NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Lau Tze Siong

MA2108 Mathematical Analysis I
AY 2005/2006 Sem 1

Question 1

(a) (i)

$$\begin{aligned}\lim_{n\to\infty}\frac{n-2n^2+3\ln n}{n^2+5-2n} &= \lim_{n\to\infty}\frac{\frac{1}{n}-2+\frac{3\ln n}{n^2}}{1+\frac{5}{n^2}-\frac{2}{n}}\\ &= -2\end{aligned}$$

(ii) Since

$$5=(5^n)^{\frac{1}{n}}\le(5^n+4^n)^{\frac{1}{n}}\le(2\cdot 5^n)^{\frac{1}{n}}=2^{\frac{1}{n}}\cdot 5$$

By Squeeze Theorem, we have $\lim_{n\to\infty}(5^n+4^n)^{\frac{1}{n}}=5$.

(iii)

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{2n^2-1}{2n^2+1}\right)^{n^2} &= \lim_{n\to\infty}\left(1-\frac{2}{2n^2+1}\right)^{n^2}\\ &= \lim_{m\to\infty}\left(\left(1-\frac{1}{m}\right)^{2m+1}\right)^{\frac{1}{2}}\\ &= \lim_{m\to\infty}\left(\left(\left(1-\frac{1}{m}\right)^{m}\right)^{2}\left(1-\frac{1}{m}\right)\right)^{\frac{1}{2}}\\ &= \left(e^{-2}\right)^{\frac{1}{2}}\\ &= e^{-1}\end{aligned}$$

(iv) Since

$$\begin{aligned} n \le & \quad 2n+\sin n & \le 3n\\ n^{\frac{1}{1+2\ln n}} \le & \quad (2n+\sin n)^{\frac{1}{1+2\ln n}} & \le (3n)^{\frac{1}{1+2\ln n}}\end{aligned}$$

Let $x_n=n^{\frac{1}{1+2\ln n}}$, $y_n=(3n)^{\frac{1}{1+2\ln n}}$, $z_n=(2n+\sin n)^{\frac{1}{1+2\ln n}}$.
Since

$$\begin{aligned}\lim_{n\to\infty}\ln x_n &= \lim_{n\to\infty}\frac{\ln n}{1+2\ln n}\\ &= \lim_{n\to\infty}\frac{1}{\frac{1}{\ln n}+2}\\ &= \frac{1}{2}.\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\ln y_n &= \lim_{n\to\infty}\frac{\ln 3n}{1+2\ln n}\\ &= \lim_{n\to\infty}\frac{\ln 3+\ln n}{1+2\ln n}\\ &= \lim_{n\to\infty}\frac{\frac{\ln 3}{\ln n}+1}{\frac{1}{\ln n}+2}\\ &= \frac{1}{2}.\end{aligned}$$

Hence by Squeeze Theorem,

$$\lim_{n\to\infty}\ln z_n = \frac{1}{2}$$

Since $\ln : \mathbb{R}_{>0} \to \mathbb{R}$ is a continuous function. we have $\lim_{n\to\infty} z_n = e^{\frac{1}{2}}$.

(b) If $n = 4m$ for some $m \in \mathbb{N}$, we have $a_n = \cos(2n\pi) + \sin(2n\pi) = 1$.
If $n = 4m+1$ for some $m \in \mathbb{N}$, we have $a_n = \cos\left(2n\pi + \frac{\pi}{2}\right) - \sin\left(2n\pi + \frac{\pi}{2}\right) = -1$.
If $n = 4m+2$ for some $m \in \mathbb{N}$, we have $a_n = \cos(2n\pi + \pi) + \sin(2n\pi + \pi) = -1$.
If $n = 4m+3$ for some $m \in \mathbb{N}$, we have $a_n = \cos\left(2n\pi + \frac{3\pi}{2}\right) - \sin\left(2n\pi + \frac{3\pi}{2}\right) = 1$.

Hence $\overline{\lim_{n\to\infty}} a_n = 1$

Question 2

(a) (i) Since,

$$\lim_{n\to\infty}\frac{\frac{2n^2-8n}{n^4+2n+1}}{\frac{1}{n^2}} = 2$$

and $\frac{2n^2-8n}{n^4+2n+1} > 0$ for $n > 4$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, we have $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2-8n}{n^4+2n+1}$ converges.

(ii) By Root Test, since

$$\begin{aligned}\lim_{n\to\infty}\left(3^n\left(\frac{n}{n+1}\right)^{n^2}\right)^{\frac{1}{n}} &= \lim_{n\to\infty}3\left(\frac{n}{n+1}\right)^n\\ &= 3\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n\\ &= 3\lim_{m\to\infty}\left(1-\frac{1}{m}\right)^m\left(1-\frac{1}{m}\right)^{-1}\end{aligned}$$

$$= 3e^{-1} > 1$$

$\displaystyle\sum_{n=1}^{\infty}3^n\left(\frac{n}{n+1}\right)^{n^2}$ diverges.

(iii) By Ratio Test,since

$$\begin{aligned}\lim_{n\to\infty}\frac{\frac{(n+1)^{3n+3}}{(3n+3)!}}{\frac{n^{3n}}{(3n)!}} &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n}\frac{(n+1)^3}{(3n+1)(3n+2)(3n+3)}\\ &= \frac{e^3}{27} < 1\end{aligned}$$

$\sum_{n=1}^{\infty} \frac{n^{3n}}{(3n)!}$ converges.

(iv) By Limit Comparison Test, since

$$\begin{aligned}\lim_{n\to\infty} \frac{\ln \frac{n^2+1}{n}}{\frac{1}{n^2}} &= \lim_{n\to\infty} \ln \left(1+\frac{1}{n^2}\right)^{n^2} \\ &= \ln e = 1\end{aligned}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} \ln \frac{n^2+1}{n}$ converges.

(b) Since $\overline{\lim_{n\to\infty}}([(-1)^n+2]^n)^{\frac{1}{n}} = \overline{\lim_{n\to\infty}}[(-1)^n+2] = 3$.
The radius of convergence $R = \frac{1}{3}$.

Question 3

(a) Since, for any given $x \in (1,\infty)$, $\lim_{n\to\infty} \frac{\sqrt{n}\ln x}{x^n + \ln x} = \lim_{n\to\infty} \frac{\sqrt{n}\frac{\ln x}{x^n}}{1+\frac{\ln x}{x^n}} = 0$ and when $x = 1$, $\lim_{n\to\infty} \frac{\sqrt{n}\ln x}{x^n+\ln x} = 0$. F_n converges to 0. Since,

$$\begin{aligned}\lim_{n\to\infty} \sup_{x\in[1,\infty)} \left|\frac{\sqrt{n}\ln x}{x^n+\ln x}\right| &= \lim_{n\to\infty} \sup_{x\in[1,\infty)} \left|\frac{\sqrt{n}\frac{\ln x}{x^n}}{1+\frac{\ln x}{x^n}}\right| \\ &= \lim_{n\to\infty} \frac{\sqrt{n}\frac{1}{e^n}}{1+\frac{1}{e^n}} \\ &= 0\end{aligned}$$

F_n is uniformly convergent.

(b) (i) Since,

$$\left|\frac{\sin nx}{\sqrt{n^3+x}}\right| \le \left|\frac{1}{n^{\frac{3}{2}}}\right|$$

and $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^3}}$ converges. By Weierstrass M-Test, $\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n^3+x}}$ converges uniformly.

(ii) Claim: $\frac{1}{\sqrt[3]{n+\ln x}}$ converges uniformly to 0 on the interval $[1,\infty)$.
Proof:
Since,

$$\begin{aligned}\lim_{n\to\infty} \sup_{x\in[1,\infty)} \left|\frac{1}{\sqrt[3]{n+\ln x}}\right| &= \lim_{n\to\infty} \left|\frac{1}{\sqrt[3]{n}}\right| \\ &= 0\end{aligned}$$

we have $\frac{1}{\sqrt[3]{n+\ln x}}$ converges uniformly to 0 on the interval $[1,\infty)$.
Let $x_n = (-1)^n$ and $s_n = \sum_{k=1}^{n} x_n$ and $y_n(x) = \frac{1}{\sqrt[3]{n+\ln x}}$.
Hence we have $s_n \le 1$ for all $n\ in\mathbb{N}$.

Hence, there exist a $N \in \mathbb{N}$ such that for all $x \in [1, \infty)$, $0 \leq y_n(x) < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}_{\geq N}$. Hence for all $m, n \in \mathbb{N}$ we have for all $x \in [1, \infty)$,

$$\begin{aligned}
\left|\sum_{k=m}^{n} x_k y_k\right| &= \left|\sum_{k=m}^{n-1} s_k(y_k - y_{k+1}) + s_n y_n - s_{m-1} y_m\right| \\
&\leq \left|\sum_{k=m}^{n-1} (y_k - y_{k+1}) + y_n + y_m\right| \\
&\leq 2y_m \leq \epsilon.
\end{aligned}$$

Hence $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[3]{n + \ln x}}$ converges uniformly in $[1, \infty)$.

(c) Claim: (a_n) is increasing

Proof:

Since $a_1 = 0.8$ and $a_2 = \frac{11}{10-0.8} > 1 > a_1$. Hence, we have $a_1 < a_2$.

Now suppose there exists a $k \in \mathbb{N}$ $a_k < a_{k+1}$, then

$$\begin{aligned}
-a_k &> -a_{k+1} \\
10 - a_k &> 10 - a_{k+1} \\
\frac{11}{10 - a_k} &< \frac{11}{10 - a_{k+1}} \\
a_{k+1} < a_{k+2}
\end{aligned}$$

. Hence, by induction, (a_n) is increasing.

Claim: (a_n) is bounded above by 2.

Proof:

We have $a_1 = 0.8 < 2$.

Now suppose there exists a $k \in \mathbb{N}$ such that $a_k < 2$, then

$$\begin{aligned}
-a_k &> -2 \\
10 - a_k &> 8 \\
\frac{11}{10 - a_k} &< \frac{11}{8} < 2.
\end{aligned}$$

Hence, by induction, (a_n) is bounded above by 2.

Since (a_n) is increasing and bounded above, by the Completeness of $\mathbb{R}$ $\lim_{n\to\infty} a_n = a$ exists and is bound above by 2. Also a satisfies the equation $a^2 - 10a + 11 = 0$. Hence $a = \frac{10-\sqrt{56}}{2}$.

Question 4

(a) Since $\displaystyle\lim_{n\to\infty} \frac{\sqrt{n+4}}{\sqrt{n+1}} = 1$. We have the radius of convergence $R = 1$. Hence the interval of convergence is,

$$\begin{aligned}
|2x - 1| &< 1 \\
-1 < 2x &- 1 < 1 \\
0 < &\ 2x < 1 \\
0 &< x < 1.
\end{aligned}$$

Since by the Alternating Series Test $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)}{\sqrt{n+3}}$ converges when $x=0$.

And by Limit Comparison Test $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)}{\sqrt{n+3}}$ diverges when $x=1$.

Hence the interval of convergence is $[0,1)$.

(b) Claim: $\frac{x^n+n\sin(nx^2)}{x^{n+1}+n\ln x}$ uniformly converges to $\frac{1}{x}$ on the interval $[2,4]$.

Proof:

For and $x \in [2,4]$, $\displaystyle\lim_{n\to\infty} \frac{x^n+n\sin(nx^2)}{x^{n+1}+n\ln x} = \lim_{n\to\infty} \frac{\frac{1}{x}+\frac{n\sin(nx^2)}{x^{n+1}}}{1+\frac{n\ln x}{x^{n+1}}} = \frac{1}{x}$. Hence $\frac{x^n+n\sin(nx^2)}{x^{n+1}+n\ln x}$ converges pointwise to $\frac{1}{x}$ on the interval $[2,4]$.

Since,

$$\begin{aligned}
\lim_{n\to\infty} \sup_{x\in[2,4]} \left| \frac{x^n-n\sin(nx^2)}{x^{n+1}+n\ln x} - \frac{1}{x} \right| &= \lim_{n\to\infty} \sup_{x\in[2,4]} \left| \frac{x^{n+1}-nx\sin(nx^2)-x^{n+1}-n\ln x}{x^{n+2}+nx\ln x} \right| \\
&= \lim_{n\to\infty} \sup_{x\in[2,4]} \left| \frac{\frac{-nx\sin(nx^2)-n\ln x}{x^{n+2}}}{1+\frac{n\ln x}{x^{n+1}}} \right| \\
&\leq \lim_{n\to\infty} \frac{\frac{4n+n\ln 4}{2^{n+2}}}{1} \\
&= \lim_{n\to\infty} \frac{4n+n\ln 4}{2^{n+2}} \\
&= 0.
\end{aligned}$$

Hence $\frac{x^n+n\sin(nx^2)}{x^{n+1}+n\ln x}$ uniformly converges to $\frac{1}{x}$ on the interval $[2,4]$.

Hence $\displaystyle\lim_{n\to\infty} \int_2^4 \frac{x^n+n\sin(nx^2)}{x^{n+1}+n\ln x}\, dx = \int_2^4 \lim_{n\to\infty} \frac{x^n+n\sin(nx^2)}{x^{n+1}+n\ln x}\, dx = \int_2^4 \frac{1}{x}\, dx = \ln 2$

(c) Yes.

Since (a_n) is positive monotone decreasing,

$$\begin{aligned}
a_n &\geq a_{n+1} \\
\frac{a_n}{n} &\geq \frac{a_{n+1}}{n+1}
\end{aligned}$$

So $\left(\frac{a_n}{n}\right)$ is also positive monotone decreasing.

By Cauchy Condensation Test,

$\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$ if and only if $\displaystyle\sum_{n=1}^{\infty} 2^n \frac{a_{2^n}}{2^n} = \sum_{n=1}^{\infty} a_{2^n}$ is convergent.

Since $\sum_{n=1}^{\infty} a_{2^n}$ is convergent by hypothesis, we deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent.

Question 5

(a) Since

$$\begin{aligned}
\ln(1+x) &= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{n}\ , \text{ for } x\in(-1,1)\\
\ln(1+\frac{x^2}{4}) &= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{4^n n}\ , \text{ for } x\in(-2,2)\\
\ln(1-\frac{x^2}{4}) &= \sum_{n=0}^{\infty}(-1)\frac{x^{2n}}{4^n n}\ , \text{ for } x\in(-2,2)
\end{aligned}$$

Hence we have,

$$\begin{aligned}
\ln\frac{4+x^2}{4-x^2} &= \ln\frac{1+\frac{x^2}{4}}{1-\frac{x^2}{4}}\\
&= \ln\left(1+\frac{x^2}{4}\right)-\ln\left(1-\frac{x^2}{4}\right)\\
&= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{4^n n}-\sum_{n=0}^{\infty}(-1)\frac{x^{2n}}{4^n n}\\
&= \sum_{n=0}^{\infty}\frac{x^{4n+2}}{2^{4n+1}(2n+1)}\ , \text{ for } x\in(-2,2)
\end{aligned}$$

Hence, $f^{(36)}=$ coefficient of $x^{36}=0$

(b) Let $a_n=\frac{(\ln n)^2}{\sqrt{n}+3}$, for sufficiently large n we have,

$$\sqrt{\frac{\sqrt{n+1}+3}{\sqrt{n}+3}}>1+\frac{1}{n\ln n}$$

Therefore,

$$\begin{aligned}
n\left(\sqrt{\frac{\sqrt{n+1}+3}{\sqrt{n}+3}}\right) &> n+\frac{1}{n\ln n}\\
\left(n\left(\sqrt{\frac{\sqrt{n+1}+3}{\sqrt{n}+3}}\right)-n\right)\ln n &> 1\\
n^{n\left(\sqrt{\frac{\sqrt{n+1}+3}{\sqrt{n}+3}}-1\right)}>e &> \left(1+\frac{1}{n}\right)^n\\
n^{n\sqrt{\frac{\sqrt{n+1}+3}{\sqrt{n}+3}}} &> (n+1)^n\\
n^{\sqrt{\frac{\sqrt{n+1}+3}{\sqrt{n}+3}}} &> n+1\\
n^{\sqrt{\sqrt{n+1}+3}} &> (n+1)^{\sqrt{\sqrt{n}+3}}\\
\sqrt{\sqrt{n+1}+3}\,(\ln n) &> \sqrt{\sqrt{n}+3}\,(\ln(n+1))\\
\left(\sqrt{n+1}+3\right)(\ln n)^2 &> \left(\sqrt{n}+3\right)(\ln(n+1))^2\\
\frac{(\ln n)^2}{\sqrt{n}+3} &> \frac{(\ln(n+1))^2}{\sqrt{n+1}+3}
\end{aligned}$$

Hence $\frac{(\ln n)^2}{\sqrt{n}+3}$ is eventually decreasing. Therefore by Alternating Series Test, $\sum_{n=1}^{\infty}(-1)^n \frac{(\ln n)^2}{\sqrt{n}+3}$ converges.
Since $\frac{(\ln x)^2}{\sqrt{x}+3} > \frac{1}{\sqrt{n}+3}$ for $n \geq 3$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+3}$ diverges, therefore by Comparison Test $\sum_{n=1}^{\infty} \frac{(\ln x)^2}{\sqrt{x}+3}$ diverges.

Therefore $\sum_{n=1}^{\infty}(-1)^n \frac{(\ln x)^2}{\sqrt{x}+3}$ is conditionally convergent.

(c) For any given $\epsilon \in \mathbb{R}_{>0}$.
Let $a_n = f_n(x)$ and $s_n = \sum_{i=n}^{m} a_i$ and $y_n = x^n$.
Since $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly, there exist $n \in \mathbb{N}$ such that $\left|\sum_{i=n}^{m} f_i(x)\right| < \frac{\epsilon}{2}$ for all $x \in [0,1]$ for all $n, m \in \mathbb{N}_{\geq N}$. Also we have $|y_n| \leq 1$.
Hence we have for all $x \in [0,1]$,

$$\begin{aligned}
\left|\sum_{k=n}^{m} a_n y_n\right| &= \left|\sum_{k=n}^{m-1} s_k(y_k - y_{k+1}) + s_m y_m - s_{n-1} y_n\right| \\
&\leq \left|\frac{\epsilon}{2}\right| \left|\sum_{k=n}^{m-1} (y_k - y_{k+1}) + y_m + y_n\right| \\
&= \epsilon y_n \leq \epsilon
\end{aligned}$$

for all $n, m \in \mathbb{N}_{\geq N}$
Hence $\sum_{k=1}^{\infty} f_n(x) x^n$ is uniformly Cauchy therefore converges uniformly.