NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Ho Chin Fung

MA1101R Linear Algebra I
AY 2006/2007 Sem 1

SECTION A

Question 1

(a) Using Gauss-Jordan Elimination, we have

$$\begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 1 & -1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 \end{pmatrix} \xrightarrow[R_4+R_1]{R_2-R_1} \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$
$$\xrightarrow[R_4-R_2]{R_3-R_2} \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1-R_3} \begin{pmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b) Let $\boldsymbol{A}$ be the matrix in part (a) and $\boldsymbol{R}$ be its reduced row-echelon form.
Observe that each $\boldsymbol{u}_i$ corresponds to the i^{th} column in $\boldsymbol{A}$. So V is the column space of $\boldsymbol{A}$. The 1^{st}, 3^{rd} and 5^{th} column in $\boldsymbol{R}$ are columns with leading entry. Therefore, $\{\boldsymbol{u}_1, \boldsymbol{u}_3, \boldsymbol{u}_5\}$ is a basis for V and its dimension is 3.

Question 2

(a) We compute

$$\begin{aligned} \det(0\boldsymbol{I} - \boldsymbol{C}) &= \begin{vmatrix} -2 & -2 & 0 \\ 1 & 1 & 0 \\ -2 & -2 & -3 \end{vmatrix} = 0. \\ \det(1\boldsymbol{I} - \boldsymbol{C}) &= \begin{vmatrix} -1 & -2 & 0 \\ 1 & 2 & 0 \\ -2 & -2 & -2 \end{vmatrix} = 0. \\ \det(3\boldsymbol{I} - \boldsymbol{C}) &= \begin{vmatrix} 1 & -2 & 0 \\ 1 & 4 & 0 \\ -2 & -2 & 0 \end{vmatrix} = 0. \end{aligned}$$

Hence, we verified that $\boldsymbol{C}$ has eigenvalues 0, 1 and 3.

(b) Note that

$$\boldsymbol{C} = \boldsymbol{P} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \boldsymbol{P}^{-1} \iff \boldsymbol{P}^{-1}\boldsymbol{C}\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

So, $\boldsymbol{P}$ is a matrix that diagonalizes $\boldsymbol{C}$.

To find $\boldsymbol{P}$, we first find the eigenvectors of $\boldsymbol{C}$ corresponding to eigenvalues 0, 1 and 3.
For $\lambda = 0$, $(0\boldsymbol{I} - \boldsymbol{C})\boldsymbol{x} = \boldsymbol{0}$.

$$\left(\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -2 & -3 & 0 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 + \frac{1}{2}R_1} \left(\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right) \xrightarrow[(-\frac{1}{3})R_3]{(-\frac{1}{2})R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Let $x_2 = s$. Then

$$\begin{aligned} x_1 &= -s. \\ x_3 &= 0. \\ \boldsymbol{x} &= \begin{pmatrix} -s \\ s \\ 0 \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}. \end{aligned}$$

Therefore, an eigenvector corresponding to $\lambda = 0$ is $(-1, 1, 0)$.
For $\lambda = 1$, $(\boldsymbol{I} - \boldsymbol{C})\boldsymbol{x} = \boldsymbol{0}$.

$$\left(\begin{array}{ccc|c} -1 & -2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -2 & -2 & -2 & 0 \end{array}\right) \xrightarrow[R_3 - 2R_1]{R_2 + R_1} \left(\begin{array}{ccc|c} -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right) \xrightarrow{R_1 + R_3} \left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right)$$
$$\xrightarrow[(\frac{1}{2})R_3]{(-1)R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Let $x_3 = s$. Then

$$\begin{aligned} x_1 &= -2s. \\ x_2 &= s. \\ \boldsymbol{x} &= \begin{pmatrix} -2s \\ s \\ s \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

So, an eigenvector corresponding to $\lambda = 1$ is $(-2, 1, 1)$.
For $\lambda = 3$, $(3\boldsymbol{I} - \boldsymbol{C})\boldsymbol{x} = \boldsymbol{0}$.

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ -2 & -2 & 0 & 0 \end{array}\right) \xrightarrow[R_3 + 2R_1]{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & -6 & 0 & 0 \end{array}\right) \xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$\xrightarrow{(\frac{1}{6})R_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 + 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Let $x_3 = s$. Then

$$\begin{aligned} x_1 &= 0. \\ x_2 &= 0. \\ \boldsymbol{x} &= \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} = s \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

So, an eigenvector corresponding to $\lambda = 3$ is $(0, 0, 1)$.

The matrix $\begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ is a solution for $\boldsymbol{P}$.

Question 3

(a) We have

$$\begin{aligned} \boldsymbol{E}_1\boldsymbol{E}_2(\boldsymbol{A}|\boldsymbol{a}) &= \boldsymbol{E}_3\boldsymbol{E}_4(\boldsymbol{B}|\boldsymbol{b}) \\ (\boldsymbol{A}|\boldsymbol{a}) &= \boldsymbol{E}_2^{-1}\boldsymbol{E}_1^{-1}\boldsymbol{E}_3\boldsymbol{E}_4(\boldsymbol{B}|\boldsymbol{b}). \end{aligned}$$

$\boldsymbol{A}$ is obtained from $\boldsymbol{B}$ by the following elementary row operations:
Firstly, $R_1 + R_3$, then, $R_2 \leftrightarrow R_3$, then, $(\frac{1}{3})R_3$, finally, $R_3 - 3R_2$.

(b) $(\boldsymbol{A}|\boldsymbol{a})$ is obtained form $(\boldsymbol{B}|\boldsymbol{b})$ by elementary row operations. $(\boldsymbol{A}|\boldsymbol{a})$ and $(\boldsymbol{B}|\boldsymbol{b})$ are row equivalent and therefore have the same solution set.
The general solution of $\boldsymbol{Ax} = \boldsymbol{a}$ is therefore

$$\boldsymbol{x} = \begin{pmatrix} -t+1 \\ t-1 \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{where t is arbitrary.}$$

The general solution of $\boldsymbol{Ax} = \boldsymbol{0}$ is

$$\boldsymbol{x} = t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad \text{where t is arbitrary.}$$

Question 4

(a) Consider the case where $p = 1$. Then $V = \text{span}\{(1,1,1)\}$.
Applying Gram-Schmidt process to $\{(1,1,1)\}$, we have

$$\boldsymbol{v}_1 = (1,1,1).$$

Then,

$$\begin{aligned} & \left\{\frac{1}{\|\boldsymbol{v}_1\|}\boldsymbol{v}_1\right\} \\ = & \left\{\frac{1}{\sqrt{3}}(1,1,1)\right\} \\ = & \left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\} \quad \text{is an orthonormal basis for } V. \end{aligned}$$

Now, consider the case where $p \neq 1$.
Then $V = \text{span}\{(1,1,1),(1,p,p)\} =$ the row space of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & p & p \end{pmatrix}$.
Performing row operations, we have

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & p & p \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & p-1 & p-1 \end{pmatrix} \xrightarrow{(\frac{1}{p-1})R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Thus, $V = \text{span}\{(1,0,0),(0,1,1)\}$. Observe that $\{(1,0,0),(0,1,1)\}$ is orthogonal.
Thus

$$\begin{aligned} & \left\{\frac{1}{\|(1,0,0)\|}(1,0,0), \frac{1}{\|(0,1,1)\|}(0,1,1)\right\} \\ = & \left\{1(1,0,0), \frac{1}{\sqrt{2}}(0,1,1)\right\} \\ = & \left\{(1,0,0), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\} \quad \text{is an orthonormal basis for } V. \end{aligned}$$

(b) For the case where $p = 1$,

$$\begin{aligned}\mathbf{proj}_V((5,3,1)) &= (5,3,1)\cdot\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\\ &= \frac{9}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\\ &= (3,3,3).\end{aligned}$$

For the case where $p \neq 1$,

$$\begin{aligned}\mathbf{proj}_V((5,3,1)) &= (5,3,1)\cdot(1,0,0)(1,0,0) + (5,3,1)\cdot\left(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\left(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\\ &= 5(1,0,0) + \frac{4}{\sqrt{2}}\left(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\\ &= (5,0,0)+(0,2,2)\\ &= (5,2,2).\end{aligned}$$

Question 5

(a) We have

$$\begin{aligned}\boldsymbol{H}^2 &= \boldsymbol{H}\\ \boldsymbol{H}\boldsymbol{H} &= \boldsymbol{H}\\ \det(\boldsymbol{H})\det(\boldsymbol{H}) &= \det(\boldsymbol{H})\\ (\det(\boldsymbol{H}))^2 - \det(\boldsymbol{H}) &= 0\\ \det(\boldsymbol{H})(\det(\boldsymbol{H})-1) &= 0.\end{aligned}$$

Therefore, $\det(\boldsymbol{H}) = 0$ or 1.
Let $\boldsymbol{H} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Then $\boldsymbol{H}^2 = \boldsymbol{H}$ and $\det(\boldsymbol{H}) = 0$.
Let $\boldsymbol{H} = \boldsymbol{I}$. Then $\boldsymbol{H}^2 = \boldsymbol{H}$ and $\det(\boldsymbol{H}) = 1$.

(b) (i) False.
When $\boldsymbol{H} = \boldsymbol{I}$, $\boldsymbol{I} - \boldsymbol{H} = \boldsymbol{0}$, which is not invertible.

(ii) True.
Otherwise, suppose $(\boldsymbol{I} + \boldsymbol{H})$ is not invertible.
Then $\exists \boldsymbol{x} \neq \boldsymbol{0}$ s.t. $(\boldsymbol{H} + \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$.

$$\begin{aligned}(\boldsymbol{H}+\boldsymbol{I})\boldsymbol{x} &= \boldsymbol{0}\\ \boldsymbol{H}\boldsymbol{x}+\boldsymbol{x} &= \boldsymbol{0}\\ \boldsymbol{H}(\boldsymbol{H}\boldsymbol{x}+\boldsymbol{x}) &= \boldsymbol{H}\boldsymbol{0}\\ \boldsymbol{H}^2\boldsymbol{x}+\boldsymbol{H}\boldsymbol{x} &= \boldsymbol{0}\\ \boldsymbol{H}\boldsymbol{x}+\boldsymbol{H}\boldsymbol{x} &= \boldsymbol{0}\\ \boldsymbol{H}\boldsymbol{x} &= \boldsymbol{0}.\end{aligned}$$

Sub $\boldsymbol{H}\boldsymbol{x} = \boldsymbol{0}$ into $\boldsymbol{H}\boldsymbol{x} + \boldsymbol{x} = \boldsymbol{0}$, we have

$$\begin{aligned}\boldsymbol{0}+\boldsymbol{x} &= \boldsymbol{0}\\ \boldsymbol{x} &= \boldsymbol{0}.\end{aligned}$$

This contradicts the condition that $\boldsymbol{x} \neq \boldsymbol{0}$. Therefore, $(\boldsymbol{I} + \boldsymbol{H})$ is always invertible.

Alt: Consider $\frac{1}{2}(2\boldsymbol{I}-\boldsymbol{H})$. We have

$$\begin{aligned}\frac{1}{2}(2\boldsymbol{I}-\boldsymbol{H})(\boldsymbol{I}+\boldsymbol{H}) &= \frac{1}{2}(2\boldsymbol{I}+2\boldsymbol{H}-\boldsymbol{H}-\boldsymbol{H}^2)\\ &= \frac{1}{2}(2\boldsymbol{I}+2\boldsymbol{H}-\boldsymbol{H}-\boldsymbol{H})\\ &= \boldsymbol{I}.\end{aligned}$$

Therefore, $(\boldsymbol{I}+\boldsymbol{H})$ is always invertible.

Question 6

(a) $\boldsymbol{M}$, the standard matrix for T, is given by $([T(\boldsymbol{e}_1)] \quad [T(\boldsymbol{e}_2)] \quad \dots \quad [T(\boldsymbol{e}_n)])$.
Consider

$$\begin{aligned}\boldsymbol{M}^T\boldsymbol{M} &= \begin{pmatrix}[T(\boldsymbol{e}_1)]^T\\ [T(\boldsymbol{e}_2)]^T\\ \vdots\\ [T(\boldsymbol{e}_n)]^T\end{pmatrix}([T(\boldsymbol{e}_1)] \quad [T(\boldsymbol{e}_2)] \quad \dots \quad [T(\boldsymbol{e}_n)])\\ &= (m_{ij}) \quad \text{, where } m_{ij}=T(\boldsymbol{e}_i)\cdot T(\boldsymbol{e}_j).\end{aligned}$$

Since $\{T(\boldsymbol{e}_1), T(\boldsymbol{e}_2), \dots, T(\boldsymbol{e}_n)\}$ is an orthonormal basis for $\mathbb{R}^n$, we have

$$T(\boldsymbol{e}_i)\cdot T(\boldsymbol{e}_j) = \begin{cases}1 & \text{, where } i=j,\\ 0 & \text{, where } i\neq j.\end{cases}$$

Then

$$\begin{aligned}(m_{ij}) &= \begin{cases}1 & \text{, where } i=j,\\ 0 & \text{, where } i\neq j.\end{cases}\\ (m_{ij}) &= \boldsymbol{I}\\ \boldsymbol{M}^T\boldsymbol{M} &= \boldsymbol{I}.\end{aligned}$$

Therefore, $\boldsymbol{M}$ is orthogonal.

(b) If $n=2$, then T is a linear transformation in $\mathbb{R}^2$ that preserves length and right angles. T can then be reflection or rotation.

SECTION B

Question 7

(a) Consider $\mathbf{0}\in\mathbb{R}^n$. We have

$$\begin{aligned}\mathbf{0}^T\boldsymbol{Q}\mathbf{0} &= \mathbf{0}^T\mathbf{0}\\ &= 0.\end{aligned}$$

Thus, $\mathbf{0}\in W$. Therefore, W is non-empty.

(b) Let $\boldsymbol{u}\in W$. Then $\boldsymbol{u}^T\boldsymbol{Q}\boldsymbol{u}=\mathbf{0}$.
Consider $(c\boldsymbol{u})\in\mathbb{R}^n$. We have

$$\begin{aligned}(c\boldsymbol{u})^T\boldsymbol{Q}(c\boldsymbol{u}) &= c^2(\boldsymbol{u}^T\boldsymbol{Q}\boldsymbol{u})\\ &= c^2(0)\\ &= 0.\end{aligned}$$

Therefore, $c\boldsymbol{u}\in W$.

(c) If $\boldsymbol{Q}$ is the $n \times n$ zero matrix, then $\forall \boldsymbol{u} \in \mathbb{R}^n$, we have

$$\begin{aligned} \boldsymbol{u}^T\boldsymbol{Q}\boldsymbol{u} &= \boldsymbol{u}^T\boldsymbol{0}\boldsymbol{u} \\ &= \boldsymbol{u}^T\boldsymbol{0} \\ &= 0. \end{aligned}$$

Thus, $\forall \boldsymbol{u} \in \mathbb{R}^n, \boldsymbol{u} \in W$. Therefore, $W = \mathbb{R}^n$.
If $\boldsymbol{Q}$ is the identity matrix, then $\forall \boldsymbol{u} \in W$, we have

$$\begin{aligned} \boldsymbol{u}^T\boldsymbol{Q}\boldsymbol{u} &= 0 \\ \boldsymbol{u}^T\boldsymbol{u} &= 0 \\ \|\boldsymbol{u}\|^2 &= 0 \\ \boldsymbol{u} &= \boldsymbol{0}. \end{aligned}$$

Therefore, $W = \{\boldsymbol{0}\}$.

(d) Let $\boldsymbol{u}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Then

$$\begin{aligned} \boldsymbol{u}_1^T\boldsymbol{Q}\boldsymbol{u}_1 &= \begin{pmatrix} 3 & 2 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -9 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \end{pmatrix}\begin{pmatrix} 12 \\ -18 \end{pmatrix} = 36 - 36 = 0. \\ \boldsymbol{u}_2^T\boldsymbol{Q}\boldsymbol{u}_2 &= \begin{pmatrix} 3 & -2 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -9 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 & -2 \end{pmatrix}\begin{pmatrix} 12 \\ 18 \end{pmatrix} = 36 - 36 = 0. \end{aligned}$$

Therefore $\boldsymbol{u}_1, \boldsymbol{u}_2 \in W$.
Next,

$$(\boldsymbol{u}_1 + \boldsymbol{u}_2)^T\boldsymbol{Q}(\boldsymbol{u}_1 + \boldsymbol{u}_2) = \begin{pmatrix} 6 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -9 \end{pmatrix}\begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 & 0 \end{pmatrix}\begin{pmatrix} 24 \\ 0 \end{pmatrix} = 144 \neq 0.$$

Therefore $(\boldsymbol{u}_1 + \boldsymbol{u}_2) \notin W$.
Since $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{W}$ and $(\boldsymbol{u}_1 + \boldsymbol{u}_2) \notin \boldsymbol{W}$, W is not a subspace of $\mathbb{R}^2$.

(e) Since $\boldsymbol{Q}$ is symmetric, we can find an orthogonal matrix $\boldsymbol{P}$ that orthogonally diagonalizes $\boldsymbol{Q}$.
The expression $\boldsymbol{u}^T\boldsymbol{Q}\boldsymbol{u}$ is called quadratic form in n variables.
Let $\boldsymbol{x} = \boldsymbol{P}^T\boldsymbol{u}$ and denote by $(x_1, x_2, \ldots, x_n)$. Then

$$\begin{aligned} \boldsymbol{u}^T\boldsymbol{Q}\boldsymbol{u} &= \boldsymbol{x}^T\boldsymbol{P}^T\boldsymbol{Q}\boldsymbol{P}\boldsymbol{x} \\ &= x_1^2\lambda_1 + x_2^2\lambda_2 + \cdots + x_n^2\lambda_n \quad \text{, where} \lambda_1, \lambda_2, \ldots, \lambda_n \text{ are the eigenvalues of } \boldsymbol{Q}. \end{aligned}$$

Suppose not all nonzero λ_i's are of the same sign. Then $\exists \lambda_k > 0$ and $\exists \lambda_l < 0$.

Let $\boldsymbol{u}_1 = \boldsymbol{P}\boldsymbol{a}$, where $a_i = \begin{cases} \sqrt{-\lambda_l} & , i = k \\ \sqrt{\lambda_k} & , i = l \\ 0 & , i \neq k, l. \end{cases}$

Let $\boldsymbol{u}_2 = \boldsymbol{P}\boldsymbol{b}$, where $b_i = \begin{cases} \sqrt{-\lambda_l} & , i = k \\ -\sqrt{\lambda_k} & , i = l \\ 0 & , i \neq k, l. \end{cases}$

Then

$$\begin{aligned} \boldsymbol{u}_1^T\boldsymbol{Q}\boldsymbol{u}_1 &= a_1^2\lambda_1 + \cdots + a_k^2\lambda_k + \cdots + a_l^2\lambda_l + \cdots + a_n^2\lambda_n \\ &= 0 + \cdots + (\sqrt{-\lambda_l})^2\lambda_k + \cdots + (\sqrt{\lambda_k})^2\lambda_l + \cdots + 0 \\ &= -\lambda_l\lambda_k + \lambda_k\lambda_l = 0. \quad \therefore \boldsymbol{u}_1 \in W. \end{aligned}$$

$$\begin{aligned}
\boldsymbol{u}_2^T\boldsymbol{Q}\boldsymbol{u}_2 &= b_1^2\lambda_1+\cdots+b_k^2\lambda_k+\cdots+b_l^2\lambda_l+\cdots+b_n^2\lambda_n \\
&= 0+\cdots+(\sqrt{-\lambda_l})^2\lambda_k+\cdots+(-\sqrt{\lambda_k})^2\lambda_l+\cdots+0 \\
&= -\lambda_l\lambda_k+\lambda_k\lambda_l=0. \quad \therefore \boldsymbol{u}_2\in W.
\end{aligned}$$

$$\begin{aligned}
(\boldsymbol{u}_1+\boldsymbol{u}_2)^T\boldsymbol{Q}(\boldsymbol{u}_1+\boldsymbol{u}_2) &= (a_1+b_1)^2\lambda_1+\cdots+(a_k+b_k)^2\lambda_k+\cdots+ \\
&\quad (a_l+b_l)^2\lambda_l+\cdots+(a_n+b_n)^2\lambda_n \\
&= 0+\cdots+(2\sqrt{-\lambda_l})^2\lambda_k+\cdots+(0)^2\lambda_l+\cdots+0 \\
&= -4\lambda_l\lambda_k\neq 0. \quad \therefore (\boldsymbol{u}_1+\boldsymbol{u}_2)\notin W.
\end{aligned}$$

W is not closed under addition and is therefore not a subspace of $\mathbb{R}^n$.
Now consider the case where all nonzero λ_i's are of the same sign.
WLOG, let all nonzero λ_i's be positive.
Let $\boldsymbol{u}\in W$. Let $\boldsymbol{x}=\boldsymbol{P}^T\boldsymbol{u}$ and denote by $(x_1,x_2,\ldots,x_n)$. Then

$$\begin{aligned}
\boldsymbol{u}^T\boldsymbol{Q}\boldsymbol{u} &= 0 \\
x_1^2\lambda_1+x_2^2\lambda_2+\cdots+x_n^2\lambda_n &= 0 \\
\Leftrightarrow \forall i,\quad x_i^2\lambda_i &= 0 \\
x_i &= \begin{cases} 0 & ,\ \text{if } \lambda_i>0, \\ c_i\in\mathbb{R} & ,\ \text{if } \lambda_i=0. \end{cases}
\end{aligned}$$

Similarly, let $\boldsymbol{v}\in W$. Let $\boldsymbol{y}=\boldsymbol{P}^T\boldsymbol{v}$ and denote by $(y_1,y_2,\ldots,y_n)$. We have

$$y_i = \begin{cases} 0 & ,\ \text{if } \lambda_i>0, \\ d_i\in\mathbb{R} & ,\ \text{if } \lambda_i=0. \end{cases}$$

Let $\boldsymbol{z}=\boldsymbol{x}+\boldsymbol{y}=(z_1,z_2,\ldots,z_n)$. Then

$$\begin{aligned}
\boldsymbol{z} &= \boldsymbol{P}^T\boldsymbol{u}+\boldsymbol{P}^T\boldsymbol{v} \\
&= \boldsymbol{P}^T(\boldsymbol{u}+\boldsymbol{v}).
\end{aligned}$$

Also,

$$\begin{aligned}
z_i &= x_i+y_i \\
&= \begin{cases} 0 & ,\ \text{if } \lambda_i>0, \\ c_i+d_i\in\mathbb{R} & ,\ \text{if } \lambda_i=0. \end{cases} \\
\Leftrightarrow \forall i,\quad z_i^2\lambda_i &= 0 \\
z_1^2\lambda_1+z_2^2\lambda_2+\cdots+z_n^2\lambda_n &= 0 \\
\boldsymbol{z}^T\boldsymbol{P}^T\boldsymbol{Q}\boldsymbol{P}\boldsymbol{z} &= 0 \\
(\boldsymbol{u}+\boldsymbol{v})^T\boldsymbol{Q}(\boldsymbol{u}+\boldsymbol{v}) &= 0.
\end{aligned}$$

Thus, W is closed under addition. Together with result from part (a)(non-empty) and (b)(closed under scalar multiplication), we have W is a subspace of $\mathbb{R}^n$.
Therefore, W is a subspace of $\mathbb{R}^n$ when all nonzero eigenvalues of $\boldsymbol{Q}$ are of the same sign.

Question 8

(a) (i) Let $\boldsymbol{x}$ be in the nullspace of $\boldsymbol{A}$. Then

$$\begin{aligned}
\boldsymbol{A}\boldsymbol{x} &= \boldsymbol{0} \\
\boldsymbol{A}\boldsymbol{A}\boldsymbol{x} &= \boldsymbol{A}\boldsymbol{0} \\
\boldsymbol{A}^2\boldsymbol{x} &= \boldsymbol{0}
\end{aligned}$$

Then $\boldsymbol{x}$ is in the nullspace of $\boldsymbol{A}^2$. Therefore, the nullspace of $\boldsymbol{A}$ is a subspace of the nullspace of $\boldsymbol{A}^2$.

By the Dimension Theorem of Matrices, we have

$$\begin{aligned}\text{rank}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{A}) &= n. \\ \text{rank}(\boldsymbol{A}^2) + \text{nullity}(\boldsymbol{A}^2) &= n.\end{aligned}$$

Taking the difference of the two equations, we have

$$\begin{aligned}\text{rank}(\boldsymbol{A}) - \text{rank}(\boldsymbol{A}^2) + \text{nullity}(\boldsymbol{A}) - \text{nullity}(\boldsymbol{A}^2) &= n - n \\ 0 + \text{nullity}(\boldsymbol{A}) - \text{nullity}(\boldsymbol{A}^2) &= 0 \\ \text{nullity}(\boldsymbol{A}) &= \text{nullity}(\boldsymbol{A}^2).\end{aligned}$$

Together with $\text{nullity}(\boldsymbol{A}) = \text{nullity}(\boldsymbol{A}^2)$, we therefore have the nullspace of $\boldsymbol{A}$ is equal to the nullspace of $\boldsymbol{A}^2$.

(ii) Let $\boldsymbol{x} \in$ (the nullspace of $\boldsymbol{A}$) $\cap$ (the column space of $\boldsymbol{A}$).
Then $\boldsymbol{Ax} = \boldsymbol{0}$ and $\exists \boldsymbol{y}$ s.t.$\boldsymbol{Ay} = \boldsymbol{x}$.

$$\begin{aligned}\boldsymbol{Ay} &= \boldsymbol{x} \\ \boldsymbol{AAy} &= \boldsymbol{Ax} \\ \boldsymbol{A}^2\boldsymbol{y} &= \boldsymbol{0} \\ \Rightarrow \quad \boldsymbol{y} &\in \text{the nullspace of } \boldsymbol{A}^2 \\ \Rightarrow \quad \boldsymbol{y} &\in \text{the nullspace of } \boldsymbol{A} \\ \Rightarrow \quad \boldsymbol{Ay} &= \boldsymbol{0} \\ \boldsymbol{x} &= \boldsymbol{0}.\end{aligned}$$

Therefore, (the nullspace of $\boldsymbol{A}$) $\cap$ (the column space of $\boldsymbol{A}$) $= \{\boldsymbol{0}\}$.

(b) Let $\boldsymbol{z} \in$ the column space of $\boldsymbol{Z}$. Then $\exists \boldsymbol{a} \in \mathbb{R}^n$ s.t. $\boldsymbol{Za} = \boldsymbol{z}$.

$$\begin{aligned}\boldsymbol{Za} &= \boldsymbol{z} \\ \boldsymbol{XYa} &= \boldsymbol{z} \\ \boldsymbol{X}(\boldsymbol{Ya}) &= \boldsymbol{z}.\end{aligned}$$

Then $\boldsymbol{z} \in$ the column space of $\boldsymbol{X}$.
Therefore, the column space of $\boldsymbol{Z}$ is a subset of the column space of $\boldsymbol{X}$.

(c) (i) Let $\boldsymbol{x} \in$ the nullspace of $\boldsymbol{B}$. Then

$$\begin{aligned}\boldsymbol{Bx} &= \boldsymbol{0} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\boldsymbol{x} &= \boldsymbol{0}\end{aligned}$$

Let $x_3 = s$. Then

$$\begin{aligned}x_1 &= 0. \\ x_2 &= 0. \\ \boldsymbol{x} &= \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} = s\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{where s is arbitrary.}\end{aligned}$$

Therefore, the nullspace of $\boldsymbol{B}$ is span $\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$.

(ii) Suppose there exists a 3×3 matrix $\boldsymbol{C}$ such that $\boldsymbol{C}^2 = \boldsymbol{B}$.
Using results from part (b), consider $\boldsymbol{CC} = \boldsymbol{B}$, we have the column space of $\boldsymbol{B}$ is a subset of the column space of $\boldsymbol{C}$. So, $\text{rank}(\boldsymbol{C}) \geq \text{rank}(\boldsymbol{B}) = 2$.
Claim: $\text{rank}(\boldsymbol{C}) \neq 3$.
Suppose not, $\text{rank}(\boldsymbol{C}) = 3$. $\boldsymbol{C}$ has full rank $\Rightarrow$ $\boldsymbol{C}$ is invertible $\Rightarrow$ $\boldsymbol{C}^2$ is invertible $\Rightarrow$ $\text{Rank}(\boldsymbol{C}^2) = 3$, a contradiction as $\text{rank}(\boldsymbol{C}^2) = \text{rank}(\boldsymbol{B}) = 2$. Thus, $\text{rank}(\boldsymbol{C}) \neq 3$ and therefore $\text{rank}(\boldsymbol{C}) = 2$.
Now, $\text{rank}(\boldsymbol{C}) = 2 = \text{rank}(\boldsymbol{C}^2)$.
Using result from part(a)(i), we have

$$\begin{aligned} \text{the nullspace of } \boldsymbol{C} &= \text{the nullspace of } \boldsymbol{C}^2 \\ &= \text{the nullspace of } \boldsymbol{B} \\ &= \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

Using result from part (b), consider $\boldsymbol{CC} = \boldsymbol{C}^2$, we have

$$\begin{aligned} \text{the column space of } \boldsymbol{C}^2 &\subseteq \text{the column space of } \boldsymbol{C} \\ \text{the column space of } \boldsymbol{B} &\subseteq \text{the column space of } \boldsymbol{C} \\ \text{the column space of } \boldsymbol{C} &\supseteq \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

So,

$$\begin{aligned} (\text{the nullspace of } \boldsymbol{C}) \cap (\text{the column space of } \boldsymbol{C}) &= \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \\ &\neq \{\mathbf{0}\}. \end{aligned}$$

This contradicts to the result of part (a)(ii).
Therefore, there does not exist any 3×3 matrix $\boldsymbol{C}$ such that $\boldsymbol{C}^2 = \boldsymbol{B}$.