NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Teo Wei Hao

ST2131/MA2216 Probability
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Question 1

(i) Let A_1 be the event that the last 4 cards drawn are aces. We count $\mathbb{P}(A_1)$ by multiplying the probability of obtaining an ace in each of the last 4 slots successively. Thus,

$$\mathbb{P}(A_1) = \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49} \quad = \quad \frac{1}{270725}.$$

(ii) Let A_2 be the event that the 4 aces are drawn consecutively. We count outcomes in A_2 by grouping the 4 aces together (under permutation), and then permutate the 49 groups (48 individual cards and 1 aces group). Thus,

$$\mathbb{P}(A_2) = \frac{49! \times 4!}{52!} \quad = \quad \frac{1}{5525}.$$

(iii) Let A_3 be the event that the first 2 cards drawn are aces, and A_4 be the event that the last 2 cards drawn are aces. Using similar argument as (1i.), we have,

$$\begin{aligned}\mathbb{P}(A_4|A_3) = \frac{\mathbb{P}(A_4A_3)}{\mathbb{P}(A_3)} \quad &= \quad \left(\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49}\right) \div \left(\frac{4}{52} \times \frac{3}{51}\right) \\ &= \quad \frac{1}{1225}.\end{aligned}$$

(iv) Let A_5 be the event that the second, third and fourth cards drawn are aces, and A_6 be the event that the first card drawn was an ace. Then similarly,

$$\begin{aligned}\mathbb{P}(A_6|A_5) = \frac{\mathbb{P}(A_6A_5)}{\mathbb{P}(A_5)} \quad &= \quad \left(\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49}\right) \div \left(\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50}\right) \\ &= \quad \frac{1}{49}.\end{aligned}$$

(v) Let A_7 be the event that 10 cards are allocated, with the the last of the 10 the only ace, and A_4 is as defined in (1iii.). Then $\mathbb{P}(A_7A_4)$ can be found by allocating 9 non-aces successively, followed by 3 aces, which a total of 12 slots are allocated. Similarly done for $\mathbb{P}(A_7)$. Thus we have,

$$\begin{aligned}\mathbb{P}(A_4|A_7) = \frac{\mathbb{P}(A_4A_7)}{\mathbb{P}(A_7)} \quad &= \quad \left(\frac{P_9^{48} \times P_3^4}{P_{12}^{52}}\right) \div \left(\frac{P_9^{48} \times P_1^4}{P_{10}^{52}}\right) \\ &= \quad \frac{3 \times 2}{42 \times 41} \\ &= \quad \frac{1}{287}.\end{aligned}$$

(vi) Using A_7 as defined in (1v.), and let A_8 be the event that the 9th card is the king of diamond. Similarly by allocating cards successively, we get

$$\begin{aligned}\mathbb{P}(A_8|A_7) = \frac{\mathbb{P}(A_8A_7)}{\mathbb{P}(A_7)} &= \left(\frac{1 \times P_8^{47} \times P_1^4}{P_{10}^{52}}\right) \div \left(\frac{P_9^{48} \times P_1^4}{P_{10}^{52}}\right) \\ &= \frac{1}{48}.\end{aligned}$$

Question 2

(i) True.
We have $\mathbb{P}(A) < \mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}$. Since $\mathbb{P}(A), \mathbb{P}(B) > 0$, we get $\mathbb{P}(B) < \frac{\mathbb{P}(AB)}{\mathbb{P}(A)} = \mathbb{P}(B|A)$.

(ii) False.
Let sample space $S = \{1, 2, 3\}$ have equally likely outcomes. Let $A = \{1, 2\}$, $B = \{3\}$, $C = \{3\}$. This give us $\mathbb{P}(A) = \frac{2}{3}$, $\mathbb{P}(B) = \frac{1}{3}$, $\mathbb{P}(A|C) = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}\{3\}} = 0$ and $\mathbb{P}(B|C) = \frac{\mathbb{P}\{3\}}{\mathbb{P}\{3\}} = 1$.

Then we have $\mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) \neq 0$, and $\mathbb{P}(A) > \mathbb{P}(B)$. However, $\mathbb{P}(A|C) < \mathbb{P}(B|C)$.

(iii) True.
We have,

$$\begin{aligned}\mathbb{P}(B|A) &= \mathbb{P}(B|A^c) \\ \frac{\mathbb{P}(AB)}{\mathbb{P}(A)} &= \frac{\mathbb{P}(A^cB)}{\mathbb{P}(A^c)} = \frac{\mathbb{P}(B) - \mathbb{P}(AB)}{1 - \mathbb{P}(A)} \\ \mathbb{P}(AB)(1 - \mathbb{P}(A)) &= (\mathbb{P}(B) - \mathbb{P}(AB))\mathbb{P}(A) \\ \mathbb{P}(AB) - \mathbb{P}(AB)\mathbb{P}(A) &= \mathbb{P}(B)\mathbb{P}(A) - \mathbb{P}(AB)\mathbb{P}(A) \\ \mathbb{P}(AB) &= \mathbb{P}(B)\mathbb{P}(A).\end{aligned}$$

Thus A and B are independent.

Note: Since $\mathbb{P}(B|A^c)$ is given in the question, it is well-defined. Therefore $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ is a valid denominator even though we are not given $\mathbb{P}(A) \neq 1$ explicitly.

Question 3

For any given $n \geq N - 1$, let A_i be the event that there is at least 1 type-i coupon obtained in the first n coupons, $i = 1, 2, \ldots, N$. Thus using the Inclusion-Exclusion Principle, we have,

$$\begin{aligned}\mathbb{P}\{T > n\} &= \mathbb{P}((A_1A_2 \cdots A_N)^c) \\ &= \mathbb{P}(A_1^c \cup A_2^c \cup \cdots \cup A_N^c) \\ &= \sum_{i=1}^{N-1} \left((-1)^{i+1} \sum_{\substack{\forall\ j \in B_i,\ s_j \in B_N \\ j<k \to s_j<s_k}} \mathbb{P}(A_{s_1}^c A_{s_2}^c \cdots A_{s_i}^c) \right),\end{aligned}$$

where $B_k = \{1, 2, \ldots, k\}$, $k = 1, 2, \ldots, N$. Since it is equally likely to obtain any of the N types of coupon, by symmetry, $A_{s_1}^c A_{s_2}^c \cdots A_{s_i}^c = A_1^c A_2^c \cdots A_i^c$. Also there are $\binom{N}{i}$ distinct $A_{s_1}^c A_{s_2}^c \cdots A_{s_i}^c$.

Lastly, $\mathbb{P}(A_1^c A_2^c \cdots A_i^c) = \mathbb{P}\{\text{all coupons are type-}(i+1)\text{ and above}\} = \left(\frac{N-i}{N}\right)^n$. Thus continuing, we have,

$$\begin{aligned}
\mathbb{P}\{T > n\} &= \sum_{i=1}^{N-1} \binom{N}{i} \mathbb{P}(A_1^c A_2^c \cdots A_i^c)(-1)^{i+1} \\
&= \sum_{i=1}^{N-1} \binom{N}{i} \left(\frac{N-i}{N}\right)^n (-1)^{i+1}, \qquad n = N-1, N, N+1, \ldots.
\end{aligned}$$

Next, let X be the r.v. of the number of distinct types of coupons in the first n selections. Let C_i be the event that there is at least 1 type-i coupon obtained in the first n coupons, where there is only x distinct types instead of N, $i = 1, 2, \ldots, x$. Thus,

$$\begin{aligned}
f_X(x) &= \sum_{\substack{\forall\ j \in B_N,\ s_j \in B_N \\ j<k \to s_j < s_k}} \mathbb{P}(A_{s_1} A_{s_2} \cdots A_{s_x} A_{s_{x+1}}^c \cdots A_{s_n}^c) \\
&= \binom{N}{x} \mathbb{P}(C_1 C_2 \cdots C_x) \\
&= \binom{N}{x} [1 - \mathbb{P}((C_1 C_2 \cdots C_x)^c)] \\
&= \binom{N}{x} \left[1 - \sum_{i=1}^{x-1} \binom{x}{i} \left(\frac{x-i}{x}\right)^n (-1)^{i+1}\right].
\end{aligned}$$

This give us the p.d.f. of X to be,

$$f_X(x) = \begin{cases} \binom{N}{x} \left[1 - \sum_{i=1}^{x-1} \binom{x}{i} \left(\frac{x-i}{x}\right)^n (-1)^{i+1}\right], & x = 1, 2, \ldots, \min(N, n); \\ 0, & \text{otherwise.} \end{cases}$$

Question 4

(i) When $0 < y < 1$, we have,

$$F_{X^2}(y) = \mathbb{P}\{X^2 < y\} = \mathbb{P}\{-\sqrt{y} < X < \sqrt{y}\} = \mathbb{P}\{X < \sqrt{y}\} = F_X(\sqrt{y}).$$

This give us,

$$\begin{aligned}
f_{X^2}(y) = \frac{d}{dy} F_{X^2}(y) &= \frac{d}{dy} F_X(\sqrt{y}) \\
&= f(\sqrt{y}) \frac{1}{2\sqrt{y}} \\
&= (1) \frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}.
\end{aligned}$$

Therefore the p.d.f. of X^2 is,

$$f_{X^2}(y) = \begin{cases} \frac{1}{2\sqrt{y}}, & 0 < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Using (4i.), we get

$$\begin{aligned} F_{X^2}(y) = \int_{-\infty}^{y} f_{X^2}(y)\ dy &= \int_0^y \frac{1}{2\sqrt{y}}\ dy \\ &= [\sqrt{y}]_0^y = \sqrt{y}. \end{aligned}$$

Let y' be the median of X^2. Then we have $\frac{1}{2} = F_{X^2}(y') = \sqrt{y'}$, and thus $y' = \frac{1}{4}$.

Question 5

(i) Notice that $\{x > 0,\ y > 0,\ x + y < 1\} = \{y \in (0,1),\ x \in (0, 1-y)\}$.
Since $f(x,y)$ is given to be the joint p.d.f. of X and Y , we have

$$\begin{aligned} 1 = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\ dx\ dy &= \int_0^1 \int_0^{1-y} cxy^2\ dx\ dy \\ &= \int_0^1 cy^2 \left[\frac{1}{2}x^2\right]_0^{1-y}\ dy \\ &= \frac{1}{2}c\int_0^1 y^2 - 2y^3 + y^4\ dy \\ &= \frac{1}{2}c\left[\frac{1}{3}y^3 - \frac{1}{2}y^4 + \frac{1}{5}y^5\right]_0^1 = \frac{c}{60}. \end{aligned}$$

Thus $c = 60$.
As a by-product, we obtain $f_Y(y) = 30y^2(1-y)^2$ for $0 < y < 1$, and $f_Y(y) = 0$ otherwise.

(ii) Notice that $\{X > Y\} = \{y \in (0, \frac{1}{2}),\ x \in (y, 1-y)\}$.
Thus,

$$\begin{aligned} \mathbb{P}\{X > Y\} = \int_0^{\frac{1}{2}}\int_y^{1-y} f(x,y)\ dx\ dy &= \int_0^{\frac{1}{2}}\int_y^{1-y} 60xy^2\ dx\ dy \\ &= \int_0^{\frac{1}{2}} 60y^2\left[\frac{1}{2}x^2\right]_y^{1-y}\ dy \\ &= 30\int_0^{\frac{1}{2}} y^2 - 2y^3\ dy \\ &= 30\left[\frac{1}{3}y^3 - \frac{1}{2}y^4\right]_0^{\frac{1}{2}} = \frac{5}{16}. \end{aligned}$$

(iii) From (5i.), we have the marginal p.d.f. of Y to be given by,

$$f_Y(y) = \begin{cases} 30y^2(1-y)^2, & 0 < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

(iv) Given that $Y = y$ with $0 < y < 1$, we have for $0 < x < 1 - y$,

$$\begin{aligned} f_{X|Y}(x|y) = \frac{f(x,y)}{f_y(y)} &= \frac{60xy^2}{30y^2(1-y)^2} \\ &= \frac{2x}{(1-y)^2}. \end{aligned}$$

Thus the conditional p.d.f. of X given that $Y = y$ with $0 < y < 1$ is,

$$f_{X|Y}(x|y) = \begin{cases} \frac{2x}{(1-y)^2}, & 0 < x < 1-y; \\ 0, & \text{otherwise.} \end{cases}$$

(v) We have,

$$\begin{aligned} E(X|Y=y) = \int_{\mathbb{R}} x f_{X|Y}(x|y)\, dx &= \int_0^{1-y} \frac{2x^2}{(1-y)^2}\, dx \\ &= \left[\frac{2x^3}{3(1-y)^2}\right]_0^{1-y} = \frac{2}{3}(1-y). \end{aligned}$$

(vi) No.
Since $f_{X|Y}(x|y)$ varies when y varies, there exists $0 < y < 1$ such that $f_{X|Y}(x|y) \neq f_X(x)$. Thus in general, $f_X(x)f_Y(y) \neq f_{X|Y}(x|y)f_Y(y) = f(x,y)$. Therefore X and Y are not independent.

Question 6

$$f_X(x) = \begin{cases} 1, & x = 0; \\ 0, & \text{otherwise} \end{cases}$$

is an example.

Note: Any even p.d.f. $f_X(x)$ will satisfy the condition, as there is no restriction on whether the function is discrete/continuous, etc. Quoting common distributions, e.g. $X \sim N(0,1)$ is possible too.

Question 7

(i) We notice that $X_1, X_2, \ldots, X_9 \sim \text{Exp}(1)$. For $i = 1, 2, \ldots, 9$, we have,

$$\begin{aligned} F_{X_i}(x) = \int_{-\infty}^{x} f(x)\, dx &= \int_0^x e^{-x}\, dx \\ &= 1 - e^{-x}. \end{aligned}$$

Thus for a fixed M, we have $Y \sim B(9, 1-e^{-M})$. Therefore the p.d.f. of Y is,

$$f_Y(y) = \begin{cases} \binom{9}{y}(1-e^{-M})^y(e^{-M})^{9-y}, & y = 0, 1, \ldots, 9; \\ 0, & \text{otherwise.} \end{cases}$$

Since Y is a binomial r.v., we can directly conclude that $E(Y) = 9(1-e^{-M})$.

(ii) We have,

$$\begin{aligned} E(Y) &= E(Y \mid M=1)\mathbb{P}\{M=1\} + E(Y \mid M=2)\mathbb{P}\{M=2\} + E(Y \mid M=3)\mathbb{P}\{M=3\} \\ &= 9(1-e^{-1})\left(\frac{1}{8}\right) + 9(1-e^{-2})\left(\frac{5}{8}\right) + 9(1-e^{-3})\left(\frac{1}{4}\right) \\ &= 9 - \frac{9}{8}e^{-1} - \frac{45}{8}e^{-2} - \frac{9}{4}e^{-3}. \end{aligned}$$