NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lau Tze Siong

MA2101 Linear Algebra II
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SECTION A

Question 1

(a) W is not a subspace of $\mathbb{R}^3$ since $(0,0,0) \notin W$.

(b) **Non-empty Subset**
For any $\boldsymbol{w} \in W$, $\boldsymbol{w} \in M_{22}(\mathbb{R}) = V$. Hence $W \subseteq V$. Also $\mathbf{0} \in W$.

Closure under Linear Combination.
For any $\begin{pmatrix} a_1 & a_1+b_1 \\ a_1-b_1 & b_1 \end{pmatrix}, \begin{pmatrix} a_2 & a_2+b_2 \\ a_2-b_2 & b_2 \end{pmatrix} \in W$, and $r \in \mathbb{R}$, we have,

$$\begin{aligned}
\begin{pmatrix} a_1 & a_1+b_1 \\ a_1-b_1 & b_1 \end{pmatrix} + r\begin{pmatrix} a_2 & a_2+b_2 \\ a_2-b_2 & b_2 \end{pmatrix} &= \begin{pmatrix} a_1+ra_2 & a_1+b_1+ra_2+rb_2 \\ a_1-b_1+ra_2-rb_2 & b_1+rb_2 \end{pmatrix} \\
&= \begin{pmatrix} a_1+ra_2 & (a_1+a_2)+r(b_1+b_2) \\ (a_1+a_2)-r(b_1+b_2) & b_1+rb_2 \end{pmatrix} \\
&\in W.
\end{aligned}$$

Hence W is a vector subspace of V.

Question 2

(a) (i) Since

$$\begin{aligned}
p_1(x)+2p_2(x)-p_3(x) &= 1+x^3+2(x+2x^3)-(x^2-3x^3) \\
&= 1+2x-x^2+8x^3,
\end{aligned}$$

$q_1(x)$ can be expressed as a linear combination of $p_1(x), p_2(x), p_3(x)$. Therefore $q_1(x) \in W$.

(ii) Claim: $q_2(x) \notin \operatorname{span}(\{p_1(x), p_2(x), p_3(x)\})$
Proof:
Suppose there exist $a, b, c \in \mathbb{R}$ such that $ap_1(x)+bp_2(x)+cp_3(x) = q_2(x)$. By comparing the coefficients of x^0 and x^2, we have $a = 0 = c$. Hence $q_2(x) = b(x+2x^2)$ for some $b \in \mathbb{R}$, which is a contradiction.

(b) Let $ap_1(x)+bp_2(x)+cp_3(x) = 0$ where $a, b, c \in \mathbb{R}$. By comparing coefficients of x^0, x and x^2, we have $a = b = c = 0$. Hence $\{p_1(x), p_2(x), p_3(x)\}$ is a linearly independent set.
Since $\operatorname{span}(\{p_1(x), p_2(x), p_3(x)\}) = W$ and $\{p_1(x), p_2(x), p_3(x)\}$ is a linearly independent set, $\{p_1(x), p_2(x), p_3(x)\}$ is a basis for W.

(c) Since $\dim(W) = 3$, we have $\dim(U) = 1$. Let $U = \operatorname{span}(\{q_2(x)\})$. Since $q_2(x) \notin W$, we have $U \cap W = \{0_{P_3(\mathbb{R})}\}$. Hence $U \oplus W = P_3(\mathbb{R})$.

Question 3

(a) Since $T_1(1) = 1 + 2x^2$,$T_1(x) = x + x^2$ and $T_1(x^2) = x + 2x^2$, we have

$$[T_1]_{\mathcal{B}_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

Since $T_2(1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$,$T_2(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,$T_3(x) = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$, we have

$$[T_2]_{\mathcal{B}_2,\mathcal{B}_1} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix}.$$

Hence

$$\begin{aligned} [T_2 \circ T_1]_{\mathcal{B}_2,\mathcal{B}_1} &= [T_2]_{\mathcal{B}_2,\mathcal{B}_1}[T_1]_{\mathcal{B}_1} \\ &= \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 2 & 4 \\ 7 & 4 & 7 \\ 7 & 4 & 7 \\ 5 & 2 & 4 \end{pmatrix}. \end{aligned}$$

(b) Hence

$$T_2 \circ T_1(a + bx + cx^2) = \begin{pmatrix} 5a + 2b + 4c & 7a + 4b + 7c \\ 7a + 4b + 7c & 5a + 2b + 4c \end{pmatrix}.$$

(c) Since $\det([T_1]_{\mathcal{B}_1}) = 1 \neq 0$, T_1 is invertible. This give us

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 2 & -1 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right).$$

Hence

$$[T_1^{-1}]_{\mathcal{B}_1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ -2 & -1 & 1 \end{pmatrix}.$$

(d) For $x \in \ker(T_2)$,

$$x = \begin{pmatrix} a_0 + 2a_2 & a_0 + a_1 + 3a_2 \\ a_0 + a_1 + 3a_2 & a_0 + 2a_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence we arrive at the following set of equations,

$$\begin{aligned} a_0 + 0a_1 + 2a_2 &= 0; \\ a_0 + 1a_1 + 3a_2 &= 0; \\ a_0 + 1a_1 + 3a_2 &= 0; \\ a_0 + 0a_1 + 2a_2 &= 0. \end{aligned}$$

Solving them, we have

$$\begin{aligned} a_0 &= -2t \\ a_1 &= -t \\ a_2 &= t. \end{aligned}$$

Hence a basis for $\ker(T_2)$ is $\{2 + x - x^2\}$.

(e) Since T_1 is invertible, $\text{null}(T_1) = 0$. Hence, by Rank-Nullity Theorem, $\text{rank}(T_1) = 3$.
Since $\text{null}(T_2) = 1$, by Rank-Nullity Theorem, $\text{rank}(T_2) = 2$.

(f) No.
Consider $T_1^{-1}(2 + x - x^2) \in P_2(\mathbb{R})\backslash\{0\}$. Since $T_2 \circ T_1(T_1^{-1}(2 + x - x^2)) = T_2(2 + x - x^2) = 0$, we have $T_1^{-1}(2 + x - x^2) \in \ker(T_2 \circ T_1)$, i.e. $T_2 \circ T_1$ is not injective. Hence $T_2 \circ T_1$ is singular.

Question 4

(i) $\dim(V) = \deg(c_T(x)) = \deg(x^3(x-1)^4) = 7$.

(ii) The possible Jordan Canonical Forms are,

$$\begin{pmatrix} 0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&1&1&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&1&1&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&1 \end{pmatrix}.$$

(iii) For either of the above Jordan Canonical Form, we may label the ordered basis that produce them, as $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6, \boldsymbol{v}_7\}$. Notice that $T(\boldsymbol{v}_3) = \boldsymbol{v}_2 \neq 0$.
We can also see that $T^2(\boldsymbol{v}_3) = T(\boldsymbol{v}_2) = 0$, and we are done.

SECTION B

Question 5

(a) (i) Applying the GramSchmidt Orthogonalisation process. We get

$$\begin{aligned} \boldsymbol{u}_1 &= F_1 \\ \boldsymbol{u}_2 &= F_2 - \frac{\langle F_1, F_2 \rangle}{\langle F_1, F_1 \rangle} F_1 = F_2 - \frac{3}{2} F_1 \\ \boldsymbol{u}_2 &= F_3 - \frac{\langle F_1, F_3 \rangle}{\langle F_1, F_1 \rangle} F_1 - \frac{\langle F_2 - \frac{3}{2}F_1, F_3 \rangle}{\langle F_2 - \frac{3}{2}F_1, F_2 - \frac{3}{2}F_1 \rangle} F_2 - \frac{3}{2} F_1 \\ &= F_3 - 3F_1 + 2(F_2 - \frac{3}{2}F_1) \\ &= F_3 - 6F_1 + 2F_2. \end{aligned}$$

Normalising $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, we have

$$\begin{aligned}
\boldsymbol{v}_1 &= \frac{\boldsymbol{u}_1}{\sqrt{\langle \boldsymbol{u}_1, \boldsymbol{u}_1\rangle}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\
\boldsymbol{v}_2 &= \frac{\boldsymbol{u}_2}{\sqrt{\langle \boldsymbol{u}_2, \boldsymbol{u}_2\rangle}} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\
\boldsymbol{v}_3 &= \frac{\boldsymbol{u}_3}{\sqrt{\langle \boldsymbol{u}_3, \boldsymbol{u}_3\rangle}} = \begin{pmatrix} -\frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \end{pmatrix}.
\end{aligned}$$

Hence $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a orthonormal basis for W.

(ii) Since $\dim V = 4$ and $\dim W = 3$, $\dim W^{\perp} = 1$.

(iii) We have,

$$\begin{aligned}
Q &= D - \langle \boldsymbol{v}_1, D\rangle \boldsymbol{v}_1 - \langle \boldsymbol{v}_2, D\rangle \boldsymbol{v}_2 - \langle D, \boldsymbol{v}_3\rangle \boldsymbol{v}_3 \\
&= D - \frac{1}{2}\boldsymbol{v}_1 + \frac{1}{2}\boldsymbol{v}_2 - \frac{1}{\sqrt{10}}\boldsymbol{v}_3 \\
&= \begin{pmatrix} 0.2 & 0.4 \\ -0.4 & -0.2 \end{pmatrix},
\end{aligned}$$

and thus

$$\begin{aligned}
P &= D - Q \\
&= \begin{pmatrix} -0.2 & 0.6 \\ 0.4 & 0.2 \end{pmatrix}.
\end{aligned}$$

(b) Since $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, ..., \boldsymbol{v}_n\}$ is an orthonormal basis for V, we can express $\boldsymbol{w}$ as $\boldsymbol{w} = \sum_{j=1}^{n} \langle \boldsymbol{w}, \boldsymbol{v}_j\rangle \boldsymbol{v}_j$. Hence,

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{w}\rangle &= \langle \mathbf{u}, \sum_{j=1}^{n} \langle \mathbf{w}, \mathbf{v}_j\rangle \mathbf{v}_j\rangle \\
&= \sum_{j=1}^{n} \langle \mathbf{w}, \mathbf{v}_j\rangle \langle \mathbf{u}, \mathbf{v}_j\rangle.
\end{aligned}$$

Question 6

(a) (i) We have

$$[T]_{\mathcal{B}} = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}.$$

Hence the characteristic equation of T is

$$\det(xI - [T]_{\mathcal{B}}) = (x-4)(x+2)^2.$$

Since

$$([T]_{\mathcal{B}} - 4)([T]_{\mathcal{B}} + 2) = \begin{pmatrix} 6 & -6 & 6 \\ 6 & -6 & 6 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the minimal polynomial of T, $m_T(x) = (x-4)(x+2)^2$. Hence T is not diagonalisable.

(ii) Since $c_T(x) = (x-4)(x+2)^2$ and $m_T(x) = (x-4)(x+2)^2$, the Jordan Canonical Form for T is

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}.$$

(b) Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be a ordered basis for $M_{22}(\mathbb{R})$.
Hence $[T]_{\mathcal{B}} = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix}$. Therefore

$$\begin{aligned} c_T(x) = \det(xI_4 - [T]_{\mathfrak{B}}) &= \det \begin{pmatrix} xI_2 - A & \mathbf{0} \\ \mathbf{0} & xI_2 - A \end{pmatrix} \\ &= (\det(xI_2 - A))^2 \\ &= (c_A(x))^2 \end{aligned}$$

Hence if $c_T(x) = 0$, we have $(c_A(x))^2 = 0$ which implies that $c_A(x) = 0$.
Also if $c_A(x) = 0$, then $c_T(x) = (c_A(x))^2 = 0$. Therefore T and A share the same eigenvalues.

Question 7

(i) For $\boldsymbol{v} \in \ker(T)$, $T(\boldsymbol{v}) = 0_V$. Hence $ST(\boldsymbol{v}) = S(0_V) = 0_V$. Therefore $\boldsymbol{v} \in \ker(ST)$.
Hence we have $\ker(T) \subseteq \ker(ST)$.

(ii) For $\boldsymbol{v} \in \mathcal{R}(ST)$, $\boldsymbol{v} = ST(\boldsymbol{w})$ for some $\boldsymbol{w} \in V$. Hence $\boldsymbol{v} = S(T(\boldsymbol{w})) \in \mathcal{R}(S)$.
Therefore $\mathcal{R}(ST) \subseteq \mathcal{R}(S)$.

(iii) From the previous parts we have $n_T \leq n_{ST}$ and $\operatorname{rank}(ST) \leq \operatorname{rank}(S)$.
Hence, by Rank Nullity Theorem, we have $n_S \leq n_{ST}$. This give us $\max(n_T, n_S) \leq n_{ST}$.

Notice that $\operatorname{rank}(ST) = \operatorname{rank}(S|_{\mathcal{R}(T)})$. By considering the linear operator $S|_{\mathcal{R}(T)} : \mathcal{R}(T) \to V$, we have $\operatorname{rank}(ST) = \dim(\mathcal{R}(T)) - \dim(\mathcal{R}(T) \cap \ker(S)) \geq \operatorname{rank}(T) - n_S$.
Hence, by Rank-Nullity Theorem, $n_{ST} = \dim V - \operatorname{rank}(ST) \leq \dim V - \operatorname{rank}(T) + n_S = n_T + n_S$.

Therefore we have $\max(n_S, n_T) \leq n_{ST} \leq n_T + n_S$.