MA2002 - Calculus Suggested Solutions
(Semester 2: AY2021/22)

Written by: Agrawal Naman
Audited by: Qi Fulin, Thang Pang Ern

December 11, 2022

Question 1

(a) Use only the ϵ, δ definition of limit, prove that $\lim\limits_{x \longrightarrow 2} \dfrac{3x^2 - x - 4}{x+1} = 2$.

Ans:
We need to find a $\delta > 0$ such that for all $\epsilon > 0$,

$$0 < |x-2| < \delta \implies \left| \frac{3x^2 - x - 4}{x+1} - 2 \right| < \epsilon$$

We can set,

$$\delta = \min \left\{ 3, \frac{\epsilon}{3} \right\}$$

Thus, for $0 < |x-2| < \delta$,

$$\begin{aligned} \left| \frac{3x^2 - x - 4}{x+1} - 2 \right| &= \left| \frac{3x^2 - x - 4 - 2x - 2}{x+1} \right| \\ &= \left| \frac{3x^2 - 3x - 6}{x+1} \right| \\ &= \left| \frac{3(x+1)(x-2)}{x+1} \right| \\ &= 3|(x-2)| \\ &< 3\left(\frac{\epsilon}{3}\right) = \epsilon \end{aligned}$$

Note we've set $\delta \leq 3$ because $\delta \leq 3 \implies |x-2| < 3 \implies -1 < x \implies x + 1 \neq 0$.

(b) Let p and q be positive constants. It is known that

$$\lim_{x \longrightarrow 0} \frac{1}{px - \sin x} \int_0^x \frac{t^2}{\sqrt{q + t^2}} dt = 3$$

Find the values of p and q.

Ans:

$$\lim_{x \longrightarrow 0} \frac{1}{px - \sin x} \int_0^x \frac{t^2}{\sqrt{q + t^2}} dt = 3$$

$$\implies \lim_{x \longrightarrow 0} \frac{\displaystyle\int_0^x \frac{t^2}{\sqrt{q+t^2}} dt}{px - \sin x} = 3$$

It is clear that:

$$x \to 0 \implies \int_0^x \frac{t^2}{\sqrt{q+t^2}}dt \to 0; \quad px - \sin x \to 0$$

Therefore, we may use L'Hôpital's rule as follows:

$$\lim_{x \to 0} \frac{\dfrac{d}{dx}\displaystyle\int_0^x \frac{t^2}{\sqrt{q+t^2}}dt}{\dfrac{d}{dx}(px - \sin x)} = 3$$

$$\implies \lim_{x \to 0} \frac{\dfrac{x^2}{\sqrt{q+x^2}}}{p - \cos x} = 3$$

$$\implies \lim_{x \to 0} \frac{x^2}{(p - \cos x)\sqrt{q+x^2}} = 3$$

Clearly, if $p \neq 1;\ q \neq 0$, the above limit will evaluate to 0, irrespective of the value of p and q. Thus, $p = 1$. So, we get:

$$\begin{aligned}
\lim_{x \to 0} \frac{1}{px - \sin x} \int_0^x \frac{t^2}{\sqrt{q+t^2}}\, dt = 3 &\implies \lim_{x \to 0} \frac{\dfrac{x^2}{\sqrt{q+x^2}}}{1 - \cos x} = 3 \\
&\implies \lim_{x \to 0} \frac{x^2(1+\cos x)}{\sqrt{q+x^2}(1-\cos x)(1+\cos x)} = 3 \\
&\implies \lim_{x \to 0} \frac{x^2}{\sin^2 x} \lim_{x \to 0} \frac{(1+\cos x)}{\sqrt{q+x^2}} = 3 \\
&\implies \left(\lim_{x \to 0} \frac{x}{\sin x}\right)^2 \lim_{x \to 0} \frac{(1+\cos x)}{\sqrt{q+x^2}} = 3 \\
&\implies 1 \cdot \frac{2}{\sqrt{q}} = 3 \\
&\implies q = \frac{4}{9}
\end{aligned}$$

Thus the solution is $p = 1;\ q = \dfrac{4}{9}$.

Question 2

Evaluate the following limits.

(a) $\lim_{x \to 0} (\cos x)^{\dfrac{1}{\ln(1+x^2)}}$

Ans:

$$\begin{aligned}
&\lim_{x \to 0} (\cos x)^{\dfrac{1}{\ln(1+x^2)}} \\
&= \lim_{x \to 0} \exp\left(\ln\left((\cos x)^{\dfrac{1}{\ln(1+x^2)}}\right)\right) \\
&= \exp\left(\lim_{x \to 0} \ln\left((\cos x)^{\dfrac{1}{\ln(1+x^2)}}\right)\right) \\
&= \exp\left(\lim_{x \to 0} \frac{\ln \cos x}{\ln(1+x^2)}\right)
\end{aligned}$$

It is known that:

$$x \to 0 \implies \ln \cos x \to \ln 1 = 0; \quad x \to 0 \implies \ln(1+x^2) \to \ln 1 = 0$$

Therefore, we may use L'Hôpital's rule as follows:

$$\exp\left(\lim_{x \to 0} \frac{\ln(\cos x)}{\ln(1+x^2)}\right) = \exp\left(\lim_{x \to 0} \frac{\dfrac{d(\ln(\cos x))}{dx}}{\dfrac{d(\ln(1+x^2))}{dx}}\right) = \exp\left(\lim_{x \to 0} -\frac{1}{2} \cdot \frac{\sin x}{x} \cdot \frac{1+x^2}{\cos x}\right)$$

$$= \exp\left(-\frac{1}{2} \cdot \lim_{x \to 0} \cdot \frac{\sin x}{x} \lim_{x \to 0} \frac{1+x^2}{\cos x}\right) = \exp\left(-\frac{1}{2}\right)$$

(b) $\lim_{x \to 1} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right)$

Ans:

$$\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x \to 1} \left(\frac{x \ln x - x + 1}{(x-1)\ln x}\right)$$

It is known that:

$$x \to 1 \implies x \ln x - x + 1 \to 0; \quad x \to 1 \implies (x-1)\ln x \to 0$$

Therefore, we may use L'Hôpital's rule as follows:

$$\begin{aligned}
\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right) &= \lim_{x \to 1} \left(\frac{x \ln x - x + 1}{(x-1)\ln x}\right) \\
&= \lim_{x \to 1} \left(\frac{\dfrac{d}{dx}(x \ln x - x + 1)}{\dfrac{d}{dx}(x \ln x - \ln x)}\right) \\
&= \lim_{x \to 1} \left(\frac{1 + \ln x - 1}{\dfrac{x-1}{x} + \ln x}\right)
\end{aligned}$$

$$= \lim_{x \to 1} \left(\frac{x \ln x}{x - 1 + x \ln x} \right)$$

It is known that:

$$x \to 1 \implies x \ln x \to 0; \quad x \to 1 \implies x - 1 + x \ln x \to 0$$

Therefore, we may use L'Hôpital's rule as follows:

$$\begin{aligned} \lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) &= \lim_{x \to 1} \left(\frac{x \ln x}{x - 1 + x \ln x} \right) \\ &= \lim_{x \to 1} \left(\frac{1 + \ln x}{1 + \ln x + 1} \right) \end{aligned}$$

Taking the limit,

$$\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) = \lim_{x \to 1} \left(\frac{1 + \ln x}{1 + \ln x + 1} \right) = 0.5$$

Question 3

(a) Let $f(x) = e^{2\sqrt{x}}$, where $x > 0$. It is known that $f''(x)$ may be expressed as:

$$f''(x) = \frac{ke^{2\sqrt{x}}(2\sqrt{x}-1)}{x\sqrt{x}}$$

where k is a constant. Find the value of k.

Ans:
Calculating the first derivative:

$$\begin{aligned} f'(x) &= \frac{d}{dx}e^{2\sqrt{x}} \\ &= e^{2\sqrt{x}}\frac{d}{dx}(2\sqrt{x}) \\ &= e^{2\sqrt{x}}\frac{2}{2\sqrt{x}} \\ &= \frac{e^{2\sqrt{x}}}{\sqrt{x}} \end{aligned}$$

Calculating the second derivative:

$$\begin{aligned} f''(x) &= \frac{d}{dx}\frac{e^{2\sqrt{x}}}{\sqrt{x}} \\ &= \frac{\sqrt{x}\frac{d}{dx}e^{2\sqrt{x}} - e^{2\sqrt{x}}\frac{d}{dx}\sqrt{x}}{x} \\ &= \frac{\sqrt{x}\frac{e^{2\sqrt{x}}}{\sqrt{x}} - e^{2\sqrt{x}}\frac{1}{2\sqrt{x}}}{x} \\ &= \frac{2\sqrt{x}e^{2\sqrt{x}} - e^{2\sqrt{x}}}{2x\sqrt{x}} \\ &= \frac{e^{2\sqrt{x}}(2\sqrt{x}-1)}{2x\sqrt{x}} \end{aligned}$$

On comparing the above equation with the equation given in question, we get:

$$k = \frac{1}{2}$$

(b) Prove that $x - \frac{x^2}{2} < \ln(1+x) < x$ for all $x > 0$.

Ans:
First, we show that $x - \frac{x^2}{2} < \ln(1+x)$ for all $x > 0$. Let,

$$g(x) = x - \frac{x^2}{2} - \ln(1+x)$$

Thus,

$$g'(x) = 1 - x - \frac{1}{1+x} = \frac{1-x^2-1}{1+x} = -\frac{x^2}{1+x}$$

For $x > 0$, $1+x > 0$ and $x^2 > 0$. Thus, $-\frac{x^2}{1+x} < 0 \implies g'(x) < 0$. Thus, $g(x)$ is a decreasing function for $x > 0$. Moreover,

$$g(0) = 0$$

Thus, $g(x)$ will have a value lesser than 0 for all $x > 0$. Hence for all $x > 0$,

$$g(x) < 0 \implies x - \frac{x^2}{2} - \ln(1+x) < 0 \implies x - \frac{x^2}{2} < \ln(1+x)$$

Next, we show that $\ln(1+x) < x$ for all $x > 0$. Let,

$$h(x) = \ln(1+x) - x$$

Thus,

$$h'(x) = \frac{1}{1+x} - 1 = \frac{1-x-1}{1+x} = -\frac{x}{1+x}$$

For $x > 0$, $1 + x > 0$. Thus, $-\frac{x}{1+x} < 0 \implies h'(x) < 0$. Thus, $h(x)$ is a decreasing function for $x > 0$. Moreover,

$$h(0) = 0$$

Thus, $h(x)$ will have a value lesser than 0 for all $x > 0$. Hence for all $x > 0$,

$$h(x) < 0 \implies \ln(1+x) - x < 0 \implies \ln(1+x) < x$$

In view of the above inequalities, we get that for all $x > 0$,

$$x - \frac{x^2}{2} < \ln(1+x) < x$$

Remark: An alternative way to prove the above inequality (taught in Analysis) Let $f(x) = \ln(1+x)$ By Maclaurin Expansion,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

By Taylor's theorem

$$\ln(1+x) = P_0(x) + R_0(x) = x + R_0(x)$$

By Mean Value Theorem $\exists c \in (0, x)$ such that,

$$f(x) - f(0) = f'(c)x \implies \ln(1+x) = f'(c)x = \frac{x}{1+c}$$

Further, $0 < c < x \implies \frac{1}{1+x} < \frac{1}{1+c} < 1$. Thus,

$$\ln(1+x) < x$$

Similarly $\exists c \in (0, x)$ such that

$$f(x) - x + \frac{x^2}{2} - 0 = x\left(\frac{1}{1+c} - 1 + c\right) = \frac{xc^2}{1+c} > 0$$

Thus,

$$\ln(1+x) > x - \frac{x^2}{2}$$

Remark: An alternative way to prove the above inequality Since, $x > 0$,

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt < \int_0^x \,dt = x$$

Also, we may show that:

$$t > 0 \implies t^2 > 0 \implies 1 > 1 - t^2 \implies \frac{1}{1+t} > 1 - t$$

Thus,

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt > \int_0^x (1-t)\,dt = x - \frac{x^2}{2}$$

Question 4

Evaluate the following definite integrals.

(a) $\displaystyle\int_1^2 x\sqrt{2-x}\,dx$

Ans:

$$\begin{aligned}
\int_1^2 x\sqrt{2-x}\,dx &= \left[x\int\sqrt{2-x}\,dx - \int (x)'\left(\int\sqrt{2-x}\,dx\right)\,dx\right]_1^2 \\
&= \left[-\frac{2x}{3}(2-x)^{3/2} - \int -\frac{2}{3}(2-x)^{3/2}\,dx\right]_1^2 \\
&= \left[-\frac{2x}{3}(2-x)^{3/2} - \frac{4}{15}(2-x)^{5/2}\right]_1^2 \\
&= \frac{2}{3}(2-1)^{3/2} + \frac{4}{15}(2-1)^{5/2} \\
&= \frac{2}{3} + \frac{4}{15} \\
&= \frac{14}{15}
\end{aligned}$$

(b) $\displaystyle\int_0^{36} |\sqrt{x}-2|dx$

Ans:

$$|\sqrt{x}-2| = \begin{cases} \sqrt{x}-2, & \sqrt{x}\geq 2 \\ 2-\sqrt{x}, & \sqrt{x}<2 \end{cases} = \begin{cases} \sqrt{x}-2, & x\geq 4 \\ 2-\sqrt{x}, & x<4 \end{cases}$$

Therefore,

$$\begin{aligned}
\int_0^{36} |\sqrt{x}-2|dx &= \int_0^4 (2-\sqrt{x})dx + \int_4^{36}(\sqrt{x}-2)dx \\
&= \left[2x - \frac{2}{3}x^{3/2}\right]_0^4 + \left[\frac{2}{3}x^{3/2} - 2x\right]_4^{36} \\
&= 8 - \frac{16}{3} + \left(\frac{2}{3}\right)6^3 - 72 - \frac{16}{3} + 8 \\
&= 16 - \frac{32}{3} - 72 + 144 \\
&= 88 - \frac{32}{3} \\
&= \frac{232}{3}
\end{aligned}$$

Question 5

(a) It is known that $\int_1^e x^2(\ln x)^2 dx = \frac{1}{27}(Ae^3+B)$, where A and B are integers. Find the values of A and B.

Ans:

$$\begin{aligned}
\int_1^e x^2(\ln x)^2\, dx &= \left[(\ln x)^2 \int x^2\, dx - \int ((\ln x)^2)' \left(\int x^2\, dx\right)\, dx\right]_1^e \\
&= \left[\frac{x^3}{3}(\ln x)^2 - 2\int \frac{\ln x}{x} \cdot \frac{x^3}{3}\, dx\right]_1^e \\
&= \left[\frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\int x^2 \ln x\, dx\right]_1^e \\
&= \left[\frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\left(\ln x \int x^2\, dx - \int (\ln x)' \left(\int x^2\, dx\right)\, dx\right)\right]_1^e \\
&= \left[\frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\left(\frac{x^3}{3}\ln x - \int \frac{1}{x} \cdot \frac{x^3}{3}\, dx\right)\right]_1^e \\
&= \left[\frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\left(\frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\, dx\right)\right]_1^e \\
&= \left[\frac{x^3}{3}(\ln x)^2 - \frac{2x^3}{9}\ln x + \frac{2x^3}{27}\right]_1^e \\
&= \left[\left(9(\ln x)^2 - 6\ln x + 2\right)\frac{x^3}{27}\right]_1^e \\
&= \frac{5e^3}{27} - \frac{2}{27} \\
&= \frac{1}{27}(5e^3 - 2)
\end{aligned}$$

Thus, we get:

$$A = 5, \quad B = -2$$

(b) Let $f(x) = (3+x)\int_1^{e^{2x}} \frac{1}{\sqrt{1+\ln t}}\, dt$. Find the value of $f'(0)$.

Ans:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[(3+x)\int_1^{e^{2x}} \frac{1}{\sqrt{1+\ln t}}\, dt\right] \\
&= (3+x)\frac{d}{dx}\left[\int_1^{e^{2x}} \frac{1}{\sqrt{1+\ln t}}\, dt\right] + \int_1^{e^{2x}} \frac{1}{\sqrt{1+\ln t}}\, dt \\
&= (3+x)\frac{d}{dx}(e^{2x})\frac{1}{\sqrt{1+\ln(e^{2x})}} + \int_1^{e^{2x}} \frac{1}{\sqrt{1+\ln t}}\, dt \\
&= \frac{2(3+x)e^{2x}}{\sqrt{1+2x}} + \int_1^{e^{2x}} \frac{1}{\sqrt{1+\ln t}}\, dt
\end{aligned}$$

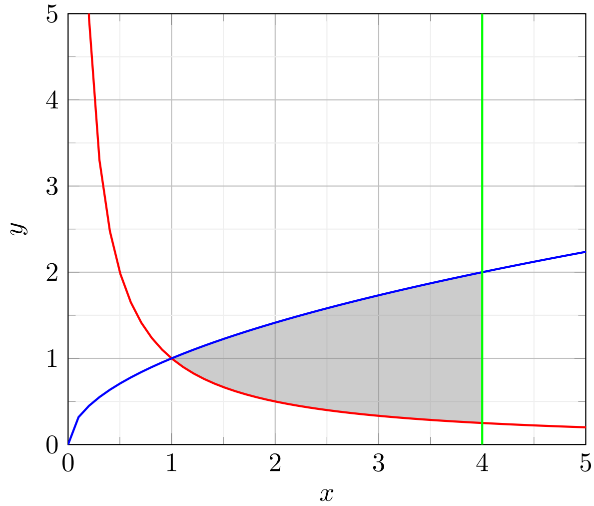
Thus,

$$f'(0) = \frac{2(3)}{\sqrt{1}} + \int_1^{e^0} \frac{1}{\sqrt{1+\ln t}}\, dt = 6 + \int_1^1 \frac{1}{\sqrt{1+\ln t}}\, dt = 6 + 0 = 6$$

Question 6

(a) Let R be the region bounded by the graphs of $y = \dfrac{1}{x}$, $y = \sqrt{x}$ and the line $x = 4$. Find the volume of solid formed by rotating R completely about the y-axis.

Ans:



Using the cylindrical shell method, the volume of the solid is given by:

$$
\begin{aligned}
V &= \int_1^4 2\pi x(\sqrt{x} - \frac{1}{x})\, dx \\
&= 2\pi \left[\frac{2}{5}x^{5/2}\right]_1^4 - 2\pi \left[x\right]_1^4 \\
&= \frac{128\pi}{5} - \frac{4\pi}{5} - 6\pi \\
&= \frac{94\pi}{5}
\end{aligned}
$$

(b) A curve C has equation $y = \sec(2x)$, where $0 \leq x \leq \dfrac{\pi}{6}$. Find the length of the curve C.

Ans: The length of the curve is given by:

$$
\begin{aligned}
L &= \int_0^{\pi/6} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^{\pi/6} \sqrt{1 + \left(\frac{d}{dx}\sec(2x)\right)^2}\, dx \\
&= \int_0^{\pi/6} \sqrt{1 + (2\sec(2x)\tan(2x))^2}\, dx \\
&= \int_0^{\pi/6} \sqrt{4\sec^2(2x)\tan^2(2x) + 1}\, dx \\
&= \int_0^{\pi/6} \sqrt{4\sec^4(2x) - 4\sec^2(2x) + 1}\, dx \\
&= \int_0^{\pi/6} \sqrt{(2\sec^2(2x) - 1)^2}\, dx
\end{aligned}
$$

Moreover, $0 \leq x \leq \frac{\pi}{6} \implies \sec 0 \leq \sec(2x) \leq \sec\frac{\pi}{3} \implies 1 \leq \sec^2(2x) \leq 4 \implies 1 \leq 2\sec^2(2x) - 1 \leq 7 \implies 2\sec^2(2x) - 1 > 0$. Thus,

$$
\begin{aligned}
L &= \int_0^{\pi/6} \sqrt{(2\sec^2(2x) - 1)^2}\, dx \\
&= \int_0^{\pi/6} (2\sec^2(2x) - 1)\, dx \\
&= 2\int_0^{\pi/6} \sec^2(2x)\, dx - \int_0^{\pi/6} dx \\
&= [\tan(2x) - x]_0^{\pi/6} \\
&= \tan\frac{\pi}{3} - \frac{\pi}{6} \\
&= \sqrt{3} - \frac{\pi}{6}
\end{aligned}
$$

Question 7

(a) Let y denote the solution of the differential equation

$$x^2\frac{dy}{dx} - xy = 1$$

with $x > 0$ that satisfies $y = 1$ when $x = 1$. Find the value of y when $x = 2$.

Ans:

$$x^2\frac{dy}{dx} - xy = 1 \implies \frac{dy}{dx} - \frac{y}{x} = \frac{1}{x^2}$$

Thus, the given differential equation is a linear differential equation, and can be calculated by finding its integrating factor:

$$\begin{aligned} I.F. &= \exp\left(\int -\frac{1}{x}\,dx\right) \\ &= \exp\left(-\int \frac{1}{x}\,dx\right) \\ &= \exp\left(\ln\frac{1}{x}\right) \\ &= \frac{1}{x} \end{aligned}$$

Thus, the solution of the differential equation is given by:

$$\frac{y}{x} = \int \frac{1}{x}\cdot\frac{1}{x^2}\,dx = \int \frac{1}{x^3}\,dx$$

$$\implies \frac{y}{x} + \frac{1}{2x^2} = c$$

Since the point $(1, 1)$ satisfies the equation,

$$1 + \frac{1}{2} = c \implies c = \frac{3}{2}$$

Thus, we get the following solution to the differential equation:

$$\frac{y}{x} + \frac{1}{2x^2} = \frac{3}{2}, \quad x > 0$$

For $x = 2$, we arrive at the following solution:

$$\begin{aligned} \frac{y}{2} + \frac{1}{8} &= \frac{3}{2} \\ \implies 4y + 1 &= 12 \\ \implies y &= \frac{11}{4} \end{aligned}$$

(b) Two chemicals A and B react to form the substance X according to the differential equation

$$\frac{dQ}{dt} = k(100 - Q)(50 - Q)$$

where $Q = Q(t)$ denotes the amount of substance X per unit volume at time t, and k is a positive constant. Initially, no amount of X is present. Derive an expression for the amount of X per unit volume at time t.

Ans:
The given differential equation is variable separable. The variables can be separated as follows:

$$\frac{dQ}{dt} = k(100 - Q)(50 - Q)$$

$$\implies \frac{dQ}{(100 - Q)(50 - Q)} = kdt$$

Let Q be the amount of substance at time t. Thus, integrating both sides gives us:

$$\int \frac{dQ}{(100 - Q)(50 - Q)} = \int kdt$$

$$\implies \int \frac{dQ}{(100 - Q)(50 - Q)} = kt + c_1$$

Solving the integral,

$$\begin{aligned}
\int \frac{dQ}{(100 - Q)(50 - Q)} &= \int \frac{dQ}{Q^2 - 150Q + 5000} \\
&= \int \frac{dQ}{Q^2 - 150Q + 5625 - 625} \\
&= \int \frac{dQ}{(Q - 75)^2 - (25)^2} \\
&= \frac{1}{50} \log\left(\frac{Q - 75 - 25}{Q - 75 + 25}\right) + c_2 \\
&= \frac{1}{50} \log\left(\frac{Q - 100}{Q - 50}\right) + c_2
\end{aligned}$$

Thus, the solution of the differential equation is:

$$\frac{1}{50} \log\left(\frac{Q - 100}{Q - 50}\right) + c = kt$$

Since, the amount is 0 at $t = 0$, we get:

$$c = k(0) - \frac{1}{50} \log\left(\frac{0 - 100}{0 - 50}\right) \implies c = -\frac{1}{50} \log 2$$

Thus, we get the following solution

$$\log\left(\frac{Q - 100}{Q - 50}\right) = 50kt + \log 2$$

$$\implies \frac{Q - 100}{Q - 50} = \exp(50kt + \log 2)$$

$$\implies Q = \frac{50 \exp(50kt + \log 2) - 100}{\exp(50kt + \log 2) - 1}$$

Thus the amount of X per unit volume at time t, is given by:

$$Q = \frac{50 \exp(50kt + \log 2) - 100}{\exp(50kt + \log 2) - 1}$$

Question 8

(a) Let $f(x)$ be an even function. Suppose $f'(0)$ exists. Prove that $f'(0) = 0$.

Ans:
Since $f'(0)$ exists, the right hand and the left hand derivatives should be equal. Thus,

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$\implies \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$\implies \lim_{h \to 0} \frac{f(h) - f(0)}{h} + \lim_{h \to 0} \frac{f(-h) - f(0)}{h} = 0$$

Moreover, since f is an even function, $f(h) = f(-h)$. Thus,

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} + \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 0$$

$$\implies 2 \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 0$$

$$\implies 2f'(0) = 0$$

$$\implies f'(0) = 0$$

(b) Let f be a function defined on a closed interval $[a, b]$. We say that f is differentiable at $x = a$ if $f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$ exists. Similarly, we say that f is differentiable at $x = b$ if $f'_-(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$ exists. Let $c \in (a, b)$ and f is said to be differentiable at $x = c$ if $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists. A function is said to be differentiable on an interval I if f is differentiable at every point in the interval I. Let f be a differentiable function on $[a, b]$. Suppose $f'_+(a) > 0, f'_-(b) > 0$ and $f(a) \geq f(b)$. Prove that the equation $f'(x) = 0$ has at least two distinct roots in (a, b). (Hint: Use Fermat Theorem.)

Ans:
We need to show that f has at least 2 local extrema in the interval (a, b). First, we assume that f has no local maxima in (a, b). Since f is differentiable in I, and hence continuous, this would mean that:

$$\nexists \ c \in [a, b] \text{ such that } f(c) > f(a) \text{ and } f(c) > f(b)$$

Since $f(a) \geq f(b)$ our assumption would imply:

$$\nexists \ c \text{ such that } f(c) > f(a)$$

This would mean that a is an absolute maximum for $f(x)$ for $x \in [a, b]$. This would mean, that for $h > 0$,

$$f(a+h) < f(a) \implies \frac{f(a+h) - f(a)}{h} < 0 \implies \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} < 0 \implies f'_+(a) < 0$$

However, this contradicts the given statement that $f'_+(a) > 0$. Thus, f has at least one local maximum in (a, b). Next, we assume that f has no local minimum in (a, b). Since f is differentiable in I, and hence continuous, this would mean that:

$$\nexists \ c \in [a, b] \text{ such that } f(c) < f(a) \text{ and } f(c) < f(b)$$

Since $f(a) \geq f(b)$ our assumption would imply:

$$\nexists \ c \text{ such that } f(c) < f(b)$$

This would mean that b is an absolute minimum for $f(x)$ for $x \in [a,b]$. This would mean, that for $h < 0$,

$$f(b+h) < f(b) \implies \frac{f(b+h)-f(b)}{h} < 0 \implies \lim_{h \to 0^-} \frac{f(b+h)-f(b)}{h} < 0 \implies f'_-(b) < 0$$

However, this contradicts the given statement that $f'_-(b) > 0$. Thus, f has at least one local minimum in (a,b).
In view of the above statements, f has at least 2 local extrema in (a,b). Thus, by Fermat's theorem (given that f is a differentiable, hence continuous function in I), there exists at least 2 points in (a,b), where $f'(x) = 0$. Thus, $f'(x) = 0$ has at least two distinct roots in (a,b).