ST2132 HW1
Ethan Keck Jun Wei (February 1, 2024)

Problem 1.

Answer: Given that for each X_i,

$$f_{X_i}(x) = \frac{1}{\Gamma(r_i/2)2^{r_i/2}} x^{r_i/2-1} e^{-x/2},$$
$$M_{X_i}(t) = E(e^{tX_i}) = \frac{1}{(1-2t)^{r_i/2}}$$

then for $Y = \sum_{i=1}^n X_i$

$$E(e^{tY}) = E(e^{t\sum_{i=1}^n X_i}) = E(e^{tX_1}e^{tX_2}\dots e^{tX_n})$$
$$= \frac{1}{(1-2t)^{r_1/2}} \cdot \frac{1}{(1-2t)^{r_2/2}} \cdot \dots \cdot \frac{1}{(1-2t)^{r_n/2}}$$
$$= \frac{1}{(1-2t)^{\sum_{i=1}^n r_i/2}}$$

Then $Y = \sum_{i=1}^n X_i$ follows a chi-square distribution with $\sum_{i=1}^n r_i$ degrees of freedom i.e. $Y \sim \chi^2(\sum_{i=1}^n r_i)$ □

Problem 2.

Answer: Let $X \sim$ Poi(1.8) be the number of floods, with $E(X) = 1.8$, $Var(X) = 1.8$ and $Y \sim$ Exp(1/3) be the time during which the ground is flooded with $E(Y) = 3$, $Var(Y) = 9$.

a) We find $P(\hat{X} = X_1 + X_2 + \ldots + X_{20} \geq 19)$, by Central Limit Theorem

$$\hat{X} \sim N(20 * 1.8, 20 * 1.8)$$

Then

$$Z = \frac{\hat{X} - 36}{\sqrt{36}} \sim N(0,1)$$
$$\Rightarrow P\left(Z \geq \frac{19-36}{6}\right)$$
$$= P\left(Z \geq -\frac{17}{6}\right) = 0.9977$$

b) We find $P(\hat{Y} = \sum_{i=1}^{120} Y_i < 365)$, by Central Limit Theorem,

$$\hat{Y} \sim N(120 * 3, 120 * 9)$$
$$Z = \frac{\hat{Y} - 360}{\sqrt{1080}} \sim N(0,1)$$
$$\Rightarrow P\left(Z < \frac{365-360}{\sqrt{1080}}\right)$$
$$= P\left(Z < \frac{5}{6\sqrt{30}}\right) = 0.5605$$

$\square$

Problem 3.

Answer: a) Given

$$M_X(t) = \frac{1}{1 - t/\lambda}$$

and

$$M_X'(0) = \frac{\lambda}{(0-\lambda)^2} = \frac{1}{\lambda}$$

Given the first moment is:

$$\alpha_1(\lambda) = E_\lambda(X) = \frac{1}{\lambda}$$

and the first sample moment:

$$\hat{\alpha}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$$
$$\Rightarrow \bar{x} = \frac{1}{\lambda}$$

Implying that $\hat{\lambda}_{mom} = 1/\bar{X}$.
b) Given the log-likelihood function:

$$l(\lambda) = n\ln\lambda - \lambda\sum_{i=1}^{n} x_i$$
$$l'(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$
$$\Rightarrow \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$
$$\lambda = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$$

Then mle $\hat{\lambda}_{mle} = 1/\bar{X}$.
c) For the $n = 6$ samples,

$$\frac{3.8 + 3.24 + 1.4 + 1.22 + 4.5 + 4.6}{6} = \frac{469}{150}$$
$$\hat{\lambda}_{mom} = \frac{150}{469} = \hat{\lambda}_{mle}$$
$$= 0.31982 \approx 0.320$$

d) Since $X_1, X_2, ..., X_i \sim i.i.d. Exp(\lambda)$
then let $Y = \sum_{i=1}^{n} x_i \sim Gamma(n, \lambda)$.
And since

$$\hat{\lambda}_{mle} = \frac{1}{\bar{x}} = \frac{n}{\sum_{i=1} x_i} = \frac{n}{y},$$

$$E(\hat{\lambda}_{mle}) = E\left(\frac{n}{Y}\right) = n \int_{y=0}^{\infty} \frac{1}{y} \cdot \frac{\lambda^n}{\Gamma(n)} e^{-\lambda y} y^{n-1} dy,$$

$$\text{Let } z = \lambda y, \frac{dy}{dz} = \frac{1}{\lambda}$$

$$\Rightarrow n \int_{z=0}^{\infty} \frac{\lambda}{z} \cdot \frac{\lambda^n}{\Gamma(n)} e^{-z} \left(\frac{z}{\lambda}\right)^{n-1} \frac{1}{\lambda} dz$$

$$= \frac{n\lambda^n}{\Gamma(n)} \int_{z=0}^{\infty} e^{-z} \left(\frac{z}{\lambda}\right)^{n-2} \frac{1}{\lambda} dz$$

$$= \frac{n\lambda}{\Gamma(n)} \int_{z=0}^{\infty} e^{-z} z^{n-2} dz$$

$$= \frac{n\lambda}{\Gamma(n)} \Gamma(n-1) = \frac{n}{n-1} \lambda \neq \lambda$$

Hence, $\hat{\lambda}_{mle} = 1/\bar{X}$ is a biased estimator of λ. □

Problem 4.

Answer: We equate first moment and first sample moment: $E(X) = \bar{X}$,

$$E(X) = \int_{x=-1}^{1} x \cdot \frac{1}{2}(1 + \theta x) dx$$

$$= \frac{1}{2} \left[\frac{x^2}{2} + \frac{\theta x^3}{3}\right]_{x=-1}^{1} = \frac{\theta}{3}$$

Then $\hat{\theta}_{mom} = 3\bar{X}$.
And since

$$E(\hat{\theta}_{mom}) = E(3\bar{X})$$

$$= 3 \cdot \frac{1}{n} \sum_{i=1}^{n} E(X_i) = \frac{3}{n} \cdot n \cdot \frac{\theta}{3} = \theta$$

Hence, $\hat{\theta}_{mom} = 3\bar{X}$ is an unbiased estimator of θ. □

Problem 5.

Answer: The likelihood and log-likelihood function are respectively given as,

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} e^{-(x_i-\theta)} = \operatorname{Exp}\left(-\sum_{i=1}^{n}(x_i-\theta)\right) \\ &= \operatorname{Exp}\left(n\theta - \frac{n}{n}\sum_{i=1}^{n} x_i\right) = \operatorname{Exp}\left(n(\theta-\bar{x})\right) \\ l(\theta) &= \ln e^{n(\theta-\bar{x})} = n(\theta-\bar{x}) \\ l'(\theta) &= n \end{aligned}$$

Let $l'(\theta) = 0$,

$$n = 0$$

Since there is no estimate, we consider:

$$L(\theta) = \operatorname{Exp}\left(n\theta - \sum_{i=1}^{n} x_i\right)$$

$L(\theta)$ is maximum when $e^{n\theta}$ is maximum, but since $x \geq \theta$, then $L(\theta)$ would be maximum if $\theta = \min\{X_1, X_2, \ldots, X_n\}$. Hence $\hat{\theta} = \min\{X_1, X_2, \ldots, X_n\}$ $\square$

Problem 6.

Answer: We find cdf of $\hat{\theta} = \max\{X_1, \ldots, X_n\}$:

$$\begin{aligned} F_{\hat{\theta}}(x) &= P(\hat{\theta} \leq x) \\ &= P(\max\{X_1, \ldots, X_n\} \leq x) \\ &= P(X_1, \ldots, X_n \leq x) = \left(\frac{x}{\theta}\right)^n \end{aligned}$$

Then the pdf is given as:

$$f_{\hat{\theta}}(x) = F'_{\hat{\theta}}(x) = \frac{nx^{n-1}}{\theta^n}$$

The mean is given as:

$$\begin{aligned} E(\hat{\theta}) &= \int_{x=0}^{\theta} x \cdot \frac{nx^{n-1}}{\theta^n} dx \\ &= \frac{n}{\theta^n}\int_{x=0}^{\theta} x^n dx = \frac{n}{\theta^n}\left[\frac{x^{n+1}}{n+1}\right]_{x=0}^{\theta} \\ &= \left(\frac{n}{n+1}\right)\theta \neq \theta \end{aligned}$$

Hence, $\hat{\theta}$ is a biased estimator of θ.
We propose $\bar{\theta} = \frac{n+1}{n}\hat{\theta}$.
Then,

$$E(\bar{\theta}) = \frac{n+1}{n}E(\hat{\theta}) = \frac{n+1}{n}\left(\frac{n}{n+1}\right)\theta = \theta$$

Then $\bar{\theta}$ is an unbiased estimator of θ. $\square$

Problem 7.

Answer: a)

$$f_{X,K}(x,k) = f_{X|K}(x|k=0)f_K(k=0) + f_{X|K}(x|k=1)f_K(k=1)$$
$$= \frac{\pi_0}{\sqrt{2\pi\sigma_0^2}}e^{-\frac{(x-\mu_0)^2}{2\sigma_0^2}} + \frac{\pi_1}{\sqrt{2\pi\sigma_1^2}}e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} = \sum_{k=0}^{1} \frac{\pi_k}{\sqrt{2\pi\sigma_k^2}}e^{-\frac{(x-\mu_k)^2}{2\sigma_k^2}}$$

The support of (X,K) is such that $(X = x, K = k) \in \{(x,k)|x \in \mathbb{R}, k \in \{0,1\}\}$
b) The likelihood function is as follows:

$$L(\pi_0, \mu_0, \sigma_0^2, \mu_1, \sigma_1^2) = \prod_{i=1}^{n} f_{X,K}(x_i, k_i)$$

Taking the log-likelihood function,

$$l(\pi_0, \mu_0, \sigma_0^2, \mu_1, \sigma_1^2) = \sum_{i=1}^{n} \ln\left(f_{X,K}(x_i, k_i)\right)$$

$$= \sum_{i=1}^{n}\left[\left(\ln \pi_0 - \ln\sqrt{2\pi\sigma_0^2} - \frac{(x_i-\mu_0)^2}{2\sigma_0^2}\right)\mathbf{1}_{\{k_i=0\}} + \left(\ln(1-\pi_0) - \ln\sqrt{2\pi\sigma_1^2} - \frac{(x_i-\mu_1)^2}{2\sigma_1^2}\right)\mathbf{1}_{\{k_i=1\}}\right]$$

Then we take the respective partial derivative, and set it to zero.
For π_0,

$$\frac{\partial l}{\partial \pi_0} = 0$$
$$\sum_{i=1}^{n} \frac{1}{\pi_0}\mathbf{1}_{\{k_i=0\}} - \sum_{i=1}^{n} \frac{1}{1-\pi_0}\mathbf{1}_{\{k_i=1\}} = 0$$
$$\frac{n_0}{\pi_0} = \frac{n_1}{1-\pi_0}$$
$$n_0 - n_0\pi_0 = n_1\pi_0$$
$$(n_0+n_1)\pi_0 = n_0$$
$$\pi_0 = \frac{n_0}{n}$$
$$\Rightarrow \hat{\pi}_0 = \frac{n_0}{n}$$

For each μ_k, where $k \in \{0, 1\}$,

$$\frac{\partial l}{\partial \mu_k} = 0$$

$$\sum_{i=1}^{n} \mathbf{1}_{\{k_i = k\}} \frac{2(x_i - \mu_k)}{2\sigma_k^2} = 0$$

$$\sum_{i=1}^{n} \mathbf{1}_{\{k_i = k\}} x_i = n_k \mu_k$$

$$\mu_k = \frac{1}{n_k} \sum_{i=1}^{n} x_i \mathbf{1}_{\{k_i = k\}}$$

$$\Rightarrow \hat{\mu}_k = \frac{1}{n_k} \sum_{i=1}^{n} x_i \mathbf{1}_{\{k_i = k\}}$$

Hence for μ_0, where $k = 0$, $\hat{\mu}_0 = \frac{1}{n_0} \sum_{i=1}^{n} x_i \mathbf{1}_{\{k_i = 0\}}$
and for μ_1, where $k = 1$, $\hat{\mu}_1 = \frac{1}{n_1} \sum_{i=1}^{n} x_i \mathbf{1}_{\{k_i = 1\}}$

For σ_k^2, where $k \in \{0, 1\}$, we take the partial derivative with respect to σ_k,

$$\frac{\partial l}{\partial \sigma_k} = 0$$

$$\sum_{i=1}^{n} \mathbf{1}_{\{k_i = k\}} \left(-\frac{1}{\sigma_k} + \frac{(x_i - \mu_k)^2}{\sigma_k^3} \right) = 0$$

$$\frac{n_k}{\sigma_k} = \sum_{i=1}^{n} \mathbf{1}_{\{k_i = k\}} \frac{(x_i - \mu_k)^2}{\sigma_k^3}$$

$$n_k \sigma_k^2 = \sum_{i=1}^{n} \mathbf{1}_{\{k_i = k\}} (x_i - \mu_k)^2$$

$$\sigma_k^2 = \frac{1}{n_k} \sum_{i=1}^{n} \mathbf{1}_{\{k_i = k\}} (x_i - \mu_k)^2$$

Hence for σ_0^2, where $k = 0$, $\hat{\sigma}_0^2 = \frac{1}{n_0} \sum_{i=1}^{n} (x_i - \mu_0)^2 \mathbf{1}_{\{k_i = 0\}}$
and for σ_1^2, where $k = 1$, $\hat{\sigma}_1^2 = \frac{1}{n_1} \sum_{i=1}^{n} (x_i - \mu_1)^2 \mathbf{1}_{\{k_i = 1\}}$ $\square$