NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Zheng Shaoxuan

MA3233 Algorithmic Graph Theory
AY 2004/2005 Sem 2

Question 1

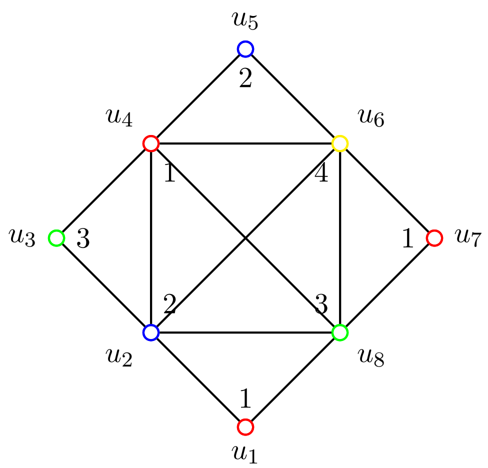
(i) Notice that the only two vertices adjacent to u_1 is u_2 and u_8. Hence in any C_5 containing u_1, u_2 and u_8 must be adjacent to u_1.

By exhausting all possibilities, the C_5s are: $u_1u_2u_3u_4u_8u_1$, $u_1u_2u_4u_6u_8u_1$, $u_1u_2u_6u_4u_8u_1$ and $u_1u_2u_6u_7u_8u_1$.

(ii) Since G has a C_3, hence $\chi(G) \geq 3$. We claim that a 3-colouring of G is not possible.

Suppose the contrary. $\forall v \in V(G)$, let $\theta(v)$ be the colouring of the vertex v. Starting from the C_3 $u_1u_2u_8u_1$, let $\theta(u_1) = 1$, $\theta(u_2) = 2$ and $\theta(u_8) = 3$. Hence, $\theta(u_4) = 1$ since u_4 is adjacent to u_2 and u_8. But now u_6 is adjacent to u_2, u_8 and u_4, three vertices of different colours! A contradiction. Hence $\chi(G) \geq 4$

A 4-colouring is possible, as shown below:

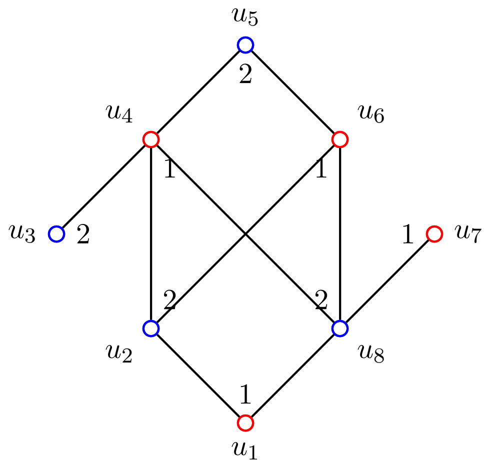


Hence $\chi(G) \leq 4$. Therefore $\chi(G) = 4$.

(iii) To obtain a bipartite subgraph of G with the most number of edges, we aim to remove as few edges as possible from G such that all odd cycles are broken.

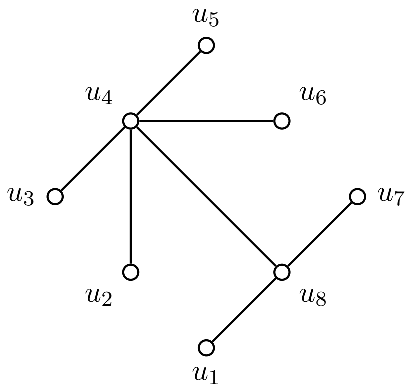
However, observe that $u_1u_2u_8u_1$, $u_3u_4u_2u_3$, $u_5u_6u_4u_5$ and $u_7u_8u_6u_7$ are C_3s whose edges are all mutually exclusive. Hence to break these C_3s, we must remove at least 4 edges, one from each C_3.

Taking into account all the other odd cycles that we need to break, with some intelligent edge choice we find that removing the 4 edges u_2u_8, u_2u_3, u_6u_4 and u_6u_7 (there can be other choices) is sufficient to leave us with a bipartite subgraph, as shown below:



(iv) Aiming to obtain a spanning tree of G with the least possible number of cut-vertices, we immediately observe that a spanning tree of G with only 1 cut vertex is not possible since there does not exist any vertex in G which is adjacent to all other vertices.

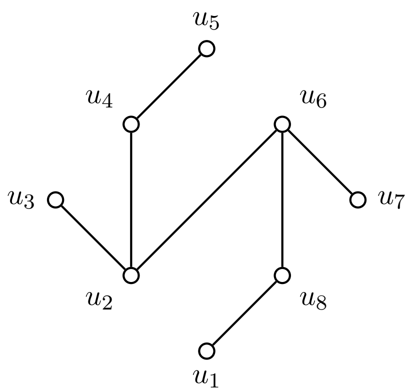
A spanning tree of G with 2 cut vertices exists and hence the below is the spanning tree of G with the least possible number of cut-vertices:



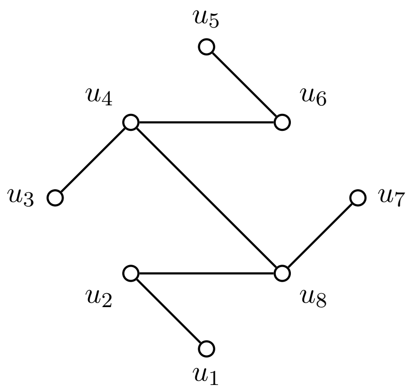
(v) We observe that to obtain a spanning tree T that is isomorphic to the spanning tree $G - E(T)$, T must have 7 edges. Also in T, u_1, u_3, u_5 and u_7 must have a degree of 1 and u_2, u_4, u_6 and u_8 must have a degree of 2.

By using the above observations and by intelligent drawing we obtain the below trees and their isomorhpic complements:

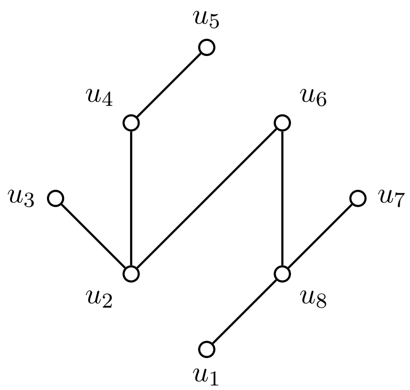
T_1



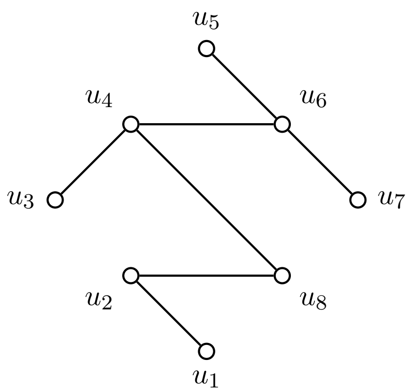
$G - E(T_1)$



T_2



$G - E(T_2)$



(vi) We observe that, in fact, T_1 from (v) is a depth-first-search tree! The DFST T_1 is constructed in the following manner: $u_1 - u_8 - u_6 - u_2 - u_4 - u_5 - u_3 - u_7$.

Since $T_1 \cong G - E(T_1)$ which is also a DFST, hence T_1 is our desired graph (there can be other answers) .

(vii) The value of the $(1, 1)$-entry in $A(G)^5$ is 46.

To explain this (even though an explanation is not needed in the question), the desired value is the number of walks of length 5 from u_1 to u_1. To cover all such possible walks without overcounting (there are many of them and some are difficult to spot!), we attempt to do it in a systemetic fashion.

Consider the walks of length 5 to be of the form u_1abcdu_1 where $a, b, c, d \in V(G)$. We observe that the first vertex that is visited after u_1 (a), and the last vertex that is visited before returning to u_1 (d), can only be either u_2 or u_8 as these are the only two vertices adjacent to u_1. By symmetry of G, the number of walks in the case where $a = u_2, d = u_2$ is the same as the number of walks where $a = u_8, d = u_8$. Similarly, the number of walks in the case where $a = u_2, d = u_8$ is the same as the number of walks where $a = u_8, d = u_2$. Hence, we would count of the number of walks in the two cases: $a = u_2, d = u_2$ and $a = u_2, d = u_8$, then add the results up and double it to obtain our final answer.

For the case where $a = u_2, d = u_2$: by meticulous examination of all possible vertices for b and c, we obtain the following 10 walks: $u_1u_2u_1u_8u_2u_1$, $u_1u_2u_3u_4u_2u_1$, $u_1u_2u_4u_3u_2u_1$, $u_1u_2u_4u_6u_2u_1$, $u_1u_2u_4u_8u_2u_1$, $u_1u_2u_6u_4u_2u_1$, $u_1u_2u_6u_8u_2u_1$, $u_1u_2u_8u_1u_2u_1$, $u_1u_2u_8u_4u_2u_1$ and $u_1u_2u_8u_6u_2u_1$.

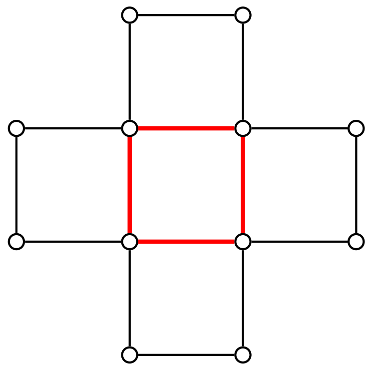
For the case where $a = u_2, d = u_8$: by meticulous examination of all possible vertices for b and c, we obtain the following 13 walks: $u_1u_2u_1u_2u_8u_1$, $u_1u_2u_3u_2u_8u_1$, $u_1u_2u_3u_4u_8u_1$, $u_1u_2u_4u_2u_8u_1$, $u_1u_2u_4u_6u_8u_1$, $u_1u_2u_6u_2u_8u_1$, $u_1u_2u_6u_4u_8u_1$, $u_1u_2u_6u_7u_8u_1$, $u_1u_2u_8u_1u_8u_1$, $u_1u_2u_8u_2u_8u_1$, $u_1u_2u_8u_4u_8u_1$, $u_1u_2u_8u_6u_8u_1$ and $u_1u_2u_8u_7u_8u_1$.

Hence the total number of walks of length 5 from u_1 to u_1 is $2 \times (10 + 13) = 46$.

Question 2

Define the following graph as such:

G



We may attempt to solve this question by performing the standard techniques of edge removal to calculate $\tau(G)$. However, due to the nature of G, a combinatorial approach will be a faster way of obtaining the answer.

Consider the 4 bolded edges in G. It is impossible for all 4 bolded edges to be present in any spanning tree of G, as they form a C_4 themselves. Therefore, we consider 4 different cases where in each case, a specific number of the bolded edges are not present in the spanning trees considered.

For the spanning trees where all 4 bolded edges are not present, just 1 edge needs to be removed from the resultant C_{12} left behind. Hence the number of spanning trees in this case is 12.

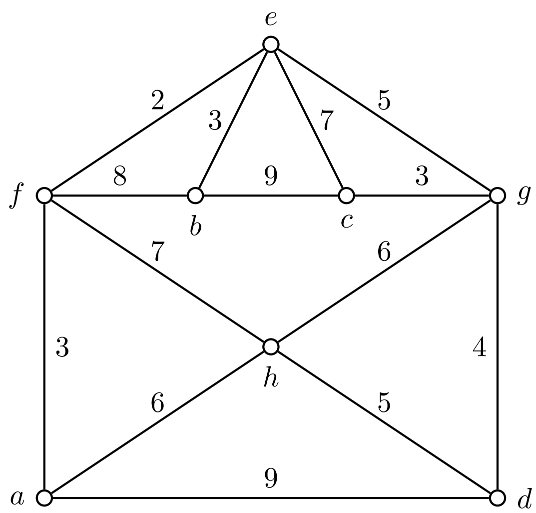
For the spanning trees where exactly 3 of the 4 bolded edges are not present, there are $\binom{4}{3}$ ways to select the 3 edges. What is left behind is a C_4 and a C_{10} with their common edge being the remaining bolded edge. Since the bolded edge has to be present in the tree, an edge needs to be removed from the C_4 and an edge needs to be removed from the C_10 to break the two cycles, and these two edges cannot be the bolded edge. There are 3 ways to do the former and 9 ways to do the latter. Hence the total number of such spanning trees is $\binom{4}{3} \times 3 \times 9 = 108$.

For the spanning trees where exactly 2 of the 4 bolded edges are not present, there are $\binom{4}{2}$ ways to select the 2 edges. What is left behind are 2 C_4s and a C_8 with their respective common edges being the remaining 2 bolded edges. Similar to before an edge needs to be removed from each of the cycles and none of these 3 edges can be a bolded edge. There are 3 ways to remove an edge from the C_4 and 6 ways to remove an edge from the C_8 (since the C_8 contains both bolded edges). Hence the total number of such spanning trees is $\binom{4}{2} \times 3 \times 3 \times 6 = 324$.

For the spanning trees where exactly 1 of the 4 bolded edges is not present, there are $\binom{4}{1}$ ways to select this edge. What is left behind are 3 C_4s and a C_6 with their respective common edges being the remaining 3 bolded edges. Similar to before an edge needs to be removed from each of the cycles and none of these 4 edges can be a bolded edge. There are 3 ways to remove an edge from the C_4 and 3 ways to remove an edge from the C_6 (since the C_6 contains all 3 bolded edges). Hence the total number of such spanning trees is $\binom{4}{1} \times 3 \times 3 \times 3 \times 3 = 324$.

Hence, the total number of spanning trees of G is $12 + 108 + 324 + 324 = 768$.

Question 3



Apply Edmond's algorithm to the above graph:

There are 4 odd vertices in the graph, a, b, c and d. The least weight and path of least weight between each pair of these vertices are:

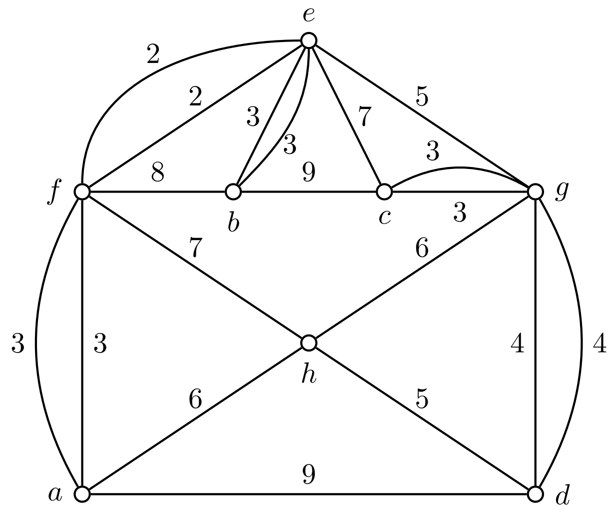
- $a - b$: 8 (via $afeb$),
- $a - c$: 12 (via $afec$),
- $a - d$: 9 (via ad),
- $b - c$: 9 (via bc),
- $b - d$: 12 (via $begd$),
- $c - d$: 7 (via cgd).

The weights of the 3 possible pairings between these 4 vertices are:

- $a - b$ and $c - d$: $8 + 7 = 15$,
- $a - c$ and $b - d$: $12 + 12 = 24$,
- $a - d$ and $b - c$: $9 + 9 = 18$.

The minimum weight pairing is $a - b$ and $c - d$.

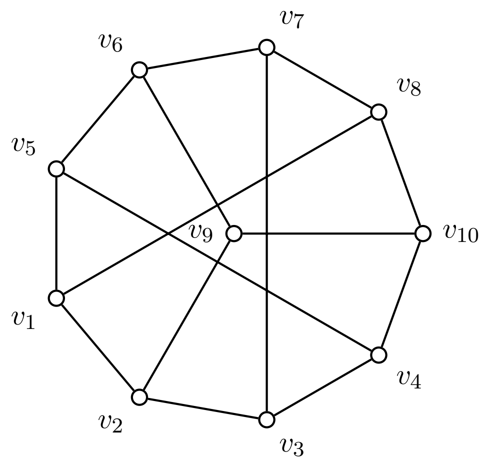
We append the paths of least weights of the two paths within the minimum weight pairing into the original graph. We obtain:

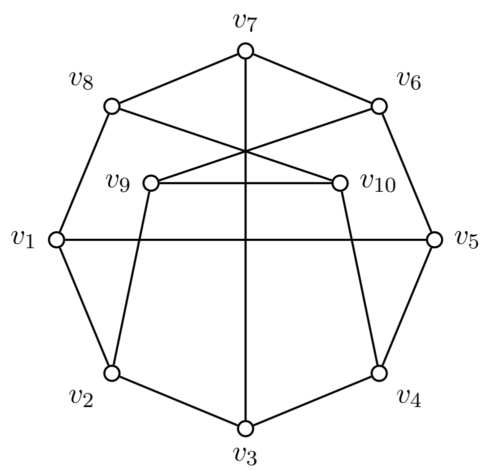


Using Fluerry's algorithm, we construct a closed trail of this new graph which contains all its edges. One such trail can be $afahfefbebcgcegdghda$, and this is also the closed walk with minimum weight which contains all the edges in the original graph. (there are many other possible answers).

Question 4

Since both graphs are 3-regular with 10 vertices, they may or may not be isomorphic. Notice that the graph on the right has a very obvious C_8. Indeed by careful observation, the graph on the left contains a C_8 too. We then proceed to label the vertices as such based on our observations (notice the C_8 in both graphs, $v_1v_2v_3v_4v_5v_6v_7v_8$):





To verify that these two graphs are isomorphic under the above labels, we check to see that each vertex is adjacent to the same vertices in both graphs.

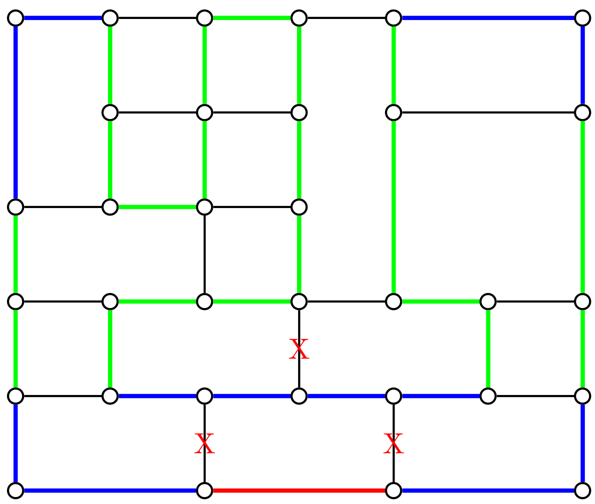
Indeed in both graphs:

- v_1 is adjacent to v_2, v_8 and v_5;
- v_2 is adjacent to v_3, v_1 and v_9;
- v_3 is adjacent to v_4, v_2 and v_7;
- v_4 is adjacent to v_5, v_3 and v_{10};
- v_5 is adjacent to v_6, v_4 and v_1;
- v_6 is adjacent to v_7, v_5 and v_9;
- v_7 is adjacent to v_8, v_6 and v_3;
- v_8 is adjacent to v_1, v_7 and v_{10};
- v_9 is adjacent to v_{10}, v_2 and v_6;
- v_{10} is adjacent to v_9, v_4 and v_8;

Therefore, these two graphs are isomorphic.

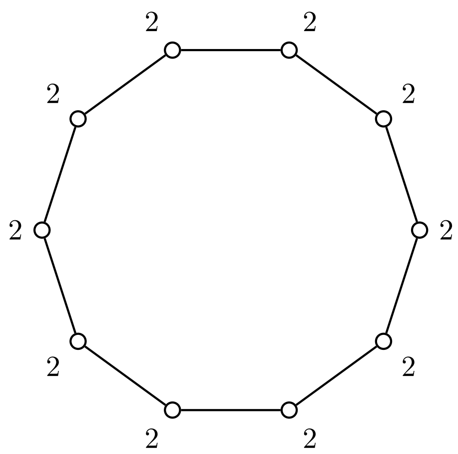
Question 5

This graph is hamiltonian. The hamiltonian cycle is as shown below, where the blue edges indicate edges that have to be included due to vertices having only two possible adjacent edges left that could be in the hamiltonian cycle, the red edge representing the edge that we guess have to be included, red crosses represent edges that are not possible to be in the hamiltonian cycle due to similar logical deductions, and green edges represents the rest of the edges filled in to complete the hamiltonian cycle (your own hamiltonian cycle can have different such green edges, as long as all vertices are in the cycle).



Question 6

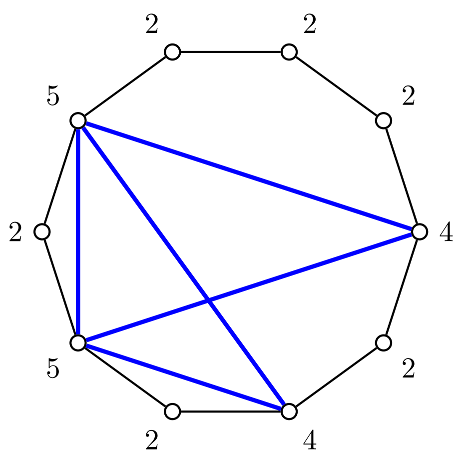
Since G is hamiltonian, it contains a spanning C_{10}, with their current degrees labelled as shown below:



Since $e(G) = 15$, 5 more edges needs to be added to the above graph to form G, and since G is semi-eulerian, G contains exactly 2 odd vertices.

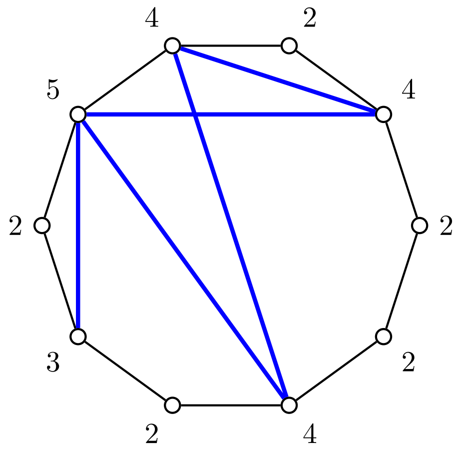
The maximum degree of any vertex in G is 7, by having the 5 additional edges all incident to one particular vertex. However, we quickly notice that the 5 additional edges, at the other end, have to be incident to 5 different vertices (otherwise G becomes a multigraph). This means that G will have 6 odd vertices, and hence is not semi-eulerian. Hence we only consider odd vertices of degrees 3 and 5 in the below cases.

Case 1: The two odd vertices are both of degree 5, i.e. two vertices are each incident to 3 additional edges. These two odd vertices must be adjacent (have an additional edge between them) since we only have 5 additional edges. In order for there not to be any more odd vertices and to preserve the nature of the (simple) graph, the remaining 4 additional edges, at the other end, have to be adjacent to exactly 2 vertices each of degree 4. We cannot have any vertex of degree 6 as all these 4 edges will have to be incident to this vertex, leading to a multigraph. Hence the only possible degree sequence in this case is $(5, 5, 4, 4, 2, 2, 2, 2, 2, 2)$. One corresponding graph is as shown:

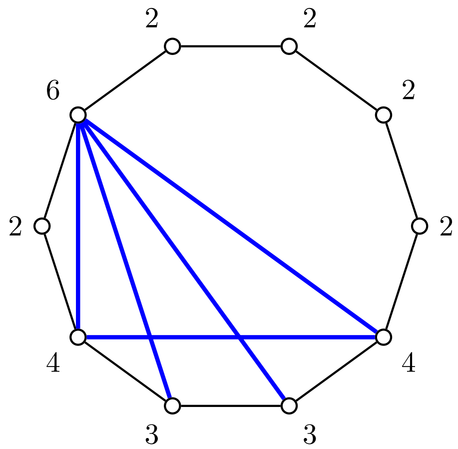


Case 2: The two odd vertices are one odd vertex of degree 3, one odd vertex of degree 5 i.e. one vertex is incident to 1 additional edge, and one vertex is incident to 3 additional edges. Among all the other even vertices, we cannot have any vertex of degree 6 (incident to 4 additional edges) as at least two of the edges incident to the vertex of degree 5 will have to be incident

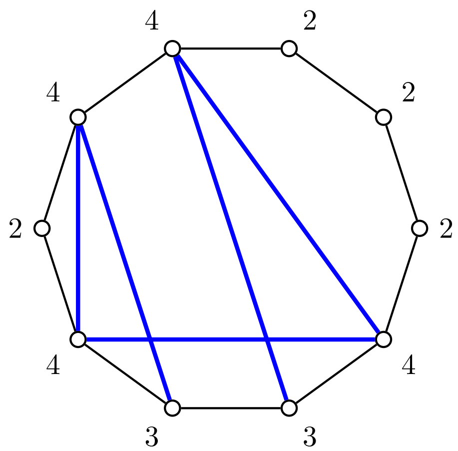
to this vertex, causing G to become a multigraph. Hence, the only possible degree sequence is $(5, 4, 4, 4, 3, 2, 2, 2, 2, 2)$. One corresponding graph is as shown:



Case 3: The two odd vertices both of degree 3 i.e. the two vertices both incident to one additional edge. This time, one vertex of degree 6 is possible. We cannot have two vertices of degree 6 as these two vertices will have multiple edges incident to both of them, causing G to be a multigraph. Hence, if a vertex of degree 6 exists, the degree sequence of G is $(6, 4, 4, 3, 3, 2, 2, 2, 2, 2)$, and one corresponding graph is as shown:



Case 3 (continued): Should there be no vertex of degree 6, then the only possible degree sequence of G is now $(4, 4, 4, 4, 3, 3, 2, 2, 2, 2)$, and one corresponding graph is as shown:



Question 7

(a) If G is disconnected, we consider the connected component of G which contains the cut-vertex and relabel the induced subgraph of this component as G.

Suppose v is a cut-vertex of G which is 3-regular. As $d(v) = 3$, $c(G - v)$ is either 2 or 3. If $c(G - v) = 3$, then each edge incident to v is a bridge as removing one edge incident to v will disconnect the respective component in $G - v$ from the rest of the graph. If $c(G - v) = 2$, there exists one component C in $G - v$ for which v is adjacent to only one of the vertices, call it u, in the component (since v is incident to exactly 3 edges and there are two components in $G - v$). The edge vu is a bridge since $G - vu$ disconnects C from the rest of the graph.

Hence G must contain at least a bridge.

(b) We claim that G has no odd cycles, and hence is bipartite.

Suppose the contrary, i.e. an odd cycle C_p exists, where p is odd. Take two vertices v_1 and v_2 which are adjacent in the induced subgraph of the cycle. Since v_1 and v_2 are adjacent, v_1v_2 is a path of length 1, an odd length. However, if you consider the path of v_1 to v_2 along C_p passing through all other vertices in the cycle, this path is of length $p-1$, an even length! A contradiction. Hence our claim is true.

(c) In any graph G of order n, consider $v \in V(G)$ where v is the vertex of largest degree, i.e. $d(v) = \Delta(G)$. In $\overline{G}$, v becomes the vertex of smallest degree, i.e. $d(v) = \delta(\overline{G}) = n - \Delta(G) - 1$ as the degrees of any vertex in a graph and its complement always sum up to $n - 1$.

Hence, $\delta(G) + \delta(\overline{G}) = (\delta(G) - \Delta(G)) + n - 1 \leq n - 1$ since $\delta(G) \leq \Delta(G)$.

Equality holds when $\delta(G) = \Delta(G)$, i.e. when G is regular.