ST2132 HW2
Ethan Keck Jun Wei (February 14, 2024)

Problem 1.

Answer: a) The confidence interval is given as:

$$\text{C.I.} = \hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$
$$\text{C.I.} = \frac{159}{314} \pm 1.645\sqrt{\frac{\frac{159}{314}(1-\frac{159}{314})}{314}}$$
$$= 0.5063 \pm 0.0464 \Rightarrow [0.460, 0.552)]$$

b) The sample size can found as:

$$n = \frac{z_{0.05}^2\hat{p}(1-\hat{p})}{\epsilon^2}$$
$$= \frac{1.645^2(0.5064)(1-0.5064)}{(0.02/2)^2}$$
$$= 6763.9 \approx 6764$$

c) The confidence interval is given as: $[0.390, 0.500]$ and $\hat{p} = (0.390 + 0.500)/2 = 0.445$

$$\epsilon = z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$
$$\Rightarrow z_{\alpha/2} = \frac{\epsilon}{\sqrt{\hat{p}(1-\hat{p})/n}}$$
$$= \frac{(0.500-0.390)/2}{\sqrt{(0.445)(1-0.445)/314}}$$
$$= 1.9611 \approx 1.96$$

Which corresponds to the friend using a 95% confidence level. □

Problem 2.

Answer: a) The two-sided confidence interval is given as

$$(\bar{X} - \bar{Y}) \pm z_{\alpha/2}\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}$$

b) The width of the interval is given as:

$$\epsilon = z_{\alpha/2}\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{6000-n}}$$

We differentiate ϵ with respect to n and equate it to 0:

$$\frac{d\epsilon}{dn} = 0$$
$$\Rightarrow z_{\alpha/2}\left(\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{6000-n}\right)^{-1/2}\left(-\frac{\sigma_x^2}{n^2} + \frac{\sigma_y^2}{(6000-n)^2}\right) = 0$$
$$\Rightarrow -\frac{\sigma_x^2}{n^2} + \frac{\sigma_y^2}{(6000-n)^2} = 0 \Rightarrow \frac{\sigma_x^2}{n^2} = \frac{\sigma_y^2}{(6000-n)^2}$$
$$\Rightarrow n = \frac{6000\sigma_x}{\sigma_x + \sigma_y} = \frac{6000 \cdot 70}{70 + 50} = 3500$$

□

Problem 3.

Answer: a) The 90% confidence interval is given as:

$$\bar{x} \pm z_{0.05}\frac{\sigma}{\sqrt{n}} = 41.83 \pm 1.645\frac{11}{\sqrt{12}}$$
$$\Rightarrow C.I. = [36.60, 47.05]$$

b) The 95% confidence interval is given as:

$$\bar{x} \pm z_{0.025}\frac{\sigma}{\sqrt{n}} = 41.83 \pm 1.960\frac{11}{\sqrt{12}}$$
$$\Rightarrow C.I. = [35.60, 48.05]$$

The 99% confidence interval is given as:

$$\bar{x} \pm z_{0.005}\frac{\sigma}{\sqrt{n}} = 41.83 \pm 2.576\frac{11}{\sqrt{12}}$$
$$\Rightarrow C.I. = [33.65, 50.01]$$

c) The 95% confidence interval, for unknown σ^2 is given as:

$$\bar{x} \pm t_{0.05}(n-1)\frac{s}{\sqrt{n}}$$
$$= 41.83 \pm 1.796\frac{11.8}{\sqrt{12}}$$
$$\Rightarrow C.I. = [35.71, 47.94]$$

□

Problem 4.

Answer: a) The distribution is given as:

$$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2(n)$$
$$\Rightarrow P\left(\chi^2_{1-\alpha/2}(n) < \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 < \chi^2_{\alpha/2}(n)\right) = 1-\alpha$$
$$\Rightarrow P\left(\frac{\chi^2_{1-\alpha/2}(n)}{\sum_{i=1}^{n}(X_i-\mu)^2} < \frac{1}{\sigma^2} < \frac{\chi^2_{\alpha/2}(n)}{\sum_{i=1}^{n}(X_i-\mu)^2}\right)$$
$$\Rightarrow P\left(\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\chi^2_{\alpha/2}(n)} < \sigma^2 < \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\chi^2_{1-\alpha/2}(n)}\right) = 1-\alpha$$

Hence the confidence interval for σ^2 is $\left[\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\chi^2_{\alpha/2}(n)}, \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\chi^2_{1-\alpha/2}(n)}\right]$.

b) The distribution is given as,

$$\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$
$$\Rightarrow P\left(\chi^2_{1-\alpha/2}(n-1) < \sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2 < \chi^2_{\alpha/2}(n-1)\right) = 1-\alpha$$
$$\Rightarrow P\left(\frac{\chi^2_{1-\alpha/2}(n-1)}{\sum_{i=1}^{n}(X_i-\bar{X})^2} < \frac{1}{\sigma^2} < \frac{\chi^2_{\alpha/2}(n-1)}{\sum_{i=1}^{n}(X_i-\bar{X})^2}\right)$$
$$\Rightarrow P\left(\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\chi^2_{\alpha/2}(n-1)} < \sigma^2 < \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\chi^2_{1-\alpha/2}(n-1)}\right) = 1-\alpha$$

Hence the confidence interval for σ^2 is $\left[\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\chi^2_{\alpha/2}(n-1)}, \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\chi^2_{1-\alpha/2}(n-1)}\right]$.

c) Similarly, the confidence interval for σ can be found by taking the square root of the bounds from (b): $\left[\sqrt{\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\chi^2_{\alpha/2}(n-1)}}, \sqrt{\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\chi^2_{1-\alpha/2}(n-1)}}\right]$. $\square$

Problem 5. *Refer to R code*