NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

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MA2108 Mathematical Analysis I

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Question 1

(i) We shall prove by induction on $n \in \mathbb{N}$ that $x_n \leq 3$ for all $n \in \mathbb{N}$, with the base case $n = 1$ being trivial. Suppose that we have $x_k \leq 3$ for some $k \in \mathbb{N}$. This implies that $x_{k+1} = \frac{\sqrt{8x_k^2+9}}{3} \leq \frac{\sqrt{8(3)^2+9}}{3} = 3$, so this completes the induction step. We are done.

(ii) We shall first prove by induction on $n \in \mathbb{N}$ that $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. Since

$$x_2 = \frac{\sqrt{8x_1^2 + 9}}{3} = \sqrt{\frac{17}{9}} \geq 1 = x_1,$$

this shows that the base case $n = 1$ is true. Now, suppose that we have $x_{k+1} \geq x_k$ for some $k \in \mathbb{N}$. This implies that $x_{k+2} = \frac{\sqrt{8x_{k+1}^2+9}}{3} \geq \frac{\sqrt{8x_k^2+9}}{3} = x_{k+1}$, so this completes the induction step, and we are done.

Since (x_n) is bounded above and monotonically increasing, it is necessarily convergent by the Monotone Convergence Theorem. Let x denote the limit of (x_n). Then we have

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{\sqrt{8x_n^2 + 9}}{3} = \frac{\sqrt{8x^2 + 9}}{3},$$

which implies that $\sqrt{8x^2 + 9} = 3x$, or equivalently, $x^2 = 9$. Hence, we have either $x = 3$ or $x = -3$. Furthermore, since $\sqrt{8x^2 + 9}$ is non-negative, we must have $x = 3$. So $\lim_{n\to\infty} x_n = 3$.

Question 2

(a) (i) For each $n \in \mathbb{N}$, define $x_n := \frac{(2n+1)!}{(n!)^2 5^n}$. Then we have $\frac{x_{n+1}}{x_n} = \frac{(2n+3)!}{((n+1)!)^2 5^{n+1}} \cdot \frac{(n!)^2 5^n}{(2n+1)!} = \frac{(2n+3)(2n+2)}{5(n+1)^2}$. This implies that $\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty} \frac{(2n+3)(2n+2)}{5(n+1)^2} = \lim_{n\to\infty} \frac{(2+\frac{3}{n})(2+\frac{2}{n})}{5(1+\frac{1}{n})^2} = \frac{4}{5} < 1$, so $\sum_{n=1}^{\infty} x_n$ converges absolutely by the Ratio Test.

(ii) For each $n \in \mathbb{N}$, define $y_n := n\left(1 + \frac{1}{4n}\right)^{-2n^2}$. Then we have $|y_n|^{1/n} = n^{\frac{1}{n}}\left(1 + \frac{1}{4n}\right)^{-2n}$. Since $\left(\left(1 + \frac{1}{4n}\right)^{4n}\right)$ is a subsequence of $\left(\left(1 + \frac{1}{n}\right)^n\right)$, we have $\lim_{n\to\infty}\left(1 + \frac{1}{4n}\right)^{4n} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$. Hence, we have $\lim_{n\to\infty} |y_n|^{1/n} = \lim_{n\to\infty} n^{\frac{1}{n}} \cdot \lim_{n\to\infty}\left(\left(1 + \frac{1}{4n}\right)^{4n}\right)^{-\frac{1}{2}} = e^{-\frac{1}{2}} < 1$, so $\sum_{n=1}^{\infty} y_n$ converges absolutely by the Root Test.

(b) Note that we have $\frac{2n+1}{n^2(n+1)^2} = \frac{(n+1)^2-n^2}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$ for all $n \in \mathbb{N}$. This implies that for all $N \in \mathbb{N}$, we have $\sum_{n=1}^{N} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{N}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = 1 - \frac{1}{(N+1)^2}$. Hence, we have

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \lim_{N\to\infty}\left(\sum_{n=1}^{N} \frac{2n+1}{n^2(n+1)^2}\right) = \lim_{N\to\infty}\left(1 - \frac{1}{(N+1)^2}\right) = 1.$$

(c) We shall prove that the series $\sum_{n=1}^{\infty}(-1)^{(n+1)}a_n$ is divergent. Arguing by contradiction, suppose that the series $\sum_{n=1}^{\infty}(-1)^{(n+1)}a_n$ converges. Let us define $b_n := \frac{1}{n}$ for all $n \in \mathbb{N}$. Noting that the series $\sum_{n=1}^{\infty}(-1)^{(n+1)}b_n$ is convergent by the Alternating Series Test, it follows that the series $\sum_{n=1}^{\infty}(-1)^{(n+1)}(a_n - b_n) = \sum_{n=1}^{\infty}(-1)^{(n+1)}a_n - \sum_{n=1}^{\infty}(-1)^{(n+1)}b_n$ is convergent.

Now, it is easy to see that $a_{2n} - b_{2n} = 0$ and $a_{2n-1} - b_{2n-1} = \frac{1}{\sqrt{2n-1}} - \frac{1}{2n-1}$ for all $n \in \mathbb{N}$. Furthermore, for each positive integer $n > 1$, we have $\sqrt{2n-1} \geq \sqrt{3} \geq \frac{3}{2}$, which implies that

$$\frac{1}{\sqrt{2n-1}} - \frac{1}{2n-1} = \frac{\sqrt{2n-1}-1}{2n-1} \geq \frac{3/2-1}{2n-1} = \frac{1}{2(2n-1)} \geq \frac{1}{4n}.$$

As the series $\sum_{n=1}^{\infty}\frac{1}{4n} = 4\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, it follows from the comparison test that the series $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{2n+1}} - \frac{1}{2n+1}\right)$ is divergent. Hence, the series

$$\sum_{n=1}^{\infty}(-1)^{(n+1)}(a_n - b_n) = \sum_{n=1}^{\infty}(-1)^{(2n-1)}\left(\frac{1}{\sqrt{2n-1}} - \frac{1}{2n-1}\right) = -\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{2n-1}} - \frac{1}{2n-1}\right)$$

diverges, which is a contradiction. So the series $\sum_{n=1}^{\infty}(-1)^{(n+1)}a_n$ is divergent as desired.

Question 3

(a) Let $\varepsilon > 0$ be given, and set $\delta = \min\left\{\frac{1}{5}, \frac{\varepsilon}{35}\right\}$. It follows that for all $x \in \mathbb{R}$ such that $0 < |x-1| < \delta$, we must have $x - 1 > -\delta \geq -\frac{1}{5}$, which implies that

$$|4x-3| \geq 4x-3 = 4(x-1)+1 > 4\left(-\frac{1}{5}\right)+1 = \frac{1}{5}.$$

Hence, for all $x \in \mathbb{R}$ such that $0 < |x-1| < \delta$, we have

$$\left|\frac{x+1}{4x-3} - 2\right| = \left|\frac{x+1-2(4x-3)}{4x-3}\right| = \left|\frac{-7(x-1)}{4x-3}\right| = \frac{7|x-1|}{|4x-3|} < \frac{7\cdot(\varepsilon/35)}{1/5} = \varepsilon.$$

By the $\varepsilon - \delta$ definition of limit, we must have $\lim_{x\to 1}\frac{x+1}{4x-3} = 2$ as desired.

(b) (i) For all non-zero $x \in \mathbb{R}$ and $n \in \mathbb{N}$, define $f(x) := \cos\left(\frac{1}{x^2}\right)$ and $x_n := \frac{1}{\sqrt{n\pi}}$. Then it is easy to see that $\lim_{n\to\infty} x_n = 0$, $x_n \neq 0$, and $f(x_n) = \cos\left(\frac{1}{x_n^2}\right) = \cos(n\pi) = (-1)^n$ for all $n \in \mathbb{N}$. Since the sequence $(f(x_n)) = ((-1)^n)$ is divergent, it follows from the divergent criterion that the limit $\lim_{x\to 0} f(x)$ does not exist.

(ii) Let $y \in \mathbb{R}$ be given. By the definition of $[y]$, we have $[y] \leq y < [y]+1$. Thus, we have $y - 1 < [y] \leq y$, so for all $x > 0$, we have

$$x^3\left(\left[\frac{1}{x^3}\right] + \left[\frac{2}{x^3}\right]\right) > x^3\left(\frac{1}{x^3} - 1 + \frac{2}{x^3} - 1\right) = x^3\left(\frac{3}{x^3} - 2\right) = 3 - 2x^3, \text{ and}$$

$$x^3\left(\left[\frac{1}{x^3}\right] + \left[\frac{2}{x^3}\right]\right) \leq x^3\left(\frac{1}{x^3} + \frac{2}{x^3}\right) = 3.$$

As $\lim_{x\to 0^+} 3 - 2x^3 = 3$, it follows from Squeeze Theorem that $\lim_{x\to 0^+} x^3\left(\left[\frac{1}{x^3}\right] + \left[\frac{2}{x^3}\right]\right) = 3$.

Question 4

(a) For each $x \in \mathbb{R}$, we see that $x^2+2-(4x-3) = x^2-4x+5 = (x-2)^2+1 > 0$, which implies that $x^2+2 \neq 4x-3$. Based on this, let us take any $a \in \mathbb{R}$, and any rational sequence (x_n) and irrational sequence (y_n) such that $\lim_{n\to\infty} x_n = a = \lim_{n\to\infty} y_n$. Then we have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n^2+2 = a^2+2$, and $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 4y_n - 3 = 4a-3$. As $a^2+2 \neq 4a-3$, we must have $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$. So f is not continuous at $x = a$. Since $a \in \mathbb{R}$ is arbitrary, we see that f is not continuous at any point of $\mathbb{R}$.

(b) Define $\varepsilon := g(a) - h(a) > 0$. As g and h are continuous at $x = a$, it follows that there exist $\delta_1, \delta_2 > 0$ such that for all $x \in \mathbb{R}$ such that $|x-a| < \delta_1$, we have $|g(x)-g(a)| < \frac{\varepsilon}{2}$, and for all $y \in \mathbb{R}$ such that $|y-a| < \delta_2$, we have $|h(y)-h(a)| < \frac{\varepsilon}{2}$. Let $\delta := \min\{\delta_1, \delta_2\} > 0$. It follows that for all $x \in (a-\delta, a+\delta)$, we have $|x-a| < \delta$, which implies that $|g(x)-g(a)| < \frac{\varepsilon}{2}$ and $|h(x)-h(a)| < \frac{\varepsilon}{2}$. This implies that $g(x)-g(a) > -\frac{\varepsilon}{2}$ and $h(a)-h(x) > -\frac{\varepsilon}{2}$, so we have

$$g(x)-h(x) = g(x)-g(a)+g(a)-h(a)+h(a)-h(x) > -\frac{\varepsilon}{2}+\varepsilon-\frac{\varepsilon}{2} = 0.$$

The desired follows.

Question 5

We shall prove that g is uniformly continuous on $[1,2)$. Note that for all $x \in \mathbb{R}$, $x \neq 2$, we have $|g(x)| = \left|(x-2)^2 \sin\left(\frac{x^2}{2-x}\right)\right| \leq (x-2)^2$. As $\lim_{x\to 2}(x-2)^2 = 0$, it follows from Squeeze Theorem that $\lim_{x\to 2}|g(x)| = 0$, and hence $\lim_{x\to 2} g(x) = 0$. Therefore, by defining $g(2) = 0$, it is easy to see that the newly defined function g is continuous on $\mathbb{R}$, and hence on $[1,2]$. Since $[1,2]$ is closed and bounded, we must have g to be uniformly continuous on $[1,2]$, and hence on $[1,2)$ as required.

Question 6

(a) Let $a := \lim_{n\to\infty} a_n$ and $b := \limsup b_n$, and $\varepsilon > 0$ be given. As (a_n) is convergent, it follows that there exists some $N_1 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{2}$ for all $n \geq N_1$. This implies that for all $n \geq N_1$, we have $a - \frac{\varepsilon}{2} < a_n < a + \frac{\varepsilon}{2}$. Furthermore, since $b = \limsup b_n$, it follows that there exists some $N_2 \in \mathbb{N}$ such that $b_n < b + \frac{\varepsilon}{2}$ for all $n \geq N_2$, and there are infinitely many n's such that $b_n > b - \frac{\varepsilon}{2}$. Let $N := \max\{N_1, N_2\}$. Then it is easy to see that $a_n + b_n < a + \frac{\varepsilon}{2} + b + \frac{\varepsilon}{2} = a+b+\varepsilon$ for all $n \geq N$, and there are infinitely many n's (greater than N_1) such that $a_n + b_n > a - \frac{\varepsilon}{2} + b - \frac{\varepsilon}{2} = a+b-\varepsilon$. Since $\varepsilon > 0$ is arbitrary, this shows that $\limsup(a_n+b_n) = a+b = \lim_{n\to\infty} a_n + \limsup b_n$ as desired.

(b) Let $\varepsilon > 0$ be given. As the series $\sum_{n=1}^{\infty} |x_{n+1} - x_n|$ is convergent, it follows that from the Cauchy criterion for series that there exists some $N \in \mathbb{N}$ greater than 1, such that for all $m, n \in \mathbb{N}$ such that $m > n \geq N$, we have $\sum_{k=n}^{m-1} |x_{k+1} - x_k| < \varepsilon$. This implies that for all $m, n \in \mathbb{N}$ such that $m > n \geq N$, we have

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots + x_{n+1} - x_n| \leq \sum_{k=n}^{m-1} |x_{k+1} - x_k| < \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, this implies that the sequence (x_n) is Cauchy, so it is convergent as desired.

Question 7

(a) (i) By setting $x = 0$, we have $f(0) + f(2(0)) = 2f(0) = 0$, so this implies that $f(0) = 0$.

(ii) Firstly, we see that for all $x \in \mathbb{R}$, we have $f\left(\frac{x}{4}\right) + f\left(\frac{x}{2}\right) = 0 = f\left(\frac{x}{2}\right) + f(x)$, which implies that $f(x) = f\left(\frac{x}{4}\right)$. This implies that $f\left(\frac{x}{4^n}\right) = f(x)$ for all $n \in \mathbb{N}$. As f is continuous at $x = 0$, and $\lim\limits_{n\to\infty} \frac{y}{4^n} = 0$ for all $y \in \mathbb{R}$, it follows from the sequential criterion for continuity that $f(x) = \lim\limits_{n\to\infty} f(x) = \lim\limits_{n\to\infty} f\left(\frac{x}{4^n}\right) = f(0) = 0$ for all $x \in \mathbb{R}$. The desired follows.

(b) Fix any $c \in (0, 1)$. We would like to show that h is continuous at $x = c$. To this end, let us take any $\varepsilon > 0$, and assume without loss of generality that $h(c) - \frac{\varepsilon}{2}, h(c) + \frac{\varepsilon}{2} \in (a, b)$. As the range of h is (a, b), it follows that there exist $c_1, c_2 \in (0, 1)$ such that $h(c_1) = h(c) - \frac{\varepsilon}{2}$, and $h(c_2) = h(c) + \frac{\varepsilon}{2}$. Furthermore, since h is increasing on $(0, 1)$, we must have $c_1 < c < c_2$.

Now, let us set $\delta := \min\{c - c_1, c_2 - c\} > 0$, and let us take any $x \in (0, 1)$ such that $0 < |x - c| < \delta$. If $x < c$, then we must have $x - c > -\delta \geq c_1 - c$, so this implies that $x > c_1$. Hence, we have $h(x) \geq h(c_1)$, which implies that $0 \leq h(c) - h(x) \leq h(c) - h(c_1) = \frac{\varepsilon}{2} < \varepsilon$, and hence $|h(x) - h(c)| < \varepsilon$. Similarly, when $x > c$, we also have $|h(x) - h(c)| < \varepsilon$. So this shows that h is continuous at $x = c$, and the desired follows.

Question 8

(a) Since $\lim\limits_{x\to\infty} f(x) = 1$, it follows that there exists some $N > 0$, such that for all $x > N$, we have $|f(x) - 1| < 1$. This implies that for all $x > N$, we have $|f(x)| \leq |f(x) - 1| + |1| < 2$. Furthermore, since f is continuous on $[0, \infty)$ (hence continuous on $[0, N]$), it is bounded on $[0, N]$, so there exists some $K > 0$, such that $|f(x)| \leq K$ for all $x \in [0, N]$. By setting $M = \max\{2, K\}$, it is easy to see that $|f(x)| \leq M$ for all $x \in [0, \infty)$ and this shows that f is bounded on $[0, \infty)$ as desired.

(b) Since g is uniformly continuous on $[1, \infty)$, it follows that there exists some $\delta > 0$, such that for all $x, y \in \mathbb{R}$ satisfying $|x - y| < \delta$, we have $|g(x) - g(y)| < 1$. Let us fix any $K \in (0, \delta)$ and $x \in [1, \infty)$. By defining $N := \left[\frac{x-1}{K}\right]$, where $[\ ]$ denotes the floor function, it follows that $N \leq \frac{x-1}{K} < N + 1$, or equivalently, $0 \leq x - (1 + NK) < K$. As $K < \delta$ by assumption, this implies that

$$
\begin{aligned}
& |g(x) - g(1)| \\
= \ & |g(x) - g(1 + NK) + g(1 + NK) - g(1 + (N - 1)K) + \cdots + g(1 + K) - g(1)| \\
\leq \ & |g(x) - g(1 + NK)| + |g(1 + NK) - g(1 + (N - 1)K)| + \cdots + |g(1 + K) - g(1)| \\
= \ & \underbrace{1 + 1 + \cdots + 1}_{(N+1) \text{ times}} \\
= \ & N + 1 \\
\leq \ & \frac{x - 1}{K} + x \\
< \ & \left(\frac{1}{K} + 1\right) x.
\end{aligned}
$$

As $|g(1)| \leq |g(1)|x$, this implies that

$$|g(x)| \leq |g(x) - g(1)| + |g(1)| < \left(\frac{1}{K} + 1\right) x + |g(1)|x = \left(\frac{1}{K} + 1 + |g(1)|\right) x.$$

Hence, the desired follows by setting $M = \frac{1}{K} + 1 + |g(1)|$.