NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Prof Lee Soo Teck

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MA2108 Mathematical Analysis I
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Question 1

(a)

$$T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = 1-2x \Rightarrow \left[T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right]_{\mathcal{B}_2} = \begin{pmatrix}1\\-2\\0\end{pmatrix}$$

$$T\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = 3x-x^2 \Rightarrow \left[T\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right]_{\mathcal{B}_2} = \begin{pmatrix}0\\3\\-1\end{pmatrix}$$

$$T\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = 3-2x^2 \Rightarrow \left[T\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]_{\mathcal{B}_2} = \begin{pmatrix}3\\0\\-2\end{pmatrix}$$

$$T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = 1-2x \Rightarrow \left[T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right]_{\mathcal{B}_2} = \begin{pmatrix}1\\-2\\0\end{pmatrix}$$

Therefore $[T]_{\mathcal{B}_2,\mathcal{B}_1} = \begin{pmatrix}1 & 0 & 3 & 1\\ -2 & 3 & 0 & -2\\ 0 & -1 & -2 & 0\end{pmatrix}$.

(b)

$$\begin{aligned}T\begin{pmatrix}a & b\\ c & d\end{pmatrix} &= (a+3c+d)+(3b-2a-2d)x-(b+2c)x^2\\ &= a(1-2x)+b(3x-x^2)+c(3-2x^2)+d(1-2x)\end{aligned}$$

Hence $\mathcal{R}(T) = \operatorname{Span}\left\{1-2x, 3x-x^2, 3-2x^2\right\}$. Now, by observation,

$$3-2x^2 = 3(1-2x)+2(3x-x^2).$$

Thus $\mathcal{R}(T) = \operatorname{Span}\left\{1-2x, 3x-x^2\right\}$. Furthermore, $\left\{1-2x, 3x-x^2\right\}$ is linearly independent since $1-2x$ is not a linear multiple of $3x-x^2$. We conclude that $\left\{1-2x, 3x-x^2\right\}$ is a basis for $\mathcal{R}(T)$.

(c) By (b), $\operatorname{rank} T = 2$. Therefore, by the dimension theorem, $\operatorname{nullity} T = 4-2 = 2$.

(d)

$$[T\circ S]_{\mathcal{B}_2} = [T]_{\mathcal{B}_2,\mathcal{B}_1}[S]_{\mathcal{B}_1,\mathcal{B}_2} = \begin{pmatrix}1 & 0 & 3 & 1\\ -2 & 3 & 0 & -2\\ 0 & -1 & -2 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 2 & 0 & 1\\ -1 & 0 & 0\\ 0 & -2 & 0\end{pmatrix} = \begin{pmatrix}-3 & -1 & 0\\ 6 & 2 & 3\\ 0 & 0 & -1\end{pmatrix}$$

Since $\mathcal{B}_2$ is the standard basis for $P_2(\mathbb{R})$, we conclude that $(T \circ S)(a + bx + cx^2) = (-3a - b) + (6a + 2b + 3c)x - x^2$.

Question 2

(a) (i) Since the eigenspace of A corresponding to the eigenvalue 3 has dimension 2, the Jordan canonical form will have two Jordan blocks corresponding to 3. In addition, since the multiplicity of eigenvalue 3 is three, the two Jordan blocks must be $J_2(3), J_1(3)$. On the other hand, the multiplicity of eigenvalue -1 is one, hence there will be a $J_1(-1)$. In conclusion, we conclude that the Jordan canonical form of A is either $\operatorname{diag}(J_2(3), J_1(3), J_1(-1), J_2(i))$ or $\operatorname{diag}(J_2(3), J_1(3), J_1(-1), J_1(i), J_1(i))$.

(ii) Respectively, $(x+1)(x-i)^2(x-3)^2$ and $(x+1)(x-i)(x-3)^2$.

(b) Let the minimal polynomial of B be $m_B(x)$.

$$c_B(x) = \det(xI - B) = \begin{vmatrix} x-4 & 0 & -1 \\ -2 & x-3 & -2 \\ -1 & 0 & x-4 \end{vmatrix} = (x-3)^2(x-5)$$

Hence $m_B(x)$ is either $(x-3)(x-5)$ or $(x-3)^2(x-5)$. Consider $(B-3I)(B-5I)$.

$$(B-3I)(B-5I) = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore $m_B(x) = (x-3)(x-5)$.

Question 3

(a) Firstly recall that $\operatorname{Tr}(B^T A) = \sum_{i=1}^2 \sum_{j=1}^2 a_{ij} b_{ij}$. Now, $\{A_1, A_2\}$ is linearly independent since A_1 is not a linear multiple of A_2. Hence $\{A_1, A_2\}$ is a basis for W_1. Let $B_1' = A_1$.

$$\Rightarrow \|B_1'\|^2 = 1$$

Define $B_1 := \frac{B_1'}{\|B_1'\|} = A_1$. Now, let $B_2' = A_2 - \langle A_2, B_1 \rangle B_1$.

$$\Rightarrow B_2' = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} - 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\Rightarrow \|B_2'\|^2 = 2$$

Define $B_2 := \frac{B_2'}{\|B_2'\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Therefore $\{B_1, B_2\}$ is an orthonormal basis for W_1.

(b)

$$\mathbf{proj}_{W_1}(F) = \langle F, B_1 \rangle B_1 + \langle F, B_2 \rangle B_2 = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \sqrt{2} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

(c) Since W_1 is a subspace of $W_1 \oplus W_2$, $\mathbf{proj}_{W_1}(F) \in W_1 \oplus W_2$. Furthermore, $W_1 \perp W_2$.

$$\Rightarrow \mathbf{proj}_{W_2}(F) = \mathbf{proj}_{W_1 \oplus W_2}(F) - \mathbf{proj}_{W_1}(F) = \begin{pmatrix} 2 & 0 \\ -2 & 2 \end{pmatrix}$$

Therefore the smallest value of the set $\{\|F - X\| \,:\, X \in W_2\}$ is $\left\|F - \mathbf{proj}_{W_2}(F)\right\| = \sqrt{30}$.

Question 4

(a) False. Consider $V = \mathbb{R}^2$ with $S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ and $S_2 = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

$$\Rightarrow S_1, S_2 \text{ are linearly independent} \quad \text{and} \quad S_1 \cap S_2 = \emptyset$$

However, $S_1 \cup S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is linearly dependent.

(b) True. Let I and $\mathbf{0}$ be the identity and zero matrix in $M_n(\mathbb{F})$ respectively. Since

$$T(I) = AI - IA = \mathbf{0},$$

$\Rightarrow T$ is not injective. Therefore T is not invertible and thus $\det(T) = 0$.

Question 5

(a) Let $0_{P(\mathbb{R})}$ be the zero operator on $P(\mathbb{R})$, p_0 be the zero polynomial in $P(\mathbb{R})$, and n be a positive integer. Suppose

$$a_0T^0 + a_1T^1 + \cdots + a_nT^n = 0_{P(\mathbb{R})}$$

where the a's are real numbers. Let $f \in P(\mathbb{R})$.

$$\Rightarrow (a_0T^0 + a_1T^1 + \cdots + a_nT^n)(f) = 0_{P(\mathbb{R})}(f)$$
$$\Rightarrow a_0T^0(f) + a_1T^1(f) + \cdots + a_nT^n(f) = p_0$$
$$\Rightarrow a_0f + a_1f' + \cdots + a_nf^{(n)} = p_0$$

Since the choice of f is arbitrary,

$$\Rightarrow a_0 = a_1 = \cdots = a_n = 0.$$

Therefore $\{T^0, T^1, T^2, \ldots, T^n\}$ is linearly independent.

(b) Let S' be a finite subset of S. Then there exist a positive integer k such that for all $T^i \in S'$, $i \leq k$. By (a), the set $\{T^0, T^1, T^2, \ldots, T^k\}$ is linearly independent. Since $S' \subseteq \{T^0, T^1, T^2, \ldots, T^k\}$,

$$\Rightarrow S' \text{ is linearly independent.}$$

Hence S is linearly independent. Furthermore, $|S| = \infty$ and $\mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ is infinite dimensional. Therefore we conclude that S is a basis for $\mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$.

Question 6

(a) Since

$$0\boldsymbol{u_1} + \cdots + 0\boldsymbol{u_n} + 0\boldsymbol{v_1} + \cdots + 0\boldsymbol{v_m} = \mathbf{0},$$

$\Rightarrow (0, \ldots, 0) \in \mathbb{F}^{n+m}$. Now, let $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m), (\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_m) \in \mathbb{F}^{n+m}$ and $x, y \in \mathbb{F}$.

$$\Rightarrow x(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) + y(\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_m) =$$
$$(x\alpha_1 + y\gamma_1, \ldots, x\alpha_n + y\gamma_n, x\beta_1 + y\delta_1, \ldots, x\beta_m + y\delta_m)$$

Now,

$$\begin{aligned}
&(x\alpha_1 + y\gamma_1)\boldsymbol{u_1} + \cdots + (x\alpha_n + y\gamma_n)\boldsymbol{u_n} + (x\beta_1 + y\delta_1)\boldsymbol{v_1} + \cdots + (x\beta_m + y\delta_m)\boldsymbol{v_m} \\
&= x(\alpha_1\boldsymbol{u_1} + \cdots + \alpha_n\boldsymbol{u_n} + \beta_1\boldsymbol{v_1} + \cdots + \beta_m\boldsymbol{v_m}) + y(\gamma_1\boldsymbol{u_1} + \cdots + \gamma_n\boldsymbol{u_n} + \delta_1\boldsymbol{v_1} + \cdots + \delta_m\boldsymbol{v_m}) \\
&= \boldsymbol{0}
\end{aligned}$$

$\Rightarrow x(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) + y(\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_m) \in W$. Therefore W is a subspace of $\mathbb{F}^{n+m}$.

(b) Firstly, observe that $\dim(U_1 \cap U_2) = \dim U_1 + \dim U_2 - \dim(U_1 + U_2) = n + m - \dim(U_1 + U_2)$. Define $T : \mathbb{F}^{n+m} \to V$ by

$$T(a_1, \ldots, a_{n+m}) = a_1\boldsymbol{u_1} + \cdots + a_n\boldsymbol{u_n} + a_{n+1}\boldsymbol{v_1} + \cdots + a_{n+m}\boldsymbol{v_m} \quad \forall (a_1, \ldots, a_{n+m}) \in \mathbb{F}^{n+m}$$

It is easily checked that T is a linear transformation. Furthermore,

$$\operatorname{Ker} T = W \quad \text{and} \quad \mathcal{R}(T) = U_1 + U_2.$$

Therefore, by the dimension theorem, $\dim W = \operatorname{nullity} T = \dim \mathbb{F}^{n+m} - \operatorname{rank} T = n + m - \dim(U_1 + U_2) = \dim(U_1 \cap U_2)$.

Question 7

(a) Observe that $\mathcal{R}(T_1) + \mathcal{R}(T_2) + \cdots + \mathcal{R}(T_k) \subseteq V$. Let $\boldsymbol{v} \in V$. Since $I_V = T_1 + \cdots + T_k$,

$$\Rightarrow \boldsymbol{v} = I_V(\boldsymbol{v}) = T_1(\boldsymbol{v}) + \cdots + T_k(\boldsymbol{v}) \in \mathcal{R}(T_1) + \mathcal{R}(T_2) + \cdots + \mathcal{R}(T_k).$$

Hence $V = \mathcal{R}(T_1) + \mathcal{R}(T_2) + \cdots + \mathcal{R}(T_k)$.
Claim: For each $1 \leq n \leq k - 1$, $[\mathcal{R}(T_1) + \mathcal{R}(T_2) + \cdots + \mathcal{R}(T_n)] \cap \mathcal{R}(T_{n+1}) = \{\boldsymbol{0}\}$.
Let $\boldsymbol{x} \in [\mathcal{R}(T_1) + \mathcal{R}(T_2) + \cdots + \mathcal{R}(T_n)] \cap \mathcal{R}(T_{n+1})$.

$$\Rightarrow \boldsymbol{x} = \boldsymbol{x_1} + \boldsymbol{x_2} + \cdots + \boldsymbol{x_n} \quad \text{where } x_i \in \mathcal{R}(T_i)\ \forall 1 \leq i \leq n$$

Then there exist $\boldsymbol{w}, \boldsymbol{w_1}, \ldots, \boldsymbol{w_n} \in V$ such that $\boldsymbol{x} = T_{n+1}(\boldsymbol{w}), \boldsymbol{x_1} = T_1(\boldsymbol{w_1}), \ldots, \boldsymbol{x_n} = T_n(\boldsymbol{w_n})$.

$$\begin{aligned}
\Rightarrow T_{n+1}(\boldsymbol{w}) &= T_1(\boldsymbol{w_1}) + T_2(\boldsymbol{w_2}) + \cdots + T_n(\boldsymbol{w_n}) \\
\Rightarrow T_{n+1}^2(\boldsymbol{w}) &= T_{n+1}T_1(\boldsymbol{w_1}) + T_{n+1}T_2(\boldsymbol{w_2}) + \cdots + T_{n+1}T_n(\boldsymbol{w_n}) \\
\Rightarrow T_{n+1}(\boldsymbol{w}) &= T_0(\boldsymbol{w_1}) + T_0(\boldsymbol{w_2}) + \cdots + T_0(\boldsymbol{w_n}) \\
\Rightarrow \boldsymbol{x} &= \boldsymbol{0}
\end{aligned}$$

Hence the claim is proven. As a consequence, $V = \mathcal{R}(T_1) \oplus \mathcal{R}(T_2) \oplus \cdots \oplus \mathcal{R}(T_k)$.

(b) For each $1 \leq i \leq k$, let $\boldsymbol{v_i} \in \mathcal{R}(T_i)$. Hence, $\exists \boldsymbol{w_i} \in V$ such that $T_i(\boldsymbol{w_i}) = \boldsymbol{v_i}$.

$$\begin{aligned}
\Rightarrow T(\boldsymbol{v_i}) &= \lambda_1 T_1(\boldsymbol{v_i}) + \cdots + \lambda_i T_i(\boldsymbol{v_i}) + \cdots + \lambda_k T_k(\boldsymbol{v_i}) \\
&= \lambda_1 T_1 T_i(\boldsymbol{w_i}) + \cdots + \lambda_i T_i^2(\boldsymbol{w_i}) + \cdots + \lambda_k T_k T_i(\boldsymbol{w_i}) \\
&= \lambda_1 T_0(\boldsymbol{w_i}) + \cdots + \lambda_i T_i(\boldsymbol{w_i}) + \cdots + \lambda_k T_0(\boldsymbol{w_i}) \\
&= \lambda_i \boldsymbol{v_i}
\end{aligned}$$

That is, the λ's are eigenvalues of T (not necessarily all). Furthermore, from above, $\mathcal{R}(T_i) \subseteq E_{\lambda_i}\ \forall 1 \leq i \leq k$.

$$\Rightarrow V = \mathcal{R}(T_1) \oplus \cdots \oplus \mathcal{R}(T_k) \subseteq E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k} \subseteq V$$

Therefore the λ's are all the eigenvalues of T and $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$. We conclude that T is diagonalizable.

Question 8

(a) Since $(I_V + iT)^* = I_V - iT$,

$$(I_V + iT)(I_V - iT) = I_V + T^2 = (I_V - iT)(I_V + iT).$$

That is, $I_V + iT$ is normal. Therefore it is orthogonally diagonalizable and thus invertible.

(b) Since $S^* = [(I_V + iT)^*]^{-1}(I_V - iT)^* = (I_V - iT)^{-1}(I_V + iT)$,

$$\begin{aligned}
\Rightarrow SS^* &= (I_V - iT)(I_V + iT)^{-1}(I_V - iT)^{-1}(I_V + iT) \\
&= (I_V - iT)[(I_V - iT)(I_V + iT)]^{-1}(I_V + iT) \\
&= (I_V - iT)[(I_V + iT)(I_V - iT)]^{-1}(I_V + iT) \\
&= (I_V - iT)(I_V - iT)^{-1}(I_V + iT)^{-1}(I_V + iT) \\
&= I_V
\end{aligned}$$

Similarly, $S^*S = I_V$. Therefore S is unitary.