NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Teo Wei Hao

**ST2131/MA2216 Probability**
AY 2004/2005 Sem 2

---

**Question 1**

(a) (i) We have $1 = \int_{\mathbb{R}} f(x)\ dx = \int_{-1}^{1} a + bx\ dx = \left[ax + \frac{bx^2}{2}\right]_{-1}^{1} = 2a$. Thus $a = \frac{1}{2}$.
Also, for all $x \in \mathbb{R}$, $f(x) \geq 0$.
Thus for $-1 \leq x < 0$, we have $\frac{1}{2} + bx \geq 0$, i.e. $b \leq \frac{-1}{2x}$. This give us $b \leq \frac{1}{2}$.
Also for $0 < x \leq 1$, we have $\frac{1}{2} + bx \geq 0$, i.e. $b \geq \frac{-1}{2x}$. This give us $b \geq -\frac{1}{2}$.

Thus the conditions are $a = \frac{1}{2}$ and $-\frac{1}{2} \leq b \leq \frac{1}{2}$.

(ii) We have,

$$\begin{aligned} E(X) = \int_{\mathbb{R}} xf(x)\ dx &= \int_{-1}^{1} ax + bx^2\ dx \\ &= \left[\frac{1}{2}ax^2 + \frac{1}{3}bx^3\right]_{-1}^{1} = \frac{2b}{3}. \end{aligned}$$

Since $-\frac{1}{2} \leq b \leq \frac{1}{2}$, $E(X)$ is maximized when $a = \frac{1}{2}$, $b = \frac{1}{2}$.

(b) Let us give an order to the employees from 1 to 300.
Let $X_i$ be the r.v. of the number of people the $i$-th employee brings to the party, $1 \leq i \leq 300$.
We have $\mathbb{P}\{X_i = 0\} = \frac{1}{6}$, $\mathbb{P}\{X_i = 1\} = \frac{1}{2}$, $\mathbb{P}\{X_i = 2\} = \frac{1}{3}$. Thus $E(X_i) = (1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{3}\right) = \frac{7}{6}$, $E(X_i^2) = (1)\left(\frac{1}{2}\right) + (4)\left(\frac{1}{3}\right) = \frac{11}{6}$ and $\mathrm{Var}(X_i) = E(X_i^2) - E(X_i)^2 = \frac{11}{6} - \frac{49}{36} = \frac{17}{36}$.

Let $X$ be the r.v. of the number of people who turn up at the party. This give us $X = \sum_{i=1}^{300} X_i$. Since the $X_i$'s are independent r.v., we have $E(X) = \sum_{i=1}^{300} E(X_i) = 300\left(\frac{7}{6}\right) = 350$, and also $\mathrm{Var}(X) = \sum_{i=1}^{300} \mathrm{Var}(X_i) = 300\left(\frac{17}{36}\right) = \frac{850}{6}$. Thus by C.L.T., we have $X \approx N\left(350, \frac{850}{6}\right)$.
Therefore, with suitable continuity correction, we have,

$$\begin{aligned} \mathbb{P}\{X \geq 330\} = \mathbb{P}\{X > 329.5\} &\approx \mathbb{P}\left\{Z > \frac{329.5 - 350}{\sqrt{850/6}}\right\} \\ &= \mathbb{P}\{Z > -1.722\} = 0.957496. \end{aligned}$$

**Question 2**

(a) The question is equivalent to finding the probability that a customer will stay in the shop for at least 0.75 hours. Let $E_1$ be the above mentioned event, and $G, A, B$ be the events that the customer is good, average and bad respectively. Let $X_G, X_A, X_B$ be the r.v. of the amount of time in hours the good, average and bad customer spent in the shop respectively.
This give us $X_G \sim \mathrm{Exp}(2)$, $X_A \sim \mathrm{Exp}(1)$ and $X_B \sim \mathrm{Exp}(0.5)$. Thus,

$$\begin{aligned} \mathbb{P}(E_1) &= \mathbb{P}(E_1 \mid G)\mathbb{P}(G) + \mathbb{P}(E_1 \mid A)\mathbb{P}(A) + \mathbb{P}(E_1 \mid B)\mathbb{P}(B) \\ &= \mathbb{P}\{X_G > 0.75\}\mathbb{P}(G) + \mathbb{P}\{X_A > 0.75\}\mathbb{P}(A) + \mathbb{P}\{X_B > 0.75\}\mathbb{P}(B) \\ &= \left(e^{-(2)(0.75)}\right)(0.35) + \left(e^{-(1)(0.75)}\right)(0.5) + \left(e^{-(0.5)(0.75)}\right)(0.15) = 0.417372. \end{aligned}$$

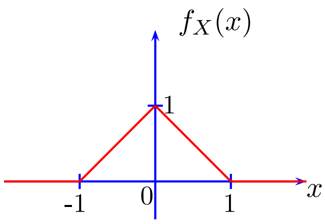
(b) Let $E_2$ be the event that a customer spent 1 to 1.5 hours in the shop. We have,

$$\begin{aligned}\mathbb{P}(G \mid E_2) &= \frac{\mathbb{P}(E_2 \mid G)\mathbb{P}(G)}{\mathbb{P}(E_2)} \\ &= \frac{\mathbb{P}(E_2 \mid G)\mathbb{P}(G)}{\mathbb{P}(E_2 \mid G)\mathbb{P}(G) + \mathbb{P}(E_2 \mid A)\mathbb{P}(A) + \mathbb{P}(E_2 \mid B)\mathbb{P}(B)} \\ &= \frac{\mathbb{P}\{1 < X_G < 1.5\}\mathbb{P}(G)}{\mathbb{P}\{1 < X_G < 1.5\}\mathbb{P}(G) + \mathbb{P}\{1 < X_A < 1.5\}\mathbb{P}(A) + \mathbb{P}\{1 < X_B < 1.5\}\mathbb{P}(B)} \\ &= \frac{\left(e^{-(2)(1)} - e^{-(2)(1.5)}\right)(0.35)}{\left(e^{-(2)(1)} - e^{-(2)(1.5)}\right)(0.35) + \left(e^{-(1)(1)} - e^{-(1)(1.5)}\right)(0.5) + \left(e^{-(0.5)(1)} - e^{-(0.5)(1.5)}\right)(0.15)} \\ &= 0.244541.\end{aligned}$$

Thus the probability that David is a good customer is 0.244541.

## Question 3

(a) We have the graph of $f_X(x)$ to be




When $x \le -1$, we have $F_X(x) = 0$.
When $-1 < x \le 0$, we have $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = 0 + \int_{-1}^{x} 1 + x\,dx = \left[\frac{(1+x)^2}{2}\right]_{-1}^{x} = \frac{(1+x)^2}{2}$.
When $0 < x < 1$, we have $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \frac{1}{2} + \int_{0}^{x} 1 - x\,dx = \frac{1}{2} + \left[\frac{-(1-x)^2}{2}\right]_{0}^{x} = 1 - \frac{(1-x)^2}{2}$.
When $x \ge 1$, we have $F_X(x) = 1$.
Therefore the c.d.f. of $X$ is,

$$F_X(x) = \begin{cases} 0, & x \le -1; \\ \frac{(1+x)^2}{2}, & -1 < x \le 0; \\ 1 - \frac{(1-x)^2}{2}, & 0 < x < 1; \\ 1, & x \ge 1. \end{cases}$$

(b) We have $F_Y(y) = \mathbb{P}\{Y \le y\} = \mathbb{P}\{1 - 2X \le y\} = \mathbb{P}\left\{X \ge \frac{1-y}{2}\right\} = 1 - F_X\left(\frac{1-y}{2}\right)$.
When $y \le -1$, we have $\frac{1-y}{2} \ge 1$, and so $F_Y(y) = 1 - 1 = 0$.
When $-1 < y < 1$, we have $0 < \frac{1-y}{2} < 1$. Thus $F_Y(y) = 1 - \left[1 - \frac{\left(1-\frac{1-y}{2}\right)^2}{2}\right] = \frac{(1+y)^2}{8}$.
When $1 \le y < 3$, we have $-1 < \frac{1-y}{2} \le 0$. Thus $F_Y(y) = 1 - \frac{\left(1+\frac{1-y}{2}\right)^2}{2} = 1 - \frac{(3-y)^2}{8}$.
When $y \ge 3$, we have $\frac{1-y}{2} \le -1$, and so $F_Y(y) = 1 - 0 = 1$.
Therefore the c.d.f. of $Y$ is,

$$F_Y(y) = \begin{cases} 0, & y \le -1; \\ \frac{(1+y)^2}{8}, & -1 < y < 1; \\ 1 - \frac{(3-y)^2}{8}, & 1 \le y < 3; \\ 1, & x \ge 3. \end{cases}$$

Thus by differentiating the c.d.f. of $Y$, we get the p.d.f. of $Y$ to be,

$$f_Y(y) = \begin{cases} \frac{1+y}{4}, & -1 < y < 1; \\ \frac{3-y}{4}, & 1 \leq y < 3; \\ 0, & \text{otherwise.} \end{cases}$$

**Question 4**

(a) For $1 < x < 3$, we have $1 < 2 < 5 - x < 4$, and so,

$$\begin{aligned} F_{X+Y}(5) = \int_{\mathbb{R}} F_Y(5-x) f_X(x)\, dx &= \int_1^3 \left( \int_1^{5-x} \frac{2}{15} y\, dy \right) \left( \frac{x}{4} \right) dx \\ &= \int_1^3 \left[ \frac{y^2}{15} \right]_1^{5-x} \left( \frac{x}{4} \right) dx \\ &= \int_1^3 \frac{x^3 - 10x^2 + 24x}{60}\, dx \\ &= \left[ \frac{x^4}{240} - \frac{x^3}{18} + \frac{x^2}{5} \right]_1^3 = \frac{22}{45}. \end{aligned}$$

Thus the probability that George's total salary increase for next year exceed \$500 is $1 - \frac{22}{45} = \frac{23}{45}$.

(b) We have,

$$\begin{aligned} E(|X-Y|) &= \int_{\mathbb{R}} E(|x-Y|) f_X(x)\, dx \\ &= \int_1^3 (E(|x-Y|,\ Y > x) + E(|x-Y|,\ Y \leq x)) \left( \frac{x}{4} \right) dx \\ &= \int_1^3 \left( \int_x^4 (y-x) \frac{2}{15} y\, dy + \int_1^x (x-y) \frac{2}{15} y\, dy \right) \left( \frac{x}{4} \right) dx \\ &= \int_1^3 \left( \left[ \frac{2}{45} y^3 - \frac{1}{15} x y^2 \right]_x^4 + \left[ \frac{1}{15} x y^2 - \frac{2}{45} y^3 \right]_1^x \right) \left( \frac{x}{4} \right) dx \\ &= \int_1^3 \left( \left( \frac{128}{45} - \frac{16}{15} x + \frac{1}{45} x^3 \right) + \left( \frac{1}{45} x^3 - \frac{1}{15} x + \frac{2}{45} \right) \right) \left( \frac{x}{4} \right) dx \\ &= \int_1^3 \frac{13}{18} x - \frac{17}{60} x^2 + \frac{1}{90} x^4\, dx \\ &= \left[ \frac{13}{36} x^2 - \frac{17}{180} x^3 + \frac{1}{450} x^5 \right]_1^3 = \frac{437}{450}. \end{aligned}$$

Thus the expected differences is $\frac{437}{450} \times \$100 \approx \$97.11$.

(c) Let event where Job $A$ and Job $B$ give a bigger raise be $E_A$ and $E_B$ respectively.
Then we see that $E_A \cap E_B = \emptyset$. Thus $\mathbb{P}(E_A E_B) = \mathbb{P}(\emptyset) = 0 \neq \mathbb{P}(E_A)\mathbb{P}(E_B)$.
Therefore $E_A$ and $E_B$ are not independent.