Logistic regression

1. Suppose we toss the coin n times. For the i^{th} toss, let $y_i = 1$ if it comes up a head and $y_i = 0$ if it comes up a tail. So we observe data $y = c(y_1, y_2, ..., y_n)$.

   (i) Assume a logistic regression model with just one parameter β_0: $P(Y = 1) = \frac{\exp(\beta_0)}{1+\exp(\beta_0)}$. Write down an expression for the log-likelihood in terms of the observed data.

   Solution: Refer to lecture notes in week 11 for the derivation. The log-likelihood function is

   $$\ln L(\beta_0) = \sum_{i=1}^{n} \{y_i\beta_0 - \ln[1 + \exp(\beta_0)]\}$$

   (ii) Find the maximum likelihood estimate (MLE) of β_0.

   Solution:

   $$\begin{aligned} &\frac{\partial}{\partial\beta_0} \sum_{i=1}^{n} \{y_i\beta_0 - \ln[1 + \exp(\beta_0)]\} = 0 \\ \rightarrow &\sum_{i=1}^{n} \left\{ y_i - \frac{\exp(\beta_0)}{1 + \exp(\beta_0)} \right\} = 0 \\ \rightarrow &n \left[\frac{\exp(\beta_0)}{1 + \exp(\beta_0)} \right] = \sum_{i=1}^{n} y_i \\ \rightarrow &\frac{\exp(\beta_0)}{1 + \exp(\beta_0)} = \bar{y} \\ \rightarrow &\text{The MLE is } \beta_0 = \ln\left(\frac{\bar{y}}{1 - \bar{y}}\right) \end{aligned}$$