NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA1102R Calculus
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Question 1

(a) (i) Since x^2 and $\cos x - 1$ are both equal to 0 when $x = 0$, we may use the L'Hôpital's Rule. So,

$$\lim_{x\to 0} \frac{x^2}{\cos x - 1} = \lim_{x\to 0} \frac{2x}{-\sin x}$$

if the latter limit exists. Similarly, since $2x$ and $-\sin x$ are both equal to 0 when $x = 0$, we may apply the L'Hôpital's Rule again.

$$\lim_{x\to 0} \frac{2x}{-\sin x} = \lim_{x\to 0} \frac{2}{-\cos x} = \frac{2}{-1} = -2$$

Therefore we have

$$\lim_{x\to 0} \frac{x^2}{\cos x - 1} = -2$$

(ii) We may use the substitution $y = 1/x$.

$$\lim_{x\to\infty} \frac{(3x^2+5)\sin\frac{2}{x}}{5x+3} = \lim_{y\to 0^+} \frac{\left(3\left(\frac{1}{y}\right)^2 + 5\right)\sin 2y}{5\left(\frac{1}{y}\right)+3} = \lim_{y\to 0^+} \frac{(3+5y^2)\sin 2y}{5y+3y^2}$$

Since both the numerator and denominator equal to 0 when $y = 0$, we may use the L'Hôpital's Rule. So,

$$\lim_{y\to 0^+} \frac{(3+5y^2)\sin 2y}{5y+3y^2} = \lim_{y\to 0^+} \frac{10y\sin 2y + (3+5y^2)(2\cos 2y)}{5+6y} = \frac{0+(3+0)(2)}{5+0} = \frac{6}{5}$$

Therefore we have

$$\lim_{x\to\infty} \frac{(3x^2+5)\sin\frac{2}{x}}{5x+3} = \frac{6}{5}$$

(iii)

$$\lim_{x\to 0}\left(\frac{1+2^x+3^x}{3}\right)^{\frac{1}{x}} = \lim_{x\to 0} e^{\ln\left(\frac{1+2^x+3^x}{3}\right)^{\frac{1}{x}}} = \lim_{x\to 0} e^{\frac{1}{x}\ln\left(\frac{1+2^x+3^x}{3}\right)} = e^{\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{1+2^x+3^x}{3}\right)}$$

The limits are the same as e is a continuous function everywhere. We may now apply L'Hôpital's Rule because $\ln\left(\frac{1+2^x+3^x}{3}\right)$ and x are both equal to 0 when $x = 0$. We have

$$\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{1+2^x+3^x}{3}\right) = \lim_{x\to 0}\frac{\left(\frac{2^x\ln 2+3^x\ln 3}{3}\right)}{\left(\frac{1+2^x+3^x}{3}\right)} = \lim_{x\to 0}\frac{2^x\ln 2+3^x\ln 3}{1+2^x+3^x} = \frac{\ln 2+\ln 3}{3} = \frac{1}{3}\ln 6 = \ln\sqrt[3]{6}$$

Therefore,

$$\lim_{x\to 0}\left(\frac{1+2^x+3^x}{3}\right)^{\frac{1}{x}} = e^{\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{1+2^x+3^x}{3}\right)} = e^{\ln\sqrt[3]{6}} = \sqrt[3]{6}$$

(b)
$$\left|\frac{x^2-2x}{x+2}-3\right| = \left|\frac{x^2-2x-3x-6}{x+2}\right| = \left|\frac{x^2-5x-6}{x+2}\right| = \left|\frac{(x+1)(x-6)}{x+2}\right| = |x+1|\left|\frac{x-6}{x+2}\right|$$

Whenever $|x+1| < \frac{1}{2}$, that is, $-\frac{1}{2} < x+1 < \frac{1}{2}$ or $-\frac{3}{2} < x < -\frac{1}{2}$, we must have:

(i) $-\frac{3}{2} < x < -\frac{1}{2} \;\Rightarrow\; -\frac{15}{2} < x-6 < -\frac{13}{2} \;\Rightarrow\; \frac{13}{2} < |x-6| < \frac{15}{2} \;\Rightarrow\; |x-6| < \frac{15}{2}$

(ii) $-\frac{3}{2} < x < -\frac{1}{2} \;\Rightarrow\; \frac{1}{2} < x+2 < \frac{3}{2} \;\Rightarrow\; \frac{1}{2} < |x+2| < \frac{3}{2} \;\Rightarrow\; \frac{1}{2} < |x+2|$

Let $\epsilon > 0$ be given. Choose $\delta = \min\{\frac{1}{2}, \frac{\epsilon}{15}\}$. Then

$$0 < |x+1| < \delta \;\Rightarrow\; \left|\frac{x^2-2x}{x+2}-3\right| = |x+1|\left|\frac{x-6}{x+2}\right| < |x+1|\frac{15/2}{1/2} = |x+1|\times 15 < \frac{\epsilon}{15}\times 15 = \epsilon$$

Question 2

(a) Let $g(x) = |f(x)|$ and let $g'(0) = L$. We have

$$L = \lim_{x\to 0}\frac{g(x)-g(0)}{x-0} = \lim_{x\to 0}\frac{g(x)-0}{x-0} = \lim_{x\to 0}\frac{g(x)}{x}$$

By taking limits for the positive side, we get

$$L = \lim_{x\to 0^+}\frac{g(x)}{x} = \lim_{x\to 0^+}\frac{|g(x)|}{|x|} = \lim_{x\to 0^+}\left|\frac{g(x)}{x}\right| = \left|\lim_{x\to 0^+}\frac{g(x)}{x}\right| = |L|$$

By taking limits for the negative side, we get

$$L = \lim_{x\to 0^-}\frac{g(x)}{x} = \lim_{x\to 0^-}\frac{|g(x)|}{-|x|} = \lim_{x\to 0^-} -\left|\frac{g(x)}{x}\right| = -\left|\lim_{x\to 0^-}\frac{g(x)}{x}\right| = -|L|$$

So we have $-|L| = |L|$, which means $L = 0$. So $g'(0) = 0$. Now we want to show that $f'(0) = 0$. Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that

$$\begin{aligned}
0 < |x-0| < \delta \;&\Rightarrow\; \left|\frac{g(x)}{x}-0\right| < \epsilon\\
&\Rightarrow\; \left|\frac{|f(x)|}{x}\right| < \epsilon\\
&\Rightarrow\; \left|\frac{f(x)}{x}\right| < \epsilon\\
&\Rightarrow\; \left|\frac{f(x)-0}{x-0}-0\right| < \epsilon\\
&\Rightarrow\; \left|\frac{f(x)-f(0)}{x-0}-0\right| < \epsilon
\end{aligned}$$

This shows that $f'(0) = 0$.

(b) Since f is continuous at 0, we have

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{f(x)}{x} \times x = \lim_{x\to 0} \frac{f(x)}{x} \times \lim_{x\to 0} x = 2 \times 0 = 0$$

By using the definition of the derivative,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x) - 0}{x} = \lim_{x\to 0} \frac{f(x)}{x} = 2$$

(c)

$$\int_{\cos x}^{5x} \sin(t^2)dt = \int_0^{5x} \sin(t^2)dt + \int_{\cos x}^{0} \sin(t^2)dt = \int_0^{5x} \sin(t^2)dt - \int_0^{\cos x} \sin(t^2)dt$$

By using the Fundamental Theorem of Calculus, we have

$$\frac{d}{dx}\int_{\cos x}^{5x} \sin(t^2)dt = \frac{d}{dx}\int_0^{5x} \sin(t^2)dt - \frac{d}{dx}\int_0^{\cos x} \sin(t^2)dt = 5\sin((5x)^2) - \sin(\cos^2 x)(-\sin x)$$

So,

$$\frac{d}{dx}\int_{\cos x}^{5x} \sin(t^2)dt = 5\sin(25x^2) + \sin x \sin(\cos^2 x)$$

Question 3

(a) Since f is differentiable at 1, the function must be continuous at 1.

$$f(1) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x\cos\left(\frac{\pi x}{2}\right) = \lim_{x\to 1^-} 1\cos\left(\frac{\pi}{2}\right) = 0$$

So we must have $f(1) = 0$. Since $0 = f(1) = a(1)^2 + b = a + b$, we have $b = -a$.

The limit of the difference quotient must also exist at 1.

$$f'(1) = \lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{ax^2 + b - 0}{x - 1} = \lim_{x\to 1^+} \frac{ax^2 + b}{x - 1}$$

We may substitute $b = -a$ into the equation to obtain

$$f'(1) = \lim_{x\to 1^+} \frac{ax^2 - a}{x - 1} = \lim_{x\to 1^+} \frac{a(x+1)(x-1)}{x - 1} = \lim_{x\to 1^+} a(x+1) = 2a$$

By looking at the left-hand limit, we get

$$f'(1) = \lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{x\cos\left(\frac{\pi x}{2}\right) - 0}{x - 1} = \lim_{x\to 1^-} \frac{x\cos\left(\frac{\pi x}{2}\right)}{x - 1}$$

We will use L'Hôpital's Rule since both the numerator and denominator are equal to 0

$$f'(1) = \lim_{x\to 1^-} \frac{\cos\left(\frac{\pi x}{2}\right) + x\left(-\frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right)\right)}{1} = \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$$

So we obtain $2a = -\frac{\pi}{2}$, giving us $a = -\frac{\pi}{4}, b = \frac{\pi}{4}$.

(b) We have $f(2) = (2-2)g(2) = 0$. So,

$$f'(x) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{(x-2)g(x)-0}{x-2} = \lim_{x\to 2} g(x) = g(2) = 1$$

since g is continuous at 2.

(c)

$$\int_0^{\frac{\pi}{4}} \ln(\sqrt{1+\tan x})dx = \int_{\frac{\pi}{4}}^0 \ln\left(\sqrt{1+\tan(\frac{\pi}{4}-t)}\right)(-1)dt = \int_0^{\frac{\pi}{4}} \ln\left(\sqrt{1+\tan(\frac{\pi}{4}-t)}\right)dt$$

We use the trigonometric addition formula, $\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$, to get

$$1+\tan(\frac{\pi}{4}-t) = 1+\frac{\tan\frac{\pi}{4}-\tan t}{1+\tan\frac{\pi}{4}\tan t} = 1+\frac{1-\tan t}{1+1\cdot\tan t} = \frac{1+\tan t+1-\tan t}{1+\tan t} = \frac{2}{1+\tan t}$$

This gives

$$\begin{aligned}
I := &\int_0^{\frac{\pi}{4}} \ln(\sqrt{1+\tan x})dx \\
= &\int_0^{\frac{\pi}{4}} \ln\left(\sqrt{1+\tan(\frac{\pi}{4}-t)}\right)dt \\
= &\int_0^{\frac{\pi}{4}} \ln\left(\sqrt{\frac{2}{1+\tan t}}\right)dt \\
= &\int_0^{\frac{\pi}{4}} \ln\left(\frac{\sqrt{2}}{\sqrt{1+\tan t}}\right)dt \\
= &\int_0^{\frac{\pi}{4}} \ln\sqrt{2} - \ln(\sqrt{1+\tan t})dt \\
= &\int_0^{\frac{\pi}{4}} \ln\sqrt{2}dt - \int_0^{\frac{\pi}{4}} \ln(\sqrt{1+\tan t})dt \\
= &\int_0^{\frac{\pi}{4}} \ln\sqrt{2}dt - I
\end{aligned}$$

where we have made the remarkable observation that the last integral is also exactly equal to I! Hence we must have

$$2I = \int_0^{\frac{\pi}{4}} \ln\sqrt{2}dt = \left[\frac{1}{2}t\ln 2\right]_0^{\frac{\pi}{4}} = \frac{\pi\ln 2}{8}$$

Hence $\int_0^{\frac{\pi}{4}} \ln(\sqrt{1+\tan x})dx = I = \frac{\pi\ln 2}{16}$.

Question 4

(a) (i)

$$\int_0^1 x(1-x)f''(x)dx = \int_0^1 xf''(x)dx - \int_0^1 x^2f''(x)dx$$

For the first integral, we can integrate by parts to get

$$\int_0^1 xf''(x)dx = \left[xf'(x)\right]_0^1 - \int_0^1 f'(x)dx = (1\cdot f'(1)-0\cdot f'(0)) - (f(1)-f(0)) = f'(1)$$

For the second integral, we may integrate by parts repeatedly on the given integral

$$\begin{aligned}
1 &= \int_0^1 f(x)dx \\
&= [xf(x)]_0^1 - \int_0^1 xf'(x)dx \\
&= (1 \cdot f(1) - 0 \cdot f(0)) - \left(\left[\frac{1}{2}x^2 f'(x)\right]_0^1 - \int_0^1 \frac{1}{2}x^2 f''(x)dx\right) \\
&= (0-0) - \left((\frac{1}{2}f'(1) - 0 \cdot f'(0)) - \int_0^1 \frac{1}{2}x^2 f''(x)dx\right) \\
&= \int_0^1 \frac{1}{2}x^2 f''(x)dx - \frac{f'(1)}{2}
\end{aligned}$$

So we have $\int_0^1 x^2 f''(x)dx = 2 + f'(1)$. Therefore,

$$\int_0^1 x(1-x)f''(x)dx = \int_0^1 xf''(x)dx - \int_0^1 x^2 f''(x)dx = f'(1) - (2 + f'(1)) = -2$$

(ii) It suffices to find a $c \in [0,1]$ such that $f''(c) = -12$. Define $g(x) = -6x^2 + 6x - f(x)$. Note that $g(x)$ is twice differentiable such that its second derivative is continuous. Let $G(x)$ be an anti-derivative of $g(x)$. Then, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
G(1) - G(0) &= \int_0^1 g(x)dx \\
&= \int_0^1 -6x^2 + 6x - f(x)dx \\
&= \left(-2x^3 + 3x^2\right)|_0^1 - \int_0^1 f(x)dx \\
&= (-2+3) - 1 \\
&= 0
\end{aligned}$$

This means that $G(0) = G(1)$. By the Mean Value Theorem, there exists $d \in (0,1)$ such that

$$g(d) = G'(d) = \frac{G(1) - G(0)}{1-0} = 0$$

Now, by direct calculations, we also have $g(0) = g(1) = 0$. So by Mean Value Theorem again, there exists $a \in (0,d)$ and $b \in (d,1)$ such that

$$g'(a) = \frac{g(d) - g(0)}{d-0} = 0, \quad g'(b) = \frac{g(1) - g(d)}{1-d} = 0$$

And by the Mean Value Theorem one more time, there exists $c \in (a,b)$ such that

$$g''(c) = \frac{g'(b) - g'(a)}{b-a} = 0$$

Now, since $g(x) = -6x^2 + 6x - f(x)$, we have $g'(x) = -12x + 6 - f'(x)$ and $g''(x) = -12 - f''(x)$. So this gives

$$0 = g''(c) = -12 - f''(c) \quad \Rightarrow \quad f''(c) = -12$$

This completes the proof.

(b) Let $F(x)$ be an anti-derivative of $f(x)$ and $G(x)$ be an anti-derivative of $F(x)$ (so $G'(x) = F(x)$ and $F'(x) = f(x)$). Integrating by parts, we get

$$0 = \int_0^1 xf(x)dx = xF(x)\Big|_0^1 - \int_0^1 F(x)dx = 1F(1) - 0F(0) - G(x)\Big|_0^1 = F(1) - (G(1) - G(0))$$

This gives $G(1) - G(0) = F(1)$. By the Mean Value Theorem, there exists $c \in (0, 1)$ such that

$$F(c) = G'(c) = \frac{G(1) - G(0)}{1 - 0} = \frac{F(1)}{1} = F(1)$$

Also, we have

$$0 = \int_0^1 f(x)dx = F(x)\Big|_0^1 = F(1) - F(0)$$

So altogether we obtain $F(0) = F(c) = F(1)$ for some $c \in (0, 1)$. By Rolle's Theorem, there exists $a \in (0, c)$ and $b \in (c, 1)$ such that $F'(a) = 0$, $F'(b) = 0$. So $f(a) = f(b) = 0$.

(c)

$$\text{Volume of solid } = \int_0^4 \pi(\sqrt{x})^2 dx = \int_0^4 \pi x dx = \frac{1}{2}\pi x^2\Big|_0^4 = 8\pi$$

Question 5

(a)

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So, by substituting this into the equation, we get

$$\begin{aligned}
0 &= (x^2 - xy)\frac{dy}{dx} + x^2 + y^2 \\
&= (x^2 - x(vx))\left(v + x\frac{dv}{dx}\right) + x^2 + (vx)^2 \\
&= x^2(1 - v)\left(v + x\frac{dv}{dx}\right) + x^2 + v^2x^2 \\
&= x^2\left((1 - v)\left(v + x\frac{dv}{dx}\right) + 1 + v^2\right)
\end{aligned}$$

But $x \neq 0$. So we may divide x^2 from both sides.

$$\begin{aligned}
0 &= (1 - v)\left(v + x\frac{dv}{dx}\right) + 1 + v^2 \\
&= v - v^2 + x\frac{dv}{dx} - vx\frac{dv}{dx} + 1 + v^2 \\
&= v + x\frac{dv}{dx} - vx\frac{dv}{dx} + 1 \\
&= x(1 - v)\frac{dv}{dx} + 1 + v
\end{aligned}$$

This gives

$$\frac{1}{x} = \left(\frac{v - 1}{v + 1}\right)\frac{dv}{dx} = \left(1 - \frac{2}{v + 1}\right)\frac{dv}{dx}$$

Integrating both sides with respect to x, we get

$$\int \frac{1}{x}dx = \int \left(1 - \frac{2}{v+1}\right) dv$$
$$\ln x = v - 2\ln(v+1) + C$$
$$\ln x = \frac{y}{x} - 2\ln\left(\frac{y}{x}+1\right) + C$$

Substituting $x = 1, y = 0$, we get $0 = 0 - 2\ln 1 + C = C$. So

$$\ln x = \frac{y}{x} - 2\ln\left(\frac{y}{x}+1\right)$$

(b) An integrating factor of the first order linear differential equation is

$$e^{\int -(1+\frac{3}{x})dx} = e^{-x-3\ln x} = e^{-x}e^{\ln x^{-3}} = x^{-3}e^{-x}$$

So we have

$$\frac{dy}{dx} - (1+\frac{3}{x})y = x+2$$
$$x^{-3}e^{-x}\frac{dy}{dx} - x^{-3}e^{-x}(1+\frac{3}{x})y = x^{-3}e^{-x}(x+2)$$
$$\frac{d}{dx}\left(x^{-3}e^{-x}y\right) = x^{-3}e^{-x}(x+2)$$

Integrating both sides with respect to x, we get

$$x^{-3}e^{-x}y = \int x^{-3}e^{-x}(x+2)dx$$

To integrate the RHS, we make the observation that $x^{-3}e^{-x}(x+2) = \frac{e^{-x}(x^2+2x)}{(x^2)^2}$ which, by using Quotient Rule, looks like some form of the derivative of $\frac{e^{-x}}{x^2}$. Indeed,

$$\frac{d}{dx}\frac{e^{-x}}{x^2} = \frac{-x^2(e^{-x}) - 2x(e^{-x})}{(x^2)^2} = -x^{-3}e^{-x}(x+2)$$

Therefore, we have

$$x^{-3}e^{-x}y = -\int -x^{-3}e^{-x}(x+2)dx$$
$$x^{-3}e^{-x}y = -\frac{e^{-x}}{x^2} + C$$
$$y = -x + Cx^3e^x$$

where C is an arbitrary constant. Substitute the initial value and we get $C = 1$. Therefore we have

$$y = x^3e^x - x$$

END OF SOLUTIONS

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