NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lin Mingyan Simon, Chang Hai Bin

MA2108 Mathematical Analysis I
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Question 1

(a) (i) We shall prove by induction that $2 \leq x_n \leq 5$ for all $n \in \mathbb{N}$.
Let $P(n)$ denote the proposition $2 \leq x_n \leq 5$ for all $n \in \mathbb{N}$. $P(1)$ is clearly true by the question. Assume that $P(k)$ holds for some $k \in \mathbb{N}$. By induction hypothesis, one has $2 \leq x_k \leq 5$. Then one has $x_{k+1} = \sqrt{6x_k - 5} \geq \sqrt{6(2) - 5} = \sqrt{7} \geq 2$, and $x_{k+1} = \sqrt{6x_k - 5} \leq \sqrt{6(5) - 5} = 5$. So one has $2 \leq x_{k+1} \leq 5$ and hence $P(k+1)$ is true as well.
Therefore, by the principle of mathematical induction, we have $P(n)$ to be true for all $n \in \mathbb{N}$.

(ii) We shall prove by induction that $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$.
Let $P(n)$ denote the proposition $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. We have $x_2 = \sqrt{6x_1 - 5} = \sqrt{6(2) - 5} = \sqrt{7} \geq 2 = x_1$. So $P(1)$ is true. Assume that $P(k)$ holds for some $k \in \mathbb{N}$. By induction hypothesis, one has $x_{k+1} \geq x_k$. Then one has $x_{k+2} = \sqrt{6x_{k+1} - 5} \geq \sqrt{6x_k - 5} = x_{k+1}$. Hence $P(k+1)$ is true as well.
Therefore, by the principle of mathematical induction, we have $P(n)$ to be true for all $n \in \mathbb{N}$.

Since (x_n) is bounded and increasing, it follows from the Monotone Convergence Theorem that (x_n) is convergent. Let x denote the limit of the sequence (x_n). Then one has

$$\begin{aligned} & \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{6x_n - 5} \\ \Rightarrow\quad & x = \sqrt{6x - 5} \\ \Rightarrow\quad & x^2 = 6x - 5 \\ \Rightarrow\quad & (x-1)(x-5) = 0 \\ \Rightarrow\quad & x = 1 \quad \text{or} \quad x = 5. \end{aligned}$$

Moreover, since $x_n \geq 2$ for all $n \in \mathbb{N}$, one has $x = \lim_{n\to\infty} x_n \geq \lim_{n\to\infty} 2 = 2$. So $x = 5$.

(b) Let $M > 0$ be given. Choose $K \in \mathbb{N}$ such that $K > M^2$. Then it follows that for all $n \in \mathbb{N}$, $n \geq K$, one has

$$\frac{3n^3 - 1}{\sqrt{n^5 + 2n^3 + 1}} \geq \frac{3n^3 - n^3}{\sqrt{n^5 + 2n^5 + n^5}} = \frac{2n^3}{2n^2\sqrt{n}} = \sqrt{n} \geq \sqrt{K} > M.$$

Hence, by definition, one has $\lim_{n\to\infty} \frac{3n^3-1}{\sqrt{n^5+2n^3+1}} = \infty$.

(c) Let $a_n = n\left(\sqrt{n^2+1} - n\right)$ for all $n \in \mathbb{N}$. Then we have $x_n = a_n \sin\frac{n\pi}{8}$, so it is easy to see that $-a_n \leq x_n \leq a_n$ for all $n \in \mathbb{N}$. Also, we note that

$$a_n = n\left(\sqrt{n^2+1} - n\right) = \frac{n\left(\sqrt{n^2+1} - n\right)\left(\sqrt{n^2+1} + n\right)}{\sqrt{n^2+1} + n} = \frac{n}{\sqrt{n^2+1} + n} = \frac{1}{\sqrt{1 + \frac{1}{n^2}} + 1}.$$

Therefore, we have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}+1}} = \frac{1}{2}$.
Hence, it follows that if (x_{n_k}) is a convergent subsequence with x as its limit, then one has $x = \lim\limits_{k\to\infty} x_{n_k} \le \lim\limits_{k\to\infty} a_{n_k} = \frac{1}{2}$. So $\frac{1}{2}$ is an upper bound on the set of cluster points of (x_n).
On the other hand, one has

$$x_{16k+4} = a_{16k+4} \sin \frac{(16k+4)\pi}{8} = a_{16k+4} \sin \left(2k + \frac{1}{2}\right)\pi = a_{16k+4}.$$

This implies that $\lim\limits_{k\to\infty} x_{16k+4} = \lim\limits_{k\to\infty} a_{16k+4} = \frac{1}{2}$. So $\limsup x_n = \frac{1}{2}$.

Question 2

(a) (i) Let $a_n = \frac{n(2n^3+1)}{7n^5-2n^2+1}$ and $b_n = \frac{1}{n}$. Then one has $\frac{a_n}{b_n} = \frac{2n^5+n^2}{7n^5-2n^2+1}$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^5+n^2}{7n^5-2n^2+1} = \lim_{n\to\infty} \frac{2+\frac{1}{n^3}}{7-\frac{2}{n^3}+\frac{1}{n^5}} = \frac{2}{7} > 0.$$

Since $\sum\limits_{n=1}^{\infty} b_n$ diverges, we have $\sum\limits_{n=1}^{\infty} a_n$ to diverge by the Limit Comparison Test.

(ii) We have

$$0 \le \sum_{n=1}^{\infty} \left| \frac{(-1)^n n^5}{3^n(n^2+1)} \right| = \sum_{n=1}^{\infty} \frac{n^5}{3^n(n^2+1)} \le \sum_{n=1}^{\infty} \frac{n^5}{3^n n^2} = \sum_{n=1}^{\infty} \frac{n^3}{3^n}.$$

Let $a_n = \dfrac{n^3}{3^n}$ for all $n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3} < 1$, it follows from the Ratio Test that the series $\sum\limits_{n=1}^{\infty} \frac{n^3}{3^n}$ converges absolutely. Hence, the series $\sum\limits_{n=1}^{\infty} \frac{n^5}{3^n(n^2+1)}$ converges by the Comparison Test, and this implies that $\sum\limits_{n=1}^{\infty} \frac{(-1)^n n^5}{3^n(n^2+1)}$ converges (absolutely).

(iii) We have

$$0 \le \sum_{n=1}^{\infty} \left| \frac{3\cos n - 2^n}{6^n} \right| \le \sum_{n=1}^{\infty} \left(\left| \frac{3\cos n}{6^n} \right| + \left| \frac{2^n}{6^n} \right| \right) \le \sum_{n=1}^{\infty} \left(\frac{3}{6^n} + \frac{1}{3^n} \right).$$

Since $\sum\limits_{n=1}^{\infty} \frac{3}{6^n}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}$ both converge, it follows that the series $\sum\limits_{n=1}^{\infty} \left(\frac{3}{6^n} + \frac{1}{3^n}\right) = \sum\limits_{n=1}^{\infty} \frac{3}{6^n} + \sum\limits_{n=1}^{\infty} \frac{1}{3^n}$ converges. Therefore it follows from the Comparison Test that $\sum\limits_{n=1}^{\infty} \left|\frac{3\cos n - 2^n}{6^n}\right|$ converges. This implies that $\sum\limits_{n=1}^{\infty} \frac{3\cos n - 2^n}{6^n}$ converges (absolutely).

(b) Suppose $\sum\limits_{n=1}^{\infty} c_n$ converges. For each $k \in \mathbb{N}$, let $s_k = \sum\limits_{n=1}^{k} c_n$, $t_k = \sum\limits_{n=1}^{k} a_n$ and $u_k = \sum\limits_{n=1}^{k} b_n$.
Note that $s_{2k} = \sum\limits_{n=1}^{2k} c_n = \sum\limits_{n=1}^{k} c_{2n-1} + \sum\limits_{n=1}^{k} c_{2n} = \sum\limits_{n=1}^{k} a_n + \sum\limits_{n=1}^{k} b_n = t_k + u_k$. Since $\sum\limits_{n=1}^{\infty} c_n$ and $\sum\limits_{n=1}^{\infty} a_n$ converges, it follows that both limits $\lim\limits_{k\to\infty} s_k$ and $\lim\limits_{k\to\infty} t_k$ exist, and thus it follows that the limit $\lim\limits_{k\to\infty} s_{2k}$ exists as well. Therefore, the limit $\lim\limits_{k\to\infty} u_k = \lim\limits_{k\to\infty} (s_{2k} - t_k) = \lim\limits_{k\to\infty} s_{2k} - \lim\limits_{k\to\infty} t_k$ exists, which implies that the series $\sum\limits_{n=1}^{\infty} b_n$ is convergent, a contradiction. So the desired holds.

Question 3

(a) Let $\varepsilon > 0$ be given. Choose $\delta = \min\{\frac{\varepsilon}{6}, \frac{1}{4}\}$. Then it follows that if $|x-2| < \delta$, then one has $x - 2 > -\delta \geq -\frac{1}{4}$, so one has $2x - 3 = 2(x-2) + 1 > 2\left(-\frac{1}{4}\right) + 1 = \frac{1}{2}$. This implies that $|2x-3| > \frac{1}{2}$, or equivalently, $\frac{1}{|2x-3|} < 2$. Hence, we have

$$\left|\frac{x}{2x-3} - 2\right| = \left|\frac{x - 2(2x-3)}{2x-3}\right| = \left|\frac{6-3x}{2x-3}\right| = \frac{3|x-2|}{|2x-3|} < 3 \cdot \delta \cdot 2 = 6\delta \leq 6 \cdot \frac{\varepsilon}{6} = \varepsilon$$

whenever $|x-2| < \delta$. Since $\varepsilon > 0$ is arbitrary, it follows from the $\varepsilon - \delta$ definition that $\lim\limits_{x \to 2} \frac{x}{2x-3} = 2$.

(b) (i) Write $f(x) = \cos\left(\frac{1}{2-\sqrt{x}}\right)$ and let $x_n = \left(2 - \frac{1}{n\pi}\right)^2$ for all $n \in \mathbb{N}$.
Since $0 < 2 - \frac{1}{n\pi} < 2$ for all $n \in \mathbb{N}$, it follows that $x_n \neq 4$. Also, we have $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \left(2 - \frac{1}{n\pi}\right)^2 = (2-0)^2 = 4$, and $f(x_n) = \cos n\pi$. From here, we see that for all $k \in \mathbb{N}$,

$$f(x_{2k}) = \cos 2k\pi = 1 \Rightarrow \lim_{k\to\infty} f(x_{2k}) = 1,$$
$$f(x_{2k+1}) = \cos(2k+1)\pi = -1 \Rightarrow \lim_{k\to\infty} f(x_{2k+1}) = -1.$$

Consequently, the sequence $(f(x_n))$ diverges so the limit $\lim\limits_{x\to 4} f(x)$ does not exist.

(ii) Take a rational sequence (x_n) and an irrational sequence (y_n) such that $\lim\limits_{n\to\infty} x_n = 0 = \lim\limits_{n\to\infty} y_n$. Then we see that

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} (x_n - 1)^3 = (0-1)^3 = -1,$$
$$\lim_{n\to\infty} F(y_n) = \lim_{n\to\infty} \frac{2^{y_n}}{1+y_n^2} = \frac{2^0}{1+0^2} = 1.$$

Since $\lim\limits_{n\to\infty} F(x_n) \neq \lim\limits_{n\to\infty} F(y_n)$, it follows that the limit $\lim\limits_{x\to 0} F(x)$ does not exist.

(c) Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to b^-} f(x) = L$, it follows that there exists some $\delta > 0$, such that $a < b - \delta$, and if $b - \delta < x < b$, then one has $|f(x) - L| < \varepsilon$ and $|h(x) - L| < \varepsilon$, or equivalently, $L - \varepsilon < f(x) < L + \varepsilon$ and $L - \varepsilon < h(x) < L + \varepsilon$. This implies that $L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$, or equivalently, $|g(x) - L| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows from definition that $\lim\limits_{x\to b^-} g(x) = L$.

Question 4

(a) Note that $f(x)^2 = 1 \Leftrightarrow f(x) = 1$ or $f(x) = -1$. Suppose on the contrary that f is not a constant function on $(0,1)$. Then necessarily, there must exist $a, b \in (0,1)$ such that $f(a) = 1$ and $f(b) = -1$. WLOG, we may assume that $a < b$. Since f is continuous on $(0,1)$, f is certainly continuous on $[a,b]$, and thus by the Intermediate Value Theorem, there exists some $c \in [a,b] \subseteq (0,1)$ such that $f(c) = 0$. This would imply that $f(c)^2 = 0$, which is a contradiction. So the desired holds.

(b) For each $n \in \mathbb{N}$, define $\varepsilon_0 = 1$, $x_n = 1 - \frac{1}{n+2}$ and $y_n = 1 - \frac{1}{n+1}$. Clearly, we have $x_n, y_n \in (0,1)$. Then we see that $\lim\limits_{n\to\infty} |x_n - y_n| = \lim\limits_{n\to\infty} \left|\frac{1}{n+1} - \frac{1}{n+2}\right| = 0$. However, we have $g(x_n) = n+2$ and $g(y_n) = n+1$, so $|g(x_n) - g(y_n)| = 1 \geq \varepsilon_0$ for all $n \in \mathbb{N}$. So g is not uniformly continuous on $(0,1)$.

Question 5

(a) Let $\limsup b_n = b$. Note that for a given bounded sequence (x_n), $\limsup x_n = x$ if and only if:
(1) There exists a convergent subsequence (x_{n_k}) of (x_n) converging to x, and
(2) For every other convergent subsequence (x_{n_ℓ}) of (x_n), one has $\lim\limits_{\ell\to\infty} x_{n_\ell} \le x$.

Firstly, we shall show that $ab \le \limsup a_n b_n$. Since $\limsup b_n = b$, by property (1) it follows that there exists a convergent subsequence (b_{n_k}) of (b_n) that converges to b. Since $\lim\limits_{n\to\infty} a_n = a$, it follows that (a_{n_k}) converges to a as well. Hence, the subsequence $(a_{n_k} b_{n_k})$ of $(a_n b_n)$ converges to ab and so by property (2) one has $ab \le \limsup a_n b_n$.

Next, we shall show that $\limsup a_n b_n \le ab$. By property (1), it follows that there exists a convergent subsequence $(a_{n_k} b_{n_k})$ of $(a_n b_n)$ that converges to $\limsup a_n b_n$. As $\lim\limits_{k\to\infty} a_{n_k} = a > 0$, it follows that there exists some $N \in \mathbb{N}$ such that $a_{n_k} > 0$ for all $k \ge N$. Henceforth, we may assume that $a_{n_k} > 0$ for all $k \in \mathbb{N}$ (else, we may simply discard the first $N-1$ terms of the sequence (a_{n_k})). We have $\lim\limits_{k\to\infty} b_{n_k} = \lim\limits_{k\to\infty} \frac{a_{n_k} b_{n_k}}{a_{n_k}} = \frac{1}{a}\limsup a_n b_n$. As $\limsup b_n = b$, by property (2) one has $\frac{1}{a}\limsup a_n b_n \le b$, so $\limsup a_n b_n \le ab$. Therefore, $\limsup a_n b_n = ab$ as desired.

(b) Let $L = \lim\limits_{n\to\infty} n^p x_n$. Then one has $\lim\limits_{n\to\infty} \frac{|x_n|}{1/n^p} = \lim\limits_{n\to\infty} |n^p x_n| = |L| \ge 0$. As $p > 1$, we see that the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges, and thus the series $\sum\limits_{n=1}^{\infty} |x_n|$ converges by the Limit Comparison Test. The desired now follows.

Question 6

(i) First Solution (by Hai Bin):
Define the function h on $[0,1]$ as follows: $h(x) = g(x) - f(x)$ for all $x \in [0,1]$. Clearly, h is continuous on $[0,1]$ with $h(x) > 0$ for all $x \in [0,1]$. Then it suffices to show that there exists some $\mu > 0$, such that $\mu f(x) \le h(x)$ for all $x \in [0,1]$.

By the Extreme Value Theorem, there exists some $c \in [0,1]$, such that $h(x) \ge h(c) > 0$ for all $x \in [0,1]$. Let $L = h(c)$. Also, since f is continuous on $[0,1]$, it follows that $|f|$ is continuous on $[0,1]$, and hence by the Extreme Value Theorem, there exists some $d \in [0,1]$, such that $|f(x)| \le |f(d)| < |f(d)| + 1$ for all $x \in [0,1]$. Let $M = |f(d)|$, and we note that $M \ge 0$. This implies that $M + 1 \ge 1$ and $\frac{|f(x)|}{M+1} < 1$ for all $x \in [0,1]$. Thus, for all $x \in [0,1]$, we have

$$\frac{L}{M+1} f(x) \le \frac{|f(x)|}{M+1} \cdot L \le L \le h(x).$$

The desired now follows by setting $\mu = \frac{L}{M+1}$ and $\lambda = 1 + \mu$.

Second Solution (by Simon):
If $f(x) \le 0$, then any $\lambda > 1$ would do since $\lambda f(x) \le f(x) < g(x)$. Henceforth, we shall consider only the case where $f(x) > 0$. Define the set $A = \left\{\frac{g(x)}{f(x)} : x \in [0,1], f(x) > 0\right\}$. Note that if $t \in A$, then $t = \frac{g(s)}{f(s)} > 1$ for some $s \in [0,1]$ such that $f(s) > 0$, so it follows that 1 is a lower bound of the set A, and hence $\inf A \ge 1$. The claim is that $\inf A > 1$.

Suppose $\inf A = 1$. This implies that for each $n \in \mathbb{N}$, $1 + \frac{1}{n}$ is not a lower bound of A, so there exists some $x_n \in [0,1]$ such that $\frac{g(x_n)}{f(x_n)} < 1 + \frac{1}{n}$, or equivalently, $g(x_n) < \left(1 + \frac{1}{n}\right) f(x_n)$. Since

$0 \leq x_n \leq 1$ for all $n \in \mathbb{N}$, it follows that (x_n) is bounded, and hence there exists a convergent subsequence (x_{n_k}) of (x_n) by the Bolzano-Weierstrass Theorem. Let $x = \lim\limits_{k\to\infty} x_{n_k}$.

Notice that $0 \leq x_{n_k} \leq 1$, so one has $0 \leq x = \lim\limits_{k\to\infty} x_{n_k} \leq 1$, i.e. $x \in [0,1]$. Since f and g are both continuous on $[0,1]$, it follows that $\lim\limits_{k\to\infty} f(x_{n_k}) = f(x)$ and $\lim\limits_{k\to\infty} g(x_{n_k}) = g(x)$. Noting that $g(x_{n_k}) < \left(1 + \frac{1}{n_k}\right) f(x_{n_k})$ for all $k \in \mathbb{N}$, we see that by taking limits one has $g(x) = \lim\limits_{k\to\infty} g(x_{n_k}) \leq \lim\limits_{k\to\infty} (1 + \frac{1}{n_k}) f(x_{n_k}) = f(x)$, which is a contradiction. Therefore, one has $\inf A > 1$ as desired. The desired now follows by setting $\lambda = \inf A$.

(ii) Here, we shall construct a counter-example where the desired conclusion could not be reached if we replace the closed interval $[0,1]$ by the open interval $(0,1)$. Define the functions f and g on $(0,1)$ as follows: $f(x) = \frac{1}{x}$ and $g(x) = 1 + \frac{1}{x}$ for all $x \in (0,1)$. Clearly, f and g are both continuous on $(0,1)$ and $0 < f(x) < g(x)$ for all $x \in (0,1)$. Let $\lambda > 1$ be given.

Noting that $\frac{g(x)}{f(x)} = \left(1 + \frac{1}{x}\right) / \frac{1}{x} = x + 1$, we shall choose an $\varepsilon \in (0,1)$, such that $1 < 1 + \varepsilon < \lambda$. Then we see that $\frac{g(\varepsilon)}{f(\varepsilon)} = \varepsilon + 1 < \lambda$. Thus one has $\lambda f(\varepsilon) > g(\varepsilon)$. Since $\lambda > 1$ is arbitrary, this would imply that such a λ could never be found, and hence the conclusion does not necessarily hold if the closed interval $[0,1]$ were to be replaced by the open interval $(0,1)$.

Question 7

(i) For each $n \in \mathbb{N}$, let $x_n = f(n)$. Since f is increasing, we must have $x_n = f(n) \leq f(n+1) = x_{n+1}$, and hence we see that the sequence (x_n) is increasing. Also, since the function f is bounded, it follows that the sequence $(x_n) = (f(n))$ is bounded as well. Hence (x_n) is convergent by the Monotone Convergence Theorem. Let the limit of (x_n) be a.

Let $\varepsilon > 0$ be given. As $\lim\limits_{n\to\infty} x_n = a$, it follows that there exists some $N \in \mathbb{N}$, such that if $n \geq N$, then one has $0 \leq a - f(n) = a - x_n < \varepsilon$. This implies that for all $x \geq N$, one has $0 \leq a - f(x) \leq a - f(N) < \varepsilon$, and thus $|f(x) - a| < \varepsilon$. Therefore, by definition, we have the limit $\lim\limits_{x\to\infty} f(x)$ to exist (and equal to a).

(ii) We shall prove that f is uniformly continuous on $(0,\infty)$.

By a similar argument above, we can also show that the limit $\lim\limits_{x\to 0} f(x)$ exists. Hence, we may define $f(0) = \lim\limits_{x\to 0} f(x)$ so that the extended function f is continuous on $[0,\infty)$.
Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to\infty} f(x) = a$, it follows that there exists some $K > 0$ such that if $x \geq K$, then one has $|f(x) - a| < \frac{\varepsilon}{4}$. Thus, for any $x_1, x_2 \geq K$, one has $|f(x_1) - f(x_2)| \leq |f(x_1) - a| + |a - f(x_2)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} < \varepsilon$.

Next, since f is (uniformly) continuous on $[0,K]$, it follows that there exists a $\delta > 0$, such that if $x_1, x_2 \in [0,K]$ and $|x_1 - x_2| < \delta$, then one has $|f(x_1) - f(x_2)| < \frac{\varepsilon}{2} < \varepsilon$.
Finally, if $0 \leq x_1 < K$, $x_2 > K$ and $|x_1 - x_2| = x_2 - x_1 < \delta$, then one has $|x_1 - K| = K - x_1 < x_2 - x_1 < \delta$, so one has $|f(x_1) - f(K)| < \frac{\varepsilon}{2}$. Thus, one has $|f(x_1) - f(x_2)| \leq |f(x_1) - f(K)| + |f(K) - f(x_2)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

The above argument now shows that if $x_1, x_2 \in [0,\infty)$ and $|x_1 - x_2| < \delta$, then one has $|f(x_1) - f(x_2)| < \varepsilon$. Hence f is uniformly continuous on $[0,\infty)$. The desired now follows.