NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Zheng Shaoxuan

MA1104 Multivariable Calculus
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Question 1

(i) The two straight lines have respective equations of

$$x = -\frac{y}{2} = -\frac{z}{5} \tag{1}$$
$$\frac{x}{2} = \frac{y}{4} = -\frac{z}{1}. \tag{2}$$

Substituting x from (1) into (2), we obtain $-y = y$ and $z = 10z$, and hence $y = 0$, $z = 0$ and $x = 0$. Hence the point of intersection of the two lines is $(0, 0, 0)$.

Alternatively consider that for any line with symmetric equation of the form $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$, (x_0, y_0, z_0) is a point on the line. Since (x_0, y_0, z_0) for both (1) and (2) are $(0, 0, 0)$, hence $(0, 0, 0)$ falls on both lines and hence is the point of intersection of the two lines.

(ii) The direction vector of the two lines are $\langle 1, -2, -5\rangle$ and $\langle 2, 4, -1\rangle$ respectively.
Hence, the normal vector of the plane which contains both lines is

$$\begin{aligned} & \langle 1, -2, -5\rangle \times \langle 2, 4, -1\rangle \\ = & \langle 22, -9, 8\rangle. \end{aligned}$$

Therefore, since $(0, 0, 0)$ lies on the plane, the equation of the plane is

$$22x - 9y + 8z = 0.$$

(iii) A vector which starts on the plane and ends on $P(2, 3, 1)$ is $\langle 2, 3, 1\rangle$ since $(0, 0, 0)$ lies on the plane. Therefore, the distance from the point P to the plane is the length of projection of $\langle 2, 3, 1\rangle$ onto the normal vector of the plane $\langle 22, -9, 8\rangle$, which is

$$\begin{aligned} & \frac{\langle 22, -9, 8\rangle \cdot \langle 2, 3, 1\rangle}{|\langle 22, -9, 8\rangle|} \\ = & \frac{25}{\sqrt{629}}. \end{aligned}$$

Question 2

(i) The domain of $f(x, y)$ is $\mathbb{R}$, as $\frac{x^2y^5}{4x^4+y^8}$ is defined for all points except $(0, 0)$, and $f(0, 0)$ is defined to be 0.

(ii) The limit exists. By conversion from rectangular coordinates to polar coordinates:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{r\to 0} \frac{r^7\cos^2\theta\sin^5\theta}{4r^4\cos^4\theta + r^8\sin^8\theta} \\
&= \lim_{r\to 0} \frac{r^3\cos^2\theta\sin^5\theta}{4\cos^4\theta + r^4\sin^8\theta} \\
&= \frac{0}{4\cos^4\theta + 0} \\
&= 0.
\end{aligned}$$

(iii) $f(x,y)$ is continuous since

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0).$$

(iv) $f_{xx}(0,0)$ exists. We first calculate $f_x(h,0)$ for all real h, by the definition of differentiation:

$$\begin{aligned}
f_x(h,0) &= \lim_{t\to 0} \frac{f(h+t,0) - f(h,0)}{t} \\
&= \lim_{t\to 0} \frac{0-0}{t} \\
&= 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
f_{xx}(0,0) &= \lim_{h\to 0} \frac{f_x(h,0) - f_x(0,0)}{h} \\
&= \lim_{h\to 0} \frac{0-0}{h} \\
&= 0.
\end{aligned}$$

Question 3

(i) The equation of the surface is $f(x,y,z) = 0$, where $f(x,y,z) = x^2 + y^2 - 2xy - x + 3y - 2z^2 + 4$. The gradient function of f at point $P(2,-3,-3)$ is

$$\begin{aligned}
\nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \\
&= \langle 2x - 2y - 1, 2y - 2x + 3, -4z \rangle \\
&= \langle 9, -7, 12 \rangle.
\end{aligned}$$

Therefore, the equation of the tangent plane of this surface at the point $P(2,-3,-3)$, is

$$\begin{aligned}
\langle x, y, z \rangle \cdot \nabla f &= \langle 2, -3, -3 \rangle \cdot \nabla f \\
9x - 7y + 12z &= 3.
\end{aligned}$$

(ii) The normal vector of the tangent plane in (i) is $\langle 9, -7, 12\rangle$. Therefore, the line in concern has a direction vector of $\langle 9, -7, 12\rangle$ and a point with position vector $\langle 2, -3, -3\rangle$ relative to the origin.

Therefore the parametric equation of the line is

$$\begin{cases} x = 2 + 9t; \\ y = -3 - 7t; \\ z = -3 + 12t. \end{cases}$$

Question 4

(i) From the given $f(x, y, z)$,

$$\begin{aligned} f(2, 5, -2) &= \sqrt[3]{4 + 25 - 2} \\ &= 3. \end{aligned}$$

Furthermore, at $(2, 5, -2)$,

$$\begin{aligned} \nabla f &= \left\langle \frac{1}{3}(x^2 + y^2 + z)^{-\frac{2}{3}}(2x), \frac{1}{3}(x^2 + y^2 + z)^{-\frac{2}{3}}(2y), \frac{1}{3}(x^2 + y^2 + z)^{-\frac{2}{3}} \right\rangle \\ &= \left\langle \frac{2}{3}x(x^2 + y^2 + z)^{-\frac{2}{3}}, \frac{2}{3}y(x^2 + y^2 + z)^{-\frac{2}{3}}, \frac{1}{3}(x^2 + y^2 + z)^{-\frac{2}{3}} \right\rangle \\ &= \left\langle \frac{4}{27}, \frac{10}{27}, \frac{1}{27} \right\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} f(x, y, z) &\approx f(2, 5, -2) + \nabla f \cdot \langle x - 2, y - 5, z + 2\rangle \\ &= 3 + \frac{4}{27}x - \frac{8}{27} + \frac{10}{27}y - \frac{50}{27} + \frac{1}{27}z + \frac{2}{27} \\ &= \frac{25}{27} + \frac{4}{27}x + \frac{10}{27}y + \frac{1}{27}z. \end{aligned}$$

(ii) By substitution of $x = 2.05$, $y = 4.96$, $z = -1.97$ into the above linear approximation,

$$\sqrt[3]{(2.05)^2 + (4.96)^2 - 1.97} \approx 2.99370 \text{ (5.d.p.)}.$$

Question 5

x, y and z are functions of t which satisfies both equations

$$\frac{x^2}{2} + \frac{y^2}{4} + z^2 = 1, \tag{3}$$
$$2z + y = 0. \tag{4}$$

By (4), $y = -2z$. Hence, by (3),

$$\begin{aligned} \frac{x^2}{2} + z^2 + z^2 &= 1 \\ x^2 &= 2 - 4z^2 \end{aligned}$$

Let $z = \frac{1}{\sqrt{2}}\cos t$. Then, $y = -2(\frac{1}{\sqrt{2}})\cos t = -\sqrt{2}\cos t$ and $x^2 = 2 - 4(\frac{1}{\sqrt{2}}\cos t)^2 = 2 - 2\cos^2 t = 2\sin^2 t$. Hence $x = \pm\sqrt{2}\sin t$.

Therefore,

$$\boldsymbol{r}(t) = \left\langle \sqrt{2}\sin t, -\sqrt{2}\cos t, \frac{1}{\sqrt{2}}\cos t \right\rangle \text{ , where } 0 \le t < 2\pi.$$

Note: For the case of $x = -\sqrt{2}\sin t$, as $-\sqrt{2}\sin t = \sqrt{2}\sin(-t)$, a parametrization of $\boldsymbol{r}(t) = \left\langle -\sqrt{2}\sin t, -\sqrt{2}\cos t, \frac{1}{\sqrt{2}}\cos t \right\rangle = \left\langle \sqrt{2}\sin(-t), -\sqrt{2}\cos(-t), \frac{1}{\sqrt{2}}\cos(-t) \right\rangle$, where $0 < (-t) \le 2\pi$, is essentially identical to the parametrization stated above. Hence we only need to consider the above stated solution.

Question 6

(i) Performing implicit partial differentiation with respect to x on the given equation, at the point $(-3, -1, 1)$,

$$\begin{aligned} (x-1)\frac{\partial z}{\partial x} + z + y\frac{1}{z}\frac{\partial z}{\partial x} &= 0 \qquad (5) \\ -5\frac{\partial z}{\partial x} &= -1 \\ \frac{\partial z}{\partial x} &= \frac{1}{5}. \end{aligned}$$

(ii) Performing implicit partial differentiation with respect to x on (5), at the point $(-3, -1, 1)$ with $\frac{\partial z}{\partial x} = \frac{1}{5}$,

$$\begin{aligned} (x-1)\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} + y\left(\frac{1}{z}\frac{\partial^2 z}{\partial x^2} - \frac{1}{z^2}\frac{\partial z}{\partial x}\right) &= 0 \\ -5\frac{\partial^2 z}{\partial x^2} &= -\frac{3}{5} \\ \frac{\partial^2 z}{\partial x^2} &= \frac{3}{25}. \end{aligned}$$

Question 7

Let $g(x, y) = x^2 - 2x + y^2 - 4y$. The gradient functions of f and g are

$$\begin{aligned} \nabla f &= \langle 2x, 2y \rangle \\ \nabla g &= \langle 2x - 2, 2y - 4 \rangle. \end{aligned}$$

By the Method of Lagrange Multiplier, to find the maximum and minimum of $f(x, y)$ subject to the constraint of $g(x, y) = 0$, $\nabla f = \lambda \nabla g$. Hence, the following system of equations must be evaluated:

$$\begin{aligned} 2x &= \lambda(2x - 2), \qquad (6) \\ 2y &= \lambda(2y - 4), \qquad (7) \\ x^2 - 2x + y^2 - 4y &= 0. \qquad (8) \end{aligned}$$

By making λ the subject of (6) and (7),

$$\begin{aligned} \frac{x}{x-1} &= \frac{y}{y-2} \\ xy-2x &= xy-y \\ 2x &= y. \end{aligned}$$

By substituting the above result into (8),

$$\begin{aligned} x^2-2x+4x^2-8x &= 0 \\ 5x(x-2) &= 0 \\ x=0 \quad &\text{or} \quad x=2. \end{aligned}$$

For $x=0$, $y=0$. For $x=2$, $y=4$. Hence the two critical points in concern are $(0,0)$ and $(2,4)$.
When $(x,y)=(0,0)$, $f(x,y)=0$.
When $(x,y)=(2,4)$, $f(x,y)=20$.
Therefore, the maximum value of $f(x,y)$ is 20 and the minimum value of $f(x,y)$ is 0, subject to the above constraint.

Question 8

Let $f(x,y)=y^2-x^2$. Hence, $f_x=-2x$ and $f_y=2y$.
The area of surface $z=f(x,y)$, over the domain $D=\{(r,\theta)|1\le r\le 2, 0\le\theta\le 2\pi\}$ is given by

$$\begin{aligned} &\iint_D \sqrt{1+f_x^2+f_y^2}\, dA \\ =& \int_0^{2\pi}\int_1^2 r\sqrt{1+4r^2}\, dr\, d\theta \\ =& [\theta]_0^{2\pi}\left[\frac{1}{12}(1+4r^2)^{\frac{3}{2}}\right]_1^2 \\ =& \frac{\pi}{6}(17^{\frac{3}{2}}-5^{\frac{3}{2}}). \end{aligned}$$

Question 9

(i) Since $\boldsymbol{F}(x,y,z)$ is a conservative vector field in $\mathbb{R}^3$,

$$\text{curl}\boldsymbol{F}=\langle 0,0,0\rangle,$$

$$\begin{cases} 3Ax^2y^2+B\cos(x+z)-9x^2y^2-B\cos(x+z)=0; \\ 2Axy^3-5y\sin(x+z)-2Axy^3+By\sin(x+z)=0; \\ 18xy^2z+B\cos(x+z)-6Axy^2z-5\cos(x+z)=0. \end{cases}$$

By inspection, $A=3$ and $B=5$.

(ii) Given that $f_x=6xy^3z+5y\cos(x+z)$, there exists a function $g(y,z)$ such that

$$\begin{aligned} f &= 3x^2y^3z+5y\sin(x+z)+g(y,z) \\ f_y &= 9x^2y^2z+5\sin(x+z)+g_y(y,z). \end{aligned}$$

Also, $f_y = 9x^2y^2z + 5\sin(x+z)$. Hence, $g_y(y,z) = 0$, i.e. there exists a function $h(z)$ such that $g(y,z) = h(z)$. Therefore,

$$\begin{aligned} f &= 3x^2y^3z + 5y\sin(x+z) + h(z) \\ f_z &= 3x^2y^3 + 5y\cos(x+z) + h_z(z). \end{aligned}$$

Also, $f_z = 3x^2y^3 + 5y\cos(x+z)$. Hence, $h_z(z) = 0$ i.e. $h(z) = D$ where D is an arbitrary constant. Therefore, a suitable $f(x,y,z)$ is

$$f = 3x^2y^3z + 5y\sin(x+z).$$

(iii) $\boldsymbol{r}(1) = \langle 2, 1, -2\rangle$ and $\boldsymbol{r}(0) = \langle 1, 0, 0\rangle$. Hence, by the Fundamental Theorem of Calculus for Line Integral,

$$\begin{aligned} \int_C \boldsymbol{F}\cdot\, d\boldsymbol{r} &= \int_C \nabla f\cdot\, d\boldsymbol{r} \\ &= f(\boldsymbol{r}(1)) - f(\boldsymbol{r}(0)) \\ &= f(2,1,-2) - f(1,0,0) \\ &= -24 - 0 \\ &= -24. \end{aligned}$$

Question 10

(i) Note: Green's Theorem cannot be used in (i) since the origin, which is enclosed by C_1, is not defined in the vector field $\boldsymbol{F}$.

C_1 can be represented by the following equation $\boldsymbol{r}(t) = \langle \cos t, \sin t\rangle$, where $0 \le t \le 2\pi$. Hence,

$$\begin{aligned} \oint_{C_1} \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_0^{2\pi} \boldsymbol{F}(\cos t, \sin t)\cdot \boldsymbol{r}'(t)\ dt \\ &= \int_0^{2\pi} \langle \cos t - \sin t, \cos t + \sin t\rangle \cdot \langle -\sin t, \cos t\rangle\ dt \\ &= \int_0^{2\pi} 1\ dt \\ &= 2\pi. \end{aligned}$$

In fact, the above answer will be the same regardless of the radius of the circle stated in C_1. Assuming instead, that C_1 has a radius of $\epsilon > 0$, then C_1 can be represented by the equation $\boldsymbol{r}(t) = \langle \epsilon\cos t, \epsilon\sin t\rangle$, where $0 \le t \le 2\pi$. Hence,

$$\begin{aligned} \oint_{C_1} \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_0^{2\pi} \boldsymbol{F}(\epsilon\cos t, \epsilon\sin t)\cdot \boldsymbol{r}'(t)\ dt \\ &= \int_0^{2\pi} \left\langle \frac{\cos t - \sin t}{\epsilon}, \frac{\cos t + \sin t}{\epsilon}\right\rangle \cdot \langle -\epsilon\sin t, \epsilon\cos t\rangle\ dt \\ &= \int_0^{2\pi} 1\ dt \\ &= 2\pi. \end{aligned}$$

This fact will be used in (ii).

(ii) The closed curve C_2 too encloses the origin. To use Green's Theorem, first define C_3 as a closed circle curve in the *clockwise* orientation, with center $(0,0)$ and radius $\epsilon > 0$, where ϵ approaches but is not equal to 0. Also define C_4 as a line connecting C_2 and C_3 together, pointing towards C_3, and define C_5 as the same line as C_4, but instead pointing towards C_2. Let C_6 be the combination of the 4 curves, C_2, C_4, C_3 and C_5. C_6 is a closed curve, where every point within the curve is defined since the origin is not enclosed within the curve. Let D be the area enclosed within C_6. We can hence apply Green's Theorem on C_6.

$$\begin{aligned}
\oint_{C_6} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_{C_6} \frac{x-y}{x^2+y^2}\, dx + \frac{x+y}{x^2+y^2}\, dy \\
&= \iint_D \frac{\partial}{\partial x}\frac{x+y}{x^2+y^2} - \frac{\partial}{\partial y}\frac{x-y}{x^2+y^2}\, dA \\
&= \iint_D \frac{(x^2+y^2)(1)-(x+y)(2x)}{(x^2+y^2)^2} - \frac{(x^2+y^2)(-1)-(x-y)(2y)}{(x^2+y^2)^2}\, dA \\
&= 0.
\end{aligned}$$

By definition,

$$\oint_{C_6} \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_4} \boldsymbol{F} \cdot d\boldsymbol{r} + \oint_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_5} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Since $-C_3$ is a circle of the anti-clockwise orientation,

$$\oint_{-C_3} \boldsymbol{F} \cdot d\boldsymbol{r} = 2\pi.$$

Since C_4 and C_5 are lines equal in magnitude and exactly opposite in direction,

$$\int_{C_4} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_5} \boldsymbol{F} \cdot d\boldsymbol{r} = 0.$$

Therefore,

$$\begin{aligned}
\oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_4} \boldsymbol{F} \cdot d\boldsymbol{r} + \oint_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_5} \boldsymbol{F} \cdot d\boldsymbol{r} &= 0 \\
\oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} &= \oint_{-C_3} \boldsymbol{F} \cdot d\boldsymbol{r} \\
&= 2\pi.
\end{aligned}$$

Question 11

Given $\boldsymbol{F}$,

$$\begin{aligned}
\text{div } \boldsymbol{F} &= (2-3x^2) + (-9y^2) + (-6z^2) \\
&= 2 - 3x^2 - 9y^2 - 6z^2.
\end{aligned}$$

By Divergence Theorem:

$$\begin{aligned}
\oiint_S \boldsymbol{F} \cdot\ d\boldsymbol{S} &= \iiint_R \text{div } \boldsymbol{F}\, dV \\
&= \iiint_R 2 - 3x^2 - 9y^2 - 6z^2\ dV.
\end{aligned}$$

Therefore, the given surface integral attains the maximum value when

$$R = \{(x,y,z) \in \mathbb{R}^3 \mid 2 - 3x^2 - 9y^2 - 6z^2 > 0\}$$

Question 12

Let P be the point $(2s_1, 4+15s_1, -2+6s_1)$ and R be the point $(t_1, 6t_1, 2t_1)$, where s_1 and t_1 are constants to be determined.
By the given cross product, $\overrightarrow{RP} = \lambda\langle -6, 2, -3\rangle$, where λ is a constant to be determined.
Since the distance from P to R is 2,

$$\begin{aligned} \lambda\sqrt{6^2+2^2+3^2} &= 2 \\ \therefore \lambda &= \frac{2}{7}. \end{aligned}$$

Since $\overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR}$,

$$\frac{2}{7}\langle -6, 2, -3\rangle = \langle 2s_1 - t_1, 4+15s_1-6t_1, -2+6s_1-2t_1\rangle$$

$$\begin{cases} 2s_1 - t_1 = -\frac{12}{7} \\ 4 + 15s_1 - 6t_1 = \frac{4}{7} \\ -2 + 6s_1 - 2t_1 = -\frac{6}{7}. \end{cases}$$

The system of equations above yield a unique solution for each of s_1 and t_1:

$$s_1 = \frac{16}{7} \qquad t_1 = \frac{44}{7}.$$

Therefore, the point P is

$$\left(\frac{32}{7}, \frac{268}{7}, \frac{82}{7}\right),$$

and the point R is

$$\left(\frac{44}{7}, \frac{264}{7}, \frac{88}{7}\right).$$

Question 13

Completing the square,

$$\begin{aligned} x^2+(x+y)^2+y^2 &= 2x^2+2xy+2y^2 \\ &= 2(x+\frac{y}{2})^2 - \frac{y^2}{2} + 2y^2 \\ &= (\sqrt{2}x + \frac{\sqrt{2}}{2}y)^2 + (\sqrt{\frac{3}{2}}y)^2. \end{aligned}$$

Implementing a change of variables,

$$\begin{cases} u = \sqrt{2}x + \frac{\sqrt{2}}{2}y & \text{where } u \in \mathbb{R}; \\ v = \frac{\sqrt{3}}{\sqrt{2}}y & \text{where } v \in \mathbb{R}. \end{cases}$$

By making x and y the subject, we have

$$\begin{cases} x = \frac{\sqrt{2}}{2}u - \frac{1}{\sqrt{6}}v; \\ y = \frac{\sqrt{2}}{\sqrt{3}}v. \end{cases}$$

The Jacobian is

$$\begin{vmatrix} \frac{\sqrt{2}}{2} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{vmatrix} = \frac{1}{\sqrt{3}}.$$

Therefore,

$$\begin{aligned} J &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)}\frac{1}{\sqrt{3}}\ du\ dv \\ &= \frac{1}{\sqrt{3}}\int_{-\infty}^{\infty} e^{-u^2}\ du\int_{-\infty}^{\infty} e^{-v^2}\ dv \\ &= \frac{\pi}{\sqrt{3}}. \end{aligned}$$