NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Teo Wei Hao

ST2131/MA2216 Probability
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Question 1

(a) Let X_A, X_B, X_{AB}, X_O be the events that the next baby born has blood A, B, AB, O respectively. With reference to the population statistic, we have

$$\begin{cases} \mathbb{P}(X_O) &= \mathbb{P}(X_A) \\ \mathbb{P}(X_B) &= \frac{1}{10}\mathbb{P}(X_A) \\ \mathbb{P}(X_B) &= 2\mathbb{P}(X_{AB}) \\ \mathbb{P}(X_A)+\mathbb{P}(X_B)+\mathbb{P}(X_{AB})+\mathbb{P}(X_O) &= 1. \end{cases}$$

Solving the system of linear equations, we get $\mathbb{P}(X_{AB}) = \frac{1}{43}$.

(b) Let X be r.v. of the number of wrongly entered values chosen. Then $X \sim H(3, 10, 4)$.
Since the error lies solely in the sign of the values, 2 wrongly signed values will cancel the error out in multiplication. Thus we have

$$\begin{aligned} \mathbb{P}\{\text{result value has no error}\} = \mathbb{P}\{X = 0\} + \mathbb{P}\{X = 2\} &= \frac{\binom{4}{0}\binom{6}{3}}{\binom{10}{3}} + \frac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}} \\ &= \frac{1}{6} + \frac{3}{10} = \frac{7}{15}. \end{aligned}$$

Question 2

(a) Let X_{IJ} denotes the event that I and J are firm friends.
Let V_1, V_2, V_3, V_4 be the events $X_{AC}X_{CD}$, $X_{AB}X_{BD}$, $X_{AB}X_{BC}X_{CD}$, $X_{AC}X_{CB}X_{BD}$ respectively.
Let Y be r.v. of the number of common couples in each outcome of an event that are firm friends.
This give us for any event X, $\mathbb{P}(X) = 0.8^Y$.
It is direct to see that $Y(V_1) = Y(V_2) = 2$, and $Y(V_3) = Y(V_4) = 3$.
For $1 \le i < j \le 4$ such that $(i, j) \ne (3, 4)$, we have $Y(V_iV_j) = 4$, while $Y(V_3V_4) = 5$.
For any distinct $1 \le i < j < k \le 4$, we have $Y(V_iV_jV_k) = 5$, and $Y(V_1V_2V_3V_4) = 5$.

Let Z denotes the event that D hears rumours that A hears.
Now Z occurs given that D quarreled with A iff least one of the events V_1, V_2, V_3, V_4 occurs.
Thus using Inclusion-Exclusion Principle, we have,

$$\begin{aligned} \mathbb{P}(Z \mid X_{AD}^c) &= \mathbb{P}(V_1 \cup V_2 \cup V_3 \cup V_4) \\ &= \sum_{i=1}^{4} \mathbb{P}(V_i) - \sum_{1\le i<j\le 4} \mathbb{P}(V_iV_j) + \sum_{1\le i<j<k\le 4} \mathbb{P}(V_iV_jV_k) - \mathbb{P}(V_1V_2V_3V_4) \\ &= \left(2\cdot 0.8^2 + 2\cdot 0.8^3\right) - \left(5\cdot 0.8^4 + 0.8^5\right) + \left(4\cdot 0.8^5\right) - \left(0.8^5\right) = 0.91136. \end{aligned}$$

(b) Similarly to (2a.), Z occurs given B and C quarreled iff at least one of the independent events X_{AD}, V_1, V_2 occurs. Thus,

$$\begin{aligned}
\mathbb{P}(Z \mid X_{BC}^c) &= 1 - \mathbb{P}(X_{AD}^c V_1^c V_2^c) \\
&= 1 - \mathbb{P}(X_{AD}^c)\mathbb{P}(V_1^c)\mathbb{P}(V_2^c) \\
&= 1 - [1 - \mathbb{P}(X_{AD})][1 - \mathbb{P}(V_1)][1 - \mathbb{P}(V_2)] \\
&= 1 - (1 - 0.8)(1 - 0.8^2)(1 - 0.8^2) = 0.97408.
\end{aligned}$$

(c) Using result of (2a.), we have,

$$\mathbb{P}(Z) = \mathbb{P}(Z \mid X_{AD})\mathbb{P}(X_{AD}) + \mathbb{P}(Z \mid X_{AD}^c)\mathbb{P}(X_{AD}^c) = (1)(0.8) + (0.91136)(0.2) = 0.982272.$$

Question 3

(a) We have $a_1 f_X(a_1) + a_2 f_X(a_2) = E(X) = E(Y) = a_1 f_Y(a_1) + a_2 f_Y(a_2)$.
Also, $f_X(a_1) + f_X(a_2) = 1 = f_Y(a_1) + f_Y(a_2)$.
Thus, $(a_2 - a_1) f_X(a_2) = (a_2 - a_1) f_Y(a_2)$, i.e. $(a_2 - a_1)[f_X(a_2) - f_Y(a_2)] = 0$.
Since $a_1 \neq a_2$, we have $f_X(a_2) = f_Y(a_2)$.
This implies that $f_X(a_1) = 1 - f_X(a_2) = 1 - f_Y(a_2) = f_Y(a_1)$.
Thus X and Y are identically distributed.

(b) We have

$$\begin{aligned}
1 = \int_{\mathbb{R}} f_n(x)\, dx &= \int_{c_n}^{\infty} \frac{c_n}{x^{n+1}}\, dx \\
&= \left[\frac{-c_n}{n x^n}\right]_{c_n}^{\infty} = \frac{1}{n} c_n^{1-n}.
\end{aligned}$$

This give us for $n > 1$, $c_n = n^{\frac{1}{1-n}}$.
Thus the only condition on c_1 comes from $f_1(x) \geq 0$ for all $x \in \mathbb{R}$, and so $c_1 \in \mathbb{R}^+$.

$$\begin{aligned}
E(X_1) = \int_{\mathbb{R}} x f_1(x)\, dx &= \int_{c_1}^{\infty} x \left(\frac{c_1}{x^{1+1}}\right) dx \\
&= \int_{c_1}^{\infty} \frac{c_1}{x}\, dx \\
&= [c_1 \ln x]_{c_1}^{\infty} = \infty.
\end{aligned}$$

Thus $E(X_1)$ does not exists.
For $n > 1$, we have,

$$\begin{aligned}
E(X_n) = \int_{\mathbb{R}} x f_n(x)\, dx &= \int_{n^{\frac{1}{1-n}}}^{\infty} n^{\frac{1}{1-n}} \left(\frac{1}{x^n}\right) dx \\
&= \left[n^{\frac{1}{1-n}} \left(\frac{-1}{(n-1)x^{n-1}}\right)\right]_{n^{\frac{1}{1-n}}}^{\infty} \\
&= n^{\frac{1}{1-n}} \left(\frac{n}{n-1}\right) = \frac{1}{n-1} \left(n^{\frac{n-2}{n-1}}\right).
\end{aligned}$$

Question 4

(a) X and Y are independent r.v., such that $X, Y \sim U(0,1)$.
For $z < 0$,

$$\begin{aligned}
\mathbb{P}\{Z \le z \mid X - Y < 0\} = \mathbb{P}\{\frac{X}{X-Y} \le z \mid X < Y\} &= \mathbb{P}\{X \ge z(X-Y) \mid X < Y\} \\
&= \mathbb{P}\{zY \ge (z-1)X \mid X < Y\} \\
&= \mathbb{P}\{\frac{z}{z-1}Y \le X \mid X < Y\} \\
&= \frac{\mathbb{P}\{\frac{z}{z-1}Y \le X \le Y\}}{\mathbb{P}\{X < Y\}}.
\end{aligned}$$

Since X and Y are identically distributed, $\mathbb{P}\{X < Y\} = \mathbb{P}\{Y > X\}$.
As X and Y are continuous r.v., $\mathbb{P}\{X = Y\} = 0$, and so $\mathbb{P}\{X < Y\} = \mathbb{P}\{Y > X\} = \frac{1}{2}$. Also,

$$\begin{aligned}
\mathbb{P}\{\frac{z}{z-1}Y \le X \le Y\} &= \int_0^1 \int_{\frac{z}{z-1}y}^{y} 1 \; dx \; dy \\
&= \int_0^1 \frac{1}{1-z} y \; dy \\
&= \left[\left(\frac{1}{1-z}\right)\left(\frac{y^2}{2}\right)\right]_0^1 = \frac{1}{2(1-z)}.
\end{aligned}$$

Therefore $\mathbb{P}\{Z \le z \mid X - Y < 0\} = \frac{1}{2(1-z)} \div \frac{1}{2} = 1 - \frac{z}{z-1}$.

Hence, for $z < 0$, we have $\mathbb{P}\{Z \le z,\ X - Y > 0\} = 0$. Thus,

$$\begin{aligned}
F_Z(z) = \mathbb{P}\{Z \le z\} &= \mathbb{P}\{Z \le z,\ X - Y > 0\} + \mathbb{P}\{Z \le z,\ X - Y < 0\} \\
&= \mathbb{P}\{Z \le z,\ X - Y < 0\} = \frac{1}{2(1-z)}.
\end{aligned}$$

(b) Continuing from (4a.), we see that given $X - Y < 0$, we must have $Z < 0$.
Thus for $z < 0$, we have $f_Z(z) = \frac{d}{dz}\left(\frac{1}{2(1-z)}\right) = \frac{1}{2(1-z)^2}$.

Now given that $X - Y > 0$, then $Z = \frac{X}{X-Y} = 1 + \frac{Y}{X-Y} \ge 1$. Thus for $0 \le z \le 1$, $f_Z(z) = 0$.
For $z > 1$, we have $\mathbb{P}\{Z \le z,\ X - Y < 0\} = 0$, and so,

$$\begin{aligned}
F_Z(z) = \mathbb{P}\{Z \le z\} &= \mathbb{P}\{Z \le z,\ X - Y > 0\} + \mathbb{P}\{Z \le z,\ X - Y < 0\} \\
&= \mathbb{P}\{Z \le z,\ X - Y > 0\} \\
&= \mathbb{P}\{X \le z(X-Y),\ X > Y\} \\
&= \mathbb{P}\{zY \le (z-1)X,\ X > Y\} \\
&= \mathbb{P}\{Y \le \frac{z-1}{z}X, X > Y\} \\
&= \mathbb{P}\{Y \le \frac{z-1}{z}X\} \\
&= \int_0^1 \int_0^{\frac{z-1}{z}x} 1 \; dy \; dx \\
&= \int_0^1 \frac{z-1}{z} x \; dx \\
&= \left[\left(\frac{z-1}{z}\right)\left(\frac{x^2}{2}\right)\right]_0^1 = \frac{1}{2}\left(1 - \frac{1}{z}\right).
\end{aligned}$$

Thus $f_Z(z) = \dfrac{1}{2z^2}$.

Therefore the p.d.f of Z is:

$$f_Z(z) = \begin{cases} \frac{1}{2(1-z)^2}, & z < 0; \\ 0, & 0 \le z \le 1; \\ \frac{1}{2z^2}, & 1 < z. \end{cases}$$

Question 5

(a) We are given that $X \sim P(10)$. Notice that the hatching of each egg is Bernoulli distributed, with probability 0.3. Therefore the conditional distribution of Y given $X = x$ is $Y|(X = x) \sim B(x, 0.3)$.

(b) We have,

$$\begin{aligned} \mathbb{P}\{X = 5, Y = 3\} = \mathbb{P}\{Y = 3 \mid X = 5\}\mathbb{P}\{X = 5\} &= \left[\binom{5}{3}(0.3)^3(0.7)^2\right]\left(e^{-10}\frac{10^5}{5!}\right) \\ &= 5.0053 \times 10^{-3}. \end{aligned}$$

Since $\mathbb{P}\{Y > X\} = 0$, we have $\mathbb{P}\{X = 8, Y = 10\} = 0$.

(c) Using the fact that $Y|(X = x) \sim B(x, 0.3)$, we have

$$\begin{aligned} E(Y \mid X = x) &= 0.3x, \\ E(Y^2 \mid X = x) = \mathrm{Var}(Y \mid X = x) + E(Y \mid X = x)^2 &= 0.21x + 0.09x^2. \end{aligned}$$

Thus using $X \sim P(10)$, we get $E(Y) = E(E(Y \mid X)) = E(0.3X) = 3$.
Also, $E(X^2) = \mathrm{Var}(X) + E(X)^2 = 110$, and so we have,

$$\begin{aligned} \mathrm{Var}(Y) = E(Y^2) - E(Y)^2 &= E(E(Y^2 \mid X)) - 9 \\ &= E(0.21X + 0.09X^2) - 9 \\ &= 0.21E(X) + 0.09E(X^2) - 9 \\ &= 2.1 + 9.9 - 9 = 3. \end{aligned}$$