IN1011 Lecture 1 Exercises

Definition: Let $a_n, a_{n-1}, \ldots, a_1, a_0$ be non-negative integers and let x be an integer such that $x \geqslant 2$ and $x > \max\{a_n, a_{n-1}, \ldots, a_1, a_0\}$. We say that the number $a_n a_{n-1} \ldots a_1 a_0$ is written in base x if this number is "shorthand" for $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 x^0$. For example, 1101 is a number written in base 3 if this is shorthand for $1 \cdot 3^3 + 1 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0$

1. How is $x + 1$ written in base x?
2. How is x written in base $x + 1$?
3. How is $2x$ written in base x?
4. How is x written in base $2x$?
5. How is x^3 written in base x?
6. How is x written in base x^3?
7. In what base is $\frac{1}{3}$ written as 0.1?
8. Why is "base 1" problematic?

Solutions: expand as a polynomial in powers of the base.

1. It is 11, since $x + 1 = 1 \cdot x^1 + 1 \cdot x^0$;
2. It is x, since $x = x \cdot (x + 1)^0$. We could have also immediately deduced this from the definition, since $x < x + 1$ so x can be a single digit in base $x + 1$;
3. It is 20, since $2x = 2 \cdot x^1 + 0 \cdot x^0$;
4. It is x, since either $x = x \cdot (2x)^0$ or using $x < 2x$;
5. It is 1000, since $x^3 = 1 \cdot x^3 + 0 \cdot x^2 + 0 \cdot x^1 + 0 \cdot x^0$;
6. It is x, since either $x = x \cdot (x^3)^0$ or using $x < x^3$;
7. We guess that decimals in a given base must be expansions in inverse powers of the base (e.g. 0.231 in base x is $\frac{2}{x} + \frac{3}{x^2} + \frac{1}{x^3}$). So, if 0.1 is how $\frac{1}{3}$ is written in some base x, we must have $\frac{1}{3} = \frac{1}{x}$. But, this strongly suggests that we choose x to be 3, so the base we are looking for is 3;

8. According to the definition, only the digit 0 is allowed in base 1. So, all polynomials in powers of 1 will be 0; that is, $0 \cdot 1^n + 0 \cdot 1^{n-1} + \ldots + 0 \cdot 1^0 = 0$ always.