NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Teo Wei Hao

MA3111S Complex Analysis (version S)
AY 2008/2009 Sem 2

Question 1

(a) Let $g_1, g_2, g_3, g_4 : \mathbb{C} \to \mathbb{C}$, $g_5 : \mathbb{C} - \{0\} \to \mathbb{C}$ be such that $g_1(z) = e^z$, $g_2(z) = \sin z$, $g_3(z) = z$, and $g_4(z) = (z-\pi)^2$ for all $z \in \mathbb{C}$; $g_5(z) = \dfrac{1}{z^2}$ for all $z \in \mathbb{C} - \{0\}$.
We notice that g_1, g_2, g_3, g_4, g_5 are all analytic functions on their respective domains.
Since $g_1 \circ g_5$ is well-defined on $\mathbb{C} - \{0\}$, $g_3(z) = 0$ only at $z = 0$, and $g_4(z) = 0$ only at $z = \pi$, we have a well-defined function $f : \mathbb{C} - \{0, \pi\} \to \mathbb{C}$ such that $f(z) = \dfrac{(g_1 \circ g_5)(z) \cdot g_2(z)}{g_3(z) \cdot g_4(z)}$.
Also, f is analytic on $\mathbb{C} - \{0, \pi\}$, i.e. f has singularities at most at $\{0, \pi\}$.

For all $z \in \mathbb{C} - \{0\}$, we have by Taylor's Theorem,

$$g_1(g_5(z)) = \sum_{k=0}^{\infty} \frac{1}{k!} g_5(z)^k = \sum_{k=0}^{\infty} \frac{1}{k! z^{2k}}.$$

By uniqueness of Laurent series expansion, the above is the Laurent series expansion of $(g_1 \circ g_5)(z)$ at 0, and so it has an essential singularity at 0. Together with the fact that $\dfrac{g_2(z)}{g_3(z) \cdot g_4(z)}$ has a removable singularity at 0 with value $\dfrac{1}{\pi^2} \neq 0$, we have f to have essential singularity at 0.

For all $z \in B(\pi, 1)$, we have,

$$f(z) = \left(\frac{(g_1 \circ g_5)(z) \cdot g_2(z)}{g_3(z)} \right) \frac{1}{(z-\pi)^2}.$$

since $\dfrac{(g_1 \circ g_5)(z) \cdot g_2(z)}{g_3(z)}$ is analytic at π, f has a pole of order 2 at π.

(b) Let us denote $\mathrm{Ann}(1,1,2) = \{z \in \mathbb{C} \mid 1 < |z-1| < 2\}$.
For all $z \in \mathrm{Ann}(1,1,2)$, we have $\left|\dfrac{1}{z-1}\right| < 1$ and $\left|\dfrac{z-1}{2}\right| < 1$, and so,

$$\begin{aligned}
\frac{1}{z^2 - z - 2} = \frac{1}{(z-2)(z+1)} &= -\frac{1}{3(z+1)} + \frac{1}{3(z-2)} \\
&= -\frac{1}{6\left(1 + \frac{z-1}{2}\right)} + \frac{1}{3(z-1)\left(1 - \frac{1}{z-1}\right)} \\
&= -\frac{1}{6} \sum_{k=0}^{\infty} \left(-\frac{z-1}{2}\right)^k + \frac{1}{3(z-1)} \sum_{k=0}^{\infty} \left(\frac{1}{z-1}\right)^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{6 \cdot 2^k} (z-1)^k + \sum_{k=1}^{\infty} \frac{1}{3(z-1)^k}.
\end{aligned}$$

By uniqueness of Laurent series expansion, the above is the Laurent series expansion of the function $\dfrac{1}{z^2 - z - 2}$ valid on the annulus $\mathrm{Ann}(1,1,2)$.

Question 2

For $u, w \in \mathbb{C}$, let us denote the line segment $[u, w] = \{z \in \mathbb{C} \mid z = (1-\alpha)u + \alpha w,\ \alpha \in [0,1]\}$.
Let $a, b, c \in U$ such that the triangle $\triangle(a,b,c)$ lies in U.
Let $\gamma : [0,3] \to \mathbb{C}$ be such that,

$$\gamma(t) = \begin{cases} (1-t)a + tb, & t \in [0,1]; \\ (2-t)b + (t-1)c, & t \in (1,2]; \\ (3-t)c + (t-2)a, & t \in (2,3]. \end{cases}$$

Then $\{\gamma\} = [a,b] \cup [b,c] \cup [c,a] = \partial\triangle(a,b,c)$.

Let $\varepsilon \in \mathbb{R}^+$.
Since $[a,b]$, $[b,c]$ and $[c,a]$ are line segments in U, there exists $N \in \mathbb{Z}^+$ such that for all $k \in \mathbb{Z}_{\geq N}$, for all $z \in \{\gamma\}$, we have $|f_k(z) - f(z)| < \dfrac{\varepsilon}{L(\gamma)}$.
Thus, by ML-estimate, and the fact that f_k is analytic on U, we have,

$$\begin{aligned} \left|\int_\gamma f(z)\ dz\right| &= \left|\int_\gamma f_k(z)\ dz - \int_\gamma f_k(z) - f(z)\ dz\right| \\ &\leq \left|\int_\gamma f_k(z)\ dz\right| + \left|\int_\gamma f_k(z) - f(z)\ dz\right| \\ &< 0 + L(\gamma)\left(\frac{\varepsilon}{L(\gamma)}\right) = \varepsilon. \end{aligned}$$

This give us $\displaystyle\int_\gamma f(z)\ dz = 0$.
Therefore by Morera's Theorem, f is analytic on U.

Question 3

Assume on the contrary that there exists $w \in \mathbb{C}$ and $r \in \mathbb{R}^+$, such that for all $z \in \mathbb{C}$, we have $f(z) \notin B(w,r)$.
This implies that for all $z \in \mathbb{C}$, we have $f(z) \neq w$.
Thus we can have a well-defined function $g : \mathbb{C} \to \mathbb{C}$ such that $g(z) = \dfrac{1}{f(z) - w}$.
Since f is entire, we have g to be entire.
Also, for all $z \in \mathbb{C}$, we have $|g(z)| = \left|\dfrac{1}{f(z)-w}\right| < \dfrac{1}{r}$, i.e. g is bounded.
Thus by Liouville Theorem, we have g to be a constant function, which implies that f is a constant function, a contradiction.

Question 4

For all $w \in S$, let the pole be of order $n_w \in \mathbb{Z}^+$. Let $M = \max\{|w| \mid w \in S\}$.
We can define an entire function $F : \mathbb{C} \to \mathbb{C}$ such that $F(z) = f(z) \displaystyle\prod_{w \in S} (z-w)^{n_w}$ for all $z \in \mathbb{C} - S$, since F has removable singularities on S now.
Let $G : \mathbb{C} - \{0\} \to \mathbb{C}$ be such that $G(z) = F\left(\dfrac{1}{z}\right)$, which is an analytic function on $\mathbb{C} - \{0\}$.
For all $z \in B'(0,M)$, we have $\dfrac{1}{z} \notin S$, and so $G(z) = f\left(\dfrac{1}{z}\right) \displaystyle\prod_{w \in S}\left(\frac{1}{z} - w\right)^{n_w} = g(z) \prod_{w \in S}\left(\frac{1}{z} - w\right)^{n_w}$.
Let g has a pole of order n at 0. Then G has a pole of order $N := n + \displaystyle\sum_{w \in S} n_w$ at 0.

Thus, we can define an analytic function $H : \mathbb{C} \to \mathbb{C}$ such that $H(z) = z^N G(z)$ for all $z \in \mathbb{C} - \{0\}$.

Let $\varepsilon \in \mathbb{R}^+$. Since H is continuous, there exists $R \in \mathbb{R}^+$ such that for all $z_1 \in B\left(0, \frac{2}{R}\right)$, we have $H(z_1) \in B(H(0), 1)$. Let $z \in \mathbb{C}$ such that $|z| > K := \max\left\{R, \frac{2^{N+1}(N+1)!(1+|H(0)|)}{\varepsilon}\right\}$.
Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the path $\gamma(t) = 2|z|e^{it}$. We have $L(\gamma) = 2\pi(2|z|) = 4\pi|z|$.
Then for all $w \in \{\gamma\}$, we have $|w| = 2|z|$, $|w - z| \geq |w| - |z| = |z|$, and $\frac{1}{w} \in B'\left(0, \frac{2}{R}\right)$, and so,

$$\left|\frac{F(w)}{(w-z)^{N+2}}\right| \leq \frac{1}{|z|^{N+2}}\left|G\left(\frac{1}{w}\right)\right| = \frac{1}{|z|^{N+2}}\left|w^N H\left(\frac{1}{w}\right)\right| = \frac{2^N|z|^N}{|z|^{N+2}}\left|H\left(\frac{1}{w}\right)\right| \leq \frac{2^N}{|z|^2}(1+|H(0)|).$$

Thus, by Cauchy Integral Formula for Derivatives and ML-estimate, we have,

$$\begin{aligned}
|F^{(N+1)}(z)| &= \left|\frac{(N+1)!}{2\pi i}\int_\gamma \frac{F(w)}{(w-z)^{N+2}}\,dw\right| \\
&\leq \frac{(N+1)!}{2\pi}(4\pi|z|)\left(\frac{2^N}{|z|^2}(1+|H(0)|)\right) \\
&= \frac{2^{N+1}(N+1)!}{|z|}(1+|H(0)|) < \varepsilon.
\end{aligned}$$

As a consequence, $F^{(N+1)}$ is bounded on $\{z \in \mathbb{C} \mid |z| > K\}$.
Since $F^{(N+1)}$ is entire, it is continuous on $\mathbb{C}$.
Since $\overline{B(0,K)}$ is a closed and bounded (compact) set, $F^{(N+1)}$ is bounded on $\overline{B(0,K)}$.
Combining with the above, $F^{(N+1)}$ is bounded on $\mathbb{C} = \overline{B(0,K)} \cup \{z \in \mathbb{C} \mid |z| > K\}$.
Thus by Liouville's Theorem, $F^{(N+1)}$ is a constant function, in particular, $|F^{(N+1)}(z)| < \varepsilon$ for all $z \in \mathbb{C}$.
This implies that $F^{(N+1)}(z) = 0$ for all $z \in \mathbb{C}$, and so F is a polynomial of degree at most N.
Thus f is a rational function, i.e. $f(z) = \frac{P(z)}{Q(z)}$ for all $z \in \mathbb{C} - S$, where $P(z) = F(z)$ and $Q(z) = \prod_{w \in S}(z-w)^{n_w}$ are polynomials.

Question 5

Let $f : \mathbb{C} \to \mathbb{C}$ be such that $f(z) = \frac{1 + iz - e^{iz}}{z^2}$ for all $z \in \mathbb{C} - \{0\}$; $f(0) = \frac{-1}{2}$.
Then f is an entire function.
Let $R \in \mathbb{R}^+$, $\gamma_{R_1} : [0, \pi] \to \mathbb{R}$ and $\gamma_{R_2} : [-R, R] \to \mathbb{R}$ be such that,

$$\begin{aligned}
\gamma_{R_1}(t) &= Re^{it}; \\
\gamma_{R_2}(t) &= t.
\end{aligned}$$

Then $\gamma_{R_1} + \gamma_{R_2}$ is a closed contour, and so $\int_{\gamma_{R_1}+\gamma_{R_2}} f(z)\,dz = 0$.
Also, we have,

$$\int_{\gamma_{R_1}} \frac{iz}{z^2}\,dz = \int_{\gamma_{R_1}} \frac{i}{z}\,dz = i(\mathrm{Log}(Re^{i\pi}) - \mathrm{Log}(R)) = i(i\pi) = -\pi.$$

Let $\varepsilon \in \mathbb{R}^+$. Let $R = \frac{4\pi}{\varepsilon}$.
For all $z \in \{\gamma_{R_1}\}$, since $\left|e^{iz}\right| \leq e^0 = 1$, we have $\left|\frac{1 - e^{iz}}{z^2}\right| \leq \frac{1 + |e^{iz}|}{|z|^2} \leq \frac{2}{R^2} \leq \frac{\varepsilon}{2\pi R}$.

By ML-estimate, we have,

$$\left|\int_{\gamma_{R_1}} \frac{1-e^{iz}}{z^2}\right| \leq (\pi R)\left(\frac{\varepsilon}{2\pi R}\right) < \varepsilon.$$

Thus, we have,

$$\begin{aligned}\left|\int_{\gamma_{R_2}} f(z)\ dz - \pi\right| &= \left|\int_{\gamma_{R_1}+\gamma_{R_2}} f(z)\ dz - \int_{\gamma_{R_1}} f(z)\ dz - \pi\right| \\ &\leq \left|\int_{\gamma_{R_1}+\gamma_{R_2}} f(z)\ dz\right| + \left|\int_{\gamma_{R_1}} \frac{1-e^{iz}}{z^2}\ dz + \int_{\gamma_{R_1}} \frac{iz}{z^2}\ dz + \pi\right| \\ &\leq 0 + \left|\int_{\gamma_{R_1}} \frac{1-e^{iz}}{z^2}\ dz\right| + \left|\int_{\gamma_{R_1}} \frac{iz}{z^2}\ dz + \pi\right| < 0 + \varepsilon = \varepsilon.\end{aligned}$$

We notice that $x \in [-R, R]$ iff $z = x + i0 \in \{\gamma_{R_2}\}$. Also, $\text{Re } f(z) = \text{Re}\left(\frac{1-ix-e^{ix}}{x^2}\right) = \frac{1-\cos x}{x^2}$.
Therefore $\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\ dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{1-\cos x}{x^2}\ dx = \text{Re}\left(\lim_{R\to\infty}\int_{\gamma_{R_2}} f(z)\ dz\right) = \pi$.

Question 6

Let $w = x_w + iy_w \in \mathbb{C} - U$. Let $u : U \to \mathbb{C}$ be such that $u(x,y) = \text{Log } \sqrt{(x-w_x)^2 + (y-x_y)^2}$.
For all $x + iy \in U$, since $w \notin U$, we have $(x-w_x)^2 + (y-w_y)^2 \neq 0$.
Thus we have $u_x(x,y) = \frac{x-w_x}{(x-w_x)^2+(y-w_y)^2}$ and $u_y(x,y) = \frac{y-w_y}{(x-w_x)^2+(y-w_y)^2}$.
So, $u_{xx}(x,y) = \frac{(y-w_y)^2-(x-w_x)^2}{((x-w_x)^2+(y-w_y)^2)^2}$ and $u_{yy} = \frac{(x-w_x)^2-(y-w_y)^2}{((x-w_x)^2+(y-w_y)^2)^2}$.
This give us $u_{xx} + u_{yy} = 0$, i.e. u is harmonic on U.

Thus, there exists analytic $F : U \to \mathbb{C}$ such that u is the real part of F.
This give us for all $z = x + iy \in U$, we have $F'(z) = u_x(x,y) - iu_y(x,y) = \frac{1}{z-w}$.
Therefore, given any closed contour $\gamma : [0,1] \to \mathbb{C}$ in U, by Fundamental Theorem of Calculus, we have $n(\gamma, w) = \frac{1}{2\pi i}\int_{\gamma} \frac{1}{z-w}\ dz = \frac{1}{2\pi i}(F(1) - F(0)) = 0$.
Therefore U is simply connected.