NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA2108 Mathematical Analysis 1
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SECTION A

Question 1

(a) (i) Since $-1 \leq \sin(n^2 - n) \leq 1$ and $-1 < \cos(2n) \leq 1 \; \forall \; n \in \mathbb{N}$
It implies that

$$-\frac{1}{3} \leq \frac{\sin(n^2 - n)}{3} \leq \frac{1}{3}$$

and

$$-\frac{1}{2} \leq \frac{\cos(n^2 - n)}{2} \leq \frac{1}{2}$$

and this yields

$$\left(-\frac{5}{6}\right)^n \leq \left(\frac{\sin(n^2 - n)}{3} + \frac{\cos(n^2 - n)}{2}\right)^n \leq \left(\frac{5}{6}\right)^n$$

By Squeeze Theorem, $\displaystyle\lim_{n\to\infty} \left(\frac{\sin(n^2 - n)}{3} + \frac{\cos(n^2 - n)}{2}\right)^n = 0$

(ii) $\left(1 + \frac{1}{4n+2}\right)^{2n} = \frac{\left(\left(1+\frac{1}{4n+2}\right)^{4n+2}\right)^{\frac{1}{2}}}{1+\frac{1}{4n+2}}$

$$\therefore \lim_{n\to\infty} \left(1 + \frac{1}{4n+2}\right)^{2n} = \lim_{n\to\infty} \frac{\left(\left(1 + \frac{1}{4n+2}\right)^{4n+2}\right)^{\frac{1}{2}}}{1 + \frac{1}{4n+2}} = e^{\frac{1}{2}}$$

(iii) By rationalizing, $\sqrt{(n + \sqrt{n})} - \sqrt{(n - \sqrt{n})}) = \frac{2\sqrt{n}}{\sqrt{(n+\sqrt{n})}+\sqrt{(n-\sqrt{n})}} = \frac{2}{\sqrt{1+\frac{1}{\sqrt{n}}}+\sqrt{1-\frac{1}{\sqrt{n}}}}$

$$\therefore \lim_{n\to\infty} \sqrt{n + \sqrt{n}} - \sqrt{n - \sqrt{n}} = \lim_{n\to\infty} \frac{2}{\sqrt{1 + \frac{1}{\sqrt{n}}} + \sqrt{1 - \frac{1}{\sqrt{n}}}} = 1$$

(b) (i) We prove by induction
Let $P(n)$ be the statement $x_n \leq 4$
x_1 is true since $x_1 = 1$
Assume that $P(k)$ is true, ie $x_k \leq 4$
then $x_k + 1 = \sqrt{2x_k + 8} \leq \sqrt{16} = 4 \therefore$ by induction, $P(n)$ is true $\forall n \in \mathbb{N}$

(ii) Let $S(n)$ be the statement $x_n + 1 \geq x_n$
$S(1)$ is true since $x_2 = \sqrt{10} > 1$
Assume that $S(k)$ is true, then $x_k + 2 = \sqrt{2x_k + 1 + 8} \geq \sqrt{2x_k + 8} = x_k + 1$
$\therefore$ By MI, $S(n)$ is true $\forall n \in \mathbb{N}$
(x_n) is increasing and monotone and bounded, $\therefore$ by Monotone Convergence Theorem, (x_n) converges.

Question 2

(a) (i) $\forall n \in \mathbb{N}$, $n^2+1>0$, $3n^4-n>0$, $\therefore \frac{n^2+1}{3n^4-n}>0$. $\lim_{n\to\infty}\frac{\frac{n^2+1}{3n^4-n}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{1+\frac{1}{n^2}}{3-\frac{1}{n^3}}=\frac{1}{3}>0$ By Limit Comparison Test, $\sum_{n=1}^{\infty}\frac{n^2+1}{3n^4-n}$ converges.

(ii) Let s_n be the nth partial sum of the series. $\therefore (s_{2n-1})=\frac{-2n+1}{2n}$ and $(s_{2n})=\frac{2n}{2n+1}$. Consider s_{2n} and s_{2n-1}. $\lim_{n\to\infty}(s_{2n}-s_{2n-1})=\lim_{n\to\infty}\frac{2n}{2n+1}=1$ Since, the odd and even terms of s_n are different, (s_n) diverges.$\therefore$ the series diverges.

Alt: Since $\lim_{n\to\infty}\frac{(-1)^n n^2}{n^2+n}$ does not exist (Consider the limits when n is odd and the limit when n is even) . The sum diverges.

(b) $\frac{b_n}{b_{n-1}}\le\frac{a_n}{a_{n-1}}\forall n\in\mathbb{N}$. $\therefore \frac{b_n}{b_{n-1}}\cdot\frac{b_{n-1}}{b_{n-2}}\cdot\ ...\ \cdot\frac{b_2}{b_1}\le\frac{a_n}{a_{n-1}}\cdot\frac{a_{n-1}}{a_{n-2}}\cdot\ ...\ \cdot\frac{a_2}{a_1}$ It follows that $\frac{b_n}{b_1}\le\frac{a_n}{a_1}$ and $b_n\le\frac{b_1}{a_1}\cdot a_n$ Since, a_n and $b_n>0\forall n\in\mathbb{N}$ and $\sum a_n$ converges ,$\therefore \sum b_n$ converges by the Comparison Test.

Question 3

(a) Given $\epsilon>0$, choose $\delta=\min\{2\epsilon,1\}$
Then

$$\begin{aligned}0<|x-2|<\delta\le 1 \quad , &\quad 0<|x-2|<1\\ \Rightarrow &\quad -1<x-2<1\\ \Rightarrow &\quad -1<x<3\\ \Rightarrow &\quad 2<3x-1<8\\ \Rightarrow &\quad |3x-1|>2\end{aligned}$$

Hence, $\left|\frac{2x-1}{3x-1}-1\right|=\left|\frac{-x+2}{3x-1}\right|=\left|\frac{x-2}{3x-1}\right|<\frac{2\epsilon}{2}=\epsilon$ Therefore, $\forall\epsilon$, $\exists\delta>0$ such that $\left|\frac{2x-1}{3x-1}-1\right|<\epsilon$ whenever $0<|x-2|<\delta$.
Hence $\lim_{x\to 2}\frac{2x+1}{3x-1}=1$

(b) Let $x_n=2+\frac{1}{(n+\frac{1}{2})\pi}$, $f(x)=\sin\left(\frac{1}{x-2}\right),\forall n\in\mathbb{N}$
Then each $x_n\ne 2$ for all n and $x_n\to 2$

$$f(x_n)=\sin\left(\frac{1}{x_n-2}\right)=\sin\left(n+\frac{1}{2}\right)\frac{\pi}{2}=\begin{cases}1 & \text{if n is even}\\ -1 & \text{if n is odd}\end{cases}$$

Then the sequence $[f(x_n)]_{n\in\mathbb{N}}$ diverges.

(c) For $x\in(3,3.1)$, we have $6<2x<6.2$ and $9<x^2<9.61$
Therefore $\lim_{x\to 3^+}\frac{\lfloor 2x\rfloor+x}{\lfloor x^2\rfloor+1}=\lim_{x\to 3^+}\frac{6+x}{9+1}=\frac{9}{10}$

(d) For $x\in(2.9,3)$,we have $5.8<2x<6$ and $8.41<x^2<9$
$\lim_{x^{3-}}\frac{\lfloor 2x\rfloor+x}{\lfloor x^2\rfloor+1}=\lim_{x\to 3^-}\frac{5+x}{9+1}=\frac{8}{9}$

Since $\lim_{x-3^-} \frac{\lfloor 2x \rfloor + x}{\lfloor x^2 \rfloor + 1} \neq \lim_{x \to 3^+} \frac{\lfloor 2x \rfloor + x}{\lfloor x^2 \rfloor + 1}$,
$\lim_{x \to 3} \frac{\lfloor 2x \rfloor + x}{\lfloor x^2 \rfloor + 1}$ does not exist.

Question 4

(a) Let $a \in \mathbb{R}$,
Since both rational and irrational numbers are dense in the Reals,we may define the following sequences.
(x_n) be a rational sequence such that $x_n \to a$ and
(y_n) be a irrational sequence such that $y_n \to a$.
$\lim_{x_n \to a} f(x_n) = \lim_{x_n \to a} 5x_n + 7 = 5a + 7$
$\lim_{y_n \to a} f(y_n) = \lim_{y_n \to a} x_n + 11 = a + 11$
If $a \neq 1$, then $\lim_{x_n \to a} f(x_n) \neq \lim_{y_n \to a} f(y_n)$,
Therefore $f(x)$ is not continuous at $x \neq 1$
If $a = 1$
Let $\epsilon > 0$ be given. Choose $\delta = \frac{\epsilon}{5}$
We have $f(a) = f(1) = 12$ and $0 < |x - 1| < \frac{\epsilon}{5}$
For rational x we have,
$|f(x) - f(a)| = |5x + 7 - 12| = 5|x - 1| < 5\left(\frac{\epsilon}{5}\right) = \epsilon$
For irrational x we have ,
$|f(x) - f(a)| = |x + 11 - 12| = |x - 1| < \frac{\epsilon}{5} < \epsilon$
$\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$
Therefore $\lim_{x \to 1} f(x) = f(1)$. Hence f is continuous at $x = 1$.

(b) Take $\epsilon = \frac{1}{11}$ since f is continous at $x = 0$, $\exists \delta > 0$ such that
$|x - 0| < \delta \Rightarrow |f(x) - f(0)| < \frac{1}{11}$
Now $|x - 0| < \delta \Rightarrow x \in (-\delta, \delta)$ and
$|f(x) - f(0)| < \frac{1}{11} \Leftrightarrow -\frac{1}{11} < f(x) - 1 < \frac{1}{11} \Rightarrow \frac{10}{11} < f(x) < \frac{12}{11}$
Hence if $x \in (-\delta, \delta)$ then $f(x) > \frac{10}{11}$

SECTION B

Question 5

(a) Since $\sum a_n$ is convergent, it is Cauchy.
Hence for any given ϵ there exist $m, n + 1 \in \mathbb{N}$ such that $|a_{n+1} + ... + a_m| < \epsilon$. Hence we consider,

$$\begin{aligned}
|x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} + ... + x_{n+1} - x_n| \\
&\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + ... + |x_{n+1} - x_n| \\
&\leq a_m + a_{m-1} + ... + a_{n+1} \\
&\leq |a_{n+1} + ... + a_m| < \epsilon
\end{aligned}$$

Therefore (x_n) is Cauchy, thus convergent.

(b) (b_n) converges, (b_n) is bounded, therefore $\exists M > 0$ such that $|b_n| \leq M$
Therefore
$\forall n \in \mathbb{N}$, $|b_n a_n| \leq M|a_n|$
Since $\sum_{n=1}^{\infty} |a_n|$ converges. By comparison test , $\sum_{n=1}^{\infty} |b_n a_n|$ converges.
Hence $\sum_{n=1}^{\infty} b_n a_n$ converges absolutely.

Question 6

(a) Given any $0 < \alpha < 1$ we have

$$\begin{aligned} f(a) = \alpha f(a) + (1-\alpha) f(a) &< \alpha f(a) + (1-\alpha) f(b) \\ &< \alpha f(b) + (1-\alpha) f(b) = f(b) \end{aligned}$$

By IVT, $\exists c \in (a,b)$ such that $f(c) = \alpha f(a) + (1-\alpha) f(b)$.

(b) $\forall x > 0$, $f(x) = f(x^2)$
Hence we get the following
$f(x) = f\left(x^{\frac{1}{2}}\right) = f\left(x^{\frac{1}{4}}\right) = ... = f\left(x^{\frac{1}{2n}}\right)$
For any $a > 0$ $f(a) = f\left(a^{\frac{1}{2}}\right) = f\left(a^{\frac{1}{4}}\right) = ... = f\left(a^{\frac{1}{2n}}\right)$
Since f is continuous,
$\lim\limits_{n\to\infty} a^{\frac{1}{2n}} = 1 \Rightarrow \lim\limits_{n\to\infty} f\left(a^{\frac{1}{2n}}\right) = f(1)$
$\Rightarrow f(a) = f(1)$
Therefore $f(x)$ is a constant.

Question 7

(i) True. Let $a_n \to a$
$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \left(\dfrac{1}{n}\right)\left(\lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} a_{n+1}\right) = 0$. Hence (b_n) converges

(ii) False. Consider $a_n = 1 + \frac{1}{n}$, $\lim\limits_{n\to\infty} a_n = 1$ but $\lim\limits_{n\to\infty} (a_n)^n = \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e \neq 1$

(iii) True. Since $0 \le |a_n b_n| \le \frac{a_n^2 + b_n^2}{2}$ and $\sum a_n^2$ and $\sum b_n^2$ converges,by Comparison Test $\sum |a_n b_n|$ converges.

(iv) False.Let,

$$f(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0 \end{cases}$$

f is not continuous at $x = 0$. But $h(x) = [f(x)]^2$ is continuous at $x = 0$.

(v) False. Let $x_n = \frac{1}{n}$, $y_n = \frac{2}{n}$ for all $n \in \mathbb{N}$
(x_n) and (y_n) are both positive and $\lim\limits_{n\to\infty} (x_n - y_n) = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ Let $f(x) = \frac{1}{x}$
$\lim\limits_{n\to\infty} f(x_n) - f(y_n) = \lim\limits_{n\to\infty} n - \dfrac{n}{2} = \lim\limits_{n\to\infty} \dfrac{n}{2} = \infty$