MA4262 - Measure and Integration Suggested Solutions

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All claims in this paper are true.

Question 1

Suppose that $x \in \{x \in \Omega : \liminf_{k\to\infty} f_k(x) > 1\}$. Then, we have

$$\begin{aligned}
\liminf_{k\to\infty} f_k(x) > 1 &\iff \exists n \in \mathbb{Z}_{\geq 1}, \liminf_{k\to\infty} f_k(x) > 1 + \frac{1}{n} \\
&\iff \exists n \in \mathbb{Z}_{\geq 1}, \lim_{m\to\infty} \inf_{k\geq m} f_k(x) > 1 + \frac{1}{n} \\
&\iff \exists n \in \mathbb{Z}_{\geq 1}, \exists m \in \mathbb{Z}_{\geq 1}, \inf_{k\geq m} f_k(x) > 1 + \frac{1}{n} \\
&\iff \exists n \in \mathbb{Z}_{\geq 1}, \exists m \in \mathbb{Z}_{\geq 1}, \forall k \in \mathbb{Z}_{\geq m}, f_k(x) > 1 + \frac{1}{n} \\
&\iff \exists r \in \mathbb{Z}_{\geq 1}, \forall k \in \mathbb{Z}_{\geq r}, f_k(x) > 1 + \frac{1}{r}
\end{aligned}$$

where the last step can be done by taking $r = \max(n, k)$.

Question 2

(i) Suppose that $E \in \Sigma^{\uparrow}_{\downarrow}$. Then, there exists $A, B \in \Sigma$ such that $A \subset E \subset B$ and $\mu(B\backslash A) = 0$. Then, $X\backslash B \subset X\backslash E \subset X\backslash A$. Since Σ is a sigma algebra, we have $X\backslash A \in \Sigma$ and $X\backslash B \in \Sigma$. Finally, $\mu((X\backslash A)\backslash(X\backslash B)) = \mu(B\backslash A) = 0$. So, $X\backslash E \in \Sigma^{\uparrow}_{\downarrow}$.

(ii) For each $i \in \mathbb{Z}_{\geq 1}$, we have $E_i \in \Sigma^{\uparrow}_{\downarrow}$. Then, there exists $A_i, B_i \in \Sigma$ such that $A_i \subset E_i \subset B_i$ and $\mu(B_i\backslash A_i) = 0$. Then, we have

$$\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} B_i.$$

Since Σ is a sigma algebra, we have $\bigcup_{i=1}^{\infty} A_i \in \Sigma$ and $\bigcup_{i=1}^{\infty} B_i \in \Sigma$. Finally,

$$\mu\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \backslash \left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty}\left(A_i\backslash\left(\bigcup_{i=1}^{\infty} B_i\right)\right)\right) \leq \mu\left(\bigcup_{i=1}^{\infty}(A_i\backslash B_i)\right) \leq \sum_{i=1}^{\infty}\mu\left(A_i\backslash B_i\right) = 0. \quad (1)$$

(iii) Suppose that $\mu(B_1) = \infty$ or $\mu(B_2) = \infty$. Without loss of generality, take it that $\mu(B_1) = \infty$. Then, $\mu(A_1) = \infty$. Since, $A_1 \subset E \subset B_2$, we also have $\mu(B_2) = \infty$. So, we have $\mu(B_1) = \mu(B_2)$.

Now, suppose that $\mu(B_1) < \infty$ and $\mu(B_2) < \infty$. Clearly, $B_2 \supset E \supset A_1$. Then

$$0 \leq \mu(B_1\backslash B_2) = \mu(B_1) - \mu(B_2) \leq \mu(B_1\backslash A_1) = 0.$$

So, $\mu(B_1) = \mu(B_2)$.

(iv) Keep the same notation as in (ii). Generally, we have

$$\sum_{i=1}^{\infty} \mu^{\uparrow}(E_i) = \sum_{i=1}^{\infty} \mu(B_i) \geq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu^{\uparrow}\left(\bigcup_{i=1}^{\infty} E_i\right).$$

Suppose that $\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \infty$. Then, $\sum_{i=1}^{\infty} \mu^{\uparrow}(E_i) \leq \infty = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu^{\uparrow}\left(\bigcup_{i=1}^{\infty} E_i\right)$. So the result clearly holds. Now, suppose that $\mu\left(\bigcup_{i=1}^{\infty} B_i\right) < \infty$. For any $\epsilon > 0$, and n large enough, we also have

$$\mu^{\uparrow}\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} B_i\right) \geq \mu\left(\bigcup_{i=1}^{n} B_i\right) - \epsilon.$$

Now, it is clear that (1) also holds for finite union and sums. So,

$$\mu\left(\bigcup_{i=1}^{n} B_i\right) - \epsilon = \mu\left(\bigcup_{i=1}^{n} A_i\right) - \epsilon = \sum_{i=1}^{n} \mu\left(A_i\right) - \epsilon = \sum_{i=1}^{n} \mu\left(B_i\right) - \epsilon = \sum_{i=1}^{n} \mu^{\uparrow}\left(E_i\right) - \epsilon.$$

Letting $n \to \infty$ and $\epsilon \to 0$, we have $\mu^{\uparrow}\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} \mu^{\uparrow}\left(E_i\right)$.

(v) Let $A, B \in \Sigma$ be such that $A \subset E_o \subset B$ and $\mu(B\backslash A) = 0$. We have $\emptyset \subset D \subset E_o \subset B$. Since $\mu(B\backslash\emptyset) = \mu(B) = \mu^{\uparrow}(E_0) = 0$, we have $\mu^{\uparrow}(D) = \mu(B) = \mu^{\uparrow}(E_0) = 0$.

Question 3

By the Chebyshev inequality, we have $\mathbf{P}(\mathbf{X} > t) \leq \frac{1}{t}\int_{\Omega} \mathbf{X} d\mathbf{P} = E(\mathbf{X})$.

Question 4

(i) Abbreviate $\mathbf{x} = (x_1, x_2, \ldots)$. We have

$$\left\{\mathbf{x} \in \Omega_{\infty} : \left|\frac{1}{N}\sum_{i=1}^{N} x_i - \frac{1}{2}\right| \leq \frac{1}{k}\right\} = P_N^{-1}\left(\left\{(x_1, \ldots, x_N) \in \Omega_N : \left|\frac{1}{N}\sum_{i=1}^{N} x_i - \frac{1}{2}\right| \leq \frac{1}{k}\right\}\right) \in \mathcal{F}_N \subset \mathcal{F}_{\text{coin}_{\infty}}.$$

(ii) We have

$$\begin{aligned}
\left\{\mathbf{x} \in \Omega_{\infty} : \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i = \frac{1}{2}\right\} &= \left\{\mathbf{x} \in \Omega_{\infty} : \lim_{N\to\infty} \left|\frac{1}{N}\sum_{i=1}^{N} x_i - \frac{1}{2}\right| = 0\right\} \\
&= \bigcap_{k=1}^{\infty} \left\{\mathbf{x} \in \Omega_{\infty} : \lim_{N\to\infty} \left|\frac{1}{N}\sum_{i=1}^{N} x_i - \frac{1}{2}\right| \leq \frac{1}{k}\right\} \\
&= \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{N=m}^{\infty} \left\{\mathbf{x} \in \Omega_{\infty} : \left|\frac{1}{N}\sum_{i=1}^{N} x_i - \frac{1}{2}\right| \leq \frac{1}{k}\right\}
\end{aligned}$$

by the $\epsilon - N$ definition of limits.

(iii) Sigma algebra is closed under countable intersections and unions.