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MA202 Project: Numerical Methods

Solving Lorenz Equations using Numerical Methods

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ABSTRACT

We have selected the Lorenz Equations to solve using the suitable numerical methods. We have analysed the Lorenz Equations using the Runge-Kutta Method, Runge-Kutta Gill Method and Euler Method and compared the solutions of these methods. We have checked the standard properties of lorenz system for different parameters of the Lorenz equations.

Introduction

Lorenz system is system of ordinary differential equations, having three set of variables, which was first studied by the Edward Norton Lorenz. We want to analyze the properties of Lorenz system of equations, which includes of Chaos, and the sensitive dependence of the initial conditions in determining the value of the function, there could be very significant change in the output values for same input values if we change the initial values of the functions.

The Lorenz Equations

The *Lorenz equations* say that

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y - x)\\ \frac{dy}{dt} &= x(\rho - z) - y\\ \frac{dz}{dt} &= xy - \beta z,\end{aligned}$$

The above equations describe the rate of change of three variables, which were: x is proportional to the rate of convection, y is proportional to horizontal temperature variation, and z is proportional to vertical temperature.

These equations can help to know the atmospheric conditions like temperature, humidity etc.., but we cannot predict these atmospheric properties for a long time, we can only predict these properties only upto a limited period, because these equations are chaotic in nature.

Even to calculate the values of these quantities for a little period of time, we need to be more accurate otherwise we cannot get the correct output values for our desired time period. We can even see that in weather apps in which they would estimate upto a week, after that we can predict. To predict the weather after this period we have to maintain very high level accuracy, with infinitely small true error.

Runge-Kutta Method

The Classical Runge-Kutta Method is a fourth order method, used to solve the IVP(initial value problem),

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t), y(t_0) = y_0 \tag{1}$$

Here, t is time, $Y = [x, y, z]$ where x,y,z are the corresponding variables for the Lorenz equations

$$y_{n+1} = y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$
$$t_{n+1} = t_n + h$$
$$k_1 = f(t_n, y_n)$$

for n = 1,2,3.., using equation

$$k_2 = f\Big(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}\Big),$$
$$k_3 = f\Big(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}\Big),$$
$$k_4 = f(t_n + h, y_n + hk_3)$$

Euler Method

Euler's Method uses iterations to come up with better estimates of y using an initial gas. It extrapolates linearly over an arbitrary step size h using the slope i.e. first derivative of y.

$$\frac{dy}{dx} = f(x,y)y_{i+1} = y_i + f(x_i, y_i)h$$

Runge-Kutta Gill Method

The Runge-Kutta-Gill method is a variant of the classical Runge-Kutta method being used for approximating the solution of the differential equation given by, $Y'(t) = f(t, Y)$

Here t is time and $Y = [x, y, z]$ where x, y and z are corresponding variables in Lorenz's equation, with the initial condition: Y = [10, 10, 10] when t = 0, and numerically evaluates the value of $f(t, Y)$ at four intermediate time-steps of t as given below.

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + (2 - (\sqrt{2})k_2) + ((2 + \sqrt{2})k_3 + k_4] + O(h^5)$$

where

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1)$$

$$k_3 = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}(-1 + \sqrt{2})k_1 + (1 - \frac{1}{2}\sqrt{2})k_2]$$

$$k_4 = hf[x_n + h, y_n - \frac{1}{2}\sqrt{2}k_2 + (1 + \frac{1}{2}\sqrt{2}k_3]$$

Here, y_{n+1} is the new coordinate, h is the small interval between two time discrete values, k1 is the slope at the initial point, k2 is the slope at a half time step, k3 is also determined at a half time step, but using the k2 slope, and k4 is the slope at a full time step using the value of k3.

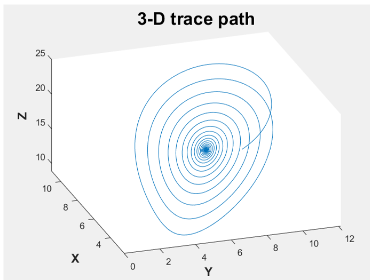
Codes

Matlab codes (Numerical solutions for Lorenz System) for the Runge-Kutta Method, Euler Method, Range-Kutta Gill Method. Click Here.

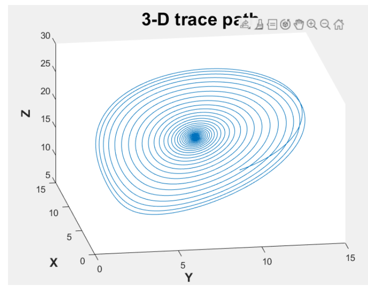
Observations

From the below graphs we can say,
For whatever initial value we plot the Graph, the inherent shape or characteristic is retained for the given values of $\sigma, \rho, and \beta$. But the paths traced by the Graphs are completely different.

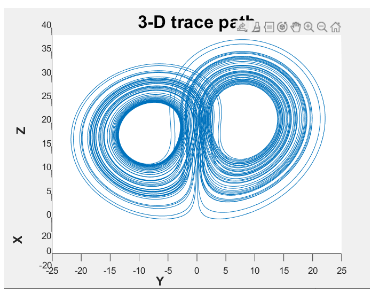
For a small change in the initial value from [10,10,10] to [10.01 , 10.01,10.1]. the maximum absolute error obtained is 34. Which is a significant error to consider. So we can say that for Solutions are sensible to the initial values.



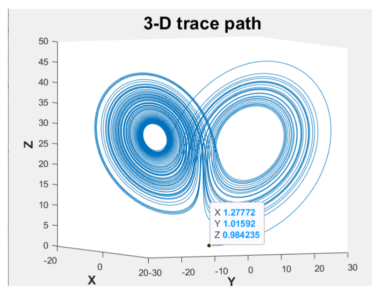
(a) $\sigma = 10; \rho = 17; \beta = 8/3;$



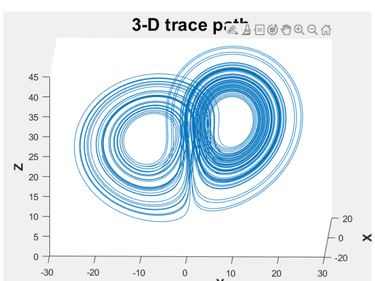
(b) $\sigma = 10; \rho = 20; \beta = 8/3;$



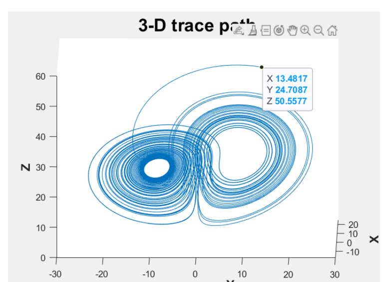
(c) $\sigma = 10; \rho = 25; \beta = 8/3$



(d) $\sigma = 10; \rho = 28; \beta = 8/3;$

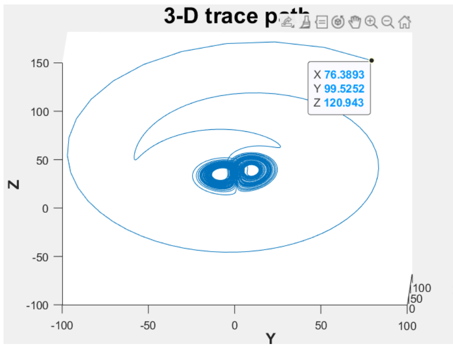


(e) $\sigma = 10; \rho = 28; \beta = 8/3; X_0 = 1; Y_0 = 1; Z_0 = 1;$

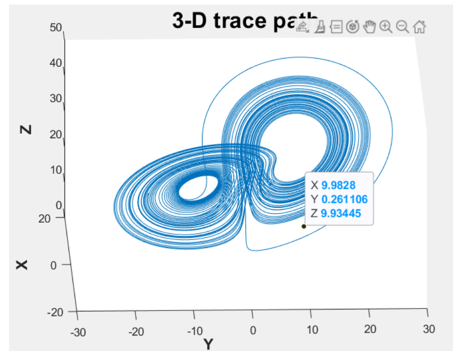


(f) $\sigma = 10; \rho = 28; \beta = 8/3; X_0 = 5; Y_0 = 5; Z_0 = 15;$

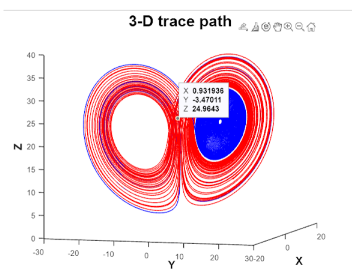
Figure 1: Graphs obtained from different initial values and parameter values



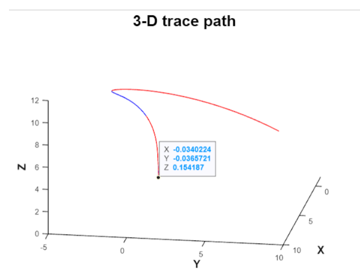
(a) $\sigma = 10; \rho = 28; \beta = 8/3; X_0 = 15; Y_0 = 25; Z_0 = 50;$



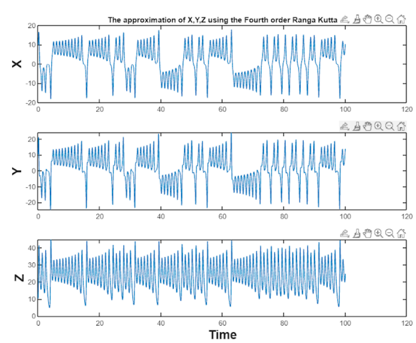
(b) Sensitivity to Initial conditions



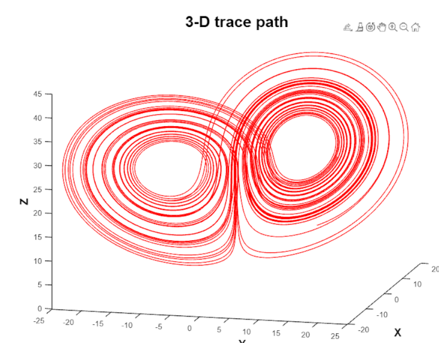
(c) Sensitivity to Initial conditions



(d) If we take the Pho value ¡ 1 then the solution will converge to zero.

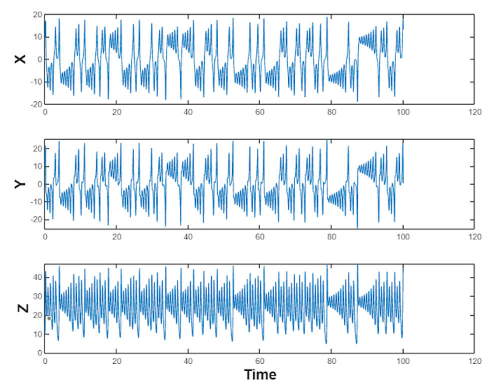


(e) Spectrum using Fourth order Ranga kutta Method

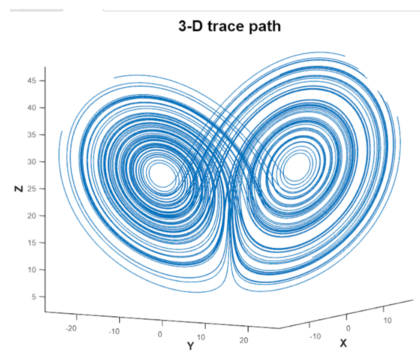


(f) Plotting the function using Fourth order Runge kutta Method

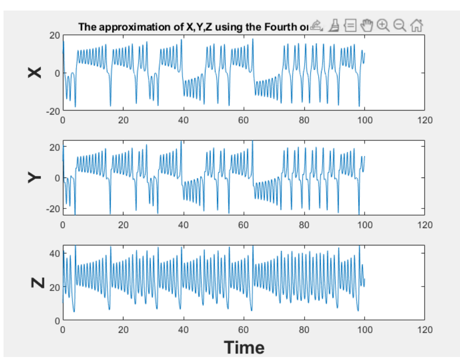
Figure 2: Graphs obtained from different initial values and parameters values



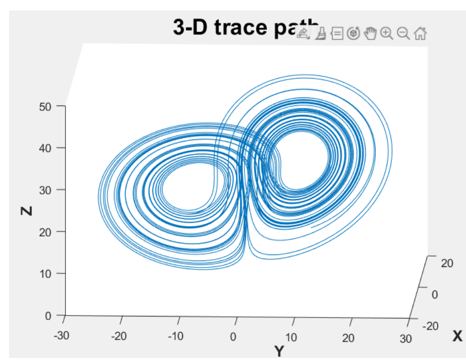
(a) Spectrum using Euler Method



(b) Plotting the function using Euler Method



(c) Spectrum using Runge-Kutta Gill Method



(d) Plotting the function using Runge-Kutta Gill Method

Figure 3: Graphs obtained from different initial values and parameter values

Conclusion

Lorenz equations hide within themselves a butterfly like structure, which reveals itself when the Rho is high enough for the graph to not collapse to the two critical points.

For small changes in the initial values, the resultant graphs trace out completely different paths eventually.(Though it will still retain the original characteristics of the graph for the given constraints of Rho, Sigma, Beta. For the same values of Sigma, Beta and Rho, even after changing the initial values the path of these points will be along the Lorenz attractor.

Thus, we have found the chaotic properties of the Lorenz equations to be true.