MA1102R-CALCULUS SUGGESTED SOLUTIONS

(SEMESTER 1 : AY2020/21)

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Question 1

(i) $f(x) = x^3e^{-x^2} \Rightarrow f'(x) = 3x^2e^{-x^2} - 2x^4e^{-x^2} = e^{-x^2}x^2(3-2x^2)$

$f'(x) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{\frac{3}{2}}$

x	$(-\infty, -\sqrt{\frac{3}{2}})$	$(-\sqrt{\frac{3}{2}}, 0)$	$(0, \sqrt{\frac{3}{2}})$	$(\sqrt{\frac{3}{2}}, +\infty)$
$f'(x)$	$-$	$+$	$+$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$

$\therefore f$ is increasing on $(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}})$ and decreasing on $(-\infty, -\sqrt{\frac{3}{2}})$ and on $(\sqrt{\frac{3}{2}}, +\infty)$.

(ii) $f(-\sqrt{\frac{3}{2}}) = (-\sqrt{\frac{3}{2}})^3e^{-\frac{3}{2}} = -(\frac{3}{2})^{\frac{3}{2}}e^{-\frac{3}{2}}$

$f(\sqrt{\frac{3}{2}}) = (\sqrt{\frac{3}{2}})^3e^{-\frac{3}{2}} = (\frac{3}{2})^{\frac{3}{2}}e^{-\frac{3}{2}}$

$\therefore f$ has local minimum point $(-\sqrt{\frac{3}{2}}, -(\frac{3}{2})^{\frac{3}{2}}e^{-\frac{3}{2}})$ and local maximum point $(\sqrt{\frac{3}{2}}, (\frac{3}{2})^{\frac{3}{2}}e^{-\frac{3}{2}})$.

(iii) $f''(x) = e^{-x^2}(4x^5 - 14x^3 + 6x) = 2e^{-x^2}x(x^2-3)(2x^2-1)$

$f''(x) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{3}$ or $x = \pm\frac{\sqrt{2}}{2}$

x	$(-\infty, \sqrt{3})$	$(-\sqrt{3}, -\frac{\sqrt{2}}{2})$	$(-\frac{\sqrt{2}}{2}, 0)$	$(0, \frac{\sqrt{2}}{2})$	$(\frac{\sqrt{2}}{2}, \sqrt{3})$	$(\sqrt{3}, +\infty)$
$f''(x)$	$-$	$+$	$-$	$+$	$-$	$+$

$\therefore f(x)$ is concave up on $(-\sqrt{3}, -\frac{\sqrt{2}}{2}), (0, \frac{\sqrt{2}}{2})$ and$(\sqrt{3}, +\infty)$,

and concave down on $(-\infty, \sqrt{3}), (-\frac{\sqrt{2}}{2}, 0)$ and $(\frac{\sqrt{2}}{2}, \sqrt{3})$.

(iv) By (iii), the inflection points are at $x = -\sqrt{3}$, $x = -\frac{\sqrt{2}}{2}$, $x = 0$, $x = \frac{\sqrt{2}}{2}$, $x = \sqrt{3}$.

Question 2

(a) Let $\epsilon > 0$. Choose $\delta = \min\{\epsilon, 1\}$.

Then whenever $0 < |x-1| < \delta$, $\quad 1 < x < 1+\delta \leqslant 2$ or $0 \leqslant 1-\delta < x < 1$,

$$\begin{aligned}
|\frac{1}{\sqrt{5-x^2}} - \frac{1}{2}| &= \frac{|2-\sqrt{5-x^2}|}{2\sqrt{5-x^2}} \leqslant \frac{|2-\sqrt{5-x^2}|}{2} \\
&= \frac{|x^2-1|}{2(2+\sqrt{5-x^2})} < \frac{|x-1||x-1+2|}{4} \leqslant \frac{|x-1|(|x-1|+2)}{4} \\
&< \frac{\delta(\delta+2)}{4} \leqslant \delta \leqslant \epsilon.
\end{aligned}$$

$$\therefore \lim_{x\to 1} \frac{1}{\sqrt{5-x^2}} = \frac{1}{2}.$$

(b)

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1+\sin x}{1+x}\right)^{\frac{1}{x^3}} &= \exp\left(\lim_{x\to 0} \frac{\ln(1+\sin x) - \ln(1+x)}{x^3}\right) \\
&= \exp\left(\lim_{x\to 0} \frac{\frac{\cos x}{1+\sin x} - \frac{1}{1+x}}{3x^2}\right) \\
&= \exp\left(\lim_{x\to 0} \frac{\frac{-\sin x(1+\sin x)-\cos^2 x}{(1+\sin x)^2} + \frac{1}{(1+x)^2}}{6x}\right) \\
&= \exp\left(\lim_{x\to 0} \frac{-\frac{1}{1+\sin x} + \frac{1}{(1+x)^2}}{6x}\right) \\
&= \exp\left(\lim_{x\to 0} \frac{1}{6} \frac{\frac{\sin x - 2x - x^2}{x}}{(1+x)^2(1+\sin x)}\right) \\
&= e^{-\frac{1}{6}}
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{2n^2+5in+2i^2} &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{\frac{i}{n}}{(2\frac{i}{n}+1)(\frac{i}{n}+2)} \\
&= \int_0^1 \frac{x}{(2x+1)(x+2)}\, dx \\
&= \frac{1}{3}\int_0^1 \left(\frac{2}{x+2} - \frac{1}{2x+1}\right) dx \\
&= \frac{1}{3}\left[2\ln|x+2| - \frac{1}{2}\ln|2x+1|\right]_0^1 \\
&= \frac{1}{2}\ln 3 - \frac{2}{3}\ln 2
\end{aligned}$$

Question 3

(a)

$$\begin{aligned}
F(x) &= \int_a^x (x-t)^2 f(t)\,dt = x^2\int_a^x f(t)\,dt - 2x\int_a^x tf(t)\,dt + \int_a^x t^2 f(t)\,dt \\
\Rightarrow F'(x) &= 2x\int_a^x f(t)\,dt + x^2 f(x) - 2\int_a^x tf(t)\,dt - 2x^2 f(x) + x^2 f(x) \\
&= 2\left(x\int_a^x f(t)\,dt - \int_a^x tf(t)\,dt\right) \\
\Rightarrow F''(x) &= 2\left(\int_a^x f(t)\,dt + xf(x) - xf(x)\right) = 2\int_a^x f(t)\,dt \\
\Rightarrow F'''(x) &= 2f(x)
\end{aligned}$$

(b)

$$\int\left(\frac{r}{x+1} - \frac{3x}{2x^2+r}\right)dx = r\ln|x+1| - \frac{3}{4}\ln|2x^2+r| + C = \ln\frac{|x+1|^r}{|2x^2+r|^{3/4}} + C$$

Let

$$F(x) = \ln\frac{|x+1|^r}{|2x^2+r|^{3/4}}$$

Then

$$F(1) = \ln\frac{2^r}{|2+r|^{3/4}}$$

We must have

$$r \neq -2$$

On the other hand, notice that

$$\lim_{x\to\infty}\frac{|x+1|^r}{|2x^2+r|^{3/4}} = 0 \text{ if } r < \frac{3}{2} \quad\text{and}\quad \lim_{x\to\infty}\frac{|x+1|^r}{|2x^2+r|^{3/4}} = \infty \text{ if } r > \frac{3}{2}$$

$$\therefore \lim_{x\to\infty} F(x) \text{ exists } \Leftrightarrow \lim_{x\to\infty}\frac{|x+1|^r}{|2x^2+r|^{3/4}} \text{ is a nonzero real number } \Rightarrow r = \frac{3}{2}.$$

When $r = \frac{3}{2}$,

$$\lim_{x\to\infty}\frac{|x+1|^{3/2}}{|2x^2+\frac{3}{2}|^{3/4}} = \left(\lim_{x\to\infty}\frac{(x+1)^2}{(2x^2+3/2)}\right)^{3/4} = \left(\lim_{x\to\infty}\frac{2(x+1)}{4x}\right)^{3/4} = \left(\lim_{x\to\infty}\frac{2}{4}\right)^{3/4} = 2^{-\frac{3}{4}}$$

$$\therefore \lim_{x\to\infty} F(x) = \ln 2^{-\frac{3}{4}} = -\frac{3}{4}\ln 2$$

$$F(1) = \ln\frac{2^{3/2}}{(2+\frac{3}{2})^{3/4}} = -\frac{3}{4}\ln 7 + \frac{9}{4}\ln 2$$

$$\therefore \int_1^\infty \left(\frac{r}{x+1} - \frac{3x}{2x^2+r}\right)dx = \lim_{t\to\infty}[F(x)]_1^t = (-\frac{3}{4}\ln 2) - (-\frac{3}{4}\ln 7 + \frac{9}{4}\ln 2) = \frac{3}{4}\ln 7 - 3\ln 2$$

Question 4

Let the storage capacity be $V(cm^3)$. Then $V = \pi r^2 h$.

It is given that $V_{outside} = 9 \times 10^6 \pi = \pi(r+15)^2(h+40)$

$$\therefore h = \frac{9 \times 10^6}{(r+15)^2} - 40 \quad \text{and} \quad V = \pi r^2 \left(\frac{9 \times 10^6}{(r+15)^2} - 40 \right) \quad (r \geqslant 0)$$

Maximize V.

Let $t = r + 15$.

$$\frac{dV}{dr} = \frac{dV}{dt}\frac{dt}{dr} = \frac{dV}{dt} = \frac{2\pi(t-15)}{t^3}(-40t^3 + 135 \times 10^6)$$

$$\frac{dV}{dt} = 0 \Rightarrow t = 15 \quad \text{or} \quad t = 150$$

Notice that $\frac{dV}{dt} > 0$ when $t \in (15, 150)$ and $\frac{dV}{dt} < 0$ when $t \in (150, +\infty)$,

$\therefore V$ attains its maximum at $t = 150$ or $r = 135$. At $r = 135$, $h = \frac{9 \times 10^6}{150^2} - 40 = 360$.

$\therefore$ The container has maximum capacity when $r = 135(cm)$ and $h = 360(cm)$.

Question 5

(i) Using the disk method,

$$V_1 = \int_0^1 \pi y^2\,dx = \frac{1}{3}\pi \int_0^1 x(1-x)^2\,dx = \frac{1}{3}\pi\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{1}{36}\pi$$

(ii) Using the cylindrical shell method,

$$V_2 = \int_0^1 2\pi x\cdot 2|y|\,dx = \int_0^1 4\pi x\cdot\frac{(1-x)\sqrt{x}}{\sqrt{3}}\,dx = \frac{4\pi}{\sqrt{3}}\int_0^1 (x^{3/2} - x^{5/2})\,dx$$
$$= \frac{4\pi}{\sqrt{3}}\left[\frac{2}{5}x^{5/2} - \frac{2}{7}x^{7/2}\right]_0^1 = \frac{4\pi}{\sqrt{3}}\cdot\frac{4}{35} = \frac{16\pi}{35\sqrt{3}}$$

(iii) Implicitly differentiate

$$(3y^2)' = [x(1-x)^2]'$$
$$\Rightarrow 6y\cdot y' = (x-1)(3x-1)$$
$$\Rightarrow \frac{dy}{dx} = \frac{(x-1)(3x-1)}{6y}$$
$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(x-1)^2(3x-1)^2}{36y^2} = 1 + \frac{(x-1)^2(3x-1)^2}{12x(1-x)^2} = \frac{(3x+1)^2}{12x}$$

Arc length $$L = 2\int_0^1 \sqrt{\frac{(3x+1)^2}{12x}} = 2\int_0^1 \frac{3x+1}{2\sqrt{3x}}\,dx = \frac{1}{\sqrt{3}}\int_0^1 (3x^{1/2} + x^{-1/2})\,dx$$
$$= \frac{1}{\sqrt{3}}\left[2x^{3/2} + 2x^{1/2}\right]_0^1 = \frac{4}{\sqrt{3}}$$

(iv) Surface area

$$S = \int_0^1 2\pi|y|\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2\pi\frac{(1-x)\sqrt{x}}{\sqrt{3}}\cdot\frac{3x+1}{2\sqrt{3x}}\,dx$$
$$= -\frac{\pi}{3}\int_0^1 (x-1)(3x+1)\,dx = -\frac{\pi}{3}\left[x^3 - x^2 - x\right]_0^1 = \frac{\pi}{3}$$

Question 6

(a) (i)

$$y = \frac{1}{z} - x^2 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx} - 2x$$

Then

$$\begin{aligned} -\frac{1}{z^2}\frac{dz}{dx} - 2x &= x^3 + \frac{2y}{x} - \frac{y^2}{x} \\ \Rightarrow \ -\frac{1}{z^2}\frac{dz}{dx} - 2x &= x^3 + \frac{\frac{2}{z} - 2x^2}{x} - \frac{(\frac{1}{z} - x^2)^2}{x} \\ \Rightarrow \frac{dz}{dx} + \left(\frac{2}{x} + 2x\right) z &= \frac{1}{x} \end{aligned}$$

(ii)

$$\int \left(\frac{2}{x} + 2x\right) dx = 2\ln|x| + x^2 + C$$

$$\text{An integrating factor} \quad v(x) = e^{2\ln|x| + x^2} = x^2 e^{x^2}$$

$$\therefore z = \frac{1}{x^2 e^{x^2}} \int x e^{x^2}\, dx = \frac{e^{x^2} + C}{2x^2 e^{x^2}} \qquad (\star)$$

$z = 1$ when $x = 1$ and $y = 0$. Substituting $z = 1$ and $x = 1$ into $(\star)$ yields

$$\begin{aligned} 1 &= \frac{e + C}{2e} \\ \Rightarrow C &= e \end{aligned}$$

$$\therefore y = \frac{1}{z} - x^2 = \frac{2x^2 e^{x^2}}{e^{x^2} + e} - x^2 = x^2 \frac{1 - e^{1-x^2}}{1 + e^{1-x^2}}$$

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(b) (i) Let the area of the triangle be S.

From the graph we can see that,

$$S = \frac{1}{2} y_0 \cdot \frac{y_0}{f'(x_0)} = \frac{y_0^2}{2f'(x_0)}$$

Since $S = 1102$ is constant,

$$\begin{aligned} \frac{[f(x)]^2}{2f'(x)} &= 1102 \\ \Leftrightarrow [f(x)]^2 &= 2204 f'(x) \end{aligned}$$

for all x.

$$\therefore K = 2204$$

(ii) Solve the ODE

$$\begin{aligned} y^2 &= 2204 \frac{dy}{dx} \\ \Rightarrow \int \frac{dx}{2204} &= \int \frac{dy}{y^2} \\ \Rightarrow \frac{x}{2204} + C &= -\frac{1}{y} \end{aligned}$$

Substituting in $x = 1, y = 1$ yields

$$\frac{1}{2204} + C = -1 \Rightarrow C = -\frac{2205}{2204}$$

Let $y = 2$,

$$\frac{x}{2204} - \frac{2205}{2204} = -\frac{1}{2} \Rightarrow x = 1103$$

$$\therefore c = 1103$$

Question 7

(a) If $f(x) = 1$ for all $x \in \mathbb{R}$, then for all $c \in \mathbb{R}, f'(c) = 0$. If not, then there exists $a \in \mathbb{R}$ such that $f(a) \neq 1$. WLOG, suppose $f(a) > 1$. Let $d = f(a) - 1$ and $\epsilon = \frac{d}{2}$.

$\because \lim_{x\to-\infty} f(x) = 1,\ \exists N > 0$ such that $\forall x < -N,\ |f(x) - 1| < \epsilon$.

Choose $b = \min\{N - 1, a - 1\}$. Then $f(b) < 1 + \epsilon = 1 + \frac{d}{2}$.

Now $f(b) < 1 + \frac{d}{2} < f(a)$ and f is continuous on $\mathbb{R}$. By the Intermediate Value Theorem, $\exists\ c_1 \in (b, a)$ such that $f(c_1) = 1 + \frac{d}{2}$.

Similarly we can find $c_2 \in (a, +\infty)$ such that $f(c_2) = 1 + \frac{d}{2}$.

Note that f is differentiable on $\mathbb{R}$. By the Mean Value Theorem, $\exists\ c \in (c_1, c_2)$ such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = 0$$

The case for $f(a) < 1$ is similar. $\square$

(b) We first assume that $f(x) > 0$.[1]

1. Let $y = nx$. Then by the substitution rule

$$\int_0^{\pi} f(nx)g(x)\,dx = \frac{1}{n}\int_0^{n\pi} f(y)g\left(\frac{y}{n}\right)\,dy.$$

2. Divide $[0, n\pi]$ into n equal sub-intervals:

$$\frac{1}{n}\int_0^{n\pi} f(y)g\left(\frac{y}{n}\right)\,dy = \frac{1}{n}\sum_{i=1}^{n}\int_{(i-1)\pi}^{i\pi} f(y)g\left(\frac{y}{n}\right)\,dy.$$

3. Use the periodicity of f. Let $z = y - (i - 1)\pi$. Then $f(z) = f(y)$ and

$$\int_{(i-1)\pi}^{i\pi} f(y)g\left(\frac{y}{n}\right)\,dy = \int_0^{\pi} f(z)g\left(\frac{z + (i-1)\pi}{n}\right)\,dz.$$

4. We want to show that for each i, there exists $z_i \in \left[\frac{(i-1)\pi}{n}, \frac{i\pi}{n}\right]$ such that

$$\int_0^{\pi} f(z)g\left(\frac{z + (i-1)\pi}{n}\right)\,dz = g(z_i)\int_0^{\pi} f(z)\,dz.$$

Let m_i be the minimum and M_i the maximum of g on the interval $\left[\frac{(i-1)\pi}{n}, \frac{i\pi}{n}\right]$. Then $f(z)m_i \leqslant f(z)g\left(\frac{z+(i-1)\pi}{n}\right) \leqslant f(z)M_i$. (Recall that $f(x) > 0$.) Integrate to get

$$m_i\int_0^{\pi} f(z)\,dz \leqslant \int_0^{\pi} f(z)g\left(\frac{z + (i-1)\pi}{n}\right)\,dx \leqslant M_i\int_0^{\pi} f(z)\,dz.$$

So

$$m_i \leqslant \frac{\int_0^{\pi} f(z)g\left(\frac{z+(i-1)\pi}{n}\right)\,dz}{\int_0^{\pi} f(z)\,dz} \leqslant M_i.$$

By the Intermediate Value Theorem, there exists $z_i \in \left[\frac{(i-1)\pi}{n}, \frac{i\pi}{n}\right]$ such that

$$g(z_i) = \frac{\int_0^{\pi} f(z)g\left(\frac{z+(i-1)\pi}{n}\right)\,dx}{\int_0^{\pi} f(z)\,dz}.$$

[1] Solution provided by Prof Wang Fei.

Equivalently,

$$\int_0^\pi f(z)g\left(\frac{z+(i-1)\pi}{n}\right)\,dz = g(z_i)\int_0^\pi f(z)\,dz.$$

5. We can express $\int_0^\pi g(z)\,dz$ as the limit of Riemann sums:

$$\int_0^\pi g(z)\,dz = \lim_{n\to\infty}\sum_{i=1}^n g(z_i)\frac{\pi}{n}.$$

6. Combine the results:

$$\begin{aligned}
\lim_{n\to\infty}\int_0^\pi \pi f(nx)g(x)\,dx &= \lim_{n\to\infty}\frac{\pi}{n}\sum_{i=1}^n\int_0^\pi f(z)\,dz\cdot g(z_i)\\
&= \int_0^\pi f(z)\,dz\cdot\lim_{n\to\infty}\sum_{i=1}^n g(z_i)\cdot\frac{\pi}{n}\\
&= \int_0^\pi f(z)\,dz\cdot\int_0^\pi g(z)\,dz.
\end{aligned}$$

For the case when f is not always positive, since f is continuous on $[0,\pi]$, it has a minimum value. Let C be a number such that $f(z)+C>0$ for all $z\in[0,\pi]$. Let $F(z)=f(z)+C$. Then F is positive, continuous and periodic with period π. Using the same argument,

$$\lim_{n\to\infty}\int_0^\pi \pi F(nx)g(x)\,dx = \int_0^\pi F(z)\,dz\cdot\int_0^\pi g(z)\,dz.$$

This gives

$$\lim_{n\to\infty}\int_0^\pi \pi f(nx)g(x)\,dx + \pi C\int_0^\pi g(x)\,dx = \int_0^\pi f(z)\,dz\cdot\int_0^\pi g(z)\,dz + \pi C\int_0^\pi g(z)\,dz.$$

The result follows. $\square$