{\sqrt{n+1}-\sqrt{n}}{2}\right| \\ &= 2\left|\cos\frac{\sqrt{n+1}+\sqrt{n}}{2}\right|\left|\sin\frac{\sqrt{n+1}-\sqrt{n}}{2}\right| \\ &\le 2\left|\frac{\sqrt{n+1}-\sqrt{n}}{2}\right| = \left|\sqrt{n+1}-\sqrt{n}\right|. \end{aligned}$$

Since $\lim_{n\to\infty}\left|\sqrt{n+1}-\sqrt{n}\right| = 0$, by Squeeze Theorem, we have $\lim_{n\to\infty}\left|\sin\sqrt{n+1}-\sin\sqrt{n}\right| = 0$.
Therefore, $\lim_{n\to\infty}\left(\sin\sqrt{n+1}-\sin\sqrt{n}\right) = 0$.

(b) Firstly, it is direct to se that $x_n \ge 0$ for all $n \in \mathbb{N}$.
Also, we have $(1+x_{n+1})(1+x_n) = 1 + x_n + x_{n+1}(1+x_n) = 1 + x_n + a$.
This give us,

$$\begin{aligned} |x_{n+2} - x_{n+1}| &= \left|\frac{a}{1+x_{n+1}} - \frac{a}{1+x_n}\right| \\ &= \frac{a}{(1+x_{n+1})(1+x_n)}|x_{n+1} - x_n| \\ &= \frac{a}{1+x_n+a}|x_{n+1} - x_n| \\ &\le \frac{a}{1+a}|x_{n+1} - x_n|. \end{aligned}$$

Since $a > 0$, we have $\frac{a}{1+a} < 1$, and so (x_n) is a contractive sequence, which is a cauchy sequence.
By Cauchy Convergent Criterion, (x_n) is convergent.

Let $\lim_{n\to\infty} x_n = x$, then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\frac{a}{1+x_n} = \frac{a}{1+x}$.
This give us $x^2 + x - a = 0$. Since $x \ge 0$, we have $x = \frac{-1+\sqrt{1+4a}}{2}$.

Question 2

(i) We shall use the established fact that for all sequence (a_n) and (b_n), we have

$$\begin{aligned} \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n &\le \liminf_{n\to\infty}(a_n + b_n) \\ &\le \liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \\ &\le \limsup_{n\to\infty}(a_n + b_n) \\ &\le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n. \end{aligned}$$

Since $\ln x$ is continuous increasing on $\mathbb{R}$, $\liminf\limits_{n\to\infty} \ln a_n = \ln \liminf\limits_{n\to\infty} a_n$ for all sequence (a_n).
Since $x_n, y_n > 0$ for all $n \in \mathbb{N}$, we can let $a_n = \ln \dfrac{x_n}{y_n}$ and $b_n = \ln y_n$ for all $n \in \mathbb{N}$.
This give us,

$$\liminf_{n\to\infty} \ln \frac{x_n}{y_n} + \liminf_{n\to\infty} \ln y_n \le \liminf_{n\to\infty} \left(\ln \frac{x_n}{y_n} + \ln y_n \right) \le \liminf_{n\to\infty} \ln \frac{x_n}{y_n} + \limsup_{n\to\infty} \ln y_n$$

$$\ln \liminf_{n\to\infty} \frac{x_n}{y_n} + \ln \liminf_{n\to\infty} y_n \le \liminf_{n\to\infty} \ln x_n \le \ln \liminf_{n\to\infty} \frac{x_n}{y_n} + \ln \limsup_{n\to\infty} y_n$$

$$\ln \left(\left(\liminf_{n\to\infty} \frac{x_n}{y_n} \right) \left(\liminf_{n\to\infty} y_n \right) \right) \le \ln \liminf_{n\to\infty} x_n \le \ln \left(\left(\liminf_{n\to\infty} \frac{x_n}{y_n} \right) \left(\limsup_{n\to\infty} y_n \right) \right)$$

$$\left(\liminf_{n\to\infty} \frac{x_n}{y_n} \right) \left(\liminf_{n\to\infty} y_n \right) \le \liminf_{n\to\infty} x_n \le \left(\liminf_{n\to\infty} \frac{x_n}{y_n} \right) \left(\limsup_{n\to\infty} y_n \right).$$

Since $y_n \ge 1$ for all $n \in \mathbb{N}$, we have $\liminf_{n\to\infty} y_n > 0$ and $\limsup_{n\to\infty} y_n > 0$.
Thus $\dfrac{\liminf_{n\to\infty} x_n}{\limsup_{n\to\infty} y_n} \le \liminf\limits_{n\to\infty} \dfrac{x_n}{y_n} \le \dfrac{\liminf_{n\to\infty} x_n}{\liminf_{n\to\infty} y_n}$.

(ii) Using the similar argument as (2i.) on the later half of the established inequality, we have

$$\liminf_{n\to\infty} \ln y_n + \limsup_{n\to\infty} \ln \frac{x_n}{y_n} \le \limsup_{n\to\infty} \left(\ln y_n + \ln \frac{x_n}{y_n} \right) \le \limsup_{n\to\infty} \ln y_n + \limsup_{n\to\infty} \ln \frac{x_n}{y_n}$$

$$\left(\liminf_{n\to\infty} y_n \right) \left(\limsup_{n\to\infty} \frac{x_n}{y_n} \right) \le \limsup_{n\to\infty} x_n \le \left(\limsup_{n\to\infty} y_n \right) \left(\limsup_{n\to\infty} \frac{x_n}{y_n} \right).$$

This give us $\dfrac{\limsup_{n\to\infty} x_n}{\liminf_{n\to\infty} y_n} \le \limsup\limits_{n\to\infty} \dfrac{x_n}{y_n} \le \dfrac{\limsup_{n\to\infty} x_n}{\liminf_{n\to\infty} y_n}$, which is what we wanted.

Question 3

(a) Since a_n is decreasing, for all $n \in \mathbb{N}$, we have $a_{n+1} \le a_i$ for all $i \in \mathbb{N}$, $i \le n$.
This give us $n \displaystyle\sum_{i=1}^{n+1} a_i = n \sum_{i=1}^{n} a_i + n a_{n+1} \le n \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} a_i = (n+1) \sum_{i=1}^{n} a_i$.
Therefore, $b_{n+1} = \dfrac{1}{n+1} \displaystyle\sum_{i=1}^{n+1} a_i \le \frac{1}{n} \sum_{i=1}^{n} a_i = b_n$, i.e. (b_n) is decreasing.
Since (a_n) is positive $b_n = \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} a_i > 0$ for all $n \in \mathbb{N}$, i.e. (b_n) is positive.
Now let $x_n = \displaystyle\sum_{i=1}^{n} a_i$, $y_n = n$ for all $n \in \mathbb{N}$. This give us $\displaystyle\lim_{n\to\infty} \left(\frac{x_{n+1} - x_n}{y_{n+1} - y_n} \right) = \lim_{n\to\infty} a_{n+1} = 0$.
Therefore by Stolz Theorem, $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{x_n}{y_n} = 0$.
Thus we can conclude by Alternating Series Test, that $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

(b) We shall work from the proven fact that $\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$ for all $n \in \mathbb{N}$.
This give us,

$$\begin{aligned} 0 < p_n &\le \left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)^{n}\right]^2 \\ &= \left(1+\frac{1}{n}\right)^{2n}\left(\frac{1}{n}\right)^2 \\ &< \frac{e^2}{n^2}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{e^2}{n^2}$ converges, by Comparison Test, $\sum_{n=1}^{\infty} p_n$ converges.

Question 4

(a) For $|x| \ge 1$, we have,

$$x^2 = \left(x^{2n}\right)^{\frac{1}{n}} \le \left(1+x^{2n}\right)^{\frac{1}{n}} \le \left(2x^{2n}\right)^{\frac{1}{n}} = 2^{\frac{1}{n}}x^2,$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} x^2 = x^2 = \lim_{n\to\infty} 2^{\frac{1}{n}}x^2$, by Squeeze Theorem, we have $\lim_{n\to\infty}\left(1+x^{2n}\right)^{\frac{1}{n}} = x^2$.

For $|x| \le 1$, we have,

$$1 \le \left(1+x^{2n}\right)^{\frac{1}{n}} \le 1 + x^{2n},$$

for all $n \in \mathbb{N}$.
Since $\lim_{n\to\infty}\left(1+x^{2n}\right) = 1$, by Squeeze Theorem, we have $\lim_{n\to\infty}\left(1+x^{2n}\right)^{\frac{1}{n}} = 1$.

Now x^2 and 1 are continuous function on $\mathbb{R}$, thus it suffice to only verify continuity of $\beta(x)$ at $x = \pm 1$. Since $\lim_{x\to 1^-} f(x) = 1 = 1^2 = \lim_{x\to 1^+} f(x)$ and $\lim_{x\to -1^-} f(x) = (-1)^2 = 1 = \lim_{x\to -1^+} f(x)$, we can conclude that $f(x)$ is continuous on $\mathbb{R}$.

(b) Let $a_n = \sqrt{2n\pi + \frac{\pi}{2}}$ and $b_n = \sqrt{2n\pi}$.
Then we have $\lim(a_n - b_n) = 0$.
However, $|f(a_n) - f(b_n)| = |\sin(2n\pi + \frac{\pi}{2}) - \sin 2n\pi| = 1$.
Therefore $f(x)$ is not uniformly continuous on $\mathbb{R}$.

(c) We shall use the established fact that $\lim_{x\to 0} \frac{c^x - 1}{x} = \ln c$ for all $c \in \mathbb{R}^+$.
Also x^n is a continuous function on $\mathbb{R}$ for all $n \in \mathbb{N}$.
Since when $x \to 0$, we have $x^n \to 0$, we get $\lim_{x\to 0} \frac{c^{x^n} - 1}{x^n} = \ln c$ for all $c \in \mathbb{R}^+$.
Thus we have,

$$\begin{aligned} \lim_{x\to 0}\left(\frac{a^{x^n} - b^{x^n}}{(a^x - b^x)^n}\right) &= \lim_{x\to 0}\left(\frac{\frac{a^{x^n}-1}{x^n} - \frac{b^{x^n}-1}{x^n}}{(\frac{a^x-1}{x} - \frac{b^x-1}{x})^n}\right) \\ &= \frac{\ln a - \ln b}{(\ln a - \ln b)^n} = \left(\ln\frac{a}{b}\right)^{1-n}, \end{aligned}$$

where $a > b > 0$ and $n \in \mathbb{N}$.

Question 5

(a) For $r \in \mathbb{R}$, let P_n be the statement that $f(nr) = nf(r)$, $n \in \mathbb{N}$.
Since $f(1 \cdot r) = f(r) = 1 \cdot f(r)$, P_1 is true.
For all $k \in \mathbb{N}$ such that P_k is true, we have,

$$f((k+1)r) = f(kr + r) = f(kr) + f(r) = kf(r) + f(r) = (k+1)f(r),$$

i.e. P_{k+1} is true.
Therefore by Mathematical Induction, P_n is true for all $n \in \mathbb{N}$.

Now, we have $f(0) = f(0+0) = f(0) + f(0)$, i.e. $f(0) = 0$.
Also $0 = f(x + (-x)) = f(x) + f(-x)$.
This give us $f(x) = -f(-x)$ for all $x \in \mathbb{R}$, i.e. f is an odd function.
This concludes for us that $f(ir) = if(r)$ for all $i \in \mathbb{Z}$.

Now let $m \in \mathbb{Z}\backslash\{0\}$.
Since $mf\left(\frac{1}{m}\right) = \left(m \cdot \frac{1}{m}\right) = f(1)$, we have $f\left(\frac{i}{m}\right) = if\left(\frac{1}{m}\right) = \frac{i}{m}f(1)$.
This implies that $f(q) = qf(1)$ for all $q \in \mathbb{Q}$.

Let $s \in \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a sequence (q_n) in $\mathbb{Q}$ such that $\lim_{n\to\infty} q_n = s$.
Since f is continuous on $\mathbb{R}$, we have

$$\begin{aligned} f(s) = f\left(\lim_{n\to\infty} q_n\right) &= \lim_{n\to\infty} f(q_n) \\ &= \lim_{n\to\infty} q_n f(1) \\ &= f(1) \lim_{n\to\infty} q_n = sf(1), \end{aligned}$$

and we are done.

(b) Let P_n be the statement that $a_n = 2\cos\left(\frac{\pi}{2^{1+n}}\right)$.
Since $a_1 = \sqrt{2} = 2\cos\left(\frac{\pi}{2^{1+1}}\right)$, P_1 is true.
For all $k \in \mathbb{N}$ such that P_k is true, we have,

$$\begin{aligned} a_{k+1} = \sqrt{2 + a_k} &= \sqrt{2 + 2\cos\left(\frac{\pi}{2^{1+k}}\right)} \\ &= \sqrt{2 + 2\left(2\cos^2\left(\frac{\pi}{2^{1+(k+1)}}\right) - 1\right)} \\ &= 2\cos\left(\frac{\pi}{2^{1+(k+1)}}\right), \end{aligned}$$

i.e. P_{k+1} is true. Therefore by Mathematical Induction, P_n is true for all $n \in \mathbb{N}$.

Now for $n \geq 2$, since $\sin x \leq x$ for $x \in \mathbb{R}^+$, we have,

$$\begin{aligned} 0 \leq b_n = \sqrt{2 - a_{n-1}} &= \sqrt{2 - 2\cos\left(\frac{\pi}{2^n}\right)} \\ &= \sqrt{2 - 2\left(1 - 2\sin^2\left(\frac{\pi}{2^{n+1}}\right)\right)} \\ &= 2\sin\left(\frac{\pi}{2^{n+1}}\right) \leq 2\left(\frac{\pi}{2^{n+1}}\right) = \frac{\pi}{2^n}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{\pi}{2^n}$ converges, by Limit Comparison Test, $\sum_{n=1}^{\infty} b_n$ converges.