MA2216/ST2131 15/16 Semester 2

Final Exam Solution

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1. (a) $X \sim \text{Exp}(\lambda), Y \sim \text{Exp}(\mu), Z = \min(X, Y), X \perp Y$

$$\begin{aligned}
&\mathbb{P}(X = Z)\\
=&\mathbb{P}(X < Y)\\
=&\int_0^\infty \int_x^\infty f_X(x) f_Y(y) dy dx\\
=&\int_0^\infty \int_x^\infty \lambda \exp(-\lambda x) \mu \exp(-\mu y) dy dx\\
=&\int_0^\infty \lambda \exp(-\lambda x) \int_x^\infty \mu \exp(-\mu y) dy dx\\
=&\int_0^\infty \lambda \exp(-\lambda x)[-\exp(-\mu y)]_x^\infty dx\\
=&\int_0^\infty \lambda \exp(-\lambda x) \exp(-\mu x) dx\\
=&\int_0^\infty \lambda \exp(-(\lambda + \mu)x) dx\\
=&[-\frac{\lambda}{\lambda + \mu} \exp(-(\lambda + \mu)x)]_0^\infty\\
=&\frac{\lambda}{\lambda + \mu}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbb{P}(Z > z) = \mathbb{P}(X > z, Y > z) = \mathbb{P}(X > z)\mathbb{P}(Y > z)\\
= \exp(-\lambda z) \exp(-\mu z) = \exp(-(\lambda + \mu)z)
\end{aligned}$$

Obviously, $Z \sim \text{Exp}(\lambda + \mu)$

(c)

$$\begin{aligned}
&\mathbb{P}(X < Y, Z > z) = \mathbb{P}(X < Y, X > z)\\
=&\int_z^\infty \int_x^\infty f_X(x) f_Y(y) dy dx\\
=&\int_z^\infty \lambda \exp(-(\lambda + \mu)x) dx\\
=&[-\frac{\lambda}{\lambda + \mu} \exp(-(\lambda + \mu)x)]_z^\infty\\
=&\frac{\lambda}{\lambda + \mu} \exp(-(\lambda + \mu)z)
\end{aligned}$$

(d) From our answer for (d),

$$\mathbb{P}(X < Y, Z > z) = \frac{\lambda}{\lambda + \mu} \exp(-(\lambda + \mu)z) = \mathbb{P}(X < Y)\mathbb{P}(Z > z)\, \forall z \geq 0$$

Hence we can conclude that Z and the event $\{X < Y\}$ are independent.

(e)

$$\mathbb{P}((X-Y)^+ = 0) = \mathbb{P}(X < Y) = \mathbb{P}(Z = X) = \frac{\lambda}{\lambda+\mu}$$

(f)

$$\begin{aligned}
&\mathbb{P}((X-Y)^+ > w) = \mathbb{P}(X - Y > w)\\
&= \int_0^\infty \int_{y+w}^\infty f_X(x) f_Y(y)\, dx\, dy\\
&= \int_0^\infty \mu \exp(-\mu x) \int_{y+w}^\infty \lambda \exp(-\lambda y)\, dx\, dy\\
&= \int_0^\infty \mu \exp(-\mu x)[-\exp(-\lambda y)]_{y+w}^\infty\, dy\\
&= \int_0^\infty \mu \exp(-\mu x) \exp(-\lambda(y+w))\, dy\\
&= \int_0^\infty \mu \exp(-(\lambda+\mu)x) \exp(-\lambda w)\, dy\\
&= \exp(-\mu w)[-\frac{\mu}{\lambda+\mu} \exp(-(\lambda+\mu)x)]_0^\infty\\
&= \frac{\mu}{\lambda+\mu} \exp(-\lambda w)
\end{aligned}$$

2. (a) $X \sim \Gamma(\alpha = 10, \lambda = 2)$, $\mathbb{E}(X) = \frac{\alpha}{\lambda} = 5$

(b)

$$\begin{aligned}
&f_Y(y) = Ce^y f_X(y)\\
=&Ce^y \frac{2^{10}}{\Gamma(10)} x^9 e^{-2x}\\
=&Dx^{10-1} e^{-x}, \text{where } D = C\frac{2^{10}}{\Gamma(10)}
\end{aligned}$$

We can determine from the form of $f_Y(y)$ that $Y \sim \Gamma(10, 1)$, and $D = \frac{1}{\Gamma(10)}$. Hence, $C = D\frac{\Gamma(10)}{2^{10}} = 2^{-10}$

(c) By Markov's inequality, $\mathbb{P}(X > 15) \leq \frac{\mathbb{E}(X)}{15} = \frac{1}{3}$

(d) Show that: $\mathbb{P}(X > 15) = \mathbb{E}[I(Y > 15)e^{-Y} 2^{10}]$

$$
\begin{aligned}
RHS &= \mathbb{E}[I(Y>15)e^{-Y}2^{10}] \\
=&\mathbb{E}[e^{-Y}2^{10} * 1 \mid Y>15] + 0 \\
=&\mathbb{E}[e^{-Y}2^{10} \mid Y>15] \\
=&\int_{15}^{\infty} e^{-y}2^{10} f_Y(y)dy \\
=&\int_{15}^{\infty} e^{-y}C^{-1} f_Y(y)dy \\
=&\int_{15}^{\infty} f_X(y)dy \\
=&\int_{15}^{\infty} f_X(x)dx \\
=&\mathbb{P}(X>15) = LHS
\end{aligned}
$$

(e)

$$
\begin{aligned}
\mathbb{P}(X>15) &= \mathbb{E}[e^{-Y}2^{10} \mid Y>15] \\
=&2^{10}\mathbb{E}[e^{-Y} \mid Y>15] \\
<&2^{10}\mathbb{E}[e^{-15} \mid Y>15] \text{ as } e^{-y} \text{ is decreasing} \\
=&2^{10}e^{-15}\mathbb{E}[1 \mid Y>15] \\
=&2^{10}e^{-15}\mathbb{P}(Y>15) \\
\leq&2^{10}e^{-15}\frac{\mathbb{E}(Y)}{15} \\
\leq&\frac{2^{11}}{3}\exp(-15)
\end{aligned}
$$

3. (a) $X \sim \mathrm{N}(0,1), Y \sim \mathrm{N}(0,1), X \perp Y$
$4X - 3Y \sim \mathrm{N}(4*0-3*0, 4*1+3*1) \sim \mathrm{N}(0,7)$

$$
\mathbb{P}(4X-3Y>2) = \mathbb{P}(\sqrt{7}\phi > 2) = \mathbb{P}(\phi > \frac{2}{\sqrt{7}}) = 1 - \phi(\frac{2}{\sqrt{7}})
$$

(b) We shall first use the moment generating function of standard normal distribution to find its moments.
$X \sim \mathrm{N}(0,1), M_X(t) = \exp(\frac{1}{2}t^2)$
$M_X^{(1)}(t) = t\exp(\frac{1}{2}t^2), M_X^{(2)}(t) = (1+t^2)\exp(\frac{1}{2}t^2), M_X^{(3)}(t) = (3t+t^3)\exp(\frac{1}{2}t^2)$
$\mathbb{E}(X) = M_X^{(1)}(0) = 0, \mathbb{E}(X^2) = M_X^{(2)}(0) = 1, \mathbb{E}(X^3) = M_X^{(3)}(0) = 0$
$\mathbb{E}((2X)^2 - 5Y + X^3 - 1) = 4\mathbb{E}(X^2) - 5\mathbb{E}(Y) + \mathbb{E}(X^3) - 1 = 4 - 0 + 0 - 1 = 3$

(c) $\mathbb{P}(|X| \leq |Y|) = \mathbb{P}(|Y| \leq |X|) = \frac{1}{2}$ as X and Y are independent standard normal random variables.

4. (a) $\mathbb{E}(S_n) = \mathbb{E}(\sum_{i=1}^{n} R_i) = \sum_{i=1}^{n} \mathbb{E}(R_i) = \frac{7}{2}n$

   (b) $\text{Var}(R_i) = \sum_{i=1}^{6}(i - \frac{7}{2})^2 = 2(2.5^2 + 1.5^2 + 0.5^2) = 17.5$
   $\text{Var}(S_n) = \text{Var}(\sum_{i=1}^{n} R_i) = \sum_{i=1}^{n} \text{Var}(R_i) = n\text{Var}(R_i) = 17.5n$

   (c) By Central Limit Theorem, S_n can be approximated as a normal distribution with expectation $3.5n$ and variance $17.5n$.
   $\mathbb{P}(S_{100} > 360) = \mathbb{P}(S_{100} \geq 360.5) = \mathbb{P}(\phi \geq \frac{360.5 - 3.5*100}{\sqrt{17.5*100}}) = 1 - \phi(\frac{10.5}{10\sqrt{17.5}}))$

5. (a) The support of both X and Y are $[1, \infty)$.

$$
\begin{aligned}
&\mathbb{P}(X \leq t) \\
=&\int_1^t \int_1^\infty f(x,y)dydx \\
=&\int_1^t \int_1^\infty \frac{1}{x^2y^2}dydx \\
=&\int_1^t x^{-2}[-y^{-1}]_1^\infty dx \\
=&\int_1^t x^{-2}dx \\
=&[-x^{-1}]_1^t = 1 - t^{-1}
\end{aligned}
$$

   $f_X(x) = \frac{d}{dx}(1 - x^{-1}) = x^{-2}$
   Similarly, since X and Y are exactly identical, $f_Y(y) = y^{-2}$
   As $f_{X,Y}(x,y) = f_X(x)f_Y(y)$, X and Y are independent.

   (b) Since X and Y are independent, $f_{X|Y}(x,y) = f_X(x) = x^{-2}$

   (c) $U = XY \geq 1$, Its distribution function is :

$$
\begin{aligned}
&\mathbb{P}(U \leq t)\ \forall t \geq 1 \\
=&\int_1^t \int_1^{\frac{t}{x}} f(x,y)dydx = \int_1^t \int_1^{\frac{t}{x}} \frac{1}{x^2y^2}dydx \\
=&\int_1^t x^{-2}[-y^{-1}]_1^{\frac{t}{x}} dx = \int_1^t x^{-2}(1 - \frac{x}{t})dx \\
=&\int_1^t x^{-2} - \frac{x^{-1}}{t}dx = [-x^{-1} - \frac{\ln x}{t}]_1^t \\
=&(-t^{-1} - \frac{\ln t}{t}) - (-1 - 0) = 1 - \frac{1 + \ln t}{t}
\end{aligned}
$$

   Its density function is :
   $f_U(u) = \frac{d}{du}(1 - \frac{1+\ln u}{u}) = \frac{d}{du}(-u^{-1} - \frac{\ln u}{u}) = u^{-2} - \frac{\frac{1}{u}u - \ln u}{u^2} = \frac{\ln u}{u^2}\ \forall u \geq 1$