ASSIGNMENT #1

① **Given** → i) $E \neq \phi$

ii) $E \subseteq S$ (S is an ordered set)

iii) α is a lower bound of E

iv) β is an upper bound of E

T.P. → $\alpha \leq \beta$

Proof → Let's assume, for contradiction that

$$\alpha > \beta \quad \text{—— ①}$$

acc. to given (iii) & (iv)

$\forall \gamma \in E$

$$\beta \geq \gamma \geq \alpha$$

when $\beta = \gamma$ and $\gamma = \alpha$

$$\beta = \alpha \quad \text{—— ②}$$

By eq^n. ① & ② we get,

$$\alpha > \alpha \quad \text{—— ③}$$

∵ S is an ordered set and $E \subseteq S$

⇒ E is an ordered set

~~By lemma~~, for an ordered set E

$\forall \alpha \in E$ $\quad \alpha > \alpha$ does not exists.

Eq^n. ③ is impossible, proving our assumpt^n wrong. Hence,

$$\alpha \leq \beta$$

② **T.P.** → There are infinite prime numbers.

Proof :→ Let's assume for contradiction that

there's a finite set of prime numbers
$\{P_1, P_2, P_3 ------ P_n\}$

let's take a new number M such that.
$M = (P_1) \times P_2 \times P_3 ---- P_n + 1$

M leaves remainder 1 when divided by P_i $\forall_{i \in \{1,2,3 --- n\}}$

which makes M a prime number as long as $M \neq 1$.

Since, 2 is a prime number
$\Rightarrow$ set $\{P_1, P_2, P_3 ---- P_n\}$ is not empty.
$\{P_1, P_2, P_3 ----- P_n\} \neq \phi$
$\Rightarrow$ M is a prime number.

$M > P_1$, $M > P_2$, $M > P_3$ ----- $M > P_n$
We can say that
$M > P_i \quad \forall_{i \in \{1, 2, 3, --- n\}}$ —— (1)

Eq^n (1) contradicts our assumption that set $\{P_1, P_2, P_3, P_4 ---- P_n\}$ is a finite set.

Hence, proving, there are infinite prime numbers.

(3) Given $\rightarrow$ i) $A \neq \phi$
ii) A is a set of Real numbers.
iii) A is bounded below.
iv) $-A$ set of all numbers $-x$ $\forall x \in A$.

To prove $\rightarrow$ $\inf(A) = -\sup(-A)$

Since A is bounded below and an ordered set

Proof $\rightarrow$ inf(A) is defined as

$\forall x \in A$ if $x \geq \alpha$
then $\alpha = \inf(A)$

We can say that
$\forall x \in A \quad x \geq \inf(A)$ —— (1)

(i) Suppose $x \in A$, then $-x \in -A$
sup(-A) is defined as

$\forall -x \in -A$, $\boxed{-x \leq \sup(-A)}$
or
$x \geq -\sup(-A)$ —— (2)

(i) By eqn (1) & (2), we get

$x \geq \inf(A)$ and $x \geq -\sup(-A)$

$\because$ A is set of Real numbers
$\Rightarrow$ A is an ordered set
this implies
$\boxed{\inf(A) = -\sup(-A)}$

(4) To prove $\rightarrow$
$|z_1 + z_2 + \cdots z_n| \leq |z_1| + |z_2| + \cdots |z_n|$

Proof :-
Using property of Theoram 1.33 (e) of complex numbers on page 14 of Ch-1 Rudin.

(e) $|z+\omega| \le |z|+|\omega|$

~~test~~

$\because$ $\bar{z}\omega$ is conjugate of $z\bar{\omega}$

$z\bar{\omega}+\bar{z}\omega = 2\,Re(z\bar{\omega})$ { where $Re(z)$ = Real part of ~~two~~ Complex no. z.

$$|z+\omega|^2 = (z+\omega)(\bar{z}+\bar{\omega}) = z\bar{z}+z\bar{\omega}+\bar{z}\omega+\omega\bar{\omega}$$
$$= |z|^2+2\,Re(z\bar{\omega})+|\omega|^2$$

$$|z+\omega|^2 \le |z|^2+2|z\bar{\omega}|+|\omega|^2 \quad \{\because 2|z\bar{\omega}| \ge 2\,Re(z\bar{\omega})\}$$

$$\le |z|^2+2|z||\omega|+|\omega|^2$$
$$|z+\omega|^2 \le (|z|+|\omega|)^2 \quad — ①$$

taking square root on both sides of ①

$$|z+\omega| \le (|z|+|\omega|) \quad — ②$$

Extending eq^n ②

let $z = |z_1+z_2+ --- z_{n-1}|$

$\omega = |z_n|$

thus using property ②

~~$|(z_1+z_2+ --- z_{n-1})+z_n| \le$~~

$$|(z_1+z_2+z_3+ --- z_{n-1})+z_n| \le |z_1+z_2+ --- z_{n-1}|+|z_n| \quad — ③$$

Hence, by applying property ② over eq^n ③ recursively leads to

$$\boxed{|z_1+z_2+z_3+ ----- z_n| \le |z_1|+|z_2|+ ----- |z_n|}$$

(5) Given :→

$$A = \{x \mid x^2+x \geq 1, x \geq 0, x \in Q\}$$

T.P. → A does not have greatest lower bound in Q.

Proof :→ Let's check domain of A

$x^2+x \geq 1$ and $x \geq 0$

$x^2+x-1 \geq 0$

$x \geq \frac{-1+\sqrt{5}}{2}$ or
$x \leq \frac{-1-\sqrt{5}}{2}$
but $x \geq 0$

so x cannot be $\leq \left(\frac{-1-\sqrt{5}}{2}\right)$

Hence, $x \geq \frac{-1+\sqrt{5}}{2}$

$$2x+1 \geq \sqrt{5} \quad \text{—— (1)}$$

$\because x \in Q$
$2x+1 \in Q$

which means left side of eqn (1) is rational. So, right side of eqn (1) must also be rational for equality to hold.

Date
Page

Let's assume for contradictn. that $\sqrt{5}$ is a rational no.

So, $\sqrt{5} = \frac{p}{q}$ (where p, q does not share 5 as common factor and p and q are co-prime numbers.

$5 = \frac{p^2}{q^2}$

$5q^2 = p^2$ —— (2)

This means p^2 is divisible by 5

hence p is divisible by 5 —— (3)

$\Rightarrow$ p^2 will also be divisible by 25

which implies that left side of 2 will also be divisible by 25.

$\Rightarrow$ q^2 is also divisibly by 5

$\Rightarrow$ q is divisible by 5 —— (4)

(3) & (4) contradicts the fact that in our assumptn. p, q did not share a factor of 5 and are co-prime which implies that our assumptn. of $\sqrt{5}$ being rational was wrong.

Hence, $\sqrt{5}$ is an irrational no.

$\Rightarrow$ ~~2x+1 is also irrational number~~.

We can also say that,

$\cancel{\forall} p \in$

let's assume set X of all positive rationals p such that

$p^2 < 5$

and let set Y of all positive rationals p such that $p^2 > 5$

let's take a number q between p and $\sqrt{5}$

$$q = p - (p - \sqrt{5})$$
$$= p - \frac{(p-\sqrt{5})(p+\sqrt{5})}{(p+\sqrt{5})}$$
$$= p - \frac{(p^2-5)}{p+\sqrt{5}}$$

$(p^2-5) \rightarrow A$

$\rightarrow$ dividing eqⁿ by $\underline{P+5}$ would make A more less

$$= p - \frac{(p^2-5)}{p+5} \quad \text{—— (5)}$$
$$= \frac{p^2+5p-p^2+5}{p+5}$$
$$q = \frac{5(p+1)}{p+5} \rightarrow \text{squaring both sides}$$
$$q^2 = \frac{25(p+1)^2}{(p+5)^2} \rightarrow \text{subtracting } -5 \text{ both sides}$$
$$q^2 - 5 = \frac{25(p^2+1+2p) - 5(p^2+25+10p)}{(p+5)^2}$$
$$= \frac{25p^2+25+\cancel{50}p - 5p^2 - 125 - \cancel{50}p}{(p+5)^2}$$
$$= \frac{20p^2 - 100}{(p+5)^2} \Rightarrow \frac{20(p^2-5)}{(p+5)^2}$$
$$\Rightarrow q^2 - 5 = \frac{20(p^2-5)}{(p+5)^2} \quad \text{—— (6)}$$

case-2

if p is in $X \rightarrow p^2 - 5 < 0$
Eqⁿ (5) shows that $q > p$ and
eqⁿ (4) shows that $q^2 < 5 \Rightarrow$ $\underline{q \text{ is also in } A}$

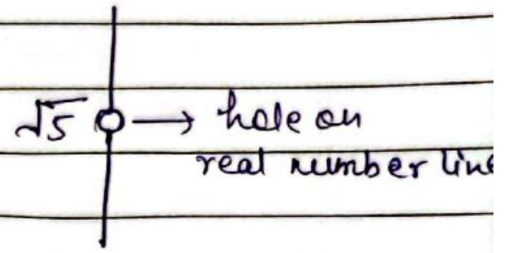
Case 2

If p is in B then $p^2-5>0$

eq^n (5) $\longrightarrow$ $0<q<p$ & eq^n (4) shows $q^2>5$

$\Rightarrow$ q is in B.

Acc. to case (1) & case (2) if an irrational number is there, you can find infinite rational/irrational numbers greater than that and infinite numbers smaller than that. —— (7)



$\therefore$ We proved that $2x+1$ is not rational

And acc. to eq^n (7) irrational number i.e. $\sqrt{5}$ does not have any lower or upper bound.

A does not have any lower bound as

$$\forall x \in A \qquad x \geq \frac{\sqrt{5}-1}{2} \qquad \therefore \inf(A) = \frac{\sqrt{5}-1}{2}$$

but $\sqrt{5}$ is an irrational number which does not have lower/upper bound.

$\therefore$ A does not have greatest lower bound in $\mathbb{Q}$.

$\Rightarrow$ $\inf(A) \notin \mathbb{Q}$.