

Problem 1.*Answer:* a)

$$\begin{aligned}
\alpha(c) &= P(T \in C; H_0) = P(X > c; \mu = 0) \\
&= P\left(\frac{X - 0}{1} > \frac{c - 0}{1}\right) \\
&= P(Z > c) = 1 - \Phi(c)
\end{aligned}$$

b)

$$\begin{aligned}
\beta(c) &= P(T \notin C; \mu = 1) \\
&= P\left(\frac{X - 1}{1} \leq \frac{c - 1}{1}\right) \\
&= P(Z \leq c - 1) = \Phi(c - 1)
\end{aligned}$$

c) Given

$$\begin{aligned}
R(c) &= 1 - \Phi(c) + \Phi(c - 1) \\
R'(c) &= -\Phi'(c) + \Phi'(c - 1) \\
&= -\frac{\exp\{-c^2/2\}}{\sqrt{2\pi}} + \frac{\exp\{-(c - 1)^2/2\}}{\sqrt{2\pi}}
\end{aligned}$$

Let $R'(c) = 0$,

$$\begin{aligned}
&-\frac{\exp\{-c^2/2\}}{\sqrt{2\pi}} + \frac{\exp\{-(c - 1)^2/2\}}{\sqrt{2\pi}} = 0 \\
&\Rightarrow \exp\{-c^2/2\} = \exp\{-(c - 1)^2/2\} \\
&\Rightarrow c^2 = (c - 1)^2 \\
&\Rightarrow c^2 = c^2 - 2c + 1 \Rightarrow 2c = 1, c = \frac{1}{2}
\end{aligned}$$

Then

$$\begin{aligned}
R''(c) &= -\frac{c \exp\{-c^2/2\}}{\sqrt{2\pi}} + \frac{(1 - c) \exp\{-(c - 1)^2/2\}}{\sqrt{2\pi}} \\
R''(c = 1/2) &= 0.352065 > 0 \text{ (local min) and,} \\
R(1/2) &= 1 - \Phi(1/2) + \Phi(-1/2) \\
&= 2\Phi(-1/2)
\end{aligned}$$

Hence, $c^* = \frac{1}{2}$, $R(c^*) = 2\Phi(-1/2)$

□

Problem 2.

Answer: a) Given the likelihood ratio:

$$\begin{aligned}
 & \frac{L(1)}{L(2)} \leq k \\
 \Rightarrow & \frac{(2\pi)^{-n/2} \exp \left[-\frac{1}{2} \sum_{i=1}^n (x_i^2) \right]}{(2\pi \cdot 2)^{-n/2} \exp \left[-\frac{1}{2 \cdot 2} \sum_{i=1}^n (x_i^2) \right]} \\
 & = 2^{n/2} \exp \left[-\frac{1}{4} \sum_{i=1}^n (x_i^2) \right] \\
 \Rightarrow & \ln 2^{n/2} - \frac{1}{4} \sum_{i=1}^n (x_i^2) \leq \ln k \\
 \Rightarrow & \frac{1}{4} \sum_{i=1}^n (x_i^2) \geq \frac{1}{2} \ln 2^n - \frac{1}{2} \ln k \\
 \Rightarrow & \sum_{i=1}^n x_i^2 \geq 2 \ln \frac{2^{10}}{k^2} = c
 \end{aligned}$$

By Neyman-Pearson Lemma, the best critical region is of the form:

$$C = \left\{ (x_1, \dots, x_{10}); \sum_{i=1}^n x_i^2 \geq c \right\}$$

where c is determined as

$$\begin{aligned}
 0.05 &= P \left(\sum_{i=1}^n x_i^2 \geq c; \sigma^2 = 1 \right) \\
 \Rightarrow 0.05 &= P \left(\chi^2(10) \geq c; \sigma^2 = 1 \right) \\
 &= 1 - P \left(\chi^2(10) \leq c \right) \\
 \Rightarrow P \left(\chi^2(10) \leq c \right) &= 0.95 \\
 \Rightarrow c &= \chi_{0.05}^2(10) = 18.31
 \end{aligned}$$

Hence for n = 10, the test with critical region $\{\sum_{i=1}^n x_i^2 \geq 18.31\}$ is the most powerful test of size $\alpha = 0.05$.

b) From R, $K(2) = 1 - P \left(\frac{1}{2} \sum_{i=1}^n x_i < \frac{18.41}{2} \right) = 0.5174595$

c) Given the log-likelihood ratio:

$$\begin{aligned}
& \frac{L(1)}{L(4)} \leq k \\
& \Rightarrow \frac{(2\pi)^{-n/2} \exp \left[-\frac{1}{2} \sum_{i=1}^n (x_i^2) \right]}{(2\pi \cdot 4)^{-n/2} \exp \left[-\frac{1}{2 \cdot 4} \sum_{i=1}^n (x_i^2) \right]} \\
& = 2^n \exp \left[-\frac{3}{8} \sum_{i=1}^n (x_i^2) \right] \\
& \Rightarrow \ln 2^n - \frac{3}{8} \sum_{i=1}^n x_i^2 \leq \ln k \\
& \Rightarrow \sum_{i=1}^n x_i^2 \geq \frac{8}{3} \ln \frac{2^{10}}{k} = c
\end{aligned}$$

By Neyman-Pearson Lemma, the best critical region is of the form:

$$C = \left\{ (x_1, \dots, x_{10}); \sum_{i=1}^n x_i^2 \geq c \right\}$$

where c is determined as

$$\Rightarrow c = \chi_{0.05}^2(10) = 18.31$$

Hence for n = 10, the test with critical region $\{\sum_{i=1}^n x_i^2 \geq 18.31\}$ is the most powerful test of size $\alpha = 0.05$.

d) From R, $K(4) = 1 - P\left(\frac{1}{4} \sum_{i=1}^n x_i < \frac{18.41}{4}\right) = 0.917559$

e) Given the likelihood ratio:

$$\begin{aligned}
& \frac{L(1)}{L(\sigma_1^2)} \leq k \\
& \Rightarrow \frac{(2\pi)^{-n/2} \exp \left[-\frac{1}{2} \sum_{i=1}^n (x_i^2) \right]}{(2\pi \cdot \sigma_1^2)^{-n/2} \exp \left[-\frac{1}{2 \cdot \sigma_1^2} \sum_{i=1}^n (x_i^2) \right]} \\
& = \sigma_1^n \exp \left[-\frac{\sigma_1^2 - 1}{2\sigma_1^2} \sum_{i=1}^n (x_i^2) \right] \\
& \Rightarrow \ln \sigma_1^n - \frac{\sigma_1^2 - 1}{2\sigma_1^2} \sum_{i=1}^n (x_i^2) \leq \ln k \\
& \Rightarrow \sum_{i=1}^n x_i^2 \geq \frac{2\sigma_1^2}{\sigma_1^2 - 1} \ln \frac{\sigma_1^{10}}{k} = c
\end{aligned}$$

By Neyman-Pearson Lemma, the best critical region is of the form:

$$C = \left\{ (x_1, \dots, x_{10}); \sum_{i=1}^n x_i^2 \geq c \right\}$$

where c is determined as

$$\Rightarrow c = \chi_{0.05}^2(10) = 18.31$$

Hence for $n = 10$, the test with critical region $\{\sum_{i=1}^n x_i^2 \geq 18.31\}$ is the most powerful test of size $\alpha = 0.05$.

f) Since the critical region C defines a test that is most powerful against each simple alternative $\sigma^2 > 1$, the test in (e) is a uniformly most powerful test, and C is a uniformly most powerful critical region of size $\alpha = 0.05$. Again, if $\alpha = 0.05$, then $c = \chi_{0.05}^2(10) = 18.31$. \square

Problem 3.

Answer: a)

$$\begin{aligned}\bar{X} &\sim N\left(\mu_1, \frac{400}{n}\right), \bar{Y} \sim N\left(\mu_2, \frac{225}{n}\right) \\ \Rightarrow \bar{X} - \bar{Y} &\sim N\left(\mu_1 - \mu_2, \frac{400}{n} + \frac{225}{n} = \frac{625}{n}\right) \\ &\Rightarrow \bar{X} - \bar{Y} \sim N\left(\theta, \frac{625}{n}\right)\end{aligned}$$

The power function is given as:

$$\begin{aligned}K(\theta) &= P(\bar{x} - \bar{y} \in C; \theta) \\ &= P\left(\frac{\bar{x} - \bar{y} - \theta}{\sqrt{625/n}} \geq \frac{c - \theta}{\sqrt{625/n}}\right) \\ &= 1 - \Phi\left(\frac{c - \theta}{25\sqrt{n}}\right)\end{aligned}$$

b) Given $\alpha = 0.05, \theta = 10$, we have

$$\begin{aligned}0.05 &= \alpha(H_0 : \theta = 0) = P(\bar{x} - \bar{y} \in C; \theta = 0) \\ &\Rightarrow 0.05 = 1 - \Phi\left(\frac{c}{25\sqrt{n}}\right) \\ &\Rightarrow \left(\frac{c}{25\sqrt{n}}\right) = 1.645 \text{ and,}\end{aligned}$$

$$\begin{aligned}0.9 &= K(10) = P(\bar{x} - \bar{y} \in C; \theta = 10) \\ &\Rightarrow 0.1 = \Phi\left(\frac{c - 10}{25\sqrt{n}}\right) \\ &\Rightarrow \left(\frac{c - 10}{25\sqrt{n}}\right) = 1.28\end{aligned}$$

Solving for n and c : $c = 45.06849, n = 1.20097 \approx 1$

\square

Problem 4.

Answer: The likelihood function for X and Y, respectively, are:

$$L_X(\mu_1) = \frac{1}{(\sqrt{2\pi})^{m/2}} \exp \left\{ -\frac{\sum_{i=1}^m (x_i - \mu_1)^2}{2} \right\},$$

$$L_Y(\mu_2) = \frac{1}{(\sqrt{2\pi})^{n/2}} \exp \left\{ -\frac{\sum_{j=1}^n (y_j - \mu_2)^2}{2} \right\}$$

Then

$$\begin{aligned} L(\mu_1, \mu_2) &= L_X(\mu_1) L_Y(\mu_2) \\ &= \frac{1}{(\sqrt{2\pi})^{m/2}} \exp \left\{ -\frac{\sum_{i=1}^m (x_i - \mu_1)^2}{2} \right\} \cdot \frac{1}{(\sqrt{2\pi})^{n/2}} \exp \left\{ -\frac{\sum_{j=1}^n (y_j - \mu_2)^2}{2} \right\} \\ &= \frac{1}{(\sqrt{2\pi})^{(m+n)/2}} \exp \left\{ -\frac{1}{2} \sum_{i=1}^m (x_i - \mu_2)^2 \right\} \exp \left\{ -\frac{1}{2} \sum_{j=1}^n (y_j - \mu_2)^2 \right\} \end{aligned}$$

b) We find the gradient of of the log-likelihood function $l(\mu_1, \mu_2)$ and set it to $\mathbf{0}$

$$\begin{aligned} & \left(\partial l(\mu_1, \mu_2) / \partial \mu_1 \quad \partial l(\mu_1, \mu_2) / \partial \mu_2 \right)^T \\ &= \begin{pmatrix} \partial \left[\ln \frac{\exp\{-\frac{1}{2} \sum_{j=1}^n (y_j - \mu_2)^2\}}{(\sqrt{2\pi})^{(m+n)/2}} - \frac{1}{2} \sum_{i=1}^m (x_i - \mu_1)^2 \right] / \partial \mu_1}{\partial \left[\ln \frac{\exp\{-\frac{1}{2} \sum_{i=1}^m (x_i - \mu_1)^2\}}{(\sqrt{2\pi})^{(m+n)/2}} - \frac{1}{2} \sum_{j=1}^n (y_j - \mu_2)^2 \right] / \partial \mu_2} \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} \sum_{i=1}^m (x_i - \mu_1) \\ \sum_{j=1}^n (y_j - \mu_2) \end{pmatrix} = \mathbf{0} \\ &\Rightarrow \begin{pmatrix} \sum_{i=1}^m x_i = m\mu_1 \\ \sum_{j=1}^n y_j = n\mu_2 \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} \hat{\mu}_1 = \frac{1}{m} \sum_{i=1}^m x_i \\ \hat{\mu}_2 = \frac{1}{n} \sum_{j=1}^n y_j \end{pmatrix} \end{aligned}$$

The Hessian of the $l(\mu_1, \mu_2)$ can be found as:

$$\begin{aligned} H_l(\mu_1, \mu_2) &= \begin{pmatrix} \partial^2 l(\mu_1, \mu_2) / \partial \mu_1^2 & \partial^2 l(\mu_1, \mu_2) / \partial \mu_1 \partial \mu_2 \\ \partial^2 l(\mu_1, \mu_2) / \partial \mu_2 \partial \mu_1 & \partial^2 l(\mu_1, \mu_2) / \partial \mu_2^2 \end{pmatrix} \Rightarrow \begin{pmatrix} -m & 0 \\ 0 & -n \end{pmatrix} \\ &\Rightarrow \det(\lambda I - H_l(\mu_1, \mu_2)) = 0 \\ &\Rightarrow \det \begin{pmatrix} \lambda + m & 0 \\ 0 & \lambda + n \end{pmatrix} \Rightarrow (\lambda + m)(\lambda + n) = 0 \\ &\lambda = -m, \lambda = -n, \text{ for all } n, m > 0 \end{aligned}$$

The Hessian and of $l(\mu_1, \mu_2)$ has negative eigenvalues $\lambda_1 = -m < 0$ and $\lambda_2 = -n < 0$ and is thus negative definite. Hence $(\hat{\mu}_1 = \bar{X} \quad \hat{\mu}_2 = \bar{Y})^T$ is a local and global maximizer of

the log-likelihood function, and thus $\hat{\mu}_{1,mle} = \frac{1}{m} \sum_{i=1}^m X_i = \bar{X}$, $\hat{\mu}_{2,mle} = \frac{1}{n} \sum_{j=1}^n Y_j = \bar{Y}$.
c) Since under $H_0 : \mu_1 = \mu_2$, we find

$$\begin{aligned} \frac{dl(\mu)}{d\mu} &= \frac{d \left[\ln \frac{1}{(\sqrt{2\pi})^{(m+n)/2}} - \frac{1}{2} \sum_{i=1}^m (x_i - \mu)^2 - \frac{1}{2} \sum_{j=1}^n (y_j - \mu)^2 \right]}{d\mu} \\ &\Rightarrow \sum_{i=1}^m (x_i - \mu) + \sum_{j=1}^n (y_j - \mu) = 0 \\ &\Rightarrow m\bar{x} - m\mu + n\bar{y} - n\mu = 0 \\ &\Rightarrow \mu(m+n) = m\bar{x} + n\bar{y} \\ &\Rightarrow \mu = \frac{m\bar{x} + n\bar{y}}{m+n} \end{aligned}$$

Hence $\hat{\mu}_{mle} = \frac{m\bar{X} + n\bar{Y}}{m+n}$

d) We find

$$\begin{aligned} \max_{\mu=\mu_1=\mu_2} L(\mu) &= \frac{1}{(\sqrt{2\pi})^{(m+n)/2}} \exp \left\{ -\frac{1}{2} \sum_{i=1}^m (x_i - \hat{\mu}_{mle})^2 \right\} \exp \left\{ -\frac{1}{2} \sum_{j=1}^n (y_j - \hat{\mu}_{mle})^2 \right\} \\ \max_{\mu_1, \mu_2} L(\mu_1, \mu_2) &= \frac{1}{(\sqrt{2\pi})^{(m+n)/2}} \exp \left\{ -\frac{1}{2} \sum_{i=1}^m (x_i - \bar{x})^2 \right\} \exp \left\{ -\frac{1}{2} \sum_{j=1}^n (y_j - \bar{y})^2 \right\} \\ &\Rightarrow \frac{\max_{\mu=\mu_1=\mu_2} L(\mu)}{\max_{\mu_1, \mu_2} L(\mu_1, \mu_2)} = \frac{\exp \left\{ -\frac{1}{2} \sum_{i=1}^m (x_i - \hat{\mu}_{mle})^2 - \frac{1}{2} \sum_{j=1}^n (y_j - \hat{\mu}_{mle})^2 \right\}}{\exp \left\{ -\frac{1}{2} \sum_{i=1}^m (x_i - \bar{x})^2 - \frac{1}{2} \sum_{j=1}^n (y_j - \bar{y})^2 \right\}} \\ &= \exp \left\{ -\frac{1}{2} \sum_{i=1}^m (x_i - \hat{\mu}_{mle})^2 - \frac{1}{2} \sum_{j=1}^n (y_j - \hat{\mu}_{mle})^2 + \frac{1}{2} \sum_{i=1}^m (x_i - \bar{x})^2 + \frac{1}{2} \sum_{j=1}^n (y_j - \bar{y})^2 \right\} \end{aligned}$$

Since

$$\begin{aligned} \sum_{i=1}^m (x_i - \hat{\mu})^2 &= \sum_{i=1}^m (x_i^2 - 2x_i\hat{\mu} + \hat{\mu}^2) \\ &= \sum_{i=1}^m (x_i^2) - 2m\bar{x}\hat{\mu} + m\hat{\mu}^2, \\ \sum_{i=1}^m (x_i - \bar{x})^2 &= \sum_{i=1}^m (x_i^2 - 2x_i\bar{x} + \bar{x}^2) \\ &= \sum_{i=1}^m (x_i^2) - 2m\bar{x}^2 + m\bar{x}^2 \\ &\Rightarrow -\sum_{i=1}^m (x_i - \hat{\mu})^2 + \sum_{i=1}^m (x_i - \bar{x})^2 \\ &= 2m\bar{x}\hat{\mu} - m\hat{\mu}^2 - m\bar{x}^2 \end{aligned}$$

Similarly

$$\begin{aligned} & -\sum_{j=1}^n (y_j - \hat{\mu})^2 + \sum_{j=1}^n (y_j - \bar{y})^2 \\ & = 2n\bar{y}\hat{\mu} - n\hat{\mu}^2 - n\bar{x}^2 \end{aligned}$$

Then

$$\begin{aligned} & -\sum_{i=1}^m (x_i - \hat{\mu})^2 + \sum_{i=1}^m (x_i - \bar{x})^2 - \sum_{j=1}^n (y_j - \hat{\mu})^2 + \sum_{j=1}^n (y_j - \bar{y})^2 \\ & = 2m\bar{x}\hat{\mu} - m\hat{\mu}^2 - m\bar{x}^2 + 2n\bar{y}\hat{\mu} - n\hat{\mu}^2 - n\bar{x}^2 \\ & = (2m\bar{x} + 2n\bar{y})\frac{m\bar{x} + n\bar{y}}{m+n} - (m+n)\left(\frac{m\bar{x} + n\bar{y}}{m+n}\right)^2 - (m\bar{x}^2 + n\bar{y}^2)\left(\frac{m+n}{m+n}\right) \\ & = \frac{mn}{m+n}(2\bar{x}\bar{y} - \bar{x}^2 - \bar{y}^2) = -\frac{mn}{m+n}(\bar{x} - \bar{y})^2 \end{aligned}$$

Hence

$$\begin{aligned} & \frac{max_{\mu=\mu_1=\mu_2} L(\mu)}{max_{\mu_1, \mu_2} L(\mu_1, \mu_2)} \\ & = \exp \left\{ -\frac{1}{2} \sum_{i=1}^m (x_i - \hat{\mu}_{mle})^2 - \frac{1}{2} \sum_{j=1}^n (y_j - \hat{\mu}_{mle})^2 + \frac{1}{2} \sum_{i=1}^m (x_i - \bar{x})^2 + \frac{1}{2} \sum_{j=1}^n (y_j - \bar{y})^2 \right\} \\ & = \exp \left\{ -\frac{1}{2} \frac{mn}{(m+n)} (\bar{x} - \bar{y})^2 \right\} \end{aligned}$$

e) Under $H_0 : \mu_1 = \mu_2$, the distribution of $\bar{X} - \bar{Y}$ and test statistic is given as:

$$\begin{aligned} \bar{X} - \bar{Y} & \sim N\left(0, \frac{1}{m} + \frac{1}{n}\right) \\ \Rightarrow T & = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{1}{m} + \frac{1}{n}}} \\ \Rightarrow T^2 & = \frac{(\bar{X} - \bar{Y})^2}{\frac{m+n}{mn}} \\ & = \frac{mn}{m+n} (\bar{X} - \bar{Y})^2 \end{aligned}$$

Set $\lambda \leq k$

$$\begin{aligned} \exp \left\{ -\frac{1}{2} \frac{mn}{(m+n)} (\bar{x} - \bar{y})^2 \right\} & \leq k \\ \Rightarrow -\frac{t^2}{2} & \leq \ln k \\ \Rightarrow t^2 & \geq 2 \ln k \\ \Rightarrow |t| & \geq \sqrt{2 \ln k} = c \end{aligned}$$

Hence the critical region is of the form

$$C = \left\{ |t| = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{1}{m} + \frac{1}{n}}} \geq c \right\}$$

To determine c, let

$$\alpha = P(|T| \geq c; \mu_1 = \mu_2) = P(|Z| \geq c)$$

$$\Rightarrow P\left(\left|\frac{\bar{x} - \bar{y}}{\sqrt{\frac{1}{m} + \frac{1}{n}}}\right| \geq z_{\alpha/2}\right) = \alpha$$

$$\Rightarrow C = \left\{ \sqrt{\frac{mn}{m+n}} |\bar{x} - \bar{y}| \geq z_{\alpha/2} \right\}$$

□