1 Prescribed Slip

We apply conservation of momemtum,

$$\int_{\Omega} \rho(\vec{x}) \frac{\partial \vec{v}}{\partial t} \, d\Omega = \int_{\Omega} \vec{f}(\vec{x},t) \, d\Omega + \int_{\Gamma} \vec{\tau}(\vec{x},t) \, d\Gamma, \tag{1}$$

to a fault interface Ω_f with boundaries Γ_{f^+} and Γ_{f^-}. For a fault interface, the body force is zero ($\vec{f}(\vec{x},t) = \vec{0}$). The tractions on the positive and negative fault faces are

$$\tau^+(\vec{x},t) = \boldsymbol{\sigma}^+ \cdot \vec{n} + \vec{\lambda} \tag{2}$$

$$\tau^-(\vec{x},t) = \boldsymbol{\sigma}^- \cdot \vec{n} - \vec{\lambda}, \tag{3}$$

where $\vec{\lambda}$ is the Lagrange multiplier that corresponds to the fault traction generating the prescribed slip. Thus, for a fault interface, we have

$$\int_{\Omega_f} \rho(\vec{x}) \frac{\partial \vec{v}}{\partial t} \, d\Omega = \int_{\Gamma_{f^+}} \boldsymbol{\sigma} \cdot \vec{n} + \vec{\lambda} \, d\Gamma + \int_{\Gamma_{f^-}} \boldsymbol{\sigma} \cdot \vec{n} - \vec{\lambda} \, d\Gamma. \tag{4}$$

1.1 Quasistatic Prescribed Slip

In the quasistatic case we neglect inertia, so we have

$$\int_{\Gamma_{f^+}} \boldsymbol{\sigma} \cdot \vec{n} + \vec{\lambda} \, d\Gamma + \int_{\Gamma_{f^-}} \boldsymbol{\sigma} \cdot \vec{n} - \vec{\lambda} \, d\Gamma = 0. \tag{5}$$

We enforce this equation on each portion of the fault interface along with our prescribed slip constraint, which leads to

$$\boldsymbol{\sigma} \cdot \vec{n} + \vec{\lambda} = \vec{0} \text{ on } \Gamma_{f^+}, \tag{6}$$

$$\boldsymbol{\sigma} \cdot \vec{n} - \vec{\lambda} = \vec{0} \text{ on } \Gamma_{f^-}, \tag{7}$$

$$\vec{u}^+ - \vec{u}^- - \vec{d} = \vec{0}, \tag{8}$$

2 Dynamic Prescribed Slip

When we integrate the weak form of the momentum equation in the dynamic case, we select numerical quadrature that yields a lumped mass matrix. That is, we have

$$\int_{\Omega} \vec{\psi}^v_{trial} \cdot \rho(\vec{x}) \frac{\partial \vec{v}}{\partial t} \, d\Omega = M_v \frac{\partial \vec{v}}{\partial t}, \tag{9}$$

where M_v is the lumped mass matrix. We can separate the integration of the weak form for negative and positive sides of the fault interface, which yields

$$M_{v^+} \frac{\partial \vec{v}^+}{\partial t} = \int_{\Gamma_{f^+}} \vec{\psi}^\lambda_{trial} \cdot \left(\boldsymbol{\sigma} \cdot \vec{n} + \vec{\lambda} \right) \, d\Gamma, \tag{10}$$

$$M_{v^-} \frac{\partial \vec{v}^-}{\partial t} = \int_{\Gamma_{f^-}} \vec{\psi}^\lambda_{trial} \cdot \left(\boldsymbol{\sigma} \cdot \vec{n} - \vec{\lambda} \right) \, d\Gamma. \tag{11}$$

Our constraint for prescribed slip is

$$\vec{u}^{+} - \vec{u}^{-} - \vec{d} = \vec{0}, \tag{12}$$

where $\vec{u}$ is the displacement vector and $\vec{d}$ is the slip vector. In order to match the order of the time derivative in the elasticity equation, we take the second derivative of the prescribed slip equation,

$$\frac{\partial \vec{v}^{+}}{\partial t} - \frac{\partial \vec{v}^{-}}{\partial t} - \frac{\partial^2 \vec{d}}{\partial t^2} = \vec{0}. \tag{13}$$

We form the weak form in the usual way,

$$\int_{\Gamma_f} \vec{\psi}^{\lambda}_{trial} \cdot \left(\frac{\partial \vec{v}^{+}}{\partial t} - \frac{\partial \vec{v}^{-}}{\partial t} - \frac{\partial^2 \vec{d}}{\partial t^2} \right) \, d\Gamma = 0. \tag{14}$$

Substituting in the expressions for the time derivative of the velocity on the negative and positive sides of the fault using the weak form of our conservation of momemtum yields

$$M_{v^+}^{-1} \int_{\Gamma_f^+} \vec{\psi}^{\lambda}_{trial} \cdot \left(\boldsymbol{\sigma} \cdot \vec{n} + \vec{\lambda} \right) \, d\Gamma + M_{v^-}^{-1} \int_{\Gamma_f^-} \vec{\psi}^{\lambda}_{trial} \cdot \left(-\boldsymbol{\sigma} \cdot \vec{n} + \vec{\lambda} \right) \, d\Gamma - \int_{\Gamma_f} \vec{\psi}^{\lambda}_{trial} \cdot \frac{\partial^2 \vec{d}}{\partial t^2} \, d\Gamma = \vec{0}. \tag{15}$$