NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA3110 Mathematical Analysis II
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Question 1

(a) We will first prove $\sup_{m\geq n}(a_m+b_m)\leq \sup_{m\geq n}a_m+\sup_{m\geq n}b_n$.
Let $\sup_{m\geq n}a_m=L$ and $\sup_{m\geq n}b_m=M$.
By definition, for all $m\geq n$, we have $a_m\leq L$ and $b_m\leq M$, which implies $a_m+b_m\leq L+M$ for all $m\geq n$, hence proving the claim.

By the property of limits, we have

$$\lim_{n\to\infty}\sup_{m\geq n}(a_m+b_m)\leq \lim_{n\to\infty}\sup_{m\geq n}a_m+\lim_{n\to\infty}\sup_{m\geq n}b_n$$

hence proving the required inequality.

(b) Let $a_n=\sin n$ and $b_n=-\sin n$.
Then $\limsup_{n\to\infty}(a_n+b_n)=0$, while $\limsup_{n\to\infty}a_n=\limsup_{n\to\infty}b_n=1$.
In this case, we have strict inequality.

Question 2

(a) We will first prove that $g'(0)$ exists implies $g'(0)=0$.
Suppose $g'(0)$ exists, but $g'(0)\neq 0$. WLOG, suppose $g'(0)>0$.
Then there exists $\delta>0$ such that

$$g(x)>0,\quad \forall x\in(0,\delta)$$

$$g(x)<0,\quad \forall x\in(-\delta,0)$$

However, by definition, $g(x)\equiv|f(x)|\geq 0$ for all x. Hence, we have a contradiction. We conclude that $g'(0)=0$.

Now, by definition of derivative,

$$g'(0)=\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}=\lim_{x\to 0}\frac{g(x)}{x}=\lim_{x\to 0}\frac{|f(x)|}{x}=0$$

Moreover, when $x>0$, we have the following inequality:

$$-\frac{|f(x)|}{x}\leq\frac{f(x)}{x}\leq\frac{|f(x)|}{x}$$

The direction of above inequality is reversed when $x<0$.
Hence, by Squeeze Lemma,

$$f'(0)=\lim_{x\to 0}\frac{f(x)}{x}=0$$

proving the statement.

(b) Suppose $f'(0)$ exists.
One direction has been proven, i.e. if $g'(0)$ exists then by part (i), $f'(0) = 0$ and $g'(0) = 0$.
We will prove the other direction.
Now, if $f'(0) = 0$, then by definition of derivative, given any $\varepsilon > 0$, there exists $\delta > 0$, such that for all x such that $0 < |x| < \delta$, $\left|\frac{f(x)}{x}\right| < \varepsilon$.
Moreover, we have for all x such that $0 < |x| < \delta$,

$$\left|\frac{|f(x)|}{x} - 0\right| = \left|\frac{f(x)}{x}\right| < \varepsilon$$

hence, proving that

$$g'(0) = \lim_{x\to 0}\frac{g(x)}{x} = \lim_{x\to 0}\frac{|f(x)|}{x} = 0$$

by definition.

Question 3

(a) We will first show that f is Riemann-integrable in $[0, 1]$, by showing that f is continuous in $[0, 1]$.
Given $\varepsilon > 0$, take $\delta = \varepsilon^2$, then for all $|x - y| < \delta$,

$$|f(x) - f(y)| \le \sqrt{|y-x|} < \sqrt{\delta} = \varepsilon$$

as required.

Now, we can apply the Riemann sum expression for $f(x)$.
Take the equal-spaced partition of f (with norm $\frac{1}{n}$), with the right end-point of each partition interval as the sample point. Then since f is integrable, and $\lim_{n\to\infty}\frac{1}{n} = 0$, by the Theorem on Convergence of Riemann Sums,

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} = \int_0^1 f(x)\, dx$$

as required.

Question 4

(a) By Taylor Expansion, we have

$$\sin y = \sin 0 + \frac{\cos c}{1!}y$$

for some c between 0 and y.
As $|\cos c| \le 1$, we have $|\sin y| \le |y|$ for all $y \in \mathbb{R}$. Therefore, we have $\left|\sin\frac{x^p}{n^p}\right| \le \left|\frac{x^p}{n^p}\right|$ for all $x \in \mathbb{R}$.

To prove convergence of the required series, we will use Weierstrass M-test. Note that if $p > 1$,

$$\sum_{n=1}^{\infty}\left|\frac{x^p}{n^p}\right| = |x^p|\sum_{n=1}^{\infty}\frac{1}{n^p}$$

converges for all $x \in \mathbb{R}$ by the p-series test.

Hence, $\displaystyle\sum_{n=1}^{\infty}\sin\frac{x^p}{n^p}$ converges uniformly on $[-r, r]$ for any $r > 0$ by Weierstrass M-test.

(b) By Taylor Expansion, we have

$$\sin y = \sin 0 + \frac{\cos 0}{1!}y - \frac{\sin c}{2!}y^2$$

for some c between 0 and y.
As $-1 \leq \sin c \leq 1$, we have $\sin y \geq y - \frac{1}{2}y^2$ for all $y \in \mathbb{R}$.
Therefore, we have

$$\sin \frac{x^p}{n^p} \geq \frac{x^p}{n^p} - \frac{1}{2}\frac{x^{2p}}{n^{2p}}$$

Now, we will show that the series

$$\sum_{n=1}^{\infty}\left(\frac{x^p}{n^p} - \frac{1}{2}\frac{x^{2p}}{n^{2p}}\right) = \sum_{n=1}^{\infty} x^p \left(\frac{1}{n^p} - \frac{1}{2}\frac{x^p}{n^{2p}}\right)$$

diverges for any $x \neq 0$.

Consider the series $\sum_{n=1}^{\infty}\left(\frac{1}{n^p} - \frac{k}{n^{2p}}\right) = \sum_{n=1}^{\infty}\frac{n^p - k}{n^{2p}}$, where $k = \frac{1}{2}x^p$.
Let $a_n = \frac{n^p - k}{n^{2p}}$ and $b_n = \frac{1}{n^p}$.

If $p \in (0, 1]$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} 1 - \frac{k}{n^p} = 1 > 0$$

Clearly, if $p = 0$, the series diverges.
Moreover, if $p \in [0, 1]$, the series $\sum_{n=1}^{\infty} b_n$ diverges by p-series test.

Hence, by Limit Comparison Test, if $p \in [0, 1]$, the series $\sum_{n=1}^{\infty}\sin\frac{x^p}{n^p}$ is divergent at any $x \neq 0$.

Question 5

(a) Let $f_n(x) = \frac{1}{n}\sin\frac{x}{\sqrt{n}}$.
Clearly, $f_n(x)$ are differentiable for all $x \in \mathbb{R}$.
Moreover, $\sum_{n=1}^{\infty} f_n(1)$ converges. This is because (by Question 4)

$$\left|\frac{1}{n}\sin\frac{1}{\sqrt{n}}\right| \leq \left|\frac{1}{n\sqrt{n}}\right|$$

and the series $\sum_{n=1}^{\infty}\left|\frac{1}{n^{\frac{3}{2}}}\right|$ converges by the p-series test.
Therefore, $\sum_{n=1}^{\infty} f_n(1)$ converges by Comparison Test.

Furthermore,

$$\sum_{n=1}^{\infty} f_n' = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}\cos\frac{x}{\sqrt{n}}$$

converges uniformly. This is because

$$\left|\frac{1}{n^{\frac{3}{2}}}\cos\frac{x}{\sqrt{n}}\right| \leq \frac{1}{n^{\frac{3}{2}}}$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ converges by p-series test.

Hence, the convergence of $\displaystyle\sum_{n=1}^{\infty} f_n'$ follows by Weierstrass M-test.

In this case, we have $f(x)$ is uniformly convergent on $\mathbb{R}$ and $f'(x) = \displaystyle\sum_{n=1}^{\infty} f_n' = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}\cos\frac{x}{\sqrt{n}}$ as required.

Question 6

(a) For $x=0$, $\displaystyle\sum_{n=1}^{\infty}\frac{\cos\sqrt{n}x}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges. Furthermore, since cosine is even, it suffices to consider $x>0$. Let $x>0$ be given. Now, we note that $\cos\sqrt{n}x \geq \frac{1}{2}$ whenever

$$2k\pi - \frac{\pi}{3} \leq \sqrt{n}x \leq 2k\pi + \frac{\pi}{3},$$

that is,

$$\left(\frac{(6k-1)\pi}{3x}\right)^2 \leq n \leq \left(\frac{(6k+1)\pi}{3x}\right)^2.$$

Let $l_k = \left(\frac{(6k-1)\pi}{3x}\right)^2$, $L_k = \left(\frac{(6k+1)\pi}{3x}\right)^2$, $I_k = [l_k, L_k]$ and $\ell_k = L_k - l_k = \frac{8k\pi^2}{3x^2}$. Since $\ell_k \to \infty$ as $k\to\infty$, we deduce that $\exists N_1 \in \mathbb{N}$ such that $I_k \cap \mathbb{N} \neq \emptyset$ for every $k \geq N_1$. In addition, let $\lfloor\cdot\rfloor$, $\lceil\cdot\rceil$ and $\#I_k$ denote the floor function, ceiling function and number of integers in I_k respectively. Then $\#I_k = \lfloor L_k\rfloor - \lceil l_k\rceil + 1 > L_k - l_k - 1 = \ell_k - 1$. Now, for every $k \geq N_1$,

$$\begin{aligned}\left|\sum_{n\in I_k}\frac{\cos\sqrt{n}x}{\sqrt{n}}\right| &\geq \sum_{n\in I_k}\frac{\cos\sqrt{n}x}{\sqrt{n}} \geq \frac{1}{2}\sum_{n\in I_k}\frac{1}{\sqrt{n}} \geq \frac{1}{2}\sum_{n\in I_k}\frac{1}{\sqrt{L_k}}\\ &= \frac{1}{2}\frac{\#I_k}{\sqrt{L_k}} > \frac{1}{2}\frac{\ell_k-1}{\sqrt{L_k}}\end{aligned}$$

Now, $\lim_{k\to\infty}\frac{1}{\sqrt{L_k}} = 0$ and

$$\lim_{k\to\infty}\frac{\ell_k}{\sqrt{L_k}} = \lim_{k\to\infty}\frac{8k\pi^2}{3x^2}\cdot\frac{3x}{(6k+1)\pi} = \frac{4\pi}{3x}.$$

Hence, $\lim_{k\to\infty}\frac{\ell_k-1}{\sqrt{L_k}} = \frac{4\pi}{3x}$. Thus $\exists N_2\in\mathbb{N}$ such that $\left|\frac{\ell_k-1}{\sqrt{L_k}} - \frac{4\pi}{3x}\right| < \frac{2\pi}{3x}$ whenever $k\geq N_2$. Therefore, for every $k \geq \max(N_1,N_2)$, we have

$$\left|\sum_{n\in I_k}\frac{\cos\sqrt{n}x}{\sqrt{n}}\right| > \frac{\pi}{3x}.$$

In conclusion, $\displaystyle\sum_{n=1}^{\infty}\frac{\cos\sqrt{n}x}{\sqrt{n}}$ fails the Cauchy Criterion and thus diverges.

(b) For $x=0$, $\displaystyle\sum_{n=1}^{\infty}\frac{\sin\sqrt{n}x}{n} = 0$.

Furthermore, since sine is an odd function, we have $-\displaystyle\sum_{n=1}^{\infty}\frac{\sin\sqrt{n}x}{n} = \sum_{n=1}^{\infty}\frac{\sin(-\sqrt{n}x)}{n}$, hence it suffices to consider the case where $x > 0$.

Let $x > 0$ be given and let $y_n = \sqrt{n}x$ for each $n \in \mathbb{N}$. Note that $\bigcup_{n=1}^{\infty}[y_n, y_{n+1})$ forms a partition on $[x, \infty)$. Moreover, $\lim_{n\to\infty} y_{n+1} - y_n = 0$ and $x^2 = y_{n+1}^2 - y_n^2$ for all $n \in \mathbb{N}$. Therefore,

$$\sum_{n=1}^{\infty} \frac{\sin \sqrt{n}x}{n} = x^2 \frac{\sin y_n}{y_n^2} = (y_{n+1} + y_n) \frac{\sin y_n}{y_n^2} (y_{n+1} - y_n).$$

Since $\lim_{n\to\infty} y_{n+1} - y_n = 0$, heuristically $y_{n+1} \approx y_n$ for large n. Hence, the tail sum of $\sum_{n=1}^{\infty} \frac{\sin \sqrt{n}x}{n}$ is the Riemann sum of $\int_I \frac{2 \sin y}{y}\, dy$ over some appropriate interval I.
Since $\int_0^{\infty} \frac{\sin t}{t}\, dt$ converges, we conclude that $\sum_{n=1}^{\infty} \frac{\sin \sqrt{n}x}{n}$ converges.