NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Stefanus Lie

**MA1102R Calculus**
AY 2011/2012 Sem 2

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### Question 1

Note that $f'(x) = -x^2e^{-x}$. Then, $f$ is decreasing everywhere and no local maximum or minimum exists.

Note that $f''(x) = x(x-2)e^{-x}$. Then, $f$ is concave up in $(-\infty, 0)$ and $(2, \infty)$ adn $f$ is concave down in $(0, 2)$. The inflection points are $(0, 2)$ and $(2, 10/e^2)$

### Question 2

(a) Note that for $x > 0$, $-\sqrt{x^3+x^2+x} \leq \sqrt{x^3+x^2+x} \cdot \sin\frac{\pi}{x} \leq \sqrt{x^3+x^2+x}$. The limit of both LHS and RHS when $x \to 0^+$ is 0. By Squeeze Theorem, the limit must be 0.

(b) Let $y = 1/x$. Then

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{x+\pi}{x+e}\right)^x &= exp\left(\lim_{x\to\infty} x \cdot \ln\left|\frac{x+\pi}{x+e}\right|\right) \\
&= exp\left(\lim_{y\to 0^+} \frac{\ln\left|\frac{1+y\pi}{1+ye}\right|}{y}\right) \\
&= exp\left(\lim_{y\to 0^+} \frac{\frac{1+ye}{1+y\pi} \cdot \frac{\pi(1+ye)-e(1+y\pi)}{(1+ye)^2}}{1}\right) \\
&= exp\left(\lim_{y\to 0^+} \frac{\pi - e}{(1+y\pi)(1+ye)}\right) \\
&= e^{e-\pi}
\end{aligned}$$

### Question 3

Note that $A(-1, 1)$ and $B(2, 4)$. Let a vertical line through $P$ cuts $AB$ at $Q(a, a+2)$. The area of triangle $APQ$ is equal to $\frac{(a+2-a^2)(a^2-1)}{2}$, by considering $PQ$ as base. In the same way, area of triangle $BPQ$ is $\frac{(a+2-a^2)(4-a^2)}{2}$ by considering $PQ$ as base. The area of triangle $ABP$ is therefore $A(a) = \frac{3}{2}(a - a^2 + 2)$.

Note that $A(a)$ is a quadratic function, attaining its maximum at $a = 1/2$, for which $P(\frac{1}{2}, \frac{1}{4})$

### Question 4

(a) Using tabular integration, the answer is $-x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$

(b) Let $x = 3\tan u$, then $\frac{dx}{du} = 3\sec^2 x$ and

$$\begin{aligned}\int \frac{1}{(x^2+9)^{3/2}}dx &= \frac{1}{9}\int \cos u\,du \\ &= \frac{1}{9}\sin u + C \\ &= \frac{x}{9\sqrt{x^2+9}} + C\end{aligned}$$

(c) Let

$$\frac{6x-2}{x^4-1} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$$

Then, multiplying both sides by $x^4-1$ and comparing coefficients of power of $x$, we get the equations : $A+C+D=0$, $B-C+D=0$, $-A+C+D=6$, and $-B-C+D=-2$. Solving this, we get $A=-3$, $B=1$, $C=2$, $D=1$. Hence,

$$\begin{aligned}\int \frac{6x-2}{x^4-1} &= -\frac{3}{2}\int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx + \int \frac{2}{x+1}dx + \int \frac{1}{x-1}dx \\ &= -\frac{3}{2}\ln|x^2+1| + \tan^{-1}x + 2\ln|x+1| + \ln|x-1| + C\end{aligned}$$

**Question 5**

(a) The volume is the same if we only consider the upper region revolved about the $x$-axis. Therefore, the volume is (using disk method)

$$\pi\int_0^1 x(1-x)^2dx = \pi\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = \pi/12$$

(b) The volume is twice as the volume of the upper region revolved about the $y$-axis. Therefore, the volume is (using shell method)

$$4\pi\int_0^1 xy\,dx = 4\pi\int_0^1 x^{3/2}(1-x)dx = 4\pi\left[\frac{2x^{2/5}}{5} - \frac{2x^{7/2}}{7}\right]_0^1 = 4\pi/35$$

**Question 6**

(a) Consider the Riemann Sum of function $f(x) = x^p$ along the interval $[0,1]$, by dividing into $n$ equal subintervals and considering the right endpoints. Then

$$\lim_{n\to\infty}\frac{1^p+2^p+\cdots+n^p}{n^{p+1}} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}f\left(\frac{i}{n}\right) = \int_0^1 f(x)dx = \frac{1}{p+1}$$

(b) Let $G(t) = \int_1 6^{t^4} \frac{\sqrt{1+u^4}}{u} du$. Then

$$F'(x) = x^{-1/2} G(2\sqrt{x}) = \frac{1}{\sqrt{x}} \int_{16}^{16x^2} \frac{\sqrt{1+u^4}}{u} du$$

So,

$$F''(x) = x^{-1/2} \cdot 32x \cdot \frac{\sqrt{1+65536x^8}}{16x^2} - \frac{x^{-3/2}}{2} \int_{16}^{16x^2} \frac{\sqrt{1+u^4}}{u} du$$

Thus, $F''(1) = 2\sqrt{65537} + 0 = 2\sqrt{65537}$

**Question 7**

(a) Note that

$$(e^{-1/x - 2\ln|x|}) \frac{dy}{dx} + y(\frac{1}{x^2} - \frac{2}{x}) = 1$$

Hence,

$$\frac{d(y(e^{-1/x-2\ln|x|}))}{dx} = e^{-1/x-2\ln|x|}$$

So,

$$y(e^{-1/x-2\ln|x|}) = \int \frac{e^{-1/x}}{x^2} dx = e^{-1/x} + C$$

When $x = 1$, $y = 2$. From this, $C = 1$. So,

$$y = \frac{e^{-1/x} + 1}{e^{-1/x-2\ln|x|}} = x^2(1 + e^{1/x})$$

(b) 5 percent of salt enters the tank at the rate of 1 liter per minute means 0.05 liter of salt enters the tank per minute. Then, 1 liter out of 100 liter of well-stirred mixture leaves the tank per minute means 0.01 of amount of salt leaves the tank. Thus,

$$\frac{dS}{dt} = 0.05 - 0.01S$$

Note that

$$\frac{dS}{0.05 - 0.01S} = dt$$

Then, $-100 \ln|0.05 - 0.01S| = t + C$ or $0.05 - 0.01S = e^{-0.01(t+C)} = D.e^{-0.01t}$. Since $S = 0$ when $t = 0$, then $D = 0.05$. So, $S = 5(1 - e^{-0.01t})$.

**Question 8**

(a) Divide $(a, b)$ into 2012 equal subintervals $(x_0, x_1), (x_1, x_2), \ldots, (x_{2011}, x_{2012})$. Here, $a = x_0$ and $b = x_{2012}$. Let also that the length of each subinterval be $l$. For each $i = 0, 1, \ldots, 2011$, using Mean Value Theorem, there exists $c_{i+1} \in (x_i, x_{i+1})$ such that $f'(c_{i+1}) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$.

Now, adding all $f'(c_{i+1})$ yields

$$f'(c_1) + f'(c_2) + \cdots f'(c_{2012}) = \sum_{i=0}^{2011} \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{1}{l} \sum_{i=0}^{2011} f(x_{i+1}) - f(x_i) = \frac{f(b) - f(a)}{l} = 0$$

(b) Let $x = \frac{x_1+x_2+\cdots+x_{2012}}{2012}$. Recall the fact that, since $f$ is concave up, then the tangent line of $f$ from $(x, f(x))$ is always lower than the curve of $f$. Then, for every $i = 1, 2, \ldots, 2012$, $f(x_i) \geq f(x)+f'(x)(x-x_i)$. Summing for all $i$, then $f(x_1)+f(x_2)+\cdots+f(x_{2012}) \geq 2012f(x)+f'(x)(2012x-(x_1+x_2+\cdots+x_{2012})) = 2012f(x)$. Q.E.D.