NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA2213 Numerical Analysis
AY 2014/2015 Sem 1
Version 1: December 10, 2014

Written by Fook Fabian
Audited by Henry Morco
Contributors

Question 1

(i)

$$\begin{aligned} x &= \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a} \\ &= \frac{70 \pm \sqrt{(70^2 - 4 \cdot 1 \cdot 1)}}{2} \\ &= 35 \pm 6\sqrt{34} \end{aligned}$$

$x_1 = 69.985711, x_2 = 0.014288631$

(ii)

$$\begin{aligned} x &= \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a} \\ &= \frac{70 \pm \sqrt{(70^2 - 4 \cdot 1 \cdot 1)}}{2} \\ &= \frac{70 \pm \sqrt{(4896)}}{2} \\ &= \frac{70 \pm 69.97}{2} \\ &= \frac{140.0}{2}, \frac{0.03}{2} \end{aligned}$$

$x_1^* = 70.00, x_2^* = 0.01500$
Relative errors:
in x_1 : $|\frac{70.00-69.985711}{69.987511}| = 0.00020$
in x_2 : $|\frac{0.01500-0.014288631}{0.014288631}| = 0.050$
To obtain more accurate x_2^*:
1)

$$\begin{aligned} x_2^* &= \frac{70 - \sqrt{4896}}{2} \\ &= \frac{70^2 - 4896}{2(70 + \sqrt{4896})} \\ &= \frac{4}{2(70 + 69.97)} \\ &= \frac{4}{2(140.0)} \\ &= 0.01429 \end{aligned}$$

2)

$$\begin{aligned} x^2 - 70x + 1 &= (x - x_1)(x - x_2) \\ &= x^2 - (x_1 + x_2)x + (x_1 x_2) \end{aligned}$$

$$\begin{aligned} \therefore x_2^* &= \frac{1}{x_1^*} \\ &= \frac{1}{70} \\ &= 0.01429 \end{aligned}$$

Question 2

$$\left(\begin{array}{ccc|c} 2.11 & -4.21 & 0.921 & 2.01 \\ 4.01 & 10.2 & -1.12 & -3.09 \\ 1.09 & 0.987 & 0.832 & 4.21 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1.09 & 0.987 & 0.832 & 4.21 \\ 4.01 & 10.2 & -1.12 & -3.09 \\ 2.11 & -4.21 & 0.921 & 2.01 \end{array}\right) \xrightarrow[R_3 - 1.94R_1]{R_2 - 3.68R_1}$$

$s_1 = 4.21, s_2 = 10.2, s_3 = 1.09$ $\quad$ $s_1 = 1.09, s_2 = 10.2, s_3 = 4.21$

$$\left(\begin{array}{ccc|c} 1.09 & 0.987 & 0.832 & 4.21 \\ 0 & 6.57 & -4.18 & -18.6 \\ 0 & -6.12 & -0.689 & -6.16 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1.09 & 0.987 & 0.832 & 4.21 \\ 0 & -6.12 & -0.689 & -6.16 \\ 0 & 6.57 & -4.18 & -18.6 \end{array}\right) \xrightarrow{R_3 + 1.07R_2}$$

$s_1 = 1.09, s_2 = 10.2, s_3 = 4.21$ $\quad$ $s_1 = 1.09, s_2 = 4.21, s_3 = 10.2$

$$\left(\begin{array}{ccc|c} 1.09 & 0.987 & 0.832 & 4.21 \\ 0 & -6.12 & -0.689 & -6.16 \\ 0 & 0 & -4.92 & -25.2 \end{array}\right)$$

$x_1 = -0.431, x_2 = 0.430, x_3 = 5.12$

Question 3

Let p_n be the interpolating polynomial of degree at most n for $f(x)$.
In Lagrange form: $f(x) - p_n(x) = \psi(x)\frac{f^{n+1}(\xi)}{(n+1)!}$
In Newton form: $f(x) - p_n(x) = \psi(x) f[x, x_0, \dots x_n]$
(By the uniqueness property of the interpolating polynomial, the p_ns above are exactly the same.)
$\therefore f[x, x_0, \dots x_n] = \frac{f^{n+1}(\xi)}{(n+1)!}$
Now consider the Newton form with a rearrangement of the order of points:

$$\begin{aligned} f(x) &= p_n(x) + (x - x_n)(x - x_{n-1}) \dots (x - x_1)(x - x_0) f[x, x_n, \dots x_0] \\ &= p_n(x) + (x - x_0)(x - x_1) \dots (x - x_{n-1})(x - x_n) f[x, x_n, \dots x_0] \\ &= p_n(x) + \psi(x) f[x, x_n, \dots x_0] \end{aligned}$$

$\therefore f[x, x_n, \dots x_0] = f[x, x_0, \dots x_n]$
$\therefore f[x, x_n, \dots x_0] = f[x, x_0, \dots x_n] = \frac{f^{n+1}(\xi)}{(n+1)!}$

Auditor's Note An alternative solution is as follows. Let the $n+2$ data points $x_0, x_1, \ldots, x_n, x$ be renamed as $a \leq x_0' \leq x_1' \leq \cdots \leq x_{n+1}' \leq b$, and let $p(x)$ be the Lagrange polynomial interpolating f at $x_0', \ldots, x_{n+1}'$. The leading term of p is $f\left[x_0', \ldots, x_{n+1}'\right] = f\left[x, x_0, \ldots, x_n\right]$. Consider $g(x) = f(x) - p(x)$ which is in $\mathrm{C}^{(n+1)}[a, b]$ and is zero at $x_0', \ldots, x_{n+1}'$. By Rolle's Theorem, there exist $\xi_{1,0}, \xi_{1,1}, \ldots, \xi_{1,n}$ such that $x_i \leq \xi_{1,i} \leq x_{i+1}$ and $g'(\xi_{1,i}) = 0$ for $i = 0, 1, \ldots, n$. Applying Rolle's Theorem again to g', g'', and so on, we obtain for $j = 2, \ldots, n+1$ the values $\xi_{j,0}, \xi_{j,1}, \ldots, \xi_{j,n-j+1}$ such that $\xi_{j-1,i} \leq \xi_{j,i} \leq \xi_{j-1,i+1}$ and $g^{(j)}(\xi_{j,i}) = 0$ for $i = 0, 1, \ldots, n-j+1$. Then set $\xi = \xi_{n+1}$ so that $0 = g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \frac{d^{n+1}}{dx^{n+1}}(p(\xi)) = f^{(n+1)}(\xi) - f\left[x_0', \ldots, x_{n+1}'\right] n!$. Then we have

$$\begin{aligned} f\left[x, x_0, \ldots, x_n\right] &= f\left[x_0', \ldots, x_{n+1}'\right] \\ &= \frac{f^{(n+1)}(\xi)}{n!} \end{aligned}$$

as desired.

Question 4

We require that $\int_{-1}^{1} f(x)dx = w_0 f(x_0) + w_1 f(x_1)$ for $f(x) = 1, x, x^2, x^3$.

$$\begin{cases} w_0 + w_1 = 2 \\ w_0 x_0 + w_1 x_1 = 0 \\ w_0 x_0^2 + w_1 x_1^2 = \frac{2}{3} \\ w_0 x_0^3 + w_1 x_1^3 = 0 \end{cases}$$

This set of equations has 4 unknowns (w_0, w_1, x_0, x_1) and 4 equations, so it is solvable. To simplify things, we will assume that $x_0 = -x_1, x_0 < x_1$.
Substituting into the third equation: $(w_0 + w_1)x_1^2 = \frac{2}{3}$
Substituting the first equation: $2x_1^2 = \frac{2}{3}$
$\therefore x_1 = \frac{1}{\sqrt{3}}, x_2 = -\frac{1}{\sqrt{3}}$
Substituting into the second equation: $w_0 - w_1 = 0 \rightarrow w_0 = w_1$
Substituting into the first equation: $2w_1 = 2$
$\therefore w_1 = 1, w_0 = 1$

Now, substitute w_0, w_1, x_0, x_1 into all equations to ensure that they are satisfied:

$$\begin{cases} 1 + 1 = 2 \\ \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} = 0 \\ \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \\ \frac{1}{3\sqrt{3}} - \frac{1}{3\sqrt{3}} = 0 \end{cases}$$

$\therefore w_0 = 1, w_1 = 1, x_1 = \frac{1}{\sqrt{3}}, x_2 = -\frac{1}{\sqrt{3}}$

Question 5

η is a solution of multiplicity 3 of $f(x) = 0$.
Newton's method: $x_{n+1} = x_n - 3\frac{f(x_n)}{f'(x_n)}$

Auditor's Note For completeness, a proof of this convergence must be supplied. It is known that a fixed-point iterative method $x_{n+1} = g(x_n)$ that converges to some η converges quadratically if $g(\eta) = g'(\eta) = 0$ and $g''(x)$ is continuous at some interval containing η. Since f is known to be a polynomial the second criterion is satisfied. For the first criterion, we note that $f(\eta) = f'(\eta) = f''(\eta) = 0$ and $f^{(3)}(\eta) \neq 0$ so we have

$$\begin{aligned}
g'(x) &= 1 - 3 + 3\frac{f(x)\,f''(x)}{f'(x)^2} \\
g'(\eta) &= -2 + 3\lim_{x\to\eta}\frac{f(x)\,f''(x)}{f'(x)^2} \\
&= -2 + 3\lim_{x\to\eta}\frac{f'(x)\,f''(x) + f(x)\,f^{(3)}(x)}{2f'(x)\,f''(x)} \\
&= -2 + \frac{3}{2} + \frac{3}{2}\lim_{x\to\eta}\frac{f(x)\,f^{(3)}(x)}{f'(x)\,f''(x)} \\
&= -\frac{1}{2} + \frac{3}{2}f^{(3)}(x)\lim_{x\to\eta}\frac{f'(x)}{f''(x)^2 + f'(x)\,f^{(3)}(x)}
\end{aligned}$$

Now note that

$$\begin{aligned}
\lim_{x\to\eta}\frac{f''(x)^2 + f'(x)\,f^{(3)}(x)}{f'(x)} &= f^{(3)}(x) + \lim_{x\to\eta}\frac{f''(x)^2}{f'(x)} \\
&= f^{(3)}(x) + \lim_{x\to\eta}\frac{2f''(x)\,f^{(3)}(x)}{f''(x)} \\
&= 3f^{(3)}(x)
\end{aligned}$$

Hence we have

$$\begin{aligned}
g'(\eta) &= -\frac{1}{2} + \frac{3}{2}f^{(3)}(x)\lim_{x\to\eta}\frac{f'(x)}{f''(x)^2 + f'(x)\,f^{(3)}(x)} \\
&= -\frac{1}{2} + \frac{1}{2} \\
&= 0
\end{aligned}$$

as desired.

END OF SOLUTIONS

Any Mistakes? *The LaTeXify Team takes great care to ensure solution accuracy. If you find any error or factual inaccuracy in our solutions, do let us know at* latexify@gmail.com*. Contributors will be credited in the next version!*

Join Us! *Want to be of service to fellow students by producing beautifully typeset Past Year Paper Solutions? Enquire at* latexify@gmail.com*.*