NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1102R Calculus
AY 2011/2012 Sem 1

Question 1

(a) $f(x) = 3x^4 - 8x^3 - 90x^2 \Rightarrow f'(x) = 12x^3 - 24x^2 - 180x = 12x(x-5)(x+3)$
When f is increasing, $f'(x) > 0 \Rightarrow -3 < x < 0$ or $x > 5$
$\therefore f$ is increasing in interval $(-3,0) \cup (5,\infty)$ and decreasing in interval $(-\infty,-3) \cup (0,5)$.

(b) The critical points are $x = -3, 0, 5$
When $x = -3$, $f(-3) = -351$
When $x = 0$, $f(0) = 0$
When $x = 5$, $f(5) = -1375$
$\therefore$ Local minima are $(-3,-351), (5,-1375)$, and local maximum is $(0,0)$.

(c) When f is concave upward, $f''(x) = 36x^2 - 48x - 180 = (3x+5)(x-3) > 0$
$\Rightarrow x < -\frac{5}{3}$ or $x > 3$
$\therefore$ The interval when f is concave upward is $\left(-\infty, -\frac{5}{3}\right) \cup (3,\infty)$.
Similarly f is concave downward on interval $\left(-\frac{5}{3}, 3\right)$.

(d) When $x = -\frac{5}{3}$, $f(-\frac{5}{3}) = -\frac{5125}{27}$
When $x = 3$, $f(3) = -783$
$\therefore$ Points of inflection are $\left(-\frac{5}{3}, -\frac{5125}{27}\right)$ and $(3,-783)$

Question 2

(a) For every $\epsilon > 0$, take $\delta = \min\left(1, \frac{5\epsilon}{7}\right)$. Then, whenever $0 < |x+3| < \delta$, we have:

$$\left|\sqrt{x^2+16} - 5\right| = \frac{x^2-9}{\sqrt{x^2+16}+5} < \frac{|x+3|\,|x+3-6|}{5} \leq \frac{|x+3|^2 + 6\,|x+3|}{5} < \frac{\delta(\delta+6)}{5} \leq \frac{7\delta}{5} \leq \epsilon$$

(b) As f is differentiable at $x = 1$, f is continuous there too.
Now, $\lim_{x\to 1^-} f(x) = a+1$ and $\lim_{x\to 1^+} f(x) = b-1$
Since f is continuous at $x = 1$, $a+1 = b-1 \Rightarrow a-b = -2$

Also, $\lim_{h\to 0^-} \frac{(1+h)^2 + a(1+h) - 1^2 - a(1)}{h} = \lim_{h\to 0^-}(2+a+h) = a+2$
$\lim_{h\to 0^+} \frac{-(1+h)^2 + b + 1^2 - b}{h} = \lim_{h\to 0^+}(h-2) = -2$
Since f is differentiable at $x = 1$, $\therefore a+2 = -2 \Rightarrow a = -4 \Rightarrow b = -2$

Question 3

(a) Notice that $\left|(x-\frac{\pi}{2})^4 \sin(\tan x)\right| \leq (x-\frac{\pi}{2})^4$ as $|\sin x| \leq 1 \; \forall x \geq \frac{\pi}{2}$. As $\lim_{x\to\frac{\pi}{2}^+}(x-\frac{\pi}{2})^4 = 0$, by Squeeze Theorem, $\lim_{x\to\frac{\pi}{2}^+}[(x-\frac{\pi}{2})^4 \sin(\tan x)] = 0$

(b) Consider the following limit

$$\lim_{x\to 0} \ln\left\{\left[\frac{(1+2x)^{\frac{1}{x}}}{e^2}\right]^{\frac{1}{x}}\right\} = \lim_{x\to 0}\frac{1}{x}\ln\left[\frac{(1+2x)^{\frac{1}{x}}}{e^2}\right] = \lim_{x\to 0}\frac{1}{x}\left[\frac{1}{x}\ln(1+2x) - 2\right]$$

By L'Hôpital's Rule, $\lim_{x\to 0}\frac{\ln(1+2x)}{x} = \lim_{x\to 0}\frac{2}{1+2x}$, so:

$$\begin{aligned}\lim_{x\to 0}\frac{1}{x}\left[\frac{1}{x}\ln(1+2x)-2\right] &= \lim_{x\to 0}\frac{1}{x}\left[\frac{2}{1+2x}-2\right]\\ &= \lim_{x\to 0}\frac{1}{x}\left(\frac{-2x}{1+2x}\right)\\ &= \lim_{x\to 0}\frac{-2}{1+2x} = -2\end{aligned}$$

Thus $\lim_{x\to 0}\left[\frac{(1+2x)^{\frac{1}{x}}}{e^2}\right]^{\frac{1}{x}} = e^{-2}$.

Question 4

(a) By considering the dimensions of the corners cut off:

$$\begin{aligned}\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{h}{5-\frac{a}{2}} &= \frac{2h}{10-a}\\ \Rightarrow h &= \frac{10-a}{2\sqrt{3}}\end{aligned}$$

As volume of box $V = \frac{1}{2}a^2 h\sin(\frac{\pi}{3}) = \frac{a^2(10-a)}{8}$, we get $\frac{dV}{da} = \frac{20a-3a^2}{8}$. To maximise V, $\frac{dV}{da} = 0$ and $\frac{d^2V}{da^2} < 0$. Solving the first equation gives $a = 0, \frac{20}{3}$. But as $a > 0, a = \frac{20}{3}$. Thus $V(\frac{20}{3}) = \frac{500}{27}$ and $\frac{d^2V}{da^2} = -10 < 0$. Hence largest volume $= \frac{500}{27}$ cubic inches with $a = \frac{20}{3}$, $h = \frac{5\sqrt{3}}{9}$.

Question 5

(a) By substituting $u = \tan\theta \Rightarrow du = \sec^2\theta \; d\theta$:

$$\begin{aligned}\int \frac{(\tan\theta+4)\sec^2\theta}{\tan\theta(\tan^2\theta+4)}\,d\theta = \int\frac{u+4}{u^3+4u}\,du &= \int\frac{1}{u}+\frac{u-1}{u^2+4}\,du \text{ (Utilising partial fractions)}\\ &= \int\frac{1}{u}+\frac{u}{u^2+4}-\frac{1}{u^2+4}\,du\\ &= \ln|u| + \frac{1}{2}\ln\left|u^2+4\right| - \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)+C\\ &= \ln|\tan\theta| + \frac{1}{2}\ln\left(\tan^2\theta+4\right) - \frac{1}{2}\tan^{-1}\left(\frac{\tan\theta}{2}\right)+C\end{aligned}$$

where C is an arbitrary constant.

(b) Applying integration by parts twice:

$$\begin{aligned}\int_1^e (x\ln x)^2\,dx = \int_1^e (x^2(\ln x)^2)\,dx &= \frac{x^3}{3}(\ln x)^2\Big|_1^e - \int_1^e \frac{2x^2}{3}\ln x\,dx \\ &= \left(\frac{x^3}{3}(\ln x)^2 - \frac{2x^3}{9}\ln x\right)\Big|_1^e + \int_1^e \frac{2x^2}{9}dx \\ &= \left(\frac{x^3}{3}(\ln x)^2 - \frac{2x^3}{9}\ln x + \frac{2x^3}{27}\right)\Big|_1^e \\ &= \frac{e^3}{3} - \frac{2e^3}{9} + \frac{2e^3}{27} - \frac{2}{27} \\ &= \frac{1}{27}(5e^3 - 2)\end{aligned}$$

Question 6

(a) Clearly the functions $\sin x$ and $\cos x$ intersect at $x = \frac{\pi}{4}$. Thus,

$$\begin{aligned}V &= \int_0^{\frac{\pi}{4}} 2\pi(2-x)(\cos^2 x - \sin^2 x)\,dx \\ &= 2\pi\int_0^{\frac{\pi}{4}} (2-x)\cos(2x)\,dx \\ &= 2\pi\int_0^{\frac{\pi}{4}} 2\cos(2x) - x\cos(2x)\,dx \\ &= 2\pi\left[\sin(2x)|_0^{\frac{\pi}{4}} - \left(\frac{x}{2}\sin(2x)\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\frac{1}{2}\sin(2x)\,dx\right)\right] \\ &= 2\pi\left[\sin(2x) - \frac{x\sin(2x)}{2} - \frac{\cos(2x)}{4}\right]\Big|_0^{\frac{\pi}{4}} \\ &= \frac{\pi(10-\pi)}{4}\end{aligned}$$

(b) The function $y = x^{\frac{1}{3}}$ can be transformed into $x = y^3$ for $1 \le y \le 2$. Then $\frac{dx}{dy} = 3y^2$, and:

$$\begin{aligned}S = \int_1^2 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = 2\pi\int_1^2 y^3\sqrt{1+9y^4}\,dx &= \frac{\pi}{27}\left(1+9y^4\right)^{\frac{3}{2}}\Big|_1^2 \\ &= \frac{\pi}{27}(145^{\frac{3}{2}} - 10^{\frac{3}{2}})\end{aligned}$$

Question 7

(a) $(x\ln x)\frac{dy}{dx} + y = 3x^3 \Rightarrow \frac{dy}{dx} + \frac{1}{x\ln x}y = \frac{3x^2}{\ln x}$ An integrating factor is $e^{\int\frac{1}{x\ln x}dx} = e^{\ln(\ln x)} = \ln x$.

Thus we have $\ln x\frac{dy}{dx} + \frac{y}{x} = 3x^2 \Rightarrow \frac{d}{dx}(y\ln x) = 3x^2 \Rightarrow y\ln x = x^3 + c$ where c is a constant to be determined.

Substituting the initial conditions gives $c = \ln 2 - 8$. Thus $y = \frac{x^3+\ln 2-8}{\ln x}$ for $x > 1$.

(b) (i) Solving the differential equation $\frac{dq}{dt} = k(M-q) \Rightarrow \frac{1}{M-q}\frac{dq}{dt} = k$, we get:
$-\ln(M-q) = kt + c$ for some constant c
$\Rightarrow M - q = Ae^{-kt}$ for $A = e^{-c} > 0$
$\Rightarrow q = M + Ae^{-kt}$
As $t = 0, q = 0$, we have $0 = M + Ae^0 \Rightarrow A = -M \Rightarrow q = M(1-e^{-kt})$

Given $t = 1, q = 140$ and $t = 2, q = 200$, we shall solve the following pair of equations:

$$140 = M(1-e^{-k}) \tag{1}$$
$$200 = M(1-e^{-2k}) \tag{2}$$

(2)/(1): $\frac{1-e^{-2k}}{1-e^{-k}} = 1 + e^{-k} = \frac{200}{140} = \frac{10}{7} \Rightarrow k = -\ln\frac{3}{7}$
Substituting $k = -\ln\frac{3}{7}$ into equation (1) gives $140 = M(1-e^{\ln\frac{3}{7}}) \Rightarrow M = 245$

Thus $q = 245\left(1 - e^{t\ln\frac{3}{7}}\right) = 245\left(1 - \left(\frac{3}{7}\right)^t\right)$

(ii) As $t \to \infty$, $\left(\frac{3}{7}\right)^t \to 0$, thus $M \to 245(1-0) = 245$
i.e. the worker is expected to finish 245 units per day eventually.

Question 8

(a) Define function $F(x) = \int_0^x f(x)\,dx$. Note that $F(0) = 0$ and $F(1) = f(0) = f(1)$.

By Mean Value Theorem, there exists $c_1 \in (0,1)$ such that $F'(c_1) = \frac{F(1)-F(0)}{1-0} = F(1)$. By First Fundamental Theorem of Calculus, $F'(c) = f(c) = f(0) = f(1)$.

We now apply Rolle's Theorem twice. There exist c_2, c_3 in $(0,c), (c,1)$ respectively such that $f'(c_2) = 0 = f'(c_3)$. Then again there exists x_0 in (c_2, c_3) such that $f''(x_0) = 0$. The fact that $(c_2, c_3) \subseteq (0,1)$ completes the proof.

(b) Given that $g'''(c) = 0$ and $g^{(4)}(c) > 0$, $\therefore g'''$ has a local minimum at c. $\therefore$ In the neighbourhood of c, for any numbers x_1 and x_2, such that $x_1 < c < x_2$, we have

$$0 = g'''(c) < g'''(x_1) \text{ and } 0 = g'''(c) < g'''(x_2)$$

By Mean Value Theorem, for any numbers x_3 and x_4 in the neighbourhood of c, such that $x_3 < c < x_4$, we can find x_1 and x_2 where $x_3 < x_1 < c < x_2 < x_4$, such that

$$0 < g'''(x_1) = \frac{g''(x_3) - g''(c)}{x_3 - c} \Rightarrow g''(x_3) < 0 \text{ and } 0 < g'''(x_2) = \frac{g''(x_4) - g''(c)}{x_4 - c} \Rightarrow g''(x_4) > 0$$

Similarly, for any number x_5 and x_6 in the neighbourhood of c, such that $x_5 < c < x_6$, we have $0 < g'(x_5)$ and $g'(x_6) < 0$.
This, coupled with the fact that $g'(c) = 0$, proves that g has a local minimum at c.