NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Teo Wei Hao

MA1104 Multivariable Calculus
AY 2006/2007 Sem 2

Question 1

(a) Let $f(x, y, z) = z^3 + xyz - 2$. This give us $f_x(x, y, z) = yz$, $f_y(x, y, z) = xz$, $f_z(x, y, z) = 3z^2 + xy$. Thus equation of tangent plane to surface $f(x, y, z) = 0$ at $(1, 1, 1)$ is

$$\begin{aligned} f_x(1,1,1)(x-1) + f_y(1,1,1)(y-1) + f_z(1,1,1)(z-1) &= 0 \\ (1)(x-1) + (1)(y-1) + 4(z-1) &= 0 \\ x + y + 4z &= 6. \end{aligned}$$

Hence, we approximate $(1.01, 0.97, c)$ to be a point on the tangent plane of the curve at $(1, 1, 1)$. This give us $c \approx \frac{1}{4}(6 - 1.01 - 0.97) = 1.005$.

(b) Let $f(x, y, z) = x+y+z$ and $g(x, y, z) = xyz^2$. This give us $f_x(x, y, z) = f_y(x, y, z) = f_z(x, y, z) = 1$ and $g_x(x, y, z) = yz^2$, $g_y(x, y, z) = xz^2$, $g_z(x, y, z) = 2xyz$.
We would like to find the minimum of $f(x, y, z)$ under the constrain $g(x, y, z) = 2500$.
Using method of Lagrange multipliers, we solve $f_x = \lambda g_x$, $f_y = \lambda g_y$ and $f_z = \lambda g_z$ for some $\lambda \in \mathbb{R}$.
Thus we have,

$$\begin{cases} 1 = \lambda yz^2 \\ 1 = \lambda xz^2 \\ 1 = 2\lambda xyz \\ xyz^2 = 2500 \end{cases}$$

Since we can see from the equations that $\lambda, x, y, z \neq 0$, it is easy to get $x = y = z/2 = 5$ as the only critical point. Since the minimum of $f(x, y, z)$ exists, it must be $f(5, 5, 10) = 20$.

Question 2

(a) Let S be the surface area we wanted. Firstly, we would like to understand how D looks like. At the intersection points of the two parabolas bounding the region D, we have $x^2 = 18 - x^2$, which give us $x = \pm 3$. Thus D is given by $x \in [-3, 3]$, $y \in [x^2, 18 - x^2]$. Therefore we have,

$$\begin{aligned} S &= \iint_D \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dA \\ &= \int_{-3}^{3} \int_{x^2}^{18-x^2} \sqrt{3}\, dy\, dx \\ &= \sqrt{3} \int_{-3}^{3} \left(18 - 2x^2\right)\, dx \\ &= \sqrt{3} \left[18x - \frac{2}{3}x^3\right]_{-3}^{3} = 72\sqrt{3}. \end{aligned}$$

(b) Notice that ∇g is continuous in $\mathbb{R}^3$. Also we have $\boldsymbol{r}(0) = \langle 0,0,0\rangle$ and $\boldsymbol{r}(1) = \langle 1,1,1\rangle$. Thus

$$\begin{aligned}
\int_C \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_C \boldsymbol{G}\cdot d\boldsymbol{r} + \int_C \boldsymbol{H}\cdot d\boldsymbol{r} \\
&= g(\boldsymbol{r}(1)) - g(\boldsymbol{r}(0)) + \int_0^1 \boldsymbol{H}(\boldsymbol{r}(t))\cdot \boldsymbol{r}'(t)\ dt \\
&= [xyz + 3e^{yz}\cos(\pi xz)]_{(0,0,0)}^{(1,1,1)} + \int_0^1 (t^2)(1) + (t^3)(2t) + (t)(3t^2)\ dt \\
&= 1 + 3e\cos(\pi) - 3\cos(0) + \left[\frac{1}{3}t^3 + \frac{2}{5}t^5 + \frac{3}{4}t^4\right]_0^1 \\
&= -3e - \frac{31}{60}.
\end{aligned}$$

(c) Let E be the region enclosed by S. We see that E is given by $x\in[0,1]$, $y\in[0,3]$, $z\in[0,5]$. Given that $\boldsymbol{F}(x,y,z) = \langle x^2y, xy^2, 5xyz\rangle$, we have $\mathrm{div}\boldsymbol{F} = 2xy + 2xy + 5xy = 9xy$.
Thus by Divergence Theorem, we have,

$$\begin{aligned}
\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} &= \iiint_E \mathrm{div}\boldsymbol{F}\ dV \\
&= \int_0^5\int_0^3\int_0^1 9xy\ dx\ dy\ dz \\
&= 9\int_0^1 x\ dx \int_0^3 y\ dy \int_0^5 1\ dz \\
&= 9\left[\frac{1}{2}x^2\right]_0^1 \left[\frac{1}{2}y^2\right]_0^3 [z]_0^5 \\
&= 9\left(\frac{1}{2}\right)\left(\frac{9}{2}\right)(5) = \frac{405}{4}.
\end{aligned}$$

Question 3

(a) We have $f_x = 2x\cos(x^2+yz)$, $f_y = z\cos(x^2+yz)$, $f_z = y\cos(x^2+yz)$.
Also $\frac{\partial x}{\partial u} = 2u$, $\frac{\partial y}{\partial u} = 2v$, $\frac{\partial z}{\partial u} = 7$, $\frac{\partial x}{\partial v} = -2v$, $\frac{\partial y}{\partial v} = 2u$, $\frac{\partial z}{\partial v} = 11$.
Thus

$$\begin{aligned}
\frac{\partial f}{\partial u} &= f_x\left(\frac{\partial x}{\partial u}\right) + f_y\left(\frac{\partial y}{\partial u}\right) + f_z\left(\frac{\partial z}{\partial u}\right) \\
&= (2x\cos(x^2+yz))(2u) + (z\cos(x^2+yz))(2v) + (y\cos(x^2+yz))(7) \\
&= (4xu + 2zv + 7y)\cos(x^2+yz),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial v} &= f_x\left(\frac{\partial x}{\partial v}\right) + f_y\left(\frac{\partial y}{\partial v}\right) + f_z\left(\frac{\partial z}{\partial v}\right) \\
&= (2x\cos(x^2+yz))(-2v) + (z\cos(x^2+yz))(2u) + (y\cos(x^2+yz))(11) \\
&= (-4xv + 2zu + 11y)\cos(x^2+yz).
\end{aligned}$$

(b) Firstly, from the cylindrical coordinates, we see that the region E bounded by the parameters is the hemisphere of radius 1, with base on the $x-y$ plane, centered at the origin, pointing in the positive z direction. Thus E can be given by rectangular coordinates $x\in[-1,1]$, $y\in[-\sqrt{1-x^2}, \sqrt{1-x^2}]$, $z\in[0,\sqrt{1-x^2-y^2}]$, or spherical coordinates $\theta\in[0,2\pi]$, $\phi\in[0,\frac{\pi}{2}]$, $\rho\in[0,1]$.

(i) Since $r\ dr\ d\theta = dy\ dx$, we can have

$$I = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} 1\ dz\ dy\ dx.$$

(ii) Since $dz\ dy\ dx = \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$, we have

$$I = \int_{0}^{2\pi}\int_{0}^{\frac{\pi}{2}}\int_{0}^{1} \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$$

(iii) Using the spherical coordinates form, we have

$$\begin{aligned} I &= \int_{0}^{2\pi} 1\ d\theta \int_{0}^{\frac{\pi}{2}} \sin\phi\ d\phi \int_{0}^{1} \rho^2\ d\rho \\ &= [\theta]_0^{2\pi}[-\cos\phi]_0^{\frac{\pi}{2}}\left[\frac{1}{3}\rho^3\right]_0^1 \\ &= (2\pi)(1)\left(\frac{1}{3}\right) = \frac{2}{3}\pi. \end{aligned}$$

(c) Let D be the area enclosed by the parameters. We see that D is the area between the x-axis, $x = 1$ and $y = x$. Thus we also have D to be given by $x \in [0, 1]$, $y \in [0, x]$. Therefore,

$$\begin{aligned} \int_0^1\int_y^1 e^{x^2}\ dx\ dy &= \int_0^1\int_0^x e^{x^2}\ dy\ dx \\ &= \int_0^1 xe^{x^2}\ dx \\ &= \left[\frac{1}{2}e^{x^2}\right]_0^1 = \frac{1}{2}(e-1). \end{aligned}$$

Question 4

(a) We notice that C can be traced by $\boldsymbol{r}(t) = \langle 2\cos t, 2\sin t, 4(\sin^2 t - \cos^2 t)\rangle$ from $t = 0$ to $t = 2\pi$. Let S be the surface with equation $z = g(x, y) = y^2 - x^2$ bounded by C, and D be the area given by the polar coordinates $r \in [0, 2]$, $\theta \in [0, 2\pi]$. Then S is a smooth surface on area D.
Since

$$\begin{aligned} \text{curl}\boldsymbol{F} &= \langle\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}\left(\frac{1}{3}x^3\right), \frac{\partial}{\partial z}(x^2y) - \frac{\partial}{\partial x}(xy), \frac{\partial}{\partial x}\left(\frac{1}{3}x^3\right) - \frac{\partial}{\partial y}(x^2y)\rangle \\ &= \langle x - 0, 0 - y, x^2 - x^2\rangle = \langle x, -y, 0\rangle, \end{aligned}$$

we have by Stokes' Theorem,

$$\begin{aligned} \int_C \boldsymbol{F}\cdot d\boldsymbol{r} &= \iint_S \text{curl}\boldsymbol{F}\cdot d\boldsymbol{S} \\ &= \iint_D -(x)(-2x) - (-y)(2y) + 0\ dA \\ &= \iint_D 2(x^2+y^2)\ dA \\ &= \int_0^2\int_0^{2\pi} 2r^3\ d\theta\ dr \\ &= 2\int_0^2 1\ d\theta \int_0^{2\pi} r^3\ dr \\ &= 2[\theta]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^2 = 16\pi. \end{aligned}$$

(b) We have $f_x(x,y) = 3x^2 - 3$ and $f_y(x,y) = -3y^2 + 12$.

When $f_x(x,y) = 0$, we have $x = \pm 1$.
When $f_y(x,y) = 0$, we have $y = \pm 2$.
Combining the above, we have $\nabla f(x,y) = \langle 0, 0\rangle$ only when $(x,y) = (\pm 1, \pm 2)$ or $(x,y) = (\pm 1, \mp 2)$.

Also $f_{xx}(x,y) = 6x$, $f_{yy}(x,y) = -6y$, $f_{xy}(x,y) = 0$, and so $D = f_{xx}f_{yy} - (f_{xy})^2 = -36xy$.
This give us $D|_{(1,2)}, D|_{(-1,-2)} < 0$ i.e. $(1,2)$ and $(-1,-2)$ are saddle points.
Also $D|_{(-1,2)}, D|_{(1,-2)} > 0$, and since $f_{xx}(-1,2) < 0$ and $f_{xx}(1,-2) > 0$, $(-1,2)$ is a local maximum point and $(1,-2)$ is a local minimum point.

Question 5

(a) We have $f_x(x,y,z) = 6x$, $f_y(x,y,z) = -10y$ and $f_z(x,y,z) = 4z$. Thus $\nabla f(1,1,2) = \langle 6, -10, 8\rangle$.
Therefore to get as cool as possible, I should set out in the direction of $-\nabla f(1,1,2) = \langle -6, 10, -8\rangle$.

Rate of change of the temperature $= 3D_{\frac{\langle -6,10,-8\rangle}{\sqrt{200}}} f(1,1,2) = \frac{3}{\sqrt{200}}(-36 - 100 - 64) = -30\sqrt{2}\ ^\circ$.

Note: The question give temperature in degrees, not degrees (Celsius/Farenheit), and thus we will follow in our solution above. =)

(b) We have,

$$\begin{aligned}
\nabla(fg) &= \langle (fg)_x, (fg)_y, (fg)_z\rangle \\
&= \langle fg_x + gf_x,\ fg_y + gf_y,\ fg_z + gf_z\rangle \\
&= \langle fg_x, fg_y, fg_z\rangle + \langle gf_x, gf_y, gf_z\rangle \\
&= f\langle g_x, g_y, g_z\rangle + g\langle f_x, f_y, f_z\rangle \\
&= f\nabla g + g\nabla f.
\end{aligned}$$

Let P_n be the statement that $\nabla(f^n) = nf^{n-1}\nabla f$, $n \in \mathbb{Z}^+$.
P_1 is $\nabla f = \nabla f$, which is immediately true.

Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $\nabla(f^k) = kf^{k-1}\nabla f$. Consider P_{k+1}.
We have $\nabla(f^{k+1}) = \nabla(f^k f) = f^k\nabla f + f\nabla(f^k) = f^k\nabla f + f(kf^{k-1}\nabla f) = (k+1)f^k\nabla f$, and so P_{k+1} is true.
Therefore by Mathematical Induction, we have P_n to be true for all $n \in \mathbb{Z}^+$.

(c) Let $P = x$ and $Q = x^3 + 3xy^2$, which give us $F(x,y) = \langle P, Q\rangle$.
Also, let C be the path of the particle, and D be the area bounded by C. We see that D is given by the polar coordinates $r \in [0,3]$, $\theta \in [0,\pi]$. Thus by Green's Theorem, we have work done on the particle by $\boldsymbol{F}$ to be

$$\begin{aligned}
\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_C P\ dx + Q\ dy &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA \\
&= \iint_D 3x^2 + 3y^2\ dA \\
&= \int_0^3 \int_0^\pi 3r^3\ d\theta\ dr \\
&= 3\int_0^\pi 1\ d\theta \int_0^3 r^3\ dr \\
&= 3[\theta]_0^\pi \left[\frac{1}{4}r^4\right]_0^3 = \frac{243}{4}\pi.
\end{aligned}$$