NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to AP Sun Rongfeng

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MA3110 Mathematical Analysis II

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Question 1

Firstly, we shall rewrite f^+ as $f^+(x) = \frac{1}{2}f(x) + \frac{1}{2}|f(x)|$. Suppose f^+ is differentiable at $x = 0$.

$$\Rightarrow |f(x)| \text{ is differentiable at } x = 0 \tag{1}$$

$$\Rightarrow \lim_{x\to 0} \frac{|f(x)|}{x} \text{ exist} \tag{2}$$

Now, since $\lim_{x\to 0^+} \frac{|f(x)|}{x} \geq 0$ and $\lim_{x\to 0^-} \frac{|f(x)|}{x} \leq 0$. We have

$$\lim_{x\to 0} \frac{|f(x)|}{x} = 0 \tag{3}$$

$$\Rightarrow \lim_{x\to 0} \left|\frac{f(x)}{x}\right| = \lim_{x\to 0} \left|\frac{|f(x)|}{x}\right| = |0| = 0 \tag{4}$$

$$\therefore f'(0) = \lim_{x\to 0} \frac{f(x)}{x} = 0 \tag{5}$$

Conversely, suppose $f'(0) = 0$.

$$\Rightarrow \lim_{x\to 0} \frac{f(x)}{x} = 0 \tag{6}$$

$$\Rightarrow \lim_{x\to 0} \left|\frac{|f(x)|}{x}\right| = \lim_{x\to 0} \left|\frac{f(x)}{x}\right| = |0| = 0 \tag{7}$$

$$\Rightarrow \lim_{x\to 0} \frac{|f(x)|}{x} = 0 \tag{8}$$

$$\Rightarrow |f(x)| \text{ is differentiable at } x = 0 \tag{9}$$

Therefore f^+ is differentiable at $x = 0$.

Question 2

Since f is twice-differentiable on $[0, 1]$, $\exists c \in (0, 1)$ such that

$$f(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{f''(c)}{2}\left(x - \frac{1}{2}\right)^2 \tag{10}$$

$$\geq f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) \tag{11}$$

$$\therefore \int_0^1 f(x)\,dx \geq \int_0^1 f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)\,dx \tag{12}$$

$$= f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left[\frac{x^2}{2} - \frac{x}{2}\right]_0^1 \tag{13}$$

$$= f\left(\frac{1}{2}\right) \tag{14}$$

Question 3

Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x f(x)\, dx$. Since f is continuous on $[a, b]$, F is differentiable on $[a, b]$.

$$\Rightarrow F'(x) = f(x)\ \forall x \in [a, b]$$

Now, F is continuous on $[a, b]$ and differentiable on (a, b). Hence, $\exists c \in (a, b)$ such that

$$F'(c) = \frac{F(b) - F(a)}{b - a} \tag{15}$$

$$\therefore f(c) = \frac{\int_a^b f(x)\, dx}{b - a} \tag{16}$$

Question 4

$$\lim_{n\to\infty} \int_0^1 \cos\frac{x}{n}\, dx = \lim_{n\to\infty} \left[n \sin\frac{x}{n} \right]_0^1 = \lim_{n\to\infty} n \sin\frac{1}{n} = 1$$

Alternatively, observe that each $\cos\frac{x}{n}$ is integrable on $[0, 1]$ and

$$\sup_{x\in[0,1]} \left| \cos\frac{x}{n} - 1 \right| = 1 - \cos\frac{1}{n} \to 0 \text{ as } n \to \infty$$

Hence $\cos\frac{x}{n} \to 1$ uniformly on $[0, 1]$.

$$\therefore \lim_{n\to\infty} \int_0^1 \cos\frac{x}{n}\, dx = \int_0^1 \lim_{n\to\infty} \cos\frac{x}{n}\, dx = \int_0^1 dx = 1$$

Question 5

(a)

$$a_{n+1}x^n = 2a_n x^n + 3a_{n-1}x^n \tag{17}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+1}x^n = 2\sum_{n=0}^{\infty} a_n x^n + 3\sum_{n=0}^{\infty} a_{n-1}x^n \tag{18}$$

$$\Rightarrow \frac{1}{x}(F(x) - x) = 2F(x) + 3xF(x) \tag{19}$$

$$\Rightarrow F(x) = \frac{x}{1 - 2x - 3x^2} \tag{20}$$

(b) Consider the partial fraction of $F(x)$ from Question 5a.

$$F(x) = \frac{1}{4}\left(\frac{1}{1 - 3x} - \frac{1}{1 + x} \right) \tag{21}$$

$$= \frac{1}{4}\left(\sum_{n=0}^{\infty} (3x)^n - \sum_{n=0}^{\infty} (-x)^n \right) \tag{22}$$

$$= \sum_{n=0}^{\infty} \frac{1}{4}(3^n - (-1)^n)x^n \tag{23}$$

Therefore $a_n = \frac{1}{4}(3^n - (-1)^n)$.

(c) $\sum_{n=0}^{\infty} a_n x^n$ converges on $\{x \in \mathbb{R} : |3x| < 1\} \cap \{x \in \mathbb{R} : |-x| < 1\} = \left\{x \in \mathbb{R} : |x| < \frac{1}{3}\right\}$.

Question 6

Firstly, observe that f is increasing as $f(x) = \lim_{n\to\infty} f_n(x) \leq \lim_{n\to\infty} f_n(y) = f(y)$ (by pointwise convergence of f_n to f) for all $0 \leq x \leq y \leq 1$. Now, let $\varepsilon > 0$ be given. Since f is continuous on $[0, 1]$, f is continuous on $[0, 1]$. Hence $\exists\delta > 0$ such that $\forall x, y \in [0, 1]$,

$$|f(x) - f(y)| < \frac{\varepsilon}{2} \text{ whenever } |x - y| < \delta$$

Let $N \in \mathbb{N}$ such that $N > \frac{1}{\delta}$. Hence $\left|\frac{k+1}{N} - \frac{k}{N}\right| = \left|\frac{1}{N}\right| < \delta$.

$$\Rightarrow f\left(\frac{k+1}{N}\right) - f\left(\frac{k}{N}\right) < \frac{\varepsilon}{2} \quad \forall k = 0, 1, \ldots, N-1$$

As $f_n \to f$ pointwise on $[0, 1]$, for each $k = 0, 1, \ldots, N$, $\exists M_k \in \mathbb{N}$ such that

$$\left|f_n\left(\frac{k}{N}\right) - f\left(\frac{k}{N}\right)\right| < \frac{\varepsilon}{2} \text{ whenever } n \geq M_k$$

Let $M = \max\{M_1, \ldots, M_k\}$. Let $x \in [0, 1]$, then $x \in \left[\frac{m}{N}, \frac{m+1}{N}\right]$ for some $m = 0, 1, \ldots, N-1$. For $n \geq M$ (note that M is independent of x),

$$\Rightarrow f_n(x) - f(x) \leq f_n\left(\frac{m+1}{N}\right) - f\left(\frac{m}{N}\right) < f\left(\frac{m+1}{N}\right) + \frac{\varepsilon}{2} - f\left(\frac{m}{N}\right) < \varepsilon \quad \text{and} \tag{24}$$

$$f_n(x) - f(x) \geq f_n\left(\frac{m}{N}\right) - f\left(\frac{m+1}{N}\right) > f\left(\frac{m}{N}\right) - \frac{\varepsilon}{2} - f\left(\frac{m+1}{N}\right) > -\varepsilon \tag{25}$$

$$\therefore |f_n(x) - f(x)| < \varepsilon \tag{26}$$