

Connection formulae
for the degenerated asymptotic solutions
of the fourth Painlevé equation

Andrei A. Kapaev

*St. Petersburg Department of Steklov Mathematical Institute
of Russian Academy of Sciences,
Fontanka 27, St. Petersburg, 191011, Russia*

Abstract

All possible 1-parametric classical and transcendent degenerated solutions of the fourth Painlevé equation with the corresponding connection formulae of the asymptotic parameters are described.

1 Introduction

We consider the general case of the fourth Painlevé equation P4 [1]

$$y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - a)y - \frac{b}{2y}, \quad (1)$$

various physical applications of which are presented in the papers [2]–[6].

Among the most important results in the P4 theory obtained without any use of inverse problem method, we mention the article of Lukashevich [7], where the basic Bäcklund transformations and rational solutions of P4 are constructed, the papers of Airault [8], Gromak and Lukashevich [9], who find the Riccati equation for the classical 1-solutions of P4, the works of Umemura and Watanabe [10] and Gromak [11] on irreducibility of the general fourth Painlevé transcendent. Survey of these and many other results is given in the paper [12].

The relation of the Painlevé equation theory with the problem of the isomonodromy deformations of the Fuchsian equations discovered in classical works of Fuchs [13], Schlesinger [14] and Garnier [15], allows Flaschka and Newell [16], Jimbo and Miwa [17] and Its and Novokshenov [18] to develop an inverse problem method for investigation of these equations, which is applied to P4 in articles of Fokas, Mugan, Ablowitz and Zhou [19, 20]. The authors parameterize the fourth Painlevé transcendent set via the monodromy data of the associated linear system of Jimbo and Miwa [17], find some Bäcklund transformations, describe the proper Riemann-Hilbert problem as some factorization problem for a piece-wise holomorphic function and reduce it to some system of integral Fredholm equations. The last allows them to prove both the Riemann-Hilbert problem solvability in the case of general position and the classical theorem of meromorphy of the Painlevé function of the fourth kind.

Using the fact revealed by Ablowitz, Ramani and Segur [2] that P4 describes the similarity reduction of the Derivative Nonlinear Schrödinger (DNS) equation, Kitaev [21] finds an alternative to [17] Lax pair by use of reduction of the Lax pair for DNS of Kaup and Newell [22]. Later, in [23], he describes the Schlesinger transformations of this linear system and the corresponding Bäcklund transformations of the fourth Painlevé transcendent, one of them coincides with the transformation found by Lukashevich [7]

and other looks new but, as it is shown in [12], is a superposition of the transformations of Lukashovich. Later, in ref. [24], Milne, Clarkson and Bassom repeat some of the calculations of Kitaev and evaluate the alternative Lax pair (in less symmetric form, though) and one of the Bäcklund transformations for P4. They also find the monodromy data for some simplest classical solutions of P4. In the work [26], the case of general 2-parametric Painlevé function is described asymptotically by use of isomonodromy method for any $\arg x$, $|x| \rightarrow \infty$, including elliptic and trigonometric asymptotic solutions with the corresponding connection formulae.

Bassom, Clarkson, Hicks and McLeod, in their paper [25], begin intensive investigation of the classical solutions of the fourth Painlevé equation. In particular, they get the connection formulae for some classical solutions of the nonlinear harmonic oscillator related to P4 (1) with $b = 0$ and integer a . Some ideas of the paper are developed further in ref. [12], where huge tables of exact solutions (rational and classical) of P4 accompanied by the numerical calculations and pictures are presented. All the solutions are obtained by applying the Bäcklund transformations to some “seed” rational and classical solutions. However, all these explicit results are almost useless for asymptotic investigation of the classical solutions and the connection formulae evaluation.

The classical solutions expressible in terms of the parabolic cylinder functions are the special cases of the so-called degenerated (in other terms, instanton, separatrix or truncated) solutions. Mostly, these solutions have the form of some asymptotically algebraic background satisfying the Painlevé equation with some additive exponentially small term depending on the initial data. Besides the classical solutions, there exist other degenerated Painlevé transcendents. In ref. [26], some of them are described as limiting cases of the transcendent 2-parametric solutions of general positions. Other transcendent degenerated solution of P4 ($b = 0$) exponentially decreasing at $+\infty$ is investigated by Its and Kapaev in ref. [27] via the Riemann-Hilbert problem method.

The main idea of the present paper consists in application of the Bäcklund transformations to the formal asymptotic solutions. Since the set of degenerated asymptotics is invariant under action of the Bäcklund transformation group and there is a finite number of essentially different degenerated asymptotic ansatzes, the action of any Bäcklund transformation can be effectively described as the superposition of permutation of the

asymptotic ansatzes and some change of the asymptotic parameters involved. This procedure allows us to show that the simplest of the classical solutions satisfying the Riccati equations, the decreasing asymptotic solution from [27] and the degenerated asymptotics found in [26], affected by the Bäcklund transformation chain, yield all the degenerated solutions of the fourth Painlevé equation and to get the complete description of any degenerated Painlevé function with the connection formulae.

In the section 2, we describe the fourth Painlevé equation as the equation of the isomonodromy deformation for the alternative linear differential system found by Kitaev and give the proper parametrization of the Painlevé transcendent set via the points of the monodromy manifold. The section 3 contains description of the group of the Bäcklund transformations of P4 generated by the group of the Schlesinger transformations for the corresponding linear system. The next section 4 gives some information on the simplest classical solutions of P4, their asymptotics and coordinates on the monodromy manifold. In the section 5, we consider the set of the formal asymptotic solutions of P4 of the separatrix class and the action of the Bäcklund transformation group on this invariant set. Using the results, we give the complete asymptotic description of the classical Painlevé functions with the connection formulae. In the section 6, we describe the transcendent degenerated solutions of P4 with the corresponding connection formulae of the asymptotic parameters with the monodromy data of the associated linear system.

2 P4 as a monodromy preserving deformation

Let us consider the Lax pair for P4 (1) of Kitaev [21], alternative to the pair found by Jimbo and Miwa [17]:

$$\begin{aligned} \frac{\partial \Psi}{\partial \lambda} = & \left\{ \left(\frac{1}{2} \lambda^3 + \lambda(x + uv) + \frac{\alpha}{\lambda} \right) \sigma_3 + i(\lambda^2 u + 2xu + u') \sigma_+ + \right. \\ & \left. + i(\lambda^2 v + 2xv - v') \sigma_- \right\} \Psi, \end{aligned} \quad (2)$$

$$\frac{\partial \Psi}{\partial x} = \left\{ \left(\frac{1}{2} \lambda^2 + uv \right) \sigma_3 + i\lambda u \sigma_+ + i\lambda v \sigma_- \right\} \Psi, \quad (3)$$

where Ψ is some 2×2 matrix valued function of the complex variable λ depending on the complex x , u , v , α , and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}.$$

Compatibility of the equations (2), (3) implies that α does not depend on x , while u and v are the functions of x satisfying the system of differential equations

$$\begin{aligned} u'' &= -(1 + 2\alpha)u - 2xu' + 4xu^2v + 2u'uv, \\ v'' &= (1 - 2\alpha)v + 2xv' + 4xuv^2 - 2uv'v. \end{aligned} \quad (4)$$

In particular, the system (4) implies the constancy of the combination

$$\beta = u'v - uv' + 2xuv - (uv)^2 \equiv \text{const}, \quad (5)$$

and yields the product

$$y = uv \quad (6)$$

satisfies the equation P4 (1)

$$y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2\left(x^2 - 2\alpha + \frac{\beta}{2}\right)y - \frac{\beta^2}{2y}, \quad (7)$$

with the parameters

$$a = 2\alpha - \frac{\beta}{2}, \quad b = \beta^2. \quad (8)$$

Remark: The relation (8) means that each Painlevé function is related to two linear systems (2) which differ from each other by the sign of parameter $\beta = \pm\sqrt{b}$ and the corresponding value of the parameter $\alpha = \frac{a}{2} \pm \frac{\sqrt{b}}{4}$.

The equation (2) has irregular singular point $\lambda = \infty$ and regular singular point $\lambda = 0$. One can introduce “canonical” solutions Ψ_k near infinity and Ψ^0 near the point zero. The canonical asymptotics near infinity are

$$\Psi_k(\lambda) = \left(I + \mathcal{O}(\lambda^{-1})\right) \exp\left\{\left(\frac{1}{8}\lambda^4 + \frac{1}{2}x\lambda^2 + (\alpha - \beta)\ln\lambda\right)\sigma_3\right\}, \quad k \in \mathbb{Z},$$

$$\lambda \rightarrow \infty, \quad \lambda \in \omega_k = \left\{\lambda \in \mathbb{C} : \arg \lambda \in \left(-\frac{3\pi}{8} + \frac{\pi}{4}k; \frac{\pi}{8} + \frac{\pi}{4}k\right)\right\},$$

moreover

$$\Psi_{k+1}(\lambda) = \Psi_k(\lambda)S_k, \quad S_{2k-1} = \begin{pmatrix} 1 & s_{2k-1} \\ 0 & 1 \end{pmatrix}, \quad S_{2k} = \begin{pmatrix} 1 & 0 \\ s_{2k} & 1 \end{pmatrix}.$$

The canonical solution near the point zero is described by

$$\Psi^0(\lambda) = \hat{\Psi}(\lambda)e^{\alpha \ln \lambda \sigma_3} P(\lambda), \quad (9)$$

where

$$P(\lambda) = \left(I + j_{\pm} \ln \lambda \cdot \sigma_{\pm} \right) \exp \left(\int^x uv \, dx \cdot \sigma_3 \right),$$

with the parameters j_{σ} , $\sigma \in \{+, -\}$, satisfying the triviality conditions

$$j_{\sigma} = 0 \quad \text{if} \quad \frac{1}{2} + \sigma\alpha \notin \mathbb{N}, \quad \text{i.e.} \quad \alpha \notin \left\{ \sigma\frac{1}{2}; \sigma\frac{3}{2}; \dots \right\}, \quad \sigma \in \{+, -\},$$

and $\hat{\Psi}(\lambda)$ holomorphic and invertible near the point zero. In fact, the parameters j_{\pm} can be expressed via the coefficients of the system (2). For example,

$$\text{if} \quad \alpha = \frac{1}{2}, \quad \text{then} \quad j_+ = i(2xu + u'), \quad (10)$$

$$\text{and if} \quad \alpha = -\frac{1}{2}, \quad \text{then} \quad j_- = i(2xv - v'). \quad (11)$$

Generally spoken, the solution $\Psi^0(\lambda)$ is not fixed by the asymptotics (9). Indeed,

- 1) the asymptotics (9) is defined up to a right multiplier of the form C^{σ_3} with the arbitrary constant C ;
- 2) if $2\alpha \in \mathbb{Z}$, one may add one of the columns of (9) multiplied by an arbitrary coefficient to another with preserving the asymptotics. It is equivalent to multiplying of (9) from the right by a triangular matrix with the unit diagonal. In the cases $\alpha \in \mathbb{Z}$ or $\frac{1}{2} \pm \alpha \in \mathbb{N}$ with $j_{\pm} \neq 0$, the arbitrariness can be eliminated by the additional condition

$$\sigma_3 \Psi^0(e^{i\pi} \lambda) \sigma_3 = \Psi^0(\lambda) M \quad (12)$$

with the jump matrix

$$M = \left(I + i\pi j_{\pm} e^{\mp 2 \int^x y(s) ds} \sigma_{\pm} \right) e^{i\pi \alpha \sigma_3}. \quad (13)$$

If $\alpha - \frac{1}{2} \in \mathbb{Z}$ and $j_{\pm} = 0$, the arbitrariness preserves.

There is also the symmetry property

$$\sigma_3 \Psi_{k+4}(e^{i\pi} \lambda) \sigma_3 = \Psi_k(\lambda) e^{i\pi(\alpha-\beta)\sigma_3}, \quad (14)$$

which yields

$$S_{k+4} = e^{-i\pi(\alpha-\beta)\sigma_3} \sigma_3 S_k \sigma_3 e^{i\pi(\alpha-\beta)\sigma_3}, \quad s_{k+4} = -s_k e^{(-1)^k 2\pi i(\alpha-\beta)}, \quad (15)$$

and, with (12) together, implies the so-called semi-cyclic relation

$$ES_1 S_2 S_3 S_4 = \sigma_3 M^{-1} E e^{i\pi(\alpha-\beta)\sigma_3} \sigma_3 \quad (16)$$

where E is the connection matrix

$$E = \Psi^0(\lambda)^{-1} \Psi_1(\lambda) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad \det E = 1. \quad (17)$$

The connection matrix is as determined by (17) as the solution Ψ^0 is determined, i.e. the matrix is fixed up to the left multiplier C^{σ_3} with the arbitrary constant C and, if $\frac{1}{2} \pm \alpha \in \mathbb{N}$, $j_{\pm} = 0$, the matrix may be multiplied by the arbitrary left upper (lower) triangular matrix.

The relation (16) can be considered as the system of four linear homogeneous equations for entries p, q, r, s , $ps - qr = 1$, of the connection matrix E . The condition of its solvability (i.e. triviality of the corresponding determinant) is equivalent to the equation

$$\left((1 + s_1 s_2)(1 + s_3 s_4) + s_1 s_4\right) e^{-i\pi(\alpha-\beta)} - (1 + s_2 s_3) e^{i\pi(\alpha-\beta)} = -2i \sin \pi \alpha, \quad (18)$$

which is called the monodromy surface, so that only three of the Stokes multipliers are independent from each other. The surface (18) has some special 1-dimensional submanifolds defined by the equations

$$\begin{aligned} \alpha &= \frac{1}{2} + n, \quad n \in \mathbb{Z}, \\ s_1 &= -s_3 e^{-i\pi\beta}, \quad s_4 = -s_2 e^{-i\pi\beta}, \quad 1 + s_2 s_3 = e^{i\pi\beta}. \end{aligned} \quad (19)$$

It can be shown, for the non-special points of the surface (18), the connection matrix E does not contain any essential free parameter, but for the special points (19), it does,

and this additional free parameter is the ratio of row components of this matrix E . Thus the manifold of the monodromy data for (2) is the surface (18) with \mathbb{CP}^1 pasted to each special point (19).

Note that the monodromy data including the Stokes matrices S_k or the Stokes multipliers s_k are functions of the coefficients of (2):

$$S_k = S_k(x, u, v, \alpha), \quad s_k = s_k(x, u, v, \alpha), \quad k \in \mathbb{Z}.$$

These functions possess the following symmetries:

$$s_k(x, -u, -v, \alpha) = -s_k(x, u, v, \alpha), \quad k \in \mathbb{Z}; \quad (20)$$

$$\bar{s}_{-k}(\bar{x}, \bar{u}, \bar{v}, \bar{\alpha}) = s_k(x, u, v, \alpha), \quad k \in \mathbb{Z}, \quad (21)$$

where the bar means the complex conjugation; the gauge symmetry

$$S_k(x, e^a u, e^{-a} v, \alpha) = e^{\frac{a}{2}\sigma_3} S_k(x, u, v, \alpha) e^{-\frac{a}{2}\sigma_3}, \quad a \in \mathbb{C}, \quad k \in \mathbb{Z}; \quad (22)$$

and the rotation symmetry

$$\begin{aligned} S_{k+n}(e^{i\frac{\pi}{2}n}x, \tau^n(e^{i\frac{\pi}{4}n}u, e^{i\frac{\pi}{4}n}v), e^{i\pi n}\alpha) = \\ = (\sigma_2)^n e^{-\frac{i\pi}{4}n(\alpha-\beta)\sigma_3} S_k(x, u, v, \alpha) e^{\frac{i\pi}{4}n(\alpha-\beta)\sigma_3} (\sigma_2)^n, \quad k, n \in \mathbb{Z}, \end{aligned} \quad (23)$$

where τ is permutation $\tau(u, v) = (v, u)$. The symmetry (23) can be treated as a Bäcklund transformation in the following sense: if some function $y(x) = f(x, \alpha, \beta, \{s_k\})$ describes the Painlevé function then the function

$$\tilde{y}(\tilde{x}) = e^{i\frac{\pi}{2}n} f\left(e^{-i\frac{\pi}{2}n}\tilde{x}, (-1)^n\tilde{\alpha}, (-1)^n\tilde{\beta}, \{(-1)^n e^{i\frac{\pi}{2}(-1)^k n(\alpha-\beta)} s_{k+n}\}\right) \quad (24)$$

describes another solution of P4 of the new variable $\tilde{x} = e^{i\frac{\pi}{2}n}x$ and the new parameters $\tilde{\alpha} = (-1)^n\alpha$, $\tilde{\beta} = (-1)^n\beta$. The symmetry (21) may be treated in the similar way.

The linear system (2) maps a set of the coefficients x, u, v, u', v' onto a manifold of the monodromy data described above. So, the general (not special) points of the 3-d complex monodromy surface (18) are in one-to-one correspondence with the 3-d complex set of the functions u, v, u', v' depending on x in accord with (4), (5). At the special points (19), the points of locally 2-d complex monodromy manifold (one of the Stokes

multipliers and a ratio of the connection matrix entries) parameterize 2-d complex set in the space of the functions u, v, u', v' with the restriction (5) and equation $j_{\pm} = 0$ together. (Note the proposed in [24] parametrization of 2-d set of the Painlevé functions y, y' by the points of 4-d monodromy manifold is less convenient.)

The equation (22) means the fourth Painlevé transcendent $y = uv$ is invariant about the gauge transformation in contrast to the coefficients u, v , so that any solution of P4 corresponds to an orbit of the 1-parametric group of the gauge transformations of the monodromy data manifold.

3 Bäcklund and Schlesinger transformations.

In what follows, the crucial role is played by the group of the Bäcklund transformations of P4 (1) considered in refs. [7, 19, 28, 29, 30] generated by the group of Schlesinger transformations (see ref. [17]) of the linear system (2). The elementary Schlesinger transformation of the function Ψ preserving the monodromy data except the exponents of the formal monodromy α near the point zero or $\alpha - \beta$ near infinity is defined as $\tilde{\Psi} = R\Psi$ with the rational matrix function R of one of the following forms:

$$\begin{aligned} R_0^+ &= I + \frac{i}{\lambda} \cdot \frac{1 + 2\alpha}{2xv - v'} \sigma_+, & R_0^- &= I + \frac{i}{\lambda} \cdot \frac{1 - 2\alpha}{2xu + u'} \sigma_-, \\ R_{\infty}^+ &= \begin{pmatrix} \lambda & iu \\ \frac{i}{u} & 0 \end{pmatrix}, & R_{\infty}^- &= \begin{pmatrix} 0 & \frac{1}{iv} \\ -iv & \lambda \end{pmatrix}. \end{aligned} \quad (25)$$

The transformed connection coefficient $\tilde{\mathcal{A}}$ is described by (2) with coefficients u, u', v, v', α replaced by $\tilde{u}, \tilde{u}', \tilde{v}, \tilde{v}', \tilde{\alpha}$:

$$\text{if} \quad R = R_0^+:$$

$$\tilde{\alpha} = -\alpha - 1, \quad \tilde{\alpha} - \tilde{\beta} = \alpha - \beta,$$

$$\tilde{u} = u - \frac{1 + 2\alpha}{2xv - v'}, \quad \tilde{v} = v,$$

$$\tilde{u}' = u' - \frac{(1 + 2\alpha)v}{2xv - v'}(u + \tilde{u}), \quad \tilde{v}' = v',$$

$$\text{so that} \quad \tilde{y} = R_0^+[y] \equiv y + \frac{2(1 + 2\alpha)y}{y' - y^2 - 2xy - \beta}, \quad (26)$$

$$\begin{aligned}
&\text{if} && R = R_0^-: \\
&&& \tilde{\alpha} = -\alpha + 1, \quad \tilde{\alpha} - \tilde{\beta} = \alpha - \beta, \\
&&& \tilde{u} = u, \quad \tilde{v} = v + \frac{1 - 2\alpha}{2xu + u'}, \\
&&& \tilde{u}' = u', \quad \tilde{v}' = v' - \frac{(1 - 2\alpha)u}{2xu + u'}(v + \tilde{v}), \\
&\text{so that} && \tilde{y} = R_0^-[y] \equiv y + \frac{2(1 - 2\alpha)y}{y' + y^2 + 2xy + \beta}, \tag{27}
\end{aligned}$$

$$\begin{aligned}
&\text{if} && R = R_\infty^+: \\
&&& \tilde{\alpha} = -\alpha, \quad \tilde{\alpha} - \tilde{\beta} = \alpha - \beta + 1, \\
&&& \tilde{u} = u' - u^2v, \quad \tilde{v} = \frac{1}{u}, \\
&&& \tilde{u}' = -2xu' - u^2v' + 4xu^2v - (1 + 2\alpha)u, \quad \tilde{v}' = -\frac{u'}{u^2}, \\
&\text{so that} && \tilde{y} = R_\infty^+[y] \equiv \frac{y'}{2y} - \frac{1}{2}y - x + \frac{\beta}{2y}, \tag{28}
\end{aligned}$$

$$\begin{aligned}
&\text{if} && R = R_\infty^-: \\
&&& \tilde{\alpha} = -\alpha, \quad \tilde{\alpha} - \tilde{\beta} = \alpha - \beta - 1, \\
&&& \tilde{u} = \frac{1}{v}, \quad \tilde{v} = -v' - uv^2, \\
&&& \tilde{u}' = -\frac{v'}{v^2}, \quad \tilde{v}' = -2xv' - u'v^2 - 4xuv^2 - (1 - 2\alpha)v, \\
&\text{so that} && \tilde{y} = R_\infty^-[y] \equiv -\frac{y'}{2y} - \frac{1}{2}y - x + \frac{\beta}{2y}, \tag{29}
\end{aligned}$$

The relations (26)–(29) are called the Bäcklund transformations of the fourth Painlevé transcendent.

The Bäcklund transformations (28), (29) are found in ref. [7]. Two transformations (26), (27) with the corresponding Schlesinger transformations are obtained in ref. [23]. Other Bäcklund transformations found in refs. [8, 19, 28] can be described as some superpositions of these basic transformations.

4 Classical solutions

It is well known [7] that P4 has two series of rational 0-solutions existing for integer and only integer value of the parameter $a = 2\alpha - \frac{\beta}{2}$:

1) $-\frac{2}{3}x + \frac{P_{n-1}(x)}{Q_n(x)}$ generated by the actions of the Bäcklund transformations on the “seed” solution of (1) $y = -\frac{2}{3}x$ for $a = 0$, $b = \frac{4}{9}$. Each of the rational solutions corresponds to the Stokes multipliers satisfying the conditions

$$1 + s_k s_{k+1} = 0, \quad k \in \mathbb{Z}. \quad (30)$$

These rational solutions are some limiting cases of 2-parametric transcendent Painlevé functions [26].

2) Other family of rational solutions described in ref. [7, 9, 28] is a collection of two forms: a) $\frac{P_{n-1}(x)}{Q_n(x)}$, and b) $-2x + \frac{P_{n-1}(x)}{Q_n(x)}$. In accord with ref. [7], all these solutions can be generated from the solution $y_0 = -2x$, existing for $a = 0$, $b = 4$ (i.e., $\beta = \pm 2$, $\alpha = \pm \frac{1}{2}$). The “seed” solution $y = -2x$ corresponds to the Stokes multipliers

$$s_{2k-1} = 0, \quad s_{2k} + s_{2k+2} = 0, \quad k \in \mathbb{Z}, \quad \text{if } \alpha = \frac{1}{2}, \quad \beta = 2, \quad (31)$$

or

$$s_{2k} = 0, \quad s_{2k-1} + s_{2k+1} = 0, \quad k \in \mathbb{Z}, \quad \text{if } \alpha = -\frac{1}{2}, \quad \beta = -2. \quad (32)$$

Its Bäcklund transformations are characterized by the same Stokes multiplier values but some other α and β . Note also, all the rational functions are some limiting cases of the classical solutions of P4 (see below). The corresponding Ψ function can be expressed via the Weber-Hermite functions.

The simplest of the so-called classical solutions of P4 satisfy the Riccati equation [8, 9] $y' = \sigma(y^2 + 2xy) + q$, where $\sigma^2 = 1$, so the parameters of (1) are $a = -\sigma(\frac{q}{2} + 1)$, $b = q^2$ (so, for $q = -2$, $a = 0$, there are two different kinds of the simplest classical solutions corresponding to both values of $\sigma = \pm 1$; for details, see also [10]). In terms of the monodromy data, the classical solutions correspond to the special linear submanifolds of the monodromy manifold. In fact, there are two kinds of such submanifolds (cf. [24]). The first kind of the submanifolds encloses all the projective complex spaces \mathbb{CP}^1 pasted to each of the special points (19) of the monodromy surface (18) and in terms of the

parameters of the λ -equation corresponding to the trivial values $j_{\pm} = 0$. The second kind of the submanifolds is distinguished by the system of equations $s_{2k-1} = 0$ or $s_{2k} = 0$.

1) Let us consider the special points corresponding to $\alpha = \frac{1}{2}$. As it is said above, these special points are characterized by the equation $j_+ = 0$, or $u' + 2xu = 0$ (10), that yields $u = Ce^{-x^2}$ with an arbitrary constant C . Now, the definition of β (5) gives the Riccati equation for the function v : $v' + uv^2 + \frac{\beta}{u} = 0$. Substitution $v = \frac{1}{u} \frac{z'}{z}$ produces the linear equation

$$z'' + 2xz' + \beta z = 0. \quad (33)$$

The Painlevé function (6) $y = uv = z'/z$ solves the Riccati equation

$$y' = -y^2 - 2xy - \beta. \quad (34)$$

As to the solution $\Psi(\lambda)$ of the equation (2), it can be expressed in terms of the Weber-Hermite functions.

2) For the opposite parameter value $\alpha = -\frac{1}{2}$, the special points are characterized by the equation $j_- = 0$, or $v' - 2xv = 0$ (11), so that $v = Ce^{x^2}$ with an arbitrary constant C . Another coefficient u satisfies the Riccati equation $u' - u^2v - \frac{\beta}{v} = 0$ which is linearized by the change $u = -\frac{1}{v} \frac{z'}{z}$:

$$z'' - 2xz' + \beta z = 0. \quad (35)$$

Now, $y = uv = -z'/z$ satisfies the Riccati equation

$$y' = y^2 + 2xy + \beta. \quad (36)$$

Similarly to the previous case, the equation (2) can be solved explicitly in terms of the parabolic cylinder functions.

3) Next linearization of the monodromy manifold takes place under lower triangular reduction of the λ -equation (2) obtained by means of the restriction $u \equiv 0$. The definition (5) gives immediately $\beta = 0$, while after (4), another coefficient v satisfies the linear equation

$$v'' - 2xv' + (2\alpha - 1)v = 0. \quad (37)$$

As easy to see, the triangular system (2) can be solved in quadratures and yields the lower triangular Ψ function and the lower triangularity of all the Stokes matrices, the

monodromy matrix and the connection matrix, so that $s_{2k-1} = 0$, $k \in \mathbb{Z}$, $j_+ = 0$, $p = s = 1$, $q = 0$, moreover the equation (18) holds identically while (16) reduces to the only equation $s_2 + s_4 = -r(1 + e^{2\pi i\alpha})$ if $\frac{1}{2} - \alpha \notin \mathbb{N}$, or $s_2 + s_4 = -i\pi j_- \exp(2 \int^x y dx)$ if $\frac{1}{2} - \alpha \in \mathbb{N}$. By definition (6), the case corresponds to the trivial Painlevé function $y \equiv 0$ which does not allow to apply the transformations (26)–(29). However, the described coefficients u , v can be affected by the Bäcklund transformations R_0^+ (26) and R_∞^- (29). Modified values are as follows:

$$R_0^+: \quad \tilde{\alpha} = -\alpha - 1, \quad \tilde{\beta} = -2\alpha - 1, \\ \tilde{u} = -\frac{1+2\alpha}{2xv-v'}, \quad \tilde{v} = v, \quad \tilde{y} = \tilde{u}\tilde{v} = -\frac{1+2\alpha}{2xv-v'}v,$$

so that

$$\tilde{y}' = \tilde{y}^2 + 2x\tilde{y} - \tilde{\beta}. \quad (38)$$

Another Bäcklund transformation R_∞^- gives $\tilde{\alpha} = -\alpha$, $\tilde{\beta} = -2\alpha + 1$, $\tilde{u} = \frac{1}{v}$, $\tilde{v} = -v'$, $\tilde{y} = \tilde{u}\tilde{v} = -\frac{v'}{v}$, and \tilde{y} satisfies the Riccati equation (38).

4) The upper-triangular reduction takes place if $v \equiv 0$, so that $\beta = 0$, $y \equiv 0$, $s_{2k} = 0$, $k \in \mathbb{Z}$, $j_- = 0$, $p = s = 1$, $r = 0$, while $s_1 + s_3 = -q(1 + e^{-2\pi i\alpha})$ if $\frac{1}{2} + \alpha \notin \mathbb{N}$, or $s_1 + s_3 = -i\pi j_+ \exp(-2 \int^x y dx)$ if $\frac{1}{2} + \alpha \in \mathbb{N}$. The coefficient function u satisfies the linear equation

$$u'' + 2xu' + (2\alpha + 1)u = 0. \quad (39)$$

The coefficients affected by the Bäcklund transformations R_0^- (27) and R_∞^+ (28) are as follows:

$$R_0^-: \quad \tilde{\alpha} = -\alpha + 1, \quad \tilde{\beta} = -2\alpha + 1, \\ \tilde{u} = u, \quad \tilde{v} = \frac{1-2\alpha}{2xu+u'}, \quad \tilde{y} = \tilde{u}\tilde{v} = \frac{1-2\alpha}{2xu+u'}u,$$

so that

$$\tilde{y}' = -\tilde{y}^2 - 2x\tilde{y} + \tilde{\beta}. \quad (40)$$

Another transformation R_∞^+ yields $\tilde{\alpha} = -\alpha$, $\tilde{\beta} = -2\alpha - 1$, $\tilde{u} = u'$, $\tilde{v} = \frac{1}{u}$, $\tilde{y} = \tilde{u}\tilde{v} = \frac{u'}{u}$, and \tilde{y} satisfies the same Riccati equation (40).

Note that the solutions of the equations (34) and (36) are related with each other. Namely, if y solves the Riccati equation (34) then its Bäcklund transformation R_∞^- reduced to $\tilde{y} = \beta/y$ solves the Riccati equation (36). Similarly, if $y = f(x, \beta)$ is a solution of

(34) then $\tilde{y} = \pm i f(\pm ix, -\beta)$ solves (36). The solutions of the Riccati equations (38) and (40) can be obtained via simple substitution in solutions of (36) and (34) the parameter $-\tilde{\beta}$ instead of β .

Let us describe the global asymptotics of the presented classical solutions. Because of the said above we can restrict ourselves in the case (34), when the substitution $y = z'/z$ yield the linear equation (33). As easy to see [9], the change

$$z = e^{-\frac{x^2}{2}} w \quad (41)$$

transform the equation (33) into the parabolic cylinder equation

$$w'' + (\beta - 1 - x^2)w = 0. \quad (42)$$

Using the results on the asymptotic properties of the Weber-Hermite functions presented in the reference book [31], we can introduce the vectors of fundamental solutions of the equation (42) $W_k = (w_1^{(k)}; w_2^{(k)})$

$$\begin{aligned} W_k &= \left(e^{\frac{x^2}{2} - \frac{\beta}{2} \ln x} (1 + \mathcal{O}(x^{-2})); e^{-\frac{x^2}{2} + (\frac{\beta}{2} - 1) \ln x} (1 + \mathcal{O}(x^{-2})) \right), \\ \arg x &\in \left(\frac{\pi}{2}(k-1); \frac{\pi}{2}k \right), \quad |x| \rightarrow \infty. \end{aligned} \quad (43)$$

The canonical vectors are related with each other by the Stokes matrices

$$W_{k+1}(x) = W_k(x) G_k, \quad (44)$$

$$\begin{aligned} G_{2k} &= \begin{pmatrix} 1 & 0 \\ g_{2k} & 1 \end{pmatrix}, \quad G_{2k-1} = \begin{pmatrix} 1 & g_{2k-1} \\ 0 & 1 \end{pmatrix}, \\ g_0 &= -i\sqrt{\pi} \frac{2^{\frac{\beta}{2}}}{\Gamma(\frac{\beta}{2})}, \quad g_1 = \sqrt{\pi} \frac{2^{1-\frac{\beta}{2}} e^{-i\pi(1-\frac{\beta}{2})}}{\Gamma(1-\frac{\beta}{2})}, \\ g_{2k+2} &= -g_{2k} e^{-i\pi\beta}, \quad g_{2k+1} = -g_{2k-1} e^{i\pi\beta}. \end{aligned}$$

In accord with (41), the corresponding fundamental vector of the solutions of (33), i.e. $Z_k = e^{-x^2/2} W_k$ yield the “generating” matrix for the function $y = z'/z$:

$$Z_k = \begin{pmatrix} z'_1 & z'_2 \\ z_1 & z_2 \end{pmatrix} = \begin{pmatrix} -\frac{\beta}{2} x^{-\frac{\beta}{2}-1} (1 + \mathcal{O}(x^{-2})) & -2e^{-x^2} x^{\frac{\beta}{2}} (1 + \mathcal{O}(x^{-2})) \\ x^{-\frac{\beta}{2}} (1 + \mathcal{O}(x^{-2})) & e^{-x^2} x^{\frac{\beta}{2}-1} (1 + \mathcal{O}(x^{-2})) \end{pmatrix}. \quad (45)$$

The classical solution of P4 satisfying the Riccati equation (34) $y' = -y^2 - 2xy - \beta$,

$$y = \frac{\nu_1 z'_1 + \nu_2 z'_2}{\nu_1 z_1 + \nu_2 z_2} \quad (46)$$

goes from (45) after multiplication in the column of constants $\nu_{1,2}$ and dividing the first entry to the second. As easy to see, this exact solution behaves asymptotically as $|x| \rightarrow \infty$, $\arg x \neq \frac{\pi}{4} + \frac{\pi}{2}n$, $n \in \mathbb{Z}$, like

$$y \sim -2x \quad \text{or} \quad y \sim -\frac{\beta}{2x} \quad (47)$$

In fact, these asymptotics are described by means of the asymptotic power series for z'_i/z_i and an exponentially small perturbation depending on the arbitrary parameter ν_1/ν_2 . (Along the excluded rays $\arg x = \frac{\pi}{4} + \frac{\pi}{2}n$, the ratio (46) is described asymptotically by some trigonometric tangent.) In more details, if $\alpha = \frac{1}{2}$, the classical solution of P4 (1) corresponding to the special point (19) satisfies the Riccati equation (34) $y' + y^2 + 2xy + \beta = 0$ and behaves asymptotically as follows:

i) $\arg x \in (-\frac{\pi}{4}; \frac{\pi}{4})$, $\nu_1 \neq 0$:

$$y = -\frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right) - 2 \left(\frac{\nu_2}{\nu_1} - \Theta(\arg x) g_0 \right) x^\beta e^{-x^2} \left(1 + \mathcal{O}(x^{-2})\right); \quad (48)$$

ii) $\arg x \in (\frac{\pi}{4}; \frac{3\pi}{4})$, $\nu_2 - \nu_1 g_0 \neq 0$:

$$y = -2x \left(1 + \mathcal{O}(x^{-2})\right) + 2 \left(\frac{\nu_1}{\nu_2 - \nu_1 g_0} - \Theta(\arg x - \frac{\pi}{2}) g_1 \right) x^{2-\beta} e^{x^2} \left(1 + \mathcal{O}(x^{-2})\right); \quad (49)$$

iii) $\arg x \in (\frac{3\pi}{4}; \frac{5\pi}{4})$, $\nu_1 e^{i\pi\beta} - \nu_2 g_1 \neq 0$:

$$y = -\frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right) - 2 \left(\frac{\nu_2 - \nu_1 g_0}{\nu_1 e^{i\pi\beta} - \nu_2 g_1} - \Theta(\arg x - \pi) g_2 \right) x^\beta e^{-x^2} \left(1 + \mathcal{O}(x^{-2})\right); \quad (50)$$

iv) $\arg x \in (\frac{5\pi}{4}; \frac{7\pi}{4})$, $\nu_2 \neq 0$:

$$y = -2x \left(1 + \mathcal{O}(x^{-2})\right) + 2 \left(\frac{\nu_1 e^{i\pi\beta} - \nu_2 g_1}{\nu_2 e^{-i\pi\beta}} - \Theta(\arg x - \frac{3\pi}{2}) g_3 \right) x^{2-\beta} e^{x^2} \left(1 + \mathcal{O}(x^{-2})\right), \quad (51)$$

where

$$\Theta(z) = \begin{cases} 0 & \text{if } z < 0, \\ \frac{1}{2} & \text{if } z = 0, \\ 1 & \text{if } z > 0. \end{cases}$$

Excluded values of the parameters correspond to the opposite algebraic terms (see (47)) and absence of any exponentially small perturbation. For example, if $\nu_1 = 0$, then $y = -2x(1 + \mathcal{O}(x^{-2}))$ for $\arg x \in (-\frac{\pi}{4}; \frac{\pi}{4})$.

On the rays $\arg x = \frac{\pi}{2}k$, $k \in \mathbb{Z}$, the asymptotic formulae (48)–(51) demonstrate the quasi-linear Stokes phenomenon imposed by the Stokes property of the Weber-Hermite functions which consists in a jump of the exponentially small term while the independent variable crosses the Stokes ray (see, e.g. [32]).

5 Degenerated asymptotic solutions and the Bäcklund transformations.

The results of the previous section on the asymptotics of the classical solutions give us a clue to look for the formal asymptotic solutions of the fourth Painlevé equation (1) in the form

$$\sum_{k=0}^{\infty} e^{\vartheta_k(x)} \sum_{n=0}^{\infty} a_{nk} x^{-n}, \quad (52)$$

with $\vartheta_k(x) = -kx^2 + b_k \ln x$ and some constants a_{nk} , b_k . The terms of the formal series can be evaluated recursively by use of any automatic system of analytic calculations. Several first terms of the formal asymptotic expansions below are calculated by use of MATHEMATICA 2.2 of Wolfram Research, Inc. (for what follows, it is enough to keep two of the successive exponentially small terms).

i) 0-parameter asymptotics of a background for 2-parameter oscillating asymptotic solution:

$$y_{2/3}(x, a, b) = -\frac{2x}{3} + \frac{a}{x} + \frac{-4 - 12a^2 + 9b}{16x^3} + \mathcal{O}\left(\frac{1}{x^5}\right); \quad (53)$$

ii) 1-parameter asymptotics approaching $-2x$:

$$\begin{aligned} y_2(x, a, b, c) = & -2x - \frac{a}{x} + \frac{4 + 12a^2 - b}{16x^3} + \frac{-44a - 36a^3 + 5ab}{32x^5} + \mathcal{O}\left(\frac{1}{x^7}\right) + \\ & + cx^{-2a} e^{-x^2} \left\{ 1 + \frac{-12 - 8a - 28a^2 + 3b}{32x^2} + \mathcal{O}\left(\frac{1}{x^4}\right) \right\} - \\ & - \frac{c^2}{2} x^{-1-4a} e^{-2x^2} \left\{ 1 + \frac{-20 - 24a - 28a^2 + 3b}{16x^2} + \mathcal{O}\left(\frac{1}{x^4}\right) \right\}; \end{aligned} \quad (54)$$

iii) 1-parameter decreasing asymptotics:

$$\begin{aligned}
y_{\pm}(x, a, \sqrt{b}, c) = & \pm \frac{\sqrt{b}}{2x} \pm \frac{(a \mp \sqrt{b})\sqrt{b}}{4x^3} \pm \\
& \pm \frac{\sqrt{b}(12 + 12a^2 \mp 32a\sqrt{b} + 17b)}{64x^5} + \mathcal{O}\left(\frac{1}{x^7}\right) + \\
& + cx^{-1+a \mp \frac{3\sqrt{b}}{2}} e^{-x^2} \left\{ 1 + \right. \\
& + \frac{1}{32x^2} (-12 + 16a - 4a^2 \mp 20\sqrt{b} \pm 24a\sqrt{b} - 15b) + \mathcal{O}\left(\frac{1}{x^4}\right) \left. \vphantom{\frac{1}{32x^2}} \right\} + \\
& + \frac{c^2}{2} x^{-3+2a \mp 3\sqrt{b}} e^{-2x^2} \left\{ 1 + \right. \\
& + \frac{-36 + 32a - 4a^2 \mp 36\sqrt{b} \pm 24a\sqrt{b} - 15b}{16x^2} + \mathcal{O}\left(\frac{1}{x^4}\right) \left. \vphantom{\frac{1}{16x^2}} \right\}. \tag{55}
\end{aligned}$$

The formal solutions $y_+(x, a, \sqrt{b}, c) = y_-(x, a, -\sqrt{b}, c)$ coincide with each other if $b = 0$. This asymptotic solution is denoted as y_0 and is given by

$$\begin{aligned}
y_0(x, a, c) = & cx^{-1+a} e^{-x^2} \left\{ 1 + \frac{-3 + 4a - a^2}{8x^2} + \mathcal{O}\left(\frac{1}{x^4}\right) \right\} + \\
& + \frac{c^2}{2} x^{-3+2a} e^{-2x^2} \left\{ 1 + \frac{-9 + 8a - a^2}{4x^2} + \mathcal{O}\left(\frac{1}{x^4}\right) \right\}. \tag{56}
\end{aligned}$$

It is readily seen that the Bäcklund transformation group generators (26)–(29) preserve the set of the formal asymptotic solutions (53)–(56). In particular,

$$\tilde{y}_{2/3}(x, a, b) \equiv R[y_{2/3}(x, a, b)] = y_{2/3}(x, \tilde{a}, \tilde{b})$$

for any Bäcklund transformation R . The general assertion is also pure algebraic and can be checked directly. Listed below actions of the Bäcklund transformations (26)–(29) are obtained by use of MATHEMATICA 2.2. Here, $b = \beta^2$, $a = 2\alpha - \frac{\beta}{2}$, $\tilde{y} = R[y]$:

$$\begin{aligned}
R_{\infty}^+: \quad & \tilde{\alpha} = -\alpha, \quad \tilde{\beta} = -2\alpha + \beta - 1, \\
& \tilde{y}_2(x, a, b, c) = y_-(x, \tilde{a}, \tilde{\beta}, \frac{c}{4}(2\alpha - \beta + 1)), \\
& \tilde{y}_-(x, a, \beta, c) = y_2(x, \tilde{a}, \tilde{b}, -c), \\
& \tilde{y}_+(x, a, \beta, c) = y_+(x, \tilde{a}, \tilde{\beta}, -\frac{4c}{\beta}), \\
& \tilde{y}_0(x, a, c) = y_2(x, \tilde{a}, \tilde{b}, -c); \tag{57}
\end{aligned}$$

$$\begin{aligned}
R_{\infty}^-: \quad & \tilde{\alpha} = -\alpha, \quad \tilde{\beta} = -2\alpha + \beta + 1, \\
& \tilde{y}_2(x, a, b, c) = y_-(x, \tilde{a}, \tilde{\beta}, -c), \\
& \tilde{y}_-(x, a, \beta, c) = y_2(x, \tilde{a}, \tilde{b}, -\frac{4c}{\beta}), \\
& \tilde{y}_+(x, a, \beta, c) = y_+(x, \tilde{a}, \tilde{\beta}, \frac{c}{4}(2\alpha - \beta - 1)), \\
& \tilde{y}_0(x, a, c) = y_+(x, \tilde{a}, \tilde{\beta}, \frac{c}{4}(2\alpha - 1)); \tag{58}
\end{aligned}$$

$$\begin{aligned}
R_0^+: \quad & \tilde{\alpha} = -\alpha - 1, \quad \tilde{\beta} = -2\alpha + \beta - 1, \\
& \tilde{y}_2(x, a, b, c) = y_-(x, \tilde{a}, \tilde{\beta}, -\frac{c}{16}(2\alpha - \beta + 1)(2\alpha + 1)), \\
& \tilde{y}_-(x, a, \beta, c) = y_2(x, \tilde{a}, \tilde{b}, -\frac{16c}{\beta(2\alpha + 1)}), \\
& \tilde{y}_+(x, a, \beta, c) = y_+(x, \tilde{a}, \tilde{\beta}, c), \\
& \tilde{y}_0(x, a, c) = y_+(x, \tilde{a}, \tilde{\beta}, c); \tag{59}
\end{aligned}$$

$$\begin{aligned}
R_0^-: \quad & \tilde{\alpha} = -\alpha + 1, \quad \tilde{\beta} = -2\alpha + \beta + 1, \\
& \tilde{y}_2(x, a, b, c) = y_-(x, \tilde{a}, \tilde{\beta}, \frac{4c}{2\alpha - 1}), \\
& \tilde{y}_-(x, a, \beta, c) = y_2(x, \tilde{a}, \tilde{b}, -\frac{c}{4}(2\alpha - 1)), \\
& \tilde{y}_+(x, a, \beta, c) = y_+(x, \tilde{a}, \tilde{\beta}, -\frac{c}{\beta}(2\alpha - \beta - 1)), \\
& \tilde{y}_0(x, a, c) = y_2(x, \tilde{a}, \tilde{b}, -\frac{c}{4}(2\alpha - 1)). \tag{60}
\end{aligned}$$

The invariance of the degenerated solution set allows us to introduce such a dependence of the multipliers c of the exponentially small terms on the parameters α and β to ensure their invariance about the Bäcklund transformations action. Let us provide the original parameters c of the asymptotic solutions y_t , $t \in \{2; +; -; 0\}$ (54)–(56) by the same indices t . Then

$$c_2(\alpha, \beta) = \frac{2^{-2\alpha + \frac{\beta}{2}}}{\Gamma(\frac{1}{2} - \alpha)\Gamma(\frac{1}{2} - \alpha + \frac{\beta}{2})} f_2(\alpha) g_2(\alpha - \frac{\beta}{2}), \tag{61}$$

$$c_-(\alpha, \beta) = \frac{2^{\alpha + \frac{\beta}{2}}}{\Gamma(\frac{1}{2} + \alpha)\Gamma(\frac{\beta}{2})} f_-(\alpha) g_-(\frac{\beta}{2}), \tag{62}$$

$$c_+(\alpha, \beta) = \frac{2^{\alpha-\beta}}{\Gamma(\frac{1}{2} + \alpha - \frac{\beta}{2})\Gamma(-\frac{\beta}{2})} f_+(\alpha - \frac{\beta}{2}) g_+(-\frac{\beta}{2}), \quad (63)$$

$$c_0(\alpha) = \frac{2^\alpha}{\Gamma(\frac{1}{2} + \alpha)} f_0(\alpha), \quad (64)$$

where the functions $f_s, g_s, s \in \{2; +; -; 0\}$, are some 1-periodic functions dependent on the initial data and $\arg x$. Now, we are ready to describe the asymptotic behavior of the classical solutions of P4.

For example, let us consider the classical solutions corresponding to the special points of the monodromy manifold (19) where $\alpha = \frac{1}{2} + n, n \in \mathbb{Z}_+$. Each solution of this series is the Bäcklund transformation of the basic solution satisfying the Riccati equation (34) $y' + y^2 + 2xy + \beta = 0$ of the form $(R_0^-(R_\infty^+))^n[y]$, since the superposition of the transformations R_∞^+ (57) and R_0^- (60) increases the value of the parameter α in 1 and preserves the parameter β . Starting from (48), we calculate the constant $f_-(\alpha)$ in (62) and obtain the asymptotics of the classical Painlevé function for $\arg x \in (-\frac{\pi}{4}; \frac{\pi}{4})$. Using the symmetry of the P4 equation (1) about rotation $x \mapsto ix, y \mapsto iy, \alpha \mapsto -\alpha, \beta \mapsto -\beta$, we transfer the asymptotics (49) to the sector $(-\frac{\pi}{4}; \frac{\pi}{4})$, obtain the product $f_2(p)g_2(q)$ in (61) and then rotate back. Similar calculations yield result for other sectors. Thus, the following assertion takes place:

Theorem 5.1 *For the parameter values*

$$\alpha = \frac{1}{2} + n, \quad n \in \mathbb{Z}_+, \quad \frac{\beta}{2} \notin \mathbb{Z}, \quad (65)$$

there exist the classical solutions described asymptotically by the equations:

$$\begin{aligned} i) \quad & \arg x \in (-\frac{\pi}{4}; \frac{\pi}{4}): \\ & y = -\frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right) + \mu_1 x^{2n+\beta} e^{-x^2} \left(1 + \mathcal{O}(x^{-2})\right), \end{aligned} \quad (66)$$

$$\mu_1 = -2(c + i\sqrt{\pi} \Theta(\arg x)) \frac{2^{n+\frac{\beta}{2}}}{n! \Gamma(\frac{\beta}{2})}, \quad c \in \mathbb{C};$$

$$\text{or} \quad y = -2x \left(1 + \mathcal{O}(x^{-2})\right), \quad \text{formally corresponding to} \quad c = \infty;$$

$$ii) \quad \arg x \in (\frac{\pi}{4}; \frac{3\pi}{4}):$$

$$\text{if} \quad c + i\sqrt{\pi} \neq 0,$$

$$\text{then} \quad y = -2x(1 + \mathcal{O}(x^{-2})) + \mu_2 x^{4n+2-\beta} e^{x^2} (1 + \mathcal{O}(x^{-2})), \quad (67)$$

$$\mu_2 = 2\sqrt{\pi} \left(\frac{i\sqrt{\pi}}{(c + i\sqrt{\pi})(e^{i\pi\beta} - 1)} + \Theta(\arg x - \frac{\pi}{2}) \right) e^{i\frac{\pi\beta}{2}} \frac{2^{2n+1-\frac{\beta}{2}}}{n! \Gamma(n+1-\frac{\beta}{2})},$$

$$\text{and if} \quad c + i\sqrt{\pi} = 0, \quad \text{then} \quad y = -\frac{\beta}{2x} (1 + \mathcal{O}(x^{-2}));$$

$$\text{iii)} \quad \arg x \in (\frac{3\pi}{4}; \frac{5\pi}{4}):$$

$$\text{if} \quad c(1 - e^{-i\pi\beta}) + i\sqrt{\pi} \neq 0,$$

$$\text{then} \quad y = -\frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})) + \mu_3 x^{2n+\beta} e^{-x^2} (1 + \mathcal{O}(x^{-2})), \quad (68)$$

$$\mu_3 = -2i\sqrt{\pi} \left(\frac{c + i\sqrt{\pi}}{c(1 - e^{-i\pi\beta}) + i\sqrt{\pi}} - \Theta(\arg x - \pi) \right) e^{-i\pi\beta} \frac{2^{n+\frac{\beta}{2}}}{n! \Gamma(\frac{\beta}{2})},$$

$$\text{and if} \quad c(1 - e^{-i\pi\beta}) + i\sqrt{\pi} = 0, \quad \text{then} \quad y = -2x(1 + \mathcal{O}(x^{-2}));$$

$$\text{iv)} \quad \arg x \in (\frac{5\pi}{4}; \frac{7\pi}{4}):$$

$$\text{if} \quad c \neq 0,$$

$$\text{then} \quad y = -2x(1 + \mathcal{O}(x^{-2})) + \mu_4 x^{4n+2-\beta} e^{x^2} (1 + \mathcal{O}(x^{-2})), \quad (69)$$

$$\mu_4 = 2\sqrt{\pi} \left(\frac{i\sqrt{\pi}}{c(1 - e^{-i\pi\beta})} + \Theta(\frac{3\pi}{2} - \arg x) \right) e^{i\frac{3\pi\beta}{2}} \frac{2^{2n+1-\frac{\beta}{2}}}{n! \Gamma(n+1-\frac{\beta}{2})},$$

$$\text{and if} \quad c = 0, \quad \text{then} \quad y = -\frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})).$$

To get the asymptotics of the classical Painlevé functions for $\alpha = -\frac{1}{2} - n$, $n \in \mathbb{Z}_+$, it is enough to substitute ix , iy , $-\alpha$, $-\beta$ instead of x , y , α and β , respectively, in all the expressions above.

In the very similar way, starting from (40) and taking into account that the combination of the Bäcklund transformations $R_\infty^\pm(R_\infty^\pm)$ changes the parameter value β in 2 and preserves α , we get the following assertions:

Theorem 5.2 *For the parameter values*

$$\alpha - \frac{\beta}{2} = \frac{1}{2} + n, \quad n \in \mathbb{Z}_+, \quad \alpha + \frac{1}{2} \notin \mathbb{Z}, \quad (70)$$

there exist the classical solutions described asymptotically by the equations:

$$i) \quad \arg x \in \left(-\frac{\pi}{4}; \frac{\pi}{4}\right):$$

$$y = \frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right) + \mu_1 x^{2n-\beta} e^{-x^2} \left(1 + \mathcal{O}(x^{-2})\right), \quad (71)$$

$$\mu_1 = -2(c + i\sqrt{\pi} \Theta(\arg x)) \frac{2^{n-\frac{\beta}{2}}}{n! \Gamma(-\frac{\beta}{2})}, \quad c \in \mathbb{C};$$

$$\text{or} \quad y = -2x \left(1 + \mathcal{O}(x^{-2})\right), \quad \text{formally corresponding to} \quad c = \infty;$$

$$ii) \quad \arg x \in \left(\frac{\pi}{4}; \frac{3\pi}{4}\right):$$

$$\text{if} \quad c + i\sqrt{\pi} \neq 0,$$

$$\text{then} \quad y = -2x \left(1 + \mathcal{O}(x^{-2})\right) + \mu_2 x^{4\alpha-\beta} e^{x^2} \left(1 + \mathcal{O}(x^{-2})\right), \quad (72)$$

$$\mu_2 = 2\sqrt{\pi} \left(\frac{i\sqrt{\pi}}{(c + i\sqrt{\pi})(e^{-i\pi\beta} - 1)} + \Theta(\arg x - \frac{\pi}{2}) \right) e^{-i\frac{\pi\beta}{2}} \frac{2^{2\alpha-\frac{\beta}{2}}}{n! \Gamma(\alpha + \frac{1}{2})},$$

$$\text{and if} \quad c + i\sqrt{\pi} = 0, \quad \text{then} \quad y = \frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right);$$

$$iii) \quad \arg x \in \left(\frac{3\pi}{4}; \frac{5\pi}{4}\right):$$

$$\text{if} \quad c(1 - e^{i\pi\beta}) + i\sqrt{\pi} \neq 0,$$

$$\text{then} \quad y = \frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right) + \mu_3 x^{2n-\beta} e^{-x^2} \left(1 + \mathcal{O}(x^{-2})\right), \quad (73)$$

$$\mu_3 = -2i\sqrt{\pi} \left(\frac{c + i\sqrt{\pi}}{c(1 - e^{i\pi\beta}) + i\sqrt{\pi}} - \Theta(\arg x - \pi) \right) e^{i\pi\beta} \frac{2^{n-\frac{\beta}{2}}}{n! \Gamma(-\frac{\beta}{2})},$$

$$\text{and if} \quad c(1 - e^{i\pi\beta}) + i\sqrt{\pi} = 0, \quad \text{then} \quad y = -2x \left(1 + \mathcal{O}(x^{-2})\right);$$

$$iv) \quad \arg x \in \left(\frac{5\pi}{4}; \frac{7\pi}{4}\right):$$

$$\text{if} \quad c \neq 0,$$

$$\text{then} \quad y = -2x \left(1 + \mathcal{O}(x^{-2})\right) + \mu_4 x^{4\alpha-\beta} e^{x^2} \left(1 + \mathcal{O}(x^{-2})\right), \quad (74)$$

$$\mu_4 = 2\sqrt{\pi} \left(\frac{i\sqrt{\pi}}{c(1 - e^{i\pi\beta})} + 1 - \Theta(\arg x - \frac{3\pi}{2}) \right) e^{-i\frac{3\pi\beta}{2}} \frac{2^{2\alpha-\frac{\beta}{2}}}{n! \Gamma(\alpha + \frac{1}{2})},$$

$$\text{and if} \quad c = 0, \quad \text{then} \quad y = \frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right).$$

As before, to get the asymptotics of the classical Painlevé functions for $\alpha - \beta/2 = -n - 1/2$, $n \in \mathbb{Z}_+$, it is enough to substitute ix , iy , $-\alpha$, $-\beta$ instead of x , y , α and β , respectively, in all the expressions above.

Next theorem follows by applying the Bäcklund transformation of the type R_∞^\pm to the previous asymptotics:

Theorem 5.3 *For the parameter values*

$$\beta = -2n, \quad n \in \mathbb{N}, \quad \alpha + \frac{1}{2} \notin \mathbb{Z}, \quad (75)$$

there exist the classical solutions described asymptotically by the equations:

$$\begin{aligned} i) \quad & \arg x \in (-\frac{\pi}{4}; \frac{\pi}{4}): \\ & y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})) + \mu_1 x^{-1+4n+2\alpha} e^{-x^2} (1 + \mathcal{O}(x^{-2})), \end{aligned} \quad (76)$$

$$\mu_1 = -2(c + i\sqrt{\pi} \Theta(\arg x)) \frac{2^{-\frac{1}{2}+2n+\alpha}}{(n-1)! \Gamma(\alpha + n + \frac{1}{2})}, \quad c \in \mathbb{C};$$

$$\text{or} \quad y = -\frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})), \quad \text{formally corresponding to} \quad c = \infty;$$

$$ii) \quad \arg x \in (\frac{\pi}{4}; \frac{3\pi}{4}):$$

$$\text{if} \quad c + i\sqrt{\pi} \neq 0,$$

$$\text{then} \quad y = -\frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})) + \mu_2 x^{2n-1-2\alpha} e^{x^2} (1 + \mathcal{O}(x^{-2})), \quad (77)$$

$$\mu_2 = -2i\sqrt{\pi} \left(\frac{i\sqrt{\pi}}{(c + i\sqrt{\pi})(e^{2i\pi\alpha} + 1)} - \Theta(\arg x - \frac{\pi}{2}) \right) \frac{(-1)^n 2^{n-\frac{1}{2}-\alpha} e^{i\pi\alpha}}{(n-1)! \Gamma(-\alpha + \frac{1}{2})},$$

$$\text{and if} \quad c + i\sqrt{\pi} = 0, \quad \text{then} \quad y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2}));$$

$$iii) \quad \arg x \in (\frac{3\pi}{4}; \frac{5\pi}{4}):$$

$$\text{if} \quad c(e^{-2i\pi\alpha} + 1) + i\sqrt{\pi} \neq 0,$$

$$\text{then} \quad y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})) + \mu_3 x^{4n-1+2\alpha} e^{-x^2} (1 + \mathcal{O}(x^{-2})), \quad (78)$$

$$\mu_3 = 2i\sqrt{\pi} \left(\frac{c + i\sqrt{\pi}}{c(e^{-2i\pi\alpha} + 1) + i\sqrt{\pi}} - \Theta(\arg x - \pi) \right) \frac{2^{2n-\frac{1}{2}+\alpha} e^{-2i\pi\alpha}}{(n-1)! \Gamma(\alpha + n + \frac{1}{2})},$$

$$\begin{aligned}
&\text{and if} \quad c(e^{-2i\pi\alpha} + 1) + i\sqrt{\pi} = 0, \quad \text{then} \quad y = -\frac{\beta}{2x}(1 + \mathcal{O}(x^{-2})); \\
&\text{iv)} \quad \arg x \in (\frac{5\pi}{4}; \frac{7\pi}{4}): \\
&\text{if} \quad c \neq 0, \\
&\text{then} \quad y = -\frac{\beta}{2x}(1 + \mathcal{O}(x^{-2})) + \mu_4 x^{2n-1-2\alpha} e^{x^2} (1 + \mathcal{O}(x^{-2})), \tag{79} \\
&\quad \mu_4 = -2i\sqrt{\pi} \left(\frac{i\sqrt{\pi}}{c(e^{-2i\pi\alpha} + 1)} + 1 - \Theta(\arg x - \frac{3\pi}{2}) \right) \frac{(-1)^n 2^{n-\frac{1}{2}-\alpha} e^{3\pi i\alpha}}{(n-1)! \Gamma(-\alpha + \frac{1}{2})}, \\
&\text{and if} \quad c = 0, \quad \text{then} \quad y = \frac{\beta}{2x}(1 + \mathcal{O}(x^{-2})).
\end{aligned}$$

As before, to get the asymptotics of the classical Painlevé functions for $\beta = 2n$, $n \in \mathbb{N}$, it is enough to substitute ix , iy , $-\alpha$, $-\beta$ instead of x , y , α and β , respectively, in all the expressions above.

The last we are going to do in this section is the description of limiting cases excluded from the theorems above. As easy to see, for any half-integer α and even β , there are two different 1-parameter families of the classical solutions. Namely,

Theorem 5.4 *If*

$$\alpha = \frac{1}{2} + n, \quad \beta = -2m, \quad n, m \in \mathbb{Z}_+, \tag{80}$$

then there exist two families of classical solutions:

1. *The first family solutions are described by the asymptotics*

$$\begin{aligned}
&i) \quad \arg x \in (-\frac{\pi}{4}; \frac{\pi}{4}): \\
&\quad y = -\frac{\beta}{2x}(1 + \mathcal{O}(x^{-2})) + \mu_1 x^{2n-2m} e^{-x^2} (1 + \mathcal{O}(x^{-2})), \tag{81} \\
&\quad \mu_1 = -c(-1)^m 2^{n-m+1} \frac{m!}{n!}, \quad c \in \mathbb{C};
\end{aligned}$$

$$\text{or} \quad y = -2x(1 + \mathcal{O}(x^{-2})), \quad \text{formally corresponding to} \quad c = \infty;$$

$$\begin{aligned}
&ii) \quad \arg x \in (\frac{\pi}{4}; \frac{3\pi}{4}): \\
&\text{if} \quad c \neq 0, \\
&\text{then} \quad y = -2x(1 + \mathcal{O}(x^{-2})) + \mu_2 x^{4n+2m+2} e^{x^2} (1 + \mathcal{O}(x^{-2})), \tag{82}
\end{aligned}$$

$$\begin{aligned}
& \mu_2 = \left(\frac{1}{c} + 2\sqrt{\pi} \Theta(\arg x - \frac{\pi}{2}) \right) (-1)^m \frac{2^{2n+m+1}}{n! (n+m)!}, \\
& \text{and if } c = 0, \quad \text{then } y = -\frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})); \\
& \text{iii) } \arg x \in (\frac{3\pi}{4}; \frac{5\pi}{4}): \\
& \text{if } 1 + 2\sqrt{\pi} c \neq 0, \\
& \text{then } y = -\frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})) + \mu_3 x^{2n-2m} e^{-x^2} (1 + \mathcal{O}(x^{-2})), \tag{83} \\
& \mu_3 = -\frac{c}{1 + 2\sqrt{\pi} c} (-1)^m 2^{n-m+1} \frac{m!}{n!}, \\
& \text{and if } 1 + 2\sqrt{\pi} c = 0, \quad \text{then } y = -2x (1 + \mathcal{O}(x^{-2})); \\
& \text{iv) } \arg x \in (\frac{5\pi}{4}; \frac{7\pi}{4}): \\
& \text{if } c \neq 0, \\
& \text{then } y = -2x (1 + \mathcal{O}(x^{-2})) + \mu_4 x^{4n+2m+2} e^{x^2} (1 + \mathcal{O}(x^{-2})), \tag{84} \\
& \mu_4 = \left(\frac{1}{c} + 2\sqrt{\pi} \Theta(\frac{3\pi}{2} - \arg x) \right) (-1)^m \frac{2^{2n+m+1}}{n! (n+m)!}, \\
& \text{and if } c = 0, \quad \text{then } y = -\frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})).
\end{aligned}$$

2. Solutions of the second family behave asymptotically as follows:

$$\begin{aligned}
& i) \quad \arg x \in (-\frac{\pi}{4}; \frac{\pi}{4}): \\
& y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})) + \mu_1 x^{2n+4m} e^{-x^2} (1 + \mathcal{O}(x^{-2})), \tag{85} \\
& \mu_1 = -\left(c + i\sqrt{\pi} \Theta(\arg x) \right) \frac{2^{n+2m+1}}{(n+m)! (m-1)!}, \quad c \in \mathbb{C}; \\
& \text{or } y = -\frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})), \quad \text{formally corresponding to } c = \infty; \\
& ii) \quad \arg x \in (\frac{\pi}{4}; \frac{3\pi}{4}): \\
& \text{if } c + i\sqrt{\pi} \neq 0, \\
& \text{then } y = -\frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})) + \mu_2 x^{2m-2n-2} e^{x^2} (1 + \mathcal{O}(x^{-2})), \tag{86}
\end{aligned}$$

$$\begin{aligned}
& \mu_2 = \frac{1}{c + i\sqrt{\pi}} (-1)^m 2^{m-n-1} \frac{n!}{(m-1)!}, \\
& \text{and if } c + i\sqrt{\pi} = 0, \quad \text{then } y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})); \\
& \text{iii)} \quad \arg x \in \left(\frac{3\pi}{4}; \frac{5\pi}{4}\right): \\
& \text{if } c \neq \infty, \\
& \text{then } y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})) + \mu_3 x^{2n+4m} e^{-x^2} (1 + \mathcal{O}(x^{-2})), \tag{87} \\
& \mu_3 = -\left(c + i\sqrt{\pi} \Theta(\pi - \arg x)\right) \frac{2^{n+2m+1}}{(n+m)! (m-1)!}, \\
& \text{and if } c = \infty, \quad \text{then } y = -\frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})); \\
& \text{iv)} \quad \arg x \in \left(\frac{5\pi}{4}; \frac{7\pi}{4}\right): \\
& \text{if } c \neq 0, \\
& \text{then } y = -\frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})) + \mu_4 x^{2m-2n-2} e^{x^2} (1 + \mathcal{O}(x^{-2})), \tag{88} \\
& \mu_4 = \frac{1}{c} (-1)^m 2^{m-n-1} \frac{n!}{(m-1)!}, \\
& \text{and if } c = 0, \quad \text{then } y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})).
\end{aligned}$$

In this case, the value $m = 0$ corresponds to the trivial solution $y \equiv 0$.

Theorem 5.5 *If*

$$\alpha = \frac{1}{2} + n, \quad \beta = 2m, \quad n, m \in \mathbb{Z}_+, \quad m \leq n, \tag{89}$$

then there exist two families of the classical solutions:

1. *The solutions of the first family are described by the asymptotic relations:*

$$\begin{aligned}
& \text{i)} \quad \arg x \in \left(-\frac{\pi}{4}; \frac{\pi}{4}\right): \\
& y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})) + \mu_1 x^{2n-4m} e^{-x^2} (1 + \mathcal{O}(x^{-2})), \tag{90} \\
& \mu_1 = -c(-1)^m 2^{n-2m+1} \frac{m!}{(n-m)!}, \quad c \in \mathbb{C};
\end{aligned}$$

$$\begin{aligned}
& \text{or} \quad y = -2x(1 + \mathcal{O}(x^{-2})), \quad \text{formally corresponding to} \quad c = \infty; \\
& \text{ii)} \quad \arg x \in (\frac{\pi}{4}; \frac{3\pi}{4}): \\
& \text{if} \quad c \neq 0, \\
& \text{then} \quad y = -2x(1 + \mathcal{O}(x^{-2})) + \mu_2 x^{4n-2m+2} e^{x^2} (1 + \mathcal{O}(x^{-2})), \quad (91) \\
& \quad \mu_2 = \left(\frac{1}{c} + 2\sqrt{\pi} \Theta(\arg x - \frac{\pi}{2}) \right) (-1)^m \frac{2^{2n-m+1}}{n! (n-m)!}, \\
& \text{and if} \quad c = 0, \quad \text{then} \quad y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})); \\
& \text{iii)} \quad \arg x \in (\frac{3\pi}{4}; \frac{5\pi}{4}): \\
& \text{if} \quad 1 + 2\sqrt{\pi} c \neq 0, \\
& \text{then} \quad y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})) + \mu_3 x^{2n-4m} e^{-x^2} (1 + \mathcal{O}(x^{-2})), \quad (92) \\
& \quad \mu_3 = -\frac{c}{1 + 2\sqrt{\pi} c} (-1)^m 2^{n-2m+1} \frac{m!}{(n-m)!}, \\
& \text{and if} \quad 1 + 2\sqrt{\pi} c = 0, \quad \text{then} \quad y = -2x(1 + \mathcal{O}(x^{-2})); \\
& \text{iv)} \quad \arg x \in (\frac{5\pi}{4}; \frac{7\pi}{4}): \\
& \text{if} \quad c \neq 0, \\
& \text{then} \quad y = -2x(1 + \mathcal{O}(x^{-2})) + \mu_4 x^{4n-2m+2} e^{x^2} (1 + \mathcal{O}(x^{-2})), \quad (93) \\
& \quad \mu_4 = \left(\frac{1}{c} + 2\sqrt{\pi} \Theta(\frac{3\pi}{2} - \arg x) \right) (-1)^m \frac{2^{2n-m+1}}{n! (n-m)!}, \\
& \text{and if} \quad c = 0, \quad \text{then} \quad y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})).
\end{aligned}$$

2. The solutions of the second family are as follows:

$$\begin{aligned}
& \text{i)} \quad \arg x \in (-\frac{\pi}{4}; \frac{\pi}{4}): \\
& y = -\frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})) + \mu_1 x^{2n+2m} e^{-x^2} (1 + \mathcal{O}(x^{-2})), \quad (94) \\
& \mu_1 = -\left(c + i\sqrt{\pi} \Theta(\arg x) \right) \frac{2^{n+m+1}}{n! (m-1)!}, \quad c \in \mathbb{C};
\end{aligned}$$

$$\begin{aligned}
& \text{or} \quad y = \frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right), \quad \text{formally corresponding to} \quad c = \infty; \\
& ii) \quad \arg x \in \left(\frac{\pi}{4}; \frac{3\pi}{4}\right): \\
& \text{if} \quad c + i\sqrt{\pi} \neq 0, \\
& \text{then} \quad y = \frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right) + \mu_2 x^{4m-2n-2} e^{x^2} \left(1 + \mathcal{O}(x^{-2})\right), \quad (95) \\
& \mu_2 = \frac{1}{c + i\sqrt{\pi}} (-1)^m 2^{2m-n-1} \frac{(n-m)!}{(m-1)!}, \\
& \text{and if} \quad c + i\sqrt{\pi} = 0, \quad \text{then} \quad y = -\frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right);
\end{aligned}$$

$$\begin{aligned}
& iii) \quad \arg x \in \left(\frac{3\pi}{4}; \frac{5\pi}{4}\right): \\
& \text{if} \quad c \neq \infty, \\
& \text{then} \quad y = -\frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right) + \mu_3 x^{2n+2m} e^{-x^2} \left(1 + \mathcal{O}(x^{-2})\right), \quad (96) \\
& \mu_3 = -\left(c + i\sqrt{\pi} \Theta(\pi - \arg x)\right) \frac{2^{n+m+1}}{n! (m-1)!},
\end{aligned}$$

$$\begin{aligned}
& \text{and if} \quad c = \infty, \quad \text{then} \quad y = \frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right); \\
& iv) \quad \arg x \in \left(\frac{5\pi}{4}; \frac{7\pi}{4}\right): \\
& \text{if} \quad c \neq 0, \\
& \text{then} \quad y = \frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right) + \mu_4 x^{4m-2n-2} e^{x^2} \left(1 + \mathcal{O}(x^{-2})\right), \quad (97) \\
& \mu_4 = \frac{1}{c} (-1)^m 2^{2m-n-1} \frac{(n-m)!}{(m-1)!}, \\
& \text{and if} \quad c = 0, \quad \text{then} \quad y = -\frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right).
\end{aligned}$$

In this case, the value $m = 0$ corresponds to the trivial solution $y \equiv 0$.

Theorem 5.6 *If*

$$\alpha = \frac{1}{2} + n, \quad \beta = 2m, \quad n, m \in \mathbb{Z}_+, \quad m \geq n + 1, \quad (98)$$

then there exist two families of the classical solutions:

1. The solutions of the first family are given by the asymptotic relations:

$$\begin{aligned}
 i) \quad & \arg x \in \left(-\frac{\pi}{4}; \frac{\pi}{4}\right): \\
 & y = -\frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right) + \mu_1 x^{2n+2m} e^{-x^2} \left(1 + \mathcal{O}(x^{-2})\right), \quad (99)
 \end{aligned}$$

$$\mu_1 = -\left(c + i\sqrt{\pi} \Theta(\arg x)\right) \frac{2^{n+m+1}}{n! (m-1)!}, \quad c \in \mathbb{C};$$

$$\text{or} \quad y = -2x \left(1 + \mathcal{O}(x^{-2})\right), \quad \text{formally corresponding to} \quad c = \infty;$$

$$ii) \quad \arg x \in \left(\frac{\pi}{4}; \frac{3\pi}{4}\right):$$

$$\text{if} \quad c + i\sqrt{\pi} \neq 0,$$

$$\text{then} \quad y = -2x \left(1 + \mathcal{O}(x^{-2})\right) + \mu_2 x^{4n-2m+2} e^{x^2} \left(1 + \mathcal{O}(x^{-2})\right), \quad (100)$$

$$\mu_2 = \frac{1}{c + i\sqrt{\pi}} (-1)^n 2^{2n-m+1} \frac{(m-n-1)!}{n!},$$

$$\text{and if} \quad c + i\sqrt{\pi} = 0, \quad \text{then} \quad y = -\frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right);$$

$$iii) \quad \arg x \in \left(\frac{3\pi}{4}; \frac{5\pi}{4}\right):$$

$$\text{if} \quad c \neq \infty,$$

$$\text{then} \quad y = -\frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right) + \mu_3 x^{2n+2m} e^{-x^2} \left(1 + \mathcal{O}(x^{-2})\right), \quad (101)$$

$$\mu_3 = -\left(c + i\sqrt{\pi} \Theta(\pi - \arg x)\right) \frac{2^{n+m+1}}{n! (m-1)!},$$

$$\text{and if} \quad c = \infty, \quad \text{then} \quad y = -2x \left(1 + \mathcal{O}(x^{-2})\right);$$

$$iv) \quad \arg x \in \left(\frac{5\pi}{4}; \frac{7\pi}{4}\right):$$

$$\text{if} \quad c \neq 0,$$

$$\text{then} \quad y = -2x \left(1 + \mathcal{O}(x^{-2})\right) + \mu_4 x^{4n-2m+2} e^{x^2} \left(1 + \mathcal{O}(x^{-2})\right), \quad (102)$$

$$\mu_4 = \frac{1}{c} (-1)^n 2^{2n-m+1} \frac{(m-n-1)!}{n!},$$

$$\text{and if} \quad c = 0, \quad \text{then} \quad y = -\frac{\beta}{2x} \left(1 + \mathcal{O}(x^{-2})\right).$$

2. The solutions of the second family are as follows:

$$\begin{aligned}
i) \quad & \arg x \in \left(-\frac{\pi}{4}; \frac{\pi}{4}\right): \\
& y = -2x(1 + \mathcal{O}(x^{-2})) + \mu_1 x^{2m-4n-2} e^{-x^2} (1 + \mathcal{O}(x^{-2})), \\
& \mu_1 = c(-1)^n 2^{m-2n} \frac{n!}{(m-n-1)!}, \quad c \in \mathbb{C},
\end{aligned} \tag{103}$$

$$\text{or} \quad y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})), \quad \text{formally corresponding to } c = \infty;$$

$$\begin{aligned}
ii) \quad & \arg x \in \left(\frac{\pi}{4}; \frac{3\pi}{4}\right): \\
& \text{if } c \neq 0, \\
\text{then} \quad & y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})) + \mu_2 x^{4m-2n-2} e^{x^2} (1 + \mathcal{O}(x^{-2})), \\
& \mu_2 = -\left(\frac{1}{c} + 2\sqrt{\pi} \Theta(\arg x - \frac{\pi}{2})\right) (-1)^n \frac{2^{2m-n-1}}{(m-1)!(m-n-1)!},
\end{aligned} \tag{104}$$

$$\text{and if } c = 0, \quad \text{then } y = -2x(1 + \mathcal{O}(x^{-2}));$$

$$\begin{aligned}
iii) \quad & \arg x \in \left(\frac{3\pi}{4}; \frac{5\pi}{4}\right): \\
& \text{if } 1 + 2\sqrt{\pi} c \neq 0, \\
\text{then} \quad & y = -2x(1 + \mathcal{O}(x^{-2})) + \mu_3 x^{2m-4n-2} e^{-x^2} (1 + \mathcal{O}(x^{-2})), \\
& \mu_3 = \frac{c}{1 + 2\sqrt{\pi} c} (-1)^n 2^{m-2n} \frac{n!}{(m-n-1)!},
\end{aligned} \tag{105}$$

$$\text{and if } 1 + 2\sqrt{\pi} c = 0, \quad \text{then } y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2}));$$

$$\begin{aligned}
iv) \quad & \arg x \in \left(\frac{5\pi}{4}; \frac{7\pi}{4}\right): \\
& \text{if } c \neq 0, \\
\text{then} \quad & y = \frac{\beta}{2x} (1 + \mathcal{O}(x^{-2})) + \mu_4 x^{4m-2n-2} e^{x^2} (1 + \mathcal{O}(x^{-2})), \\
& \mu_4 = -\left(\frac{1}{c} + 2\sqrt{\pi} \Theta\left(\frac{3\pi}{2} - \arg x\right)\right) (-1)^n \frac{2^{2m-n-1}}{(m-n-1)!(m-1)!}, \\
\text{and if } c = 0, \quad & \text{then } y = -2x(1 + \mathcal{O}(x^{-2})).
\end{aligned} \tag{106}$$

As easy to see, Theorems 5.4, 5.5 and 5.6 allow some classical solutions to have

uniform asymptotics of the type $\pm\beta/2x$ or $-2x$ on the x complex plane. In fact, these special cases correspond to the rational solutions of the equation P4 described above. Furthermore, using these theorems, we can specify the parameters values which allow the rational solutions to exist:

- a) the parameters $\alpha = \pm(n + \frac{1}{2})$, $\beta = \mp 2m$, $n, m \in \mathbb{Z}_+$, allow the rational solutions with the asymptotics $y = -\frac{\beta}{2x} + \mathcal{O}(x^{-3}) = \pm\frac{m}{x} + \mathcal{O}(x^{-3})$ to exist;
- b) the parameters $\alpha = \pm(n + \frac{1}{2})$, $\beta = \pm 2m$, $n, m \in \mathbb{Z}_+$, $m \leq n$, allow the rational solutions with the asymptotics $y = \frac{\beta}{2x} + \mathcal{O}(x^{-3}) = \pm\frac{m}{x} + \mathcal{O}(x^{-3})$ to exist;
- c) the parameters $\alpha = \pm(n + \frac{1}{2})$, $\beta = \pm 2m$, $n, m \in \mathbb{Z}_+$, $m \geq n + 1$, allow the rational solutions with the asymptotics $y = -2x + \mathcal{O}(x^{-1})$ to exist.

6 Transcendent degenerated Painlevé functions

All the classical solutions are *truncated* in the interior of all of the complex sectors $\arg x \in (\frac{\pi}{4}(m-1) + \frac{\pi}{2}n; \frac{\pi}{4}m + \frac{\pi}{2}n)$, $\forall n, m: n \in \mathbb{Z}, m \in \{0; 1\}$. In fact, this is the characteristic property of the classical Painlevé functions. Indeed, it is shown in the work [26] that in the case of general position distinguished by the inequality $s_{2+n}(s_{1+n} + s_{3+n} + s_{1+n}s_{2+n}s_{3+n})(1 + s_{1+n+m}s_{2+n+m}) \neq 0$, the Painlevé function is described asymptotically by means of some elliptic function of the periods depending on $\arg x$ in the interior of the indicated sector. Hence, the condition for the non-elliptic asymptotic behavior in the interior of any such sector is reduced to the system

$$s_{2+n}(s_{1+n} + s_{3+n} + s_{1+n}s_{2+n}s_{3+n})(1 + s_{1+n+m}s_{2+n+m}) = 0 \quad \forall n, m: n \in \mathbb{Z}, m \in \{0; 1\}.$$

Direct checking shows that the last system of equations leads either to the constraint (19), or to (30), or to $s_1 = s_3 = 0$, or to $s_2 = s_4 = 0$, any of which corresponds to a classical Painlevé function (see above).

In contrast to the case of the classical solutions, the Painlevé function which has the elliptic asymptotics inside at least one of the sectors can not be expressed via the classical special functions and should be called transcendent. The function is parameterized by the Stokes multipliers s_k , and the monodromy surface equation (18) with the symmetry (15) together yields the connection formulae for it.

Besides the asymptotic solutions of general position for any $\arg x$, the work [26] contains description of some transcendent degenerated cases corresponding to the following Stokes multiplier values (for simplicity, we omit here the number shift n): 1) $s_2 = 0$, $s_1 + s_3 \neq 0$; 2) $s_2 \neq 0$, $s_1 + s_3 + s_1 s_2 s_3 = 0$, $\frac{1}{2} + \alpha \notin \mathbb{Z}$; and 3) $1 + s_2 s_3 = 0$. Below, the rest of the transcendent degenerated cases is described. As in ref. [26], the results are formulated for any of the opened sectors $\arg x \in (-\frac{\pi}{4} + \frac{\pi}{2}n; \frac{\pi}{4} + \frac{\pi}{2}n)$, $n \in \mathbb{Z}$.

Let us begin with the case

$$s_2 = s_1 + s_3 = 0. \quad (107)$$

The equation (107) with the equation of the monodromy surface (18) imply the restriction $\cos \pi(\alpha - \frac{\beta}{2}) \sin \pi \frac{\beta}{2} = 0$, i.e. β is even or $2\alpha - \beta$ is odd. Theorem 6.1 below is obtained in the paper [27] for the case (107) with the additional conditions $\beta = 0$ and $\alpha + \frac{1}{2} \notin \mathbb{N}$ for $x \rightarrow +\infty$. Below, using the symmetry (23), (24) we give its elementary generalization.

Theorem 6.1 *If $x \rightarrow \infty$, $\arg(x) \in (-\frac{\pi}{4} + \frac{\pi}{2}n; \frac{\pi}{4} + \frac{\pi}{2}n)$ for some $n \in \mathbb{Z}$, and*

$$s_{2+n} = s_{1+n} + s_{3+n} = 0, \quad \beta = 0, \quad (-1)^n \alpha + \frac{1}{2} \notin \mathbb{N}, \quad (108)$$

then the corresponding solution $y(x)$ of the fourth Painlevé equation P_4 possesses the following asymptotic behavior:

$$y = (-1)^n \frac{s_{1+n} s_{4+n}}{\pi^{3/2}} e^{-i\pi(1+n)(-1)^n \alpha} \Gamma\left(\frac{1}{2} - (-1)^n \alpha\right) 2^{(-1)^n \alpha - \frac{3}{2}} \times \\ \times x^{2(-1)^n \alpha - 1} e^{-(-1)^n x^2} (1 + \mathcal{O}(x^{-1})). \quad (109)$$

Theorem 6.1 is proved in [27] by means of direct asymptotic investigation of the Riemann-Hilbert problem for the function $\Psi(\lambda)$ and affirms the existence of the genuine Painlevé function described asymptotically via the formal expression (52), (56) for y_0 with the main term (109) depending on the Stokes multipliers. Next Theorem 6.2 follows from Theorem 6.1 after applying the Bäcklund transformations (26)–(29) in the way described in the previous section.

Theorem 6.2 *Let $x \rightarrow \infty$, $\arg(x) \in (-\frac{\pi}{4} + \frac{\pi}{2}n; \frac{\pi}{4} + \frac{\pi}{2}n)$ for some $n \in \mathbb{Z}$, and*

$$s_{2+n} = s_{1+n} + s_{3+n} = 0. \quad (110)$$

(i) Let

$$(-1)^n \beta = 2k, \quad k \in \mathbb{Z}_+, \quad (-1)^n \alpha + \frac{1}{2} - k \notin \mathbb{N}, \quad (111)$$

then the corresponding solution $y(x)$ of the fourth Painlevé equation P_4 possesses the following asymptotic behavior:

$$\begin{aligned} y &= e^{i\frac{\pi}{2}n} y_+(e^{-i\frac{\pi}{2}n} x, (-1)^n a, (-1)^n \beta, c_n) = \\ &= \frac{\beta}{2x} + \frac{\beta(2\alpha - \frac{3\beta}{2})}{4x^3} + \mathcal{O}\left(\frac{1}{x^5}\right) + \\ &+ (-1)^n \frac{s_{1+n} s_{4+n}}{\pi^{3/2}} e^{-i\pi(n+1)(-1)^n \alpha} \Gamma\left(\frac{1}{2} + k - (-1)^n \alpha\right) \times \\ &\times 2^{(-1)^n \alpha - \frac{3}{2} - 2k} k! x^{2(-1)^n \alpha - 1 - 4k} e^{-(-1)^n x^2} (1 + \mathcal{O}(x^{-2})); \end{aligned} \quad (112)$$

(ii) Let

$$(-1)^n \beta = -2k, \quad k \in \mathbb{Z}_+, \quad (-1)^n \alpha + \frac{1}{2} \notin \mathbb{N}, \quad (113)$$

then the corresponding solution $y(x)$ of the fourth Painlevé equation P_4 possesses the following asymptotic behavior:

$$\begin{aligned} y &= e^{i\frac{\pi}{2}n} y_-(e^{-i\frac{\pi}{2}n} x, (-1)^n a, (-1)^n \beta, c_n) = \\ &= -\frac{\beta}{2x} - \frac{\beta(2\alpha + \frac{\beta}{2})}{4x^3} + \mathcal{O}\left(\frac{1}{x^5}\right) + \\ &+ (-1)^{n+k+nk} \frac{s_{1+n} s_{4+n}}{\pi^{3/2}} e^{-i\pi(n+1)(-1)^n \alpha} \Gamma\left(\frac{1}{2} - (-1)^n \alpha\right) \times \\ &\times 2^{(-1)^n \alpha - \frac{3}{2} - k} k! x^{2(-1)^n \alpha - 1 - 2k} e^{-(-1)^n x^2} (1 + \mathcal{O}(x^{-2})); \end{aligned} \quad (114)$$

(iii) Let

$$(-1)^n (2\alpha - \beta) = 2l - 1, \quad l \in \mathbb{N}, \quad 1 - l - (-1)^n \frac{\beta}{2} \notin \mathbb{N}, \quad (115)$$

then the corresponding solution $y(x)$ of the fourth Painlevé equation P_4 possesses the following asymptotic behavior:

$$\begin{aligned} y &= e^{i\frac{\pi}{2}n} y_2(e^{-i\frac{\pi}{2}n} x, (-1)^n a, b, c_n) = \\ &= -2x - \frac{2\alpha - \frac{\beta}{2}}{x} + \mathcal{O}\left(\frac{1}{x^3}\right) - \\ &- \frac{s_{1+n} s_{4+n}}{\pi^{3/2}} e^{i\frac{\pi}{2}(n+1)(-1)^n \beta + i\frac{\pi}{2}(1-n)} \Gamma\left(l + (-1)^n \frac{\beta}{2}\right) \times \\ &\times 2^{(-1)^n \frac{\beta}{2} - 2l} (l-1)! x^{-(1)^n \beta + 2 - 4l} e^{-(-1)^n x^2} (1 + \mathcal{O}(x^{-2})); \end{aligned} \quad (116)$$

(iv) Let

$$(-1)^n(2\alpha - \beta) = 1 - 2l, \quad l \in \mathbb{N}, \quad -(-1)^n \frac{\beta}{2} \notin \mathbb{N}, \quad (117)$$

then the corresponding solution $y(x)$ of the fourth Painlevé equation P_4 possesses the following asymptotic behavior:

$$\begin{aligned} y &= e^{i\frac{\pi}{2}n} y_+(e^{-i\frac{\pi}{2}n} x, (-1)^n a, (-1)^n \beta, c_n) = \\ &= \frac{\beta}{2x} + \frac{\beta(2\alpha - \frac{3\beta}{2})}{4x^3} + \mathcal{O}\left(\frac{1}{x^5}\right) + \\ &+ (-1)^{l(n-1)} \frac{s_{1+n} s_{4+n}}{\pi^{3/2}} e^{i\frac{\pi}{2}(n+1)(-1)^n \beta + i\frac{\pi}{2}(n-1)} \Gamma\left(1 + (-1)^n \frac{\beta}{2}\right) \times \\ &\times 2^{-(-1)^n \frac{\beta}{2} - 1 - l} (l-1)! x^{-(-1)^n \beta - 2l} e^{-(-1)^n x^2} (1 + \mathcal{O}(x^{-2})). \end{aligned} \quad (118)$$

Proof. Application of any one of the Bäcklund transformations to the genuine solution of P_4 generates another genuine solution of P_4 with the different parameters a , b , or, equivalently, α , β . For $n = 0$, applying (57)–(60) to the “seed” solution y_0 (109), we check the resulting asymptotic solutions are of the type y_2 , y_+ given by (116) and (118). Applying the same Bäcklund transformations to the asymptotic formulae (112)–(118), we see they transform into each other. The last gives the assertion of the theorem for $n = 0$, i.e. for the sector $\arg x \in (-\frac{\pi}{4}; \frac{\pi}{4})$. To complete the proof, it is enough to use the rotation symmetry (24).

The cases excluded from Theorems 6.1 and 6.2 above correspond to the linear submanifolds of the monodromy surface (18), i.e. to the classical solutions of P_4 described above.

To complete the description of the class of the degenerated solutions, let us present the assertions concerning “less degenerated” solutions from the article [26].

Theorem 6.3 *If $x \rightarrow \infty$, $\arg x \in (-\frac{\pi}{4} + \frac{\pi}{2}n; \frac{\pi}{4} + \frac{\pi}{2}n)$ for some $n \in \mathbb{Z}$, and*

$$s_{2+n} = 0, \quad s_{1+n} + s_{3+n} \neq 0, \quad (119)$$

then

$$\begin{aligned} y(x) &= e^{i\frac{\pi}{2}n} y_+(e^{-i\frac{\pi}{2}n} x, (-1)^n a, (-1)^n \beta, c) = \\ &= \frac{\beta}{2x} + \mathcal{O}(x^{-3}) + ax^{-1+2(-1)^n(\alpha-\beta)} e^{-(-1)^n x^2} (1 + \mathcal{O}(x^{-2})), \end{aligned} \quad (120)$$

where

$$a = (-1)^n e^{-i\pi n(-1)^n(\alpha-\beta)} \frac{2^{\frac{1}{2}+(-1)^n(\alpha-\beta)} \sqrt{\pi}}{\Gamma(\frac{1}{2} + (-1)^n(\alpha - \frac{\beta}{2})) \Gamma(-(-1)^n \frac{\beta}{2})} \times \\ \times \frac{i}{s_{1+n} + s_{3+n}} \left(\Theta(\frac{\pi}{2}n - \arg x) s_{1+n} - \Theta(\arg x - \frac{\pi}{2}n) s_{3+n} \right).$$

Before the next theorem will be formulated, it is necessary to note that the assumption

$$s_{1+n} + s_{3+n} + s_{1+n}s_{2+n}s_{3+n} = 0 \quad (121)$$

with the equation of the monodromy surface (18) together implies that

$$1 + s_{2+n}s_{3+n} = e^{i\pi(-1)^n\beta} \quad (122)$$

or

$$1 + s_{2+n}s_{3+n} = -e^{-i\pi(-1)^n(2\alpha-\beta)} \quad (123)$$

Theorem 6.4 *If $x \rightarrow \infty$, $\arg x \in (-\frac{\pi}{4} + \frac{\pi}{2}n; \frac{\pi}{4} + \frac{\pi}{2}n)$ for some $n \in \mathbb{Z}$, the equations (121), (122) hold, and*

$$s_{2+n} \neq 0, \quad \frac{1}{2} + (-1)^n\alpha \notin \mathbb{N}, \quad (124)$$

then

$$y = e^{i\frac{\pi}{2}n} y_-(e^{-i\frac{\pi}{2}n} x, (-1)^n a, (-1)^n \beta, c_n) = \\ -\frac{\beta}{2x} + \mathcal{O}(x^{-3}) + cx^{-1+2(-1)^n(\alpha+\frac{\beta}{2})} e^{-(1)^n x^2} (1 + \mathcal{O}(x^{-2})), \quad (125)$$

where

$$c = (-1)^n e^{-i\pi n(-1)^n(\alpha+\frac{\beta}{2})} 2^{-\frac{1}{2}+(-1)^n(\alpha+\frac{\beta}{2})} i \frac{\Gamma(\frac{1}{2} - (-1)^n\alpha)}{\sqrt{\pi} \Gamma((-1)^n \frac{\beta}{2})} \times \\ \times \frac{1}{s_{2+n}} \left\{ \Theta(\frac{\pi}{2}n - \arg x) e^{i\pi(-1)^n\alpha} (s_n + s_{2+n} + s_n s_{1+n} s_{2+n}) - \right. \\ \left. - \Theta(\arg x - \frac{\pi}{2}n) e^{-i\pi(-1)^n\alpha} (s_{2+n} + s_{4+n} + s_{2+n} s_{3+n} s_{4+n}) \right\}. \quad (126)$$

Theorem 6.5 *If $x \rightarrow \infty$, $\arg x \in (-\frac{\pi}{4} + \frac{\pi}{2}n; \frac{\pi}{4} + \frac{\pi}{2}n)$, for some $n \in \mathbb{Z}$, the equations (121), (123) hold, and*

$$s_{2+n} \neq 0, \quad \frac{1}{2} - (-1)^n\alpha \notin \mathbb{N}, \quad (127)$$

then

$$\begin{aligned} y &= e^{i\frac{\pi}{2}n} y_2(e^{-i\frac{\pi}{2}n} x, (-1)^n a, (-1)^n b, c_n) = \\ &= -2x + \mathcal{O}(x^{-1}) - cx^{-(1)^n(4\alpha-\beta)} e^{-(-1)^n x^2} (1 + \mathcal{O}(x^{-2})), \end{aligned} \quad (128)$$

where

$$\begin{aligned} c &= 2^{-(-1)^n(2\alpha-\frac{\beta}{2})} e^{i\pi n(\frac{1}{2}+(-1)^n(2\alpha-\frac{\beta}{2}))} i \frac{\Gamma(\frac{1}{2} + (-1)^n \alpha)}{\sqrt{\pi} \Gamma(\frac{1}{2} - (-1)^n(\alpha - \frac{\beta}{2}))} \times \\ &\times \frac{1}{s_{2+n}} \left\{ \Theta\left(\frac{\pi}{2}n - \arg x\right) e^{-i\pi(-1)^n \alpha} (s_n + s_{2+n} + s_n s_{1+n} s_{2+n}) - \right. \\ &\left. - \Theta\left(\arg x - \frac{\pi}{2}n\right) e^{i\pi(-1)^n \alpha} (s_{2+n} + s_{4+n} + s_{2+n} s_{3+n} s_{4+n}) \right\}. \end{aligned}$$

Because the equation

$$s_{1+n} + s_{3+n} + s_{1+n} s_{2+n} s_{3+n} = 0 \quad (129)$$

with the equation of the monodromy surface (18) and the condition $\alpha - \frac{1}{2} \in \mathbb{Z}$ together imply

$$1 + s_{2+n} s_{3+n} = e^{i\pi(-1)^n \beta}, \quad 1 + s_{1+n} s_{2+n} = e^{-i\pi(-1)^n \beta}, \quad s_{3+n} + s_{1+n} e^{i\pi(-1)^n \beta} = 0,$$

it is important to formulate assertion for this limiting case of the Theorems 6.4 and 6.5.

Theorem 6.6 *Let $x \rightarrow \infty$, $\arg x \in (-\frac{\pi}{4} + \frac{\pi}{2}n; \frac{\pi}{4} + \frac{\pi}{2}n)$ for some $n \in \mathbb{Z}$, and*

$$s_{2+n} \neq 0, \quad s_{1+n} + s_{3+n} + s_{1+n} s_{2+n} s_{3+n} = 0; \quad (130)$$

(i) if

$$(-1)^n \alpha = \frac{1}{2} - k, \quad k \in \mathbb{N}, \quad (131)$$

then

$$\begin{aligned} y &= e^{i\frac{\pi}{2}n} y_-(e^{-i\frac{\pi}{2}n} x, (-1)^n a, (-1)^n \beta, c_n) = -\frac{\beta}{2x} + \mathcal{O}\left(\frac{1}{x^3}\right) - \\ &- e^{i\frac{\pi}{2}n} (-1)^{k(1-n)} \left(\frac{s_{4+n}}{s_{2+n}} + e^{-i\pi(-1)^n \beta} \right) \frac{e^{-i\frac{\pi}{2}(-1)^n \beta(n-2)}}{\sqrt{\pi} \Gamma((-1)^n \frac{\beta}{2})} \times \\ &\times 2^{-k+(-1)^n \frac{\beta}{2}} (k-1)! x^{-2k+(-1)^n \beta} e^{-(-1)^n x^2} \left(1 + \mathcal{O}\left(\frac{1}{x^2}\right) \right); \end{aligned}$$

(ii) if

$$(-1)^n \alpha = -\frac{1}{2} + l, \quad l \in \mathbb{N}, \quad (132)$$

then

$$\begin{aligned} y = e^{i\frac{\pi}{2}n} y_2(e^{-i\frac{\pi}{2}n} x, (-1)^n a, b, c_n) = -2x + \mathcal{O}\left(\frac{1}{x}\right) + \\ + e^{-i\frac{\pi}{2}n} (-1)^l \left(\frac{s_{4+n}}{s_{2+n}} + e^{-i\pi(-1)^n \beta} \right) \frac{e^{-i\frac{\pi}{2}(-1)^n \beta(n-2)}}{\sqrt{\pi} \Gamma((-1)^n \frac{\beta}{2} + 1 - l)} \times \\ \times 2^{-2l+1+(-1)^n \frac{\beta}{2}} (l-1)! x^{-4l+2+(-1)^n \beta} e^{-(-1)^n x^2} \left(1 + \mathcal{O}\left(\frac{1}{x^2}\right) \right). \end{aligned} \quad (133)$$

Acknowledgments. The author thanks the Russian Foundation for Fundamental Investigations for its support under grant number 96-01-00668.

References

- [1] E. L. INCE, *Ordinary differential equations*, New York: Dover, 1959.
- [2] M. J. ABLOWITZ, A. RAMANI AND H. SEGUR, A connection between nonlinear evolution equations and ordinary differential equations of p-type. *J. Math. Phys.* **21** (1980) 1006-1015.
- [3] P. WINTERITZ, Physical applications of Painlevé type equations quadratic in the highest derivative. — in: *Painlevé Transcendents, Their Asymptotics and Physical Applications*, (P. Winteritz and D. Levi, eds.), NATO ASI Series B, Plenum, New York, 1992, 425-431.
- [4] M. FLORJAŃCZYK AND L. GAGNON, Exact solutions for a higher-order nonlinear Schrödinger equation, *Phys. Rev. A* **41** (1990) 4478-4485.
- [5] A. S. FOKAS, A. R. ITS AND A. V. KITAEV, Discrete Painlevé equations and their appearance in quantum gravity, — *Comm. Math. Phys.* **142** (1991) 313-344.
- [6] A. B. SHABAT AND A. P. VESELOV, Dressing chain and the spectral theory of the Schrödinger operator. *Functional Analysis and Its Applications*, **27** # 2 (1993) 1 (in Russian).
- [7] N. A. LUKASHEVICH, Theory of the fourth Painlevé equation. *Diff. Eqns.*, **3** (1967) 395-399.
- [8] H. AIRAULT, Rational solutions of Painlevé equations. *Stud. Appl. Math.* **61** (1979) 31-53.
- [9] V. I. GROMAK AND N. A. LUKASHEVICH, Special classes of solutions of Painlevé equations. *Diff. Eqns.* **18** (1982) 317-326.
- [10] H. UMEMURA AND H. WATANABE, Solutions of the second and fourth Painlevé equations, I. *Nagoya Math. J.* **148** (1997) 151-198.

- [11] V. I. GROMAK, On the transcendence of the Painlevé equations. *Differ. Uravnen.* **32** (1996) 154-160 (in Russian).
- [12] A. P. BASSOM, P. A. CLARKSON AND A. C. HICKS, Bäcklund transformations and solution hierarchies for the fourth Painlevé equation, *Stud. Appl. Math.* **95** (1995) 1-71.
- [13] R. FUCHS, Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen, *Math. Ann.* **63** (1907) 301-321.
- [14] L. SCHLESINGER, Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten. *J. R. Angew. Math.* **141** (1912) 96-145.
- [15] R. GARNIER, Sur les équationes différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équationnes nouvelles d'ordre supérieur dont l'intégrale générale a ses points critique fixes. *Ann. Sci. Ec. Norm. Supér.* **29** (1912) 1-126.
- [16] H. FLASCHKA AND A. C. NEWELL, Monodromy- and spectrum-preserving deformations I. *Commun. Math. Phys.* **76** (1980) 65-116.
- [17] M. JIMBO AND T. MIWA, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, II. *Physica* **2 D** (1981) 407-448.
- [18] A. R. ITS AND V. YU. NOVOKSHENOV, The Isomonodromic Deformation Method in the Theory of Painlevé Equations, *Lect. Not. Math.* **1181** 1986.
- [19] A. S. FOKAS, U. MUGAN AND M. J. ABLOWITZ, A method of linearization for Painlevé equations: Painlevé IV, V. *Physica* **30 D** (1988) 247-283.
- [20] A. S. FOKAS AND X. ZHOU, On the Solvability of Painlevé II and IV. *Commun. Math. Phys.* **144** (1992) 601-622.
- [21] A. V. KITAEV, On the self-similar solutions of the modified nonlinear Schrödinger equation. *Teor. Mat. Fizika*, **64** (1985) 347-369 (in Russian).

- [22] D. J. KAUP AND A. C. NEWELL, An exact solution for a derivative nonlinear Schrödinger equation. *J. Math. Phys.* **19** (1978) 798-801.
- [23] A. V. KITAEV, Asymptotic Description of the Fourth Painlevé Equation Solutions on the Stokes Rays Analogies. – *Zap. Nauch. Semin. LOMI*, **169**, (1988) 84 (in Russian).
- [24] A. E. MILNE, P. A. CLARKSON AND A. P. BASSOM, Application of the isomonodromy deformation method to the fourth Painlevé equation. *Inverse Problems* **13** (1997) 421-439.
- [25] A. P. BASSOM, P. A. CLARKSON, A. C. HICKS AND J. B. MCLEOD, Integral equations and exact solutions for the fourth Painlevé equation. *Proc. Royal Soc. London A* **437** (1992) 1-24.
- [26] A. A. KAPAEV, Global asymptotics of the fourth Painlevé transcendent. Steklov Math. Inst. and IUPUI Preprint # 96-5, 1996.
- [27] A. R. ITS AND A. A. KAPAEV, Connection formulae for Clarkson-McLeod solution of the fourth Painlevé equation. *J. Phys. A: Math. Gen.* **31** (1998) 1-41.
- [28] YOSHIHIRO MURATA, Rational solutions of the second and the fourth Painlevé equation, *Funkcial. Ekvac.* **28** (1985) 878-894.
- [29] A. S. FOKAS AND M. J. ABLOWITZ, On unified approach to transformation and elementary solutions of Painlevé equations. *J. Math. Phys.* **23** (1982) 2033-2042.
- [30] V. I. GROMAK, Theory of the fourth Painlevé equation. *Diff. Eqns.*, **23** (1987) 506-513.
- [31] H. BATEMAN, A. ERDELYI, Higher Transcendental Functions, McGraw-Hill, NY, 1953.
- [32] M. V. BERRY, Uniform asymptotic smoothing of Stokes discontinuities. *Proc. R. Soc. London A* **422** (1989) 7-21.