NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS

MA2213 Numerical Analysis
AY 2010/2011 Sem 2
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Contributors –

Question 1

(a)
$$\left(\begin{array}{cc|c} 0.003000 & 59.14 & 59.17 \\ 5.291 & -6.130 & 46.78 \end{array}\right) \xrightarrow{R_2 - 1764R_1} \left(\begin{array}{cc|c} 0.003000 & 59.14 & 59.17 \\ 0 & -104300 & -104400 \end{array}\right)$$

$x_1 = -10.00, x_2 = 1.001$

(b)
$$\left(\begin{array}{cc|c} 0.003000 & 59.14 & 59.17 \\ 5.291 & -6.130 & 46.78 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 5.291 & -6.130 & 46.78 \\ 0.003000 & 59.14 & 59.17 \end{array}\right)$$
$$\xrightarrow{R_2 - 0.0005670R_1} \left(\begin{array}{cc|c} 5.291 & -6.130 & 46.78 \\ 0 & 59.14 & 59.14 \end{array}\right)$$

$x_1 = 10.00, x_2 = 1.000$

Question 2

(a) $P(x) - 165x^5 + 0.0025x^4 - 986x^3 - 321x^2 + 0.0001$ interpolates $f(x) - 165x^5 + 0.0025x^4 - 986x^3 - 321x^2 + 0.0001$ at the specified points.
By the uniqueness property of the interpolating polynomial, the polynomial we are looking for is given by $P(x) - 165x^5 + 0.0025x^4 - 986x^3 - 321x^2 + 0.0001$, i.e. $-165x^5 + 111.0025x^4 - 986x^3 - 316x^2 - 45x + 15.0001$.

(b) Obviously, $f(x)$ interpolates itself at the specified points. Also, $f(x)$ is of degree 7 (<11).
By the uniqueness property of the interpolating polynomial, $P(x) = f(x) = -2x^7 + 54321x^5 + 50x^3 + 2000x^2 - 10x + 0.000001$.

Question 3

Let $x_0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ and $x_0 = -x_6; x_1 = -x_5; x_2 = -x_4; x_3 = 0$ (since the nodes are symmetrically placed about the origin).
We require $\int_{-1}^{1} f(x)dx = \sum_{i=0}^{6} A_i f(x_i)$ for $f(x) = 1, x, x^2 \ldots x^6$.
Thus, we have

$$\begin{cases} A_0 + A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = \int_{-1}^{1}(1)dx = 2 \\ A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 + A_5x_5 + A_6x_6 = \int_{-1}^{1} xdx = 0 \\ A_0x_0^2 + A_1x_1^2 + A_2x_2^2 + A_3x_3^2 + A_4x_4^2 + A_5x_5^2 + A_6x_6^2 = \int_{-1}^{1} x^2dx = \frac{2}{3} \\ A_0x_0^3 + A_1x_1^3 + A_2x_2^3 + A_3x_3^3 + A_4x_4^3 + A_5x_5^3 + A_6x_6^3 = \int_{-1}^{1} x^3dx = 0 \\ A_0x_0^4 + A_1x_1^4 + A_2x_2^4 + A_3x_3^4 + A_4x_4^4 + A_5x_5^4 + A_6x_6^4 = \int_{-1}^{1} x^4dx = \frac{2}{5} \\ A_0x_0^5 + A_1x_1^5 + A_2x_2^5 + A_3x_3^5 + A_4x_4^5 + A_5x_5^5 + A_6x_6^5 = \int_{-1}^{1} x^5dx = 0 \\ A_0x_0^6 + A_1x_1^5 + A_2x_2^6 + A_3x_3^6 + A_4x_4^6 + A_5x_5^6 + A_6x_6^6 = \int_{-1}^{1} x^6dx = \frac{2}{7} \end{cases}$$

Rearranging the second, fourth and sixth equations:

$$\begin{cases} x_0(A_0-A_6)+x_1(A_1-A_5)+x_2(A_2-A_4)=0 \\ x_0^3(A_0-A_6)+x_1^3(A_1-A_5)+x_2^3(A_2-A_4)=0 \\ x_0^5(A_0-A_6)+x_1^5(A_1-A_5)+x_2^5(A_2-A_4)=0 \end{cases}$$

This may be expressed in the form $xA=0$, where $x=\begin{pmatrix} x_0 & x_1 & x_2 \\ x_0^3 & x_1^3 & x_2^3 \\ x_0^5 & x_1^5 & x_2^5 \end{pmatrix}$ and $A=\begin{pmatrix} A_0-A_6 \\ A_1-A_5 \\ A_2-A_4 \end{pmatrix}$

Consider $V=\begin{pmatrix} 1 & x_0^2 & (x_0^2)^2 \\ 1 & x_1^2 & (x_1^2)^2 \\ 1 & x_2^2 & (x_2^2)^2 \end{pmatrix}$: This is a Vandermonde matrix.

$\det(V)=(x_2^2-x_0^2)(x_2^2-x_1^2)(x_1^2-x_0^2)\neq 0$, since x_0,x_1,x_2 are all distinct and have the same sign.

$V^T=\begin{pmatrix} 1 & 1 & 1 \\ x_0^2 & x_1^2 & x_2^2 \\ (x_0^2)^2 & (x_1^2)^2 & (x_2^2)^2 \end{pmatrix}$

$\det(V^T)=\det(V)\neq 0$

Multiplying the columns of V^T by x_0,x_1,x_2: $X=\begin{pmatrix} x_0 & x_1 & x_2 \\ x_0^3 & x_1^3 & x_2^3 \\ x_0^5 & x_1^5 & x_2^5 \end{pmatrix}$

$\det(X)=x_0x_1x_2\det(V^T)\neq 0$, since $x_0,x_1,x_2\neq 0$

Since $\det(X)\neq 0$, $xA=0$ has only the trivial solution for A.

i.e. $A_0-A_6=0, A_1-A_5=0, A_2-A_4=0$

$$\begin{aligned} E(p(x)) &= \int_{-1}^{1} p(x)dx=\sum_{i=0}^{6} A_i p(x_i) \\ &= 0-k[x_0^7(A_0-A_6)+x_1^7(A_1-A_5)+x_2^7(A_2-A_4)] \\ &= 0 \end{aligned}$$

Note: We showed that $E(p(x))=0$ for $p(x)$, a polynomial of degree at most 7. This result may be generalized to any nonzero odd function $g(x)$, i.e. if $g(-x)=-g(x)$ and $g(x)\neq 0$, then $E(g(x))=0$. The proof is exactly the same as above.

Question 4

(a) This question requires the following lemma: any divided difference of f is independent of the order of its terms.

Consider the Newton form of $P(x)$:

$$P(x)=f(x_0)+(x-x_0)f[x_0,x_1]+\cdots+\left(\prod_{i=0}^{6}(x-x_i)\right)f[x_0,x_1,\ldots,x_7]$$

Thus $f[x_6,x_3,x_5,x_2,x_1,x_4,x_0,x_7]=f[x_0,x_1,\ldots,x_7]=-15$

$f[x_6,x_3,x_5,x_2,x_1,x_4,x_0]=f[x_0,x_1,\ldots,x_6]=0$

$f[x_0,x_1,\ldots,x_7]=\frac{f[x_1,x_2,\ldots,x_7]-f[x_0,x_1,\ldots,x_6]}{x_7-x_0}$

$-15=\frac{f[x_1,x_2,\ldots,x_7]}{8+2}$

$f[x_1,x_2,\ldots,x_7]=-150$

$\therefore f[x_6,x_3,x_5,x_2,x_1,x_4,x_7]=-150$

$f[x_0,x_1,\ldots,x_6]=\frac{f[x_1,x_2,\ldots,x_6]-f[x_0,x_1,\ldots,x_5]}{x_6-x_0}$

Since $f[x_0, x_2, \ldots, x_6] = f[x_1, x_2, \ldots, x_5] = 0$ (they are coefficients of the given polynomial),
$0 = \frac{f[x_1, x_2, \ldots, x_6]}{6+2}$
$f[x_1, x_2, \ldots, x_6] = 0$
$\therefore f[x_6, x_3, x_5, x_2, x_1, x_4] = 0$

(b) Note that $p(x) = q(x)$ for $x = 1, 2, \ldots, 5$, and $q(x)$ is of degree 4.
Thus we may have $q(x) = p(x) + k(x-1)\ldots(x-5)$. The second term cancels to 0 for $x = 1, 2, \ldots, 5$.
Substitute $x = 6$ to find k:
$p(6) = q(6) + 120k$
$10 = 2777 + 120k$
$\therefore k = -\frac{2767}{120}$
$p(x) = 2x^4 + x^3 - x^2 + 5 - \frac{2767}{120}(x-1)\ldots(x-5)$

Question 5

Let $f(x) = e^x, I(f) = \int_0^2 e^x dx = e^2 - 1$

(a)

$$\begin{aligned} E_T(f) &= -\frac{(\frac{b-a}{n})^2(b-a)}{12} f''(\eta) \\ &\leq \frac{2^3}{12n^2} e^2 \end{aligned}$$

We need to solve $\frac{E_T(f)}{I(f)} < 10^{-4}$
i.e. $\frac{2^3}{12n^2(e^2-1)} e^2 < 10^{-4}$
$n > 87.8$
$\therefore n \geq 88$

(b)

$$\begin{aligned} E_S(f) &= -\frac{(\frac{b-a}{n})^4(b-a)}{180} f^(4)(\eta) \\ &\leq \frac{2^5}{180n^4} e^2 \end{aligned}$$

We need to solve $\frac{E_S(f)}{I(f)} < 10^{-4}$
i.e. $\frac{2^5}{180n^4(e^2-1)} e^2 < 10^{-4}$
$n > 6.7$
$\therefore n \geq 8$ (n must be even)

END OF SOLUTIONS

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