MA2104 - Multivariable Calculus Suggested Solutions

(Semester 2: AY2018/19)

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Question 1.

(a) Notice the line passes through the origin.

$$\text{Distance} = \left| \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \times \frac{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}{\sqrt{2^2+1^2+2^2}} \right| = \frac{1}{3} \left| \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} \right| = \sqrt{2}$$

(b) (i) Let $f(x,y) = x^2 - y^2 + 3$.

$$\begin{aligned} f_x &= 2x \qquad f_y = -2y \\ f_x(3,3) &= 6 \qquad f_y(3,3) = -6 \\ z - 3 &= 6(x-3) - 6(y-3) \\ \pi : z &= 6x - 6y + 3 \end{aligned}$$

(ii) To find the intersection, we solve the equation : $x^2 - y^2 + 3 = 6x - 6y + 3$
$(x+y)(x-y) = 6(x-y)$ is true when $x = y$.
$\forall (x,y) \in \mathbb{R}^2,\ x = y \implies z = 3$ for both S and π. Thus ℓ_1 lies in both S and π.

(iii) If $x \neq y$, then the intersection yields $x + y = 6$.
Let $x = 6$ and $y = 0$. Then $z = 39$.

$$\ell_2 : \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 6-3 \\ 0-3 \\ 39-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 12 \end{pmatrix} \implies x - 3 = -y + 3 = \frac{z-3}{12}.$$

Question 2

(a) We first locate the critical points:

$$\begin{aligned} f_x &= 3x^2 + 3y \qquad & f_y &= 3y^2 + 3x \\ 3x^2 + 3y = 0 &\implies y = -x^2 \qquad & 3x^4 + 3x = 0 &\implies 3x(x^3+1) \end{aligned}$$

There are 2 critical points $(0,0)$ and $(-1,-1)$. Next, we calculate the second partial derivatives $D(x,y)$:

$$\begin{gathered} f_{xx} = 6x \qquad f_{yy} = 6y \qquad f_{xy} = 3 \\ D(x,y) = (6x)(6y) - 9 = 36xy - 9 \\ D(0,0) = -9 \implies \text{saddle point} \qquad D(-1,-1) = 27 \implies \text{local max} \end{gathered}$$

(b) By Lagrange Multipliers,

$$\begin{gathered} 2 = \lambda(2x + 2y) \\ 1 = \lambda(2x + 4y) \\ 2\lambda(x+y) = 4\lambda(x+2y) \implies x + y = 2x + 4y \implies x = -3y \\ \implies 5 = 9y^2 - 6y^2 + 2y^2 = 5y^2 \implies y = \pm 1, x = \mp 3 \\ f(-3,1) = -5 \implies min \qquad f(3,-1) = 5 \implies max \end{gathered}$$

Question 3

(a) Using the change of variables $u = x + y,\ v = y - 2x$, the vertices are transformed to $A(0,0),\ B(3,-6),\ C(3,3)$. AB is the line $v = -2u$ and AC is the line $v = u$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3 \implies \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3}.$$

$$\begin{aligned} \iint_R \sqrt{x+y}\ (y-2x)^2\ dxdy &= \int_0^3 \int_{-2u}^{u} \sqrt{u} v^2 \frac{1}{3}\ dvdu \\ &= \frac{1}{3} \int_0^3 \left[\frac{1}{3} \sqrt{u} v^3 \right]_{-2u}^{u} du \\ &= \int_0^3 u^{\frac{7}{2}}\ du \\ &= \left[\frac{2}{9} u^{\frac{9}{2}} \right]_0^3 \\ &= \frac{2}{9} \cdot 3^{\frac{9}{2}} \end{aligned}$$

(b) Find potential function f such that $\mathbf{F} = \nabla f$:

$f_x(x,y,z) = y\sin(z) \implies f(x,y,z) = xy\sin(z) + g(y,z) \implies f_y(x,y,z) = x\sin(z) + g_y(y,z)$
But $f_y(x,y,z) = x\sin(z)$ so $g_y(y,z) = 0 \implies g(y,z) = h(z) \implies f(x,y,z) = xy\sin(z) + h(z)$
$\implies f_z(x,y,z) = xy\cos(z) + h'(z) = xy\cos(z) \implies h'(z) = 0 \implies h(z) = K,$ a constant.
Hence, $f = xy\sin(z)$ (taking $K = 0$).

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}) - f(0,0,0) = \frac{\pi^2}{4}.$$

Question 4

(a) Let $P(x,y) = 7y - e^{\sin x}$, $Q(x,y) = 9x - \cos(y^3+7y)$. Then :

$$\begin{aligned}\int_C P\ dx\ + Q\ dy &= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA\\ &= \iint_D 9 - 7\ dA\\ &= 2\cdot\pi(2)^2\\ &= 8\pi\end{aligned}$$

(b) Using spherical coordinates, the equation of the cone can be written as

$$\sqrt{3}\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} = \rho\sin\phi$$

This gives $\tan\phi = \sqrt{3}$, or $\phi = \frac{\pi}{3}$. The equation of the sphere can be written as

$$\rho^2 = 2\rho\cos\phi \implies \rho = 2\cos\phi$$

Therefore the description of the solid E in spherical coordinates is

$$E = \{(\rho,\theta,\phi) \mid 0\le\theta\le 2\pi,\ 0\le\phi\le\frac{\pi}{3},\ 0\le\rho\le 2\cos\phi\}$$

$$\begin{aligned}\iiint_E dV &= \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{2\cos\phi} \rho^2\sin\phi\ d\rho d\phi d\theta\\ &= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{3}} \sin\phi\left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=2\cos\phi} d\phi\\ &= \frac{16\pi}{3}\int_0^{\frac{\pi}{3}} \sin\phi\cos^3\phi\ d\phi\\ &= \frac{16\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_0^{\frac{\pi}{3}}\\ &= \frac{5\pi}{4}\end{aligned}$$

Question 5

(a)

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)dzdydx = \iiint_E f(x,y,z)\ dV$$

where $E = \{(x,y,z) \mid -1 \leq x \leq 1, x^2 \leq y \leq 1, 0 \leq z \leq 1-y\}$. This description of E enables us to write projections onto the three coordinate planes as follows:

$$\begin{aligned}
\text{on the } xy\text{-plane:} \quad & \{(x,y) \mid -1 \leq x \leq 1,\ x^2 \leq y \leq 1\} \\
& = \{(x,y) \mid 0 \leq y \leq 1,\ -\sqrt{y} \leq x \leq \sqrt{y}\} \\
\text{on the } yz\text{-plane:} \quad & \{(y,z) \mid x^2 \leq y \leq 1,\ 0 \leq z \leq 1-y\} \\
& = \{(y,z) \mid 0 \leq z \leq 1,\ x^2 \leq y \leq 1-z\} \\
\text{on the } xz\text{-plane:} \quad & \{(x,z) \mid -1 \leq x \leq 1,\ 0 \leq z \leq 1-y\} \\
& = \{(x,z) \mid 0 \leq z \leq 1,\ -\sqrt{1-z} \leq x \leq \sqrt{1-z}\}
\end{aligned}$$

$$E = \{(x,y,z) \mid 0 \leq z \leq 1,\ -\sqrt{1-z} \leq x \leq \sqrt{1-z},\ x^2 \leq y \leq 1-z\}$$

Thus,

$$\iiint_E f(x,y,z)\ dV = \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} f(x,y,z)dydxdz$$

(b) A parametrisation of the curve is given by :

$$\mathbf{r}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \cos\theta\sin\theta \end{pmatrix},\ 0 \leq \theta \leq 2\pi.$$

Then $\mathbf{F}(\mathbf{r}(\theta)) = \langle \sin^3\theta, \cos\theta, \cos^3\theta\sin^3\theta \rangle$ and $\mathbf{r}'(\theta) = \langle -\sin\theta, \cos\theta, \cos^2\theta - \sin^2\theta \rangle$.

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(\theta)) \cdot \mathbf{r}'(\theta)\ d\theta \\
&= \int_0^{2\pi} \begin{pmatrix} \sin^3\theta \\ \cos\theta \\ \cos^3\theta\sin^3\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \\ \cos^2 - \sin^2\theta \end{pmatrix} d\theta \\
&= \int_0^{2\pi} -\sin^4\theta + \cos^2\theta + (\cos^3\theta\sin^3\theta)(\cos^2\theta - \sin^2\theta)\ d\theta \\
&= \int_0^{2\pi} -\sin^4\theta + \cos^2\theta + \frac{1}{8}\big(\sin^3(2\theta)\cos(2\theta)\big)\ d\theta \\
&= \frac{1}{64}\left[8\theta + 32\sin(2\theta) - 2\sin(4\theta) + \sin^4(2\theta)\right]_0^{2\pi} \\
&= \frac{\pi}{4}
\end{aligned}$$

Question 6

(a)

$$\begin{aligned}
\int_C g\nabla f \cdot d\mathbf{r} &= \iint_\Sigma \text{curl } (g\nabla f) \cdot d\mathbf{\Sigma} = \iint_\Sigma \text{curl } \left(\begin{pmatrix} gf_x \\ gf_y \\ gf_z \end{pmatrix} \right) \cdot d\mathbf{\Sigma} \\
&= \iint_\Sigma \begin{pmatrix} g_y f_z + g f_{zy} - g_z f_y - g f_{yz} \\ g_z f_x + g f_{xz} - g_x f_z - g f_{zx} \\ g_x f_y + g f_{yx} - g_y f_x - g f_{xy} \end{pmatrix} \cdot d\mathbf{\Sigma} \\
&= \iint_\Sigma \begin{pmatrix} g_y f_z - g_z f_y \\ g_z f_x - g_x f_z \\ g_x f_y - g_y f_x \end{pmatrix} \cdot d\mathbf{\Sigma}
\end{aligned}$$

On the other hand :

$$\begin{aligned}
\int_C f\nabla g \cdot d\mathbf{r} &= \iint_\Sigma \text{curl } (f\nabla g) \cdot d\mathbf{\Sigma} = \iint_\Sigma \text{curl } \left(\begin{pmatrix} fg_x \\ fg_y \\ fg_z \end{pmatrix} \right) \cdot d\mathbf{\Sigma} \\
&= \iint_\Sigma \begin{pmatrix} f_y g_z + f g_{zy} - f_z g_y - f g_{yz} \\ f_z g_x + f g_{xz} - f_x g_z - f g_{zx} \\ f_x g_y + f g_{yx} - f_y g_x - f g_{xy} \end{pmatrix} \cdot d\mathbf{\Sigma} \\
&= \iint_\Sigma \begin{pmatrix} f_y g_z - f_z g_y \\ f_z g_x - f_x g_z \\ f_x g_y - f_y g_x \end{pmatrix} \cdot d\mathbf{\Sigma} \\
&= -\int_C g\nabla f \cdot d\mathbf{r} \\
&= \int_{-C} g\nabla f \cdot d\mathbf{r}.
\end{aligned}$$

(b) Let $\mathbf{F} = \mathbf{F_1} + \mathbf{F_2}$ where

$$\mathbf{F_1} = \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\langle x, y, z\rangle \qquad \mathbf{F_2} = \langle 0, 0, z^2\rangle$$

S is the boundary of the ellipsoid E given by $x^2 + \frac{y^2}{4} + \frac{z^2}{9} \leq 1$.

$$\iint_S \mathbf{F_2} \cdot d\mathbf{S} = \iiint_E 2z \; dV = 0 \text{ by symmetry of } E$$

Since $\mathbf{F_1}$ is undefined at $(0, 0, 0)$, we introduce a unit sphere T centered at $(0, 0, 0)$ and calculate a modified flux. First note that

$$\begin{aligned}\operatorname{div}(\mathbf{F}) &= \frac{3}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{3x^2+3y^2+3z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} \\ &= 0.\end{aligned}$$

Then

$$\iint_S \mathbf{F_1} \cdot d\mathbf{S} - \iint_T \mathbf{F_1} \cdot d\mathbf{T} = \iiint_{E'} \operatorname{div}(\mathbf{F_1}) \; dV = 0$$
$$\therefore \iint_S \mathbf{F_1} \cdot d\mathbf{S} = \iint_T \mathbf{F_1} \cdot d\mathbf{T}$$

A parametrisation of T is given by

$$R(\phi, \theta) = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}, \; 0 \leq \phi \leq \pi, \; 0 \leq \theta \leq 2\pi.$$

and so

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_T \mathbf{F_1} \cdot d\mathbf{T} &= \int_0^{2\pi} \int_0^{\pi} \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} \; d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} 1 \; d\phi d\theta \\ &= 4\pi.\end{aligned}$$

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