NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Zheng Shaoxuan

MA2101 Linear Algebra II
AY 2008/2009 Semester 1

Question 1

(a) False. $\boldsymbol{A} = \boldsymbol{I}_2$ is diagonalisable, since it is already diagonal. However the two eigenvalues $\boldsymbol{A}$ has are both 1.

(b) False. Let $V = \left\{ \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \right\}$, where $f(x)$ and $g(x)$ are polynomials.

Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \dots \right\}$. S is an infinite linearly independent set, but S is not a basis of V since $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in V$ but $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin \text{Span}S$.

(c) False. Consider $V = \mathbb{R}^2$, $U = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$, $W = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, $W' = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

$U \oplus W = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\} = \mathbb{R}^2$.

$U \oplus W' = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\} = \mathbb{R}^2$.

$\therefore U \oplus W = U \oplus W'$, but $W \neq W'$.

(d) True. First we prove the claim $(\boldsymbol{A}^T)^2 = (\boldsymbol{A}^2)^T$ for any square matrix $\boldsymbol{A}$.

Write $\boldsymbol{A}$ as $(a_{ij})_{n\times n}$. Then, $(\boldsymbol{A}^T)^2 = (a_{ji})^2_{n\times n} = (c_{ij})_{n\times n}$, where $c_{ij} = \sum_{k=1}^n a_{ki}a_{jk}$. Also, $(\boldsymbol{A}^2)^T = (d_{ij})^T_{n\times n} = (d_{ji})_{n\times n}$, where $d_{ij} = \sum_{k=1}^n a_{ik}a_{kj}$. Since $d_{ji} = \sum_{k=1}^n a_{jk}a_{ki} = c_{ij}$, our claim $(\boldsymbol{A}^T)^2 = (\boldsymbol{A}^2)^T$ is true.

For any orthogonal $\boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}$. Hence,

$$\begin{aligned} \boldsymbol{A}^2(\boldsymbol{A}^2)^T &= \boldsymbol{A}^2(\boldsymbol{A}^T)^2 \\ &= \boldsymbol{A}\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{A}^T \\ &= \boldsymbol{A}\boldsymbol{I}\boldsymbol{A}^T \\ &= \boldsymbol{A}\boldsymbol{A}^T \\ &= \boldsymbol{I}. \end{aligned}$$

Therefore, $\boldsymbol{A}^2$ is also orthogonal.

Question 2

(i) The additive identity (a_0, b_0) is such that for any $(a, b) \in \mathbb{R}^2$, $(a, b) + (a_0, b_0) = (a, b)$. So, $(a + a_0, b + b_0) = (a, b)$, and hence, $(a_0, b_0) = (0, 0)$.

The multiplicative identity (a_1, b_1) is such that for any $(a, b) \in \mathbb{R}^2$, $(a, b) \times (a_1, b_1) = (a, b)$. So, $(a \times a_1, b \times b_1) = (a, b)$, and hence, $(a_1, b_1) = (1, 1)$.

(ii) Consider $(0,1)$, which is not the additive identity $(0,0)$. For all (a,b), $(0,1)\times(a,b) = (0,b) \neq (1,1)$. Hence, $(0,1)$ has no multiplicative identity. This violates (M5) and therefore, $(\mathbb{R}^2, +, \times)$ is not a field.

Question 3

(i) First of all, $\boldsymbol{A0} = \boldsymbol{0} = \boldsymbol{0A}$. Therefore $\boldsymbol{0} \in W$ and hence W is non-empty.

For any $\boldsymbol{W}_1, \boldsymbol{W}_2 \in W$, $\alpha, \beta \in \mathbb{F}$,

$$\begin{aligned}\boldsymbol{A}(\alpha\boldsymbol{W}_1 + \beta\boldsymbol{W}_2) &= \alpha\boldsymbol{AW}_1 + \beta\boldsymbol{AW}_2 \\ &= \alpha\boldsymbol{W}_1\boldsymbol{A} + \beta\boldsymbol{W}_2\boldsymbol{A} \\ &= (\alpha\boldsymbol{W}_1 + \beta\boldsymbol{W}_2)\boldsymbol{A}.\end{aligned}$$

Hence, $\alpha\boldsymbol{W}_1 + \beta\boldsymbol{W}_2 \in W$ and therefore, W is a subspace of $\mathcal{M}_{nn}(\mathbb{R})$

(ii) For any $\boldsymbol{A}^k$, $k = 0, 1, \ldots, r-1$, $\boldsymbol{AA}^k = \boldsymbol{A}^{k+1} = \boldsymbol{A}^k\boldsymbol{A}$. Therefore, $\boldsymbol{A}^k \in W$.

Suppose $\boldsymbol{I}, \boldsymbol{A}, \ldots, \boldsymbol{A}^{r-1}$ are not linearly independent vectors. Then there exists $\alpha_0, \ldots, \alpha_{r-1} \in \mathbb{F}$, not all 0, such that $\alpha_0\boldsymbol{I} + \alpha_1\boldsymbol{A} + \ldots + \alpha_{r-1}A^{r-1} = \boldsymbol{0}$.

Let i be the biggest subscript such that $\alpha_i \neq 0$. Note that $i < r$. Then, $\alpha_0\boldsymbol{I} + \alpha_1\boldsymbol{A} + \ldots + \alpha_i\boldsymbol{A}^i = \boldsymbol{0}$. Since $\alpha_i \neq 0$, $\boldsymbol{A}^i + \dfrac{\alpha_{i-1}}{\alpha_i}\boldsymbol{A}^{i-1} + \ldots + \dfrac{\alpha_0}{\alpha_i}\boldsymbol{I} = \boldsymbol{0}$. Let $m'_{\boldsymbol{A}}(x) = x^i + \dfrac{a_{i-1}}{a_i}x^{i-1} + \ldots + \dfrac{a_0}{a_i}$. Then $m'_{\boldsymbol{A}}(\boldsymbol{A}) = \boldsymbol{0}$.

Now, $m_{\boldsymbol{A}}(x)$ is the minimum polynomial of $\boldsymbol{A}$ and it has degree r. However, $m'_{\boldsymbol{A}}(x)$ is a polynomial satisfying $m'_{\boldsymbol{A}}(\boldsymbol{A}) = \boldsymbol{0}$, and with degree i, less than that of the degree of $m_{\boldsymbol{A}}(x)$, a contradiction.

Hence, $\boldsymbol{I}, \boldsymbol{A}, \ldots, \boldsymbol{A}^{r-1}$ are linearly independent vectors in W.

Question 4

We claim that the set $A = \text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$ is a subspace of V of dimension r.

First of all, $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly independent as B is a basis of V. Therefore, since $r < n$, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ are also linearly independent. Hence $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ is a basis of A and therefore A has dimension r.

Furthermore, $\boldsymbol{0} \in A$, hence A is non-empty.

For any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in A$, let $\boldsymbol{x}_1 = \alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \ldots + \alpha_r\boldsymbol{v}_r$ and $\boldsymbol{x}_2 = \beta_1\boldsymbol{v}_1 + \beta_2\boldsymbol{v}_2 + \ldots + \beta_r\boldsymbol{v}_r$, for some $\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2, \ldots, \beta_r \in \mathbb{F}$. Then, for all $p, q \in \mathbb{F}$,

$$\begin{aligned}p\boldsymbol{x}_1 + q\boldsymbol{x}_2 &= p(\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \ldots + \alpha_r\boldsymbol{v}_r) + q(\beta_1\boldsymbol{v}_1 + \beta_2\boldsymbol{v}_2 + \ldots + \beta_r\boldsymbol{v}_r) \\ &= (p\alpha_1 + q\beta_1)\boldsymbol{v}_1 + (p\alpha_2 + q\beta_2)\boldsymbol{v}_2 + \ldots + (p\alpha_r + q\beta_r)\boldsymbol{v}_r \in \text{Span}\ \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}.\end{aligned}$$

Hence, $p\boldsymbol{x}_1 + q\boldsymbol{x}_2 \in A$, and therefore A is a subspace of V of dimension r.

Question 5

(i) $\dim(\mathbb{R}^n) = n$, $\dim(P_m(\mathbb{R})) = m + 1$.

(ii) If $m + 1 < n$, then $\dim(P_m(\mathbb{R})) < \dim(\mathbb{R}^n)$. Also, since $P_m(\mathbb{R})$ is the codomain of T, $\text{rank}(T) \leq \dim(P_m(\mathbb{R}))$.

By the Dimension Theorem, $\text{rank}(T) + \text{nullity}(T) = \dim(\mathbb{R}^n)$. Hence,

$$\begin{aligned} \text{nullity}(T) &= \dim(\mathbb{R}^n) - \text{rank}(T) \\ &\geq \dim(\mathbb{R}^n) - \dim(P_m(\mathbb{R})) \\ &> 0. \end{aligned}$$

Therefore, $\text{Ker}(T) \neq \{\mathbf{0}\}$.

(iii) Since $m+1 = n$, $\dim(P_m(\mathbb{R})) = \dim(\mathbb{R}^n)$. Therefore, to show T is bijective it suffices to show that T is injective.

Since the null space of $[T]_{B_2,B_1} = \{\mathbf{0}\}$, the only $\boldsymbol{v} \in \mathbb{R}^n$ such that $T(v) = 0$ is $\boldsymbol{v} = \mathbf{0}$. Hence $\text{Ker}(T) = \{\mathbf{0}\}$ and therefore T is injective. From the above, T is bijective.

Question 6

(i) From the given formula of T_1 in the question, we have: $T_1(1) = -1 + x$, $T_1(x) = 1 - x + x^3$, $T_1(x^2) = x + x^2 + 2x^3$. Since $B_2 = \{1, x, x^2, x^3\}$, we have

$$[T_1(1)]_{B_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, [T_1(x)]_{B_2} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, [T_1(x^2)]_{B_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

Therefore, since $B_1 = \{1, x, x^2\}$,

$$\begin{aligned} [T_1]_{B_2,B_1} &= \begin{pmatrix} [T_1(1)]_{B_2} & [T_1(x)]_{B_2} & [T_1(x^2)]_{B_2} \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}. \end{aligned}$$

(ii) By formula,

$$\begin{aligned} [T_2 \circ T_1]_{B_1} &= [T_2]_{B_1,B_2}[T_1]_{B_2,B_1} \\ &= \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Question 7

(i) For any linear transformation T, $T(\mathbf{0}) = \mathbf{0}$. Note that T^{j-i} is defined since $j > i$.

So, $\forall \boldsymbol{x} \in \text{Ker}T^i$,

$$\begin{aligned} & T^i(\boldsymbol{x}) = \mathbf{0} \\ \Rightarrow\ & T^{j-i} \circ T^i(\boldsymbol{x}) = \mathbf{0} \\ \Rightarrow\ & T^j(x) = \mathbf{0} \end{aligned}$$

Hence, $\boldsymbol{x} \in \text{Ker}T^j$. Therefore, $\text{Ker}T^i \subset \text{Ker}T^j$.

(ii) For any $\boldsymbol{x} \in V$, $\boldsymbol{x} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$. Now,

$$\begin{aligned} T^3(\boldsymbol{x}) &= T \circ T \circ T(\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3) \\ &= T \circ T(\alpha_1 T(\boldsymbol{u}_1) + \alpha_2 T(\boldsymbol{u}_2) + \alpha_3 T(\boldsymbol{u}_3)) \\ &= T \circ T(\alpha_1 \boldsymbol{u}_2 + \alpha_2 \boldsymbol{u}_3) \\ &= T(\alpha_1 \boldsymbol{u}_3) \\ &= \boldsymbol{0}. \end{aligned}$$

Hence, $\mathrm{Ker}T^3 = V$.

Note that $\boldsymbol{u}_1 \in \mathrm{Ker}T^3$ but $T^2(\boldsymbol{u}_1) = \boldsymbol{u}_3 \neq 0$. Hence, $\boldsymbol{u}_1 \notin \mathrm{Ker}T^2$. Therefore, $\mathrm{Ker}T^2 \neq \mathrm{Ker}T^3$.

Similarly, $\boldsymbol{u}_2 \in \mathrm{Ker}T^2$ but $T(\boldsymbol{u}_2) = \boldsymbol{u}_3 \neq 0$. Hence, $\boldsymbol{u}_2 \notin \mathrm{Ker}T$. Therefore, $\mathrm{Ker}T \neq \mathrm{Ker}T^2$.

Therefore, $\mathrm{Ker}T \neq \mathrm{Ker}T^2 \neq \mathrm{Ker}T^3 = V$.

(iii) Since $\mathrm{Ker}T^3 = V$, a basis for $\mathrm{Ker}T^3$ is $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$.

We claim that $\mathrm{Ker}T^2 = \mathrm{Span}\{\boldsymbol{u}_2, \boldsymbol{u}_3\}$.

To prove this, first we note that for any $\alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3 \in \mathrm{Span}\{\boldsymbol{u}_2, \boldsymbol{u}_3\}$, $T^2(\alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3) = 0$. Hence, $\alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3 \in \mathrm{Ker}T^2$ and therefore $\mathrm{Span}\{\boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \mathrm{Ker}T^2$.

Subsequently, for any $\boldsymbol{x} \in \mathrm{Ker}T^2$, let $\boldsymbol{x} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$. Then, $\boldsymbol{0} = T^2(\boldsymbol{x}) = \alpha_1 \boldsymbol{u}_3$. Hence, since $\boldsymbol{u}_3 \neq \boldsymbol{0}$, $\alpha_1 = 0$. So, $x = \alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3$ and hence, $x \in \mathrm{Span}\{\boldsymbol{u}_2, \boldsymbol{u}_3\}$. Therefore, $\mathrm{Ker}T^2 \subseteq \mathrm{Span}\{\boldsymbol{u}_2, \boldsymbol{u}_3\}$. This, with the above, proves the claim $\mathrm{Ker}T^2 = \mathrm{Span}\{\boldsymbol{u}_2, \boldsymbol{u}_3\}$.

Similarly, we claim that $\mathrm{Ker}T = \mathrm{Span}\{\boldsymbol{u}_3\}$.

To prove this, first we note that for any $\alpha_3 \boldsymbol{u}_3 \in \mathrm{Span}\{\boldsymbol{u}_3\}$, $T(\alpha_3 \boldsymbol{u}_3) = 0$. Hence, $\alpha_3 \boldsymbol{u}_3 \in \mathrm{Ker}T$ and therefore $\mathrm{Span}\{\boldsymbol{u}_3\} \subseteq \mathrm{Ker}T$.

Subsequently, for any $\boldsymbol{x} \in \mathrm{Ker}T$, let $\boldsymbol{x} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$. Then, $\boldsymbol{0} = T(\boldsymbol{x}) = \alpha_1 \boldsymbol{u}_2 + \alpha_2 \boldsymbol{u}_3$. Hence, since $\{\boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a subset of the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, $\{\boldsymbol{u}_2, \boldsymbol{u}_3\}$ is linearly independent and hence $\alpha_1 = \alpha_2 = 0$. So, $x = \alpha_3 \boldsymbol{u}_3$ and hence, $x \in \mathrm{Span}\{\boldsymbol{u}_3\}$. Therefore, $\mathrm{Ker}T \subseteq \mathrm{Span}\{\boldsymbol{u}_3\}$. This, with the above, proves the claim $\mathrm{Ker}T = \mathrm{Span}\{\boldsymbol{u}_3\}$.

Finally we note that since $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a basis and hence a linearly independent set, we have $\{\boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $\{\boldsymbol{u}_3\}$ to be linearly independent sets as well. Therefore, a basis of $\mathrm{Ker}T^2$ is $\{\boldsymbol{u}_2, \boldsymbol{u}_3\}$ and a basis of $\mathrm{Ker}T$ is $\{\boldsymbol{u}_3\}$.

Question 8

(i) The characteristic polynomial of $\boldsymbol{A}$ is

$$\begin{aligned} c_{\boldsymbol{A}}(\lambda) &= \det(\lambda \boldsymbol{I} - \boldsymbol{A}) \\ &= \det \begin{pmatrix} \lambda - 1 & -1 & -a \\ 0 & \lambda - 1 & 0 \\ 0 & -1 & \lambda - 2 \end{pmatrix} \\ &= (\lambda - 1) \det \begin{pmatrix} \lambda - 1 & 0 \\ -1 & \lambda - 2 \end{pmatrix} \\ &= (\lambda - 1)^2 (\lambda - 2). \end{aligned}$$

(ii) From the characteristic polynomial, the dimension of the eigenspace E_2 is 1 and the dimension of the eigenspace E_1 is either 1 or 2. For A to be diagonalisable, it suffices to check that the dimension of E_1 is 2.

For $\lambda = 1$, E_1 is the null space of $(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{pmatrix} 0 & -1 & -a \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}$. For E_1 to have dimension 2, row operations from row 1 onto row 3 must cause row 3 to be all 0s. Therefore $a = 1$.

(iii) Since A is not diagonalisable, A must contain the Jordan block $J_2(1)$ rather than the two Jordan blocks $J_1(1)$, $J_1(1)$. Therefore, the only Jordan form of A, up to rearrangement of the Jordan blocks, is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Question 9

(i) The characteristic polynomial of $\boldsymbol{A}$ is

$$\begin{aligned} c_{\boldsymbol{A}}(\lambda) &= \det(\lambda \boldsymbol{I} - \boldsymbol{A}) \\ &= \det \begin{pmatrix} \lambda & 0 & -1 & 1 \\ -2 & \lambda - 2 & 1 & -1 \\ 0 & 0 & \lambda + 1 & -1 \\ 0 & 0 & 1 & \lambda - 1 \end{pmatrix} \\ &= (\lambda - 2) \det \begin{pmatrix} \lambda & -1 & 1 \\ 0 & \lambda + 1 & -1 \\ 0 & 1 & \lambda - 1 \end{pmatrix} \\ &= \lambda(\lambda - 2) \det \begin{pmatrix} \lambda + 1 & -1 \\ 1 & \lambda - 1 \end{pmatrix} \\ &= \lambda(\lambda - 2)(\lambda^2 - 1 + 1) \\ &= \lambda^3(\lambda - 2). \end{aligned}$$

Therefore, its roots are $\lambda = 0, 2$.

Hence, the possible minimal polynomials of $\boldsymbol{A}$ are: $\lambda(\lambda - 2)$, $\lambda^2(\lambda - 2)$, and $\lambda^3(\lambda - 2)$.

(ii) We check for minimal polynomials of $\boldsymbol{A}$ by substituting $\boldsymbol{A}$ for λ into the various possible minimal polynomials of $\boldsymbol{A}$.

Observe that $\boldsymbol{A} - 2\boldsymbol{I} = \begin{pmatrix} -2 & 0 & 1 & -1 \\ 2 & 0 & -1 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$. Hence $\boldsymbol{A}(\boldsymbol{A} - 2\boldsymbol{I}) = \begin{pmatrix} 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 2 & -2 \end{pmatrix} \neq \boldsymbol{0}$.

Therefore $\lambda(\lambda - 2)$ is not the minimal polynomial of $\boldsymbol{A}$.

Furthermore observe $\boldsymbol{A}^2(\boldsymbol{A} - 2\boldsymbol{I}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \boldsymbol{0}$. Therefore $\lambda^2(\lambda - 2)$ is the minimal polynomial of $\boldsymbol{A}$.

From the minimal polynomial of $\boldsymbol{A}$, we can tell that the largest Jordan block corresponding to $\lambda = 0$ is $J_2(0)$. Hence the Jordan canonical form of $\boldsymbol{A}$ is constructed with the Jordan blocks $J_2(0)$, $J_1(0)$ and $J_1(2)$. Therefore, a Jordan canonical form of $\boldsymbol{A}$ would be $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

Question 10

(i) For any $\boldsymbol{u} = (u_1, u_2, u_3, u_4), \boldsymbol{v} = (v_1, v_2, v_3, v_4), \boldsymbol{w} = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$, and for any $\alpha \in \mathbb{R}$, we check for the validity of (IP1) to (IP4).

For (IP1),

$$\begin{aligned} \langle \boldsymbol{u}, \boldsymbol{v} \rangle &= u_1v_1 + 2u_2v_2 + 3u_3v_3 + 4u_4v_4 \\ &= v_1u_1 + 2v_2u_2 + 3v_3u_3 + 4v_4u_4 \\ &= \langle \boldsymbol{v}, \boldsymbol{u} \rangle. \end{aligned}$$

For (IP2),

$$\begin{aligned} \langle \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{w} \rangle &= (u_1+v_1)w_1 + 2(u_2+v_2)w_2 + 3(u_3+v_3)w_3 + 4(u_4+v_4)w_4 \\ &= u_1w_1 + v_1w_1 + 2u_2w_2 + 2v_2w_2 + 3u_3w_3 + 3v_3w_3 + 4u_4w_4 + 4v_4w_4 \\ &= (u_1w_1 + 2u_2w_2 + 3u_3w_3 + 4u_4w_4) + (v_1w_1 + 2v_2w_2 + 3v_3w_3 + 4v_4w_4) \\ &= \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle. \end{aligned}$$

For (IP3),

$$\begin{aligned} \langle \alpha\boldsymbol{u}, \boldsymbol{v} \rangle &= \alpha u_1v_1 + 2\alpha u_2v_2 + 3\alpha u_3v_3 + 4\alpha u_4v_4 \\ &= \alpha(u_1v_1 + 2u_2v_2 + 3u_3v_3 + 4u_4v_4) \\ &= \alpha\langle \boldsymbol{u}, \boldsymbol{v} \rangle. \end{aligned}$$

For (IP4), $\langle \mathbf{0}, \mathbf{0} \rangle = \mathbf{0}$, and for $\boldsymbol{v} \neq \mathbf{0}$,

$$\begin{aligned} \langle \boldsymbol{v}, \boldsymbol{v} \rangle &= v_1v_1 + 2v_2v_2 + 3v_3v_3 + 4v_4v_4 \\ &= v_1^2 + 2v_2^2 + 3v_3^2 + 4v_4^2 \\ &> 0. \end{aligned}$$

Therefore, $\langle ., . \rangle$ defines an inner product on $\mathbb{R}^4$.

(ii) Let $\boldsymbol{x}_1 = (1, -2, 1, -1)$, $\boldsymbol{x}_2 = (2, -3, 2, -3)$, $\boldsymbol{x}_3 = (3, -5, 3, -4)$, $\boldsymbol{x}_4 = (-1, 1, -1, 2)$, where $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$ spans W. Then, for any $\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$, $v \in W^\perp$ if and only if $\langle v, u_i \rangle = 0$ for $i = 1, 2, 3, 4$.

Hence we have: $\left\{ \begin{array}{rcrcrcrcl} v_1 & - & 4v_2 & + & 3v_3 & - & 4v_4 & = & 0 \\ 2v_1 & - & 6v_2 & + & 6v_3 & - & 12v_4 & = & 0 \\ 3v_1 & - & 10v_2 & + & 9v_3 & - & 16v_4 & = & 0 \\ -v_1 & + & 2v_2 & - & 3v_3 & + & 8v_4 & = & 0 \end{array} \right.$

Solving by Gaussian elimination,

$$\left(\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 0 \\ 2 & -6 & 6 & -12 & 0 \\ 3 & -10 & 9 & -16 & 0 \\ -1 & 2 & -3 & 8 & 0 \end{array} \right) \xrightarrow[R_4+R_1]{\substack{R_2-2R_1 \\ R_3-3R_1}} \left(\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 0 \\ 0 & 2 & 0 & -4 & 0 \\ 0 & 2 & 0 & -4 & 0 \\ 0 & -2 & 0 & 4 & 0 \end{array} \right) \xrightarrow[R_4+R_2]{R_3-R_2} \left(\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 0 \\ 0 & 2 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

Let $v_4 = s$, $v_3 = t$. Then, $v_2 = 2s$, $v_1 = 4v_2 - 3v_3 + 4v_4 = 12s - 3t$. So, $v = \begin{pmatrix} 12 \\ 2 \\ 0 \\ 1 \end{pmatrix} s + \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} t.$

Therefore, $W^\perp = \mathrm{Span}\left\{ \begin{pmatrix} 12 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$

Question 11

(i) Since for any $\boldsymbol{x} \in E_1, \boldsymbol{y} \in E_2$, $\langle x, y\rangle = 0$, and that $E_1 \cup E_2 = \mathbb{R}^4$, to find an orthonormal basis of $\mathbb{R}^4$ consisting of eigenvectors of A, it suffices for us to find an orthonormal basis of E_1 and an orthonormal basis of E_2 (whose elements are all eigenvectors of $\boldsymbol{A}$), and take the union of these two orthonormal bases together to form the desired orthonormal basis of $\mathbb{R}^4$.

The orthonormal basis of E_2 is $\left\{ \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$.

For the orthonormal basis of E_1, we perform the Gram-Schmidt algorithm.

Let $\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\boldsymbol{u}_3 = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}$. Then,

$$\boldsymbol{v}_1 = \frac{1}{\|\boldsymbol{u}_1\|}\boldsymbol{u}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$\begin{aligned}
\boldsymbol{v}_2' &= \boldsymbol{u}_2 - \langle \boldsymbol{u}_2, \boldsymbol{v}_1\rangle \boldsymbol{v}_1 \\
&= \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{4}\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \\
\boldsymbol{v}_2 &= \frac{1}{\|\boldsymbol{v}_2'\|}\boldsymbol{v}_2' = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{v}_3' &= \boldsymbol{u}_3 - \langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle \boldsymbol{v}_1 - \langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle \boldsymbol{v}_2 \\
&= \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix} - \frac{1}{4} \left\langle \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \left\langle \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix} - \frac{1}{4}(-2) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2}(3) \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{3}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \\
\boldsymbol{v}_3 &= \frac{1}{\|\boldsymbol{v}_3'\|} \boldsymbol{v}_3' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.
\end{aligned}
$$

Therefore, an orthonormal basis of E_1 is $\left\{ \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$.

Therefore, the desired orthonormal basis is $\left\{ \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$.

(ii) A matrix that diagonalises $\boldsymbol{A}$ is one which has columns of its linearly independent eigenvectors. Furthermore, an orthogonal matrix has columns which form an orthonormal basis of $\mathbb{R}^4$. Hence, to obtain an orthogonal matrix which diagonalises $\boldsymbol{A}$, we take the vectors in the basis found in (i) as its columns. We obtain $\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} \end{pmatrix}$.