NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Chang Hai Bin

MA1101R Linear Algebra I
AY 2009/2010 Sem 1

Question 1

(a) (i) Assume the number of orange, grapefruit, and mango bought be x, y, z respectively.

$$\begin{cases} x & +y & +z & = 100 \\ 0.5x & +1.1y & +1.5z & = 200 \end{cases}$$

(ii) Note that the most expensive item is mango, and even if Dr. Ng spend all of his \$ 200 on mango, he still could not spend all of his money. So, it is not possible for Dr. Ng to make his purchases.
In terms of mathematics, if $x, y, z \in \mathbb{N}, x + y + z = 100$, then $0.5x + 1.1y + 1.5z \leq 1.5x + 1.5y + 1.5z = 1.5(x + y + z) = 1.5 \times 100 = 150 < 200$ which is a contradiction.

(b) $$\left(\begin{array}{ccc|c} k & 1 & 1 & 0 \\ 1 & 2k & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow{row-swap} \left(\begin{array}{ccc|c} 1 & 2k & 1 & 1 \\ 0 & 1 & 1 & 1 \\ k & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 - kR_1} \left(\begin{array}{ccc|c} 1 & 2k & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1-2k^2 & 1-k & -k \end{array}\right) \xrightarrow{R_3-(1-2k^2)R_2}$$
$$\left(\begin{array}{ccc|c} 1 & 2k & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2k^2-k & 2k^2-k-1 \end{array}\right)$$

(i) The system is inconsistent when $2k^2 - k = 0$, because from the third row of the REF, we will get the equation $0x_1 + 0x_2 + 0x_3 = -1$.
So, the system has no solution when $k = 0$ or $k = \frac{1}{2}$

(ii) For $k \neq 0, \frac{1}{2}$, there will be 3 pivot-columns, and hence there is a unique solution.

(iii) There is no $k \in \mathbb{R}$ such that the linear system has infinitely many solutions, since it is either inconsistent ($k = 0, \frac{1}{2}$) or has a unique solution ($k \neq 0, \frac{1}{2}$)

Alternatively, $\det \begin{pmatrix} k & 1 & 1 \\ 1 & 2k & 1 \\ 0 & 1 & 1 \end{pmatrix} = 2k^2 - k$.

Case 1: $k \neq 0$ or $\frac{1}{2}$. Then the inverse of the matrix exists and is unique, and the system has a unique solution of $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} k & 1 & 1 \\ 1 & 2k & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Case 2: $k = 0$. Then the original linear system becomes $\begin{cases} & & x_2 & + & x_3 & = & 0 \\ x_1 & + & & + & x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 1 \end{cases}$ which is inconsistent.

Case 3: $k = \frac{1}{2}$. Then $\begin{cases} \frac{1}{2}x_1 & + & x_2 & + & x_3 & = & 0 \\ x_1 & + & x_2 & + & x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 1 \end{cases}$ Substituting Row 3 into Row 1 and 2, we get: $\begin{cases} \frac{1}{2}x_1 & +1 & = & 0 \\ x_1 & +1 & = & 1 \end{cases}$ which is inconsistent.

(c) (i) $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 2 & 1 & 3 \end{array}\right) \overset{Gaussian Elimination}{\rightarrow} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{5}{2} \end{array}\right)$

So, the linear system is inconsistent.

(ii) $A^TAv = A^Tb$

$\Rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3/14 \\ 13/7 \end{pmatrix}.$

(iii) if $\boldsymbol{y}$ is a least square solution to $\boldsymbol{Ax} = \boldsymbol{b}$

$\Rightarrow \boldsymbol{A}^T\boldsymbol{Ay} = \boldsymbol{A}^T\boldsymbol{b}$

$\Rightarrow r\boldsymbol{A}^T\boldsymbol{Ay} = r\boldsymbol{A}^T\boldsymbol{b}$ for non-zero r.

$\Rightarrow \boldsymbol{A}^T\boldsymbol{A}(r\boldsymbol{y}) = \boldsymbol{A}^T(r\boldsymbol{b})$

$\Rightarrow$ r$\boldsymbol{y}$ is a least square solution to $\boldsymbol{Ax}$=r$\boldsymbol{b}$

Question 2

(a) (i) $\boldsymbol{E}_1 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\boldsymbol{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\boldsymbol{E}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\boldsymbol{E}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

(ii) So, $\boldsymbol{A}^{-1} = \boldsymbol{E}_4\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1$

To solve $\boldsymbol{Ax}$=$\boldsymbol{b}$

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{A}^{-1}\boldsymbol{b} = \boldsymbol{E}_4\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{b} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= ... = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix} \end{aligned}$$

(b) (i)

$$\begin{aligned} T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) &= T\left(2\begin{pmatrix} 2 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = 2\,T\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) - 3\,T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) \\ &= 2\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) &= T\left(2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = 2\,T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) - T\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) \\ &= 2\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \end{aligned}$$

So the standard matrix for T is $\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$

(ii) A basis for the range of T is $\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}\right\}$

Note: We can use $\left\{\begin{pmatrix}2\\3\\-1\end{pmatrix}, \begin{pmatrix}3\\5\\-1\end{pmatrix}\right\}$ as well.

(iii) $\text{rank}(T) = 2$,

Since $\begin{pmatrix}0 & 1\\1 & 1\\1 & -1\end{pmatrix}$ has 2 columns,

$\text{null}(T) = \text{no. of columns - rank}(T) = 0$

(c) Note that $(1,1,1,0)\cdot(0,x,-x,0) = 0 = (0,-1,-1,-1)\cdot(0,x,-x,0)$

So, if we define S such that:

$$S\left(\begin{pmatrix}a\\b\\c\\d\end{pmatrix}\right) = \begin{pmatrix}0 & 0 & 0 & 0\\1 & 0 & 0 & 0\\-1 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}0\\a\\-a\\0\end{pmatrix}$$

Then the range, which is span $\{(0,1,-1,0)\}$, is orthogonal to span $\{(1,1,1,0),(0,-1,-1,-1)\}$

Note: There are many other ways to make your S such that it satisfies the condition in the question.

Question 3

(a) $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix}\lambda-1 & -2 & 0\\-2 & \lambda+1 & 2\\0 & 2 & \lambda-1\end{vmatrix} = \ldots = (\lambda-1)(\lambda-3)(\lambda+3)$

So the eigenvalues are $1, 3, -3$.

For the eigenvalue 1, $\begin{pmatrix}\lambda-1 & -2 & 0\\-2 & \lambda+1 & 2\\0 & 2 & \lambda-1\end{pmatrix}\boldsymbol{v} = \begin{pmatrix}0 & -2 & 0\\-2 & 2 & 2\\0 & 2 & 0\end{pmatrix}\boldsymbol{v} = \boldsymbol{0}$

So $\boldsymbol{E}_1 = \text{span}\{(1,0,1)\} = \text{span}\left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right\}$

For the eigenvalue 3, $\begin{pmatrix}\lambda-1 & -2 & 0\\-2 & \lambda+1 & 2\\0 & 2 & \lambda-1\end{pmatrix}\boldsymbol{v} = \begin{pmatrix}2 & -2 & 0\\-2 & 4 & 2\\0 & 2 & 2\end{pmatrix}\boldsymbol{v} = \boldsymbol{0}$

So $\boldsymbol{E}_3 = \text{span}\{(1,1,-1)\} = \text{span}\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)\right\}$

For the eigenvalue -3, $\begin{pmatrix}\lambda-1 & -2 & 0\\-2 & \lambda+1 & 2\\0 & 2 & \lambda-1\end{pmatrix}\boldsymbol{v} = \begin{pmatrix}-4 & -2 & 0\\-2 & -2 & 2\\0 & 2 & -4\end{pmatrix}\boldsymbol{v} = \boldsymbol{0}$

So $\boldsymbol{E}_{-3} = \text{span}\{(-1,2,1)\} = \text{span}\left\{\left(\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\right\}$

So let $\boldsymbol{P} = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}}\\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{pmatrix}$

$\boldsymbol{AP} = \boldsymbol{P}\begin{pmatrix}1 & 0 & 0\\0 & 3 & 0\\0 & 0 & -3\end{pmatrix}$

So, for this matrix $\boldsymbol{P}$, $\boldsymbol{P}^T\boldsymbol{AP} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

(b) (i) $\boldsymbol{B}_2 = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}$, $\boldsymbol{B}_3 = \begin{pmatrix} k & 1 & 0 \\ 0 & k & 1 \\ 0 & 0 & k \end{pmatrix}$

(ii) There is only one eigenvalue: k. This is because that $\boldsymbol{B}_n$ is upper triangular, and hence the eigenvalues are the diagonal entries.

(iii) If $\boldsymbol{B}_n$ has an eigenvector, its corresponding eigenvalue must be k. The eigenspace of the eigenvalue k is span{(1,0,0,...,0)}, since:

$$(k\boldsymbol{I} - \boldsymbol{B}_n)\boldsymbol{v} = \left[\begin{pmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & k \end{pmatrix} - \begin{pmatrix} k & 1 & 0 & \cdots & 0 \\ 0 & k & 1 & \cdots & 0 \\ 0 & 0 & k & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & k \end{pmatrix}\right]\boldsymbol{v}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}\boldsymbol{v} = \boldsymbol{0}$$

So $\boldsymbol{v} \in \text{span}\{(x_1, x_2, \ldots, x_n) | x_2 = x_3 = \ldots = x_n = 0\} = \text{span}\{(1, 0, \ldots, 0)\}$
Hence, $\boldsymbol{B}_n$ has at most one linearly independent eigenvectors, but not n linearly independent eigenvectors.
$\boldsymbol{B}_n$ is not diagonalizable for $n \geq 2$

(c) No. We need to show that whenever $\boldsymbol{C}$ is symmetric, there exists an $x \in \mathbb{R}$, such that $\det(\boldsymbol{C}) - x\boldsymbol{I} = 0$, and hence the rows of $\boldsymbol{C} - x\boldsymbol{I}$ is not a basis for $\mathbb{R}^n$
By theorem 6.3.4, in the real number field $\mathbb{R}$, a square matrix is orthogonally diagonalizable if and only if it is symmetric.

So, if $\boldsymbol{C}$ is symmetric, then it is (orthogonally) diagonalizable.
In particular, there exists $\lambda \in \mathbb{R}$ which is an eigenvalue for $\boldsymbol{C}$, with a corresponding eigenvector v such that $\boldsymbol{C}v = \lambda v = \lambda \boldsymbol{I} v$
And so, $(\boldsymbol{C} - \lambda\boldsymbol{I})x = \boldsymbol{0}$ has a non-trivial solution v.
Hence, the row-space of $(\boldsymbol{C} - \lambda\boldsymbol{I})$ does not form a basis for $\mathbb{R}^n$.

Question 4

(i) $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = (1, 2, -1, 0) \cdot (-1, 1, 1, 3) = -1 + 2 + (-1) + 0 = 0$
$\boldsymbol{u}_1 \cdot \boldsymbol{u}_3 = (1, 2, -1, 0) \cdot (2, -1, 0, 1) = 2 + (-2) + 0 + 0 = 0$
$\boldsymbol{u}_2 \cdot \boldsymbol{u}_3 = (-1, 1, 1, 3) \cdot (2, -1, 0, 1) = -2 + (-1) + 0 + 3 = 0$

(ii) Using the hints given, $\boldsymbol{u}_4 = (-2)\boldsymbol{u}_1 + (1)\boldsymbol{u}_2 + (-1)\boldsymbol{u}_3$
$\boldsymbol{u}_5 = (1)\boldsymbol{u}_1 + (-1)\boldsymbol{u}_2 + (1)\boldsymbol{u}_3$
$\boldsymbol{u}_6 = (-2)\boldsymbol{u}_1 + (-2)\boldsymbol{u}_2 + (5)\boldsymbol{u}_3$

(iii) Since $\boldsymbol{u}_6$ is a linear combination of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_5$ (or to be specific, a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$),
$\text{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6\} = \text{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_5\}$

Similarly, span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_5\}$ = span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_4\}$ = span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_3\}$
Since $\boldsymbol{S}$=$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is orthogonal, and hence linearly independent,
So $\boldsymbol{S}$ is a basis for span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6\}$

(iv) If we express $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6\}$ as row vectors,

$$\begin{pmatrix} -2 & 1 & -1 \\ 1 & -1 & 1 \\ -2 & -2 & 5 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}_4 \\ \boldsymbol{u}_5 \\ \boldsymbol{u}_6 \end{pmatrix}$$

Since $\det \begin{pmatrix} -2 & 1 & -1 \\ 1 & -1 & 1 \\ -2 & -2 & 5 \end{pmatrix} = 3 \neq 0$, $\begin{pmatrix} -2 & 1 & -1 \\ 1 & -1 & 1 \\ -2 & -2 & 5 \end{pmatrix}^{-1}$ exists, and

$$\begin{pmatrix} -2 & 1 & -1 \\ 1 & -1 & 1 \\ -2 & -2 & 5 \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{u}_4 \\ \boldsymbol{u}_5 \\ \boldsymbol{u}_6 \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \end{pmatrix}$$

So $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ can be expressed in linear combination of $\boldsymbol{u}_4, \boldsymbol{u}_5, \boldsymbol{u}_6$
By using similar argument in part (iii), we get span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6\}$ = span$\{\boldsymbol{u}_4, \boldsymbol{u}_5, \boldsymbol{u}_6\}$
Since span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6\}$ has dimension 3 (from part (iii)), and span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6\}$ = span$\{\boldsymbol{u}_4, \boldsymbol{u}_5, \boldsymbol{u}_6\}$,
So $\boldsymbol{T}$ is also a basis for span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6\}$

(v) The transition matrix from $\boldsymbol{T}$ to $\boldsymbol{S}$ is:

$$\boldsymbol{P} = \begin{pmatrix} [\boldsymbol{u}_4]_S & [\boldsymbol{u}_5]_S & [\boldsymbol{u}_6]_S \end{pmatrix} = \begin{pmatrix} -2 & 1 & -2 \\ 1 & -1 & -2 \\ -1 & 1 & 5 \end{pmatrix}$$

(vi) If we are given the assumption that $(1, 0, 0, 1)$ and $(0, 1, 0, 1)$ is in span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6\}$,
then we can choose $(a, b, c, d) = (-1, 1, 1, 3) = \boldsymbol{u}_2$
If we observe the third entry, we know that $(-1, 1, 1, 3)$ is not a linear combination of $(1, 0, 0, 1)$ and $(0, 1, 0, 1)$,
Hence, $\{(-1, 1, 1, 3)$, $(1, 0, 0, 1)$ and $(0, 1, 0, 1)\}$ is linearly independent, and has 3 elements.
By part (iii), span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6\}$ has dimension 3,
So, $\{(-1, 1, 1, 3)$, $(1, 0, 0, 1)$ and $(0, 1, 0, 1)\}$ is a basis for span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6\}$.
In particular, $\{(-1, 1, 1, 3)$, $(1, 0, 0, 1)$ and $(0, 1, 0, 1)\}$ spans span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6\}$
If we are not given the assumption that $(1, 0, 0, 1)$ and $(0, 1, 0, 1)$ is in span$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6\}$, then we need to prove that $(1, 0, 0, 1)$ and $(0, 1, 0, 1)$ is a linear combination of $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6\}$.
We can either solve for it through tedious computations, or make some clever observation/guesses.
Hint: $\boldsymbol{u}_1 + \boldsymbol{u}_2 = (1, 2, -1, 0) + (-1, 1, 1, 3) = (0, 3, 0, 3)$,
$11 \cdot \boldsymbol{u}_3 - \boldsymbol{u}_6 = (22, -11, 0, 11) - (10, -11, 0, -1) = (12, 0, 0, 12)$.
After that, we can follow the arguments above.