MA3233: Combinatorics and Graphs II
AY21/22 Semester II Suggested Solutions

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Question 1

Let $r \geq 1$ be an integer and G be an r-regular graph. Show that G is bipartite if and only if $E(G)$ can be decomposed into edge-disjoint copies of $K_{1,r}$.

Solution. Let $r \geq 1$ be an integer and G be an r-regular graph.
$(\implies)$
Suppose G is bipartite. Let $G = (A \cup B, E)$ where A and B are the partite sets.
Since $\sum_{v \in A} d_G(v) = \sum_{v \in B} d_G(v)$, we have $r|A| = r|B|$ which means $|A| = |B|$.
Every vertex in B has degree r. Thus we can decompose $E(G)$ into edge-disjoint copies of $K_{1,r}$ based on the end of each edge on B. In other words, $E(G)$ can be expressed as a disjoint union:

$$\bigcup_{v \in B} E_v, \quad \text{where } E_v = \{e \in E(G) \mid v \text{ is incident to } e\}$$

Every edge in G is incident to one $v \in B$, so it will belong to one subset E_v. Also, if we let $v \in B$ be arbitrary, the subgraph G_v formed by the edges in E_v is isomorphic to $K_{1,r}$, with v as the central vertex.
Therefore, we have found a decomposition of $E(G)$ where G is r-regular and bipartite, into edge-disjoint copies of $K_{1,r}$.

$(\impliedby)$

Method 1

Suppose G is an r-regular graph whose edges can be decomposed into edge-disjoint copies of $K_{1,r}$. Let A be the set of vertices which are centres of the copies of $K_{1,r}$ while B be the set of all other vertices. We will show that G is bipartite with A and B as partite sets.
Suppose there is an edge connecting two vertices in A, which we will denote as a_1 and a_2. This means that the edge belongs to a copy of $K_{1,r}$ with a_1 as the centre vertex, and another with a_2 as the centre vertex. This means that the two copies of $K_{1,r}$ are not vertex disjoint, violating the given condition.
Suppose there is an edge connecting two vertices in B, which we will denote as b_1 and b_2. This means that the edge is not incident to any centre vertex of $K_{1,r}$, and thus does not belong to any copy of $K_{1,r}$. This also violates the given condition that the edges can be decomposed into copies of $K_{1,r}$.
Therefore, every edge in G is incident to one vertex in A and another in B, and G is bipartite with A and B as partite sets.

Method 2

Suppose G is not bipartite, so there is an odd cycle $C_{2k+1}, k \in \mathbb{N}$ in G. Let the vertices of the cycle be labelled in clockwise order as $v_1, v_2, c_3, \cdots, v_{2k}, v_{2k+1}$.
Claim: If the graph can be decomposed into edge-disjoint copies of $K_{1,r}$, every copy of $K_{1,r}$ contains exactly 2 consecutive edges on C_{2k+1}.
Proof of Claim.

- We cannot include 3 edges from C_{2k+1} in one copy of $K_{1,r}$
  - If $k = 1$, then C_3 is not a subgraph of $K_{1,r}$
  - If $k > 1$, then in the three edges, at least two are vertex-disjoint. But every pair of edges in $K_{1,r}$ are not vertex-disjoint.
- We cannot include 2 vertex-disjoint edges from C_{2k+1} in one copy of $K_{1,r}$, because every pair of edges in $K_{1,r}$ are not vertex-disjoint.
- Suppose we try to only include one edge from C_{2k+1} in a copy of $K_{1,r}$, without loss of generality denote the edge by v_1v_2. Since in $K_{1,r}$, there is a central vertex such that every edge is incident to it, v_1 or v_2 must correspond to the central vertex. Without loss of generality let v_1 be the central vertex. Since $d_G(v_1) = r$, all r edges in G incident to v_1 must be included in the copy of $K_{1,r}$, including the edge in C_{2k+1} which is incident to v_1 but not v_2. Therefore, we can't include only 1 edge from the cycle in one copy of $K_{1,r}$.

Therefore, suppose we work on distributing the edges of C_{2k+1} into copies of $K_{1,r}$, there must have 1 edge (which we will denote as v_iv_{i+1}) which belongs to a different copy of $K_{1,r}$ compared to the other $2k$ edges of the cycle. Since every edge in $K_{1,r}$ is incident to a central vertex, v_i or v_{i+1} must correspond to the central vertex. Suppose, without loss of generality, that v_i corresponds to the central vertex, we will need to find another $r-1$ edges incident to v_i to be included in this copy of $K_{1,r}$. But since $v_{i-1}v_i$ belonged to another copy of $K_{1,r}$ and $d_G(v_i) = r$, we can only find at most $r-2$ more edges to be included. Thus, v_i (and similarly v_{i+1} cannot be the central vertex). Therefore, we cannot decompose $E(G)$ into edge-disjoint copies of $K_{1,r}$.

Thus, given an r-regular graph G, G is bipartite if and only if $E(G)$ can be decomposed into edge-disjoint copies of $K_{1,r}$.

Question 2

For every $d \geq 1$, denote by $f(d)$ the maximum integer t such that every graph G with $\delta(G) \geq d$ contains a path of length at least t. Determine $f(d)$.

Solution. We claim that $f(d) = d$ for all $d \geq 1$.

- $f(d) \leq d$. There exists a graph G such that $\delta(G) \geq d$ and the maximum length of a path in G is d. Indeed if $G \cong K_{d+1}$, then $\delta(G) = d \geq d$ and there is a Hamiltonian path of G. The path has $d+1$ vertices and d edges so the length is d. We cannot extend it to a larger length because the path is already Hamiltonian. Thus, $f(d) \leq d$.
- $f(d) \geq d$. For all graphs with minimum degree $\delta(G) \geq d$, there is a path in G of at least length d.
  Claim: For all graphs G, there exists a path of length at least $\delta(G)$.
  Proof of Claim. Let P be the longest path found in G and $v_0 \in V(G)$ be an endpoint of P. Observe that $N_G(v_0) \subseteq V(P)$. Otherwise if w is a neighbor of v_0 but $v_0w \notin E(P)$, then we can extend P by v_0w to form a path longer than P, which contradicts the assumption that P is the longest path in G. Thus,

$$\begin{aligned} N_G(v_0) \subseteq V(P) &\implies |V(P)| \geq d_G(v_0) + 1 \geq \delta(G) + 1 \\ &\implies P \text{ has length at least } \delta(G) \end{aligned}$$

  From the claim, for all G such that $\delta(G) \geq d$, there is a path in G of at least length $\delta(G)$ which is at least d.

Therefore, we have shown that $f(d) = d$ for all $d \geq 1$.

Question 3

A Latin square of order n is an $n \times n$ array filled with symbols from $\{1, \cdots, n\}$ such that no symbol is repeated twice in any row or column. Suppose in an $n \times n$ array, the first k rows are already filled by symbols from $\{1, \cdots, n\}$

such that no symbol is repeated twice in any row or column. Show that one can always fill the remaining $n-k$ rows to complete a Latin square. Below is an example for $n=4, k=2$:

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 3 & 4 & 1 \\ & & & \\ & & & \end{bmatrix} \implies \begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 1 & 4 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

Solution. Label the columns in the incomplete $n \times n$ Latin square as c_1 to c_n. Construct a bipartite graph G with partite sets:

$$\begin{aligned} &A = \{v_1, v_2, \cdots, v_n\} \\ &B = \{w_1, w_2, \cdots, w_n\} \\ &\forall 1 \le i, j \le n, v_i \sim w_j \text{ in } G \iff \text{the number } i \text{ has not appeared in } c_j \text{ yet.} \end{aligned}$$

Claim 1: G is $(n-k)$-regular.
Proof of Claim:
Given any $i \in \{1, 2, \cdots, n\}$, in the first k rows, i has appeared in k different columns (i.e. in each row, i is in exactly 1 different column) because no symbol appears twice on any row or column. Thus, i has yet to appear in $n-k$ columns, so $d_G(v_i) = n-k$.
Given any $j \in \{1, 2, \cdots, n\}$, c_j should have k different numbers from $\{1, 2, \cdots, n\}$ in the current incomplete Latin square. Thus, there are $n-k$ more numbers that have not appeared in c_j, so $d_G(w_j) = n-k$.

Claim 2: Every way to fill the next row of the Latin square corresponds to a perfect matching in G.
Proof of Claim:

- For every column, a new number (that hasn't been used in the column yet) is to be assigned. Corresponding to G, we pick an edge incident to w_j for each w_j in B. So we pick n edges.
- Among the edges picked, no two can be incident to one same vertex in A because no number appears twice in one row of the square. Every vertex in A must be incident to one of the n edges to be picked.

We are picking n edges in G such that every vertex in A and B is incident to exactly one of them. This corresponds to a perfect matching in G.

Claim 3: Given a d-regular ($d \ge 1$) bipartite graph G with partite sets A and B, a perfect matching exists.
Proof of Claim:
Let $S \subseteq A$ be arbitrary. Since $d_G(v) = d$ for all $v \in S$, $\sum_{v \in S} d_G(v) = d|S|$.
In $N(S)$, a vertex may also be adjacent to vertices in $A \backslash S$,

$$\begin{aligned} d|S| \le d|N(S)| &\implies |S| \le |N(S)| \\ \forall S \subseteq A, |N(S)| \ge |S| &\implies G \text{ has a matching perfect to } A \\ &\implies G \text{ has a perfect matching.} \end{aligned}$$

With that, we can show that for any number of filled rows $1 \le k \le n$, given an $n \times n$ Latin square with $n-k$ unfilled rows, there is a way to complete it.
We prove by induction on $n-k$. If $n-k=1$, the corresponding bipartite graph G constructed has exactly one perfect matching. Fill in the last row according to the matching (assign i to column c_j if $v_i \sim w_j$) and we can complete the square.
Assume the statement is true with $m-1$ unfilled rows ($1 \le m-1 \le n$). Now, suppose we have m unfilled rows. By claim 3, the corresponding bipartite graph G will be m-regular, and a perfect matching of G exists. Fill in the next row according to the matching (assign i to column c_j if $v_i \sim w_j$) and we can get a square with $m-1$ unfilled rows left. By the induction hypothesis, there is a way to continue and complete the entire Latin square. Thus, by Mathematical Induction, every incomplete $n \times n$ Latin square with k filled rows ($1 \le k < n$) can be completed.

Question 4

(a) Show that for all $k \geq 2$, every connected k-regular bipartite graph is 2-connected.
(b) Suppose we remove the "bipartite" condition in (a), is the conclusion still true?

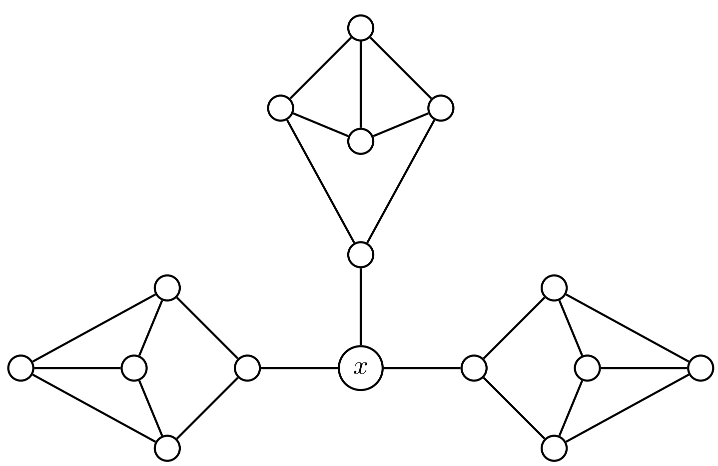
Solution.
(a)
Let $G = (V, E)$ be a k-regular bipartite graph with partite sets A and B. ($k \geq 2$). Since $\sum_{v \in A} d_G(v) = \sum_{v \in B} d_G(v)$, we have $k|A| = k|B|$, which means $|A| = |B|$.
Suppose, without loss of generality, that G has a cut-vertex v in A. Removal of v results in at least two connected components in $G - v$. Let G_1 be one of the connected components, such that its partite sets are $A_1 \subset A \backslash \{v\}, B_1 \subseteq B, B_1 \cap N_G(v) \neq \emptyset$. Then, G_1 is a bipartite subgraph of $G - v$, which is also bipartite.

Since:

- Removing v will not affect the degrees of vertices in A_1, $\sum_{w \in A_1} d_{G_1}(w) = k|A_1|$.
- There are at most $k - 1$ and at least 1 neighbor of v in B_1 for G (Otherwise, if $B_1 \cap N_G(v) = \emptyset$, G_1 is already initially disconnected from the rest of G, and if $N_G(v) \subseteq B_1$, $G_1 + v$ is already initially disconnected from the rest of G. So, $\sum_{w \in B_1} d_{G_1}(w) = k|B_1| - |B_1 \cap N_G(v)|$, where $|B_1 \cap N_G(v)|$ is not divisible by k.

We observe that $\sum_{w \in B_1} d_{G_1}(w)$ is not divisible by k, so $\sum_{w \in A_1} d_{G_1}(w) \neq \sum_{w \in B_1} d_{G_1}(w)$. However, this is contradictory given that G_1 is bipartite.
Therefore, for all $k \geq 2$, every connected k-regular bipartite graph has no cut-vertex and is thus 2-connected.

(b)



The graph above contains triangles ($C_3 \subseteq G$) so it is not bipartite. Also, the vertex x as labelled is a cut-vertex, so G is not 2-connected. Thus, the conclusion is false.

Question 5

Determine the maximum and minimum of $\chi(G \cup H)$, over all pairs of graphs (G, H) with $V(G) = V(H), \chi(G) = 20$, and $\chi(H) = 22$. Justify your answers.

Solution.
The minimum value of $\chi(G \cup H)$ is 22. Since $H \subseteq G \cup H$, $\chi(H) = 22 \leq \chi(G \cup H)$. Indeed this can be achieved if G is a subgraph of H. Then, $H = G \cup H$ and $\chi(G \cup H) = 22$.
For example, take H to be K_{22} and G to be a K_{20} subgraph on 20 of the vertices, together with 2 more isolated vertices.
The maximum value of $\chi(G \cup H)$ is $\chi(G)\chi(H)$ $20 \times 22 = 440$.

Claim: $\forall G, H, V(G) = V(H) \implies \chi(G \cup H) \leq \chi(G)\chi(H)$.
Proof of Claim:
Let $c : V(G) \to [\chi(G)]$ be a proper $\chi(G)$-coloring of G, and $d : V(H) \to [\chi(H)]$ be a proper $\chi(H)$-coloring of H.
(*Note. Here* $[n]$ *represents* $\{1, 2, \cdots, n\}$ *where* $n \in \mathbb{N}$)
Define a coloring $f : V(G) \to [\chi(G)] \times [\chi(H)]$ such that $\forall v \in V(G), f(v) = (c(v), d(v))$. We claim that f is a proper coloring of $G \cup H$.

For all $u, v \in V(G)$, if $u \sim v$ in $G \cup H$, then they are either adjacent in G or H.

$$\begin{aligned} u \sim v \text{ in } G &\implies c(u) \neq c(v) \\ &\implies (c(u), d(u)) \neq (c(v), d(v)) \\ &\implies f(u) \neq f(v) \\ u \sim v \text{ in } H &\implies d(u) \neq d(v) \\ &\implies (c(u), d(u)) \neq (c(v), d(v)) \\ &\implies f(u) \neq f(v) \end{aligned}$$

Thus, f is a proper coloring of $G \cup H$, which uses $|[\chi(G)] \times [\chi(H)]| = \chi(G)\chi(H)$ colors.
Therefore, $\chi(G \cup H) \leq \chi(G)\chi(H)$ When $\chi(G) = 20, \chi(H) = 22$, $\chi(G \cup H) \leq 440$.

It remains to define $V(G)$ and G, H such that $\chi(G \cup H) = 440$.
Let $V(G) = \{(i, j) | 1 \leq i \leq 20, 1 \leq j \leq 22, i, j \in \mathbb{N}\}$ and $E(G) = \{vw | v = (i_1, j_1), w = (i_2, j_2), j_1 = j_2\}$.
Then G is isomorphic to the disjoint union of 22 K_{20}. Since $\chi(K_{20}) = 20$ and each copy of K_{20} is disconnected from others in G, $\chi(G) = 20$.

Let $V(H) = V(G)$ and $E(H) = \{vw | v = (i_1, j_1), w = (i_2, j_2), j_1 \neq j_2\}$.
Then H is isomorphic to a complete 22-partite graph, where every partite set have 20 vertices (i, j) of the same first index i.

- For all $1 \leq j \leq k \leq 22, (1, j) \sim (1, k)$ in H. Thus $K_{22} \subseteq H$ and $\chi(H) \geq 22$.
- If we color the vertices of H by the coloring $f : V(H) \to [22]$ s.t. $f((i, j)) = j$ for all $1 \leq i \leq 20, 1 \leq j \leq 22$, then for all $v, w \in V(H)$:

  $$\begin{aligned} v \sim w \text{ in } H &\implies \text{the two vertices have different index in the second component} \\ &\implies f(v) \neq f(w) \end{aligned}$$

  So, f is a proper 22-coloring of H.

$\chi(H) = 22$ thus holds.
We now claim that $G \cup H$ is isomorphic to K_{440}. First, the graph has $20 \times 22 = 440$ vertices. Next, for any $v, w \in V(G)$,

- If v, w have the same index in their second component, then they are adjacent in G.
- If v, w have distinct indices in their second component, then they are adjacent in H.

Thus, for any two vertices in $G \cup H$, they are adjacent in $G \cup H$. So, $G \cup H$ is isomorphic to K_{440}, and $\chi(G \cup H) = 440$.

Therefore, the maximum and minimum values of $\chi(G \cup H)$ are 440 and 22 respectively.

Question 6

Show that if a simple undirected graph G has a Hamiltonian graph, then for every $S \subset V(G)$, the number of connected components in $G - S$ is at most $|S| + 1$.

Solution. Adapted from West, Introduction to Graph Theory.
Let $c(H)$ denote the number of connected components in a graph H. Let P be a Hamiltonian path of G and $S \subset V(G)$ be arbitrary.
From P, every time we delete a vertex from it, the number of components will increase by at most one. So after removing the vertices in S, $c(P - S) \leq 1 + |S|$. Since P is a spanning subgraph of G and adding edges will not increase the number of components, we have $c(G - S) \leq c(P - S) \leq 1 + |S|$.

Question 7: True or False

7(a) For any given graph G, its vertex covering number $\tau(G)$ is always equal to its matching number $\mu(G)$.
False. For example, a triangle (C_3) has matching number 1 but vertex covering number 2.

7(b) In a bipartite graph G on $A \cup B$, if every vertex in A has degree $d_1 \geq 1$ and every vertex in B has degree $d_2 \geq d_1$, then there exists a matching complete to B.
True. If G is disconnected, we shall just consider each component. Thus, assume G is connected. For every edge $xy \in E(G)$ with $x \in B, y \in A$, we have $d_G(y) = d_1 \leq d_2 = d_G(x)$. Thus for a subset $S \subseteq B$,

$$\begin{aligned}
|S| &= \sum_{x \in S} 1 \\
&= \sum_{x \in S, y \in N(S), xy \in E(G)} \frac{1}{d_2} \\
&\leq \sum_{x \in S, y \in N(S), xy \in E(G)} \frac{1}{d_1} \\
&= \sum_{y \in N(S)} \sum_{x : x \in S, xy \in E(G)} \frac{1}{d_1} \\
&\leq \sum_{y \in N(S)} 1 \quad (\because \text{ some neighbors of } y \text{ might not be in } S) \\
&= |N(S)|
\end{aligned}$$

By Hall's Theorem, G has a matching perfect to B.

7(c) For every graph G, its chromatic number cannot exceed the square of its clique number.
False. By the Mycielskian construction, we can obtain a triangle-free graph (which means its clique number is 2) with any chromatic number, including chromatic numbers greater than 4.

7(d) Every tree has at most one perfect matching.
True.
Suppose there exists an n-vertex tree T with two distinct perfect matchings S_1, S_2. Then n must be even, since any graph with odd number of vertices can never have a perfect matching.

Consider n being even. Let V_{12} be the vertices incident to an edge in $S_1 \cap S_2$ in the tree. Since each edge in $S_1 \cap S_2$ is incident to two vertices, and the edges are vertex-disjoint (since the edges belong to a matching), thus $|V_{12}|$ is even.

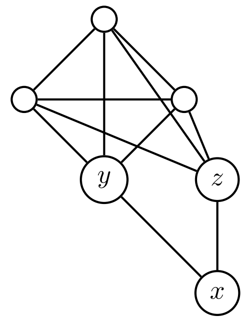
Let $V' = V(T) \backslash V_{12}$. Then $|V'| = n - |V_{12}|$ is even. Let $E' = (S_1 \backslash S_2) \cup (S_2 \backslash S_1)$. Then every edge in E' is not incident to any vertex in V_{12} and can only be incident to two vertices in V'.

With that, consider the subgraph of T, $G = (V', E')$. Since S_1 and S_2 are perfect matchings, every vertex in G (which is in V' but not in V_{12}) can only be incident to one edge in $(S_1 \backslash S_2)$ and one different edge in $(S_2 \backslash S_1)$. Every vertex in G will have degree 2. G is a 2-regular graph, and its components can only be cycles. However, this contradicts the fact that G is the subgraph of a tree (which has to be acyclic).

7(e) Given any n-vertex graph G whose minimum degree is at least $n/2$, G always contains a Hamiltonian cycle.
True. This is essentially Dirac's Theorem.

7(f) Given any n-vertex graph G whose average degree is at least $n/2$, G always contains a Hamiltonian cycle.
False. Consider the graph formed by adding a new, isolated vertex to K_{10}. The average degree of the graph is $\frac{9 \times 10}{11} = \frac{90}{11} \geq \frac{11}{2}$, but due to the isolated vertex, there is no Hamiltonian cycle in the graph.

7(g) If a graph G doesn't contain K_5 or $K_{3,3}$ as a subgraph, then G is always planar.
False. Even if a graph doesn't contain K_5 or $K_{3,3}$ as a subgraph, it may contain a topological minor of K_5 or $K_{3,3}$, in which case it's not planar. Here's an example:



The graph is itself a topological minor of K_5. By removing x and drawing the edge yz we obtain K_5. It is obvious that the graph is not planar.

7(h) There exists a 6-connected planar graph.
False. A six-connected planar graph must have at least seven vertices. Let $n \in \mathbb{N}$ be the number of vertices in a planar graph G. If the minimum degree of G is at least 6, then there will be at least $\frac{6n}{2} = 3n$ edges in the graph. But since G is planar, it must have at most $3n - 6$ edges (this statement holds for all $n \geq 7 \geq 3$), so its minimum degree must be less than 6. Since a graph's connectivity is not more than its minimum degree, the connectivity of a planar graph must be less than 6 too.

7(i) If a tournament contains a directed cycle, then it must contain a directed triangle.
True. Suppose a tournament T does not have a directed triangle. Let $n \in \mathbb{N}$ be the number of vertices in the smallest directed cycle in T, and label the vertices in the cycle as $v_1 \to v_2 \to v_3 \to \cdots \to v_n \to v_1$. There is a directed edge between v_1 and v_3. If the directed edge is $v_1 \to v_3$, then by replacing $v_1 \to v_2 \to v_3$ with $v_1 \to v_3$ in the directed cycle, we obtain a smaller directed cycle in T, which is a contradiction. If the directed edge is $v_1 \to v_3$, then $v_1 \to v_2 \to v_3 \to v_1$ is a directed triangle in T. Therefore, we conclude that the statement is true.

7(j) Every tournament contains a directed Hamiltonian cycle.
False. A transitive tournament is acyclic.