NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Teo Wei Hao

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SECTION A

Question 1

(a) (i) Let X_i be i.i.d. r.v. of the lifetime in hours of the i-th component used, $i \in \mathbb{Z}^+$.
We are also given that $\mu = 100$ and $\sigma = 30$. Thus by Central Limit Theorem, we have $\sum_{i=1}^{30} X_i \approx N(3000, 27000)$. As such,

$$\begin{aligned} \mathbb{P}\{\sum_{i=1}^{30} X_i \geq 2900\} \approx \mathbb{P}\left\{Z \geq \frac{2900-3000}{\sqrt{27000}}\right\} &= \mathbb{P}\{Z \geq -0.61\} \\ &= 1 - 0.2709 = 0.7291. \end{aligned}$$

(ii) Let n be the number of components required. Then we see that $n > 30$, and thus we can use Central Limit Theorem to get $\sum_{i=1}^{n} X_i \approx N(100n, 900n)$. By referring to the statistical table, we obtain,

$$\begin{aligned} \mathbb{P}\{Z \geq -1.6449\} = 0.95 &\leq \mathbb{P}\{\sum_{i=1}^{n} X_i \geq 2900\} \\ &\approx \mathbb{P}\left\{Z \geq \frac{2900-100n}{\sqrt{900n}}\right\}. \end{aligned}$$

Thus we conclude that $-1.6449 \geq \frac{2900-100n}{\sqrt{900n}}$. Let $u = \sqrt{900n}$, i.e. $n = \frac{u^2}{900}$. This give us,

$$\begin{aligned} -1.6449 &\geq \frac{2900 - \frac{1}{9}u^2}{u} \\ \frac{1}{9}u^2 - 1.6449u - 2900 &\geq 0. \end{aligned}$$

We solve the above quadratic inequality with $u \geq 0$, and substituting back to get $n \geq 31.7820$. Therefore at least 32 components must be in stock.

(b) Let A_1, A_2, A_3 be the event that the two-headed coin, the fair coin, and the biases coin is selected. Let B be the event that the selected coin shows head when flipped. Then by Baye's rule,

$$\begin{aligned} \mathbb{P}(A_1 \mid B) = \frac{\mathbb{P}(A_1 B)}{\mathbb{P}(B)} &= \frac{\mathbb{P}(B \mid A_1)\mathbb{P}(A_1)}{\mathbb{P}(B \mid A_1)\mathbb{P}(A_1) + \mathbb{P}(B \mid A_2)\mathbb{P}(A_2) + \mathbb{P}(B \mid A_3)\mathbb{P}(A_3)} \\ &= \frac{(1)\left(\frac{1}{3}\right)}{(1)\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{3}\right)} = \frac{4}{9}. \end{aligned}$$

Question 2

(a) We have for $x > 0$,

$$\begin{aligned} f_X(x) = \int_{\mathbb{R}} f(x,y)\ dy &= \int_0^\infty xe^{-x(y+1)}\ dy \\ &= \left[-e^{-x(y+1)}\right]_0^\infty = e^{-x}. \end{aligned}$$

Now for $y > 0$, we have,

$$\begin{aligned} f_Y(y) = \int_{\mathbb{R}} f(x,y)\ dx &= \int_0^\infty xe^{-x(y+1)}\ dx \\ &= \left[\frac{-x}{y+1}e^{-x(y+1)}\right]_0^\infty - \int_0^\infty \frac{-1}{y+1}e^{-x(y+1)}\ dx \\ &= \left[\frac{-1}{(y+1)^2}e^{-x(y+1)}\right]_0^\infty = \frac{1}{(y+1)^2}. \end{aligned}$$

Thus the marginal p.d.f. of X and Y are given by,

$$\begin{aligned} f_X(x) &= \begin{cases} e^{-x}, & x > 0; \\ 0, & \text{otherwise}, \end{cases} \\ f_Y(y) &= \begin{cases} \frac{1}{(y+1)^2}, & y > 0; \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

(b) When $x > 0$, $y > 0$, we have, $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)} = \dfrac{xe^{-x(y+1)}}{e^{-x}} = xe^{-xy}$.
Thus the conditional p.d.f. of Y given that $X = x$ is,

$$f_{Y|X}(y|x) = \begin{cases} xe^{-xy}, & x > 0, y > 0; \\ 0, & \text{otherwise}. \end{cases}$$

(c) We have,

$$\begin{aligned} E(Y \mid X) = \int_{\mathbb{R}} y f_{Y|X}(y|x)\ dy &= \int_0^\infty xye^{-xy}\ dy \\ &= \left[-ye^{-xy}\right]_0^\infty - \int_0^\infty -e^{-xy}\ dy \\ &= \left[-\frac{1}{x}e^{-xy}\right]_0^\infty = \frac{1}{x}. \end{aligned}$$

(d) Let $W = XY$. Then for $w > 0$, we have,

$$\begin{aligned}
F_W(w) = \mathbb{P}\{XY \le w\} &= \int_{\mathbb{R}} \mathbb{P}\{xY \le w \mid X = x\} f_X(x)\ dx \\
&= \int_{\mathbb{R}} \mathbb{P}\left\{Y \le \frac{w}{x} \mid X = x\right\} f_X(x)\ dx \\
&= \int_0^\infty \left(\int_0^{\frac{w}{x}} x e^{-xy}\ dy \right) e^{-x}\ dx \\
&= \int_0^\infty \left[-e^{-xy}\right]_0^{\frac{w}{x}} e^{-x}\ dx \\
&= \int_0^\infty \left(1 - e^{-w}\right) e^{-x}\ dx \\
&= \left(1 - e^{-w}\right) \left[-e^{-x}\right]_0^\infty = 1 - e^{-w}.
\end{aligned}$$

Thus $f_W(w) = \frac{d}{dw} F_W(w) = e^{-w}$.
Therefore the p.d.f. of XY is,

$$f_{XY}(w) = \begin{cases} e^{-w}, & w > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Question 3

(a) We are given that $U \sim U(0, 2\pi)$ and $V \sim \text{Exp}(1)$.
Let $x = \sqrt{2v}\cos u$ and $y = \sqrt{2v}\sin u$. This give us $u = \tan^{-1}\frac{y}{x}$ and $v = \frac{x^2+y^2}{2}$.
Now $\frac{\partial x}{\partial u} = -\sqrt{2v}\sin u$, $\frac{\partial x}{\partial v} = \frac{1}{\sqrt{2v}}\cos u$, $\frac{\partial y}{\partial u} = \sqrt{2v}\cos u$ and $\frac{\partial y}{\partial v} = \frac{1}{\sqrt{2v}}\sin u$.
Thus $J(u,v) = \left(-\sqrt{2v}\sin u\right)\left(\frac{1}{\sqrt{2v}}\sin u\right) - \left(\frac{1}{\sqrt{2v}}\cos u\right)\left(\sqrt{2v}\cos u\right) = -1$, and so $|J(u,v)| = 1$.
Together with the fact that U and V are independent, we have,

$$\begin{aligned}
f_{(X,Y)}(x,y) &= \frac{1}{|J(u,v)|} f_{(U,V)}(u,v) \\
&= (1) f_U(u) f_V(v) \\
&= \frac{1}{2\pi} e^{-v} \\
&= \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}.
\end{aligned}$$

Thus the joint p.d.f. of X and Y is given by,

$$f_{(X,Y)}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}, \quad x, y \in \mathbb{R}.$$

(b) Notice that $f_{(X_1,X_2)}(x_1,x_2) = \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_2^2}\right) = \frac{1}{2\pi} e^{-\frac{x_1^2+x_2^2}{2}}$.
Then from what we found in (3a.), we see that $W = V$, and thus $W \sim \text{Exp}(1)$.
This give us the p.d.f. of W to be,

$$f_W(w) = \begin{cases} e^{-w}, & w > 0; \\ 0, & \text{otherwise.} \end{cases}$$

SECTION B

Question 4

(a) We are given that X and Y are independent r.v. such that $X, Y \sim U(0,1)$.
When $0 < w \leq 1$, we have,

$$\begin{aligned} f_W(w) = \int_{\mathbb{R}} f_X(w-y) f_Y(y)\ dy &= \int_0^1 f_X(w-y)(1)\ dy \\ &= \int_0^w f_X(w-y)\ dy + \int_w^1 f_X(w-y)\ dy \\ &= \int_0^w 1\ dy \\ &= w. \end{aligned}$$

This give us $F_W(w) = \int_{-\infty}^w f_W(w)\ dw = \int_0^w f_W(w)\ dw = \frac{1}{2}w^2$.
When $1 < w \leq 2$, we have,

$$\begin{aligned} f_W(w) = \int_{\mathbb{R}} f_X(w-y) f_Y(y)\ dy &= \int_0^1 f_X(w-y)(1)\ dy \\ &= \int_0^{w-1} f_X(w-y)\ dy + \int_{w-1}^1 f_X(w-y)\ dy \\ &= \int_{w-1}^1 1\ dy \\ &= 2-w. \end{aligned}$$

This give us $F_W(w) = \int_{-\infty}^w f_W(w)\ dw = \frac{1}{2} + \int_1^w f_W(w)\ dw = 2w - \frac{1}{2}w^2 - 1$.
Therefore the c.d.f. of W is,

$$F_W(w) = \begin{cases} 0, & w \leq 0; \\ \frac{1}{2}w^2, & 0 < w \leq 1; \\ 2w - \frac{1}{2}w^2 - 1, & 1 < w \leq 2; \\ 1, & w > 2. \end{cases}$$

(b) (i) Let $W = X_2 + X_3$. Using the result of (4a.) and the fact that X_1, X_2, X_3 are i.i.d. r.v., we get,

$$\begin{aligned} \mathbb{P}\{X_1 > X_2 + X_3\} = \mathbb{P}\{W < X_1\} &= \int_{\mathbb{R}} \mathbb{P}\{W < x_1\} f_{X_1}(x_1)\ dx_1 \\ &= \int_0^1 F_W(x_1)(1)\ dx_1 \\ &= \int_0^1 \frac{1}{2}x_1^2\ dx_1 \\ &= \left[\frac{1}{6}x_1^3\right]_0^1 = \frac{1}{6}. \end{aligned}$$

(ii) Let A_i be the events that X_i is larger than the sum of the other two respectively, for $i = 1, 2, 3$. Let A be the event that the largest of the three is larger than the sum of the other two. Notice that $A = A_1 \cup A_2 \cup A_3$ and the A_i's are mutually exclusive. This give us,

$$\mathbb{P}(A) = \sum_{i=1}^3 \mathbb{P}(A_i) = \sum_{i=1}^3 \frac{1}{6} = \frac{1}{2}.$$

Question 5

(a) Let $f_X(1) = p$ and $\mathbb{P}\{X > 1\} = q$. Notice that $0 < p, q < 1$ such that $p + q = 1$.
We would like to prove that $\mathbb{P}\{X > x\} = q^x$, $x \in \mathbb{Z}^+$.

Let P_n be the statement that $\mathbb{P}\{X > n\} = q^n$, $n \in \mathbb{Z}^+$.
We have P_1 given to be true.
Assume that P_k is true for some $k \in \mathbb{Z}^+$. Then,

$$\begin{aligned}\mathbb{P}\{X > k+1\} &= \mathbb{P}\{X > k+1 \mid X > k\}\mathbb{P}\{X > k\} \\ &= \mathbb{P}\{X > 1\}\mathbb{P}\{X > k\} \\ &= q \cdot q^k = q^{k+1},\end{aligned}$$

i.e. P_{k+1} is true.
Therefore by Mathematical Induction, we have $\mathbb{P}\{X > n\} = q^n$ for all $n \in \mathbb{Z}^+$.

Thus, $f_X(x) = \mathbb{P}\{X > x\} - \mathbb{P}\{X > x+1\} = q^x(1-q) = q^x p$, i.e. $X \sim \text{Geom}(p)$.
Therefore the mean of X is $\frac{1}{p}$.

(b) Let Y_i be the r.v. of the number of rolls required to get the i-th new number. For example, if in 5 rolls the numbers obtained are $(5, 2, 2, 1, 1)$, then $Y_3 = 4$, since 1 is the third number appearing.
Let $X_1 = Y_1$, and $X_i = Y_i - Y_{i-1}$ for $i = 2, 3, 4, 5, 6$.
Since we have a fair dice, $X_i \sim \text{Geom}(p)$, where $p = \dfrac{\text{number of sides not obtained yet}}{\text{total sides}} = \dfrac{7-i}{6}$.
Also let X be the r.v. of the number of rolls before all 6 sides appeared at least once.
Then we have $X = \sum_{i=1}^{6} X_i$. Therefore,

$$\begin{aligned}E(X) = E\left(\sum_{i=1}^{6} X_i\right) &= \sum_{i=1}^{6} E(X_i) \\ &= \sum_{i=1}^{6} \frac{6}{7-i} = \frac{147}{10},\end{aligned}$$

i.e. 14.7 rolls are expected to get the result we wanted.

Question 6

(i) Since each rolls are mutually independent, we have $\mathbb{P}\{X_i = 1\} = \mathbb{P}\{Y_i = 1\} = \frac{1}{6}$, $i = 1, 2, \ldots, n$.

(ii) If $i \neq j$, then similarly as above, we get $\mathbb{P}\{X_i = 1, Y_j = 1\} = \mathbb{P}\{X_i = 1\}\mathbb{P}\{Y_j = 1\} = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$.
If $i = j$, then since it is impossible to get 1 and 2 together in a roll, $\mathbb{P}\{X_i = 1, Y_j = 1\} = 0$.

(iii) We have, $\text{Cov}(X_i, Y_j) = E(X_iY_j) - E(X_i)E(Y_j) = \mathbb{P}\{X_i = 1, Y_j = 1\} - \mathbb{P}\{X_i = 1\}\mathbb{P}\{Y_i = 1\}$.
Thus when $i \neq j$, then $\text{Cov}(X_i, Y_j) = 0$, else $\text{Cov}(X_i, Y_j) = -\frac{1}{36}$.

$$\begin{aligned}\text{Cov}(X, Y) = \text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j\right) &= \left(\sum_{i=1}^{n} \text{Cov}(X_i, Y_i)\right) + \left(\sum_{\substack{1 \le i,j \le n \\ i \neq j}} \text{Cov}(X_i, Y_j)\right) \\ &= \sum_{i=1}^{n}\left(-\frac{1}{36}\right) + 0 = -\frac{n}{36}.\end{aligned}$$