NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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MA2202 Algebra I
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Question 1

Note that

$$T = \{(1), (12), (13), (23)\}$$

T is not a subgroup of S_3. $(12), (23) \in T$ as $(12)(12) = (1)$ and $(23)(23) = (1)$, but $(12)(23) = (123) \notin T$ as $(123)(123) = (132) \neq (1)$.

Question 2

(i) $G = \langle g \rangle$, and $|G| = 30$. Let H be a subgroup of G, such that $|H| = 6$. Now, since H is a subgroup of G which is a cyclic group, $H = \langle g^s \rangle$, where $|H| = \circ(g^s) = \frac{30}{\gcd(30,s)}$. Hence, $\gcd(30, s) = 5$, which implies that $s = 5$ or 25. Thus, $g^s = g^5$ or $g^{25} = g^{-5}$, and $H = \left\langle g^5 \right\rangle$ or $H = \left\langle g^{-5} \right\rangle$. Since $\left\langle g^5 \right\rangle = \left\langle g^{-5} \right\rangle$, the only subgroup of order 6 is $\left\langle g^5 \right\rangle$.

(ii) Let $h \in G$ such that $\circ(h) = 6$. Since G is cyclic and generated by g, $h = g^s$ for some integer s. Now, $\circ(h) = \circ(g^s) = \frac{30}{\gcd(s,30)} = 6 \Leftrightarrow \gcd(s, 30) = 5 \Leftrightarrow s = 25$ or 5. Thus, the only elements of order 6 are g^5 and $g^{25} = g^{-5}$.

Question 3

Let $k \in K$ such that $\circ(k) = n$. Since η is surjective, there exists $g \in G$ such that $k = \eta(g)$. Thus, $k^n = 1_K = (\eta(g))^n = \eta(g^n)$, which implies that $g^n \in \text{Ker}(\eta)$. Now, if $g^i \in \text{Ker}(\eta)$ for some $0 \leq i < n$, then $\eta(g^i) = (\eta(g))^i = k^i = 1_K$, and hence $\circ(k) < n$, which is a contradiction. Thus, $g^i \notin \text{Ker}(\eta)$ for all $0 \leq i < n$.

Choose an integer m, $0 < m \leq |G|$ such that $(g^n)^m = 1_G$. We claim that $\circ(g) = nm$. Suppose that $g^s = 1_G$ for some s. Then, $\eta(g^s) = (\eta(g))^s = k^s = 1_K$, and thus $n|s$. Since $1_G = (g^n)^{\frac{s}{n}}$, by our choice of m, $m \leq \frac{s}{n}$, and hence $mn \leq s$. Thus, mn is the least positive integer s such that $g^s = 1_G$, and we conclude that $\circ(g) = nm$. Thus, $\circ(g^m) = \frac{mn}{\gcd(mn,m)} = \frac{mn}{m} = n$, and we are done.

Question 4

Let $g \in \mathbb{Z}/(6)$, where $g \neq [0]_6$. Then, $g = [x]_6 = \underbrace{[1]_6 + ...[1]_6}_{x \text{ times}}$, and hence $\sigma(g) = \sigma([x]_6) = \underbrace{\sigma([1]_6) + ... + \sigma([1]_6)}_{x \text{ times}}$. Thus, σ is completely determined by the image of $[1]_6$.

Let $g = [1]_6 \in \mathbb{Z}/(6)$. Then, $\circ(g)|6$. Since $\sigma(g) \in \mathbb{Z}/(4)$, we have that $\circ(\sigma(g))|4$. Moreover, since $(\sigma(g))^6 = \sigma(g^6) = \sigma([0]_6) = [0]_4$, we have that $\circ(\sigma(g))|6$. Hence, $\circ(\sigma(g)) = 1$ or 2.

If $\circ(\sigma(g)) = 1$, then $\sigma(g) = [0]_4$, and thus σ is just the trivial group homomorphism defined by $\sigma : \mathbb{Z}/(6) \to \mathbb{Z}/(4)$, where $\sigma(h) = [0]_4$ for all $h \in \mathbb{Z}/(6)$.

If $\circ(\sigma(g)) = 2$, then $\sigma(g) = [2]_4$. Thus, σ is defined by $\sigma : \mathbb{Z}/(6) \to \mathbb{Z}/(4)$, where $\sigma([1]_6) = [2]_4$, and $\sigma([x]_6) = [2x]_4$. We shall prove that this mapping is well-defined. If $[x]_6 = [y]_6$, then $6|x - y \to 4|2(x - y) \to [2x]_4 = [2y]_4$. For all $[x]_6, [y]_6 \in \mathbb{Z}/(6)$, $\sigma([x]_6 + [y]_6) = \sigma([x + y]_6) = [2x + 2y]_4 = [2x]_4 + [2y]_4 = \sigma([x]_6) + \sigma([y]_6)$.

Question 5

It is given that H is a subgroup of A_6. It suffices to show that A_6 is a subset of H. We admit the result that A_6 is generated by 3-cycles, and in fact $A_6 = \langle(123), (124), (125), (126)\rangle$. Thus, it suffices to show that $(123), (124), (125), (126) \in H$ to prove that $A_6 \subseteq H$.

It is given that $(123) \in H$. Since $(345) \in A_6$ and H is a normal subgroup of A_6, $(345)(123)(345)^{-1} = (124) \in H$ as well. Similarly, since $(356), (365) \in A_6$, we have that $(356)(123)(356)^{-1} = (125) \in H$ and $(365)(123)(365)^{-1} = (126) \in H$, and hence $A_6 \subseteq H$, and we conclude that $H = A_6$.

Question 6

(i) By direct computation, it is easily verified that $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $A^4 = I$, and thus the order of A is 4. Similarly, $B^2 = I$, and thus the order of B is 2.

Now, $BAB = BAB^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -A = A^{-1}$, and hence the order of BAB is 4.

(ii) $G_1 = \{A^i B^j | i = 0, 1, 2, 3, j = 0, 1\}$, and thus $|G_1| = 8$.

(iii) $\tau\sigma\tau = (24)(1234)(24) = (1432)$, and thus the order of $\tau\sigma\tau$ is 4. Moreover, since the order of τ and σ is 2 and 4 respectively, and $\tau\sigma\tau = (1432) = \sigma^{-1}$, by similar calculations as in parts (i) and (ii), we conclude that $G_2 = \{\sigma^i \tau^j | i = 0, 1, 2, 3, j = 0, 1\}$, and hence $|G_2| = 8$.

(iv) We construct $\eta : G_1 \to G_2$, defined by $\eta(A^i B^j) = \sigma^i \tau^j$, for $i = 0, 1, 2, 3,\ j = 0, 1$. It is clear that η constructed in this manner is a well-defined group homomorphism. Note that both $|G_1| = |G_2| = 8$. Moreover, we have that $\tau\sigma\tau = \sigma^{-1}$, $BAB = A^{-1}$, $\circ(A) = \circ(\sigma) = 4$ and $\circ(B) = \circ(\tau) = 2$. Thus, η is an isomorphism.

Question 7

We consider the map $\sigma : G/(H \cap K) \to G/H \times G/K$, defined by $\sigma(g(H \cap K)) = (gH, gK)$ for all $g \in G$. Firstly, we shall show that this map is well-defined. Suppose we have that $g(H \cap K) = h(H \cap K)$. Then, $gh^{-1} \in H \cap K$, which implies that $gh^{-1} \in H$ and $gh^{-1} \in K$, and thus $gH = hH$ and $gK = hK$. Hence, $(gH, gK) = (hH, hK)$, and σ is well-defined.

Next, we shall show that σ is injective. Suppose that $\sigma(g(H \cap K)) = \sigma(h(H \cap K))$, i.e. $(gH, gK) = (hH, hK)$. Then, $gH = hH$ and $gK = hK$, which implies that $gh^{-1} \in H$ and $gh^{-1} \in K$, and thus $g(H \cap K) = h(H \cap K)$. Hence, σ is an injective map, and we conclude that $|G/(H \cap K)| \leq |G/H \times G/K| = |G : H||G : K|$.

Question 8

We first show that G is a finite cyclic group. Let $g \in G$, where $g \neq e_G$. Then, $\{e_G\} \subset \langle g\rangle \subseteq G$. Since G and $\{e_G\}$ are the only subgroups of G, we conclude that $G = \langle g\rangle$, and thus G is cyclic. If G is infinite, then $G \cong \mathbb{Z}$, and $2\mathbb{Z}$ is a subgroup of $\mathbb{Z}$, but $2\mathbb{Z} \neq \{0\}$ and $2\mathbb{Z} \neq \mathbb{Z}$, which is a contradiction. Hence, G is a finite cyclic group.

If $\circ(g) = mn$, where $m, n \in \mathbb{Z}^+$, $m \geq 2$, $n \geq 2$, then $\circ(g^m) = \frac{mn}{\gcd(mn,m)} = \frac{mn}{m} = n$. Hence, $\{e_G\} \subset \langle g^m \rangle \subset G$, and thus $\langle g^m \rangle$ is a non-trivial subgroup of G, which is a contradiction. Thus, $\circ(g) = p$, where p is a prime number, and thus $G \equiv \mathbb{Z}/(p)$, where p is a prime.

Question 9

(i) True.

For all $g \in G$, $(g^{-1})^{-1} = g \in G$, and $g^{-1} \in G$. Hence, $G \subseteq \{x^{-1} | x \in G\}$.

(ii) False.

Consider S_3. Then, $(12), (23) \in T$, but $(12)(23) = (123) \notin T$ as (123) is an even permutation. Hence, T is not a subgroup of S_3.

(iii) True.

By the First Isomorphism Theorem, $\mathbb{Z}/(\mathrm{Ker}(\xi)) \cong \mathbb{Z}$. Suppose that ξ is not injective. Then, $\mathrm{Ker}(\xi) \neq \{0\}$, and thus $\mathrm{Ker}(\xi) = m\mathbb{Z}$ for some $m \in \mathbb{Z}^+$, $m \geq 2$. Then, $|\mathbb{Z}/\mathrm{Ker}(\xi)| = |\mathbb{Z}/m\mathbb{Z}| < \infty$, which contradicts the fact that $\mathbb{Z}/(\mathrm{Ker}(\xi)) \cong \mathbb{Z}$. Hence, ξ is injective, and thus ξ is an isomorphism.

(iv) False.

Consider $H = \langle (12) \rangle$ being a subgroup of S_3, and let $g_1 = (1)$, $g_2 = (23)$. Then, $g_1 H g_2 H = \{(23), (132), (13)\} \neq gH$ for all $g \in S_3$, as gH will contain only two elements for all $g \in S_3$.

(v) True.

Let G be a group of order $2p$, where p is an odd prime. By Cauchy's Theorem, there exists $g \in G$ such that $\circ(g) = p$. Consider $H = \langle g \rangle$, which is a cyclic subgroup of G, and $|H| = \circ(g) = p$. Moreover, the index of H, $|G : H| = |G/H| = 2$, and thus H is a normal subgroup of G, order p.

(vi) False.

Consider S_4. Then, we have that $H = \langle (12)(34) \rangle$ is a normal subgroup of $V = \langle (12)(34), (13)(24) \rangle$, which is in turn a normal subgroup of S_4. However, H is not a normal subgroup of S_4, as $g = (123) \in S_4$, $h = (12)(34) \in H$, but $ghg^{-1} = (23)(14) \notin H$.

(vii) False.
Let $G_1 = G_2 = C_4 \times C_2$. $N_1 = C_2$ and N_2 is the unique subgroup of order 2 in C_4. Then G_1/N_1 is isomorphic to C_4 and G_2/N_2 is isomorphic to $C_2 \times C_2$, but we know that $C_4 \ncong C_2 \times C_2$.

(viii) False.
Consider the Dihedral group D_n. It has C_2 which is normal in D_n and abelian, and $D_n/C_2 \cong C_n$ is abelian as well. But D_n is not an Abelian group.