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GLOBAL ASYMPTOTICS OF THE FOURTH PAINLEVÉ TRANSCENDENT

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ABSTRACT

Global asymptotic properties of the classical fourth Painlevé transcendent obtained via the isomonodromy deformation method are announced. Connection formulae are presented as well.

1. Introduction

We consider the general case of the fourth Painlevé equation P4 [1]

$$
w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4xw^2 + (-4\alpha + \beta + 2x^2)w - \frac{\beta^2}{2w} \tag{1}
$$

where the prime means the differentiation with respect to x. It is well known that the equation (1) arises as a similarity reduction of the Derivative Nonlinear Schrödinger equation [2]. Recently there has been considerable interest in P4 due to the possibility to solve the connection problem for the special case of a nonlinear harmonic oscillator which is equivalent to (1) with $\beta = 0$ and some half-integer α [3], and due to remarkable role of the P4-hierarchy in the generalizations of the harmonic oscillator with the creation-annihilation operators of higher orders [4].

The connection of the Painlevé equation theory with the problem of the isomonodromy deformation of the Fuchs type equations [5] allows the authors of refs. [6, 7] to develop an inverse problem method for investigation these equations, which was applied to P4 in refs. [8, 9]. It was shown there that the Cauchy problem for P4 in general admits global meromorphic in x solution. In refs. [10, 11], ideas of refs. [6, 7] have been used to develop the isomonodromy deformation technique for investigation of the asymptotic properties of the Painlevé transcendents. The first result in this direction concerning P4 was obtained in ref. [12] where the real decreasing solutions of the equation

$$
\tilde{w}'' = \frac{(\tilde{w}')^2}{2\tilde{w}} - \frac{3}{2}\tilde{w}^3 - 4\tilde{x}\tilde{w}^2 + (-4i\alpha + i\beta - 2\tilde{x}^2)\tilde{w} - \frac{\beta^2}{2\tilde{w}} \, , \tag{2}
$$

for $\alpha = 0$, $\tilde{x} \to +\infty$, were considered. Note that this equation goes from (1) after the changes

$$
w = e^{i\pi/4}\tilde{w} \, , \qquad x = e^{i\pi/4}\tilde{x} \, . \tag{3}
$$

The present paper describes the solution of the connection problem for P4 solved by means of the advanced version of the isomonodromy deformation method [10, 11]. Similar results for P2 and P1 were listed in refs. [13, 14]. All necessary proves are going to be published in ref. [15].

2. P4 as a monodromy preserving deformation

There are several ways to get for P4 (1) the representation of zero curvature. One of the Lax pairs was found in ref. [7], however the pair from ref. [12] seems a little more convenient for our aims:

$$
\frac{\partial \Psi}{\partial \xi} = \left\{ \Bigl(\frac{1}{2}\xi^3 + \xi(x+uv) + \frac{\alpha}{\xi}\Bigr)\sigma_3 + i\Bigl(\xi^2 u + 2xu + u'\Bigr)\sigma_+ + i\Bigl(\xi^2 v + 2xv - v'\Bigr)\sigma_- \right\}\Psi \, , \tag{4}
$$

$$
\frac{\partial \Psi}{\partial x} = \left\{ \Bigl(\frac{1}{2}\xi^2 + uv\Bigr)\sigma_3 + i\xi u\sigma_+ + i\xi v\sigma_- \right\}\Psi \, , \tag{5}
$$

where Ψ is some 2×2-matrix-valued function of the complex variable ξ, and x, u, v, α are some complex parameters,

$$
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_+ = \begin{pmatrix} & 1 \\ 0 & , \end{pmatrix} \qquad \sigma_- = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}.
$$

Compatibility of the equations (4), (5) implies that α does not depend on x, while u and v are such functions of x that

$$\beta = u'v - uv' + 2xuv - (uv)^2 \equiv \text{const} \tag{6}$$

and the product

$$w = uv \tag{7}$$

satisfies the equation P4 (1). Unfortunately, in the paper [12], the monodromy data of the equation (4) are not described completely even for the special case of real reduction for the equation (2) with $\alpha = 0$ discussed there. So we need to give some attention to this question. The equation (4) is rational with respect to independent variable ξ. Its singular points are irregular point $\xi = \infty$ and regular singular point $\xi = 0$. Monodromy data of this equation are Stokes matrices S_k connecting the canonical solutions Ψ_k uniquely determined by their asymptotics near infinity, a monodromy matrix M near zero point and a connection matrix E for both singularities. Here

$$\Psi_k(\xi) = \Big(I + O(\xi^{-1})\Big) \exp\Big\{\big(\frac{1}{8}\xi^4 + \frac{1}{2}x\xi^2 + (\alpha - \beta)\ln\xi\big)\sigma_3\Big\}$$

$$k \in \mathbb{Z}, \qquad \xi \to \infty$$

for $\xi \in \omega_k = \Big\{\xi \in \mathbb{C}:\ \arg\xi \in \big(-\frac{3\pi}{8} + \frac{\pi}{4}k;\ \frac{\pi}{8} + \frac{\pi}{4}k\big)\Big\}$, moreover

$$\Psi_{k+1}(\xi) = \Psi_k(\xi)S_k\ , \qquad S_{2k-1} = \begin{pmatrix} 1 & s_{2k-1} \\ 0 & 1 \end{pmatrix}\ , \qquad S_{2k} = \begin{pmatrix} 1 & 0 \\ s_{2k} & 1 \end{pmatrix}\ ;$$

a canonical solution near zero point is

$$\Psi^0(\xi) = \hat{\Psi}(\xi)e^{\alpha\ln\xi\sigma_3}P(\xi), \qquad P(\xi) = \begin{pmatrix} 1 & j_+\ln\xi \\ j_-\ln\xi & 1 \end{pmatrix}\exp\Big(\int^x uv\ dx\sigma_3\Big)$$

where $\hat{\Psi}(\xi)$ is holomorphic and invertible near zero point, and $j_+ = 0$ if $\alpha - \frac{1}{2} \notin \mathbb{Z}_+$, while $j_- = 0$ if $\alpha - \frac{1}{2} \notin \mathbb{Z}_-\backslash\{0\}$. There are the symmetry properties

$$\sigma_3\Psi_{k+4}(e^{i\pi}\xi)\sigma_3 = \Psi_k(\xi)e^{i\pi(\alpha-\beta)\sigma_3},$$

$$\sigma_3\Psi^0(e^{i\pi}\xi)\sigma_3 = \Psi^0(\xi)M$$

where

$$M = \Big(I + i\pi j_\pm\exp\Big(\mp 2\int^x uv\ dx\Big)\sigma_\pm\Big)e^{i\pi\alpha\sigma_3},$$

which yield

$$S_{k+4} = e^{-i\pi(\alpha-\beta)\sigma_3}\sigma_3 S_k\sigma_3 e^{i\pi(\alpha-\beta)\sigma_3}, \quad s_{k+4} = -s_k e^{(-1)^k 2\pi i(\alpha-\beta)}, \tag{8}$$

and so-called semi-cyclic relations

$$S_1S_2S_3S_4 = E^{-1}\sigma_3 M^{-1}Ee^{i\pi(\alpha-\beta)\sigma_3}\sigma_3 \tag{9}$$

where E is the connection matrix defined by the equation

$$E = \Psi^0(\xi)^{-1}\Psi_1(\xi).$$

Note that the monodromy data including the Stokes matrices S_k or the Stokes multipliers s_k are functions of the coefficients of (4):

$$S_k = S_k(x, u, v, \alpha), \quad s_k = s_k(x, u, v, \alpha), \quad k \in \mathbb{Z}.$$

These functions possess the following symmetries:

$$s_k(x, -u, -v, \alpha) = -s_k(x, u, v, \alpha), \quad k \in \mathbb{Z}; \tag{10}$$

$$\bar{s}_{-k}(\bar{x}, \bar{u}, \bar{v}, \bar{\alpha}) = s_k(x, u, v, \alpha), \quad k \in \mathbb{Z}, \tag{11}$$

where the bar means complex conjugation; the gauge symmetry

$$S_k(x, e^a u, e^{-a} v, \alpha) = e^{\frac{a}{2}\sigma_3} S_k(x, u, v, \alpha) e^{-\frac{a}{2}\sigma_3}, \quad a \in \mathbb{C}, \quad k \in \mathbb{Z}; \tag{12}$$

and symmetry

$$S_{k-m}(e^{-i\pi m/2}x, \tau^m(e^{-i\pi m/4}u, e^{-i\pi m/4}v), e^{-i\pi m}\alpha) =$$
$$= (\sigma_2)^m e^{\frac{i\pi}{4}m(\alpha-\beta)\sigma_3} S_k(x, u, v, \alpha) e^{-\frac{i\pi}{4}m(\alpha-\beta)\sigma_3} (\sigma_2)^m, \quad k, m \in \mathbb{Z}, \tag{13}$$

where τ is permutation

$$\tau(u, v) = (v, u). \tag{14}$$

Schlesinger transformations of (4) preserving the monodromy data (except the exponents of the formal monodromy α near zero and $\alpha - \beta$ near infinity) and generating Bäcklund transformations of the Painlevé transcendents (see [16–18, 8]) can be treated as some symmetries of the functions considered:

$$S_k(x, \tilde{u}, \tilde{v}, \tilde{\alpha}) = S_k(x, u, v, \alpha), \quad k \in \mathbb{Z} \tag{15}$$

We omit the formulae for $\tilde{u}$, $\tilde{v}$, and present the results for the Painlevé function itself:

$$\tilde{\alpha} = -\alpha - \sigma, \qquad \sigma = \pm 1, \quad \tilde{\alpha} - \tilde{\beta} = \alpha - \beta,$$

$$\tilde{w} = w + \frac{2(1 + 2\sigma\alpha)w}{w' - \sigma(w^2 + 2xw + \beta)} \tag{16}$$

and another transformation

$$\tilde{\alpha} = -\alpha, \qquad \tilde{\alpha} - \tilde{\beta} = \alpha - \beta + \sigma, \qquad \sigma = \pm 1,$$

$$\tilde{w} = \sigma\frac{w'}{2w} - \frac{1}{2}w - x + \frac{\beta}{2w}. \tag{17}$$

The linear system (4) maps a set of the coefficients x, u, v, α onto a manifold of the monodromy data. The structure of this manifold can be described in the following way. From (9), the equation of a monodromy surface follows:

$$\Big((1 + s_{1+m}s_{2+m})(1 + s_{3+m}s_{4+m}) + s_{1+m}s_{4+m}\Big) e^{-i\pi(-1)^m(\alpha-\beta)} -$$
$$-(1 + s_{2+m}s_{3+m}) e^{i\pi(-1)^m(\alpha-\beta)} = -2i(-1)^m \sin\pi\alpha\ , \tag{18}$$

so that only three of the Stokes multipliers are independent. The surface (18) has got some special one-dimensional submanifolds defined by the equations

$$\alpha = \frac{1}{2} + n, \quad n \in \mathbb{Z}, \quad 1 + s_{2+m}s_{3+m} = e^{i\pi(-1)^m\beta},$$

$$s_{1+m} = -s_{3+m}e^{-i\pi(-1)^m\beta}, \quad s_{4+m} = -s_{2+m}e^{-i\pi(-1)^m\beta}. \tag{19}$$

It can be shown, that, for non-special points of the surface (18), the connection matrix E does not contain any essential free parameter, but for the special points (19), it does, and this additional free parameter is the ratio of row components of this matrix E. Thus the manifold of the monodromy data for (4) is the surface (18) with $\mathbb{C}P^1$ which must be pasted to each special point (19).

Note that (13) means that fourth Painlevé transcendent is invariant about the gauge transformation of Ψ-function in contrast to the coefficients u, v, so that any solution of P4 corresponds to an orbit of the one-parametric group of the gauge transformations of the monodromy data manifold.

In the case of general position, the asymptotic solutions of P4 depend on two complex parameters which are functions of the first integrals of P4, i.e. the products of the Stokes multipliers $s_{2k-1}s_{2m}$, or the ratios s_k/s_m where k and m are some integers of the common parity. Therefore, the equations (8), (18) yield the connection formulae for the parameters of asymptotic description desired.

3. List of the asymptotic solutions of the P4 equation in the general case

Let

$$\mu^2 = z(z - z_1)(z - z_3)(z - z_5) \equiv z^4 + 4e^{i\varphi_0}z^3 + 4e^{2i\varphi_0}z^2 + 4A(\varphi_0)z, \tag{20}$$

where $\varphi_0 \in (-\pi/4; 0) \cup (0; \pi/4)$ is a real parameter, be the elliptical curve defined on the Riemann surface which is two complex planes pasted along the straight line cuts $(z_5; z_3)$ and $(0; z_1)$, moreover $\mu(z) \to z^2 + 2e^{i\varphi}z$ as $z \to \infty^+$, i.e. on the upper sheet. Define a- and b-cycles on this surface in such a way that the contours

$$\frac{1}{2}a = (z_5; z_3), \quad \frac{1}{2}b = (z_3; z_1)$$

lie on the upper sheet, moreover the contour $(z_5; z_3)$ lies to the left of the corresponding cut.

Problem A. *Find the differentiable function $A(\varphi_0)$ of the parameter φ_0 such that the curve (20) satisfies the following system of Boutroux equations:*

$$\mathrm{Re} \oint_{a,b} \frac{\mu(z)}{z} dz = 0 \tag{21}$$

while indexing of the branch points z_1, z_3, z_5 is fixed by the boundary conditions

$$z_{3,1} \to -\frac{2}{3} \quad \text{as} \quad \varphi_0 \to 0,$$

$$z_1 \to 0, \quad z_{3,5} \to -2e^{\pm i\pi/4} \quad \text{as} \quad \varphi_0 \to \pm\frac{\pi}{4}. \tag{22}$$

For the first time, the similar problem was formulated in ref. [19] for the P1 equation, discussions of such problems for P1 and P2 one can find in refs. [20–22].

Theorem 1. *The boundary-value Problem A for the branch points $z_k(\varphi_0)$, $k = 1, 3, 5$, of the curve (20) is uniquely solvable.*

This theorem provide a nontrivial contents for the following assertion.

Theorem 2. *If $x \to \infty$,*

$$\arg(x) \in \Big(\frac{\pi}{2}(n + \frac{m-1}{2}); \frac{\pi}{2}(n + \frac{m}{2})\Big)$$

$$n \in \mathbb{Z}, \qquad m = 0, 1, \tag{23}$$

and

$$(1 + s_{1+n}s_{2+n})(1 + s_{2+n}s_{3+n}) - 1 \neq 0, \qquad 1 + s_{1+m+n}s_{2+m+n} \neq 0, \tag{24}$$

then the asymptotic form of the solution of (1) is described by means of the inversion of the elliptical integral

$$\int_0^r \frac{dz}{\mu(z)} = e^{i\varphi_0}t + \chi, \tag{25}$$

where

$$\chi = p\Omega_a + q\Omega_b + O(t^{-1})$$

moreover

$$p = -\frac{1}{2\pi i}\ln\big((1 + s_{1+n}s_{2+n})(1 + s_{2+n}s_{3+n}) - 1\big),$$

$$q = -(-1)^m\frac{1}{2\pi i}\ln\Big((1 + s_{1+m+n}s_{2+m+n})\exp\big((-1)^{n+m+1}\frac{2\pi i}{3}(\alpha - \beta)\big)\Big),$$

$$\Omega_{a,b} = \oint_{a,b}\frac{dz}{\mu(z)}.$$

Here the function $\mu(z)$ is uniquely defined as the solution of the boundary Problem A. The connection between $r(t)$ and the Painlevé function $w(x)$ is given by

$$t = \frac{1}{2}(e^{-i\varphi}x)^2,$$

$$r = e^{-i\pi n/2}\big(e^{-i\varphi}x\big)^{-1}w,$$

$$\varphi = \varphi_0 + \frac{\pi}{2}n = \text{const},$$

$$|\mathrm{Im}\, t| < \text{const}, \quad \mathrm{Re}\, t \to +\infty. \tag{26}$$

Theorem 3. *If $x \to \infty$, and*

$$\arg(x) \in \Big(\frac{\pi}{2}(n + \frac{m-1}{2}) + \varepsilon; \frac{\pi}{2}(n + \frac{m}{2}) - \varepsilon\Big)$$

$$n \in \mathbb{Z}, \qquad m = 0, 1,$$

and

$$|(1 + s_{1+n}s_{2+n})(1 + s_{2+n}s_{3+n}) - 1| > \varepsilon, \qquad |1 + s_{1+m+n}s_{2+m+n}| > \varepsilon,$$

where $\varepsilon > 0$ is small, then the asymptotic solution (23)–(26) is uniform about $\arg(x)$.

Theorem 4. *If* $x \to \infty, \quad \arg(x) \to \frac{\pi}{2}n, \quad n \in \mathbb{Z}$,

$$(1+s_{1+n}s_{2+n})(1+s_{2+n}s_{3+n})-1 \neq 0, \tag{27}$$

then the asymptotic solution of the P4 equation is as follows:

(i) *if* $\arg\big(1-(1+s_{1+n}s_{2+n})(1+s_{2+n}s_{3+n})\big) \in \left(-\frac{\pi}{3};\frac{\pi}{3}\right)$ *and*

$$(1+s_{1+n}s_{2+n})(1+s_{2+n}s_{3+n}) \neq 0,$$

then

$$r+\frac{2}{3}=\frac{1}{\sqrt{t}}\Big(ae^{i\chi}+be^{-i\chi}\Big)+O\big(t^{-1+2|\mathrm{Re}\,\rho|}\big), \tag{28}$$

where

$$\chi=\frac{2t}{\sqrt{3}}+i\rho\ln 4\sqrt{3}t-(-1)^n\frac{2\pi}{3}(\alpha-\beta)-\frac{3\pi}{4},$$

$$ab=-i\rho/\sqrt{3}, \quad |\mathrm{Re}\,\rho|<\frac{1}{6},$$

moreover

$$a=-\frac{1}{1+s_{2+n}s_{3+n}}\frac{\sqrt{2\pi}}{\sqrt[4]{3}\Gamma(-\rho)}e^{i\pi\rho/2}, \quad b=-\frac{1}{1+s_{1+n}s_{2+n}}\frac{\sqrt{2\pi}}{\sqrt[4]{3}\Gamma(\rho)}e^{i\pi\rho/2},$$

$$\rho=\frac{1}{2\pi i}\ln\Big(1-(1+s_{1+n}s_{2+n})(1+s_{2+n}s_{3+n})\Big); \tag{29}$$

(ii) *if* $\arg\big((1+s_{1+n}s_{2+n})(1+s_{2+n}s_{3+n})-1\big) \in (-\pi;\pi)$ *then*

$$\frac{1}{r+\frac{2}{3}}=\frac{1}{2}+ae^{i\chi}+be^{-i\chi}+O\big(t^{-1+3|\mathrm{Re}\,\rho|}\big), \tag{30}$$

where

$$\chi=\frac{2t}{\sqrt{3}}+i\rho\ln 4\sqrt{3}t-(-1)^n\frac{2\pi}{3}(\alpha-\beta),$$

$$ab=1/4, \quad |\mathrm{Re}\,\rho|<1/2,$$

moreover

$$a=\frac{1}{1+s_{2+n}s_{3+n}}\sqrt{\frac{\pi}{2}}\frac{e^{i\pi\rho/2}}{\Gamma(\frac{1}{2}-\rho)}, \quad b=\frac{1}{1+s_{1+n}s_{2+n}}\sqrt{\frac{\pi}{2}}\frac{e^{i\pi\rho/2}}{\Gamma(\frac{1}{2}+\rho)},$$

$$\rho=\frac{1}{2\pi i}\ln\Big((1+s_{1+n}s_{2+n})(1+s_{2+n}s_{3+n})-1\Big); \tag{31}$$

The connection between $r(t)$ *and the Painlevé function* $w(x)$ *is given by (26) with* $\varphi=\pi n/2$.

The asymptotic solutions of the type (28) for $x \to \pm\infty$ in real reduction are discussed in refs. [3] (for $\beta=0$) and [4].

Theorem 5. *If* $x \to \infty, \quad \arg(x) \to \frac{\pi}{4} + \frac{\pi}{2}n, \quad n \in \mathbb{Z}$,

$$1 + s_{2+n}s_{3+n} \neq 0, \tag{32}$$

and

$$s_{3+n}(s_{2+n} + s_{4+n} + s_{2+n}s_{3+n}s_{4+n}) \neq 0,$$
$$s_{2+n}(s_{1+n} + s_{3+n} + s_{1+n}s_{2+n}s_{3+n}) \neq 0,$$

then the asymptotic behavior of the fourth Painlevé transcendent is as follows:

(i) *if* $\arg\Big((-1 - s_{2+n}s_{3+n})e^{i\frac{2\pi}{3}(-1)^n(\alpha-\beta)}\Big) \in \big(-\frac{\pi}{3}; \frac{\pi}{3}\big)$, *then*

$$r + 2 = \frac{1}{\sqrt{t}}\Big(ae^{i\chi} + be^{-i\chi}\Big) + O\Big(t^{-1+2|\mathrm{Re}\,\gamma|}\Big), \tag{33}$$

where

$$\chi = 2t + i\gamma \ln 4t + \frac{\pi}{4},$$
$$\gamma = -3\rho + (-1)^n(\alpha - \beta)$$
$$ab = i\Big(\rho - (-1)^n(\alpha - \frac{\beta}{2})\Big), \quad |\mathrm{Re}\,\gamma| < \frac{1}{2},$$

moreover

$$\rho = -\frac{1}{2\pi i}\ln(-1 - s_{2+n}s_{3+n}),$$
$$a = -\frac{1}{s_{3+n}(s_{2+n} + s_{4+n} + s_{2+n}s_{3+n}s_{4+n})}\times$$
$$\times\frac{(2\pi)^{3/2}e^{i\frac{\pi}{2}\gamma}}{\Gamma\big(\frac{1}{2} + \rho\big)\Gamma\big(-(-1)^n(\alpha - \frac{\beta}{2}) + \rho\big)\Gamma\big(\frac{1}{2} + (-1)^n\frac{\beta}{2} + \rho\big)},$$
$$b = -\frac{1}{s_{2+n}(s_{1+n} + s_{3+n} + s_{1+n}s_{2+n}s_{3+n})}\times$$
$$\times\frac{(2\pi)^{3/2}e^{i\frac{\pi}{2}\gamma}}{\Gamma\big(\frac{1}{2} - \rho\big)\Gamma\big((-1)^n(\alpha - \frac{\beta}{2}) - \rho\big)\Gamma\big(\frac{1}{2} - (-1)^n\frac{\beta}{2} - \rho\big)}; \tag{34}$$

(ii) *if* $\arg\Big((-1 - s_{2+n}s_{3+n})e^{i\frac{2\pi}{3}(-1)^n(\alpha-\beta)}\Big) \in \big(-\frac{2\pi}{3}; 0\big)$, *then*

$$\frac{2}{r} = ae^{i\chi} - 1 + O\Big(t^{-\frac{1}{2}+\mathrm{Re}\,\gamma}\Big), \tag{35}$$

where

$$\chi = 2t + i\gamma \ln 4t,$$
$$\gamma = -3\rho + (-1)^n(\alpha - \beta) + \frac{1}{2},$$
$$|\mathrm{Re}\,\gamma| < \frac{1}{2},$$

moreover

$$\rho = -\frac{1}{2\pi i}\ln(-1 - s_{2+n}s_{3+n}),$$

$$a=\frac{1}{s_{3+n}(s_{2+n}+s_{4+n}+s_{2+n}s_{3+n}s_{4+n})}\times$$

$$\times\frac{(2\pi)^{3/2}e^{i\frac{\pi}{2}\gamma}}{\Gamma\big(\frac{1}{2}+\rho\big)\Gamma\big(-(-1)^n(\alpha-\frac{\beta}{2})+\rho\big)\Gamma\big(\frac{1}{2}+(-1)^n\frac{\beta}{2}+\rho\big)}\ ; \tag{36}$$

(iii) *if* $\arg\Big((-1-s_{2+n}s_{3+n})e^{i\frac{2\pi}{3}(-1)^n(\alpha-\beta)}\Big)\in\big(0;\frac{2\pi}{3}\big)$, *then*

$$\frac{2}{r}=be^{-i\chi}-1+O\Big(t^{-\frac{1}{2}-\mathrm{Re}\,\gamma}\Big), \tag{37}$$

where

$$\chi=2t+i\gamma\ln 4t,$$

$$\gamma=-3\rho+(-1)^n(\alpha-\beta)-\frac{1}{2}$$

$$|\mathrm{Re}\,\gamma|<\frac{1}{2},$$

moreover

$$\rho=-\frac{1}{2\pi i}\ln(-1-s_{2+n}s_{3+n}),$$

$$b=\frac{1}{s_{2+n}(s_{1+n}+s_{3+n}+s_{1+n}s_{2+n}s_{3+n})}\times$$

$$\times\frac{(2\pi)^{3/2}e^{i\frac{\pi}{2}\gamma}}{\Gamma\big(\frac{1}{2}-\rho\big)\Gamma\big((-1)^n(\alpha-\frac{\beta}{2})-\rho\big)\Gamma\big(\frac{1}{2}-(-1)^n\frac{\beta}{2}-\rho\big)}\ ; \tag{38}$$

(iv) *if* $\arg\Big((1+s_{2+n}s_{3+n})e^{i\frac{2\pi}{3}(-1)^n(\alpha-\beta)}\Big)\in\big(-\frac{2\pi}{3};\frac{2\pi}{3}\big)$, *then*

$$r=\frac{1}{2it}\Big(2\rho+(-1)^n\frac{\beta}{2}-ae^{i\chi}-be^{-i\chi}\Big)+O\big(t^{-2+2|\mathrm{Re}\,\gamma|}\big), \tag{39}$$

where

$$\chi=2t+i\gamma\ln 4t,$$

$$\gamma=-3\rho+(-1)^n(\alpha-\beta)$$

$$ab=\rho\Big(\rho+(-1)^n\frac{\beta}{2}\Big),\quad |\mathrm{Re}\,\gamma|<1,$$

moreover

$$\rho=-\frac{1}{2\pi i}\ln(1+s_{2+n}s_{3+n}),$$

$$a=\frac{1}{s_{3+n}(s_{2+n}+s_{4+n}+s_{2+n}s_{3+n}s_{4+n})}\times$$

$$\times\frac{(2\pi)^{3/2}e^{i\frac{\pi}{2}\gamma}}{\Gamma(\rho)\Gamma\big(\frac{1}{2}-(-1)^n(\alpha-\frac{\beta}{2})+\rho\big)\Gamma\big((-1)^n\frac{\beta}{2}+\rho\big)}\ ,$$

$$b=-\frac{1}{s_{2+n}(s_{1+n}+s_{3+n}+s_{1+n}s_{2+n}s_{3+n})}\times$$

$$\times \frac{(2\pi)^{3/2} e^{i\frac{\pi}{2}\gamma}}{\Gamma(-\rho)\Gamma\big(\frac{1}{2}+(-1)^n(\alpha-\frac{\beta}{2})-\rho\big)\Gamma\big(-(-1)^n\frac{\beta}{2}-\rho\big)}\,. \tag{40}$$

The connection between $r(t)$ and the Painlevé function $w(x)$ is as follows:

$$t = \frac{1}{2}(e^{-i\varphi}x)^2,$$

$$r = e^{-i\frac{\pi}{4}-i\frac{\pi}{2}n}\big(e^{-i\varphi}x\big)^{-1}y,$$

$$\varphi = \frac{\pi}{4} + \frac{\pi}{2}n = \mathrm{const},$$

$$|\mathrm{Im}\,t| < \mathrm{const}, \quad \mathrm{Re}\,t \to +\infty\,.$$

The asymptotic solution of the equation (1) with $\alpha = 0$ and $e^{-i\pi/4}x \to +\infty$ of the form (39) in a reduction, which is equivalent to real reduction in the equation (2), including the connection formulae of the asymptotic parameters like $a,\ b,\ \rho$ with the monodromy data of the equation (4) was found in ref. [12].

Remark 1. The formulae (28), (29) endure the limit transition $\big(1+s_{1+n}s_{2+n}\big)\big(1+s_{2+n}s_{3+n}\big) \to 0$. If $\alpha = \pm\frac{1}{6}+k$, and $\beta = \pm\frac{2}{3}+2l$ where $k, l \in \mathbb{Z}$, the limit transition $1+s_{1+n}s_{2+n} = 1+s_{2+n}s_{3+n} = 0$ in (28), (29) will give the asymptotic description of the rational solution of P4 generated from the solution $w = -\frac{2}{3}x$ existing for $\alpha = \pm 1/6$, $\beta = \pm 2/3$ by means of a Bäcklund transformation chain constructed from (16), (17) (see refs. [16, 18]). Similarly, the formulae (30)–(40) endure the limit transition $s_{2+n}(s_{1+n}+s_{3+n}+s_{1+n}s_{2+n}s_{3+n}) \to 0$ or $s_{3+n}(s_{2+n}+s_{4+n}+s_{2+n}s_{3+n}s_{4+n}) \to 0$. These simultaneous limits by suitable α and β can lead to families of 0-parametric rational solutions (see refs. [16, 18]) and to families of 1-parametric solutions expressed by means of Weber-Hermite functions (see refs. [18, 3]).

Remark 2. To obtain the real or imaginary reduction in the asymptotic formulae listed in this paper one has to use the symmetries (10)–(12).

Remark 3. The asymptotic formulae (28), (29) and (30), (31) can be recounted into each other in the common subset of the monodromy data. Similar assertion is correct for the pairs (33), (34) and (35), (36) (or (37), (38)) as well as for (35), (36) (or (37), (38)) and (39), (40) .

Theorem 6. *If $x \to \infty$, $\arg(x) \in \big(\frac{\pi}{2}n; \frac{\pi}{2}+\frac{\pi}{2}n\big)$, $n \in \mathbb{Z}$, and*

$$1 + s_{2+n}s_{3+n} = 0, \tag{41}$$

then

$$r(t) = r_-(t) + \frac{a}{\sqrt{t}}\exp\big(-\frac{2t}{\sqrt{3}} - i\frac{2\pi}{3}(-1)^n(\alpha-\beta)\big)\Big(1+O\big(t^{-1}\big)\Big), \tag{42}$$

where

$$r_-(t) = -\frac{2}{3} + O(t^{-1})$$

depends on t, α, and β only, while

$$a = -\frac{i}{\sqrt{2\pi}\sqrt[4]{3}}\Big(\Theta\big(\frac{\pi}{4}+\frac{\pi}{2}n-\arg(x)\big)(1+s_{1+n}s_{2+n}) - \Theta\big(\arg(x)-\frac{\pi}{4}-\frac{\pi}{2}n\big)(1+s_{3+n}s_{4+n})\Big), \tag{43}$$

where

$$\Theta(z)=\begin{cases}0 \text{ if } z<0,\\ \frac{1}{2} \text{ if } z=0,\\ 1 \text{ if } z>0,\end{cases}$$

and the connection between $r(t)$ and the Painlevé function $w(x)$ is as follows:

$$t=\frac{1}{2}\big(e^{-i\frac{\pi}{4}-i\frac{\pi}{2}n}x\big)^2,\quad r=w/x\ . \tag{44}$$

Theorem 7. *If $x\to\infty$, $\arg(x)\in\big(-\frac{\pi}{4}+\frac{\pi}{2}n;\frac{\pi}{4}+\frac{\pi}{2}n\big)$, $n\in\mathbb{Z}$,*

$$s_{2+n}=0 \tag{45}$$

and

$$s_{1+n}+s_{3+n}\neq 0,$$

then

$$r(t)=r_+(t)+\frac{a}{2t}\exp\big(-2t+(-1)^n(\alpha-\beta)\ln 4t\big)\Big(1+O\big(t^{-1}\big)\Big), \tag{46}$$

where

$$r_+(t)=(-1)^n\frac{\beta}{4t}+O(t^{-2})$$

depend on α, β and t only,

$$\begin{aligned}a=&\frac{\sqrt{2\pi}}{\Gamma\big(\frac{1}{2}+(-1)^n(\alpha-\frac{\beta}{2})\big)\Gamma\big(-(-1)^n\frac{\beta}{2}\big)}\frac{i}{s_{1+n}+s_{3+n}}\times\\ &\times\Big(\Theta\big(\frac{\pi}{2}n-\arg(x)\big)s_{1+n}-\Theta\big(\arg(x)-\frac{\pi}{2}n\big)s_{3+n}\Big),\end{aligned} \tag{47}$$

where

$$\Theta(z)=\begin{cases}0 \text{ if } z<0,\\ \frac{1}{2} \text{ if } z=0,\\ 1 \text{ if } z>0,\end{cases}$$

and the connection between $r(t)$ and the Painlevé function $w(x)$ is as follows:

$$t=\frac{1}{2}\big(e^{-i\pi n/2}x\big)^2,\ r=w/x. \tag{48}$$

Before the next theorem will be formulated, it is necessary to note that the assumption

$$s_{1+n}+s_{3+n}+s_{1+n}s_{2+n}s_{3+n}=0 \tag{49}$$

with the equation of the monodromy surface (18) together implies that

$$1+s_{2+n}s_{3+n}=e^{i\pi(-1)^n\beta} \tag{50}$$

or

$$1+s_{2+n}s_{3+n}=-e^{-i\pi(-1)^n(2\alpha-\beta)} \tag{51}$$

Theorem 8. *If* $x \to \infty$, $\arg(x) \in \left(-\frac{\pi}{4} + \frac{\pi}{2}n; \frac{\pi}{4} + \frac{\pi}{2}n\right)$, $n \in \mathbb{Z}$, *the equations* (49), (50) *hold, and*

$$s_{2+n} \neq 0, \quad \frac{1}{2} + (-1)^n \alpha \notin \mathbb{N}$$

then

$$r(t) = r_+(t) + \frac{c}{2t} \exp\Big(-2t + (-1)^n \big(\alpha + \frac{\beta}{2}\big) \ln 4t\Big) \Big(1 + O\big(t^{-1}\big)\Big), \tag{52}$$

where

$$r_+(t) = -(-1)^n \frac{\beta}{4t} + O\big(t^{-2}\big)$$

depends on t, α *and* β *only,*

$$c = i \frac{\Gamma\big(\frac{1}{2} - (-1)^n \alpha\big)}{\sqrt{2\pi}\, \Gamma\big((-1)^n \frac{\beta}{2}\big)} \times$$

$$\times \frac{1}{s_{2+n}} \Big\{ \Theta\big(\frac{\pi}{2}n - \arg(x)\big) e^{i\pi(-1)^n \alpha} \big(s_n + s_{2+n} + s_n s_{1+n} s_{2+n}\big) -$$

$$-\Theta\big(\arg(x) - \frac{\pi}{2}n\big) e^{-i\pi(-1)^n \alpha} \big(s_{2+n} + s_{4+n} + s_{2+n} s_{3+n} s_{4+n}\big) \Big\} \tag{53}$$

where

$$\Theta(z) = \begin{cases} 0 \text{ if } z < 0, \\ \frac{1}{2} \text{ if } z = 0, \\ 1 \text{ if } z > 0, \end{cases}$$

and the connection between $r(t)$ *and the Painlevé function* $w(x)$ *is given by* (48).

Theorem 9. *If* $x \to \infty$, $\arg(x) \in \left(-\frac{\pi}{4} + \frac{\pi}{2}n; \frac{\pi}{4} + \frac{\pi}{2}n\right)$, $n \in \mathbb{Z}$, *the equations* (49), (51) *hold, and*

$$s_{2+n} \neq 0, \quad \frac{1}{2} - (-1)^n \alpha \notin \mathbb{N}$$

then

$$r(t) = r_+(t) - \frac{c}{\sqrt{t}} \exp\Big(-2t - (-1)^n \big(2\alpha - \frac{\beta}{2}\big) \ln 4t\Big) \Big(1 + O\big(t^{-1}\big)\Big), \tag{54}$$

where

$$r_+(t) = -2 + O\big(t^{-1}\big)$$

depends on t, α *and* β *only,*

$$c = i \frac{\Gamma\Big(\frac{1}{2} + (-1)^n \alpha\Big)}{\sqrt{2\pi}\, \Gamma\Big\{\frac{1}{2} - (-1)^n \big(\alpha - \frac{\beta}{2}\big)\Big)} \times$$

$$\times \frac{1}{s_{2+n}} \Big\{ \Theta\big(\frac{\pi}{2}n - \arg(x)\big) e^{-i\pi(-1)^n \alpha} \big(s_n + s_{2+n} + s_n s_{1+n} s_{2+n}\big) -$$

$$-\Theta\big(\arg(x) - \frac{\pi}{2}n\big) e^{i\pi(-1)^n \alpha} \big(s_{2+n} + s_{4+n} + s_{2+n} s_{3+n} s_{4+n}\big) \Big\} \tag{55}$$

where

$$\Theta(z) = \begin{cases} 0 \text{ if } z < 0, \\ \frac{1}{2} \text{ if } z = 0, \\ 1 \text{ if } z > 0, \end{cases}$$

and the connection between $r(t)$ and the Painlevé function $w(x)$ is given by (48).

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