NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA1102R Calculus
AY 2006/2007 Sem 2

Question 1

(a) Since $\lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$, we have $\lim_{n\to\infty} 3\cdot\left(\frac{2}{3}\right)^n + 2 = 2$ and $\lim_{n\to\infty} 2\cdot\left(\frac{2}{3}\right)^n + 3 = 3$. Hence,

$$\lim_{n\to\infty}\frac{3\cdot 2^n + 2\cdot 3^n}{2\cdot 2^n + 3\cdot 3^n} = \lim_{n\to\infty}\frac{3\cdot\left(\frac{2}{3}\right)^n + 2}{2\cdot\left(\frac{2}{3}\right)^n + 3} = \frac{\lim_{n\to\infty} 3\cdot\left(\frac{2}{3}\right)^n + 2}{\lim_{n\to\infty} 2\cdot\left(\frac{2}{3}\right)^n + 3} = \frac{2}{3}.$$

(b) Consider $f(x) = \ln(e^x + 2x^2)^{\frac{1}{x}} = \frac{\ln(e^x+2x^2)}{x}$. Since $\lim_{x\to 0}\ln(e^x + 2x^2) = 0$ and $\lim_{x\to 0} x = 0$, we can apply L'Hôpital's Rule to get $\lim_{x\to 0}\frac{\ln(e^x + 2x^2)}{x} = \lim_{x\to 0}\frac{e^x + 4x}{e^x + 2x^2} = \frac{1}{1} = 1$. Since e^x is a continuous function, we have $\lim_{x\to 0}(e^x + 2x^2)^{\frac{1}{x}} = \lim_{x\to 0} e^{f(x)} = e^{\left(\lim_{x\to 0} f(x)\right)} = e^1 = e$.

Question 2

(a) Integrating by parts, we have

$$\begin{aligned}\int_0^\pi e^x\cos x\ dx &= [e^x\cos x]_0^\pi - \int_0^\pi e^x(-\sin x)\ dx\\ &= [e^x\cos x]_0^\pi - \left([e^x(-\sin x)]_0^\pi - \int_0^\pi e^x(-\cos x)\ dx\right)\\ &= [e^x\cos x]_0^\pi + [e^x\sin x]_0^\pi - \int_0^\pi e^x\cos x\ dx.\end{aligned}$$

Rearranging, we have

$$\begin{aligned}2\int_0^\pi e^x\cos x\ dx &= [e^x\cos x]_0^\pi + [e^x\sin x]_0^\pi\\ &= -e^\pi - 1.\end{aligned}$$

Therefore $\int_0^\pi e^x\cos x\ dx = -\left(\frac{1+e^\pi}{2}\right)$.

(b) We have, from partial fraction,

$$\begin{aligned}\int_0^2 \frac{3x^2+5x+6}{(x+2)(x^2+4)}\ dx &= \int_0^2 \frac{2x}{x^2+4} + \frac{1}{x^2+4} + \frac{1}{x+2}\ dx\\ &= \left[\ln|x^2+4| + \frac{1}{2}\tan^{-1}\frac{x}{2} + \ln|x+2|\right]_0^2\\ &= (\ln 8 + \frac{\pi}{8} + \ln 4) - (\ln 4 + \ln 2) = 2\ln 2 + \frac{\pi}{8}.\end{aligned}$$

Question 3

Firstly, by doing implicit differentiation, we obtain $2x + 2(y-2)\frac{dy}{dx} = 0$, i.e. $\frac{dy}{dx} = \frac{-x}{y-2}$. Since the equation of circle can be split into 2 equations, $y = 2 + \sqrt{1-x^2}$ and $y = 2 - \sqrt{1-x^2}$, we have

$$\begin{aligned}
\text{Surface area} &= 2\pi \int_{-1}^{1} (2+\sqrt{1-x^2})\sqrt{1+(\frac{dy}{dx})^2} + (2-\sqrt{1-x^2})\sqrt{1+(\frac{dy}{dx})^2}\ dx \\
&= 2\pi \int_{-1}^{1} (2+\sqrt{1-x^2})\sqrt{\frac{1}{1-x^2}} + (2-\sqrt{1-x^2})\sqrt{\frac{1}{1-x^2}}\ dx \\
&= 2\pi \int_{-1}^{1} 4\sqrt{\frac{1}{1-x^2}}\ dx \\
&= 2\pi \int_{0}^{1} 8\sqrt{\frac{1}{1-x^2}}\ dx \qquad \text{(by symmetry)} \\
&= 16\pi \int_{0}^{\frac{\pi}{2}} \sqrt{\frac{1}{1-(\sin y)^2}} \cos y\ dy \qquad (\text{sub } x = \sin y) \\
&= 16\pi \int_{0}^{\frac{\pi}{2}} 1\ dy = 8\pi^2.
\end{aligned}$$

Question 4

(a) For all $n \in \mathbb{Z}_{\geq 2}$, we have

$$\frac{1}{n^2-1} = \frac{1}{(n-1)^2 + 2n - 2} \leq \frac{1}{(n-1)^2}.$$

Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \sum_{m=1}^{\infty} \frac{1}{m^2}$ is convergent, by Comparison Test, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ is convergent.

(b) Let $a_n = \frac{1}{b_n}$, where $b_n = n \ln n$, $n \in \mathbb{Z}^+$.
This give us $b_{n+1} - b_n = (n+1)\ln(n+1) - n\ln n \geq n\ln(n+1) - n\ln n = n\ln(\frac{n+1}{n}) \geq 0$.
Hence we have $b_{n+1} \geq b_n$ and therefore $a_{n+1} \leq a_n$.
Together with $\lim\limits_{n\to\infty} a_n = 0$, by Alternating Series Test, this series converges.

(c) Since $\displaystyle\lim_{x\to\infty} \int_2^x \frac{1}{x\ln x}\ dx = \lim_{x\to\infty} [\ln(\ln x)]_2^x = \lim_{x\to\infty} \ln(\ln x) - \ln(\ln 2) = \infty$, we have the sum $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ to diverges by using the Integral Test.
We also have,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{1}{\ln n}\sin\frac{1}{n}}{\frac{1}{n\ln n}} &= \lim_{n\to\infty} n \sin\frac{1}{n} \\
&= \lim_{n\to 0} \frac{1}{n}\sin n \\
&= 1.
\end{aligned}$$

Thus by Limit Comparison Test, we have $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}\sin\frac{1}{n}$ to diverges.

Question 5

(a) Consider Maclarin's expansion for e^x

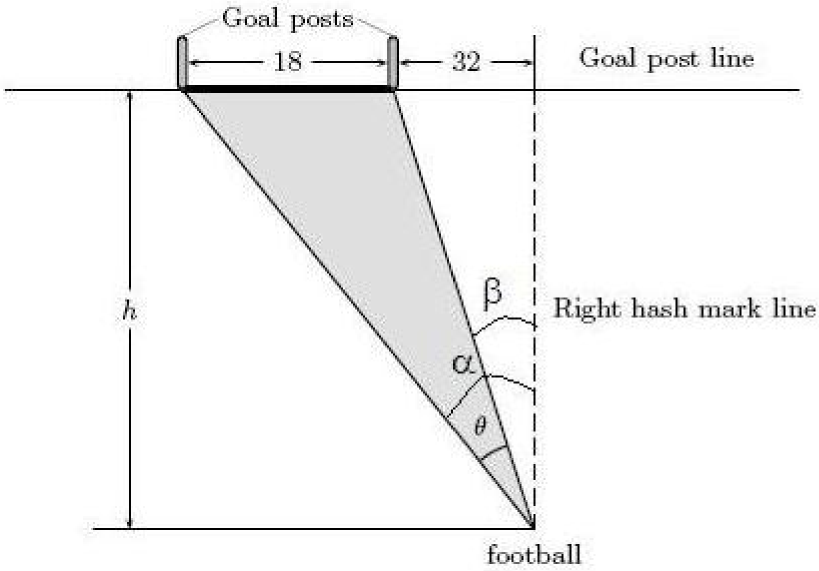
$$\begin{aligned} e^x &= \sum_{r=0}^{\infty} \frac{x^r}{r!} \\ e^{x^2} &= \sum_{r=0}^{\infty} \frac{(x^2)^r}{r!} = \sum_{r=0}^{\infty} \frac{x^{2r}}{r!} \\ x^5 e^{x^2} &= \sum_{r=0}^{\infty} \frac{x^{2r+5}}{r!}. \end{aligned}$$

Hence the coefficient of $x^{2007} = x^{2(1001)+5}$ is $\frac{1}{1001!}$ which is also $\frac{f^{(2007)}(0)}{2007!}$, i.e. $f^{(2007)}(0) = \frac{2007!}{1001!}$.

(b) Since $\lim_{x\to 0} \int_0^x \frac{t^2}{\sqrt{a+3t}}\, dt = 0$ and $\lim_{x\to 0} x - \sin x = 0$, we apply L'Hôpital's Rule to get

$$\begin{aligned} 1 = \lim_{x\to 0} \frac{\int_0^x \frac{t^2}{\sqrt{a+3t}}\, dt}{x - \sin x} &= \lim_{x\to 0} \frac{\frac{x^2}{\sqrt{a+3x}}}{1-\cos x} \\ &= \lim_{x\to 0} \frac{1}{\sqrt{a+3x}} \cdot \lim_{x\to 0} \frac{x^2}{1-\cos x} \\ &= \frac{1}{\sqrt{a}} \cdot \lim_{x\to 0} \frac{2x}{\sin x} \\ &= \frac{1}{\sqrt{a}} \cdot 2 = \frac{2}{\sqrt{a}}. \end{aligned}$$

Hence we have $a = 4$.



Question 6

(a) Using trigonometric ratios we have $\tan\alpha = \frac{50}{h}$ and $\tan\beta = \frac{32}{h}$. Hence

$$\begin{aligned}\tan\theta &= \tan(\alpha-\beta)\\ &= \frac{\tan\alpha-\tan\beta}{1-\tan\alpha\tan\beta}\\ &= \frac{\frac{50}{h}-\frac{32}{h}}{1+\frac{1600}{h^2}}\\ &= \frac{18h}{h^2+1600}.\end{aligned}$$

(b) For $0 \le \theta \le \frac{\pi}{2}$, $\tan\theta$ is a increasing function. To maximize θ, it suffices to maximize $\tan\theta$. Differentiating $\frac{18h}{1600+h^2}$ with respect to h,

$$\frac{d}{dh}\left(\frac{18h}{1600+h^2}\right) = \frac{-18(h^2-1600)}{(h^2+1600)^2} = \frac{-18(h-40)(h+40)}{(h^2+1600)^2}.$$

Since $h \in \mathbb{R}^+$, when $\frac{d}{dh}(\tan\theta) = 0$, we have $h - 40 = 0$, i.e. $h = 40$.
Since $\frac{d}{dh}(\tan\theta)\big|_{h=40^-} > 0$ and $\frac{d}{dh}(\tan\theta)\big|_{h=40^+} < 0$, by first derivative test, $h = 40$ is a local maximum. Checking the end points $\left.\frac{18h}{h^2+1600}\right|_{h=0} = 0$, $\lim_{x\to\infty} \left.\frac{18h}{h^2+1600}\right|_{h=x} = 0$.
Hence $\tan\theta$, and thus also θ, is maximum when $h = 40$.

Question 7

(a) Let $f(x) = \ln(x+1)$. Since $f'(x)$ is decreasing on the interval $-1 < x < 1$, we have $\frac{f(x)-f(0)}{x-0} \le f'(0)$ as a consequence of Mean Value Theorem, i.e. $\ln(x+1) \le x$. Let $x_1 = -x$, then for the interval $-1 < x_1 < 1$, we get $\ln(1-x_1) \le -x_1$, i.e. $x_1 \le -\ln(1-x_1)$. Hence for $-1 < x < 1$, we have $\ln(x+1) \le x \le -\ln(1-x)$.

(b) For any integer $k > 1$, we have $-1 < \frac{1}{k} < 1$. Hence $\ln\left(1+\frac{1}{k}\right) \le \frac{1}{k} \le -\ln\left(1-\frac{1}{k}\right)$. Simplifying which, we will have $\ln\left(\frac{k+1}{k}\right) \le \frac{1}{k} \le \ln\left(\frac{k}{k-1}\right)$.

(c) Hence from (7b.), for all integer $n > 1$,

$$\begin{aligned}\sum_{k=n}^{2n} \ln\left(\frac{k+1}{k}\right) \le & \sum_{k=n}^{2n} \frac{1}{k} \le \sum_{k=n}^{2n} \ln\left(\frac{k}{k-1}\right) \\ \ln\left(\frac{2n+1}{n}\right) \le & \sum_{k=n}^{2n} \frac{1}{k} \le \ln\left(\frac{2n}{n-1}\right).\end{aligned}$$

Since $\lim_{n\to\infty} \ln\left(\frac{2n+1}{n}\right) = \lim_{n\to\infty} \ln\left(\frac{2n}{n-1}\right) = \ln 2$, by Squeeze Theorem, $\lim_{n\to\infty} \sum_{k=n}^{2n} \frac{1}{k} = \ln 2$.

Question 8

(a) We have,

$$\begin{aligned}\int_0^x \frac{\cos x}{1+x}\,dx + 2\int_0^x \frac{\cos x}{(1+x)^3}\,dx &= \left[\frac{\sin x}{1+x}\right]_0^x + \int_0^x \frac{\sin x}{(1+x)^2} + 2\int_0^x \frac{\cos x}{(1+x)^3}\,dx \\ &= \frac{\sin x}{1+x} - \left[\frac{\cos x}{(1+x)^2}\right]_0^x - 2\int_0^x \frac{\cos x}{(1+x)^3}\,dx + 2\int_0^x \frac{\cos x}{(1+x)^3}\,dx \\ &= \frac{\sin x}{1+x} - \frac{\cos x}{(1+x)^2} + \frac{1}{1^2}.\end{aligned}$$

For all $x \in \mathbb{R}$, we have $\frac{-1}{1+x} \le \frac{\sin x}{1+x} \le \frac{1}{1+x}$ and $\frac{-1}{(1+x)^2} \le \frac{\cos x}{(1+x)^2} \le \frac{1}{(1+x)^2}$, and so by Squeeze Theorem, $\lim_{x\to\infty} \frac{\sin x}{1+x} = \lim_{x\to\infty} \frac{\cos x}{(1+x)^2} = 0$.

Hence taking limits, we get $\int_0^\infty \frac{\cos x}{1+x}\,dx + 2\int_0^\infty \frac{\cos x}{(1+x)^3}\,dx = \lim_{x\to\infty}\left(\frac{\sin x}{1+x} - \frac{\cos x}{(1+x)^2} + 1\right) = 1.$

(b) Consider the function $h(x) = \left(\int_a^x f(x)\,dx\right)\left(\int_b^x g(x)\,dx\right)$.

Since $\int_a^x f(x)\,dx$ and $\int_b^x g(x)\,dx$ are both differentiable on $[a,b]$, $h(x)$ is continuous.

Since $h(a) = 0 = h(b)$, applying Rolle's Theorem, there exists $c \in (a,b)$ such that $h'(c) = 0$.

Hence $f(c)\int_b^c g(x)\,dx + g(c)\int_a^c f(x)\,dx = 0$.

Therefore there exist $c \in (a,b)$ such that $g(c)\int_a^c f(x)\,dx = f(c)\int_c^b g(x)\,dx$.