NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
solutions prepared by Wei Boyan, Tay Jun Jie

MA2108 Mathematical Analysis I
AY 2009/2010 Sem 2

Question 1

(a) Let $P(n) : x_n \leqslant 2$. When $n = 1, x_1 = 1 \leqslant 2$, so $P(1)$ is true. Suppose $P(k)$ is true, thus $x_k \leqslant 2$. Then $x_{n+1} = \frac{1}{5}({x_k}^2 + 6) \leqslant \frac{1}{5}(4 + 6) = 2$. By Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(b) Claim: x_n is increasing. Proof:

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{5}(x_n^2 - 5x_n + 6) \\ &= \frac{1}{5}(x_n - 2)(x_n - 3) \end{aligned}$$

Since $x_n \leqslant 2$, we have $x_{n+1} > x_n$. Therefore x_n is increasing. By Monotone Convergence Theorem, x_n is converges. Let x be the limit of x_n.

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{5}(x_n^2 + 6) = \frac{1}{5}(x^2 + 6)$$
$$(x - 2)(x - 3) = 0$$

Thus $x = 2$ or $x = 3$. Since $x_n \leqslant 2$, we obtain $x = 2$. we conclude that x_n is convergent and its limit is 2.

Question 2

(a) (i) Firstly, observe that $\left(\frac{1}{2n+\sqrt{n}+1}\right)$ is a decreasing sequence of strictly positive terms with

$$\lim_{n\to\infty} \frac{1}{2n + \sqrt{n} + 1} = 0.$$

Therefore the series converges by Alternating Series Test.

(ii)

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left| \frac{n^2}{3^n} \left(1 + \frac{1}{3n}\right)^{6n^2} \right|^{\frac{1}{n}} \\ &= \frac{1}{3} \lim_{n\to\infty} n^{\frac{1}{n}} \lim_{n\to\infty} n^{\frac{1}{n}} \lim_{n\to\infty} \left(1 + \frac{1}{3n}\right)^{3n} \lim_{n\to\infty} \left(1 + \frac{1}{3n}\right)^{3n} \\ &= \frac{1}{3}e^2 > 1 \end{aligned}$$

Therefore the series diverges by Root Test.

(b) Observe that $\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}(\frac{1}{2n-1} - \frac{1}{2n+1})$. Thus

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} &= \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\end{aligned}$$

(c) Since the $a_n, b_n > 0$ for all $n \in \mathbb{N}$, $a_n b_n > 0$ for all $n \in \mathbb{N}$. In addition, $b_n \to 0$ as $\sum b_n$ converges.

$$\rho = \lim_{n\to\infty} \frac{a_n b_n}{a_n} = \lim_{n\to\infty} b_n = 0.$$

Since $\sum a_n$ converges, $\sum a_n b_n$ converges by Limit Comparison Test.

Question 3

(a) Given $\varepsilon > 0$, choose $\delta = \min\left(\frac{1}{6}, \frac{3}{20}\varepsilon\right)$. Suppose $0 < |x - 0| < \delta$,

$$\begin{aligned}\left|\frac{(2x+1)(x-2)}{3x+1} + 2\right| &= \left|\frac{2x^2+3x}{3x+1}\right| \\ &= \frac{|x|\,|2x+3|}{|3x+1|} \\ &< \frac{\delta\,|2x+3|}{|3x+1|}\end{aligned}$$

Since $0 < |x| < \frac{1}{6}$, we have $\frac{|2x+3|}{|3x+1|} \leqslant \frac{20}{3}$. Then,

$$\begin{aligned}\left|\frac{(2x+1)(x-2)}{3x+1} + 2\right| &< \frac{\delta\,|2x+3|}{|3x+1|} \\ &\leqslant \frac{20}{3}\delta \\ &= \varepsilon\end{aligned}$$

(b) (i) Let $f(x) = (x^2+x+1)\sin(\frac{3}{\lambda})$. Let $x_n = \frac{3}{(2n+1)\pi}$, $y_n = \frac{3}{2n\pi}$. Then $x_n \neq 0, x_n \to 0, y_n \neq 0$ and $y_n \to 0$.

$$\lim_{n\to\infty} f(y_n) = 0$$
$$\lim_{n\to\infty} f(x_n) \neq 0$$

Therefore $\lim_{n\to\infty} f(x)$ does not exist by the Divergent Criterion.

(ii)

$$\frac{6}{x} - 1 < \left[\frac{6}{x}\right] \leqslant \frac{6}{x}$$
$$3 - \frac{x}{2} < \frac{x}{2}\left[\frac{6}{x}\right] \leqslant 3 \qquad \because x > 0$$

Since $\lim_{x\to 0^+} 3 - \frac{x}{2} = \lim_{x\to 0^+} 3 = 3$, $\lim_{x\to 0^+} \frac{x}{2}\left[\frac{6}{x}\right] = 3$ by Squeeze Theorem.

Question 4

Let $\varepsilon > 0$ be given. Since $\lim_{x\to a} g(x) = 0$, $\exists\delta > 0$ such that

$$|g(x)| < \frac{\varepsilon}{M} \text{ whenever } 0 < |x-a| < \delta.$$

Let $\delta_1 = \min(\delta, h) > 0$. If $0 < |x-a| < \delta_1$, then

$$|f(x)g(x)| < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

Therefore $\lim_{x\to a} f(x)g(x) = 0$.

Question 5

Let $a \in \mathbb{R}$, take a rational sequence (x_n) and an irrational sequence (y_n) such that $x_n \to a$, and $y_n \to a$. Then

$$\begin{aligned} f(x_n) &= -x_n \to -a \\ f(y_n) &= 3y_n - 8 \to 3a - 8. \end{aligned}$$

If f is continuous at $x = a$, then

$$\begin{aligned} -a &= 3a - 8 \\ a &= 2. \end{aligned}$$

It follows that if $a \neq 2$, then f is not continuous at $x = a$. At $x = 2$, given $\varepsilon > 0$, we choose $\delta = \frac{\varepsilon}{3}$, then for $|x-2| < \delta$, we have

$$\begin{aligned} |-x+2| &= |x-2| < \delta < \varepsilon \\ |3x-8+2| &= 3|x-2| < 3\delta = \varepsilon \end{aligned}$$

Therefore, $|f(x) - f(2)| < \varepsilon$, so f is continuous at $x = 2$.

Question 6

Let $\varepsilon > 0$, since f and g are uniformly continuous on $\mathbb{R}$, there exists $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} x, y \in \mathbb{R}, |x-y| < \delta_1 &\Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{4} \\ x, y \in \mathbb{R}, |x-y| < \delta_2 &\Rightarrow |g(x) - g(y)| < \frac{\varepsilon}{4} \end{aligned}$$

Let $\delta = \min(\delta_1, \delta_2)$, then for $x, y \in \mathbb{R}$, with $|x-y| < \delta$, we have

$$\begin{aligned} |F(x) - F(y)| &= |f(x)g(x) - f(y)g(y)| \\ &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\ &\leqslant |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| \\ &= |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)| \\ &< |f(x)|\,\frac{\varepsilon}{4} + |g(y)|\,\frac{\varepsilon}{4} \\ &\leqslant \frac{1}{2}\frac{\varepsilon}{4} + 2\frac{\varepsilon}{4} \\ &= \frac{5\varepsilon}{8} < \varepsilon \end{aligned}$$

Thus, F is also uniformly continuous.

Question 7

(a) Let $m = \liminf(y_n), M = \limsup(x_n)$ and $\varepsilon > 0$ be given. Thus $\exists K \in \mathbb{N}$ such that for $n \geqslant K$,

$$m - \varepsilon < y_n \qquad \text{and} \qquad x_n < M + \varepsilon.$$

Hence $M - m > x_n - y_n$ for $n \geq K$. Let $x \in C(x_n - y_n)$, so there exist subsequence $(x_{n_k} - y_{n_k})$ such that $x_{n_k} - y_{n_k} \to x$. Thus $\exists K_1 \in \mathbb{N}$ such that $|x_{n_k} - y_{n_k} - x| < \varepsilon$ whenever $k \geq K_1$.

$$x_{n_k} - y_{n_k} - \varepsilon < x < x_{n_k} - y_{n_k} + \varepsilon \qquad \forall k \geq K_1$$

Now, $\exists K_2 \in \mathbb{N}$ such that $K_2 \geq K_1$ and $n_k \geq K$ whenever $k \geq K_2$. Hence,

$$x < x_{n_k} - y_{n_k} + \varepsilon < M - m + \varepsilon \qquad k \geq K_2$$

Therefore $x < M - m + \varepsilon$ for all $\varepsilon > 0$, that is, $x \leq M - m$. In conclusion, $M - m$ is an upper bound of $C(x_n - y_n)$ and $\limsup(x_n - y_n) = \sup C(x_n - y_n) \leq M - m$.

(b) (i) Since $b_n > 0\ \forall n \in \mathbb{N}$, $S_n > S_{n-1}$. Then $S_n^2 > S_n S_{n-1}$. Therefore,

$$\begin{aligned} \frac{b_n}{S_n^2} &< \frac{b_n}{S_n S_{n-1}} \\ &= \frac{S_n - S_{n-1}}{S_n S_{n-1}} \\ &= \frac{1}{S_{n-1}} - \frac{1}{S_n} \end{aligned}$$

(ii) Let $T_n = \sum_{k=1}^{n} \frac{b_k}{S_k^2}$, then

$$\begin{aligned} T_n &< \frac{b_1}{S_1^2} + \frac{1}{S_1} - \frac{1}{S_2} + \frac{1}{S_2} - \frac{1}{S_3} + \cdots + \frac{1}{S_{n-1}} - \frac{1}{S_n} \\ &= \frac{2}{S_1} - \frac{1}{S_n} \\ &< \frac{2}{S_1} \end{aligned}$$

So (T_n) is bounded, since $\frac{b_n}{S_n^2} > 0$, (T_n) is increasing. Therefore, $\sum_{n=1}^{\infty} \frac{b_n}{S_n^2}$ is convergent.

Question 8

(a) Let $\varepsilon > 0$ be given, $\exists \mu > 0$ such that $x > \mu$ implies

$$|f(x) - L| < \varepsilon$$

Since $\lim_{n\to\infty} x_n = \infty$, $\exists K \in \mathbb{N}$ such that $n \geq K$ implies $x_n > \mu$. Therefore, $n \geq K$ implies

$$|f(x_n) - L| < \varepsilon$$

Therefore, $\lim_{n\to\infty} f(x_n) = L$.

(b) Let $\varepsilon > 0$ be given. By assumption, $\exists M \in \mathbb{R}$ such that

$$|g(x) - g(x')| < \frac{\varepsilon}{3} \text{ whenever } x, x' > M.$$

Let (x_n) be a sequence in $\mathbb{R}$ such that $x_n \to \infty$. Now, $\exists N \in \mathbb{N}$ such that $x_n > M$ whenever $n \geq N$. Hence

$$|g(x_n) - g(x_m)| < \frac{\varepsilon}{3} \text{ whenever } n, m \geq N.$$

That is, $(g(x_n))$ is Cauchy and whence it converges to some $L \in \mathbb{R}$. Let (y_n) be another sequence in $\mathbb{R}$ such that $y_n \to \infty$. By the above argument, $g(y_n) \to L'$ for some $L' \in \mathbb{R}$. Now, $\exists K_1 \in \mathbb{N}$ such that

$$|g(x_n) - L| < \frac{\varepsilon}{3} \text{ whenever } n \geq K_1.$$

Similarly, $\exists K_2 \in \mathbb{N}$ such that

$$|g(y_m) - L'| < \frac{\varepsilon}{3} \text{ whenever } m \geq K_2.$$

Lastly, $\exists K_3 \in \mathbb{N}$ such that $x_n, y_m > M$ whenever $n, m \geq K_3$. Hence

$$|g(x_n) - g(y_m)| < \frac{\varepsilon}{3} \text{ whenever } n, m \geq K_3.$$

Let $K = \max\{K_1, K_2, K_3\}$. If $n, m \geq K$,

$$\begin{aligned} |L - L'| &\leq |L - g(x_n)| + |g(x_n) - g(y_m)| + |g(y_m) - L'| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

Thus $|L - L'| < \varepsilon$ for all $\varepsilon > 0$, that is, $L = L'$. In conclusion, for every sequence (z_n) in $\mathbb{R}$ such that $z_n \to \infty$, the sequence $(g(z_n))$ converges to L. Therefore $\lim_{x\to\infty} g(x) = L$.

Question 9

(a) Let $a = \min\{x_1, \ldots, x_n\}$ and $b = \max\{x_1, \ldots, x_n\}$. If $a = b$, then $\frac{1}{n}\sum_{k=1}^{n} f(x_k) = f(x_1)$ and we are done. Suppose $a < b$, hence $[a, b] \subset (0, 1)$ and f is continuous on $[a, b]$. By Extreme Value Theorem, $\exists c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.

$$f(c) \leq f(x_k) \leq f(d) \qquad \forall k = 1, \ldots, n$$

$$f(c) \leq \frac{1}{n}\sum_{k=1}^{n} f(x_k) \leq f(d)$$

If $f(c) = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$ or $f(d) = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$ then we are done. Suppose $f(c) < \frac{1}{n}\sum_{k=1}^{n} f(x_k) < f(d)$, applying Intermediate Value Theorem to f on $[c, d]$ or $[d, c]$, $\exists e \in (c, d)$ or (d, c) such that $f(e) = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$.

(b) Firstly, $\exists \delta > 0$ such that for all $x, y \in [0, \infty)$,

$$|g(x) - g(y)| < 1 \text{ whenever } |x - y| < \delta.$$

Now, since $\left|\frac{k\delta}{2} - \frac{(k-1)\delta}{2}\right| < \delta$ for all $k \in \mathbb{N}$, we have

$$\left|g\left(\frac{k\delta}{2}\right) - g\left(\frac{(k-1)\delta}{2}\right)\right| < 1 \qquad \forall k \in \mathbb{N}$$

$$\left|g\left(\frac{k\delta}{2}\right)\right| < 1 + \left|g\left(\frac{(k-1)\delta}{2}\right)\right| \qquad \forall k \in \mathbb{N}$$

$$\left|g\left(\frac{k\delta}{2}\right)\right| < k \qquad \forall k \in \mathbb{N}$$

Let $C = \frac{2}{\delta} > 0$. Now, $\bigcup_{k\in\mathbb{N}} \left[\frac{(k-1)\delta}{2}, \frac{k\delta}{2}\right)$ forms a partition for $[0, \infty)$. Let $x \in (0, \infty)$, then $x \in \left[\frac{(m-1)\delta}{2}, \frac{m\delta}{2}\right)$ for some $m \in \mathbb{N}$. Furthermore, $\left|x - \frac{(m-1)\delta}{2}\right| < \delta$. If $m = 1$, then

$$|g(x) - g(0)| < 1$$
$$|g(x)| < 1 < 1 + Cx$$

If $m > 1$, since $\frac{(m-1)\delta}{2} \leq x$, we have $\frac{1}{x} \leq \frac{2}{(m-1)\delta}$. Therefore,

$$\left|g(x) - g\left(\frac{(m-1)\delta}{2}\right)\right| < 1$$
$$|g(x)| < 1 + \left|g\left(\frac{(m-1)\delta}{2}\right)\right|$$
$$|g(x)| < 1 + (m-1) = 1 + \frac{1}{x}(m-1)x$$
$$|g(x)| < 1 + \frac{2}{(m-1)\delta}(m-1)x$$
$$|g(x)| < 1 + \frac{2}{\delta}x = 1 + Cx$$

In conclusion, $|g(x)| < 1 + Cx$ for all $x \in (0, \infty)$.