ST2132 HW4
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Problem 1.

Answer: a) We find cdf of Y:

$$F_Y(y) = \int_0^y 2y dy = y^2, 0 \leq y \leq 1$$
$$\Rightarrow F_Y(y) = \begin{cases} 0, y < 0 \\ y^2, 0 \leq y \leq 1 \\ 1, y > 1 \end{cases}$$

Then,

$$F_{Y_{(10)}}(y) = F_Y(y)^{10} = (y^2)^{10} = \begin{cases} y^{20}, 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases}$$
$$\Rightarrow f_Y(y) = 20y^{19}, 0 \leq y \leq 1$$

Take $U_1, \ldots, U_{20} \sim_{i.i.d} Uni(0, 1)$, with cdf and pdf:
$F_{U_1}(x) = x, f_{U_1} = 1, 0 \leq x \leq 1$. Then the largest order statistic then,

$$f_{U_{(20)}}(x) = \frac{20!}{(20-1)!1!(20-20)!} x^{20-1}(1-x)^{20-20}$$
$$= 20x^{19} \sim Beta(20, 20 - 20 + 1)$$

Hence, $Y_{(10)} \sim Beta(20, 1)$, thus

$$f_{Y(20)}(y) = 20y^{19}$$
$$= \frac{\Gamma(21)}{\Gamma(20)\Gamma(1)} y^{20-1}(1-y)^{20-20+1-1}, 0 \leq y \leq 1$$
$$\Rightarrow f_{Y(10)}(y) = \begin{cases} 20y^{19}, 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases}$$

b) We find

$$P(Y_{(10)} > 0.9) = 1 - P(Y_{(10)} \leq 0.9)$$
$$= 1 - F_{Y_{(10)}}(0.9) = 1 - 0.9^{20} = 0.87842 \approx 0.88$$

$\square$

Problem 2.

Answer: The sample in order of magnitude is as follows:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$X_{(k)}$	5.2	5.5	6.3	7.1	7.5	8.7	9.2	9.8	10.5	10.9	11.1	11.8	12.7	14.4

a) Given

$$\begin{aligned} a &= \lfloor p(14+1) \rfloor \\ b &= p(14+1) - a \\ \hat{\pi}_p &= X_{(a)} + b(X_{(a+1)} - X_{(a)}) \end{aligned}$$

For $p = 0.25$,

$$\begin{aligned} a = 3, b &= 3.75 - 3 = 0.75 \\ \hat{\pi}_{0.25} &= X_{(3)} + (0.75)(X_{(4)} - X_{(3)}) \\ &= 6.3 + 0.75(7.1 - 6.3) = 6.9 \end{aligned}$$

For $p = 0.35$,

$$\begin{aligned} a = 5, b &= 5.25 - 5 = 0.25 \\ \hat{\pi}_{0.35} &= X_{(5)} + (0.25)(X_{(6)} - X_{(5)}) \\ &= 7.5 + 0.25(8.7 - 7.5) = 7.8 \end{aligned}$$

For m, $p = 0.5$, since $n = 14$ is even,

$$\begin{aligned} m &= 0.5X_{(7)} + 0.5X_{(8)} \\ &= 0.5(9.2) + 0.5(9.8) = 9.5 \end{aligned}$$

For $p = 0.75$,

$$\begin{aligned} a = 11, b &= 11.25 - 11 = 0.25 \\ \hat{\pi}_{0.75} &= X_{(11)} + (0.25)(X_{(12)} - X_{(11)}) \\ &= 11.1 + 0.25(11.8 - 11.1) = 11.275 \end{aligned}$$

(b) i) Given $W = \sum_{i=1}^{14} \mathbf{1}_{(X_i < \pi_{0.25})} \sim Bin(14, 0.25)$,

$$\begin{aligned} P(X_{(1)} < \pi_{0.25} < X_{(5)}) &= P(W = 1, \ldots, 4) \\ &= \sum_{k=1}^{4} \binom{14}{k} (0.25)^k (0.75)^{14-k} \\ &= P(W \leq 4) - P(W \leq 0) = 0.7415 - 0.0178 = 0.7237 \end{aligned}$$

Thus, $(X_{(1)}, X_{(5)}) = (5.2, 7.5)$ is a 72.37% C.I. for $\pi_{0.25}$.
ii) Given $W = \sum_{i=1}^{14} \mathbf{1}_{(X_i < \pi_{0.35})} \sim Bin(14, 0.35)$,

$$\begin{aligned} P(X_{(3)} < \pi_{0.35} < X_{(8)}) &= P(W = 3, \ldots, 7) \\ &= \sum_{k=3}^{7} \binom{14}{k} (0.35)^k (0.65)^{14-k} \\ &= P(W \le 7) - P(W \le 2) = 0.9247 - 0.0839 = 0.8408 \end{aligned}$$

Thus, $(X_{(3)}, X_{(8)}) = (6.3, 9.8)$ is a 84.08% C.I. for $\pi_{0.35}$.
iii) Given $W = \sum_{i=1}^{14} \mathbf{1}_{(X_i < \pi_{0.5})} \sim Bin(14, 0.5)$,

$$\begin{aligned} P(X_{(5)} < \pi_{0.5} < X_{(10)}) &= P(W = 5, \ldots, 9) = \sum_{k=5}^{9} \binom{14}{k} (0.5)^n \\ &= P(W \le 9) - P(W \le 4) = 0.9102 - 0.0898 = 0.8204 \end{aligned}$$

Thus, $(X_{(5)}, X_{(10)}) = (7.5, 10.9)$ is a 82.04% C.I. for m.
(c) i) Let $N^- \sim Bin(14, 0.5)$, we have $n^- = 5$

$$P(N^- \ge 5; H_0) = 0.91021 > \alpha = 0.05$$

Hence, we fail to reject H_0 at 5% significance level.
ii)

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$X_{(k)}$	5.2	5.5	6.3	7.1	7.5	8.7	9.2	9.8	10.5	10.9	11.1	11.8	12.7	14.4
$x_{(k)} - m_0$	-2.8	-2.5	-1.7	-0.9	-0.5	0.7	1.2	1.8	2.5	2.9	3.1	3.8	4.7	6.4
$\lvert x_{(k)} - m_0 \rvert$	2.8	2.5	1.7	0.9	0.5	0.7	1.2	1.8	2.5	2.9	3.1	3.8	4.7	6.4
$R_{(k)}$	9	7.5	5	3	1	2	4	6	7.5	10	11	12	13	14
Signed $R_{(k)}$	-9	-7.5	-5	-3	-1	2	4	6	7.5	10	11	12	13	14

Then, $W = -9 - 7.5 - 5 - 3 - 1 + 2 + 4 + 6 + 7.5 + 10 + 11 + 12 + 13 + 14 = 54$

$$n = 14, \frac{n(n+1)(2n+1)}{6} = 1015$$
$$P(W \le 54; H_0) \approx \Phi\left(\frac{54+1}{\sqrt{1015}}\right)$$
$$\Phi(1.726) = 0.9573 \ge \alpha = 0.05$$

Hence, we fail to reject H_0 at 5% significance level.
iii) Assuming $H_0 : \mu = m = 8, H_1 : \mu < 8$

$$\bar{x} = 9.34, S^2 = 7.69$$
$$t = \frac{9.34 - 8}{\sqrt{7.69/14}} = 1.8080$$
$$-t_{0.05}(14 - 1) = -1.771$$

Since $t = 1.808 > -t_{0.05}(13) = -1.771$, we fail to reject H_0 at 5% significance level. □

Problem 3.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$x_{(k)}$	6	6	7	8	8	9	10	10	10	11	11	12	12	13	15
$x_{(k)} - m_0$	-3	-3	-2	-1	-1	-	1	1	1	2	2	3	3	4	6
Signed $R_{(k)}$	-10.5	-10.5	-7	-3	-3	-	3	3	3	7	7	10.5	10.5	13	14

Answer: i) Let $N^+ \sim Bin(14, 0.5)$,
We have $n^+ = 9$

$$P(N^+ \geq 9; H_0) = 0.21197 \geq \alpha = 0.1$$

Hence, we fail to reject H_0 at 10% significance level.
ii) $W = -10.5 - 10.5 - 7 - 3 - 3 + 3 + 3 + 3 + 7 + 7 + 10.5 + 10.5 + 13 + 14 = 37$

$$n = 14, \frac{n(n+1)(2n+1)}{6} = 1015$$
$$P(W \geq 37; H_0) \approx 1 - \Phi\left(\frac{37+1}{\sqrt{1015}}\right)$$
$$1 - \Phi(1.192) = 1 - 0.8830$$
$$= 0.117 \geq \alpha = 0.1$$

Hence, we fail to reject H_0 at 10% significance level.
iii) Assuming $H_0 : \mu = m = 9, H_1 : \mu > 9$

$$\bar{x} = 9.87, S^2 = 6.70$$
$$t = \frac{9.87 - 9}{\sqrt{6.7/15}} = 1.3017$$
$$t_{0.1}(15-1) = 1.345$$

Since $t = 1.3017 < t_{0.1}(14) = 1.345$, we fail to reject H_0 at 10% significance level. □

Problem 4. *Refer to code*

Problem 5. *Refer to code*