NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
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ST2131/MA2216 Probability
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Question 1

(a) We are given that $W \sim N(400, 40^2)$. Let n cars be required to cause structural damage to the bridge with probability exceeding 0.1. Let X_i be the r.v. of the weight of each car, $i = 1, 2, \ldots, n$. Since n is large, by C.L.T., we have $\sum_{i=1}^n X_i \approx N(3n, 0.3^2 n)$. Thus $(\sum_{i=1}^n X_i) - W \approx N(3n - 400, 0.3^2 n + 40^2)$. By referring to the statistical table, we obtain,

$$\begin{aligned} \mathbb{P}\{Z > 1.2816\} = 0.1 \quad &\leq \quad \mathbb{P}\left\{\left(\sum_{i=1}^n X_i\right) - W > 0\right\} \\ &\approx \quad \mathbb{P}\left\{Z > \frac{400 - 3n}{\sqrt{0.3^2 n + 40^2}}\right\}. \end{aligned}$$

Thus we conclude that $1.2816 \geq \dfrac{400 - 3n}{\sqrt{0.3^2 n + 40^2}}$. Let $u = \sqrt{0.3^2 n + 40^2}$, i.e. $n = \dfrac{u^2 - 40^2}{0.3^2}$.
This give us,

$$\begin{aligned} 1.2816 \quad &\geq \quad \frac{1}{u}\left(400 - 3\left(\frac{u^2 - 40^2}{0.3^2}\right)\right) \\ (1.2816)(0.3^2)u \quad &\geq \quad (400)(0.3^2) - 3(u^2 - 40^2) \\ 3u^2 + (1.2816)(0.3^2)u - (400)(0.3^2) - (3)(40^2) \quad &\geq \quad 0. \end{aligned}$$

We solve the above quadratic inequality with $u \geq 0$, and substitute back to get $n \geq 116.2$.
Thus $n \geq 117$.

(b) Φ can be treated as a function of 1 variable, thus together with X and Z independent, we have,

$$E(\Phi(X)) = \int_{\mathbb{R}} \Phi(x) f_X(x)\, dx \quad = \quad \int_{\mathbb{R}} \mathbb{P}\{Z \leq x\} f_X(x)\, dx = \mathbb{P}\{Z \leq X\}.$$

Question 2

(i) Since X_1, X_2, X_3 are independent r.v., we have $X \sim P(\lambda_1 + \lambda_2)$ and $Y \sim P(\lambda_2 + \lambda_3)$.

(ii) We have $E(X) = E(X_1) + E(X_2) = \lambda_1 + \lambda_2$, and $E(Y) = E(X_2) + E(X_3) = \lambda_2 + \lambda_3$.

(iii) We have $\text{Cov}(X_i, X_j) = 0$ if $i \neq j$. Thus

$$\begin{aligned} \text{Cov}(X, Y) \quad &= \quad \text{Cov}(X_1 + X_2, X_2 + X_3) \\ &= \quad \text{Cov}(X_1, X_2) + \text{Cov}(X_2, X_2) + \text{Cov}(X_1, X_3) + \text{Cov}(X_2, X_3) \\ &= \quad \text{Var}(X_2) \\ &= \quad \lambda_2. \end{aligned}$$

(iv) Since X_1, X_2, X_3 are independent r.v., we have,

$$\begin{aligned}
\mathbb{P}\{X=j, Y=k\} &= \sum_{i=0}^{j} \mathbb{P}\{X=j, Y=k \mid X_2=i\}\mathbb{P}\{X_2=i\} \\
&= \sum_{i=0}^{j} \mathbb{P}\{X_1=j-i, X_3=k-i \mid X_2=i\}\mathbb{P}\{X_2=i\} \\
&= \sum_{i=0}^{j} \mathbb{P}\{X_1=j-i\}\mathbb{P}\{X_3=k-i\}\mathbb{P}\{X_2=i\} \\
&= \sum_{i=0}^{j} \left(e^{-\lambda_1}\frac{\lambda_1^{j-i}}{(j-i)!}\right)\left(e^{-\lambda_3}\frac{\lambda_3^{k-i}}{(k-i)!}\right)\left(e^{-\lambda_2}\frac{\lambda_2^{i}}{i!}\right) \\
&= e^{-(\lambda_1+\lambda_2+\lambda_3)}\sum_{i=0}^{j}\frac{\lambda_1^{j-i}\lambda_2^{i}\lambda_3^{k-i}}{(j-i)!i!(k-i)!}.
\end{aligned}$$

Question 3

(i) We have,

$$\begin{aligned}
f_U(u) = \int_{\mathbb{R}} f_{(U,V)}(u,v)\ dv &= \int_{u^{-1}}^{u} \frac{1}{2u^2v}\ dv \\
&= \left[\frac{1}{2u^2}\ln v\right]_{u^{-1}}^{u} \\
&= \frac{1}{2u^2}(\ln u - \ln u^{-1}) = \frac{1}{u^2}\ln u, \quad 1 < u.
\end{aligned}$$

Thus the marginal p.d.f. of U is given by,

$$f_U(u) = \begin{cases} \frac{1}{u^2}\ln u, & 1 < u; \\ 0, & \text{otherwise.} \end{cases}$$

For $0 < v < 1$, we have $1 < v^{-1} \leq u$. Therefore,

$$\begin{aligned}
f_V(v) = \int_{\mathbb{R}} f_{(U,V)}(u,v)\ du &= \int_{v^{-1}}^{\infty} \frac{1}{2u^2v}\ du \\
&= \left[\frac{-1}{2uv}\right]_{v^{-1}}^{\infty} \\
&= \frac{1}{2}.
\end{aligned}$$

For $v \geq 1$, we have $1 \leq v \leq u$. Therefore,

$$\begin{aligned}
f_V(v) = \int_{\mathbb{R}} f_{(U,V)}(u,v)\ du &= \int_{v}^{\infty} \frac{1}{2u^2v}\ du \\
&= \left[\frac{-1}{2uv}\right]_{v}^{\infty} \\
&= \frac{1}{2v^2}.
\end{aligned}$$

Thus the marginal p.d.f. of V is given by,

$$f_V(v) = \begin{cases} \frac{1}{2}, & 0 < v < 1; \\ \frac{1}{2v^2}, & v \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) No.
Since $f_{(U,V)}(u,v) \neq f_U(u)f_V(v)$ for some $u, v \in \mathbb{R}$, we have U and V to be not independent.

(iii) Let $x = \sqrt{uv}$, $y = \sqrt{\frac{u}{v}}$. This give us $u = xy$, and $v = \frac{x}{y}$.
For $u \in (1, \infty)$ and $v \in (0, \infty)$, we have $x, y \in (0, \infty)$. Now since $1 < u$, $u^{-1} \leq v \leq u$, we have $\frac{1}{xy} \leq \frac{x}{y} \leq xy$. This implies that $1 \leq x^2, y^2$ and since $x, y > 0$, we have $1 \leq x, y$.
Next, we obtain $\frac{\delta x}{\delta u} = \frac{1}{2}\sqrt{\frac{v}{u}}$, $\frac{\delta x}{\delta v} = \frac{1}{2}\sqrt{\frac{u}{v}}$, $\frac{\delta y}{\delta u} = \frac{1}{2}\sqrt{\frac{1}{uv}}$ and $\frac{\delta y}{\delta v} = \frac{-1}{2}\sqrt{\frac{u}{v^3}}$.
Thus $J(u,v) = \left(\frac{1}{2}\sqrt{\frac{v}{u}}\right)\left(\frac{-1}{2}\sqrt{\frac{u}{v^3}}\right) - \left(\frac{1}{2}\sqrt{\frac{u}{v}}\right)\left(\frac{1}{2}\sqrt{\frac{1}{uv}}\right) = -\frac{1}{2v}$.
Since $v \geq 0$, we have $|J(u,v)| = \frac{1}{2v}$. Therefore,

$$\begin{aligned} f_{(X,Y)}(x,y) &= \frac{1}{|J(u,v)|} f_{(U,V)}(u,v) \\ &= (2v)\left(\frac{1}{2u^2 v}\right) \\ &= \frac{1}{u^2} = \frac{1}{x^2 y^2}, \quad x, y \geq 1. \end{aligned}$$

Thus the joint p.d.f. of X and Y is given by,

$$f_{(X,Y)}(x,y) = \begin{cases} \frac{1}{u^2} = \frac{1}{x^2y^2}, & x, y \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

(iv) Since $f_{(X,Y)}(x,y) = \left(\frac{1}{x^2}\right)\left(\frac{1}{y^2}\right)$, we have the marginal p.d.f. of X and Y to be given by,

$$f_X(x) = \begin{cases} \frac{1}{x^2}, & x \geq 1; \\ 0, & \text{otherwise.} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{y^2}, & y \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

X and Y are thus independent.

Question 4

(i) We are given that X and Y are r.v. such that $Y \sim N(\mu, \sigma^2)$ and $X \mid (Y = y) \sim N(y, 1)$.
This give us $E(X \mid Y = y) = y$ Thus $E(X) = E(E(X \mid Y)) = E(Y) = \mu$.

(ii) We have

$$E(X^2 \mid Y = y) = \text{Var}(X \mid Y = y) + E(X \mid Y = y)^2 = 1 + y^2.$$

(iii) Using (4ii.), we get

$$\begin{aligned} \text{Var}(X) = E(X^2) - E(X)^2 &= E(E(X^2 \mid Y)) - \mu^2 \\ &= E(1 + Y^2) - \mu^2 \\ &= 1 + E(Y^2) - \mu^2 \\ &= 1 + \text{Var}(Y) + E(Y)^2 - \mu^2 \\ &= 1 + \sigma^2. \end{aligned}$$

(iv) We have,

$$\begin{aligned} E(XY) = E(E(XY \mid Y)) &= E(YE(X \mid Y)) \\ &= E(Y^2) \\ &= \text{Var}(X) + E(Y)^2 \\ &= \sigma^2 + \mu^2. \end{aligned}$$

Thus $\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \sigma^2$.

(v) We have the covariance matrix, $\Sigma = \begin{pmatrix} \text{Var}(X) & \text{Cov}(X,Y) \\ \text{Cov}(X,Y) & \text{Var}(Y) \end{pmatrix} = \begin{pmatrix} 1+\sigma^2 & \sigma^2 \\ \sigma^2 & \sigma^2 \end{pmatrix}$.
We thus get $\det(\Sigma) = (1+\sigma^2)(\sigma^2) - (\sigma^2)(\sigma^2) = \sigma^2$.

(vi) We have,

$$\begin{aligned} f_{(X,Y)}(x,y) = f_{X|Y}(x|y)f_Y(y) &= \left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-y)^2}\right)\left(\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}(\frac{y-\mu}{\sigma})^2}\right) \\ &= \frac{1}{2\pi\sigma}e^{-\frac{1}{2}(x-y)^2}e^{-\frac{1}{2}(\frac{y-\mu}{\sigma})^2}, \quad x,y \in \mathbb{R}. \end{aligned}$$

Now with what we determined in (4v.), let $\boldsymbol{r} = \begin{pmatrix} x-\mu \\ y-\mu \end{pmatrix}$. Then we have,

$$f_{(X,Y)}(x,y) = \frac{1}{2\pi(\det\Sigma)^{\frac{1}{2}}}e^{-\frac{1}{2}(\boldsymbol{r}^T\Sigma^{-1}\boldsymbol{r})}, \quad x,y \in \mathbb{R}.$$

Thus (X,Y) is a bivariate normal distribution.

(vii) Yes.
We have $\text{Cov}(X-Y,Y) = \text{Cov}(X,Y) - \text{Cov}(Y,Y) = \sigma^2 - \sigma^2 = 0$.
Since $(X-Y,Y)$ is bivariate normal, we can thus conclude that $X-Y$ and Y are independent.

Question 5

(a) (i) We have,

$$\begin{aligned} \mathbb{P}\{J_k = 1\} &= \mathbb{P}\{J_k = 1 \mid I_k = 0\}\mathbb{P}\{I_k = 0\} + \mathbb{P}\{J_k = 1 \mid I_k = 1\}\mathbb{P}\{I_k = 1\} \\ &= \mathbb{P}\{I_{k-1} = 1 \mid I_k = 0\}\mathbb{P}\{I_k = 0\} + \mathbb{P}\{I_{k-1} = 0 \mid I_k = 1\}\mathbb{P}\{I_k = 1\} \\ &= \mathbb{P}\{I_{k-1} = 1\}\mathbb{P}\{I_k = 0\} + \mathbb{P}\{I_{k-1} = 0\}\mathbb{P}\{I_k = 1\} \\ &= p(1-p) + p(1-p) = 2p(1-p). \end{aligned}$$

(ii) By definition of our J_ks, we have $X = \sum_{k=2}^n J_k$.

(iii) Thus, $E(X) = E(\sum_{k=2}^n J_k) = \sum_{k=2}^n E(J_k) = \sum_{k=2}^n 2p(1-p) = 2p(1-p)(n-1)$.

(b) (i) We have

$$\begin{aligned} f_X(x) = \int_{\mathbb{R}} f(x,y)\ dy &= \int_0^x \frac{2}{x}e^{-2x}\ dy \\ &= 2e^{-2x}, \quad x > 0. \end{aligned}$$

Thus the marginal p.d.f. of X is given by,

$$f_X(x) = \begin{cases} 2e^{-2x}, & x > 0; \\ 0, & \text{otherwise,} \end{cases}$$

i.e. $X \sim \text{Exp}(2)$.

(ii) We have $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)} = \dfrac{1}{x}$. Thus,

$$\begin{aligned} E(Y \mid X = x) = \int_{\mathbb{R}} y f_{Y|X}(y|x)\, dy &= \int_0^x \frac{y}{x}\, dy \\ &= \left[\frac{y^2}{2x}\right]_0^x = \frac{x}{2}. \end{aligned}$$

(iii) By result of (5bii.), and using the fact that $X \sim \text{Exp}(2)$ gives us $E(X) = \frac{1}{2}$, we have

$$\begin{aligned} E(Y) = E(E(Y \mid X)) &= E\left(\frac{X}{2}\right) \\ &= \frac{1}{2}E(X) \\ &= \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}. \end{aligned}$$

(iv) Now using the additional fact that $\text{Var}(X) = \frac{1}{4}$,

$$\begin{aligned} E(XY) = E(E(XY \mid X)) &= E(XE(Y \mid X)) \\ &= E\left(X\left(\frac{X}{2}\right)\right) \\ &= \frac{1}{2}E(X^2) \\ &= \frac{1}{2}(\text{Var}(X) + E(X)^2) \\ &= \frac{1}{2}\left(\frac{1}{4} + \left(\frac{1}{2}\right)^2\right) = \frac{1}{4}. \end{aligned}$$

Alt: The above method uses solutions of all the parts prior to it. Thus in the situation where we do not have the solution to any of the parts, we still have a short solution that directly obtain the answer to (5biv.) independently, by using,

$$\begin{aligned} E(XY) = \int_0^\infty \int_y^\infty xyf(x,y)\, dx\, dy &= \int_0^\infty \int_y^\infty 2ye^{-2x}\, dx\, dy \\ &= \int_0^\infty y\left[-e^{-2x}\right]_y^\infty\, dy \\ &= \int_0^\infty ye^{-2y}\, dy \\ &= \left[-\frac{1}{2}ye^{-2y}\right]_0^\infty - \int_0^\infty -\frac{1}{2}e^{-2y}\, dy \\ &= 0 + \left[-\frac{1}{4}e^{-2y}\right]_0^\infty = \frac{1}{4}. \end{aligned}$$