PDMI PREPRINT — 13/1996

Degenerated asymptotic solutions of the fourth Painlevé equation

Andrei A. KAPAEV

Russian Academy of Sciences
Steklov Mathematical Institute
St. Petersburg Department
Fontanka 27, St. Petersburg 191011, Russia

December 1996

ABSTRACT

Deeply degenerated asymptotic solutions of the classical fourth Painlevé equation are described. The corresponding connection formulae of the asymptotic parameters with the monodromy data are presented as well. Bibliography–4.

Below, we continue discussion [1] of the global asymptotic behavior of the fourth Painlevé transcendent, i.e. general solution of the fourth Painlevé equation

$$y_{xx}=\frac{(y_x)^2}{2y}+\frac{3}{2}y^3+4xy^2+2(x^2-a)y-\frac{b}{2y},\qquad a,\,b=\mathrm{const},\tag{P4}$$

where

$$a=2\alpha-\frac{\beta}{2},\qquad b=\beta^2.$$

Bäcklund and Schlesinger transformations.

The elementary Schlesinger transformation of the function Ψ preserving the monodromy data except the exponents of the formal monodromy α near point zero or $\alpha-\beta$ near infinity is defined as

$$\widetilde{\Psi}=R\Psi\tag{1}$$

with the rational matrix function R of one of the following forms:

$$R^0_+=I+\frac{i}{\lambda}\cdot\frac{1+2\alpha}{2xv-v'}\,\sigma_+,\tag{2a}$$

$$R^0_-=I+\frac{i}{\lambda}\cdot\frac{1-2\alpha}{2xu+u'}\,\sigma,\tag{2b}$$

$$R^\infty_+=\begin{pmatrix}\lambda & iu\\ \frac{i}{u} & 0\end{pmatrix},\tag{2c}$$

$$R^\infty_-=\begin{pmatrix}0 & \frac{1}{iv}\\ -iv & \lambda\end{pmatrix}.\tag{2d}$$

The connection coefficient $\mathcal{A}$ (see [1]) is transformed as follows:

$$\widetilde{\mathcal{A}}=R\mathcal{A}R^{-1}+\frac{dR}{d\lambda}R^{-1}=$$

$$=\Big(\frac{1}{2}\lambda^3+\lambda(x+\widetilde{u}\widetilde{v})+\frac{\widetilde{\alpha}}{\lambda}\Big)\sigma_3+i\Big(\lambda^2\widetilde{u}+2x\widetilde{u}+\widetilde{u}'\Big)\sigma_++i\Big(\lambda^2\widetilde{v}+2x\widetilde{v}-\widetilde{v}'\Big)\sigma_-,\tag{3}$$

with the parameters

if $\quad R=R^0_+$:

$$\widetilde{\alpha}=-\alpha-1,\quad \widetilde{\alpha}-\widetilde{\beta}=\alpha-\beta,$$

$$\widetilde{u}=u-\frac{1+2\alpha}{2xv-v'},\quad \widetilde{v}=v,$$

$$\widetilde{u}'=u'-\frac{(1+2\alpha)v}{2xv-v'}(u+\widetilde{u}),\quad \widetilde{v}'=v',$$

so that $\quad \widetilde{y}=y+\dfrac{2(1+2\alpha)y}{y'-y^2-2xy-\beta},$ (4a)

if $\quad R=R^0_-$:

$$\widetilde{\alpha}=-\alpha+1,\quad \widetilde{\alpha}-\widetilde{\beta}=\alpha-\beta,$$

$$\widetilde{u}=u,\quad \widetilde{v}=v+\frac{1-2\alpha}{2xu+u'},$$

$$\widetilde{u}'=u',\quad \widetilde{v}'=v'-\frac{(1-2\alpha)u}{2xu+u'}(v+\widetilde{v}),$$

so that $\quad \widetilde{y}=y+\dfrac{2(1-2\alpha)y}{y'+y^2+2xy+\beta},$ (4b)

$$
\begin{aligned}
\text{if} \quad R = & R_+^\infty : \\
& \widetilde{\alpha} = -\alpha, \quad \widetilde{\alpha} - \widetilde{\beta} = \alpha - \beta + 1, \\
& \widetilde{u} = u' - u^2 v, \quad \widetilde{v} = \frac{1}{u}, \\
& \widetilde{u}' = -2xu' - u^2 v' + 4xu^2 v - (1+2\alpha)u, \quad \widetilde{v}' = -\frac{u'}{u^2}, \\
\text{so that} \quad & \widetilde{y} = \frac{y'}{2y} - \frac{1}{2}y - x + \frac{\beta}{2y},
\end{aligned} \tag{4c}
$$

$$
\begin{aligned}
\text{if} \quad R = & R_-^\infty : \\
& \widetilde{\alpha} = -\alpha, \quad \widetilde{\alpha} - \widetilde{\beta} = \alpha - \beta - 1, \\
& \widetilde{u} = \frac{1}{v}, \quad \widetilde{v} = -v' - uv^2, \\
& \widetilde{u}' = -\frac{v'}{v^2}, \quad \widetilde{v}' = -2xv' - u'v^2 - 4xuv^2 - (1-2\alpha)v, \\
\text{so that} \quad & \widetilde{y} = -\frac{y'}{2y} - \frac{1}{2}y - x + \frac{\beta}{2y},
\end{aligned} \tag{4d}
$$

The relations (4a)–(4d) are called the Bäcklund transformations of the fourth Painlevé transcendent. In fact, we have got twice more Bäcklund transformations for each Painlevé transcendent because of the possibility to use two Lax pairs which differs in the sign of β (see ref. [1]), so the formulae (4a)–(4d) with the parameter β and α replaced by $\hat{\beta} = -\beta$ and $\hat{\alpha} = \alpha - \frac{1}{2}\beta$ give us the Bäcklund transformations of P4 as well. However, if we talk about some concrete parameterization of the fourth Painlevé transcendent via the monodromy data, we have to remember that the transformations (4a)–(4d) preserve the Stokes multipliers, while the Bäcklund transformations derived from (4a)–(4d) by the change $\beta \mapsto -\beta$ do not.

The Bäcklund transformations (4c), (4d) are found in ref. [2]. Other two transformations (4a), (4b) with the corresponding Schlesinger transformations are obtained in ref. [3].

Degenerated asymptotic solutions of P4 and Bäcklund transformations.

In this subsection we describe some of 1-parameter (in complex sense) asymptotic as $x \to \infty$, $x \in \mathbb{R}$, solutions of P4 interesting from the point of view of their physical applications.

i) 0-parameter asymptotics of a background for 2-parameter oscillating asymptotic solution:

$$
\begin{aligned}
y_{2/3}(x,a,b) = & -\frac{2x}{3} + \frac{a}{x} + \frac{-4 - 12a^2 + 9b}{16x^3} + \frac{3(-28a + 12a^3 + 9ab)}{32x^5} + \\
& + \frac{27(112 + 224a^2 - 80a^4 - 264b + 72a^2 b + 27b^2)}{1024x^7} + \\
& + \frac{9a(16688 - 7200a^2 + 1008a^4 - 7704b + 216a^2 b + 891b^2)}{2048x^9} + \mathcal{O}(\frac{1}{x^{11}});
\end{aligned} \tag{5}
$$

ii) 1-parameter asymptotics approaching $-2x$:

$$
\begin{aligned}
y_2(x,a,b,c) = {} & -2x - \frac{a}{x} + \frac{4+12a^2-b}{16x^3} + \frac{-44a-36a^3+5ab}{32x^5} + \\
& + \frac{1072+6240a^2+2160a^4-296b-408a^2b+7b^2}{1024x^7} + \mathcal{O}\Big(\frac{1}{x^9}\Big) + \\
& + cx^{-2a}e^{-x^2}\Big\{1 + \frac{-12-8a-28a^2+3b}{32x^2} + \\
& + \frac{400+2432a+1632a^2+1920a^3+784a^4-136b-288ab-168a^2b+9b^2}{2048x^4} + \mathcal{O}\Big(\frac{1}{x^6}\Big)\Big\} - \\
& - \frac{c^2}{2}x^{-1-4a}e^{-2x^2}\Big\{1 + \frac{-20-24a-28a^2+3b}{16x^2} + \mathcal{O}\Big(\frac{1}{x^4}\Big)\Big\};
\end{aligned}
\tag{6}
$$

iii) 1-parameter decreasing asymptotics:

$$
\begin{aligned}
y_\pm(x,a,\sqrt{b},c) = {} & \pm\frac{\sqrt{b}}{2x} \pm \frac{(a\mp\sqrt{b})\sqrt{b}}{4x^3} \pm \frac{\sqrt{b}\,(12+12a^2\mp 32a\sqrt{b}+17b)}{64x^5} \pm \\
& \pm \frac{\sqrt{b}\,(100a+20a^3\mp 112\sqrt{b}\mp 96a^2\sqrt{b}+125ab\mp 46b^{3/2})}{128x^7} + \mathcal{O}\Big(\frac{1}{x^9}\Big) + \\
& + cx^{-1+a\mp\frac{3\sqrt{b}}{2}}e^{-x^2}\Big\{1 + \frac{1}{32x^2}(-12+16a-4a^2\mp 20\sqrt{b}\pm 24a\sqrt{b}-15b) + \\
& + \frac{1}{2048x^4}(912-1600a+864a^2-192a^3+16a^4\pm 2208\sqrt{b}\mp \\
& \mp 3136a\sqrt{b}\pm 1696a^2\sqrt{b}\mp 192a^3\sqrt{b}+1944b-2832ab+ \\
& + 696a^2b\pm 1176b^{3/2}\mp 720ab^{3/2}+225b^2) + \mathcal{O}\Big(\frac{1}{x^6}\Big)\Big\} + \\
& + \frac{c^2}{2}x^{-3+2a\mp 3\sqrt{b}}e^{-2x^2}\Big\{1 + \frac{-36+32a-4a^2\mp 36\sqrt{b}\pm 24a\sqrt{b}-15b}{16x^2} + \mathcal{O}\Big(\frac{1}{x^4}\Big)\Big\}.
\end{aligned}
\tag{7}
$$

Note, that $y_+(x,a,\sqrt{b},c) = y_-(x,a,-\sqrt{b},c)$, so they coincide with each other if $b=0$. This asymptotic solution is denoted as

$$
\begin{aligned}
y_0(x,a,c) = {} & cx^{-1+a}e^{-x^2}\Big\{1 + \frac{-3+4a-a^2}{8x^2} + \frac{57-100a+54a^2-12a^3+a^4}{128x^4} + \mathcal{O}\Big(\frac{1}{x^6}\Big)\Big\} + \\
& + \frac{c^2}{2}x^{-3+2a}e^{-2x^2}\Big\{1 + \frac{-9+8a-a^2}{4x^2} + \mathcal{O}\Big(\frac{1}{x^4}\Big)\Big\}.
\end{aligned}
\tag{8}
$$

Next table gathers actions of the Bäcklund transformations (4a)–(4d) and their counterparts with other sign of β:

$$
R^\infty_\pm y = \pm\frac{y'}{2y} - \frac{1}{2}y - x + \frac{\beta}{2y}, \qquad R^0_\pm y = y \pm \frac{1\pm 2\alpha}{\pm\frac{y'}{2y} - \frac{1}{2}y - x - \frac{\beta}{2y}},
$$

$$
\widehat{R}^\infty_\pm y = \pm\frac{y'}{2y} - \frac{1}{2}y - x - \frac{\beta}{2y}, \qquad \widehat{R}^0_\pm y = y \pm \frac{1\pm 2\alpha}{\pm\frac{y'}{2y} - \frac{1}{2}y - x + \frac{\beta}{2y}},
$$

corresponding to the Schlesinger transformations (2a)–(2d) in the alternative parameterization (hatted transformations do not preserve the Stokes multipliers). Here,

$$\sqrt{b} = \beta, \quad a = 2\alpha - \frac{\beta}{2}.$$

Resulting parameters and transformed functions are marked by tilde (i.e. $R[y(x,a,b,c)] \equiv \widetilde{y}(x,a,b,c)$).

The Bäcklund transformation actions preserving the Stokes multipliers are given by:

R^{∞}_{+}:

$$2\widetilde{\alpha} - \widetilde{\beta} = -\beta + 1, \quad \widetilde{\beta} = -2\alpha + \beta - 1, \quad \widetilde{a} = -\frac{a-1}{2} - \frac{3\sqrt{b}}{4}, \quad \widetilde{b} = \big(-a - 1 + \frac{\sqrt{b}}{2}\big)^2,$$
$$\widetilde{y}_{2/3}(x,a,b) = y_{2/3}(x,\widetilde{a},\widetilde{b}), \quad \widetilde{y}_2(x,a,b,c) = y_-\big(x,\widetilde{a},\widetilde{\beta},-\frac{1}{4}c(-a-1+\frac{\beta}{2})\big),$$
$$\widetilde{y}_-(x,a,\beta,c) = y_2(x,\widetilde{a},\widetilde{b},-c), \quad \widetilde{y}_+(x,a,\beta,c) = y_+\big(x,\widetilde{a},\widetilde{\beta},-\frac{4c}{\beta}\big),$$
$$\widetilde{y}_0(x,a,c) = y_2(x,\widetilde{a},\widetilde{b},-c);$$

R^{∞}_{-}:

$$2\widetilde{\alpha} - \widetilde{\beta} = -\beta - 1, \quad \widetilde{\beta} = -2\alpha + \beta + 1, \quad \widetilde{a} = -\frac{a+1}{2} - \frac{3\sqrt{b}}{4}, \quad \widetilde{b} = \big(-a + 1 + \frac{\sqrt{b}}{2}\big)^2,$$
$$\widetilde{y}_{2/3}(x,a,b) = y_{2/3}(x,\widetilde{a},\widetilde{b}), \quad \widetilde{y}_2(x,a,b,c) = y_-(x,\widetilde{a},\widetilde{\beta},-c),$$
$$\widetilde{y}_-(x,a,\beta,c) = y_2\big(x,\widetilde{a},\widetilde{b},-\frac{4c}{\beta}\big), \quad \widetilde{y}_+(x,a,\beta,c) = y_+\big(x,\widetilde{a},\widetilde{\beta},-\frac{1}{4}c(-a+1+\frac{\beta}{2})\big),$$
$$\widetilde{y}_0(x,a,c) = y_+\big(x,\widetilde{a},\widetilde{\beta},\frac{1}{4}c(a-1)\big);$$

R^{0}_{+}:

$$2\widetilde{\alpha} - \widetilde{\beta} = -\beta - 1, \quad \widetilde{\beta} = -2\alpha + \beta - 1, \quad \widetilde{a} = -\frac{a+3}{2} - \frac{3\sqrt{b}}{4}, \quad \widetilde{b} = \big(-a - 1 + \frac{\sqrt{b}}{2}\big)^2,$$
$$\widetilde{y}_{2/3}(x,a,b) = y_{2/3}(x,\widetilde{a},\widetilde{b}), \quad \widetilde{y}_2(x,a,b,c) = y_-\big(x,\widetilde{a},\widetilde{\beta},\frac{1}{16}c(-a-1+\frac{\beta}{2})(a+1+\frac{\beta}{2})\big),$$
$$\widetilde{y}_-(x,a,\beta,c) = y_2\left(x,\widetilde{a},\widetilde{b},-\frac{16c}{\beta(a+1+\frac{\beta}{2})}\right), \quad \widetilde{y}_+(x,a,\beta,c) = y_+\big(x,\widetilde{a},\widetilde{\beta},c\big),$$
$$\widetilde{y}_0(x,a,c) = y_+(x,\widetilde{a},\widetilde{\beta},c);$$

R^{0}_{-}:

$$2\widetilde{\alpha} - \widetilde{\beta} = -\beta + 1, \quad \widetilde{\beta} = -2\alpha + \beta + 1, \quad \widetilde{a} = -\frac{a-3}{2} - \frac{3\sqrt{b}}{4}, \quad \widetilde{b} = \big(-a + 1 + \frac{\sqrt{b}}{2}\big)^2,$$
$$\widetilde{y}_{2/3}(x,a,b) = y_{2/3}(x,\widetilde{a},\widetilde{b}), \quad \widetilde{y}_2(x,a,b,c) = y_-\big(x,\widetilde{a},\widetilde{\beta},\frac{4c}{a-1+\frac{\beta}{2}}\big),$$
$$\widetilde{y}_-(x,a,\beta,c) = y_2\big(x,\widetilde{a},\widetilde{b},\frac{1}{4}c(-a+1-\frac{\beta}{2})\big), \quad \widetilde{y}_+(x,a,\beta,c) = y_+\big(x,\widetilde{a},\widetilde{\beta},\frac{c}{\beta}(-a+1+\frac{\beta}{2})\big),$$
$$\widetilde{y}_0(x,a,c) = y_2\big(x,\widetilde{a},\widetilde{\beta},-\frac{1}{4}c(a-1)\big)$$

The Bäcklund transformation actions not preserving the Stokes multipliers are given by:

$\widehat{R}^{\infty}_{+}$:

$$2\widetilde{\alpha}-\widetilde{\beta}=\beta+1,\quad \widetilde{\beta}=-2\alpha-1,\quad \widetilde{a}=-\frac{a-1}{2}+\frac{3\sqrt{b}}{4},\quad \widetilde{b}=\big(-a-1-\frac{\sqrt{b}}{2}\big)^2,$$
$$\widetilde{y}_{2/3}(x,a,b)=y_{2/3}(x,\widetilde{a},\widetilde{b}),\quad \widetilde{y}_2(x,a,b,c)=y_-\big(x,\widetilde{a},\widetilde{\beta},-\frac{1}{4}c(-a-1-\frac{\beta}{2})\big),$$
$$\widetilde{y}_-(x,a,\beta,c)=y_+(x,\widetilde{a},\widetilde{\beta},\frac{4c}{\beta}),\quad \widetilde{y}_+(x,a,\beta,c)=y_2(x,\widetilde{a},\widetilde{b},-c);$$

$\widehat{R}^{\infty}_{-}$:

$$2\widetilde{\alpha}-\widetilde{\beta}=\beta-1,\quad \widetilde{\beta}=-2\alpha+1,\quad \widetilde{a}=-\frac{a+1}{2}+\frac{3\sqrt{b}}{4},\quad \widetilde{b}=\big(-a+1-\frac{\sqrt{b}}{2}\big)^2,$$
$$\widetilde{y}_{2/3}(x,a,b)=y_{2/3}(x,\widetilde{a},\widetilde{b}),\quad \widetilde{y}_2(x,a,b,c)=y_-(x,\widetilde{a},\widetilde{\beta},-c),$$
$$\widetilde{y}_-(x,a,\beta,c)=y_+\big(x,\widetilde{a},\widetilde{\beta},-\frac{1}{4}c(-a+1-\frac{\beta}{2})\big),\quad \widetilde{y}_+(x,a,\beta,c)=y_2\big(x,\widetilde{a},\widetilde{b},\frac{4c}{\beta}\big);$$

$\widehat{R}^{0}_{+}$:

$$2\widetilde{\alpha}-\widetilde{\beta}=\beta-1,\quad \widetilde{\beta}=-2\alpha-1,\quad \widetilde{a}=-\frac{a+3}{2}+\frac{3\sqrt{b}}{4},\quad \widetilde{b}=\big(-a-1-\frac{\sqrt{b}}{2}\big)^2,$$
$$\widetilde{y}_{2/3}(x,a,b)=y_{2/3}(x,\widetilde{a},\widetilde{b}),\quad \widetilde{y}_2(x,a,b,c)=y_-\big(x,\widetilde{a},\widetilde{\beta},\frac{1}{16}c(-a-1+\frac{\beta}{2})(a+1+\frac{\beta}{2})\big),$$
$$\widetilde{y}_-(x,a,\beta,c)=y_+(x,\widetilde{a},\widetilde{\beta},c),\quad \widetilde{y}_+(x,a,\beta,c)=y_2\Big(x,\widetilde{a},\widetilde{b},\frac{16c}{\beta(a+1-\frac{\beta}{2})}\Big);$$

$\widehat{R}^{0}_{-}$:

$$2\widetilde{\alpha}-\widetilde{\beta}=\beta+1,\quad \widetilde{\beta}=-2\alpha+1,\quad \widetilde{a}=-\frac{a-3}{2}+\frac{3\sqrt{b}}{4},\quad \widetilde{b}=\big(-a+1-\frac{\sqrt{b}}{2}\big)^2,$$
$$\widetilde{y}_{2/3}(x,a,b)=y_{2/3}(x,\widetilde{a},\widetilde{b}),\quad \widetilde{y}_2(x,a,b,c)=y_-(x,\widetilde{a},\widetilde{\beta},\frac{4c}{a-1-\frac{\beta}{2}}),$$
$$\widetilde{y}_-(x,a,\beta,c)=y_+\big(x,\widetilde{a},\widetilde{\beta},-\frac{c}{\beta}(-a+1-\frac{\beta}{2})\big),\quad \widetilde{y}_+(x,a,\beta,c)=y_2\big(x,\widetilde{a},\widetilde{b},\frac{1}{4}c(-a+1+\frac{\beta}{2})\big);$$

Results of this subsection are obtained with help of computing MATHEMATICA package and are used for getting some of the connection formulae below.

Asymptotics. Connection formulae.

In the cited paper [1], the case of more deep degeneration $s_{2+n}=1+s_{1+n}s_{3+n}=0$ is omitted. Here, we give the complete description of the special situation.

Note here that these equations with the equation of the monodromy surface (eq. (18) of ref. [1]) implies the restriction

$$\cos\pi\big(\alpha-\frac{\beta}{2}\big)\sin\pi\frac{\beta}{2}=0,$$

i.e. β is even or $2\alpha-\beta$ is odd. Theorem 1 below is obtained in the paper [4].

Theorem 1. *If $x \to \infty$,*

$$\arg(x) \in \Big(-\frac{\pi}{4} + \frac{\pi}{2}n; \frac{\pi}{4} + \frac{\pi}{2}n\Big), \quad n \in \mathbb{Z}, \tag{9}$$

and

$$s_{2+n} = s_{1+n} + s_{3+n} = 0, \quad \beta = 0, \quad \alpha + \frac{1}{2} \notin \mathbb{N}, \tag{10}$$

then the corresponding solution $y(x)$ of the fourth Painlevé equation P4 *possesses the following asymptotic behavior:*

$$\begin{aligned} y = (-1)^n \frac{s_{1+n}s_{4+n}}{\pi^{3/2}} e^{-i\pi(1+n)(-1)^n\alpha}\Gamma\Big(\frac{1}{2} - (-1)^n\alpha\Big)2^{(-1)^n\alpha-\frac{3}{2}}\times \\ \times \exp\big\{-(-1)^n x^2 + \big(2(-1)^n\alpha - 1\big)\ln x\big\}\big(1 + \mathcal{O}(x^{-1})\big). \end{aligned} \tag{11}$$

Next Theorem 2 follows from Theorem 1 after applying the Bäcklund transformations (4a)–(4d).

Theorem 2. *Let $x \to \infty$,*

$$\arg(x) \in \Big(-\frac{\pi}{4} + \frac{\pi}{2}n; \frac{\pi}{4} + \frac{\pi}{2}n\Big), \quad n \in \mathbb{Z}, \tag{12}$$

and

$$s_{2+n} = s_{1+n} + s_{3+n} = 0. \tag{13}$$

(i) *Let*

$$(-1)^n\beta = 2k, \quad k \in \mathbb{Z}_+, \quad (-1)^n\alpha + \frac{1}{2} - k \notin \mathbb{N}, \tag{14}$$

then the corresponding solution $y(x)$ of the fourth Painlevé equation P4 *possesses the following asymptotic behavior:*

$$\begin{aligned} y = y_+(x, a, \beta, c) = \frac{\beta}{2x} + \frac{\beta(2\alpha - \frac{3\beta}{2})}{4x^3} + \mathcal{O}\big(\frac{1}{x^5}\big) + \\ +(-1)^n \frac{s_{1+n}s_{4+n}}{\pi^{3/2}} e^{-i\pi(n+1)(-1)^n\alpha}\,\Gamma\big(\frac{1}{2} + k - (-1)^n\alpha\big) \times 2^{(-1)^n\alpha-\frac{3}{2}-2k}\,k\times \\ \times \exp\big\{-(-1)^n x^2 + \big(2(-1)^n\alpha - 1 - 4k\big)\ln x\big\}\big(1 + \mathcal{O}(x^{-2})\big); \end{aligned} \tag{15}$$

(ii) *Let*

$$(-1)^n\beta = -2k, \quad k \in \mathbb{Z}_+, \quad (-1)^n\alpha + \frac{1}{2} \notin \mathbb{N}, \tag{16}$$

then the corresponding solution $y(x)$ of the fourth Painlevé equation P4 *possesses the following asymptotic behavior:*

$$\begin{aligned} y = y_-(x, a, \beta, c) = -\frac{\beta}{2x} - \frac{\beta(2\alpha + \frac{\beta}{2})}{4x^3} + \mathcal{O}\big(\frac{1}{x^5}\big) + \\ +(-1)^{n+k+nk} \frac{s_{1+n}s_{4+n}}{\pi^{3/2}} e^{-i\pi(n+1)(-1)^n\alpha}\,\Gamma\big(\frac{1}{2} - (-1)^n\alpha\big)\times \end{aligned} \tag{17}$$

$$\times 2^{(-1)^n\alpha-\frac{3}{2}-k}\, k \times \exp\bigl\{-(-1)^n x^2 + \bigl(2(-1)^n\alpha - 1 - 2k\bigr)\ln x\bigr\}\bigl(1+\mathcal{O}(x^{-2})\bigr);$$

(iii) *Let*

$$(-1)^n(2\alpha-\beta) = 2l-1, \quad l\in\mathbb{N}, \quad 1-l-(-1)^n\frac{\beta}{2}\notin\mathbb{N}, \tag{18}$$

then the corresponding solution $y(x)$ *of the fourth Painlevé equation* P4 *possesses the following asymptotic behavior:*

$$y = y_2(x,a,b,c) = -2x - \frac{2\alpha-\frac{\beta}{2}}{x} + \mathcal{O}(\frac{1}{x^3}) -$$

$$-\frac{s_{1+n}s_{4+n}}{\pi^{3/2}}\, e^{i\frac{\pi}{2}(n+1)(-1)^n\beta + i\frac{\pi}{2}(1-n)}\,\Gamma\bigl(l+(-1)^n\frac{\beta}{2}\bigr)\times \tag{19}$$

$$\times 2^{-(-1)^n\frac{\beta}{2}-2l}\,(l-1)\times\exp\bigl\{-(-1)^n x^2 + \bigl(-(-1)^n\beta + 2 - 4l\bigr)\ln x\bigr\}\bigl(1+\mathcal{O}(x^{-2})\bigr);$$

(iv) *Let*

$$(-1)^n(2\alpha-\beta) = 1-2l, \quad l\in\mathbb{N}, \quad -(-1)^n\frac{\beta}{2}\notin\mathbb{N}, \tag{20}$$

then the corresponding solution $y(x)$ *of the fourth Painlevé equation* P4 *possesses the following asymptotic behavior:*

$$y = y_+(x,a,\beta,c) = \frac{\beta}{2x} + \frac{\beta(2\alpha-\frac{3\beta}{2})}{4x^3} + \mathcal{O}(\frac{1}{x^5}) +$$

$$+(-1)^{l(n-1)}\frac{s_{1+n}s_{4+n}}{\pi^{3/2}}\, e^{i\frac{\pi}{2}(n+1)(-1)^n\beta + i\frac{\pi}{2}(n-1)}\,\Gamma\bigl(1+(-1)^n\frac{\beta}{2}\bigr)\times \tag{21}$$

$$\times 2^{-(-1)^n\frac{\beta}{2}-1-l}\,(l-1)\times\exp\bigl\{-(-1)^n x^2 + \bigl(-(-1)^n\beta - 2l\bigr)\ln x\bigr\}\bigl(1+\mathcal{O}(x^{-2})\bigr).$$

The cases excluded from the Theorems 1 and 2 above correspond to the special points (see formula (19) of ref. [1]) of the monodromy surface (18) of [1] and to the so-called classical solutions of P4.

Note for the following that the equation

$$s_{1+n} + s_{3+n} + s_{1+n}s_{2+n}s_{3+n} = 0 \tag{22}$$

with the equation of the monodromy surface ((18), [1]) and the condition

$$\alpha - \frac{1}{2} \in \mathbb{Z}$$

together imply

$$1 + s_{2+n}s_{3+n} = e^{i\pi(-1)^n\beta},$$

$$1 + s_{1+n}s_{2+n} = e^{-i\pi(-1)^n\beta},$$

$$s_{3+n} + s_{1+n}e^{i\pi(-1)^n\beta} = 0.$$

The following assertion deals with the special case.

Theorem 3. *Let* $x \to \infty$,

$$\arg(x) \in \big(-\frac{\pi}{4} + \frac{\pi}{2}n; \frac{\pi}{4} + \frac{\pi}{2}n\big), \quad n \in \mathbb{Z}, \tag{23}$$

and

$$s_{2+n} \neq 0, \quad s_{1+n} + s_{3+n} + s_{1+n}s_{2+n}s_{3+n} = 0; \tag{24}$$

(i) *if*

$$(-1)^n \alpha = \frac{1}{2} - k, \quad k \in \mathbb{N}, \tag{25}$$

then

$$y = y_-(x, a, \beta, c) = -\frac{\beta}{2x} + \mathcal{O}\big(\frac{1}{x^3}\big) -$$

$$-e^{i\frac{\pi}{2}n}(-1)^{k(1-n)}\Big(\frac{s_{4+n}}{s_{2+n}} + e^{-i\pi(-1)^n\beta}\Big)\frac{e^{-i\frac{\pi}{2}(-1)^n\beta(n+2)}}{\sqrt{\pi}\,\Gamma\big((-1)^n\frac{\beta}{2}\big)}\times$$

$$\times 2^{-k+(-1)^n\frac{\beta}{2}}\,(k-1) \times \exp\big\{-(-1)^n x^2 + \big(-2k + (-1)^n\beta\big)\ln x\big\}\Big(1 + \mathcal{O}\big(\frac{1}{x^2}\big)\Big); \tag{26}$$

(ii) *if*

$$(-1)^n \alpha = -\frac{1}{2} + l, \quad l \in \mathbb{N}, \tag{27}$$

then

$$y = y_2(x, a, b, c) = -2x + \mathcal{O}\big(\frac{1}{x}\big) +$$

$$+e^{-i\frac{\pi}{2}n}(-1)^l\Big(\frac{s_{4+n}}{s_{2+n}} + e^{-i\pi(-1)^n\beta}\Big)\frac{e^{-i\frac{\pi}{2}(-1)^n\beta(n+2)}}{\sqrt{\pi}\,\Gamma\big((-1)^n\frac{\beta}{2} + 1 - l\big)}\times$$

$$\times 2^{-2l+1+(-1)^n\frac{\beta}{2}}\,(l-1) \times \exp\big\{-(-1)^n x^2 + \big(-4l + 2 + (-1)^n\beta\big)\ln x\big\}\Big(1 + \mathcal{O}\big(\frac{1}{x^2}\big)\Big). \tag{28}$$

Acknowledgments. The work was partially supported by the Russian Foundation for Fundamental Investigations under grant number 96-01-00668.

References

1 A. A. Kapaev, "Global asymptotics of the fourth Painlevé transcendent,", Preprint, P–6–96.
2 N. A. Lukashevich, "Theory of the fourth Painlevé equation," *Diff. Eqns.*, **3**, 395-399 (1967).
3 A. V. Kitaev, "Asymptotic description of the fourth Painleve equation solutions on the Stokes rays analogies," *Zap. Nauchn. Semin. LOMI*, **169**, 84 (1988).
4 A. R. Its and A. A. Kapaev, "Connection formulae for Clarkson–McLeod solution of the fourth Painlevé equation," *Proc. Royal Soc. London* (to appear).