NATIONAL UNIVERSITY OF SINGAPORE
MATHEMATICS SOCIETY

PAST YEAR PAPER SOLUTIONS
with credits to Chang Hai Bin

MA2101S Linear Algebra II (S)
AY10/11 Sem 1

Question 1

(a) Since α satisfy the polynomial $x^2 - x = x(x-1), \gcd(x, x-1) = 1$,
V =Im(α)$\oplus$Im($\alpha$$-Id_V$) =Im($\alpha$) $\oplus$ ker(α)

Note:
This is based on the proposition:
if α is a linear operator on vector space V, and α satisfies $p(x)q(x)$ for some monic $p(x), q(x) \in F[x]$, and $\gcd(p(x), q(x)) = 1$, then:

(i) $\ker(p(\alpha))$ =Im$(q(x))$, $\ker(q(\alpha))$ =Im$(p(x))$,

(ii) $p(\alpha)|_{\mathrm{Im}(p(\alpha))}$ and $q(\alpha)|_{\mathrm{Im}(q(\alpha))}$ are bijective linear operators on their respective domain.

(iii) hence, we can show that V =Im$(p(\alpha))$$\oplus$Im$(q(\alpha))$.

(b) Let W =Im(α), $\dim(W) = n, \dim(L(W,W)) = n^2$ (the vector space of operators from W to W)
So $\left\{\mathrm{Id}_W, \alpha, \alpha^2, \ldots, \alpha^{(n^2)}\right\}$ is linearly independent, $p(\alpha) = \lambda_0 \ \mathrm{Id}_W + \ldots + (\lambda_{n^2})\alpha^{(n^2)} = 0_{L(W,W)}$ for some $\lambda_0, \ldots, \lambda_{n^2} \in F$.
So, $q(\alpha) = \alpha p(\alpha) = \alpha\left(\lambda_0 \mathrm{Id}_W + \ldots + (\lambda_{n^2})\alpha^{(n^2)}\right) = 0_{L(V,V)}$
Since α satisfy some non-trivial $q(x) \in F[x]$, the minimal polynomial exists.

Im(α^3) =Im$\left(\alpha|_{\mathrm{Im}(\alpha^2)}\right) = \left(\alpha|_{\mathrm{Im}(\alpha)}\right) = \mathrm{Im}(\alpha^2)$,
we can use induction to show that Im(α^i) =Im$(\alpha)\forall i \in \mathbb{N}$

Assume for a contradiction that $m_\alpha(x)$ is divisible by x^2,
so $\exists w, v \in V, k \in \mathbb{N}, \alpha(v) = w \neq 0$, but $\alpha^k(w) = 0$
But this means that if $\{w, a_1, a_2, \ldots, a_{n-1}\}$ is a basis for Im(α),
Im(α^{k+1}) =span$\{\alpha^k(w), \alpha(a_1), \ldots, \alpha(a_{n-1})\}$ =span$\{\alpha(a_1), \ldots, \alpha(a_{n-1})\}$
and so $\dim[\mathrm{Im}(\alpha^{k+1})] < \dim[\mathrm{Im}(\alpha)]$, a contradiction.

So, $m_\alpha(x)$ is not divisible by x^2

Using the Notes for part (a), as shown above, if $m_\alpha(x) = xp(x)$ for some monic $p(x) \in F[x]$, and $\gcd(x, p(x)) = 1$,,
So V =Im$(\alpha)$$\oplus$Im$(p(\alpha))$ =Im$(\alpha) \oplus \ker(\alpha)$ (Since Im$(p(\alpha)) = \ker(\alpha)$, according to the notes above)

(c) It does not hold. Counterexample: Let $V = \{$infinite sequences in $\mathbb{C}\}$,
define $\alpha : V \to V, \alpha[(a_1, a_2, a_3, \ldots)] = (a_2, a_3, a_4, \ldots)$ (the left shift operator)
So Im$(\alpha) = V$ (since $\forall v_1 = (b_1, b_2, \ldots) \in V, \exists v_2 = (0, b_1, b_2, \ldots) \in V$, such that $\alpha(v_2) = v_1$)
But $\ker(\alpha)$ =span$\{(1, 0, 0, \ldots)\}$.
So, $\ker(\alpha)$$\cap$Im$(\alpha) \neq \{\mathbf{0}\}$

Question 2

(a) Choose the ordered basis $\mathcal{B} = \{b_{11}, b_{21}, \ldots, b_{n1}, b_{12}, b_{22}, \ldots, b_{n2}, \ldots\ldots\ldots, b_{1n}, \ldots, b_{nn}\}$
where $b_{ij} = \begin{cases} 1, & \text{the i-th row, j-th column entry;} \\ 0, & \text{otherwise.} \end{cases}$
Let c_i is the i-th column of A.
One can easily show that $\phi(b_{ji}) = c_j$ for all i, j.
and $[\phi]_{\mathcal{B}} = \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix}$
So $\det(\phi) = \det([\phi]_{\mathcal{B}}) = [\det(A)]^n$

(b) Define $\alpha : \mathcal{H}_n \to \mathcal{H}_n$, $\alpha(W) = \overline{B}^T W$
So $\det(\alpha) = \left[\det(\overline{B}^T)\right]^n$ (by result from (a))
Define $\beta : \mathcal{H}_n \to \mathcal{H}_n$, $\beta(W) = WB$
Similar to part (a), we can show that $\det(\beta) = [\det(B)]^n$.
Alternatively, since $\beta = \tau \circ \alpha \circ \tau$ (where τ is the transpose function)
$\det(\beta) = \det(\tau)\det(\alpha)\det(\tau) = \det(\tau \circ \tau)\det(\alpha) = \det(\mathrm{Id})\det(\alpha) = 1 \cdot \det(\alpha) = \det(\alpha)$
So $\Phi = \alpha \circ \beta$
$\det(\Phi) = \det(\alpha \circ \beta) = \det(\alpha)\det(\beta) = \left[\det(\overline{B}^T)\right]^n [\det(B)]^n$
$= \left[\det(\overline{B}^T)\det(B)\right]^n = \left[\det(\overline{B})\det(B)\right]^n$
$= \left[\overline{\det(B)}\det(B)\right]^n = |\det(B)|^{2n}$

Question 3

Let α represents any particular linear operator on a finite-dimensional vector space V. (eg. if $V = F^n$, then $\alpha(v) = Av$ for any n-dimensional column vector v)
One can easily show that, for the minimal polynomial and characteristic polynomial of α and α^*, $\chi_\alpha(x) = \chi_{\alpha^*}(x)$ and $m_\alpha(x) = m_{\alpha^*}(x)$

Next we need the lemma (proven later):
If α is a linear operator on W (finite dimensional), and if $m_\alpha(x) = (f(x))^k$ for some monic irreducible $f(x) \in F[x], k \in \mathbb{N}$, and W is α-cyclic, then W^* is α^*-cyclic.

Then, since (according to assumption in the question) $V = \bigoplus_{i=1}^k U_i$, where each of the U_i is α-cyclic, and the minimal polynomial of $\alpha|_{U_i}$ is $(f_i(x))^{k_i}, k_i \in \mathbb{N}, f_i(x)$ irreducible,
There exist α-cyclic basis $B_i = \{v_i, \alpha(v_i), \ldots\}$ for U_i, α^*-cyclic basis ${B_i}^* = \{g_i, \alpha^*(g_i), \ldots\}$ for ${U_i}^*$,
$m_{v_i}(x) = m_{\alpha|_{U_i}} = m_{\alpha^*|_{U_i^*}} = m_{g_i}(x)$
So $[\alpha|_{U_i}]_{B_i} = C(m_{v_i}(x)) = C(m_{g_i}(x)) = [\alpha^*|_{U_i^*}]_{B_i^*}$

Where $C(f(x)) = \begin{bmatrix} 0 & \ldots & \ldots & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & \ldots & 0 & -a_1 \\ 0 & 1 & 0 & \ldots & 0 & -a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 & 0 & -a_{n-2} \\ 0 & \ldots & \ldots & 0 & 1 & -a_{n-1} \end{bmatrix}$
If $f(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$

Let $B_1 = \{v_1, \alpha(v_1), \ldots\}, B_2 = \{v_2, \alpha(v_2), \ldots\}$ be the α-cyclic basis for each of the U_is.
Let $B = B_1 \cup B_2 \cup \ldots \cup B_k$

Let $B_1^* = \{g_1, \alpha^*(g_1), \ldots\}, B_2^* = \{g_2, \alpha^*(g_2), \ldots\}$ be the α-cyclic basis for each of the U_is.
Let $B^* = B_1^* \cup B_2^* \cup \ldots \cup B_k^*$
then

$$[\alpha]_B = \begin{bmatrix} C(m_{v_1}(x)) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & C(m_{v_n}(x)) \end{bmatrix}$$
$$= \begin{bmatrix} C(m_{g_1}(x)) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & C(m_{g_n}(x)) \end{bmatrix} = [\alpha^*]_{B^*}$$

So, if $\alpha(x) = Ax$, then for the standard basis S, $[\alpha]_S = A, [\alpha^*]_{S^*} = A^T$,
but α, α^* are represented by the matrix above.
So, A, A^T are both similar to that matrix,
So, A is similar to A^T.

Lemma 1:
If α is a linear operator on W (finite dimensional), and if $m_\alpha(x) = (f(x))^k$ for some monic irreducible $f(x) \in F[x], k \in \mathbb{N}$, and W is α-cyclic, then W^* is α^*-cyclic.
Proof: If W is finite dimensional, then $\dim(W^*) = \dim(W)$

Since $m_{\alpha^*}(x) = m_\alpha(x) = (f(x))^k$,
$\exists g \in W^*, g \in \ker(f(x))^k \setminus \ker(f(x))^{k-1}$
Let $Z = \langle g \rangle_{\alpha^*}$
Using Lemma 2 (provided below), $m_g(x) = m_{\alpha^*|_Z}(x)$

Note that $m_{\alpha^*|_Z}(x) | m_{\alpha^*}(x) = f(x)^k$, and $f(x)$ is irreducible,
So $m_{\alpha^*|_Z}(x) = f(x)^i$ for some $i \in \mathbb{N}, i \leq k$.

But $\alpha^*|_Z$ does not satisfy $(f(x))^{k-1}$ (consider $g \in Z \setminus \ker(f(\alpha^*))^{k-1}$)
So $m_{\alpha^*|_Z}(x) = f(x)^k$
$\dim(\langle g \rangle_{\alpha^*}) = \deg(m_g(x)) = \deg(f(x)^k) = \deg(\chi_{\alpha^*}(x)) = \dim(W^*)$
So $\langle g \rangle_{\alpha^*} = W^*$, W^* is α-cyclic.

Lemma 2:
Let $\alpha : V \to V$, let $v \in V \setminus \{0\}$, and $\dim(\langle v \rangle_\alpha) = n, n \in \mathbb{N}$
Then

(i) $\deg(m_v(x)) = n$

(ii) $\chi_{\alpha|_{\langle v \rangle_\alpha}}(x) = m_{\alpha|_{\langle v \rangle_\alpha}}(x) = m_v(x)$

Question 4

(a) Not true. Counter-example: $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, the principle minors of A are all negative (both are -1), but consider $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,
then $\mathbf{x}^T A \mathbf{x} = 1 > 0$

Note: For arbitrary field F, let A=$(a_{ij}) \in M_n(F)$,

then the principal minors of A are $\det(A_1), \ldots, \det(A_n)$, where $A_r = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{rr} \end{bmatrix}$

(b) Not true. Counter-example: $V = \{infinite sequences with finite number of non-zero elements.\}$
Let $B_v = \{e_1, e_2, \ldots\}$ be the "standard basis" for V.
Let $U = span\{e_2\}$
Define $\phi(e_i, e_j) = \begin{cases} 1, & \text{i=1 or j=1;} \\ 0, & \text{otherwise} \end{cases}$
and define $\phi(u, v)$ according to their bilinear sum with respect to the basis B.

So, ϕ can be "represented" with the infinite matrix: $\begin{bmatrix} 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

$U^\perp = span\{e_2, e_3, \ldots\}$ (Since $\phi(e_i, e_2) = 0 \forall i \in \{2, 3, 4, \ldots\}$, and $\phi(e_1, e_2) \neq 0$)
and $(U^\perp)^\perp = span\{e_2, e_3, \ldots\}$ (Since $\phi(e_i, e_j) = 0 \forall i, j \in \{2, 3, 4, \ldots\}$, and $\phi(e_1, e_j) \neq 0$ for any particular j)
So, in this case, $U \neq (U^\perp)^\perp$.

(c) Not true. Counter-example: $[\phi]_{\mathfrak{B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathcal{B} = \{\text{standard basis} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.\}$
Let $U = \left\{ \begin{pmatrix} r \\ r \end{pmatrix} : r \in \mathbb{R} \right\}$ with the basis $B_1 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.
Then $[\phi|_{U \times U}]_{B_1} = \left[\phi\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right] = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = [0]$, which is degenerate.

(d) True. We will try to prove a more general statement:
If ϕ is a bilinear form on V, U is a vector subspace of V, U is finite dimensional, $\phi, \phi|_{U \times U}$ are both non-degenerate, then $\phi|_{U^{\perp,L} \times U^{\perp,L}}, \phi|_{U^{\perp,R} \times U^{\perp,R}}$ are both non-degenerate.
We will only prove for the case $\phi|_{U^{\perp,L} \times U^{\perp,L}}$.
In particular, if ϕ is symmetric bilinear form, then $U^{\perp,L} = U^{\perp,R} = U^\perp$
Proof: We need the following proposition:
if ϕ is a bilinear form on $V, U \subseteq V$, U finite dimensional, then the following are equivalent:

(i) $\phi|_{U \times U}$ is non-degenerate
(ii) $U \oplus U^{\perp,L} = V$
(iii) $U \cap U^{\perp,L} = \{\mathbf{0}\}$
(iv) $U \oplus U^{\perp,R} = V$
(v) $U \cap U^{\perp,R} = \{\mathbf{0}\}$

Fix any $x \in U^{\perp,L} \setminus \{\mathbf{0}\}$, then $\phi(x, u) = 0 \forall u \in U$.
Since ϕ is non-degenerate, $\exists y \in V, \phi(x, y) \neq 0$,

Using the proposition above, let $y = y_u + y', y_u \in U, y' \in U^{\perp,L}$
Then $0 \neq \phi(x, y) = \phi(x, y_u) + \phi(x, y') = 0 + \phi(x, y') = \phi(x, y')$
So, $\forall x \in U^{\perp,L}, \exists y' \in U^{\perp,L}$, such that $\phi(x, y') \neq 0$

Now, to show that $\exists w \in U^{\perp,L}$, such that $\phi(w,x) \neq 0$.
(Note: In the special case that ϕ is symmetric, then this part is trivial by letting $w = y'$. We are proving for the general case, that is, ϕ may not be symmetric)
Assume (for a contradiction) that $\forall w \in U^{\perp,L}, \phi(w,x) = 0$

Let $\mathcal{B}_u = \{u_1, \ldots, u_n\}$ be a basis for U,
Lemma: $\exists! u \in U, \phi(u_i, u) = \phi(u_i, x) \forall i \in \{1, \ldots, n\}$
Proof: Since $[\phi|_{U\times U}]_{\mathcal{B}_u} = \begin{bmatrix} \phi(u_1,u_1) & \phi(u_1,u_2) & \cdots & \phi(u_1,u_n) \\ \phi(u_2,u_1) & \phi(u_2,u_2) & \cdots & \phi(u_2,u_n) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(u_n,u_1) & \phi(u_n,u_2) & \cdots & \phi(u_n,u_n) \end{bmatrix}$ is invertible, ($\phi|_{U\times U}$ is non-degenerate)
Let $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = [\phi|_{U\times U}]^{-1}_{\mathcal{B}_u} \begin{bmatrix} \phi(u_1,x) \\ \vdots \\ \phi(u_n,x) \end{bmatrix}$
And let $u = \lambda_1 u_1 + \ldots + \lambda_n u_n$,
Then $\begin{bmatrix} \phi(u_1,x) \\ \vdots \\ \phi(u_n,x) \end{bmatrix} = [\phi|_{U\times U}]_{\mathcal{B}_u} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n \lambda_j \phi(u_1,u_j) \\ \vdots \\ \sum_{i=1}^n \lambda_j \phi(u_n,u_j) \end{bmatrix} = \begin{bmatrix} \phi(u_1,u) \\ \vdots \\ \phi(u_n,u) \end{bmatrix}$
if $\exists v \in U, \phi(u_i, v) = \phi(u_i, x) \forall i \in \{1, \ldots, n\}$, and $v = \mu_1 u_1 + \ldots + \mu_n u_n$, then
$\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = [\phi|_{U\times U}]^{-1}_{\mathcal{B}_u} \begin{bmatrix} \phi(u_1,x) \\ \vdots \\ \phi(u_n,x) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}$
Let $x = x' - u$, then $\forall w \in U^{\perp,L}$,

$$\begin{aligned} \phi(w,x') &= \phi(w,x) - \phi(w,u) \\ &= \phi(w,x) - 0 \quad \text{Since } w \in U^{\perp,L} \\ &= 0 \quad \text{assumption (for contradiction)} \end{aligned}$$

So, $\forall v \in V$, let $v = v_u + v', v \in U, v' \in U^{\perp,L}$

$$\begin{aligned} \phi(v,x') &= \phi(v_u,x) - \phi(v',u) \\ &= \phi(v_u,x) - 0 \quad \text{as shown above} \\ &= 0 \quad (\text{Since } \phi(u_i,x') = 0 \forall i \in \{1,\ldots,n\}) \end{aligned}$$

$x' \in V^{\perp,R} = \{\mathbf{0}\}$ (Since ϕ is non-degenerate)
$x = u$. Since $x \in U^{\perp,L}, x = u \in U$
$x \in U^{\perp,L} \cap U = \{\mathbf{0}\}$ (Using the Proposition above)
This is a contradiction, since we assumed $x \neq \mathbf{0}$.

So, $\forall x \in U^{\perp,L} \setminus \{\mathbf{0}\}, \exists y', w \in U^{\perp,L}$, such that $\phi(w,x), \phi(x,y') \neq 0$.
Then $\phi|_{U^{\perp,L}\times U^{\perp,L}}$ is non-degenerate.
Similarly, we can prove that $\phi|_{U^{\perp,R}\times U^{\perp,R}}$ is non-degenerate.