ST2132 HW6
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Problem 1.

Answer: a)

$$\begin{aligned}\alpha(c) &= P(T \in C; H_0) = P(X > c; \mu = 0) \\ &= P\left(\frac{X-0}{1} > \frac{c-0}{1}\right) \\ &= P(Z > c) = 1 - \Phi(c)\end{aligned}$$

b)

$$\begin{aligned}\beta(c) &= P(T \notin C; \mu = 1) \\ &= P\left(\frac{X-1}{1} \leq \frac{c-1}{1}\right) \\ &= P\left(Z \leq c-1\right) = \Phi(c-1)\end{aligned}$$

c) Given

$$\begin{aligned}R(c) &= 1 - \Phi(c) + \Phi(c-1) \\ R'(c) &= -\Phi'(c) + \Phi'(c-1) \\ &= -\frac{\exp\{-c^2/2\}}{\sqrt{2\pi}} + \frac{\exp\{-(c-1)^2/2\}}{\sqrt{2\pi}}\end{aligned}$$

Let $R'(c) = 0$,

$$\begin{aligned}-\frac{\exp\{-c^2/2\}}{\sqrt{2\pi}} + \frac{\exp\{-(c-1)^2/2\}}{\sqrt{2\pi}} &= 0 \\ \Rightarrow \exp\{-c^2/2\} &= \exp\{-(c-1)^2/2\} \\ \Rightarrow c^2 &= (c-1)^2 \\ \Rightarrow c^2 = c^2 - 2c + 1 \Rightarrow 2c &= 1, c = \frac{1}{2}\end{aligned}$$

Then

$$\begin{aligned}R''(c) &= -\frac{c\exp\{-c^2/2\}}{\sqrt{2\pi}} + \frac{(1-c)\exp\{-(c-1)^2/2\}}{\sqrt{2\pi}} \\ R''(c = 1/2) &= 0.352065 > 0 \text{ (local min) and,} \\ R(1/2) &= 1 - \Phi(1/2) + \Phi(-1/2) \\ &= 2\Phi(-1/2)\end{aligned}$$

Hence, $c^* = \frac{1}{2}, R(c^*) = 2\Phi(-1/2)$ □

Problem 2.

Answer: a) Given the likelihood ratio:

$$\frac{L(1)}{L(2)} \leq k$$
$$\Rightarrow \frac{(2\pi)^{-n/2}\exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i^2)\right]}{(2\pi\cdot 2)^{-n/2}\exp\left[-\frac{1}{2\cdot 2}\sum_{i=1}^{n}(x_i^2)\right]}$$
$$= 2^{n/2}\exp\left[-\frac{1}{4}\sum_{i=1}^{n}(x_i^2)\right]$$
$$\Rightarrow \ln 2^{n/2} - \frac{1}{4}\sum_{i=1}^{n}(x_i^2) \leq \ln k$$
$$\Rightarrow \frac{1}{4}\sum_{i=1}^{n}(x_i^2) \geq \frac{1}{2}\ln 2^n - \frac{1}{2}\ln k$$
$$\Rightarrow \sum_{i=1}^{n} x_i^2 \geq 2\ln\frac{2^{10}}{k^2} = c$$

By Neyman-Pearson Lemma, the best critical region is of the form:

$$C = \left\{(x_1,\ldots x_{10}); \sum_{i=1}^{n} x_i^2 \geq c\right\}$$

where c is determined as

$$0.05 = P\left(\sum_{i=1}^{n} x_i^2 \geq c; \sigma^2 = 1\right)$$
$$\Rightarrow 0.05 = P\left(\chi^2(10) \geq c; \sigma^2 = 1\right)$$
$$= 1 - P\left(\chi^2(10) \leq c\right)$$
$$\Rightarrow P\left(\chi^2(10) \leq c\right) = 0.95$$
$$\Rightarrow c = \chi^2_{0.05}(10) = 18.31$$

Hence for n = 10, the test with critical region $\{\sum_{i=1}^{n} x_i^2 \geq 18.31\}$ is the most powerful test of size $\alpha = 0.05$.

b) From R, $K(2) = 0.5174595$

c) Given the log-likelihood ratio:

$$\frac{L(1)}{L(4)} \leq k$$
$$\Rightarrow \frac{(2\pi)^{-n/2}\exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i^2)\right]}{(2\pi\cdot 4)^{-n/2}\exp\left[-\frac{1}{2\cdot 4}\sum_{i=1}^{n}(x_i^2)\right]}$$
$$= 2^n \exp\left[-\frac{3}{8}\sum_{i=1}^{n}(x_i^2)\right]$$
$$\Rightarrow \ln 2^n - \frac{3}{8}\sum_{i=1}^{n}x_i^2 \leq \ln k$$
$$\Rightarrow \sum_{i=1}^{n}x_i^2 \geq \frac{8}{3}\ln\frac{2^{10}}{k} = c$$

By Neyman-Pearson Lemma, the best critical region is of the form:

$$C = \left\{(x_1, \ldots x_{10}); \sum_{i=1}^{n}x_i^2 \geq c\right\}$$

where c is determined as

$$\Rightarrow c = \chi^2_{0.05}(10) = 18.31$$

Hence for n = 10, the test with critical region $\{\sum_{i=1}^{n} x_i^2 \geq 18.31\}$ is the most powerful test of size $\alpha = 0.05$.

d) From R, $K(4) = 0.917559$

e) Given the likelihood ratio:

$$\frac{L(1)}{L(\sigma_1^2)} \leq k$$
$$\Rightarrow \frac{(2\pi)^{-n/2}\exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i^2)\right]}{(2\pi\cdot\sigma_1^2)^{-n/2}\exp\left[-\frac{1}{2\cdot\sigma_1^2}\sum_{i=1}^{n}(x_i^2)\right]}$$
$$= \sigma_1^n \exp\left[-\frac{\sigma_1^2-1}{2\sigma_1^2}\sum_{i=1}^{n}(x_i^2)\right]$$
$$\Rightarrow \ln\sigma_1^n - \frac{\sigma_1^2-1}{2\sigma_1^2}\sum_{i=1}^{n}(x_i^2) \leq \ln k$$
$$\Rightarrow \sum_{i=1}^{n}x_i^2 \geq \frac{2\sigma_1^2}{\sigma_1^2-1}\ln\frac{\sigma_1^{10}}{k} = c$$

By Neyman-Pearson Lemma, the best critical region is of the form:

$$C = \left\{(x_1, \ldots x_{10}); \sum_{i=1}^{n}x_i^2 \geq c\right\}$$

where c is determined as

$$\Rightarrow c = \chi^2_{0.05}(10) = 18.31$$

Hence for n = 10, the test with critical region $\{\sum_{i=1}^{n} x_i^2 \geq 18.31\}$ is the most powerful test of size $\alpha = 0.05$.
f) Since the critical region C defines a test that is most powerful against each simple alternative $\sigma^2 > 1$, the test in (e) is a uniformly most powerful test, and C is a uniformly most powerful critical region of size $\alpha = 0.05$. Again, if $\alpha = 0.05$, then $c = \chi^2_{0.05}(10) = 18.31$. $\square$

Problem 3.

Answer: a)

$$\bar{X} \sim N\left(\mu_1, \frac{400}{n}\right), \bar{Y} \sim N\left(\mu_2, \frac{225}{n}\right)$$
$$\Rightarrow \bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \frac{400}{n} + \frac{225}{n} = \frac{625}{n}\right)$$
$$\Rightarrow \bar{X} - \bar{Y} \sim N\left(\theta, \frac{625}{n}\right)$$

The power function is given as:

$$K(\theta) = P(\bar{x} - \bar{y} \in C; \theta)$$
$$= P\left(\frac{\bar{x} - \bar{y} - \theta}{\sqrt{625/n}} \geq \frac{c - \theta}{\sqrt{625/n}}\right)$$
$$= 1 - \Phi\left(\frac{c - \theta}{25\sqrt{n}}\right)$$

b) Given $\alpha = 0.05, \theta = 10$, we have

$$0.05 = \alpha(H_0 : \theta = 0) = P(\bar{x} - \bar{y} \in C; \theta = 0)$$
$$\Rightarrow 0.05 = 1 - \Phi\left(\frac{c}{25\sqrt{n}}\right)$$
$$\Rightarrow \left(\frac{c}{25\sqrt{n}}\right) = 1.645 \text{ and,}$$

$$0.9 = K(10) = P(\bar{x} - \bar{y} \in C; \theta = 10)$$
$$\Rightarrow 0.1 = \Phi\left(\frac{c - 10}{25\sqrt{n}}\right)$$
$$\Rightarrow \left(\frac{c - 10}{25\sqrt{n}}\right) = 1.28$$

Solving for n and c: $c = 45.06849, n = 1.20097 \approx 1$ $\square$

Problem 4.

Answer: The likelihood function for X and Y, respectively, are:

$$
\begin{aligned}
L_X(\mu_1) &= \frac{1}{(\sqrt{2\pi})^{m/2}} \exp\left\{-\frac{\sum_{i=1}^m (x_i-\mu_1)^2}{2}\right\}, \\
L_Y(\mu_2) &= \frac{1}{(\sqrt{2\pi})^{n/2}} \exp\left\{-\frac{\sum_{j=1}^n (y_j-\mu_2)^2}{2}\right\}
\end{aligned}
$$

Then

$$
\begin{aligned}
L(\mu_1,\mu_2) &= L_X(\mu_1)L_Y(\mu_2) \\
&= \frac{1}{(\sqrt{2\pi})^{m/2}} \exp\left\{-\frac{\sum_{i=1}^m (x_i-\mu_1)^2}{2}\right\} \cdot \frac{1}{(\sqrt{2\pi})^{n/2}} \exp\left\{-\frac{\sum_{j=1}^n (y_j-\mu_2)^2}{2}\right\} \\
&= \frac{1}{(\sqrt{2\pi})^{(m+n)/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^m (x_i-\mu_2)^2\right\} \exp\left\{-\frac{1}{2}\sum_{j=1}^n (y_j-\mu_2)^2\right\}
\end{aligned}
$$

b) We find the gradient of of the log-likelihood function $l(\mu_1,\mu_2)$ and set it to $\mathbf{0}$

$$
\begin{aligned}
&\begin{pmatrix}\partial l(\mu_1,\mu_2)/\partial\mu_1 & \partial l(\mu_1,\mu_2)/\partial\mu_2\end{pmatrix}^T \\
&= \begin{pmatrix}
\partial\left[\ln \frac{\exp\left\{-\frac{1}{2}\sum_{j=1}^n (y_j-\mu_2)^2\right\}}{(\sqrt{2\pi})^{(m+n)/2}} - \frac{1}{2}\sum_{i=1}^m (x_i-\mu_1)^2\right]/\partial\mu_1 \\
\partial\left[\ln \frac{\exp\left\{-\frac{1}{2}\sum_{i=1}^m (x_i-\mu_1)^2\right\}}{(\sqrt{2\pi})^{(m+n)/2}} - \frac{1}{2}\sum_{j=1}^n (y_j-\mu_2)^2\right]/\partial\mu_2
\end{pmatrix} \\
&\Rightarrow \begin{pmatrix}\sum_{i=1}^m (x_i-\mu_1) \\ \sum_{j=1}^n (y_j-\mu_2)\end{pmatrix} = \mathbf{0} \\
&\Rightarrow \begin{pmatrix}\sum_{i=1}^m x_i = m\mu_1 \\ \sum_{j=1}^n y_j = n\mu_2\end{pmatrix} \\
&\Rightarrow \begin{pmatrix}\hat{\mu}_1 = \frac{1}{m}\sum_{i=1}^m x_i \\ \hat{\mu}_2 = \frac{1}{n}\sum_{j=1}^n y_j\end{pmatrix}
\end{aligned}
$$

The Hessian of the $l(\mu_1,\mu_2)$ can be found as:

$$
\begin{pmatrix}
\partial^2 l(\mu_1,\mu_2)/\partial\mu_1^2 & \partial^2 l(\mu_1,\mu_2)/\partial\mu_1\partial\mu_2 \\
\partial^2 l(\mu_1,\mu_2)/\partial\mu_2\partial\mu_1 & \partial^2 l(\mu_1,\mu_2)/\partial\mu_2^2
\end{pmatrix} \Rightarrow \begin{pmatrix} m & 0 \\ 0 & n\end{pmatrix}
$$

The Hessian and thus of $l(\mu_1,\mu_2)$ has positive eigenvalues $\lambda_1 = m > 0$ and $\lambda_2 = n > 0$ and is thus positive definite. Hence $\begin{pmatrix}\hat{\mu}_1 = \bar{X} & \hat{\mu}_2 = \bar{Y}\end{pmatrix}^T$ is a local minimiser of the log-likelihood function, and thus $\hat{\mu}_{1,mle} = \frac{1}{m}\sum_{i=1}^m X_i = \bar{X}$, $\hat{\mu}_{2,mle} = \frac{1}{n}\sum_{j=1}^n Y_i = \bar{Y}$.

c) Since under $H_0 : \mu_1 = \mu_2$, we find

$$\frac{dl(\mu)}{d\mu} = \frac{d\left[\ln \frac{1}{(\sqrt{2\pi})^{(m+n)/2}} - \frac{1}{2}\sum_{i=1}^{m}(x_i-\mu)^2 - \frac{1}{2}\sum_{j=1}^{n}(y_j-\mu)^2\right]}{d\mu}$$

$$\Rightarrow \sum_{i=1}^{m}(x_i-\mu) + \sum_{j=1}^{n}(y_j-\mu) = 0$$

$$\Rightarrow m\bar{x} - m\mu + n\bar{y} - n\mu = 0$$

$$\Rightarrow \mu(m+n) = m\bar{x} + n\bar{y}$$

$$\Rightarrow \mu = \frac{m\bar{x}+n\bar{y}}{m+n}$$

Hence $\hat{\mu}_{mle} = \frac{m\bar{X}+n\bar{Y}}{m+n}$

d) We find

$$max_{\mu=\mu_1=\mu_2} L(\mu) = \frac{1}{(\sqrt{2\pi})^{(m+n)/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{m}(x_i-\hat{\mu}_{mle})^2\right\} \exp\left\{-\frac{1}{2}\sum_{j=1}^{n}(y_j-\hat{\mu}_{mle})^2\right\}$$

$$max_{\mu_1,\mu_2} L(\mu_1,\mu_2) = \frac{1}{(\sqrt{2\pi})^{(m+n)/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{m}(x_i-\bar{x})^2\right\} \exp\left\{-\frac{1}{2}\sum_{j=1}^{n}(y_j-\bar{y})^2\right\}$$

$$\Rightarrow \frac{max_{\mu=\mu_1=\mu_2} L(\mu)}{max_{\mu_1,\mu_2} L(\mu_1,\mu_2)} = \frac{\exp\left\{-\frac{1}{2}\sum_{i=1}^{m}(x_i-\hat{\mu}_{mle})^2 - \frac{1}{2}\sum_{j=1}^{n}(y_j-\hat{\mu}_{mle})^2\right\}}{\exp\left\{-\frac{1}{2}\sum_{i=1}^{m}(x_i-\bar{x})^2 - \frac{1}{2}\sum_{j=1}^{n}(y_j-\bar{y})^2\right\}}$$

$$= \exp\left\{-\frac{1}{2}\sum_{i=1}^{m}(x_i-\hat{\mu}_{mle})^2 - \frac{1}{2}\sum_{j=1}^{n}(y_j-\hat{\mu}_{mle})^2 + \frac{1}{2}\sum_{i=1}^{m}(x_i-\bar{x})^2 + \frac{1}{2}\sum_{j=1}^{n}(y_j-\bar{y})^2\right\}$$

Since

$$\sum_{i=1}^{m}(x_i-\hat{\mu})^2 = \sum_{i=1}^{m}(x_i^2 - 2x_i\hat{\mu} + \hat{\mu}^2)$$

$$= \sum_{i=1}^{m}(x_i^2) - 2m\bar{x}\hat{\mu} + m\hat{\mu}^2,$$

$$\sum_{i=1}^{m}(x_i-\bar{x})^2 = \sum_{i=1}^{m}(x_i^2 - 2x_i\bar{x} + \bar{x}^2)$$

$$= \sum_{i=1}^{m}(x_i^2) - 2m\bar{x}^2 + m\bar{x}^2$$

$$\Rightarrow -\sum_{i=1}^{m}(x_i-\hat{\mu})^2 + \sum_{i=1}^{m}(x_i-\bar{x})^2$$

$$= 2m\bar{x}\hat{\mu} - m\hat{\mu}^2 - m\bar{x}^2$$

Similarly

$$
\begin{aligned}
&-\sum_{j=1}^{n}(y_j-\hat{\mu})^2+\sum_{j=1}^{n}(y_j-\bar{y})^2 \\
&=2n\bar{y}\hat{\mu}-n\hat{\mu}^2-n\bar{x}^2
\end{aligned}
$$

Then

$$
\begin{aligned}
&-\sum_{i=1}^{m}(x_i-\hat{\mu})^2+\sum_{i=1}^{m}(x_i-\bar{x})^2-\sum_{j=1}^{n}(y_j-\hat{\mu})^2+\sum_{j=1}^{n}(y_j-\bar{y})^2 \\
&=2m\bar{x}\hat{\mu}-m\hat{\mu}^2-m\bar{x}^2+2n\bar{y}\hat{\mu}-n\hat{\mu}^2-n\bar{x}^2 \\
&=(2m\bar{x}+2n\bar{y})\frac{m\bar{x}+n\bar{y}}{m+n}-(m+n)\left(\frac{m\bar{x}+n\bar{y}}{m+n}\right)^2-(m\bar{x}^2+n\bar{y}^2)\left(\frac{m+n}{m+n}\right) \\
&=\frac{mn}{m+n}(2\bar{x}\bar{y}-x^2-y^2=-\frac{mn}{m+n}(\bar{x}-\bar{y})^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{max_{\mu=\mu_1=\mu_2}L(\mu)}{max_{\mu_1,\mu_2}L(\mu_1,\mu_2)} \\
&=\exp\left\{-\frac{1}{2}\sum_{i=1}^{m}(x_i-\hat{\mu}_{mle})^2-\frac{1}{2}\sum_{j=1}^{n}(y_j-\hat{\mu}_{mle})^2+\frac{1}{2}\sum_{i=1}^{m}(x_i-\bar{x})^2+\frac{1}{2}\sum_{j=1}^{n}(y_j-\bar{y})^2\right\} \\
&=\exp\left\{-\frac{1}{2}\frac{mn}{(m+n)}(\bar{x}-\bar{y})^2\right\}
\end{aligned}
$$

e) Under $H_0:\mu_1=\mu_2$, the distribution of $\bar{X}-\bar{Y}$ and test statistic is given as:

$$
\begin{aligned}
&\bar{X}-\bar{Y}\sim N\left(0,\frac{1}{m}+\frac{1}{n}\right) \\
&\Rightarrow T=\frac{\bar{X}-\bar{Y}}{\sqrt{\frac{1}{m}+\frac{1}{n}}} \\
&\Rightarrow T^2=\frac{(\bar{X}-\bar{Y})^2}{\frac{m+n}{mn}} \\
&=\frac{mn}{m+n}(\bar{X}-\bar{Y})^2
\end{aligned}
$$

Set $\lambda\leq k$

$$
\begin{aligned}
&\exp\left\{-\frac{1}{2}\frac{mn}{(m+n)}(\bar{x}-\bar{y})^2\right\}\leq k \\
&\Rightarrow-\frac{t^2}{2}\leq\ln k \\
&\Rightarrow t^2\geq 2\ln k \\
&\Rightarrow|t|\geq\sqrt{2\ln k}=c
\end{aligned}
$$

Hence the critical region is of the form

$$C = \left\{ |t| = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{1}{m} + \frac{1}{n}}} \geq c \right\}$$

To determine c, let

$$\begin{aligned} \alpha &= P(|T| \geq c; \mu_1 = \mu_2) = P(|Z| \geq c) \\ &\Rightarrow P\left(\left| \frac{\bar{x} - \bar{y}}{\sqrt{\frac{1}{m} + \frac{1}{n}}} \right| \geq z_{\alpha/2} \right) = \alpha \\ &\Rightarrow C = \left\{ \sqrt{\frac{mn}{m+n}}\, |\bar{x} - \bar{y}| \geq z_{\alpha/2} \right\} \end{aligned}$$

$\square$