MA1101R - Linear Algebra I Suggested Solutions

AY19/20 Semester 2

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Question 1

(a) For any $\mathbf{w} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2\} \cap \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, we have $w = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3$ and so

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 - b_1\mathbf{v}_1 - b_2\mathbf{v}_2 - b_3\mathbf{v}_3 = 0.$$

Setting up the augmented matrix and row reducing yields

$$\begin{pmatrix} 1 & 0 & -2 & -1 & -2 \\ 0 & -1 & 3 & -1 & -2 \\ 1 & -2 & 4 & -2 & -3 \\ -2 & 1 & 1 & 2 & 3 \\ 2 & 0 & -4 & 0 & 2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, a solution to the system is given by span$\{(-1,\, 1,\, 0,\, -3,\, 1)\,, (2,\, 3,\, 1,\, 0,\, 0)\}$. This represents **coefficients to the system above, not the vectors themselves**. Set $a_1 = -1$, $a_2 = 1$ and our vector is $(-1,\, -1,\, -3,\, 3,\, -2)$. Put $a_1 = 2, a_2 = 3$ and our vector is $(2,\, -3,\, -4,\, -1,\, 4)$. Thus $S = \{(-1,\, -1,\, -3,\, 3,\, -2)\,,\, (2,\, -3,\, -4,\, -1,\, 4)\}$.

(b) We aim to solve the system

$$\begin{aligned} (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3) \cdot \mathbf{u}_1 &= 8a_1 + 7a_2 + 7a_3 &= 0 \\ (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3) \cdot \mathbf{u}_2 &= 10a_1 - 7a_2 - 11a_3 &= 0. \end{aligned}$$

Setting up the augmented matrix and row reducing yields

$$\begin{pmatrix} 8 & 7 & 7 \\ 10 & -7 & -11 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -\frac{2}{9} \\ 0 & 1 & \frac{79}{63} \end{pmatrix}$$

Thus, the system has a solution given by span$\left\{\begin{pmatrix} 14 \\ -79 \\ 63 \end{pmatrix}\right\}$. Thus, one such vector is given by $14\mathbf{v}_1 - 79\mathbf{v}_2 + 63\mathbf{v}_3 = (75, 5, -25, -45, -70)$.

Question 2

(a) Using Gram-Schmidt process, we have

$$\begin{aligned} \mathbf{u}_1 &= (5,2,6,-4) = \mathbf{v}_1 \\ \mathbf{u}_2 &= (-12,-3,-12,6) - \frac{(-12,-3,-12,6)\cdot(5,2,6,-4)}{5^2+2^2+6^2+(-4)^2}(5,2,6,-4) \\ &= \mathbf{v}_2 - 2\mathbf{u}_1 \\ &= (-2,1,0,-2) \\ \mathbf{u}_3 &= (2a+3, 8a+3, -3a+6, 2a-6) - \frac{(2a+3,8a+3,-3a+6,2a-6)\cdot(5,2,6,-4)}{5^2+2^2+6^2+(-4)^2}(5,2,6,-4) \\ &- \frac{(2a+3,8a+3,-3a+6,2a-6)\cdot(-2,1,0,-2)}{(-2)^2+1^2+0^2+(-2)^2}(-2,1,0,-2) \\ &= \mathbf{v}_3 - \mathbf{u}_1 - \mathbf{u}_2 \\ &= (2a, 8a, -3a, 2a) \end{aligned}$$

Provided $a \neq 0$, the required orthonormal basis $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ has $\mathbf{w}_1 = \left(\frac{5}{9},\, \frac{2}{9},\, \frac{2}{3},\, \frac{-4}{9}\right), \mathbf{w}_2 \left(-\frac{2}{3},\, \frac{1}{3},\, 0,\, -\frac{2}{3}\right)$ and $\mathbf{w}_3 = \left(\frac{2}{9},\, \frac{8}{9},\, \frac{1}{3},\, \frac{2}{9}\right)$. If $a = 0$, then $T = \{\mathbf{w}_1, \mathbf{w}_2\}$ will do.

(b) Note that if $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent, then the length of $\mathbf{u}_3$ must be 0, which can only happen when $a = 0$. Thus, for $a = 0$, the required orthogonal basis is $\{\mathbf{u}_1, \mathbf{u}_2\}$, while for $a \neq 0$, the orthogonal basis is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Hence the possible values for the dimension is $\dim(V) = 2$ or $\dim(V) = 3$.

(c) Firstly, from (a), we have

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{v}_1 \\ \mathbf{u}_2 &= 2\mathbf{v}_1 + \mathbf{v}_2 \\ \mathbf{u}_3 &= \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3. \end{aligned}$$

For the orthonormal basis,

$$\begin{aligned} \mathbf{w}_1 &= \frac{1}{9}\mathbf{v}_1 \\ \mathbf{w}_2 &= \frac{1}{3}(2\mathbf{v}_1 + \mathbf{v}_2) \\ \mathbf{w}_3 &= \frac{1}{9a}(\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3). \end{aligned}$$

Hence, the transition matrix **from** T **to** S is given by

$$\mathbf{P}_{S\to T} = \begin{pmatrix} \frac{1}{9} & \frac{2}{3} & \frac{1}{9a} \\ 0 & \frac{1}{3} & -\frac{1}{9a} \\ 0 & 0 & \frac{1}{9a} \end{pmatrix}$$

To find the transition matrix from S to T, we only need to find the inverse of the matrix above.

$$\mathbf{P}_{T\to S} = \begin{pmatrix} 9 & -18 & -27 \\ 0 & 3 & 3 \\ 0 & 0 & 9a \end{pmatrix}$$

(d) **Way 1**

We first find the projection of $(4,7,-9,-5)$ onto $V := \text{span}\{(5,2,6,-4), (-12,-3,-12,6), (5,11,3,-4)\}$. Using the orthogonal basis found in part (a), we have

$$\begin{aligned} \text{Proj}_V((4,7,-9,-5)) &= \frac{(5,2,6,-4)\cdot(4,7,-9,-5)}{5^2+2^2+6^2+(-4)^2}(5,2,6,-4) + \frac{(-2,1,0,-2)\cdot(4,7,-9,-5)}{(-2)^2+1^2+0^2+(-2)^2}(-2,1,0,-2) \\ &+ \frac{(2,8,-3,2)\cdot(4,7,-9,-5)}{2^2+8^2+(-3)^2+2^2}(2,8,-3,2) \\ &= (0,9,-3,0). \end{aligned}$$

Now, we aim to solve the system

$$\begin{pmatrix} 5 & -12 & 5 \\ 2 & -3 & 11 \\ 6 & -12 & 3 \\ -4 & 6 & -4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 9 \\ -3 \\ 0 \end{pmatrix}.$$

Observe that $-(5,2,6,-4) + (5,11,3,-4) = (0,9,-3,0)$, so a least square solution is given by $\mathbf{x} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Way 2

Solving for $\mathbf{A}^T\mathbf{A}x = \mathbf{A}^T b$, we get the augmented matrix,

$$\left(\begin{array}{ccc|c} 81 & -162 & 81 & 0 \\ -162 & 333 & -153 & 9 \\ 81 & -153 & 171 & 90 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Question 3

(a) We have

$$\begin{pmatrix}1 & -2 & 1 & 3\\ 1 & -1 & 0 & 4\\ 1 & 0 & -1 & 5\end{pmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{pmatrix}1 & -2 & 1 & 3\\ 0 & 1 & -1 & 1\\ 0 & 2 & -2 & 2\end{pmatrix} \xrightarrow[R_3-2R_2]{R_1+2R_2} \begin{pmatrix}1 & 0 & -1 & 5\\ 0 & 1 & -1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Hence, a basis for the row space is given by $\{(1,0,-1,5),(0,1,-1,1)\}$.

On the other hand, a basis for the null space is given by $\{(1,1,1,0)^T,(-5,-1,0,1)^T\}$.

(b) For any matrix $\mathbf{A}$, denote the row space and null space of $\mathbf{A}$ by $R(\mathbf{A})$ and $N(\mathbf{A})$ respectively. For any subspace W, define

$$W^\perp = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v}\cdot\mathbf{u} = 0 \;\forall\, \mathbf{u} \in W\}.$$

We first show that $(R(\mathbf{A}))^\perp = N(\mathbf{A})$. Indeed, we have

$$\begin{aligned}\mathbf{u} \in N(\mathbf{A}) \quad &\iff \quad \mathbf{Au} = \mathbf{0}\\ &\iff \quad \text{for any row } \mathbf{a} \text{ of } \mathbf{A},\ \mathbf{a}\cdot\mathbf{u} = 0.\\ &\iff \quad \mathbf{u} \in (R(\mathbf{A}))^\perp.\end{aligned}$$

Observe that the column space of $\mathbf{A}^T$ is equal to the row space of $\mathbf{A}$. Thus, we have

$$N(\mathbf{B}) = R(\mathbf{A}) \implies R(\mathbf{B}) = (N(\mathbf{B}))^\perp = (R(\mathbf{A}))^\perp = N(\mathbf{A}).$$

Hence, it suffices to pick the matrix $\mathbf{B} = \begin{pmatrix}-5 & -1 & 0 & 1\\ 1 & 1 & 1 & 0\end{pmatrix}$.

For completeness sake, we verify that the matrix $\begin{pmatrix}-5 & -1 & 0 & 1\\ 1 & 1 & 1 & 0\end{pmatrix}$ works. Indeed, we have

$$\begin{pmatrix}-5 & -1 & 0 & 1\\ 1 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\\-1\\5\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \text{ and } \begin{pmatrix}-5 & -1 & 0 & 1\\ 1 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

The proof is complete.

Question 4

(a) We first find the characteristic polynomial of $\mathbf{A}$. We have

$$\begin{aligned}\det(\mathbf{A}-x\mathbf{I}_3) \quad &= \quad \begin{pmatrix}\frac{5}{2}-x & 1 & -2\\ 1 & 1-x & -1\\ 2 & 1 & -\frac{3}{2}-x\end{pmatrix}\\ &= \quad \left(\frac{5}{2}-x\right)\left((1-x)\left(-\frac{3}{2}-x\right)-(-1)\times 1\right) - 1\left(1\left(-\frac{3}{2}-x\right)-(-1)\times 2\right)\\ &+ \quad (-2)(1\times 1 - 2(1-x))\\ &= \quad -x^3+2x^2-\frac{5}{4}x+\frac{1}{4}\\ &= \quad -\frac{1}{4}(2x-1)^2(x-1).\end{aligned}$$

By right, the characteristic polynomial have 1 as the coefficient of the highest term. The characteristic polynomial of $\mathbf{A}$ is $c(x) = \frac{1}{4}(2x-1)^2(x-1)$. Thus, the eigenvalues of $\mathbf{A}$ are $\dfrac{1}{2}$ and 1. A matrix which has the same characteristic polynomial is

$$\begin{pmatrix}\frac{1}{2} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & 1\end{pmatrix} \tag{1}$$

(b) To find the eigenspace E_1, we solve the system

$$\left(\begin{array}{ccc|c} \frac{3}{2} & 1 & -2 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 1 & -\frac{5}{2} & 0 \end{array}\right) \xrightarrow[2R_3]{2R_1} \left(\begin{array}{ccc|c} 3 & 2 & -4 & 0 \\ 1 & 0 & -1 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right) \xrightarrow[-R_1+R_3]{-R_2+R_3} \left(\begin{array}{ccc|c} 3 & 2 & -4 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-3R_2+R_1} \left(\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

It follows that $E_1 = \operatorname{span}\left\{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\right\}$.

(c) As for the eigenspace $E_{\frac{1}{2}}$, we have

$$\left(\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 1 & \frac{1}{2} & -1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right) \xrightarrow[-R_1+R_3]{-\frac{1}{2}R_1+R_2} \left(\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Hence, we have $E_{\frac{1}{2}} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}\right\}$

(d) We have

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}^{-1} \mathbf{A} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} =: D.$$

Then,

$$\lim_{n\to\infty} D^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}^{-1},$$

we have

$$\lim_{n\to\infty} \mathbf{A}^n = \lim_{n\to\infty} (\mathbf{PDP}^{-1})^n = \lim_{n\to\infty} \mathbf{PD}^n\mathbf{P}^{-1} = \mathbf{P}\left(\lim_{n\to\infty} \mathbf{D}^n\right)\mathbf{P}^{-1} = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ 4 & 2 & -4 \end{pmatrix}.$$

Question 5

(a) Clearly, $\mathbf{0} \in V$.

For $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{pmatrix} \in V$ and a scalar $c \in \mathbb{R}$ we have

$$c\mathbf{a} + \mathbf{b} = c\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \\ ca_3 \\ 0 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{pmatrix} = \begin{pmatrix} ca_1 + b_1 \\ ca_2 + b_2 \\ ca_3 + b_3 \\ 0 \end{pmatrix} \in V.$$

Thus, V is a subspace of $\mathbb{R}^4$.

A basis for V is given by $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$.

(b) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

(c) $\text{rank}(\mathbf{A}) = 3$, $\text{nullity}(\mathbf{A}) = 3 - \text{rank}(\mathbf{A}) = 3 - 3 = 0$.

(d) Take $\mathbf{B} = \mathbf{A}^T$. The matrix $\mathbf{B}$ is not unique. In fact, for any $\mathbf{u} \in \mathbb{R}^3$, the matrix $(\mathbf{I}_3\ \mathbf{u})$ satisfies the relation. For instance, take $\mathbf{u} = (-1, 0, 1)$. Put $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Then $\mathbf{BA} = \mathbf{I}_3$.

(e) Note that $4 = \text{rank}(\mathbf{I}_4) = \text{rank}(\mathbf{AD}) \leq \text{rank}(\mathbf{A}) = 3$, which is a contradiction.

Question 6

(a) Note that we have

$$c_{\mathbf{A}}(x) = \det(\mathbf{I_3} - x\mathbf{A}) = \begin{pmatrix} x-2 & a & b \\ 0 & x-c & d \\ 0 & 0 & x-e \end{pmatrix} = (x-2)(x-c)(x-e)$$

and so $x = 2$ is an eigenvalue of $\mathbf{A}$. In particular, $e_1 = (1,0,0)^T$ is an eigenvector associated with 2.

(b) **Way 1:**

By Vieta's formula, the product of roots of the polynomial is -18. It follows that $\frac{-18}{2 \times 9} = -1$ is a root of the polynomial too.

Way 2:

$c(x) = (x-2)(x-c)(x-e)$. Either c or e must be 9. WLOG, put $e = 9$. To get 18 as the constant term, $c = -1$, which means that -1 is a root of $c(x)$.

Since the characteristic polynomial has three distinct roots, the dimension of the union of the eigenspaces must be ≥ 3. Since the union of the eigenspaces cannot exceed $\dim = 3$, this forces the dimension of the union of the eigenspaces to be 3 and $\mathbf{A}$ must be diagonalizable.

Question 7

(a) Since S is an orthonormal basis of V, it follows from triangle inequality that

$$\begin{aligned} ||\mathbf{v}|| &= ||c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k|| \\ &\leq |c_1|\ ||\mathbf{u}_1|| + |c_2|\ ||\mathbf{u}_2|| + \cdots |c_k|\ ||\mathbf{u}_k|| \\ &= |c_1| + |c_2| + \cdots + |c_k|. \end{aligned}$$

(b) Write

$$||v|| = ||c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k|| = \sqrt{c_1^2 + c_2^2 + \cdots + c_k^2}.$$

Then, it follows that $||v||^2 = c_1^2 + c_2^2 + \cdots + c_k^2$ and so for each positive integer $1 \leq i \leq k$, we have $c_i^2 \leq ||v||^2$, which implies that $|c_i| \leq ||v|| \leq 1$. Hence, $|c_i| \leq 1$.

(d) We will prove the result from part (d) only because the proof for part (c) is similar. From part (b), we have

$$\begin{aligned} ||\mathbf{Av}|| &= ||\mathbf{A}(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k)|| \\ &\leq |c_1|\ ||\mathbf{Au}_1|| + |c_2|\ ||\mathbf{Au}_2|| + \cdots + |c_k|\ ||\mathbf{Au}_k|| \\ &\leq ||\mathbf{v}||\,(||\mathbf{Au}_1|| + ||\mathbf{Au}_2|| + \cdots + ||\mathbf{Au}_k||) \\ &= M||\mathbf{v}||. \end{aligned}$$

Setting $||v|| \leq 1$ gives the result for (c).